Mathematical Foundations for Probabilistic and Fuzzy Data Analysis

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Chapter 1.

Sentences

1.1. Basic Characteristics of Sentences

Colloquially, we observe the occurrences in nature via our senses and brain, possibly aided by instruments. The concepts we have about the world enable us to specify and communicate via speech or exposition the various phenomena. The current exposition contains *constants* (representing specific things within a specified context) in form of symbols and words which have a generally understood meaning by themselves, and *variables*, which do not have sufficient meaning by themselves.

Example 1.1. We will treat the number π as a mathematical constant.

Example 1.2. The symbol x in 'x pebbles' and the symbol φ in 'We say that φ.' are variables.

In this exposition, we typically include mathematical constants in a *sentence* (such as the one in Example 1.1), and we will require sentences to satisfy *two-valued logic* in the following sense.

Principle 1.1 (Principle of Bivalence, truth value, true, false). "[T]he *principle of bivalence* [...] allows only two *truth values*, namely *true*[1] and *false*" (Rautenberg, 2009, p.2).

We will use the symbol T for *true* and F for *false*. In contrast to a sentence, a *formula* contains one[2] or more variables, and thus does not have enough meaning by itself to be discernible as true or false. Writing the formula

$$\varphi(x, y, \ldots),$$

[1] A more thorough characterization of the concept of *truth* requires a theory far more fundamental than the one stated in this exposition (see, for instance, Kirkham, 1995).

[2] Until the definition of the numbers 0,1,2 in Section 4, we will use 'one' in the sense of 'a' and 'two' in the sense of 'a [something] and another [something]'

we thus express that x, y, etc. are variables, and $\varphi(x, y, \ldots)$ becomes a sentence (which we then also write as φ) when these variables are replaced by constants.

Example 1.3. Within the context of physics, the formula 'After throwing x in the air, it fell to the ground.' becomes a sentence after replacing x by the specific object *the pebble in my hand.*

We say that $\alpha(\varphi, \psi, \ldots)$ is a *scheme* in case

1. $\alpha(\varphi, \psi, \ldots)$ is a formula with variables φ,ψ, etc. and
2. φ,ψ, etc. are to be replaced by sentences.

Example 1.4. In order for the formula 'After throwing the pebble in the air, φ.' to become a meaningful sentence it is evidently necessary to replace the variable φ by a sentence, such as 'it fell to the ground'.

We will frequently encounter a specific formula $\varphi(x, y, \ldots)$, which becomes a true sentence no matter by which conceivable constants its variables x, y, etc. are replaced. In this connection, we call

$$\forall x\, \forall y, \ldots\ (\varphi(x, y, \ldots)) \tag{1.1}$$

a *universal sentence* in case

1. $\varphi(x, y, \ldots)$ is a formula and
2. $\forall x\, \forall y, \ldots\ (\varphi(x, y, \ldots))$ is a sentence which states that φ is a true sentence for any constants replacing x, y, etc. Here, $\forall$ abbreviates *for any* (alternatively, *for all*), which symbol we call the *universal quantifier*.

When a universal quantifier occurs within a sentence, we also speak of *universal quantification.* We write the universal sentence (1.1) also as

$$\begin{aligned} &\forall x, y, \ldots\ (\varphi(x, y, \ldots)),\\ &\forall x\, (\forall y\, (\cdots (\varphi(x, y, \ldots)) \cdots)). \end{aligned}$$

It may happen that a given formula becomes a true sentence when its variables are replaced by certain constants, and that it may become a false sentence when the variables are replaced by other constants. We say that

$$\exists x\, \exists y, \ldots\ (\varphi(x, y, \ldots)) \tag{1.2}$$

is an *existential sentence* in case

1. $\varphi(x, y, \ldots)$ is a formula and

2. $\exists x\,\exists y, \ldots\,(\varphi(x, y, \ldots))$ is a sentence which states that φ is a true sentence for some constants replacing x, y, etc. Here, $\exists$ abbreviates *for some* (alternatively, *there exists a* or *there is a*), which symbol we refer to as the *existential quantifier*.

When using an existential quantifier within a sentence, we also speak of *existential quantification*. We write the existential sentence (1.2) also as

$$\exists x, y, \ldots\,(\varphi(x, y, \ldots)),$$
$$\exists x\,(\exists y\,(\cdots(\varphi(x, y, \ldots))\cdots)).$$

Variables are usually replaced by constants of a certain type and occurring thus within a certain context, which we will refer to as the *domain of discourse*[3].

It is possible for universal and/or existential quantifiers to occur within a formula, which thus contains one or more variables not being replaced by constants (from the domain of discourse) in connection with the universal/existential quantification. We say that a variable is *bound* in case it occurs in connection with a universal or existential quantifier. Otherwise, we will call the variable *free*.

Example 1.5. Let $\varphi(a, x, y, \ldots)$ be a given formula. Then, the following two formulas have the bound variables x, y and the free variable a.

$$\forall x, y\,(\varphi(a, x, y))$$
$$\forall x\,\exists y\,(\varphi(a, x, y))$$

Once the free variable a is replaced by a constant (from the domain of discourse), these formulas become sentences.

We end this first section with two rules about the application of multiple universal and of multiple existential quantifiers.

Principle 1.2 (Commutativity of universal & of existential quantification). "From the meaning of the universal quantifier it follows that in an expression $\forall x\,\forall y\,(\varphi(x, y))$ the two universal quantifiers may be interchanged without altering the sense of the sentence. This also holds for the existential quantifiers in an expression such as $\exists x\,\exists y\,(\varphi(x, y))$." (Hilbert and Ackermann, 2008, p.60)[4].

[3] We will further characterize this concept after introducing the notion of a *set* in Chapter 2.

[4] In this citation, we omitted the original italic type of the first sentence and replaced the originally stated expressions $(x)(y)A(x, y)$ as well as $(Ex)(Ey)A(x, y)$ by $\forall x\,\forall y\,(\varphi(x, y))$ and $\exists x\,\exists y\,(\varphi(x, y))$, respectively.

When universal and existential quantifiers are applied jointly, their interchange may alter the meaning and the truth value of the sentence.

Example 1.6. To demonstrate the truth of the principle of commutativity with respect to universal quantification, we first consider the two sentences

$$\forall x \, \forall y \, (\varphi(x, y))$$
$$\forall y \, \forall x \, (\varphi(x, y))$$

where the domain of discourse concerning the variable x is given by the list of all planets of the Solar System, where the domain of discourse regarding the variable y is the list of all natural satellites within the Solar System, and where the formula $\varphi(x, y)$ stands for 'x as well as y orbits the Sun'. Evidently, the sentences 'Any planet as well as any moon orbits the Sun' and 'Any moon as well as any planet orbits the Sun' express the same (true) fact.

To verify that the principle of commutativity does not necessarily apply to a combination of universal and existential quantification, let us consider now the two sentences

$$\forall x \, \exists y \, (\psi(x, y))$$
$$\exists y \, \forall x \, (\psi(x, y))$$

where the domain of discourse concerning x is modified to be now the list of all outer planets of the Solar System, and where the formula $\psi(x, y)$ represents 'y orbits x'. The first of the considered sentences may be colloquially written as 'Every outer planet has a moon that revolves it', which is true. The second sentence may be stated as 'There is a moon which revolves all outer planets', which is false.

Note that using the original context of all planets of the Solar System (regarding x) leads to the false sentence 'Every planet has a moon that revolves it'; the second sentence reads then 'There is a moon which revolves all planets', which is also false. Although both sentences have the same truth value, their meanings are qualitatively different.

1.2. Combination of Sentences

In this section we will see how certain connective symbols can be used to obtain a new sentence from one or more given sentences. The following principle lays the foundation for establishing the truth value of a newly assembled sentence.

Principle 1.3 (Two-value logic, Principle of Extensionality). "*Two-valued logic* is based on two foundational principles: the principle of bivalence [...] and the *principle of extensionality*, according to which the truth value of a connected sentence depends only on the truth values of its parts, not on their meaning." (Rautenberg, 2009, p.2)[5].

Before considering mechanisms for combining two (or multiple) sentences, we inspect a natural method for altering the meaning and truth value of a single given sentence.

Definition 1.1 (Negation, not). We form the *negation* of an arbitrary given sentence φ by writing

$$\neg\varphi, \tag{1.3}$$

for which expression we say '*not* φ' or 'it is not true that φ'. In case φ is true, we will take its negation to be a false sentence, and in case φ is false, we consider $\neg\varphi$ as true (thus, the truth value of $\neg\varphi$ depends only on the truth value of φ, in accordance with the principle of extensionality). This definition is encoded concisely in the following table.

φ	$\neg\varphi$
T	F
F	T

The first column of this so-called *truth table* contains all possible truth values that the original sentence φ can attain according to the principle of bivalence, *viz.* T (in the second row) and F (in the third row); the second column then contains the assigned truth values with respect to the negation of the original sentence, according to the two possible cases, *viz.* φ is true (second row) so that $\neg\varphi$ is false, and φ is false (third row) so that $\neg\varphi$ is true.

Example 1.7. Letting φ stand for 'The feather fell to the ground', we could express the negation $\neg\varphi$ verbally as 'It is not true that the feather fell to the ground' or as 'The feather didn't fall to the ground'.

[5] We changed the italic formatting of the original citation.

In the following, we will consider various ways of combining two given sentences φ and ψ. A simple way is to use the word *and* to fuse two sentences.

Definition 1.2 (Conjunction, and). We form the *conjunction* of two given sentences φ and ψ by writing

$$\varphi \wedge \psi, \tag{1.4}$$

for which expression we say 'φ *and* ψ'. We take the conjunction of φ and ψ to be a true sentence only in case both φ is true and ψ is true, as shown in the following truth table.

φ	ψ	$\varphi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Notice that a truth value for the new sentence is defined for all possible combinations of truth values that the original sentences can take by themselves.

Example 1.8. We consider, as the domain of discourse, the collection of all objects in my left hand, consisting of a pebble and a feather. Moreover, we let $\varphi(O)$ stand for the formula 'I threw O in the air'. Then, the universal sentence $\forall O\,(\varphi(O))$ may be stated as 'I threw all objects (in my left hand) in the air'. Since there are only two objects in my left hand ('one object and another object'), we may also say that 'I threw the pebble in the air and I threw the feather in the air'. Thus, the preceding universal sentence may also be written in the form of the conjunction $\varphi_p \wedge \varphi_f$ where φ_p stands for the sentence 'I threw the pebble in the air' and where φ_f stands for the sentence 'I threw the feather in the air'.

A third basic method is based on fusing two sentences with the word *or* in the following sense.

Definition 1.3 (Disjunction, or). We form the *disjunction* of two given sentences φ and ψ by writing

$$\varphi \vee \psi, \tag{1.5}$$

for which expression we say 'φ *or* ψ'. This conjunction is defined to be true in case at least one of the original sentences φ and ψ is true, according to the following truth table.

φ	ψ	$\varphi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

The fourth basic method is the logical counterpart of an 'entailment'.

Definition 1.4 (Implication/conditional sentence, if ... then, antecedent/premise/hypothesis, consequent/conclusion). We form an *implication* (alternatively, a *conditional sentence*) based on two given sentences φ and ψ by writing

$$\varphi \Rightarrow \psi, \tag{1.6}$$

for which expression we say '*if* φ, *then* ψ'. This implication is to be considered as false only in in case φ is true and ψ false, as shown by the following truth table.

φ	ψ	$\varphi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Here, we call φ the *antecedent* (alternatively, the *premise* or the *hypothesis*) and ψ the *consequent* (alternatively, the *conclusion*).

The truth table for the implication shows that, whenever a given implication $\varphi \Rightarrow \psi$ is true and the antecedent φ happens to be also true, the consequent ψ must be true as well (see the second row, where all three truth values are T). We will refer to this characteristic feature of an implication as the *law of detachment* or as *Modus Ponens*. This law allows us to state a first method for establishing the truth of a sentence (we will encounter further methods in Section 1.3).

Method 1.1 (Inference). In case a given implication $\varphi \Rightarrow \psi$ and its antecedent φ are true, we may in view of the Law of Detachment *infer* (the truth of) ψ from (the truth of) φ. We alternatively say that 'φ implies ψ', or that '(the truth of) ψ follows from (the truth of) φ' (or simply that 'ψ follows to be true').

Example 1.9. The sentence

$$\forall P\,(P \text{ is an outer planet } \Rightarrow \exists S\,(S \text{ orbits } P)),$$

where the universal quantification is with respect to the listed planets within the Solar System and the existential quantification with respect to the listed natural satellites (also within the Solar System) involves an implication and could be phrased as 'If a planet (within the Solar System) is an outer planet, then it has a moon'.

The fifth basic method extends an entailment to form a *bidirectional entailment*.

Definition 1.5 (Equivalence, if and only if/iff, equivalent sentences, necessary and sufficient condition, brackets). We form an *equivalence* of two given sentences φ and ψ by writing

$$\varphi \Leftrightarrow \psi, \tag{1.7}$$

for which we say 'φ *if, and only if* ψ' (or shorter 'φ *iff* ψ'). This implication is defined to be true when the conjunction

$$(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi) \tag{1.8}$$

is true, and defined to be false when the preceding conjunction is false. Here, we use *brackets*[6] to indicate which sentences are connected first (here the implications); the unbracketed connection (i.e., the conjunction) is formed last. We then call φ and ψ *equivalent sentences*, and we say that φ is a *necessary and sufficient condition* for ψ and vice versa.

Proposition 1.1. *The equivalence $\varphi \Leftrightarrow \psi$ of two given sentences φ and ψ has the following truth table.*

φ	ψ	$\varphi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Proof. To construct the truth table given below, we write down all possible combinations of truth values of the sentences φ and ψ (in the first and the second column), determine then row by row for each combination the truth values of the implication $\varphi \Rightarrow \psi$ according to the truth table for an implication (in the third column), similarly the truth values for the second implication $\psi \Rightarrow \varphi$ (in the fourth column), and finally the truth values of the conjunction of the two previously evaluated implications according to the truth table for a conjunction (see the fifth column). Since the equivalence

[6] We will also the bracket symbols [...] and {...}

$\varphi \Leftrightarrow \psi$ was defined to take the same truth values as that conjunction, we immediately obtain the last column, which shows that the proposed truth table applies indeed to the stated equivalence.

φ	ψ	$\varphi \Rightarrow \psi$	$\psi \Rightarrow \varphi$	$(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$	$\varphi \Leftrightarrow \psi$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$\square$

Note 1.1*.* The truth table in Proposition 1.1 shows that an equivalence turns out to be true in case the original sentences are both true or both false. Moreover, an equivalence is false when one of the original sentences is false and the other true. We also notice that an equivalence is true only in the case that both implications are true; in this case each of the sentences implies the other one.

We are now in a position to define a general *combination of sentences*.

Definition 1.6 (Combination of sentences, logical connectives). We say that α is a *combination of sentences* in case α is a sentence consisting of other given sentences φ, ψ, etc. (including sentences negated by means of $\neg$) that are connected by means of any of the so-called *logical connectives* $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ (where brackets are used whenever necessary).

Let us now summarize the symbols previously introduced to aid the formulation of sentences.

Notation 1.1 (Logical symbols)*.* The following symbols are called the *logical symbols*:

- x, y, O, P, S, ... (the variables),
- $\forall$, $\exists$ (the quantifiers),
- $=$ (the equality symbol),
- $\neg$ (the negation symbol),
- $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$ (the logical connectives), and
- (...),[...], {...} (the brackets).

1.3. Establishment of True Sentences

In this section we will encounter methods for establishing a newly formed sentence as being true. There are various ways for doing this: For instance, we will frequently 'define', 'assume' or 'prove' a sentence. To bring out the distinction between these 'activities' more clearly, we first characterize different types of sentences.

1.3.1. Types of true sentences

A *definition*, which will always be considered as a true sentence, is characteristically stated in a form such as

$$\text{For any } x \text{ we say that } \varphi(x) \text{ iff } \psi(x). \tag{1.9}$$

Here, x is a variable with respect to a certain domain of discourse (bound via universal quantification), $\varphi(x)$ is a formula which contains besides the bound variable x a new concept (i.e., a new kind of constant in form of a symbol, word or phrase), and $\psi(x)$ is a formula which explains the new concept in terms only of previously introduced concepts/constants.

Example 1.10. Let us take the list of all planets within the Solar System as the domain of discourse, and let us assume that we have previously established (at least) the concepts of a 'planet', the Solar System, and the Asteroid Belt. Then, the following sentence may serve as a (true) definition of an *outer planet*.

$$\begin{aligned}&\text{For any } P \text{ we say that } P \text{ is an outer planet iff}\\ &\quad P \text{ is beyond the Asteroid Belt.}\end{aligned}$$

We could evidently find various other ways of expressing this definition, for instance, 'All planets (of the Solar System) beyond the Asteroid Belt are called outer planets'. Although such formulations may appear to be more concise or elegant, they could obscure the logical structure and thus obstruct their use in subsequent 'logical activities'. We will therefore usually stick to the more formalized structure.

The next kind of true sentence is the *principle* (recall, for instance, the Principle of Bivalence stated in Section 1.1), which we consider as a description of the way that a mathematical method or theory 'works'.

Note 1.2. Recall that the principles of Bivalence and of Extensionality determined the 'working' of a truth table.

Whereas a principle is invariably taken to be a true sentence, an *assumption* is a sentence whose truth is not assigned absolutely, but only temporarily or tentatively within a line of reasoning. By contrast, an *axiom* will be regarded as an 'invariably true assumption'. Thus, an axiom is often quite similar to a principle, but the reason for establishing an axiom is usually less obvious than the reason for adopting a principle.

We will now consider types of sentences whose truth follows necessarily from other facts. To demonstrate the truth of such sentences, we formulate a line of arguments and express it by means of true sentences. We will call such a piece of writing a *proof*. The most relevant results proved throughout this exposition will be called *theorems*. A *lemma* is utilized in a much narrower context than a theorem, usually as a significant stepping stone towards proving a (more widely applied) theorem. Whereas theorems and lemmas are well-known results, regarded to be of such relevance as to deserve their own name (e.g., the *Recursion Theorem*), the *propositions* and *corollaries* represent more ordinary and less standardized stepping stones in developing the mathematical exposition. Whereas the proof of a proposition may be intricate and long (i.e., 'technical'), the proof of a corollary is typically accomplished in a direct, easy-to-follow way by applying a small number of arguments.

1.3.2. Proof techniques for logical connectives and basic sentential laws

In this section we review some standardized procedures for establishing a proof of a given (supposedly true) sentence, which methods we will apply frequently. We first consider combinations of sentences which do not involve quantification.

Definition 1.7 (Tautology, contradiction).

(1) Given an arbitrary combination α of sentences, we say that α is a *tautology* iff α is true irrespective of the truth values that the connected sentences take.

(2) Given an arbitrary combination β of sentences, we say that β is a *contradiction* iff β is false irrespective of the truth values that the connected sentences have.

The following basic technique makes use in particular of the definition of a tautology and of the truth tables regarding the negation and regarding the logical connectives.

Method 1.2 (Proof via truth table). To establish the truth of a given sentence α which is combined of other given sentences φ, ψ, etc. by means of any of the logical connectives, the negation symbol and the brackets, we may show that α is a tautology. To do this, we write down in a systematic way all possible combinations of truth values that the connected sentences φ, ψ, etc. may take according to the Principle of Bivalence. We then determine for each of these combinations of truth values the resulting truth value of α. We may do this successively by considering within one step only one of the negations or logical connectives used to form α, beginning with the innermost bracketed expression(s) and ending with the unbracketed expression.

Example 1.11. Let φ be the sentence 'Today is Monday'. Then, the disjunction

$$\varphi \vee \neg\varphi \tag{1.10}$$

('Today is Monday or today isn't Monday.') should be logically true, irrespective of whether it is actually true or false that today is Monday. Thus, we may anticipate that (1.10) is a law in the form of a tautology. Similarly, the conjunction

$$\varphi \wedge \neg\varphi \tag{1.11}$$

('Today is Monday and today isn't Monday.') should be logically false, again irrespective of whether it is actually true or false that today is Monday. Therefore, we expect (1.11) to be a contradiction. Let us now construct the corresponding truth table(s). In the first column we write down the two possible truth values that φ can have (viz. T and F according to the Principle of Bivalence). In a first step, we use the truth table that characterizes the negation to write down the column of truth values regarding $\neg\varphi$. In a second step, we apply the truth table for the disjunction to determine the truth values of the suspected law (1.10). Since (1.10) turns out to be true for both possible combinations of truth values that φ and $\neg\varphi$ may simultaneously take, (1.10) is a tautology by definition and thus a true sentence. Similarly, we obtain the column of truth values for (1.11), which sentence indeed turns out to be a contradiction.

φ	$\neg\varphi$	$\varphi \vee \neg\varphi$	$\varphi \wedge \neg\varphi$
T	F	T	F
F	T	T	F

Since the law (1.10) we just established will be frequently applied, we state it again as the following theorem.

Theorem 1.2 (Law of the Excluded Middle). *The following disjunction is true for an arbitrary given sentence* φ.

$$\varphi \vee \neg\varphi. \tag{1.12}$$

Note 1.3. The procedure for establishing the truth of the disjunction $\varphi \vee \neg\varphi$ via the preceding truth table simply consists in the demonstration that (at least) one part of the disjunction is true.

We may apply this approach to establish the truth of an arbitrary disjunction $\varphi \vee \psi$.

Method 1.3 (Proof of a disjunction). To prove that the disjunction $\varphi \vee \psi$ of two given sentences φ and ψ is true, we may show that one of the two sentences is true by itself. After this demonstration, we may argue that the disjunction is true, because it is defined to be true when (at least) one of its parts is true.

We now establish further useful facts about tautologies and contradictions, using again the method of truth table.

Theorem 1.3 (Tautology & Contradiction Laws). *The following equivalences are true for an arbitrary given sentence* φ, *an arbitrary tautology* α, *and an arbitrary contradiction* β.

a) **Tautology Law for the conjunction:**

$$\varphi \;\Leftrightarrow\; (\alpha \wedge \varphi). \tag{1.13}$$

b) **Tautology Law for the implication:**

$$\varphi \Leftrightarrow (\alpha \Rightarrow \varphi). \tag{1.14}$$

c) **Contradiction Law:**

$$\varphi \Leftrightarrow (\beta \vee \varphi). \tag{1.15}$$

Proof. Concerning a) – c), we obtain the following truth table, taking account of the definitions of a tautology and of a contradiction, so that the columns of truth values for α and β contains only Ts and Fs, respectively.

φ	α	β	$\alpha \wedge \varphi$	(1.13)	$\alpha \Rightarrow \varphi$	(1.14)	$\beta \vee \varphi$	(1.15)
T	T	F	T	T	T	T	T	T
F	T	F	F	T	F	T	F	T

□

We now demonstrate Method 1.2 by proving the following more intricate theorem.

Theorem 1.4 (Law of the Hypothetical Syllogism). *The following implication is true for arbitrary given sentences α, φ and ψ.*

$$[(\varphi \Rightarrow \alpha) \wedge (\alpha \Rightarrow \psi)] \;\Rightarrow\; [\varphi \Rightarrow \psi]. \tag{1.16}$$

Proof. We form the first three columns of the truth table (given at the end of this paragraph) by writing down all possible combinations of truth values that the three given sentences α, φ and ψ can take. In the fourth and fifth column, we determine the truth values of the innermost logical connections $\varphi \Rightarrow \alpha$ and $\alpha \Rightarrow \psi$, respectively, using the truth table that defines an implication. In the sixth column, we evaluate the conjunction of the previous two implications (using now the truth table regarding the conjunction). At the same level of bracketing is the subsequently evaluated implication $\varphi \Rightarrow \psi$ (second last column). Finally, we write in the last column the truth values of the proposed implication (1.16), based on the various combinations of truth values regarding the previously evaluated antecedent (sixth column) and consequent (seventh column).

φ	α	ψ	$\varphi \Rightarrow \alpha$	$\alpha \Rightarrow \psi$	$(.) \wedge (.)$	$\varphi \Rightarrow \psi$	(1.16)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the last column with respect to the implication to be proven contains only Ts, we see that this implication is true for any combination of truth values that the connected sentences α, φ and ψ can take. Notice that the truth values of the newly formed connections in Columns 4 – 8 depend only on the truth values of the connected parts and on the kind of connective used, and not on the specific meaning of the connected sentences. Thus, the proof is consistent with the Principle of Extensionality. □

Note 1.4. The preceding truth table hints at a general method for proving a conjunction. Let us observe that the occurrence of the truth value T in the sixth column regarding $(.) \wedge (.)$ stems from the fact that we previously established (within the fourth and the fifth column) the truth value T for both parts $\varphi \Rightarrow \alpha$ and $\alpha \Rightarrow \psi$ of that conjunction.

Method 1.4 (Proof of a conjunction). To prove that a conjunction $\varphi \wedge \psi$ of two given sentences φ and ψ is true, we may show by separate proofs that each of the sentences φ and ψ is true by itself. After this demonstration, we may argue that the conjunction is true, because it is defined to be true precisely when both of its parts are true.

Since an equivalence is defined as a conjunction, this method may be applied for the purpose of establishing the truth of an equivalence.

Method 1.5 (Proof of an equivalence). To prove an equivalence $\varphi \Leftrightarrow \psi$ for given sentences φ and ψ, we may carry out a proof of the conjunction

$$(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi),$$

where each of the two parts of this conjunction involves a proof of an implication.

Since implications frequently occur within combinations of sentences, we now describe various ways of proving them. The truth table regarding the proof of the Law of the Hypothetical Syllogism constitutes a suitable starting point for these tasks.

Note 1.5. In the truth table regarding the proof of the Law of the Hypothetical Syllogism, we evaluated four different implications. In view of the definition of an implication, these evaluations can be carried out quite economically. Since an implication is defined to take the truth value T whenever the antecedent is false, we need not care about the truth value of the consequent. Thus, in all four instances in which $(\varphi \Rightarrow \alpha) \wedge (\alpha \Rightarrow \psi)$ takes the value F (within the sixth column), we obtain the truth value T in the corresponding rows for the implication (1.16) (within the last column). In the other four instances in which the antecedent $(\varphi \Rightarrow \alpha) \wedge (\alpha \Rightarrow \psi)$ takes the value T, we actually need to inspect the truth values of the consequent $\varphi \Rightarrow \psi$ in the corresponding rows (within the seventh column). As these values are all T, the implication (1.16) also takes the value T in these instances.

Thus, the definition of an implication gives rise to the following technique for establishing the truth of an implication.

Method 1.6 (Direct proof of an implication). To prove in a direct manner that an implication $\varphi \Rightarrow \psi$ based on two given sentences φ and ψ is true, we may begin by assuming the antecedent φ to be a true (since the implication is automatically true when the antecedent φ is false, so that nothing is to be shown in that case). Then, as an implication is only true in case the consequent is also true, it remains to demonstrate that ψ is true.

Since the Law of the Hypothetical Syllogism states that an implication $\varphi \Rightarrow \psi$ (which we intend to prove) is true precisely when the equivalent sentence $(\varphi \Rightarrow \alpha) \wedge (\alpha \Rightarrow \psi)$ is true, this law may be applied to carry out a direct proof of an implication in two consecutive steps.

Method 1.7 (Direct proof of an implication in steps). To prove in a direct manner that an implication $\varphi \Rightarrow \psi$ based on two given sentences φ and ψ is true, we may carry out a proof of the equivalent conjunction

$$(\varphi \Rightarrow \alpha) \wedge (\alpha \Rightarrow \psi),$$

which then involves two separate proofs of the implications $\varphi \Rightarrow \alpha$ and $\alpha \Rightarrow \psi$.

In situations where the previously established sentences and the antecedent φ of an implication $\varphi \Rightarrow \psi$ to be proven do not provide enough information to demonstrate in the course of a direct proof that the consequent ψ is true, we may introduce a new sentence α jointly with its negation $\neg\alpha$ as additional *case assumptions*. Due to the Law of the Excluded Middle, such two cases are *exhaustive* in the sense that one these cases is necessarily true. The next exercise shows that a given implication may indeed be written equivalently in a form with additional cases and thus justifies the subsequent method.

Exercise 1.1. The following equivalence is true for arbitrary given sentences φ, ψ and α.

$$[\varphi \Rightarrow \psi] \;\Leftrightarrow\; [([\varphi \wedge \alpha] \Rightarrow \psi) \wedge ([\varphi \wedge \neg\alpha] \Rightarrow \psi)]. \tag{1.17}$$

(Hint: Apply a proof via truth table.)

Method 1.8 (Proof of an implication by cases). To prove that an implication $\varphi \Rightarrow \psi$ for given sentences φ and ψ is true, we may establish the equivalent conjunction

$$[(\varphi \wedge \alpha) \Rightarrow \psi] \wedge [(\varphi \wedge \neg\alpha) \Rightarrow \psi].$$

To do this, we assume first $\varphi \wedge \alpha$ (the *first case*) to prove the first implication directly, then $\varphi \wedge \neg\alpha$ (the *second case*) to prove the second implication directly.

We may also consider *cases* to establish the truth of a given sentence ψ which is not in the form of an implication.

Exercise 1.2. Verify the following implication for arbitrary given sentences ψ, α and β.

$$[\alpha \vee \beta] \;\Rightarrow\; [\psi \Leftrightarrow ([\alpha \Rightarrow \psi] \wedge [\beta \Rightarrow \psi])]. \tag{1.18}$$

(Hint: Construct a truth table.)

Since the truth of a given disjunction $\alpha \vee \beta$ implies with (1.18) that a given sentence ψ is equivalent to the conjunction of the implications $\alpha \Rightarrow \psi$ and $\beta \Rightarrow \psi$, we may apply the following method.

Method 1.9 (Proof of a sentence by cases). To show that a given sentence ψ is true, where we have already established the truth of the disjunction $\alpha \vee \beta$ of given sentences α and β, we may carry out a proof of the conjunction

$$(\alpha \Rightarrow \psi) \wedge (\beta \Rightarrow \psi).$$

For this purpose, we assume first α (the *first case*) to prove the first implication directly, then β (the *second case*) to prove the second implication directly.

The following law motivates and justifies an indirect way of proving an implication.

Theorem 1.5 (Law of Contraposition). *The following equivalence is true for arbitrary given sentences φ and ψ.*

$$[\varphi \Rightarrow \psi] \Leftrightarrow [(\neg\psi) \Rightarrow (\neg\varphi)]. \tag{1.19}$$

Proof. We evidently obtain the following truth table by using the truth tables that characterize the negation, the implication, and the equivalence.

φ	ψ	$\neg\psi$	$\neg\varphi$	$(\neg\psi) \Rightarrow (\neg\varphi)$	$\varphi \Rightarrow \psi$	$[\varphi \Rightarrow \psi] \Leftrightarrow [(\neg\psi) \Rightarrow (\neg\varphi)]$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

The proposed equivalence is seen to be true since it takes the truth value T for all possible combinations of truth values that the connected given sentences φ and ψ can have. $\square$

Method 1.10 (Proof by contraposition). To prove that an implication $\varphi \Rightarrow \psi$ for given sentences φ and ψ is true, we may establish the equivalent implication

$$(\neg\psi) \Rightarrow (\neg\varphi).$$

A second widely applied indirect procedure for establishing implications is prepared by the next exercise.

Exercise 1.3. Show that the following equivalence holds for arbitrary given sentences φ, ψ and β, where β is a contradiction.

$$[\varphi \Rightarrow \psi] \Leftrightarrow [(\varphi \wedge \neg\psi) \Rightarrow \beta]. \tag{1.20}$$

(Hint: Apply a proof via truth table.)

Method 1.11 (Proof of an implication by contradiction). To prove that an implication $\varphi \Rightarrow \psi$ for given sentences φ and ψ is true, we may choose an arbitrary contradiction β and establish the implication

$$(\varphi \wedge \neg\psi) \Rightarrow \beta.$$

Before proceeding with a variant of the proof by contradiction, we establish four more laws.

Theorem 1.6 (Substitution Rules for sentences). *The following sentences are true for arbitrary given sentences φ, ψ and δ.*

a) **Substitution Rule for negations:***:*

$$[\varphi \Leftrightarrow \psi] \;\Rightarrow\; [(\neg\varphi) \Leftrightarrow (\neg\psi)]. \tag{1.21}$$

b) **Substitution Rules for conjunctions:***:*

$$(\varphi \Leftrightarrow \psi) \;\Rightarrow\; ([\delta \wedge \varphi] \Leftrightarrow [\delta \wedge \psi]). \tag{1.22}$$
$$(\varphi \Leftrightarrow \psi) \;\Rightarrow\; ([\varphi \wedge \delta] \Leftrightarrow [\psi \wedge \delta]). \tag{1.23}$$

c) **Substitution Rules for disjunctions:***:*

$$(\varphi \Leftrightarrow \psi) \;\Rightarrow\; ([\delta \vee \varphi] \Leftrightarrow [\delta \vee \psi]). \tag{1.24}$$
$$(\varphi \Leftrightarrow \psi) \;\Rightarrow\; ([\varphi \vee \delta] \Leftrightarrow [\psi \vee \delta]). \tag{1.25}$$

d) **Substitution Rules for implications:***:*

$$(\varphi \Leftrightarrow \delta) \;\Rightarrow\; ([\varphi \Rightarrow \psi] \Leftrightarrow [\delta \Rightarrow \psi]). \tag{1.26}$$
$$(\psi \Leftrightarrow \delta) \;\Rightarrow\; ([\varphi \Rightarrow \psi] \Leftrightarrow [\varphi \Rightarrow \delta]). \tag{1.27}$$

e) **Substitution Rules for equivalences:***:*

$$[(\varphi \Leftrightarrow \delta) \wedge (\psi \Leftrightarrow \delta)] \;\Rightarrow\; [\varphi \Leftrightarrow \psi]. \tag{1.28}$$
$$[(\varphi \Leftrightarrow \delta) \wedge (\delta \Leftrightarrow \psi)] \;\Rightarrow\; [\varphi \Leftrightarrow \psi]. \tag{1.29}$$

Exercise 1.4. Prove the Substitution Rules for sentences.

Note 1.6. The Substitution Rules for sentences show that we may always replace within a given true negation, conjunction, disjunction, implication or equivalence one of the involved sentences by an equivalent sentence to obtain again a true negation, conjunction, disjunction, implication and equivalence, respectively. These rules should be within our grasp, so that we will usually not mention them in applications.

Corollary 1.7. *The following equivalence is true for an arbitrary sentence ψ and an arbitrary tautology α.*

$$\psi \;\Leftrightarrow\; [(\neg\psi) \Rightarrow (\neg\alpha)]. \tag{1.30}$$

Proof. On the one hand, the equivalence $\psi \Leftrightarrow [\alpha \Rightarrow \psi]$ holds according to the Tautology Law for the implication. On the other hand, the equivalence $[\alpha \Rightarrow \psi] \Leftrightarrow [(\neg\psi) \Rightarrow (\neg\alpha)]$ is true because of the Law of Contraposition. Thus, the conjunction

$$(\psi \Leftrightarrow [\alpha \Rightarrow \psi]) \wedge ([\alpha \Rightarrow \psi] \Leftrightarrow [(\neg\psi) \Rightarrow (\neg\alpha)])$$

is also true. Then, this conjunction implies the truth of (1.30) with the Substitution Law for equivalences (1.29). □

The preceding corollary shows that we may demonstrate the truth of a given sentence ψ by carrying out a direct proof of the equivalent implication $(\neg\psi) \Rightarrow (\neg\alpha)$, where α is a sentence known to be true.

Method 1.12 (Proof of a sentence by contradiction). To prove that a given sentence ψ is true, we may assume $\neg\psi$ to be true, choose an arbitrary tautology α (i.e., an already established true sentence), and show then that $\neg\alpha$ is also true.

In the remainder of the current Section 1.3.2, we state further necessary laws, which we may all be proved by means of truth tables. To begin with, we see that an implication may be explained solely by means of a negation and a disjunction.

Theorem 1.8 (Conditional Laws). *The following equivalences are true for arbitrary given sentences φ and ψ.*

$$[\varphi \Rightarrow \psi] \;\Leftrightarrow\; [(\neg\varphi) \vee \psi]. \tag{1.31}$$
$$[\neg(\varphi \Rightarrow \psi)] \;\Leftrightarrow\; [\varphi \wedge (\neg\psi)]. \tag{1.32}$$

Proof. Regarding (1.31), we evidently obtain the following truth table by using the truth tables that characterize the negation, the implication, and the equivalence.

φ	ψ	$\neg\varphi$	$(\neg\varphi) \vee \psi$	$\varphi \Rightarrow \psi$	$[\varphi \Rightarrow \psi] \Leftrightarrow [(\neg\varphi) \vee \psi]$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

The proposed equivalence is seen to be true since it takes the truth value T for all possible combinations of truth values that the connected given sentences φ and ψ can have. $\square$

Exercise 1.5. Prove (1.32) via truth table.

Combining a sentence with itself via conjunction or disjunction, we see that one of the two is redundant.

Theorem 1.9 (Idempotent Laws for sentences). *The following equivalences are true for an arbitrary given sentence φ.*

a) **Idempotent Law for the conjunction:**

$$(\varphi \wedge \varphi) \ \Leftrightarrow \ \varphi. \tag{1.33}$$

b) **Idempotent Law for the disjunction:**

$$(\varphi \vee \varphi) \ \Leftrightarrow \ \varphi. \tag{1.34}$$

Proof. Since the given sentence φ is combined with itself, we duplicate for greater clarity the column of truth values that φ may take when constructing the truth table.

φ	φ	$\varphi \wedge \varphi$	$(\varphi \wedge \varphi) \Leftrightarrow \varphi$	$\varphi \vee \varphi$	$(\varphi \vee \varphi) \Leftrightarrow \varphi$
T	T	T	T	T	T
F	F	F	T	F	T

The proposed equivalences are true since they invariably take the truth value T for both possible combinations of truth values that φ may take. $\square$

We obtain an equivalent version of a sentence by negating it twice.

Theorem 1.10 (Double Negation Law). *The following equivalence holds for an arbitrary given sentence φ.*

$$\varphi \ \Leftrightarrow \ \neg(\neg\varphi). \tag{1.35}$$

Exercise 1.6. Prove the Double Negation Law via truth table.

The following commutative laws show that, in case two sentences are combined via conjunction, disjunction or equivalence, we may interchange the sentences without altering the truth value of the combination.

Theorem 1.11 (Commutative Laws for sentences). *The following equivalences are true for arbitrary given sentences φ and ψ.*

a) **Commutative Law for the conjunction:**

$$(\varphi \wedge \psi) \Leftrightarrow (\psi \wedge \varphi). \tag{1.36}$$

b) **Commutative Law for the disjunction:**

$$(\varphi \vee \psi) \;\Leftrightarrow\; (\psi \vee \varphi). \tag{1.37}$$

c) **Commutative Law for the equivalence:**

$$(\varphi \Leftrightarrow \psi) \;\Leftrightarrow\; (\psi \Leftrightarrow \varphi). \tag{1.38}$$

Exercise 1.7. Verify the Commutative Laws for sentences.

The subsequent associative laws indicate that it is irrelevant in which order we combine three sentences via conjunctions/disjunctions.

Theorem 1.12 (Associative Laws for sentences). *The following equivalences are true for arbitrary given sentences φ, ψ and ϑ.*

a) **Associative Law for the conjunction:**

$$[(\varphi \wedge \psi) \wedge \vartheta] \;\Leftrightarrow\; [\varphi \wedge (\psi \wedge \vartheta)]. \tag{1.39}$$

b) **Associative Law for the disjunction:**

$$[(\varphi \vee \psi) \vee \vartheta] \;\Leftrightarrow\; [\varphi \vee (\psi \vee \vartheta)]. \tag{1.40}$$

Exercise 1.8. Prove the Associative Laws for sentences.

Notation 1.2. The Associative Laws for sentences show that the truth value of the conjunction or disjunction of three sentences does not depend on the position of the brackets (.). We will therefore usually write for the left-hand and the right-hand side of (1.39) and of (1.40)

$$\varphi \wedge \psi \wedge \vartheta, \tag{1.41}$$

$$\varphi \vee \psi \vee \vartheta. \tag{1.42}$$

This notation allows for a straightforward modification of the method for proving a sentence by cases.

Exercise 1.9. Verify the following implication for arbitrary given sentences ψ, α, β and γ.

$$[\alpha \vee \beta \vee \gamma] \;\Rightarrow\; [\psi \Leftrightarrow ([\alpha \Rightarrow \psi] \wedge [\beta \Rightarrow \psi] \wedge [\gamma \Rightarrow \psi])]. \tag{1.43}$$

Method 1.13 (Proof by three cases). To show that a given sentence ψ is true, where we have already established the truth of the disjunction $\alpha \vee \beta \vee \gamma$ of given sentences α, β and γ, we may prove that each of the three parts of the multiple conjunction

$$(\alpha \Rightarrow \psi) \wedge (\beta \Rightarrow \psi) \wedge (\gamma \Rightarrow \psi)$$

is true. For this purpose, we assume first α (the *first case*) to prove the first implication directly, then β (the *second case*) to prove the second implication directly, and finally γ (the *third case*) to prove the third implication directly.

There are various useful distributive laws regarding the formation of sentences, which involve two different logical connectives.

Theorem 1.13 (Distributive Laws for sentences). *The following equivalences are true for arbitrary given sentences α, β and φ.*

a) **Distributivity of the conjunction over the disjunction:**

$$[\varphi \wedge (\alpha \vee \beta)] \;\Leftrightarrow\; [(\varphi \wedge \alpha) \vee (\varphi \wedge \beta)]. \tag{1.44}$$

b) **Distributivity of the disjunction over the conjunction:**

$$[\varphi \vee (\alpha \wedge \beta)] \;\Leftrightarrow\; [(\varphi \vee \alpha) \wedge (\varphi \vee \beta)]. \tag{1.45}$$

c) **Distributivity of the implication over the conjunction:**

$$[\varphi \Rightarrow (\alpha \wedge \beta)] \;\Leftrightarrow\; [(\varphi \Rightarrow \alpha) \wedge (\varphi \Rightarrow \beta)]. \tag{1.46}$$

d) **Distributivity of the implication over the disjunction:**

$$[\varphi \Rightarrow (\alpha \vee \beta)] \;\Leftrightarrow\; [(\varphi \Rightarrow \alpha) \vee (\varphi \Rightarrow \beta)]. \tag{1.47}$$

Exercise 1.10. Establish the Distributive Laws for sentences.

Proof. □

The following frequently applied laws do not strictly fit into the preceding scheme of distributivity, which we therefore state separately.

Exercise 1.11. Establish the following equivalences for arbitrary given sentences α, β and φ.

$$[(\alpha \wedge \beta) \Rightarrow \varphi] \;\Leftrightarrow\; [(\alpha \Rightarrow \varphi) \vee (\beta \Rightarrow \varphi)]. \tag{1.48}$$

$$\Leftrightarrow\; [\alpha \Rightarrow (\beta \Rightarrow \varphi)]. \tag{1.49}$$

$$[(\alpha \vee \beta) \Rightarrow \varphi] \;\Leftrightarrow\; [(\alpha \Rightarrow \varphi) \wedge (\beta \Rightarrow \varphi)]. \tag{1.50}$$

De Morgan's laws show that negating a conjunction yields a disjunction and vice versa.

Theorem 1.14 (De Morgan's Laws for sentences). *The following equivalences are true for arbitrary given sentences φ and ψ.*

a) **De Morgan's Law for the conjunction:**

$$[\neg(\varphi \wedge \psi)] \ \Leftrightarrow \ [(\neg\varphi) \vee (\neg\psi)]. \tag{1.51}$$

b) **De Morgan's Law for the disjunction:**

$$[\neg(\varphi \vee \psi)] \ \Leftrightarrow \ [(\neg\varphi) \wedge (\neg\psi)]. \tag{1.52}$$

Exercise 1.12. Prove De Morgan's Laws for sentences via truth table.

1.3.3. Basic laws for existential and universal quantification

Theorem 1.15 (Quantifier Negation Laws). *The following equivalences are true for an arbitrary given formula $\varphi(y)$.*

a) **Negation Law for universal sentences:**

$$\neg\forall y\,(\varphi(y)) \ \Leftrightarrow \ \exists y\,(\neg\varphi(y)). \tag{1.53}$$

b) **Negation Law for existential sentences:**

$$\neg\exists y\,(\varphi(y)) \ \Leftrightarrow \ \forall y\,(\neg\varphi(y)). \tag{1.54}$$

Proof. Concerning a), we notice that the negation $\neg\forall y\,(\varphi(y))$ means that

- it is not true that $\varphi(y)$ holds for all y, that is,
- there is a y for which $\varphi(y)$ does not hold, that is,
- there is a y for which it is not true that $\varphi(y)$,

which means $\exists y\,(\neg\varphi(y))$. Since all of these sentences have the same meaning, they are either all false or all true. Thus, the two sentences $\neg\forall y\,(\varphi(y))$ and $\exists y\,(\neg\varphi(y))$ either both take the truth value T or both the truth value F, so that they are indeed equivalent.

Concerning b), the negation $\neg\exists y\,(\varphi(y))$ means that

- it is not true that $\varphi(y)$ holds for some (i.e., at least one) y, that is,

- $\varphi(y)$ does not hold for any y, that is,
- for any y it is not true that $\varphi(y)$,

which means $\forall y\,(\neg\varphi(y))$. Because all of these sentences mean the same thing and thus share the same truth value, we have in particular that the two sentences $\neg\exists y\,(\varphi(y))$ and $\forall y\,(\neg\varphi(y))$ are equivalent. □

Corollary 1.16. *The following equivalences hold for an arbitrary formula* $\varphi(x, y)$*.*

$$\neg\forall x, y\,(\varphi(x, y)) \;\Leftrightarrow\; \exists x, y\,(\neg\varphi(x, y)), \tag{1.55}$$

$$\neg\exists x, y\,(\varphi(x, y)) \;\Leftrightarrow\; \forall x, y\,(\neg\varphi(x, y)). \tag{1.56}$$

Proof. We obtain the following equivalences immediately with the notations for universal/existential sentences and with the Quantifier Negation Laws.

$$\begin{aligned}
\neg\forall x, y\,(\varphi(x, y)) &\Leftrightarrow \neg\forall x\,(\forall y\,(\varphi(x, y)))\\
&\Leftrightarrow \exists x\,(\neg\forall y\,(\varphi(x, y)))\\
&\Leftrightarrow \exists x\,(\exists y\,(\neg\varphi(x, y)))\\
&\Leftrightarrow \exists x, y\,(\neg\varphi(x, y)),
\end{aligned}$$

$$\begin{aligned}
\neg\exists x, y\,(\varphi(x, y)) &\Leftrightarrow \neg\exists x\,(\exists y\,(\varphi(x, y)))\\
&\Leftrightarrow \forall x\,(\neg\exists y\,(\varphi(x, y)))\\
&\Leftrightarrow \forall x\,(\forall y\,(\neg\varphi(x, y)))\\
&\Leftrightarrow \forall x, y\,(\neg\varphi(x, y)).
\end{aligned}$$

The proposed equivalences follow then to be true with the Substitution Rule for equivalences (1.29). □

We were able to reveal the truth of the previous two relationships (1.53) and (1.54) between universal and existential sentences by analyzing the meanings of universal and existential quantification in connection with negation. We now formulate a method for establishing the truth of a universal sentence based on the following logical connection between the words 'for any' and 'for an arbitrary'. The idea will be to demonstrate that a given formula $\varphi(x)$ becomes a true sentence no matter which constant we take from the universe to replace the variable x. The practical approach to doing this will be to take an arbitrary constant $\bar{x}$, or in other words, to 'let $\bar{x}$ be arbitrary', and to prove then the sentence $\varphi(\bar{x})$. Having accomplished this task, $\varphi(\bar{x})$ thus holds for an arbitrary constant $\bar{x}$, which means that we may just as well take any of the other constants to obtain that true sentence. We therefore infer from the truth of $\varphi(\bar{x})$ for the arbitrary $\bar{x}$ the truth of $\varphi(x)$ for any x.

Method 1.14 (Proof of a universal sentence). To prove that a universal sentence $\forall x\,(\varphi(x))$ is true, we take an arbitrary constant $\bar{x}$ from the domain of discourse and show that $\varphi(\bar{x})$ is a true sentence in that particular instance. Then, since $\varphi(\bar{x})$ is true for an arbitrary $\bar{x}$, we may conclude that $\varphi(x)$ holds for any x, i.e. that the universal sentence $\forall x\,(\varphi(x))$ is true.

The idea for proving an existential sentence is less intricate.

Method 1.15 (Proof of an existential sentence). To prove that an existential sentence $\exists x\,(\varphi(x))$ is true, we seek within the given domain of discourse a particular constant, say $\bar{x}$, for which we can show that $\varphi(\bar{x})$ is true. After this demonstration, we may argue that $\varphi(x)$ is true for some x.

The proofs of the following sentences demonstrate the application of some of the previously established proof techniques.

Theorem 1.17 (Neutrality Laws for quantification). *The following sentences are true for an arbitrary formula $\psi(y)$.*

a) *The conjunction involving an arbitrary given sentence φ is neutral with respect to existential quantification in the sense that*

$$[\varphi \wedge \exists y\,(\psi(y))] \;\Leftrightarrow\; \exists y\,(\varphi \wedge \psi(y)), \tag{1.57}$$

$$[\exists y\,(\psi(y)) \wedge \varphi] \;\Leftrightarrow\; \exists y\,(\psi(y) \wedge \varphi). \tag{1.58}$$

b) *The disjunction involving an arbitrary given sentence φ is neutral with respect to existential quantification in the sense that*

$$[\varphi \vee \exists y\,(\psi(y))] \;\Leftrightarrow\; \exists y\,(\varphi \vee \psi(y)), \tag{1.59}$$

$$[\exists y\,(\psi(y)) \vee \varphi] \;\Leftrightarrow\; \exists y\,(\psi(y) \vee \varphi). \tag{1.60}$$

c) *The conjunction involving an arbitrary given sentence φ is neutral with respect to universal quantification in the sense that*

$$[\varphi \wedge \forall y\,(\psi(y))] \;\Leftrightarrow\; \forall y\,(\varphi \wedge \psi(y)), \tag{1.61}$$

$$[\forall y\,(\psi(y)) \wedge \varphi] \;\Leftrightarrow\; \forall y\,(\psi(y) \wedge \varphi). \tag{1.62}$$

d) *The disjunction involving an arbitrary given sentence φ is neutral with respect to universal quantification in the sense that*

$$[\varphi \vee \forall y\,(\psi(y))] \;\Leftrightarrow\; \forall y\,(\varphi \vee \psi(y)), \tag{1.63}$$

$$[\forall y\,(\psi(y)) \vee \varphi] \;\Leftrightarrow\; \forall y\,(\psi(y) \vee \varphi) \tag{1.64}$$

e) The implication involving an arbitrary given antecedent φ is neutral with respect to universal quantification in the sense that

$$[\varphi \Rightarrow \forall y\,(\psi(y))] \;\Leftrightarrow\; \forall y\,(\varphi \Rightarrow \psi(y)). \tag{1.65}$$

Proof. Concerning a), we prove the equivalence (1.57) with two direct proofs of an implication, the first of these in combination with a proof of an existential sentence, the second in combination with a proof of a conjunction. To prove the first implication ('$\Rightarrow$') directly, we assume that

$$\varphi \,\wedge\, \exists y(\psi(y)) \tag{1.66}$$

is true. On the one hand, the second part $\exists y(\psi(y))$ of this conjunction is thus true in particular, which means that there exists some y within the domain of discourse, say $\bar{y}$, such that $\psi(\bar{y})$ holds. On the other hand, the conjunction (1.66) implies in particular the truth of φ, so that φ and $\psi(\bar{y})$ are both true. Thus, the conjunction $\varphi \wedge \psi(\bar{y})$ holds, which shows that $\varphi \wedge \psi(y)$ is true for some y. This finding establishes the truth of the existential sentence $\exists y(\varphi \wedge \psi(y))$, so that the proof of the first implication is complete.

To prove the second implication ('$\Leftarrow$') directly, we now assume the existential sentence $\exists y(\varphi \wedge \psi(y))$, which means that there is some y, say $\bar{y}$, such that

$$\varphi \,\wedge\, \psi(\bar{y}) \tag{1.67}$$

is true. This conjunction implies, on the one hand, especially $\psi(\bar{y})$, which shows that $\psi(y)$ holds for some y, so that the existential sentence $\exists y(\psi(y))$ is true. On the other hand, the conjunction (1.67) implies especially φ. Thus, φ and $\exists y(\psi(y))$ are both true, so that the conjunction $\varphi \wedge \exists y(\psi(y))$ holds, which proves the second implication. Consequently, the equivalence (1.57) is true.

Concerning b), we prove the first implication ('$\Rightarrow$') of (1.59) directly, assuming

$$\varphi \,\vee\, \exists y(\psi(y)) \tag{1.68}$$

to be true. In case the first part φ of this disjunction is true, we replace the variable y in $\psi(y)$ by an arbitrary $\bar{y}$ from the domain of discourse and form the disjunction $\varphi \vee \psi(\bar{y})$; since the first part of this disjunction is true, the disjunction itself is also true, irrespective of the truth value of $\psi(\bar{y})$. In case the second part $\exists y(\psi(y))$ of the disjunction (1.68) is true, there is some y, say $\bar{y}$, such that $\psi(\bar{y})$ holds; then, the disjunction $\varphi \vee \psi(\bar{y})$ is also true, irrespective of the truth value of φ. Thus, the disjunction $\varphi \vee \psi(\bar{y})$ is true in any case, and the truth of this disjunction shows that there is some

y for which $\varphi \vee \psi(y)$ becomes true. This means that $\exists y(\varphi \vee \psi(y))$ holds, so that the proof of the first part of the equivalence (1.59) is complete.

To prove the second part ('$\Leftarrow$') directly, we now assume the existential sentence $\exists y(\varphi \vee \psi(y))$ to be true, so that there is some y, say $\bar{y}$, satisfying

$$\varphi \ \vee \ \psi(\bar{y}). \tag{1.69}$$

If the first part φ of this disjunction is true, then the disjunction $\varphi \vee \exists y(\psi(y))$ also holds (irrespective of the truth value of $\exists y(\psi(y))$). If the second part $\psi(\bar{y})$ of the disjunction (1.69) is true, then there evidently is some y such that $\psi(y)$ holds, which means that the existential sentence $\exists y(\psi(y))$ is true. Then, the disjunction $\varphi \vee \exists y(\psi(y))$ holds again (irrespective of the truth value of φ), so that the proof of the second implication is complete. Thus, the equivalence (1.59) is true.

Concerning c), we prove the first part ('$\Rightarrow$') of the equivalence (1.61) directly, assuming

$$\varphi \ \wedge \ \forall y\,(\psi(y)) \tag{1.70}$$

to hold. We now take an arbitrary $\bar{y}$. As the preceding conjunction implies especially $\forall y\,(\psi(y))$, we have that $\psi(\bar{y})$ is true in particular. Moreover, the conjunction (1.70) implies especially also the truth of φ. Thus, φ and $\psi(\bar{y})$ are both true, so that the conjunction $\varphi \wedge \psi(\bar{y})$ holds (for the arbitrary $\bar{y}$). We may therefore conclude that the conjunction $\varphi \wedge \psi(y)$ is true for any y, which means $\forall y(\varphi \wedge \psi(y))$. Thus, the first part of the equivalence (1.61) holds.

Regarding the second part ('$\Leftarrow$'), we now assume $\forall y(\varphi \wedge \psi(y))$ to be true. Taking an arbitrary constant $\bar{y}$ from the domain of discourse, we then have that $\varphi \wedge \psi(\bar{y})$ is true in particular, which conjunction in turn implies especially the truth of φ. Next, we establish the universal sentence $\forall y(\psi(y))$ and let $\bar{\bar{y}}$ be arbitrary. In view of the assumed universal sentence, we see that $\varphi \wedge \psi(\bar{\bar{y}})$ is then true in particular, and that this conjunction now further implies the truth especially of $\psi(\bar{\bar{y}})$. Since $\bar{\bar{y}}$ is arbitrary, we may therefore conclude that the universal sentence $\forall y(\psi(y))$ is true, besides φ. Thus, the conjunction $\varphi \wedge \forall y(\psi(y))$ holds, which proves the implication '$\Leftarrow$', so that the proof of the first equivalence in c) is complete.

Concerning d), we prove the first part ('$\Rightarrow$') of the equivalence (1.63) directly, making the assumption that

$$\varphi \ \vee \ \forall y\,(\psi(y)) \tag{1.71}$$

is true. To prove the universal sentence on the right-hand side of the equivalence (1.63), we let $\bar{y}$ be arbitrary within the domain of discourse. On the one hand, in case the first part φ of the preceding disjunction is true, the disjunction $\varphi \vee \psi(\bar{y})$ is then also true (irrespective of the truth value of $\psi(\bar{y})$). On the other hand, in case the second part $\forall y(\psi(y))$ of the disjunction (1.71) is true, then $\psi(\bar{y})$ is in particular true for the arbitrarily selected $\bar{y}$, so that the disjunction $\varphi \vee \psi(\bar{y})$ is also true (irrespective of the truth value of φ). We thus showed that $\varphi \vee \forall y(\psi(y))$ implies $\varphi \vee \psi(\bar{y})$ for an arbitrary $\bar{y}$, so that we may conclude that this disjunction is true for any y. Thus, $\forall y(\varphi \vee \psi(y))$ holds, completing the proof of the first implication.

We prove the second part ('$\Leftarrow$') of the equivalence by contraposition, i.e. we prove (directly) the implication

$$\neg[\varphi \vee \forall y\,(\psi(y))] \Rightarrow \neg\forall y\,(\varphi \vee \psi(y)).$$

For this purpose we assume that the antecedent $\neg[\varphi \vee \forall y(\psi(y))]$ is true, which then implies the conjunction $\neg\varphi \wedge \neg\forall y(\psi(y))$ with De Morgan's Law for sentences (1.52), and therefore especially $\neg\varphi$ as well as $\neg\forall y(\psi(y))$. The latter negation now further implies $\exists y(\neg\psi(y))$ with the Quantifier Negation Law (1.53). Since $\neg\varphi$ is also true, the conjunction $\neg\varphi \wedge \exists y(\neg\psi(y))$ holds. This in turn implies $\exists y(\neg\varphi \wedge \neg\psi(y))$ with a), so that there exists some y, say $\bar{y}$, such that $\neg\varphi \wedge \neg\psi(\bar{y})$ holds. Applying now again De Morgan's Law for sentences (1.52), we obtain the true negation $\neg[\varphi \vee \psi(\bar{y})]$, which shows that $\neg[\varphi \vee \psi(y)]$ is true for some y. This means $\exists y(\neg[\varphi \vee \psi(y)])$, which implies the desired consequent $\neg\forall y(\varphi \vee \psi(y))$ with the Quantifier Negation Law (1.53). This completes the proof of the second part of the proposed equivalence, which is therefore true.

Concerning e), we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\varphi \Rightarrow \forall y(\psi(y))$ to be true, which implication implies with the Conditional Law (1.31) the true disjunction $\neg\varphi \vee \forall y(\psi(y))$, which in turn implies $\forall y(\neg\varphi \vee \psi(y))$ with d). To establish the right-hand side $\forall y(\varphi \Rightarrow \psi(y))$ of the equivalence (1.65), we let $\bar{y}$ be arbitrary, so that the previously obtained universal sentence gives in particular the true sentence $\neg\varphi \vee \psi(\bar{y})$, which implies $\varphi \Rightarrow \psi(\bar{y})$ again with the Conditional Law (1.31). Since $\bar{y}$ is arbitrary, we may therefore conclude that the universal sentence $\forall y(\varphi \Rightarrow \psi(y))$ holds, which completes the proof of the first part of the proposed equivalence e).

To prove the second part ('$\Leftarrow$'), we now assume the universal sentence $\forall y(\varphi \Rightarrow \psi(y))$ to be true. To show that this implies the implication $\varphi \Rightarrow \forall y(\psi(y))$, we assume that φ holds and demonstrate that $\forall y(\psi(y))$ follows to be true. For this purpose, we take an arbitrary $\bar{y}$, so that the assumed universal sentence yields in particular $\varphi \Rightarrow \psi(\bar{y})$. Since φ is by assumption true, this implies $\psi(\bar{y})$ with the preceding implication. As $\bar{y}$ was arbitrary, we may therefore conclude that the universal sentence $\forall y(\psi(y))$ is indeed true. This in turn proves the implication $\varphi \Rightarrow \forall y(\psi(y))$ and thus the second part of the equivalence e). Consequently, (1.65) is also true. $\square$

Exercise 1.13. Prove (1.58), (1.60), (1.62), and (1.64).
(Hint: Proceed in analogy to the proofs of (1.57), (1.59), (1.61) and (1.63).)

Strict neutrality of the implication with respect to universal quantification does not hold in case the sentence φ appears as the consequent.

Exercise 1.14. Verify the following sentences for an arbitrary sentence φ and an arbitrary formula $\psi(y)$.

$$[\exists y\,(\psi(y)) \Rightarrow \varphi] \;\Leftrightarrow\; \forall y\,(\psi(y) \Rightarrow \varphi). \tag{1.72}$$

$$\exists y\,(\psi(y)) \;\Rightarrow\; [\varphi \Leftrightarrow \forall y\,(\psi(y) \Rightarrow \varphi)]. \tag{1.73}$$

(Hint: regarding (1.72), prove both parts of the equivalence directly. Prove the implication (1.73) and both parts of the equivalence therein directly.)

Theorem 1.18 (Distributive Laws for quantification). *The following sentences are true for arbitrary formulas $\varphi(y)$ and $\psi(y)$.*

a) Universal quantification distributes over the conjunction in the sense that

$$\forall y\,(\varphi(y) \,\wedge\, \psi(y)) \;\Leftrightarrow\; [\forall y\,(\varphi(y)) \,\wedge\, \forall y\,(\psi(y))]. \tag{1.74}$$

b) Existential quantification distributes over the disjunction in the sense that

$$\exists y\,(\varphi(y) \,\vee\, \psi(y)) \;\Leftrightarrow\; [\exists y\,(\varphi(y)) \,\vee\, \exists y\,(\psi(y))]. \tag{1.75}$$

Proof. Regarding a), we carry out a proof of an equivalence by proving the two implications directly, the first in combination with a proof of a conjunction, the second in combination with a proof of a universal sentence. To prove the first implication ('$\Rightarrow$'), we make the assumption that $\forall y(\varphi(y) \wedge \psi(y))$ is true. To prove the universal sentence on the right-hand side of the equivalence (1.74), we let $\bar{y}$ be arbitrary. Because of the assumed universal sentence,

$$\varphi(\bar{y}) \wedge \psi(\bar{y}) \tag{1.76}$$

is then in particular true. This conjunction implies, on the one hand, that $\varphi(\bar{y})$ is especially true (for the arbitrary $\bar{y}$), so that we may conclude that the universal sentence $\forall y(\varphi(y))$ holds. On the other hand, the conjunction (1.76) implies especially that $\psi(\bar{y})$ is true (for the arbitrary $\bar{y}$), from which we may infer that the universal sentence $\forall y(\psi(y))$ holds as well. Since $\forall y(\varphi(y))$ and $\forall y(\psi(y))$ are both true, the conjunction $\forall y(\varphi(y)) \wedge \forall y(\psi(y))$ holds, which proves the first part of the proposed equivalence.

To prove the second implication ('$\Leftarrow$') directly, we now assume that $\forall y(\varphi(y)) \wedge \forall y(\psi(y))$ is true, which means that $\forall y\,(\varphi(y))$ and $\forall y\,(\psi(y))$ are both true. To prove the universal sentence on the left-hand side of the equivalence (1.74), we take an arbitrary $\bar{y}$. In view of the previous two universal sentences, we then see that $\varphi(\bar{y})$ and $\psi(\bar{y})$ are both true in particular, which means that the conjunction $\varphi(\bar{y}) \wedge \psi(\bar{y})$ holds. As $\bar{y}$ was arbitrary, we therefore conclude that $\forall y(\varphi(y) \wedge \psi(y))$ is true, which proves the second part of the equivalence. As both implications are true, the proposed equivalence a) holds.

Concerning b), we first prove ('$\Rightarrow$') directly, by assuming $\exists y\,(\varphi(y) \vee \psi(y))$, which means that there is some y, say $\bar{y}$, such that

$$\varphi(\bar{y}) \;\vee\; \psi(\bar{y}) \tag{1.77}$$

is true. If, on the one hand, $\varphi(\bar{y})$ is true, we have that $\varphi(y)$ holds for some y, which means that $\exists y(\varphi(y))$ holds; then, the disjunction $\exists y(\varphi(y)) \vee \exists y(\psi(y))$ is also true, irrespective of the truth value of $\exists y(\psi(y))$. If, on the other hand, the second part $\psi(\bar{y})$ of the disjunction (1.77) is true, then $\psi(y)$ holds for some y, which means that $\exists y(\psi(y))$ holds; consequently, the disjunction $\exists y(\varphi(y)) \vee \exists y(\psi(y))$ is true as well, irrespective of the truth value of $\exists y(\psi(y))$. Thus, the first part of the equivalence holds.

To prove ('$\Leftarrow$'), we now assume

$$\exists y(\varphi(y)) \vee \exists y(\psi(y)). \tag{1.78}$$

On the one hand, if $\exists y(\varphi(y))$ is true, this means that there is some y, say $\bar{y}$, such that $\varphi(\bar{y})$ holds, which gives the true disjunction $\varphi(\bar{y}) \vee \psi(\bar{y})$ (irrespective of the truth value of $\psi(\bar{y})$); this shows that $\varphi(y) \vee \psi(y)$ holds for some y, so that $\exists y(\varphi(y) \vee \psi(y))$ holds. On the other hand, if the second part $\exists y(\psi(y))$ of the disjunction in (1.78) is true, there is some y, say $\bar{y}$, such that $\psi(\bar{y})$ holds, which yields again the true disjunction $\varphi(\bar{y}) \vee \psi(\bar{y})$ (irrespective of the truth value of $\varphi(\bar{y})$); as before, this means that $\exists y(\varphi(y) \vee \psi(y))$ is true, so that we proved the second implication and thus the equivalence. $\square$

Note 1.7. We see in (1.74) that $\forall$ is in a certain sense 'compatible' with $\wedge$, and similarly in (1.75) that $\exists$ is compatible with $\vee$. These kinds of compatibility are also reflected by the fact that, on the one hand, $\forall y(\varphi(y))$ means that $\varphi(y)$ is true for a constant $\bar{a}$ of the domain of discourse, and is true also for another constant $\bar{b}$, and for another $\bar{c}$, etc., so that $\forall$ connects all the true sentences $\varphi(\bar{a})$, $\varphi(\bar{b})$, $\varphi(\bar{c})$, etc., with conjunctions.

Similarly, $\exists y(\varphi(y))$ means that $\varphi(y)$ may be found to be true after replacing the variable y by some particular constant $\bar{a}$, or $\bar{b}$, or $\bar{c}$, etc., so that

$\exists$ connects the possibly true sentences $\varphi(\bar{a})$, $\varphi(\bar{b})$, $\varphi(\bar{c})$, etc., by means of disjunctions.

Thus, we consider the equivalences

$$\forall y\,(\varphi(y)) \;\Leftrightarrow\; [\varphi(\bar{a}) \,\wedge\, \varphi(\bar{b}) \,\wedge\, \varphi(\bar{c}) \,\wedge\, \cdots], \tag{1.79}$$

$$\exists y\,(\varphi(y)) \;\Leftrightarrow\; [\varphi(\bar{a}) \,\vee\, \varphi(\bar{b}) \,\vee\, \varphi(\bar{c}) \,\vee\, \cdots] \tag{1.80}$$

to be true.

Theorem 1.19 (Negation Law for existential conjunctions & for universal implications). *The following equivalences are true for arbitrary formulas $\varphi(y)$ and $\vartheta(y)$.*

a) **Negation Law for existential conjunctions:**

$$\neg\exists y\,(\varphi(y) \,\wedge\, \vartheta(y)) \;\Leftrightarrow\; \forall y\,(\varphi(y) \Rightarrow \neg\vartheta(y)). \tag{1.81}$$

b) **Negation Law for universal implications:**

$$\neg\forall y\,(\varphi(y) \Rightarrow \neg\vartheta(y)) \;\Leftrightarrow\; \exists y\,(\varphi(y) \,\wedge\, \vartheta(y)). \tag{1.82}$$

Proof. Concerning (1.81), we first observe that $\neg\exists y(\varphi(y) \wedge \vartheta(y))$ is equivalent to

$$\forall y\,(\neg[\varphi(y) \,\wedge\, \vartheta(y)]) \tag{1.83}$$

because of the Quantifier Negation Law (1.54). We now prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\neg\exists y(\varphi(y) \wedge \vartheta(y))$ to be true, which then implies the truth of (1.83). To establish the universal sentence

$$\forall y\,(\varphi(y) \Rightarrow \neg\vartheta(y)) \tag{1.84}$$

on the right-hand side of the proposed equivalence (1.81), we take an arbitrary $\bar{y}$, so that the previously established universal sentence (1.83) yields in particular the true sentence

$$\neg[\varphi(\bar{y}) \,\wedge\, \vartheta(\bar{y})], \tag{1.85}$$

which in turn implies

$$\neg\varphi(\bar{y}) \,\vee\, \neg\vartheta(\bar{y}) \tag{1.86}$$

with De Morgan's Law for sentences (1.51). This disjunction now further implies

$$\varphi(\bar{y}) \Rightarrow \neg\vartheta(\bar{y}) \tag{1.87}$$

with the Conditional Law (1.31), and since $\bar{y}$ was arbitrary, we may therefore conclude that the universal sentence (1.84) is true. Thus, the proof of the first part of the equivalence (1.81) is complete.

To prove the second part ('$\Leftarrow$'), we now assume the universal sentence (1.84) holds, and we show that the universal sentence (1.83) follows then to be true. To do this, we let $\bar{y}$ be arbitrary, so that the assumption (1.84) gives the true sentence (1.87). This implication then implies (1.86) with the Conditional Law (1.31), and furthermore (1.85) with De Morgan's Law for sentences (1.51). Because $\bar{y}$ is arbitrary, we may therefore conclude that the universal sentence (1.83) is indeed true, which in turn implies the truth of the equivalent negation $\neg\exists y(\varphi(y) \wedge \vartheta(y))$. This completes the proof of the second part of the equivalence, which is thus true.

Concerning (1.82), we observe in light of the true equivalence (1.81) that the implications

$$\neg\exists y\,(\varphi(y)\ \wedge\ \vartheta(y))\ \Rightarrow\ \forall y\,(\varphi(y) \Rightarrow \neg\vartheta(y))$$
$$\forall y\,(\varphi(y) \Rightarrow \neg\vartheta(y))\ \Rightarrow\ \neg\exists y\,(\varphi(y)\ \wedge\ \vartheta(y))$$

are true. Next, we apply the Law of Contraposition to both implications and obtain the true implications

$$\neg\forall y\,(\varphi(y) \Rightarrow \neg\vartheta(y))\ \Rightarrow\ \neg\neg\exists y\,(\varphi(y)\ \wedge\ \vartheta(y))$$
$$\neg\neg\exists y\,(\varphi(y)\ \wedge\ \vartheta(y))\ \Rightarrow\ \neg\forall y\,(\varphi(y) \Rightarrow \neg\vartheta(y)),$$

so that the equivalence

$$\neg\forall y\,(\varphi(y) \Rightarrow \neg\vartheta(y))\ \Leftrightarrow\ \neg\neg\exists y\,(\varphi(y)\ \wedge\ \vartheta(y))$$

holds. Since the equivalence

$$\exists y\,(\varphi(y)\ \wedge\ \vartheta(y))\ \Leftrightarrow\ \neg\neg\exists y\,(\varphi(y)\ \wedge\ \vartheta(y))$$

also holds because of the Double Negation Law, it follows with the Substitution Rule for equivalences (1.28) that the proposed equivalence (1.82) is true. $\square$

Exercise 1.15. Establish the following equivalences for arbitrary formulas $\varphi(x, y)$ and $\vartheta(x, y)$.

$$\neg\exists x, y\,(\varphi(x,y)\ \wedge\ \vartheta(x,y))\ \Leftrightarrow\ \forall x, y\,(\varphi(x,y) \Rightarrow \neg\vartheta(x,y)), \tag{1.88}$$
$$\neg\forall x, y\,(\varphi(x,y) \Rightarrow \neg\vartheta(x,y))\ \Leftrightarrow\ \exists x, y\,(\varphi(x,y)\ \wedge\ \vartheta(x,y)). \tag{1.89}$$

(Hint: Proceed in analogy to the proof of Theorem 1.19.)

Exercise 1.16. Prove the following equivalence for arbitrary formulas $\varphi(x)$, $\psi(y)$ and $\vartheta(x, y)$.

$$\forall x, y\,([\varphi(x)\ \wedge\ \psi(y)] \Rightarrow \vartheta(x,y))\ \Leftrightarrow\ \forall x\,(\varphi(x) \Rightarrow \forall y\,(\psi(y) \Rightarrow \vartheta(x,y))). \tag{1.90}$$

(Hint: Prove both parts of the equivalence directly.)

1.3.4. Equality

The following notion of 'sameness' will be fundamental.

Principle 1.4 (Leibniz' Law). Two given things x and y are equal or identical, symbolically

$$x = y, \tag{1.91}$$

iff x has every property that y has and y has every property that x has.

Definition 1.8 (Equation, inequality). For any x and y we call the sentence $x = y$ an *equation*. Furthermore, we say that x and y are *unequal* iff $\neg x = y$ is true, in which case we also write

$$x \neq y. \tag{1.92}$$

We then call this sentence an *inequality*.

We may consider the equality symbol $=$ as establishing a kind of *relation*[7] between x and y. The following notation is convenient in case there is another thing z besides y that related to x by equality.

Notation 1.3. We will also write

$$x = y = z \tag{1.93}$$

instead of

$$x = y \land y = z. \tag{1.94}$$

When an equation is given we may apply the fundamental method of *substitution*.

Method 1.16 (Substitution). Let $x = y$ be a true equation and $\varphi(x)$ a given sentence containing the constant x. Then we may substitute for x the identical y in $\varphi(x)$ and write $\varphi(y)$. Similarly, when $\psi(y)$ is another given sentence containing y, we may substitute for y the equal x in $\psi(y)$ and write $\psi(x)$.

The formation of equations follows four frequently applied rules, which are so basic that we will usually not refer to them explicitly.

Exercise 1.17. Show that the following sentences are true.

a) $=$ is *reflexive* in the sense that

$$\forall x\, (x = x), \tag{1.95}$$

[7] This notion will be given a formal definition in Chapter 3.

b) $=$ is *symmetrical* in the sense that

$$\forall x, y \,(x = y \Leftrightarrow y = x), \tag{1.96}$$

c) $=$ is *Euclidean* in the sense that

$$\forall x, y, z \,([x = z \,\wedge\, y = z] \Rightarrow x = y), \tag{1.97}$$

d) $=$ is *transitive* in the sense that

$$\forall x, y, z \,([x = y \,\wedge\, y = z] \Rightarrow x = z), \tag{1.98}$$

(Hint: Apply Method 1.14 and Method 1.16.)

Exercise 1.18. Show that the following sentence is true.

$$\forall x, y, z \,([x = y \Leftrightarrow x = z] \Leftrightarrow y = z). \tag{1.99}$$

(Hint: To prove '$\Rightarrow$', apply (1.8) and Method 1.11)

Exercise 1.19. Show that $\neq$ is *symmetrical* in the sense that

$$\forall x, y \,(x \neq y \Leftrightarrow y \neq x). \tag{1.100}$$

1.3.5. Unique existence

We will frequently encounter existential sentences which are true for *precisely one* constant.

Definition 1.9 (There exists a unique/there is a unique, uniquely existential sentence). For an arbitrary formula $\varphi(X)$, we say that *there exists a unique* (or that *there is a unique*) X such that $\varphi(X)$ holds, symbolically

$$\exists! X \,(\varphi(X)) \tag{1.101}$$

iff there exists a set X such that φ holds and there is no constant which also satisfies φ but is different from X, that is,

$$\exists X \,(\varphi(X) \,\wedge\, \neg\exists X^* \,(\varphi(X^*) \,\wedge\, X^* \neq X)). \tag{1.102}$$

In this case, we speak of a *uniquely existential sentence*.

Based on the following proposition, we will see how such a sentence can be proved conveniently.

Theorem 1.20 (Criterion for unique existence). *The following equivalence is true for an arbitrary formula $\varphi(X)$.*

$$\exists! X\,(\varphi(X)) \;\Leftrightarrow\; [\exists X\,(\varphi(X)) \;\wedge\; \forall X, X'\,([\varphi(X) \;\wedge\; \varphi(X')] \Rightarrow X = X')]\,. \tag{1.103}$$

Proof. To prove the first part ('$\Rightarrow$') of the proposed equivalence directly, we assume $\exists! X\,(\varphi(X))$, so that the existential sentence (1.102) holds by definition of a uniquely existential sentence. Thus, there exists a constant, say $\bar{X}$, such that the conjunction

$$\varphi(\bar{X}) \;\wedge\; \neg\exists X^*\,(\varphi(X^*) \;\wedge\; X^* \neq \bar{X})$$

is true. This means that the sentence $\varphi(\bar{X})$ and the negation

$$\neg\exists X^*\,(\varphi(X^*) \;\wedge\; X^* \neq \bar{X})$$

both hold. Here, we may write the latter equivalently as

$$\forall X^*\,(\varphi(X^*) \Rightarrow \neg X^* \neq \bar{X}) \tag{1.104}$$

by applying (1.81). The truth of $\varphi(\bar{X})$ shows that the first part of the conjunction in (1.103) to be proven is true. We now establish the second part

$$\forall X, X'([\varphi(X) \;\wedge\; \varphi(X')] \Rightarrow X = X'). \tag{1.105}$$

To do this, we let X and X' be arbitrary, and we prove the implication in (1.104) directly, assuming the conjunction $\varphi(X) \wedge \varphi(X')$ to be true. On the one hand, this conjunction implies the truth of $\varphi(X)$, which in turn implies $\neg X \neq \bar{X}$ with (1.104), and therefore $X = \bar{X}$ with the Double Negation Law. On the other hand, the preceding conjunction implies in particular $\varphi(X')$ and then $\neg X' \neq \bar{X}$ with (1.104), so that $X' = \bar{X}$ $[= X]$ follows to be true with the Double Negation Law. These equations yield $X = X'$, which proves the implication in (1.105). Since X and X' are arbitrary, we may therefore conclude that the universal sentence (1.105) holds, completing the proof of the conjunction in (1.103) and completing thus the proof of the first part of the proposed equivalence.

Next, we prove the second part ('$\Leftarrow$') of the equivalence (1.103) directly. For this purpose, we assume

$$\exists X\,(\varphi(X)) \;\wedge\; \forall X, X'\,([\varphi(X) \;\wedge\; \varphi(X')] \Rightarrow X = X')$$

is true. Then, both parts of this conjunction, $\exists X\,(\varphi(X))$ and (1.105) are true by themselves. Thus, there is a constant, say $\bar{\bar{X}}$, such that $\varphi(\bar{\bar{X}})$ is true. Let us now prove the universal sentence

$$\forall X^*\,(\varphi(X^*) \Rightarrow \neg X^* \neq \bar{\bar{X}}). \tag{1.106}$$

To do this, we take an arbitrary X^* and assume that $\varphi(X^*)$ holds. Thus, the conjunction $\varphi(X^*) \wedge \varphi(\bar{\bar{X}})$ is true, which further implies $X^* = \bar{\bar{X}}$ with the second part of the assumed conjunction. This equation gives with the Double Negation Law $\neg X^* \neq \bar{\bar{X}}$, which negation proves the implication in (1.106). Since X^* is arbitrary, we therefore conclude that the universal sentence (1.106) holds indeed, which in turn implies

$$\neg \exists X^* \, (\varphi(X^*) \wedge X^* \neq \bar{\bar{X}}))$$

with (1.81). Together with the true sentence $\varphi(\bar{\bar{X}})$, this shows that the conjunction

$$\varphi(\bar{\bar{X}}) \wedge \neg \exists X^* \, (\varphi(X^*) \wedge X^* \neq \bar{\bar{X}})$$

holds, so that the existential sentence

$$\exists X \, (\varphi(X) \wedge \neg \exists X^* \, (\varphi(X^*) \wedge X^* \neq X))$$

is evidently true. This means by definition of a uniquely existential sentence that $\exists! X \, (\varphi(X))$ holds, so that the proof of the equivalence (1.103) is complete. $\square$

The Criterion for unique existence yields the following proof technique.

Method 1.17 (Proof of a uniquely existential sentence). To carry out a proof of a uniquely existential sentence (1.102), we may in light of the Criterion for unique existence show by two separate proofs that

1. there exists a constant X for which $\varphi(X)$ is a true sentence, i.e.

$$\exists X \, (\varphi(X)), \tag{1.107}$$

   which we call the so-called *existential part*), and that

2. whenever two constants X and X' turn the given formula into true sentences they follow to be identical, i.e.

$$\forall X, X' \, ([\varphi(X) \wedge \varphi(X')] \Rightarrow X = X'), \tag{1.108}$$

   which we call the *uniqueness part*.

We demonstrate the application of this method by establishing another basic property of '='.

Proposition 1.21. *For any y there exists a unique x which is identical with y holds, that is,*

$$\forall y \, (\exists! x \, (x = y)). \tag{1.109}$$

Proof. To prove the universal sentence, we let $\bar{y}$ be arbitrary and apply then Method 1.17, letting $\varphi(x)$ be the formula $x = \bar{y}$. Regarding the existential part, we observe that $\bar{y}$ satisfies $\bar{y} = \bar{y}$ due to the reflexivity (1.95) of '='; thus, the existential sentence $\exists y(\varphi(y))$ is true. Regarding the uniqueness part, we verify

$$\forall x, x' \left([\varphi(x) \wedge \varphi(x')] \Rightarrow x = x' \right). \tag{1.110}$$

To do this, we let x and x' be arbitrary and prove the implication directly, assuming $\varphi(x) \wedge \varphi(x')$ to be true sentences, which mean that the equations $x = \bar{y}$ and $x' = \bar{y}$ are both true. This conjunction implies the desired $x = x'$ since '=' is Euclidean according to (1.97). Since x and x' are arbitrary, we may therefore conclude that the universal sentence (1.110) is true, so that the uniqueness part holds as well. Thus, the proof of the uniquely existential sentence (1.109) is true, and as Y was arbitrary, it follows that the proposition is true. □

If a certain constant X^* turns a given formula $\varphi(X)$ into a true sentence and if we can demonstrate that any other constant X for which $\varphi(X)$ is true turns out to be identical with X^*, we would naturally say that X^* is the only constant for which that sentence holds. Evidently then, such a demonstration should be a suitable method for proving a uniquely existential sentence.

Proposition 1.22. *The following implication is true for an arbitrary given formula $\varphi(X)$ and an arbitrary given constant $\bar{X}$.*

$$[\varphi(\bar{X}) \wedge \forall X \left(\varphi(X) \Rightarrow \bar{X} = X \right)] \;\Rightarrow\; \exists! X \, \varphi(X). \tag{1.111}$$

Proof. We prove the implication directly, assuming that $\varphi(\bar{X})$ and

$$\forall X \left(\varphi(X) \Rightarrow \bar{X} = X \right) \tag{1.112}$$

are true sentences. The first of these two assumptions shows that the existential sentence $\exists X \, \varphi(X)$ holds, which proves the existential part of the proof of the uniquely existential sentence $\exists! X \, \varphi(X)$. To prove the uniqueness part, we verify

$$\forall X, X' \left([\varphi(X) \wedge \varphi(X')] \Rightarrow X = X' \right). \tag{1.113}$$

For this purpose, we take arbitrary constants X and X' and assume $\varphi(X) \wedge \varphi(X')$ to be true. This conjunction implies, on the one hand, especially the truth of $\varphi(X)$, which in turn implies

$$\bar{X} = X \tag{1.114}$$

with the assumption (1.112). On the other hand, the preceding conjunctions shows that the sentence $\varphi(X')$ holds, which further implies

$$\bar{X} = X' \tag{1.115}$$

with (1.112). Clearly, combining the equations (1.114) and (1.115) via substitution yields $X = X'$, which finding proves the implication in (1.113). Then, as X and X' were arbitrary, we may therefore conclude that the universal sentence (1.113) holds. Thus, the proof of the uniquely existential sentence $\exists! X\, \varphi(X)$ is complete, and the truth of this sentence establishes the proposed implication. □

Method 1.18 (Proof of a uniquely existential sentence (variant)). To carry out a proof of a uniquely existential sentence $\exists! X\, \varphi(X)$ for a given formula $\varphi(X)$, we may find a particular constant $\bar{X}$ for which $\varphi(X)$ becomes a true sentence and prove then the universal sentence

$$\forall X' \,(\varphi(X') \Rightarrow \bar{X} = X'). \tag{1.116}$$

We define uniquely an existential sentence involving two variables in analogy to the Criterion for unique existence (for a single variable).

Notation 1.4. We will also write

$$\exists! X, Y\, (\varphi(X, Y)) \tag{1.117}$$

instead of

$$\begin{aligned} &\exists X, Y\, (\varphi(X, Y)) \\ &\quad \wedge\ \forall X, Y, X', Y'\, ([\varphi(X, Y)\ \wedge\ \varphi(X', Y')] \Rightarrow [X = X'\ \wedge\ Y = Y']). \end{aligned} \tag{1.118}$$

Exercise 1.20. Establish the equivalence of the existential sentence

$$\exists X, Y\, (\varphi(X, Y)\ \wedge\ \neg\exists X^*, Y^*\, (\varphi(X^*, Y^*)\ \wedge\ [X^* \neq X\ \vee\ Y^* \neq Y]))$$

and the conjunction (1.118).

(Hint: Use similar arguments as in the proof of Theorem 1.20, replacing (1.35) by (1.52) and replacing (1.81) by (1.88).)

Chapter 2.

Sets

2.1. Basic Characteristics of Sets

Definition 2.1 (Set, element, set system). In Georg Cantor's words:

> Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unsrer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen. (Cantor, 1895, p.481).

That is, we understand a *set* to be any aggregation M of certain and clearly distinct objects within our perception or conception (which are called the *elements* of M) to a whole.

We also say that a set *contains* elements. More specifically, we say that a set *consists of* certain elements in case the set contains precisely these elements and no other elements. To express that y is an element of a set A, we write

$$y \in A \tag{2.1}$$

In case we consider multiple elements x, y, etc. of A, we write instead of $(y \in A \land z \in A) \land \cdots$ the shorter sentence

$$y, z, \ldots \in A \tag{2.2}$$

Furthermore, we write

$$y \notin A \tag{2.3}$$

iff it is false that y is in A, that is, iff $\neg y \in A$ is true.

In certain situations, we wish to highlight the fact that a given set consists itself of certain sets, in which case we will also speak of a *set system*.

We may use the concept of non-membership '$\notin$' to characterize distinctness '$\neq$'.

Proposition 2.1. *If x is an element of a set A and if y is not in A, then x and y are distinct, that is,*

$$\forall A, x, y\,([x \in A \,\wedge\, y \notin A] \Rightarrow x \neq y). \tag{2.4}$$

Proof. We let x and y be arbitrary and prove the implication by contradiction, assuming that the conjunction of $x \in A$, $y \notin A$ and $x = y$ holds. Because of the equation, we may substitute y for x in $x \in A$, which gives $y \in A$. Thus, $y \in A$ and the assumption $y \notin A$ are both true, so that we have a contradiction according to (1.11). This completes the proof by contradiction of the implication in (2.4). As x and y were arbitrary, we therefore conclude that the universal sentence (2.4) is true. □

Conversely, we may characterize non-membership '$\notin$' by means of distinctness '$\neq$'.

Proposition 2.2. *A y is not an element of a set A iff any x in A is different from y, that is,*

$$\forall A, y\,(y \notin A \Leftrightarrow \forall x\,(x \in A \Rightarrow x \neq y)). \tag{2.5}$$

Proof. We let A and y be arbitrary and assume first $y \notin A$ to be true. To show that this implies

$$\forall x\,(x \in A \Rightarrow x \neq y), \tag{2.6}$$

we let x be arbitrary and assume $x \in A$ to be true. The conjunction of this assumption and the initially assumed $y \notin A$ then implies the desired consequent $x \neq y$ with (2.4). Since x is arbitrary, we may therefore conclude that the universal sentence (2.6) is true, so that the first part ('$\Rightarrow$') of the proposed equivalence holds. We now prove the second part ('$\Leftarrow$') by contradiction, assuming (2.6) and $\neg y \notin A$ to be both true. The latter assumption yields with the Double Negation Law $y \in A$, which in turn implies with the assumed (2.6) that $y \neq y$, in contradiction to the fact that $y = y$ holds. Thus, the proof of the equivalence is complete, and since A and y were arbitrary, we may therefore conclude that the proposed universal sentence (2.5) is true. □

Notation 2.1. We will generally use calligraphic letters (e.g. $\mathcal{A}$, $\mathcal{X}$) to denote set systems, upper case letters (e.g. A, X) to represent either the set elements of a set system or just sets without reference to a set system, and lower case letters (e.g. a, x, etc.) for elements of a set.

We now describe the situation in which all elements of a given set can be found in another set.

Definition 2.2 (Subset). For any sets A and B we say that A is a *subset* of B, symbolically

$$A \subseteq B, \tag{2.7}$$

if, and only if,

$$\forall y\,(y \in A \Rightarrow y \in B). \tag{2.8}$$

We then also say that B *includes* A.

Proposition 2.3. *If an y is not element of a set B, then y is not element of any subset of B either, that is,*

$$\forall A, B, y\,([A \subseteq B \,\wedge\, y \notin B] \Rightarrow y \notin A). \tag{2.9}$$

Proof. We let A, B and y be arbitrary, assume that A is a subset of B, and we further assume that y is not in B. Here, the assumption $A \subseteq B$ implies with the definition of a subset

$$\forall x\,(x \in A \Rightarrow x \in B),$$

so that $y \in A \Rightarrow y \in B$ is true in particular. We may write this implication equivalently as $y \notin B \Rightarrow y \notin A$, so that the initial assumption $y \notin B$ implies $y \notin A$. This proves the implication in (2.9), and since A, B and y were arbitrary, we may therefore conclude that the proposed universal sentence (2.9) is true. □

Proposition 2.4. *Any set is a subset of itself, that is,*

$$\forall B\,(B \subseteq B). \tag{2.10}$$

Proof. We let B be an arbitrary set and prove $B \subseteq B$ by verifying the equivalent universal sentence (using the definition of a subset)

$$\forall y\,(y \in B \Rightarrow y \in B). \tag{2.11}$$

For this purpose, we let y be arbitrary and observe then the truth of the disjunction $y \in B \,\vee\, y \notin B$. If $y \in B$ is true, both the antecedent and the consequent of the implication in (2.11) are true, so that the implication itself also holds. If $y \notin B$ is true, then the antecedent of the implication is false, and therefore the implication itself is again true. Since y was arbitrary, we may therefore conclude that the universal sentence (2.11) holds, which means $B \subseteq B$ by definition of a subset. As B was an arbitrary set, we may then further conclude that the proposed universal sentence (2.10) is true. □

The concepts of a subset and equality of sets involve universal quantification, requiring the specification of a domain of discourse. From now on, this domain will consist only of sets and, for maximum flexibility, consist of all conceivable sets.

Definition 2.3 (Universe). We say that U is the *universe* iff U is the domain of discourse consisting of all possible sets[1], symbolically

$$U = \{A : A = A\}. \tag{2.12}$$

Note 2.1. Since $A = A$ is a tautology, the sets within the universe do not have any specific and thereby limiting property.

Proposition 2.5. *The inclusion $\subseteq$ is transitive in the sense that*

$$\forall A, B, C\,([A \subseteq B \,\wedge\, B \subseteq C] \Rightarrow A \subseteq C). \tag{2.13}$$

Proof. To prove the stated universal sentence (using Method 1.14) we let A, B, C be arbitrary sets within the universe. Then, to prove the implication directly (applying Method 1.6), we assume $A \subseteq B \,\wedge\, B \subseteq C$, which means that

$$\forall y\,(y \in A \Rightarrow y \in B) \tag{2.14}$$

and

$$\forall y\,(y \in B \Rightarrow y \in C) \tag{2.15}$$

are both true by Definition 2.2. To show that this implies $A \subseteq C$, that is,

$$\forall y\,(y \in A \Rightarrow y \in C), \tag{2.16}$$

we let y be arbitrary and and prove the implication directly, assuming $y \in A$ to be true. This assumption implies with (2.14) the truth of $y \in B$, which in turn implies $y \in C$ with (2.15). Thus, the direct proof of the implication in (2.16) is complete, so that the inclusion $A \subseteq C$ follows to be by definition of a subset true. This finding completes the direct proof of the implication in (2.13), and since A, B, C were arbitrary sets, we may conclude that the proposed universal sentence (2.13) is true. $\square$

Note 2.2. Proposition 2.5 may be viewed as a set-theoretical analog to the Law of the Hypothetical Syllogism.

[1] It can be shown by means of a proof by contradiction that this domain of all possible sets is itself not a set, which is known as *Russel's paradox*. Such an 'improper' set is sometimes called a *class*.

We now apply the principle of equality of general things specifically to sets. In case every element of set A is in a set B and also every element of B is in A, we clearly have two sets consisting of the same elements. The following axiom states that we then consider the two sets to be equal (so that there are no other properties that A and B must share in order to be identical).

Axiom 2.1 (Axiom of Extension). For any sets A and B (within the universe) it is true that, if A is a subset of B and also B a subset of A, then A and B are equal, i.e.,

$$\forall A, B\,([A \subseteq B \,\land\, B \subseteq A] \Rightarrow A = B). \tag{2.17}$$

The following proposition includes this implication and also shows that we do not need another assumption to justify the reversed implication.

Theorem 2.6 (Equality Criterion for sets). *Any two sets A and B are identical if, and only if, every element of A is also an element of B and each element in B is also in A, i.e.*

$$\forall A, B\,(A = B \Leftrightarrow \forall y\,(y \in A \Leftrightarrow y \in B)). \tag{2.18}$$

Proof. To prove the universal sentence, we let A and B be arbitrary sets (within the universe). To verify the equivalence, we first prove the implication ('$\Rightarrow$')

$$A = B \;\Rightarrow\; \forall y\,(y \in A \Leftrightarrow y \in B) \tag{2.19}$$

directly, assuming $A = B$. To show that the consequent

$$\forall y\,(y \in A \Leftrightarrow y \in B) \tag{2.20}$$

holds, we let y be arbitrary and observe in light of Principle 1.4 that $A = B$ means that A and B share all of their properties, in particular that of consisting of certain elements. Clearly, this means that y is an element of A if, and only if, y is also an element of B. Since y is arbitrary, we may therefore conclude that the universal sentence (2.20) is true, so that the first implication (2.19) holds.

To prove the second implication ('$\Leftarrow$')

$$\forall y\,(y \in A \Leftrightarrow y \in B) \;\Rightarrow\; A = B \tag{2.21}$$

directly, we now assume the universal sentence (2.20) to be true. Let us now observe the truth of the equivalences

$$\begin{aligned} A \subseteq B \,\land\, B \subseteq A \;&\Leftrightarrow\; \forall y\,(y \in A \Rightarrow y \in B) \,\land\, \forall y\,(y \in B \Rightarrow y \in A) \\ &\Leftrightarrow\; \forall y\,([y \in A \Rightarrow y \in B] \,\land\, [y \in A \Rightarrow y \in B]). \end{aligned}$$

in light of the definition of a subset in connection with the Substitution Rules for conjunctions and the Distributive Law for quantification (1.74). Evidently, the preceding universal sentence

$$\forall y\,([y \in A \Rightarrow y \in B] \wedge [y \in A \Rightarrow y \in B])$$

is equivalent to the antecedent in (2.21) due to (1.8), since the conjunction

$$[y \in A \Rightarrow y \in B] \wedge [y \in A \Rightarrow y \in B]$$

defines for an arbitrary y the equivalence $y \in A \Leftrightarrow y \in B$. In view of the previously established equivalences, the conjunction $A \subseteq B \wedge B \subseteq$ is then according to the Substitution Rules for equivalences also equivalent to the antecedent in (2.21). Therefore, the assumed truth of the latter implies the truth of the former, i.e. the truth of $A \subseteq B \wedge B \subseteq$, which in turn implies $A = B$ with the Axiom of Extension. This completes the proof of the second part ('$\Leftarrow$') of the equivalence in (2.18).

Since both parts of the equivalence in (2.18) are true and since A and B are arbitrary sets, we may finally conclude that the proposition holds. □

Method 2.1 (Verification that two given sets are identical). To verify that two given sets A and B are equal, we may show that $y \in A$ implies $y \in B$ and that $y \in B$ implies $y \in A$ for any y.

Corollary 2.7. *Two sets A and B are identical iff A is a subset of B and B a subset of A, that is,*

$$\forall A, B\,(A = B \Leftrightarrow [A \subseteq B \wedge B \subseteq A]). \tag{2.22}$$

Proof. We let A and B be arbitrary sets and recall from the proof of the Equality Criterion for sets that the equivalence

$$\forall y\,(y \in A \Leftrightarrow y \in B) \Leftrightarrow [A \subseteq B \wedge B \subseteq A]$$

holds. Since the equivalence

$$A = B \Leftrightarrow \forall y\,(y \in A \Leftrightarrow y \in B)$$

is also true according to the Equality Criterion for sets, we obtain the equivalence in (2.22) by applying the Substitution Rule for equivalences (1.29). As A and B are arbitrary, we may therefore conclude that the corollary is true. □

Exercise 2.1. Show that two sets A and B are unequal iff there is an element which is in A but not in B or in B but not in A, that is,

$$\forall A, B\,(A \neq B \Leftrightarrow \exists y\,([y \in A \wedge y \notin B] \vee [y \in B \wedge y \notin A])). \tag{2.23}$$

(Hint: Use Definition 1.8, (2.18), (1.53), (1.8), (1.51), (1.32), and (2.1).)

We will occasionally apply the concept of a subset in a version which excludes the possibility that the two sets are equal.

Definition 2.4 (Proper subset). For any sets A and B we say that A is a *proper subset* of B, symbolically

$$A \subset B, \tag{2.24}$$

iff A is a subset of but not equal to B, that is,

$$A \subseteq B \wedge A \neq B. \tag{2.25}$$

We may then characterize a subset also

Proposition 2.8. *A is a subset of B iff A is a proper subset of or equal to B, that is,*

$$\forall A, B\,(A \subseteq B \Leftrightarrow [A \subset B \vee A = B]). \tag{2.26}$$

Proof. We let A and B be arbitrary sets and observe first the truth of the equivalences

$$\begin{aligned}
A \subset B \vee A = B \;&\Leftrightarrow\; (A \subseteq B \wedge A \neq B) \vee A = B \\
&\Leftrightarrow\; A = B \vee (A \subseteq B \wedge A \neq B) \\
&\Leftrightarrow\; (A = B \vee A \subseteq B) \wedge (A = B \vee A \neq B) \\
&\Leftrightarrow\; A = B \vee A \subseteq B
\end{aligned}$$

using (2.25) in connection with the Substitution Rule for disjunctions (1.25), the Commutative Law for the disjunction, the Distributive Law for sentences (1.45), and (1.13) in connection with the fact that the disjunction $A = B \vee A \neq B$ is a tautology according to (1.10). Consequently, the equivalence

$$[A \subset B \vee A = B] \Leftrightarrow [A = B \vee A \subseteq B] \tag{2.27}$$

follows to be true with the Substitution Rule for equivalences (1.29).

We now prove the first part ('$\Rightarrow$') of the equivalence in (2.26) directly, assuming that $A \subseteq B$ holds. Then, the disjunction $A = B \vee A \subseteq B$ is also true (irrespective of the truth value of $A = B$), which in turn implies $A \subset B \vee A = B$ with (2.27), so that the proof of ('$\Rightarrow$') is complete.

To prove the second part ('$\Leftarrow$') directly, we assume now the disjunction $A \subset B \vee A = B$ to be true, which implies with (2.27) the disjunction $A = B \vee A \subseteq B$. If the first part $A = B$ holds, then it follows with the Equality Criterion for sets that $A \subseteq B \wedge B \subseteq A$ is also true, which conjunction implies in particular the desired inclusion $A \subseteq B$. If the second part $A \subseteq B$ of the preceding disjunction holds, then we arrived already at the desired consequent. Thus, $A \subseteq B$ holds in any case, and the proof of ('$\Leftarrow$') is complete.

Therefore, the equivalence in (2.26) is true, and since A and B were arbitrary, we may therefore conclude that the proposition holds, as claimed. $\square$

Note 2.3. We see clearly in (2.26) that the fact $B = B$ implies $B \subseteq B$ for any set B, which we established already in Proposition 2.4.

In the remainder of this section, we will introduce two key concepts which, applied together, allow us to define sets with particular properties. First, we give the concept of a property of a general thing a narrower set-theoretical meaning.

Definition 2.5 (Definite property). Letting $\varphi(X, Y, \ldots)$ be an arbitrary formula where the variables X, Y, etc., are sets, we say that $\varphi(X, Y, \ldots)$ is a *definite property* of X iff X is a free variable and $Y, \ldots$ are bound variables.

Example 2.1. The formula

$$\forall y, A\,([y \in A \,\wedge\, A \in \mathcal{S}] \Rightarrow y \in \mathcal{S}) \tag{2.28}$$

is a definite property of the set system $\mathcal{S}$; in this instance $\mathcal{S}$ is said to have the definite property of being a 'transitive set'.

The second concept for defining a particular set claims the existence of a set obtained by 'picking out' all those elements from a given set which have a certain property.

Axiom 2.2 (Axiom of Specification/comprehension scheme). For an arbitrary formula $\varphi(y, G, A, B, \ldots)$ and for any sets G, A, B, etc., there exists a set X which contains all the elements of G for which φ is a true sentence, that is,

$$\exists X\, \forall y(y \in X \Leftrightarrow [y \in G \,\wedge\, \varphi(y, G, A, B, \ldots)]). \tag{2.29}$$

Note 2.4. (2.29) constitutes a scheme when φ is not fixed beforehand.

In the following section we will apply the previously introduced concepts to define various basic types of sets. Our main goal will be the construction of the 'set of natural numbers', which set will allow us to count computational quantities such as numerical observations.

2.2. The Empty Set $\emptyset$

Let us now apply the Axiom of Specification in connection with the Equality Criterion for sets (which is essentially based on the Axiom of Extension), to establish the concept of an 'empty set'.

Theorem 2.9 (Unique existence of a set without elements). *There exists a unique set which does not contain any elements, that is,*

$$\exists! X\, \forall y\, (y \notin X), \tag{2.30}$$

Proof. First, we establish the uniqueness part and prove accordingly

$$\exists X\, \forall y\, (y \notin X). \tag{2.31}$$

For this purpose, we let $\varphi(y)$ be the formula

$$y \neq y$$

and let G be an arbitrary set within the universe. We may then apply the Axiom of Specification to obtain the true sentence

$$\exists X\, \forall y\, (y \in X \Leftrightarrow [y \in G \,\wedge\, \varphi(y)]).$$

Thus, there exists a set, say $\bar{X}$, which contains every element of G for which $\varphi(y)$ is true, which means that the set $\bar{X}$ satisfies

$$\forall y\, (y \in \bar{X} \Leftrightarrow [y \in G \,\wedge\, y \neq y]). \tag{2.32}$$

We now prove that the set $\bar{X}$ satisfies also

$$\forall y\, (y \notin \bar{X}). \tag{2.33}$$

To do this, we let $\bar{y}$ be arbitrary, so that the equivalence

$$\bar{y} \in \bar{X} \Leftrightarrow [\bar{y} \in G \,\wedge\, \bar{y} \neq \bar{y}]$$

holds in view of (2.32). Since $\bar{y} \neq \bar{y}$ is false due to (1.95), the conjunction

$$\bar{y} \in G \,\wedge\, \bar{y} \neq \bar{y}$$

is then also false. Consequently, the preceding equivalence shows that $\bar{y} \in \bar{X}$ is also false, which we may also write as $\bar{y} \notin \bar{X}$ because of (2.3). As $\bar{y}$ was arbitrary, we may therefore conclude that the universal sentence (2.33) is true. We thus showed that there exists a set X satisfying $\forall y\, (y \notin X)$, which proves the existential sentence (2.31) and thus the existential part of the uniquely existential sentence to be proven.

To establish the uniqueness part, we now prove

$$\forall X, X' \, ([\forall y \, (y \notin X) \, \wedge \, \forall y \, (y \notin X')] \Rightarrow X = X'). \tag{2.34}$$

For this purpose, we let X and X' be arbitrary and prove the implication in (2.34) directly, assuming the antecedent

$$\forall y \, (y \notin X) \, \wedge \, \forall y \, (y \notin X')$$

to be true. As a conjunction is only true when both of its parts are true, we have that

$$\forall y \, (y \notin X) \tag{2.35}$$

and

$$\forall y \, (y \notin X') \tag{2.36}$$

both hold. To show that this implies $X = X'$, we verify

$$\forall y \, (y \in X \Leftrightarrow y \in X'). \tag{2.37}$$

To prove this universal sentence, we let $\bar{y}$ be arbitrary, so that $\bar{y} \notin X$ is true because of (2.35), and futhermore $\bar{y} \notin X'$ holds due to (2.36). Thus, the negations $\neg \bar{y} \in X$ and $\neg \bar{y} \in X'$ are both true, which means that $\bar{y} \in X$ and $\bar{y} \in X'$ are both false. Therefore, the equivalence in (2.37) takes the truth value 'T', and since $\bar{y}$ was arbitrary, we may further conclude that this equivalence holds for any y. This completes the proof of the universal sentence (2.37), which in turn implies $X = X'$ with (2.18). This equation proves the implication in (2.34), and as X and X' were arbitrary, we may therefore conclude that the universal sentence (2.34) is also true. This completes the proof of the uniqueness part of the proposed uniquely existential sentence (2.30), which is thus true. □

Note 2.5. The reason for choosing the formula $y \neq y$ is that it becomes a false sentence for any y, so that the sentence $y \in \bar{X}$ in (2.32) is also false, as desired. It is therefore possible to replace $y \neq y$ by any other formula which turns into a false sentence for any y.

Definition 2.6 (Empty set). For any set A we say that A is the *empty set*, symbolically

$$A = \emptyset \quad \text{or} \quad A = \{\,\} \quad \text{or} \quad A = \{y : y \neq y\}, \tag{2.38}$$

iff A does not contain any elements, that is, iff

$$\forall y \, (y \notin A). \tag{2.39}$$

Note 2.6. "We say that a set A (...)" abbreviates "For any set A we say that A (...)".

Note 2.7. Clearly, the equation $\emptyset = \emptyset$ holds, so that the preceding definition implies

$$\forall y\,(y \notin \emptyset). \tag{2.40}$$

Corollary 2.10. *There exists no element in the empty set, that is,*

$$\neg \exists y\,(y \in \emptyset). \tag{2.41}$$

Proof. Since $\forall y\,(y \notin \emptyset)$ holds by definition of the empty set, which we may also write as $\forall y\,(\neg y \in \emptyset)$, we may apply the Quantifier Negation Law (1.54) to obtain the equivalently true sentence (2.41). □

Proposition 2.11. *A set A is nonempty iff there is an y in A, that is,*

$$\forall A\,(A \neq \emptyset \Leftrightarrow \exists y\,(y \in A)). \tag{2.42}$$

Proof. We let A be an arbitrary set and first observe the truth of the equivalences

$$\begin{aligned} A \neq \emptyset \;&\Leftrightarrow\; \neg A = \emptyset \\ &\Leftrightarrow\; \neg \forall y\,(y \notin A) \\ &\Leftrightarrow\; \neg(\neg \exists y\,(y \in A)) \\ &\Leftrightarrow\; \exists y\,(y \in A) \end{aligned}$$

due to the definition of an inequality, the definition of the empty set, the Quantifier Negation Law (1.54), and the Double Negation Law. Then, the Substitution Rule for equivalences (1.29) gives the desired equivalence in (2.42), and since A is arbitrary, we may therefore conclude that the proposition holds. □

Proposition 2.12. *The empty set is included any set, that is,*

$$\forall A\,(\emptyset \subseteq A). \tag{2.43}$$

Proof. We let A be arbitrary and prove the inclusion $\emptyset \subseteq A$ by verifying the equivalent (using the definition of a subset)

$$\forall y\,(y \in \emptyset \Rightarrow y \in A). \tag{2.44}$$

Letting y be arbitrary, we now see that $\neg y \in \emptyset$ holds by definition of the empty set, so that $y \in \emptyset$ is false. Thus, the implication in (2.44) has a false antecedent and is therefore true. Since y is arbitrary, we may therefore conclude that the universal sentence (2.44) is true. Then, $\emptyset \subseteq A$ follows to be true by definition of a subset, and as A was also arbitrary, we may further conclude that the proposed universal sentence (2.43) holds. □

Note 2.8. The preceding Proposition 2.12 shows in particular that the empty set is included in itself, that is,

$$\emptyset \subseteq \emptyset. \tag{2.45}$$

Proposition 2.13. *Any subset of the empty set is identical with the empty set, that is,*

$$\forall A \, (A \subseteq \emptyset \Leftrightarrow A = \emptyset). \tag{2.46}$$

Proof. We let A be an arbitrary set and prove the first implication '$\Rightarrow$' directly, assuming $A \subseteq \emptyset$ to be true. Since $\emptyset \subseteq A$ also holds according to (2.43), we see that the conjunction $A \subseteq \emptyset \wedge \emptyset \subseteq A$ is true. It then follows with the Axiom of Extension that the desired equation $A = \emptyset$ holds. Assuming conversely $A = \emptyset$ to be true, we obtain from the (2.45) via substitution the desired consequent $A \subseteq \emptyset$ also of the second implication '$\Leftarrow$', so that the proof of the equivalence in (2.46) is complete. Since A was an arbitrary set, we may therefore conclude that the proposition is true. $\square$

Proposition 2.14. *There is no proper subset of the empty set, that is,*

$$\neg \exists A \, (A \subset \emptyset). \tag{2.47}$$

Proof. We first observe that (2.47) is equivalent to

$$\forall A \, (\neg A \subset \emptyset) \tag{2.48}$$

due to the Quantifier Negation Law (1.54). To prove that universal sentence, we now let A be arbitrary and prove the negation $\neg A \subset \emptyset$ by contradiction, assuming $A \subset \emptyset$ to be true. By definition of a proper subset, this means that the conjunction $A \subseteq \emptyset \wedge A \neq \emptyset$ holds. This implies in particular $A \subseteq \emptyset$, which in turn implies $A = \emptyset$ with (2.46). Thus, $A = \emptyset$ and $A \neq \emptyset$ are both true, so that we obtained a contradiction according to (1.11), and the proof of $\neg A \subset \emptyset$ is complete. As A was arbitrary, we therefore conclude that the universal sentence (2.48) is true, which means that the equivalent negated existential sentence (2.47) also holds. $\square$

We use the empty set to define our first *number*.

Definition 2.7 (Zero). We say that a set A is the (*number*) *zero*, symbolically $A = 0$, iff $A = \emptyset$.

Note 2.9. The fact that $A = 0 \Leftrightarrow A = \emptyset$ (for any A) is equivalent to $0 = \emptyset$ due to (1.99) justifies the following alternative, shorter definition of 0.
"We call

$$0 = \emptyset. \tag{2.49}$$

the number zero."

2.3. Intersections of Sets

With the Axiom of Specification and the Equality Criterion for sets, we may also form a unique set by extracting from two given sets all elements that they have in common. Such a set may be viewed as the 'overlap' of two sets.

Theorem 2.15. *For any sets A and B there exists a unique set X such that an element y is in X iff y is in A and in B, i.e.*

$$\exists! X\, \forall y\, (y \in X \Leftrightarrow [y \in A \,\wedge\, y \in B]). \tag{2.50}$$

Proof. We let A and B be arbitrary sets and define $\varphi(y)$ to be the formula

$$y \in B.$$

Then, the Axiom of Specification yields the true existential sentence

$$\exists X\, \forall y\, (y \in X \Leftrightarrow [y \in A \,\wedge\, \varphi(y)]),$$

which proves the existential part of the uniquely existential sentence (2.50) to be proven. Regarding the uniqueness part, we now prove

$$\begin{aligned}&\forall X, X' \, ([\forall y\, (y \in X \Leftrightarrow [y \in A \,\wedge\, y \in B]) \,\wedge\, \forall y\, (y \in X' \Leftrightarrow [y \in A \,\wedge\, y \in B])] \\ &\quad \Rightarrow X = X').\end{aligned} \tag{2.51}$$

For this, purpose, we let X and X' be arbitrary and prove the implication directly, assuming that

$$\forall y\, (y \in X \Leftrightarrow [y \in A \,\wedge\, y \in B]) \,\wedge\, \forall y\, (y \in X' \Leftrightarrow [y \in A \,\wedge\, y \in B])$$

holds. Thus, the two parts

$$\forall y\, (y \in X \Leftrightarrow [y \in A \,\wedge\, y \in B]) \tag{2.52}$$

and

$$\forall y\, (y \in X' \Leftrightarrow [y \in A \,\wedge\, y \in B]) \tag{2.53}$$

of the preceding conjunction are true. Next, we establish the truth of

$$\forall y\, (y \in X \Leftrightarrow y \in X'), \tag{2.54}$$

letting $\bar{y}$ be arbitrary. Then, in view of (2.52) and (2.53), the equivalences

$$\bar{y} \in X \Leftrightarrow [\bar{y} \in A \,\wedge\, \bar{y} \in B]$$

and

$$\bar{y} \in X' \Leftrightarrow [\bar{y} \in A \wedge \bar{y} \in B]$$

are both true. Consequently, the conjunction of these two equivalences implies with the Substitution Rule for equivalences the truth of

$$\bar{y} \in X \Leftrightarrow \bar{y} \in X'.$$

As $\bar{y}$ was arbitrary, the universal sentence (2.54) follows therefore to be true, which then further implies $X = X'$ with the Equality Criterion for sets. Then, since X and X' were also arbitrary, we may therefore conclude that (2.51) holds, which proves the uniqueness part. Thus, the proof of the uniquely existential sentence (2.50) is complete. Finally, because the sets A and B were arbitrary as well, we may finally conclude that the proposed universal sentence is true. □

Note 2.10. Thus, the set X is (for any A and any B) uniquely specified by

$$\forall y \, (y \in X \Leftrightarrow [y \in A \wedge y \in B]),$$

which property we will use in the following definition.

Definition 2.8 (Intersection of two sets). For any sets A and B we say that a set X is the *intersection* of A and B, symbolically

$$X = A \cap B \quad \text{or} \quad X = \{y : y \in A \wedge y \in B\}, \tag{2.55}$$

iff X contains precisely all the elements in A and B in the sense that

$$\forall y \, (y \in X \Leftrightarrow [y \in A \wedge y \in B]). \tag{2.56}$$

Note 2.11. For any sets A and B, the evident truth of the equation $A \cap B = A \cap B$ implies by definition of the intersection of two sets

$$\forall y \, (y \in A \cap B \Leftrightarrow [y \in A \wedge y \in B]). \tag{2.57}$$

Theorem 2.16 (Commutative Law for the intersection of two sets). *The intersection of a set A and a set B is identical with the intersection of B and A, that is,*

$$\forall A, B \, (A \cap B = B \cap A). \tag{2.58}$$

Proof. We let A and B be arbitrary sets and prove the stated equation by verifying

$$\forall y \, (y \in A \cap B \Leftrightarrow y \in B \cap A). \tag{2.59}$$

To do this, we let y be arbitrary and observe the truth of the equivalences

$$\begin{aligned} y \in A \cap B &\Leftrightarrow (y \in A \wedge y \in B) \\ &\Leftrightarrow (y \in B \wedge y \in A) \\ &\Leftrightarrow y \in B \cap A, \end{aligned}$$

applying (2.57), then the Commutative Law for the conjunction, and then again (2.57). Thus, the Substitution Rule for equivalences (1.29) yields the desired equivalence $y \in A \cap B \Leftrightarrow y \in B \cap A$. Since y is arbitrary, we may therefore conclude that the universal sentence (2.59) is true, which then yields the desired equation $A \cap B = B \cap A$ with the Equality Criterion for sets. As A and B were also arbitrary, we may then further conclude that the proposed commutative law is indeed true. □

Theorem 2.17 (Idempotent Law for the intersection of two sets). *The pairwise intersection is idempotent in the sense that*

$$\forall A\,(A \cap A = A). \tag{2.60}$$

Proof. We let A be an arbitrary set and prove the stated equation by verifying

$$\forall y\,(y \in A \cap A \Leftrightarrow y \in A). \tag{2.61}$$

To do this, we let y be arbitrary and observe the truth of the equivalences

$$\begin{aligned} y \in A \cap A &\Leftrightarrow (y \in A \wedge y \in A) \\ &\Leftrightarrow y \in A, \end{aligned}$$

applying (2.57) and then the Idempotent Law for the conjunction. Thus, the Substitution Rule for equivalences (1.29) yields the desired equivalence $y \in A \cap A \Leftrightarrow y \in A$. Since y is arbitrary, we may therefore conclude that the universal sentence (2.61) is true, which then yields the desired equation $A \cap A = A$ with the Equality Criterion for sets. As A was also arbitrary, we may then further conclude that the proposed law is indeed true. □

Proposition 2.18. *The intersection of any set and the empty set as well as the intersection of the empty set and any set is empty, that is,*

$$\forall A\,(A \cap \emptyset = \emptyset \wedge \emptyset \cap A = \emptyset). \tag{2.62}$$

Proof. Letting A be an arbitrary set, we prove the first part of the conjunction by verifying

$$\forall y\,(y \notin A \cap \emptyset). \tag{2.63}$$

To do this, we take an arbitrary y and apply a proof by contradiction, assuming $y \in A \cap \emptyset$ to be true. This assumption implies the conjunction $y \in A \wedge y \in \emptyset$ with (2.57) and thus in particular $y \in \emptyset$. Since $y \notin \emptyset$ is also true according to (2.40), we obtained a contradiction according to (1.11), completing the proof of $y \notin A \cap \emptyset$. As y was arbitrary, we therefore conclude that (2.63) holds, which then further implies the desired equation $A \cap \emptyset = \emptyset$ by definition of the empty set.

To prove the second part of the conjunction in (2.62), we observe the truth of the equation

$$\emptyset \cap A = A \cap \emptyset \ (= \emptyset)$$

in light of the Commutative Law for the intersection of two sets, so that $\emptyset \cap A = \emptyset$ also holds. Since A was arbitrary, the proposed universal sentence (2.62) follows then to be true. □

Clearly, replacing the variable A in the preceding proposition by the constant $\emptyset$ gives the following equation.

Corollary 2.19. *The intersection of the empty set and the empty set is also empty, that is,*

$$\emptyset \cap \emptyset = \emptyset. \tag{2.64}$$

Since it happens frequently that the intersection of two given sets is empty, the following definition will be useful.

Definition 2.9 (Disjoint sets). We say that sets A and B are *disjoint* iff the intersection of the two sets is empty, that is, iff

$$A \cap B = \emptyset. \tag{2.65}$$

Corollary 2.20. *Any disjoint sets A and B where A is nonempty are distinct sets, that is,*

$$\forall A, B \, (A \neq \emptyset \Rightarrow [A \cap B = \emptyset \Rightarrow A \neq B]). \tag{2.66}$$

Proof. Letting A and B be arbitrary sets such that A is nonempty, we use the Law of Contraposition to prove the implication $A \cap B = \emptyset \Rightarrow A \neq B$, by assuming $\neg A \neq B$ to be true and by demonstrating that $A \cap B \neq \emptyset$ also holds. The preceding assumption implies now $A = B$ with the Double Negation Law, so that we obtain

$$A \cap B = A \cap A = A \quad (\neq \emptyset)$$

by applying substitution and the Idempotent Law for the intersection of two sets (and recalling the initial assumption $A \neq \emptyset$). We thus find $A \cap B \neq \emptyset$, as desired. Since A and B are arbitrary, we may infer from this finding the truth of the stated universal sentence. □

Proposition 2.21. *If two sets A and B are disjoint and if a set C is included in B, then A and C are also disjoint, that is,*

$$\forall A, B, C\,([A \cap B = \emptyset \,\wedge\, C \subseteq B] \Rightarrow A \cap C = \emptyset). \tag{2.67}$$

Proof. Letting A, B and C be arbitrary sets, we assume the equation $A \cap B = \emptyset$ and the inclusion $C \subseteq B$ to be true. Thus,

$$\forall y\,(y \notin A \cap B) \tag{2.68}$$

holds by definition of the empty set, and

$$\forall y\,(y \in C \Rightarrow y \in B) \tag{2.69}$$

holds by definition of a subset. We use the former definition also to establish now $A \cap C = \emptyset$, by verifying according the universal sentence

$$\forall y\,(y \notin A \cap C). \tag{2.70}$$

We let y be arbitrary, and we prove the negation $y \notin A \cap C$ by establishing the contradiction

$$y \in A \cap B \,\wedge\, y \notin A \cap B, \tag{2.71}$$

assuming the negation of $y \notin A \cap C$ to be true. Then, $y \in A \cap C$ follows to be true with the Double Negation Law. Consequently, the definition of the intersection of two sets gives us $y \in A$ and $y \in C$. The latter implies with (2.69) $y \in B$, and since $y \in A$ is also true, we thus find $y \in A \cap B$ (using again the definition of the intersection of two sets). Because y satisfies also $y \notin A \cap B$ in view of (2.68), we arrived at the desired contradiction (2.71). We thus proved $y \notin A \cap C$, so that the equivalent $A \cap C = \emptyset$ is also true. This finding in turn proves the implication in (2.67), in which the sets A, B and C are arbitrary. We may therefore conclude that the proposed universal holds. □

The Associative Law for sentences regarding the conjunction constitutes the foundation for the following

Theorem 2.22 (Associative Law for the intersection of two sets). *The intersection of two sets is associative in the sense that*

$$\forall A, B, C\,(A \cap [B \cap C] = [A \cap B] \cap C). \tag{2.72}$$

Proof. We let A, B and C be arbitrary sets and verify first

$$\forall y\,(y \in A \cap [B \cap C] \Leftrightarrow y \in [A \cap B] \cap C). \tag{2.73}$$

Letting y be arbitrary, we now obtain the true equivalences

$$\begin{aligned} y \in A \cap [B \cap C] &\Leftrightarrow (y \in A \wedge y \in B \cap C) \\ &\Leftrightarrow (y \in A \wedge [y \in B \wedge y \in C]) \\ &\Leftrightarrow ([y \in A \wedge y \in B] \wedge y \in C) \\ &\Leftrightarrow (y \in A \cap B \wedge y \in C) \\ &\Leftrightarrow (y \in [A \cap B] \cap C) \end{aligned}$$

by applying (2.57), again (2.57) in connection with the Substitution Rule for conjunctions (1.22), the Associative Law for the conjunction, again (2.57) in connection with the Substitution Rule for conjunctions (1.23) and the Commutative Law for the conjunction, and finally again (2.57). Consequently, the equivalence in (2.73) follows to be true with the Substitution Rule for equivalences (1.29). Because y is arbitrary, we may therefore conclude that (2.73) is true, so that the desired equation $A \cap [B \cap C] = [A \cap B] \cap C$ also holds in view of the Equality Criterion for sets. Then, as A, B and C were also arbitrary, it follows that the proposed universal sentence (2.72) is true, as claimed. □

Proposition 2.23. *The intersection of two sets is included in each of the two sets, i.e.*

$$\forall A, B\, (A \cap B \subseteq A \wedge A \cap B \subseteq B). \tag{2.74}$$

Proof. We let A and B be arbitrary sets and first observe the truth of the disjunction

$$A \cap B = \emptyset \vee A \cap B \neq \emptyset.$$

In the first case that $A \cap B = \emptyset$ is true, we notice that $\emptyset \subseteq A$ and $\emptyset \subseteq B$ because of (2.43). Therefore, substitution based on the preceding equation yields the inclusions $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so that the conjunction in (2.74) is true in the first case.

In the second case that $A \cap B \neq \emptyset$ is true, we now see that there exists an element in $A \cap B$, according to (2.42). Let us next observe that the conjunction in (2.74) is equivalent to

$$\forall y\, (y \in A \cap B \Rightarrow y \in A) \wedge \forall y\, (y \in A \cap B \Rightarrow y \in B)$$

because of the definition of a subset and the Substitution Rules for conjunctions. Then, this conjunction is also equivalent to

$$\forall y\, ([y \in A \cap B \Rightarrow y \in A] \wedge [y \in A \cap B \Rightarrow y \in B])$$

according to the Distributive Law for quantification (1.74), and moreover equivalent to

$$\forall y\, (y \in A \cap B \Rightarrow [y \in A \wedge y \in B]) \tag{2.75}$$

due to the Distributive Law for sentences (1.46). To prove this universal sentence, we let y be arbitrary. We may now prove the implication directly, assuming $y \in A \cap B$ to be true, since $A \cap B$ is not empty in the current second case. Then, the assumed antecedent $y \in A \cap B$ immediately implies the truth of the consequent $y \in A \wedge y \in B$ by definition of the intersection of two sets. As y is arbitrary, we may therefore conclude that (2.75) holds, which in turn implies the truth of the equivalent conjunction in (2.74) also for the second case.

Since A and B were arbitrary sets, it then follows that the proposed universal sentence (2.74) holds. □

Exercise 2.2. Establish the following universal sentence.

$$\forall A, B, C \, (B \subseteq C \Rightarrow A \cap B \subseteq A \cap C). \tag{2.76}$$

(Hint: Use the definition of a subset in connection with the definition of the intersection of two sets.)

Proposition 2.24. *The intersection of a set B with a subset of B is identical with that subset, that is,*

$$\forall A, B \, (A \subseteq B \Rightarrow A \cap B = A). \tag{2.77}$$

Proof. We let A and B be arbitrary and prove the implication by cases.

In the first case, we assume $A \subseteq B \wedge A = \emptyset$ to be true, which implies in particular $A = \emptyset$ and therefore

$$A \cap B = \emptyset \cap B = \emptyset$$

by applying substitution and (2.62). These equations yield $A \cap B = \emptyset$ and then the desired $A \cap B = A$ via substitution.

In the second case, we assume now that $A \subseteq B \wedge A \neq \emptyset$ holds. The second part of this conjunction shows in light of (2.42) that there exists an element in A, say $\bar{y}$. Since the first part $A \subseteq B$ of the assumed conjunction means by definition of a subset that

$$\forall y \, (y \in A \Rightarrow y \in B) \tag{2.78}$$

holds, we have that $\bar{y} \in A$ implies $\bar{y} \in B$. Thus, the conjunction $\bar{y} \in A \wedge \bar{y} \in B$ holds, which in turn implies $\bar{y} \in A \cap B$ by definition of the intersection of two sets, so that there exists an element in the intersection $A \cap B$. Based on these findings, we now verify

$$\forall y \, (y \in A \cap B \Leftrightarrow y \in A), \tag{2.79}$$

by letting y be arbitrary and by proving both parts of the equivalence directly. Regarding the first part ('$\Rightarrow$'), we may assume $y \in A \cap B$ (recalling that $A \cap B$ is not empty), and therefore $y \in A$ follows in particular to be true by definition of the intersection of two sets. Regarding the second part ('$\Leftarrow$'), we may now assume $y \in A$ to be true (since A is also nonempty), which then implies $y \in B$ with (2.78). Therefore, the conjunction $y \in A \wedge y \in B$ is also true, which now gives $y \in A \cap B$ (using again the definition of the intersection of two sets). This completes the proof of the equivalence in (2.79), and since y is arbitrary, we therefore conclude that (2.79) holds.

This universal sentence in turn implies $A \cap B = A$ with the Equality Criterion for sets, completing the proof of the implication in (2.77) by cases. As A and B were arbitrary, we may now finally conclude that the proposed sentence is true. □

Proposition 2.25. *If a set A is included in a set C and a set B included in a set D, then the intersection of the two subsets A and B is included in the intersection of C and D, that is,*

$$\forall A, B, C, D\,([A \subseteq C \wedge B \subseteq D] \Rightarrow A \cap B \subseteq C \cap D). \tag{2.80}$$

Proof. We let A, B, C and D be arbitrary sets and prove the implication directly, assuming that the conjunction of $A \subseteq C$ and $B \subseteq D$ holds. We now prove the inclusion $A \cap B \subseteq C \cap D$ by verifying the equivalent (using the definition of a subset)

$$\forall y\,(y \in A \cap B \Rightarrow y \in C \cap D). \tag{2.81}$$

Letting y be arbitrary, we assume that $y \in A \cap B$ holds, so that $y \in A$ and $y \in B$ follow to be both true by definition of the intersection of two sets. We notice in light of the definition of a subset that the initial assumptions $A \subseteq C$ and $B \subseteq D$ are equivalent to, respectively,

$$\forall y\,(y \in A \Rightarrow y \in C) \tag{2.82}$$

and

$$\forall y\,(y \in B \Rightarrow y \in D). \tag{2.83}$$

Thus, the previously established $y \in A$ implies $y \in C$ with (2.82), and $y \in B$ implies $y \in D$ with (2.83). Since $y \in C$ and $y \in D$ are both true, it follows (again by definition of the intersection of two sets) that $y \in C \cap D$ holds, proving the implication in (2.81).

As y is arbitrary, we may therefore conclude that the universal sentence (2.81) holds, which then further implies $A \cap B \subseteq C \cap D$ by definition of a subset. This in turn proves the implication in (2.80), and since A, B, C and D were arbitrary, we may finally conclude that the proposition is true. □

Corollary 2.26. *If a set A is subset both of a set C and a set D, then A is subset of the intersection of C and D, that is,*

$$\forall A, C, D\,([A \subseteq C \,\wedge\, A \subseteq D] \Rightarrow A \subseteq C \cap D). \tag{2.84}$$

Proof. Letting A, C and D be arbitrary and assuming $A \subseteq C \,\wedge\, A \subseteq D$ to be true, it follows with (2.80) that $A \cap A \subseteq C \cap D$ holds. Since $A \cap A$ is identical with A according to the Idempotent Law for the intersection of two sets, substitution yields the desired $A \subseteq C \cap D$. As A, C and D were arbitrary, we may therefore conclude that (2.84) is true. □

Exercise 2.3. Show that if two sets A and B are both included in a set C, then the intersection of A and B is also included in X, that is,

$$\forall A, B, C\,([A \subseteq C \,\wedge\, B \subseteq C] \Rightarrow A \cap B \subseteq C). \tag{2.85}$$

Proposition 2.27. *If a nonempty set A and a set B are disjoint, then A is not included in B, that is,*

$$\forall A, B\,([A \neq \emptyset \,\wedge\, A \cap B = \emptyset] \Rightarrow \neg A \subseteq B). \tag{2.86}$$

Proof. We let A and B be arbitrary sets and prove then the implication by contradiction, assuming $A \neq \emptyset$, $A \cap B = \emptyset$ and $\neg(\neg A \subseteq B)$ to be true. The latter implies $A \subseteq B$ with the Double Negation Law and then

$$\forall y\,(y \in A \Rightarrow y \in B) \tag{2.87}$$

by definition of a subset. Furthermore, the initial assumption $A \neq \emptyset$ implies with (2.42) that there exists an element in A, say $\bar{y}$. Now, $\bar{y} \in A$ further implies $\bar{y} \in B$ with (2.87), so that $\bar{y} \in A$ and $\bar{y} \in B$ are both true. Consequently, $\bar{y} \in A \cap B$ holds by definition of the intersection of two sets, so that the existential sentence $\exists y\,(y \in A \cap B)$ is true, which then yields $A \cap B \neq \emptyset$ with (2.42). As we assumed $A \cap B = \emptyset$ to be also true, we evidently obtained a contradiction, which proves the implication in (2.86). Since A and B were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

The idea of specifying the intersection to two sets is readily generalized to the intersection of an arbitrary (nonempty) system of sets.

Theorem 2.28. *The following sentences are true for any set system $\mathcal{S} \neq \emptyset$ and any set $\bar{A} \in \mathcal{S}$.*

a) *There exists a unique set $\bigcap \mathcal{S}$ such that an element y is in $\bigcap \mathcal{S}$ iff y is element of $\bar{A}$ and moreover element of all sets in $\mathcal{S}$, i.e.*

$$\exists! \bigcap \mathcal{S}\, \forall y\,(y \in \bigcap \mathcal{S} \Leftrightarrow [y \in \bar{A} \,\wedge\, \forall A\,(A \in \mathcal{S} \Rightarrow y \in A)]). \tag{2.88}$$

b) The set $\bigcap \mathcal{S}$ *satisfies also*

$$\forall y\,(y \in \bigcap \mathcal{S} \Leftrightarrow \forall A\,(A \in \mathcal{S} \Rightarrow y \in A)). \tag{2.89}$$

Proof. We let $\mathcal{S}$ be an arbitrary set system and assume then that $\mathcal{S} \neq \emptyset$ holds, so that there exists an element in the set system $\mathcal{S}$, according to (2.42). We now let $\bar{A} \in \mathcal{S}$ be arbitrary.

Concerning a), we now define $\varphi(y)$ to be the formula

$$\forall A\,(A \in \mathcal{S} \Rightarrow y \in A).$$

Then, we may apply the Axiom of Specification to obtain the true existential sentence

$$\exists \bigcap \mathcal{S}\, \forall y\,(y \in \bigcap \mathcal{S} \Leftrightarrow [y \in \bar{A} \,\wedge\, \varphi(y)]),$$

which proves the existential part regarding the uniquely existential sentence (2.88). The uniqueness part is established by applying the Equality Criterion for sets. Thus, the proof of the uniquely existential sentence (2.88) is complete, and a) is therefore true.

Concerning b), we let y be arbitrary and observe in light of (2.88) that the set $\bigcap \mathcal{S}$ and y satisfy

$$y \in \bigcap \mathcal{S} \Leftrightarrow [y \in \bar{A} \,\wedge\, \forall A\,(A \in \mathcal{S} \Rightarrow y \in A)]. \tag{2.90}$$

To prove the first part ('$\Rightarrow$') of the equivalence in (2.89) directly, we now assume $y \in \bigcap \mathcal{S}$ to be true, which implies with (2.90) in particular the universal sentence with respect to A in (2.89), as desired. Thus, the proof of the first part of the equivalence in (2.89) is complete. To prove the second part ('$\Leftarrow$'), we assume that

$$\forall A\,(A \in \mathcal{S} \Rightarrow y \in A)$$

is true and observe in view of the previously established $\bar{A} \in \mathcal{S}$ that this gives $y \in \bar{A}$. Thus, the conjunction in (2.90) holds, so that $y \in \bigcap \mathcal{S}$ follows to be true, which already proves the second part of the equivalence in (2.89). Since y is arbitrary, we may therefore conclude that the set $\bigcap \mathcal{S}$ satisfies the universal sentence (2.89).

As the set $\mathcal{S}$ was arbitrary in the proof of a) and b), the theorem follows then to be true. $\square$

Exercise 2.4. Prove the uniqueness part of (2.88).

(Hint: Proceed in analogy to the proof of (2.50).)

Since it is clear from the preceding proposition that $\bigcap \mathcal{S}$ represents the uniquely specified set with the characteristic definite property (2.89), we provide a simplified form of definition.

Definition 2.10 (Intersection of a set system). For any set system $\mathcal{S} \neq \emptyset$ we call the set $\bigcap \mathcal{S}$ consisting of those elements which are in all sets in $\mathcal{S}$, in the sense that

$$\forall y\,(y \in \bigcap \mathcal{S} \Leftrightarrow \forall A\,(A \in \mathcal{S} \Rightarrow y \in A)),$$

the *intersection of* $\mathcal{S}$. This set is also symbolized by

$$\bigcap_{A \in \mathcal{S}} A \quad \text{or} \quad \{y : \forall A\,(A \in \mathcal{S} \Rightarrow y \in A)\}. \tag{2.91}$$

Proposition 2.29. *The intersection of a set system $\mathcal{S}$ is included in all of the sets in $\mathcal{S}$, that is,*

$$\forall \mathcal{S}, A\,(A \in \mathcal{S} \Rightarrow \bigcap \mathcal{S} \subseteq A)). \tag{2.92}$$

Proof. We let $\mathcal{S}$ be an arbitrary set system, we let A be an arbitrary set, and we assume that $A \in \mathcal{S}$ is true. To show that this implies $\bigcap \mathcal{S} \subseteq A$, we verify

$$\forall y\,(y \in \bigcap \mathcal{S} \Rightarrow y \in A). \tag{2.93}$$

Letting y be arbitrary, we assume $y \in \bigcap \mathcal{S}$ to be true, which assumption implies with (2.89) that

$$\forall A\,(A \in \mathcal{S} \Rightarrow y \in A)$$

holds. Therefore, the initially assumed $A \in \mathcal{S}$ implies $y \in A$, proving the implication in (2.93). As y is arbitrary, we may therefore conclude that (2.93) is true, which means $\bigcap \mathcal{S} \subseteq A$ by definition of a subset. This in turn proves the implication in (2.92), and since A was also arbitrary, we may then further conclude that (2.92) holds. Finally, because $\mathcal{S}$ and $\bar{A}$ were arbitrary, the proposed sentence follows then to be true. □

Definition 2.11 (System of pairwise disjoint sets, pairwise disjoint sets). We say for any set system $\mathcal{S}$ that $\mathcal{S}$ is a *system of pairwise disjoint sets* (alternatively, that $\mathcal{S}$ has *pairwise disjoint sets*) iff any two distinct sets in $\mathcal{S}$ are disjoint, i.e. iff

$$\forall A, B\,([A, B \in \mathcal{S} \land A \neq B] \Rightarrow A \cap B = \emptyset). \tag{2.94}$$

2.4. Set Differences

Another straightforward utilization of the Axiom of Specification in combination with the Equality Criterion for sets consists in the definition of a unique set which contains precisely every element which is in a given set A but not in another set B.

Theorem 2.30. *For any sets A and B there exists a unique set $A \setminus B$ such that an element y is in $A \setminus B$ iff y is element of A and not element of B, i.e.*

$$\exists! A \setminus B \, \forall y \, (y \in A \setminus B \Leftrightarrow [y \in A \, \wedge \, y \notin B]). \tag{2.95}$$

Exercise 2.5. Prove Theorem 2.30.

(Hint: Proceed in analogy to the proof of Theorem 2.15.)

Definition 2.12 (Set difference). For any sets A and B we call the set $A \setminus B$ containing precisely all elements in A which are not in B, in the sense that

$$\forall y \, (y \in A \setminus B \Leftrightarrow [y \in A \, \wedge \, y \notin B]), \tag{2.96}$$

the *set difference* of A and B. This set is also symbolized by

$$\{y : y \in A \, \wedge \, y \notin B\}. \tag{2.97}$$

The definition of the difference of two sets now gives rise to a variety of useful relationships with the previously established types of sets. We begin with two basic results

Proposition 2.31. *The set difference of a set A and a set B is identical with the set difference of A and the intersection of A and B, that is,*

$$\forall A, B \, (A \setminus B = A \setminus [A \cap B]). \tag{2.98}$$

Proof. Letting A and B be arbitrary sets, we verify

$$\forall y \, (y \in A \setminus B \Leftrightarrow y \in A \setminus [A \cap B]). \tag{2.99}$$

Letting y be arbitrary, we obtain the true equivalences

$$\begin{aligned}
y \in A \setminus (A \cap B) &\Leftrightarrow y \in A \, \wedge \, \neg y \in A \cap B \\
&\Leftrightarrow y \in A \, \wedge \, \neg(y \in A \, \wedge \, y \in B) \\
&\Leftrightarrow y \in A \, \wedge \, (y \notin A \, \vee \, y \notin B) \\
&\Leftrightarrow (y \in A \, \wedge \, y \notin A) \, \vee \, (y \in A \, \wedge \, y \notin B) \\
&\Leftrightarrow y \in A \, \wedge \, y \notin B \\
&\Leftrightarrow y \in A \setminus B
\end{aligned}$$

using the definition of a set difference together with Definition 2.1 concerning $\notin$, the definition of the intersection of two sets, De Morgan's Law for sentences (1.51), the Distributive Law for sentences (1.44), (1.15) together with the fact that $y \in A \wedge y \notin A$ is a contradiction as in (1.11), and finally again the definition of a set difference. As y is arbitrary, we may therefore conclude that (2.99) is true, which in turn implies the equation in (2.98) with the Equality Criterion of sets. Since A and B are also arbitrary, we may finally conclude that the universal sentence (2.98) holds. □

The following exercise shows that the definitions of the intersection of two sets and of the set difference are in a certain sense compatible, since both are specified in terms of a conjunction.

Exercise 2.6. Verify that the intersection of a set A with the set difference of a set B and a set C can be written as the set difference of the intersection of A with B and C, that is,

$$\forall A, B, C\,(A \cap [B \setminus C] = [A \cap B] \setminus C). \tag{2.100}$$

(Hint: Proceed similarly as in the proof of Proposition 2.31, and apply in particular (1.39).)

Proposition 2.32. *The pairwise intersection is distributive over the set difference in the sense that*

$$\forall A, B, C\,(A \cap [B \setminus C] = [A \cap B] \setminus [A \cap C]). \tag{2.101}$$

Proof. Letting A, B and C be arbitrary, we obtain for an arbitrary y the true equivalences

$$\begin{aligned}
y \in (A \cap B) \setminus (A \cap C) &\Leftrightarrow y \in A \cap B \,\wedge\, y \notin A \cap C \\
&\Leftrightarrow (y \in A \,\wedge\, y \in B) \,\wedge\, \neg(y \in A \,\wedge\, y \in C) \\
&\Leftrightarrow (y \in A \,\wedge\, y \in B) \,\wedge\, (y \notin A \,\vee\, y \notin C) \\
&\Leftrightarrow ([y \in A \,\wedge\, y \in B] \,\wedge\, y \notin A) \\
&\qquad \vee\, ([y \in A \,\wedge\, y \in B] \,\wedge\, y \notin C) \\
&\Leftrightarrow (y \in B \,\wedge\, [y \in A \,\wedge\, y \notin A]) \\
&\qquad \vee\, (y \in A \,\wedge\, [y \in B \,\wedge\, y \notin C]) \\
&\Leftrightarrow (y \in A \,\wedge\, [y \in B \,\wedge\, y \notin C]) \\
&\Leftrightarrow y \in A \,\wedge\, y \in B \setminus C \\
&\Leftrightarrow y \in A \cap (B \setminus C)
\end{aligned}$$

using the definition of a set difference, the definition of the intersection of two sets, De Morgan's Law for the conjunction, the Distributivity of the conjunction over the disjunction, the Commutative Law for the intersection together with the Associative Law for the intersection, then the Contradiction Law with the fact that $y \in A \wedge y \notin A$ is a contradiction and consequently also $y \in B \wedge [y \in A \wedge y \notin A]$, subsequently the definition of a set difference, and finally the definition of the intersection of two sets. Because y is arbitrary, we may apply the Equality Criterion for sets to infer from the truth of the preceding equivalences the truth of the equation in (2.101). Since A, B and C are also arbitrary, we may therefore conclude that the proposed universal sentence holds. □

2.4.1. Basic relationships of set differences with the empty set

Proposition 2.33. *The difference of a set and the empty set is the set itself, that is,*

$$\forall A\,(A \setminus \emptyset = A). \tag{2.102}$$

Proof. Letting A be an arbitrary set, we apply the Equality Criterion for sets and prove the equation by verifying the equivalent

$$\forall y\,(y \in A \setminus \emptyset \Leftrightarrow y \in A). \tag{2.103}$$

For this purpose, we let y be arbitrary and observe that $y \in A \setminus \emptyset$ is equivalent to the conjunction $y \in A \wedge y \notin \emptyset$ according to (2.96). Since $y \notin \emptyset$ is a tautology by definition of the empty set, the preceding conjunction is equivalent to $y \in A$ because of (1.13). Consequently, $y \in A \setminus \emptyset$ is equivalent to $y \in A$, and since y is arbitrary, we may therefore conclude that the universal sentence (2.103) holds, which gives the equation $A \setminus \emptyset = A$ with the Equality Criterion for sets. Because the set A was also arbitrary, we may then further conclude that the proposed universal sentence (2.102) is also true. □

Exercise 2.7. Show for any set A that

a) the difference of A and A itself is empty, that is,

$$\forall A\,(A \setminus A = \emptyset). \tag{2.104}$$

b) the difference of the empty set and A is empty, that is,

$$\forall A\,(\emptyset \setminus A = \emptyset). \tag{2.105}$$

(Hint: Take the same approach as in the proof of Proposition 2.18, using (2.39) and (2.96).)

Exercise 2.8. Show for any sets A and B that the set difference of A and B and the set difference of B and A are disjoint, that is,

$$\forall A, B\,([A \setminus B] \cap [B \setminus A] = \emptyset). \tag{2.106}$$

Proposition 2.34. *Two sets A and B are disjoint iff the set difference of A and B is identical with A, that is,*

$$\forall A, B\,(A \cap B = \emptyset \Leftrightarrow A \setminus B = A). \tag{2.107}$$

Proof. We let A and B be arbitrary sets and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $A \cap B = \emptyset$ to be true, which equation implies by definition of the empty set

$$\forall y\,(y \notin A \cap B). \tag{2.108}$$

To establish the truth of $A \setminus B = A$, we apply the Equality Criterion for sets and prove the equivalent

$$\forall y\,(y \in A \setminus B \Leftrightarrow y \in A). \tag{2.109}$$

For this purpose, we take an arbitrary y, so that $\neg y \in A \cap B$ holds in view of (2.108). This negation further implies $\neg(y \in A \wedge y \in B)$ with the definition of the intersection of two sets, and then De Morgan's Law for sentences (1.51) gives the true disjunction

$$y \notin A \vee y \notin B. \tag{2.110}$$

In case that $y \notin A$ holds, we have that $y \in A$ is false, and the conjunction of $y \in A$ and $y \notin B$ is then also false. In light of the definition of a set difference, we now see that $y \in A \cap B$ is false. Together with the false $y \in A$, this shows that both sides of the equivalence in (2.109) are false, so that the equivalence itself is true.

In case that the second part $y \notin B$ of the disjunction (2.110) holds, we notice that the conjunction $y \in A \wedge y \notin B$ is equivalent to $y \in A$ because of (1.13) in connection with the fact that $y \notin B$ is then a tautology. Since $y \in A \setminus B$ is equivalent to $y \in A \wedge y \notin B$ (by definition of a set difference), it follows with the Substitution Rule for equivalences (1.29) that $y \in A \setminus B$ is equivalent to A.

Thus, the equivalence in (2.109) holds in any case, and as y was arbitrary, we therefore conclude that (2.109) is true. This universal sentence yields then $A \setminus B = A$ with the Equality Criterion for sets, so that the proof of the first part of the equivalence in (2.107) is complete.

We now prove the second part ('$\Leftarrow$') of the equivalence by contradiction, assuming $A \setminus B = A$ and $A \cap B \neq \emptyset$ to be true. The former assumption implies (2.109) again with the Function Criterion for sets, and the latter assumption implies with (2.42) that there exists an element in $A \cap B$, say $\bar{y}$. Here, $\bar{y} \in A \cap B$ means (by definition of the intersection of two sets) that $\bar{y} \in A$ and $\bar{y} \in B$ are both true. Let us now observe that (2.109) yields the true equivalence

$$\bar{y} \in A \setminus B \Leftrightarrow \bar{y} \in A.$$

Because $\bar{y} \in A$ is true, the left-hand side $\bar{y} \in A \setminus B$ of this equivalence also holds, which means (by definition of a set difference) that $\bar{y} \in A$ and $\bar{y} \notin B$ are both true. The latter contradicts the previously established $\bar{y} \in B$, which completes the proof of the second part of the equivalence in (2.107). Since A and B were arbitrary, we may therefore conclude that the proposed universal sentence holds, as claimed. □

We will occasionally apply the following lemma as a tool to turn two given sets A and B into disjoint sets by removing all the elements from B which are in A.

Lemma 2.35 (Generation of two disjoint sets). *Any set A and the difference of a set B and A are disjoint, that is,*

$$\forall A, B\,(A \cap [B \setminus A] = \emptyset). \tag{2.111}$$

Proof. We let A and B be arbitrary sets and prove

$$\forall y\,(y \notin A \cap [B \setminus A]). \tag{2.112}$$

Letting y be arbitrary, we now obtain the equivalences

$$\begin{aligned} y \in A \cap [B \setminus A] &\Leftrightarrow y \in A \wedge y \in B \setminus A \\ &\Leftrightarrow y \in A \wedge (y \in B \wedge \neg y \in A) \\ &\Leftrightarrow y \in A \wedge (\neg y \in A \wedge y \in B) \\ &\Leftrightarrow (y \in A \wedge \neg y \in A) \wedge y \in B, \end{aligned}$$

using the definition of the intersection of two sets, the definition of a set difference, the Commutative Law for sentences (1.36), and the Associative Law for sentences (1.39). Since $y \in A \wedge \neg y \in A$ is a contradiction as in (1.11), the multiple conjunction on the right-hand side is false. Consequently, the left-hand side $y \in A \cap (B \setminus A)$ is also false. Thus, the negation $y \notin A \cap [B \setminus A]$ is true, and since y is arbitrary, we may therefore conclude that (2.112) holds. This universal sentence then implies $A \cap [B \setminus A] = \emptyset$ by definition of the empty set, and as A and B were arbitrary sets, we may finally conclude that the proposed universal sentence (2.111) is true. □

Proposition 2.36. *If a set A is not included in a set B, then the set difference of A and B is nonempty, that is,*

$$\forall A, B\,(\neg A \subseteq B \Rightarrow A \setminus B \neq \emptyset). \tag{2.113}$$

Proof. We let A and B be arbitrary sets and observe the truth of the equivalences

$$\begin{aligned}
\neg A \subseteq B \;&\Leftrightarrow\; \neg\forall y\,(y \in A \Rightarrow y \in B) \\
&\Leftrightarrow\; \neg\forall y\,(y \in A \Rightarrow \neg y \notin B) \\
&\Leftrightarrow\; \exists y\,(y \in A \,\wedge\, y \notin B) \\
&\Leftrightarrow\; \exists y\,(y \in A \setminus B) \\
&\Leftrightarrow\; A \setminus B \neq \emptyset,
\end{aligned}$$

using the definition of a subset, the Double Negation Law, (1.82), the definition of a set difference, and finally (2.42). Therefore, the equivalence in (2.113) holds. Since A and B are arbitrary, we therefore conclude that the proposed universal sentence is true. □

Note 2.12. The proof of Proposition 2.36 shows that, if a set A is not included in a set B, then there exists an element in the set difference of A and B, i.e., we may write (2.113) equivalently as

$$\forall A, B\,(\neg A \subseteq B \Rightarrow \exists y\,(y \in A \setminus B)). \tag{2.114}$$

Corollary 2.37. *If a nonempty set A and a set B are disjoint, then the set difference of A and B is nonempty, and thus there exists an element in the set difference of A and B, that is,*

$$\forall A, B\,([A \neq \emptyset \,\wedge\, A \cap B = \emptyset] \Rightarrow A \setminus B \neq \emptyset), \tag{2.115}$$

$$\forall A, B\,([A \neq \emptyset \,\wedge\, A \cap B = \emptyset] \Rightarrow \exists y\,(y \in A \setminus B)). \tag{2.116}$$

Proof. Letting A and B be arbitrary sets such that $A \neq \emptyset$ and $A \cap B = \emptyset$ are true, we obtain $\neg A \subseteq B$ with (2.86). This negation then implies on the one hand $A \setminus B \neq \emptyset$ with (2.113), and on the other hand $\exists y\,(y \in A \setminus B)$ with (2.114). As A and B were arbitrary, the proposed sentences then follow to be true. □

2.4.2. Basic relationships of set differences with subsets

Proposition 2.38. *If a set A is a subset of the set difference of two sets B and C, then A and C are disjoint sets, that is,*

$$\forall A, B, C\,(A \subseteq B \setminus C \Rightarrow A \cap C = \emptyset). \tag{2.117}$$

Proof. Letting A, B and C be arbitrary sets we now prove the implication by contradiction, assuming that $A \subseteq B \setminus C$ and $A \cap C \neq \emptyset$ are both true. The latter implies with (2.42) that there exists an element in $A \cap C$, say $\bar{y}$. This in turn implies by definition of the intersection of two sets that $y \in A$ and $y \in C$ are both true. The former implies $y \in B \setminus C$ with the assumption $A \subseteq B \setminus C$ and the definition of a subset, so that $y \in B$ and $y \notin C$ are true by definition of a set difference. Here, $y \notin C$ contradicts the previously established $y \in C$, which completes the proof by contradiction of the implication in (2.117). Since A, B and C are arbitrary, we therefore conclude that the proposition is true. □

Exercise 2.9. Prove that the difference of a set A and a set B is empty if A is included in B, that is,

$$\forall A, B\, (A \subseteq B \Rightarrow A \setminus B = \emptyset). \tag{2.118}$$

(Hint: Proceed similarly as in the proof of Proposition 2.38.)

Exercise 2.10. Show that, if two sets B and C are disjoint and if, then any subset of B and the set C are also disjoint, that is,

$$\forall A, B, C\, ([B \cap C = \emptyset \ \wedge\ A \subseteq B] \Rightarrow A \cap C = \emptyset). \tag{2.119}$$

(Hint: Prove the implication by contradiction, using (2.42).)

Proposition 2.39. *Any subset A of a set B is identical with the difference of B and the difference of B and A, that is,*

$$\forall A, B\, (A \subseteq B \Rightarrow B \setminus [B \setminus A] = A). \tag{2.120}$$

Proof. We let A and B be arbitrary sets, assume $A \subseteq B$, and apply the Equality Criterion for sets to prove the stated equation, by verifying the equivalent universal sentence

$$\forall y\, (y \in B \setminus [B \setminus A] \Leftrightarrow y \in A). \tag{2.121}$$

For this purpose, we let y be arbitrary and observe the truth of the equivalences

$$\begin{aligned}
y \in B \setminus [B \setminus A] \ &\Leftrightarrow\ y \in B \ \wedge\ \neg y \in B \setminus A \\
&\Leftrightarrow\ y \in B \ \wedge\ \neg(y \in B \ \wedge\ y \notin A) \\
&\Leftrightarrow\ y \in B \ \wedge\ (y \notin B \ \vee\ \neg y \notin A) \\
&\Leftrightarrow\ (y \in B \ \wedge\ y \notin B) \ \vee\ (y \in B \ \wedge\ y \in A) \\
&\Leftrightarrow\ y \in B \ \wedge\ y \in A \\
&\Leftrightarrow\ y \in B \cap A \\
&\Leftrightarrow\ y \in A \cap B \\
&\Leftrightarrow\ y \in A
\end{aligned}$$

in light of the definition of a set difference (twice), De Morgan's Law for sentences (1.51), the Distributive Law for sentences (1.44) together with the Double Negation Law, (1.15) in connection with the fact that $y \in B \land y \notin B$ constitutes a contradiction according to (1.11), the definition of the intersection of two sets, the Commutative Law for the intersection of two sets, and finally substitution based on the equation $A \cap B = A$ implied by the initially assumed inclusion $A \subseteq B$ with (2.77). Consequently, the equivalence in (2.121) is true, and since y is arbitrary, we may therefore conclude that (2.121) holds. This universal sentence in turn implies the desired equation $B \setminus [B \setminus A] = A$ with the Equality Criterion for sets. This proves the implication in (2.120), and as A and B were arbitrary, we therefore conclude that the proposed universal sentence (2.120) holds, as claimed. □

Proposition 2.40. *If a set A is a subset of a set B, then the difference of a set C and B is included in the difference of C and A, that is,*

$$\forall A, B, C\,(A \subseteq B \Rightarrow C \setminus B \subseteq C \setminus A). \tag{2.122}$$

Proof. We let A, B and C be arbitrary, assume $A \subseteq B$, that is (applying the definition of a subset),

$$\forall y(y \in A \Rightarrow y \in B), \tag{2.123}$$

and show that this implies $C \setminus B \subseteq C \setminus A$, that is,

$$\forall y\,(y \in C \setminus B \Rightarrow y \in C \setminus A). \tag{2.124}$$

To do this, we let y be arbitrary and assume that the antecedent $y \in C \setminus B$ holds, which means by definition of a set difference that $y \in C$ and $y \notin B$ are both true. Let us now observe in view of (2.123) that the implication $y \in A \Rightarrow y \in B$ is true, so that its contraposition $y \notin B \Rightarrow y \notin A$ is also true (by the Law of Contraposition). With this, the previously established $y \notin B$ implies $y \notin A$. Thus, $y \in C$ and $y \notin A$ are both true, which means that $y \in C \setminus A$ holds. As y is arbitrary, we therefore conclude that (2.124) is true, so that $C \setminus B \subseteq C \setminus A$ holds. Since A, B and C were also arbitrary, it follows that the proposed universal sentence (2.122) is true. □

Exercise 2.11. Show that the difference of a set B and a set A is included in B, that is,

$$\forall A, B\,(B \setminus A \subseteq B). \tag{2.125}$$

(Hint: Proceed similarly as in the proof of Proposition 2.40.)

The result of this exercise immediately yields the following finding with the transitivity of $\subseteq$.

Corollary 2.41. *If a set B is included in a set C, then the difference of B and any set A is also included in C, that is,*

$$\forall A, B, C\,(B \subseteq C \Rightarrow B \setminus A \subseteq C). \tag{2.126}$$

Proof. Letting A, B and C be arbitrary sets such that $B \subseteq C$ holds, we first observe that $B \setminus A \subseteq B$ holds according to (2.125). Thus, the conjunction of $B \setminus A \subseteq B$ and $B \subseteq C$ is true, which in turn implies the desired inclusion $B \setminus A \subseteq C$ with (2.13). This proves the implication in (2.126), and since A, B and C were arbitrary, we may therefore conclude that the corollary is true. □

Proposition 2.42. *The difference of a set B and a proper subset of B is nonempty, that is,*

$$\forall A, B\,(A \subset B \Rightarrow B \setminus A \neq \emptyset). \tag{2.127}$$

Proof. We let A and B be arbitrary sets and prove the implication by contradiction, assuming that $A \subset B$ and $B \setminus A = \emptyset$ both hold. Here, the latter assumption implies with (2.41)

$$\neg \exists y\,(y \in B \setminus A), \tag{2.128}$$

and the former assumption that $A \subseteq B$ and $A \neq B$ are both true. Then, the former inclusion implies $A \setminus B = \emptyset$ with (2.118), and the latter inequality implies

$$\neg \forall y\,(y \in A \Leftrightarrow y \in B)$$

with (2.18). This negation further implies

$$\exists y\,(\neg[y \in A \Rightarrow y \in B] \;\vee\; \neg[y \in B \Rightarrow y \in A])$$

with the Quantifier Negation Law (1.53), (1.8) and De Morgan's Law (1.51), and then

$$\exists y\,(y \in A \,\wedge\, y \notin B) \,\vee\, \exists y\,(y \in B \,\wedge\, y \notin A)$$

with the Distributive Law for quantification (1.75) and the Conditional Law 1.32). This shows by definition of a set difference that $\exists y\,(y \in A \setminus B)$ or $\exists y\,(y \in B \setminus A)$ holds. The second part of this disjunction is false because of the truth of (2.128), so that the first part $\exists y\,(y \in A \setminus B)$ is true. This sentence implies $A \setminus B \neq \emptyset$ with (2.42), which evidently contradicts the previously established $A \setminus B = \emptyset$ according to (2.18). Thus, the proof of the implication in (2.127) is complete. Since A and B were arbitrary, it then follows that the proposed universal sentence is true. □

Proposition 2.43. *It is true that, if a nonempty set A is a proper subset of a nonempty set B, then the set difference of B and A is also a proper subset of B, that is,*

$$\forall A, B\,([A \neq \emptyset \land A \subset B] \Rightarrow B \setminus A \subset B). \tag{2.129}$$

Proof. We let A and B be arbitrary sets and consider the two cases $B = \emptyset$ and $B \neq \emptyset$. In the first case of $B = \emptyset$, we observe that (2.47) implies $\neg A \subset B$ with the Quantifier Negation Law (1.54). Thus, $A \subset B$ is false, so that the conjunction $A \neq \emptyset \land A \subset B$ is also false. Consequently, the implication in (2.129) is then true in the first case. In the second case of $B \neq \emptyset$, we may prove this implication directly, assuming that A is a nonempty, proper subset of B. To show that this implies $B \setminus A \subset B$, we first verify $B \setminus A \neq A$. For this purpose, we apply Exercise 2.1 and show that there exists an element y satisfying

$$(y \in B \setminus A \land y \notin B) \lor (y \in B \land y \notin B \setminus A). \tag{2.130}$$

Now, the initial assumption $A \neq \emptyset$ implies with (2.42) that there exists an element in A, say $\bar{y}$. Then, the disjunction $\bar{y} \notin B \lor \bar{y} \in A$ is also true, which we may write equivalently as $\neg(\bar{y} \in B \land \bar{y} \notin A)$ by applying De Morgan's Law (1.51) and the Double Negation Law. This negation means in connection with the definition of a set difference that $\bar{y} \notin B \setminus A$ holds. Furthermore, the initial assumption $A \subset B$ implies (by definition of a proper subset) in particular that $A \subseteq B$ is true. Therefore, the previously established $\bar{y} \in A$ implies $\bar{y} \in B$ with the definition of a subset, so that the conjunction $\bar{y} \in B \land \bar{y} \notin B \setminus A$ holds. Then, the disjunction

$$(\bar{y} \in B \setminus A \land \bar{y} \notin B) \lor (\bar{y} \in B \land \bar{y} \notin B \setminus A)$$

is also true, so that we proved the existence of an element y which satisfies (2.130). This existential sentence then implies with Exercise 2.1 that $B \setminus A \neq A$ holds. Since $B \setminus A \subseteq A$ is also true in view of (2.125), it follows from these findings by definition of a proper subset that the desired $B \setminus A \subset A$ holds. As A and B were arbitrary, we may therefore conclude that the proposed universal sentence (2.129) is true. □

In certain situations, there is a fundamental set X which includes all of the occurring sets. In such situations, it will be useful to have the following concept at out disposal.

Definition 2.13 (Complement). For any set X and any subset A of X we call

$$A^c = X \setminus A \tag{2.131}$$

the *complement of A with respect to X.*

Proposition 2.44. *For any set X and any subset B of X it is true that an element y of X is in the complement of B (with respect to X) iff y is not in B, that is,*

$$\forall X, B\,(B \subseteq X \Rightarrow \forall y\,(y \in X \Rightarrow [y \in B^c \Leftrightarrow y \notin B])). \tag{2.132}$$

Proof. Letting X and B be arbitrary, assuming $B \subseteq X$, and letting furthermore y be arbitrary in X, we see that $y \in B^c$ is by definition of a complement equivalent to $y \in X \setminus B$, which in turn is equivalent to $y \in X \wedge y \notin B$ by definition of a set difference. Since $y \in X$ is true and thus a tautology, the conjunction $y \in X \wedge y \notin B$ is equivalent to $y \notin B$ according to (1.13). Consequently, $y \in B^c$ is equivalent to $y \notin B$, and as y is arbitrary, we may therefore conclude that the universal sentence in (2.132) holds. Since X and B were also arbitrary, the universal sentence (2.132) follows then to be true. □

Exercise 2.12. Verify the following sentences for any set X.

a) The complement of the empty set with respect to X is identical with X, and the complement of X (with respect to X) is empty, i.e.

$$\emptyset^c = X, \tag{2.133}$$
$$X^c = \emptyset. \tag{2.134}$$

b) The intersection of a set A and its complement (with respect to X) is empty, i.e.

$$\forall A\,(A \subseteq X \Rightarrow A \cap A^c = \emptyset). \tag{2.135}$$

c) A set $A \subseteq X$ is identical with the complement of its complement, i.e.

$$\forall A\,(A \subseteq X \Rightarrow [A^c]^c = A). \tag{2.136}$$

d) The complement of a set $A \subseteq X$ is included in X, i.e.

$$\forall A\,(A \subseteq X \Rightarrow A^c \subseteq X). \tag{2.137}$$

(Hint: Apply Proposition 2.33, Exercise 2.7, Lemma 2.35, Proposition 2.39, and Exercise 2.11.)

We end this section with the demonstration of the useful fact that the set difference may be expressed by means of the complement and the intersection of two sets.

Theorem 2.45. *The intersection of the set A with the complement of the set B is equal to the set difference of A and B, that is,*

$$\forall X, A, B\,([A \subseteq X \;\wedge\; B \subseteq X] \Rightarrow A \setminus B = A \cap B^c). \tag{2.138}$$

Proof. Letting X, A and B be arbitrary sets and assuming $A \subseteq X$ as well as $B \subseteq X$ to be true, we first prove

$$\forall y\,(y \in A \setminus B \Leftrightarrow y \in A \cap B^c). \tag{2.139}$$

Letting y be arbitrary, we now obtain the true equivalences

$$\begin{aligned}
y \in A \cap B^c \;&\Leftrightarrow\; y \in A \;\wedge\; y \in B^c \\
&\Leftrightarrow\; y \in A \;\wedge\; y \in X \setminus B \\
&\Leftrightarrow\; y \in A \;\wedge\; (y \in X \;\wedge\; y \notin B) \\
&\Leftrightarrow\; y \in A \;\wedge\; (y \notin B \;\wedge\; y \in X) \\
&\Leftrightarrow\; (y \in A \;\wedge\; y \notin B) \;\wedge\; y \in X \\
&\Leftrightarrow\; (y \in A \setminus B) \;\wedge\; y \in X \\
&\Leftrightarrow\; y \in (A \setminus B) \cap X \\
&\Leftrightarrow\; y \in (A \setminus B)
\end{aligned}$$

by applying the definition of the intersection of two sets, the definition of a complement, the definition of a set difference, the Commutative Law for Sentences (1.36), the Associative Law for sentences (1.39), again the definition of a set difference, again the definition of the intersection of two sets, and finally (2.77) based the fact that the initially assumed $A \subseteq X$ implies $A \setminus B \subseteq X$ with (2.126). These equivalences evidently imply the truth of the equivalence in (2.139), and since y is arbitrary, it then follows that the universal sentence (2.139) also holds. Consequently, the Equality Criterion for sets gives the equation $A \setminus B = A \cap B^c$, which in turn proves the implication in (2.138). Finally, as X, A and B were also arbitrary, we conclude that the stated theorem is true. □

Corollary 2.46. *It is true for any subset A of a subset B of a set X that A and the complement of B (with respect to X) are disjoint, that is,*

$$\forall A, B, X\,([A \subseteq B \;\wedge\; B \subseteq X] \Rightarrow A \cap B^c = \emptyset). \tag{2.140}$$

Proof. Letting A, B and C be arbitrary sets such that A is included in B and such that B included in X, we obtain the inclusion $A \subseteq X$ with (2.13). Since the assumed inclusion $A \subseteq B$ implies with (2.118) $A \setminus B = \emptyset$, for which set difference the other two inclusions $A \subseteq X$ and $B \subseteq X$ imply

$A \setminus B = A \cap B^c$, we obtain the desired consequent $A \cap B^c = \emptyset$ of the implication in (2.140) via substitution. Here, the sets A, B and X were arbitrary, so that the corollary follows indeed to be true. □

Proposition 2.47. *For any disjoint sets A and B both included in a set X, it is true that A is included in the complement of B, that is,*

$$\forall A, B, X\,([A \subseteq X \,\wedge\, B \subseteq X \,\wedge\, A \cap B = \emptyset] \Rightarrow A \subseteq B^c). \tag{2.141}$$

Proof. We take arbitrary sets A, B and X such that $A \subseteq X$, $B \subseteq X$ and $A \cap B = \emptyset$ are all true. To establish the inclusion $A \subseteq B^c$, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall y\,(y \in A \Rightarrow y \in B^c). \tag{2.142}$$

We let y be arbitrary and assume $y \in A$ to be true, so that the assumed inclusion $A \subseteq X$ gives $y \in X$ (by definition of a subset). Since the disjointness assumption $A \cap B = \emptyset$ implies the negation $\neg y \in A \cap B$ by definition of the empty set, which implies the negation $\neg(y \in A \wedge y \in B)$ with the definition of the intersection of two sets, we obtain the true disjunction $y \notin A \vee y \notin B$ with De Morgan's Law for the conjunction. Because we assumed $y \in A$ to be true, so that the negation $y \notin A$ is false, the second part $y \notin B$ of the preceding disjunction is true. Consequently, the truth of $B \subseteq X$, of $y \in X$ and of $y \notin B$ implies with (2.132) the truth of $y \in B^c$, proving the implication in (2.142). Since y is arbitrary, we may therefore conclude that the universal sentence (2.142) is true, which then gives the desired inclusion $A \subseteq B^c$. This finding completes the proof of the implication in (2.141), and as A, B and X were initially arbitrary sets, we may now finally conclude that the proposed universal sentence is true. □

Exercise 2.13. Prove for any disjoint sets A and B both included in a set X that B is included in the complement of A^c, that is,

$$\forall A, B, X\,([A \subseteq X \,\wedge\, B \subseteq X \,\wedge\, A \cap B = \emptyset] \Rightarrow B \subseteq A^c). \tag{2.143}$$

2.5. Pairs and Singletons

We now introduce another axiom which will enable us to define a new type of set, which would not be possible with the axioms of specification and extension alone.

Axiom 2.3 (Axiom of pairing). For any a and any b there exists a set Z which contains a and b, that is,

$$\forall a, b \, \exists Z \, (a \in Z \wedge b \in Z). \tag{2.144}$$

As we intend to derive from this axiom a precise and unambiguous definition of a 'pair' of sets, we will in analogy to the establishment of the intersection of two sets and of the set difference exclude the possibility that this new set contains additional elements aside from a and b.

Theorem 2.48. *The following sentences are true for any a, b and any set Z containing a and b.*

a) There exists a unique set $\{a,b\}$ such that an element y is in $\{a,b\}$ iff y is in Z and moreover if y is identical with a or b, i.e.

$$\exists! \{a,b\} \, \forall y \, (y \in \{a,b\} \Leftrightarrow [y \in Z \wedge (y = a \vee y = b)]). \tag{2.145}$$

b) The set $\{a,b\}$ is nonempty, that is,

$$\{a,b\} \neq \emptyset. \tag{2.146}$$

c) The set $\{a,b\}$ satisfies also

$$\forall y \, (y \in \{a,b\} \Leftrightarrow [y = a \vee y = b]). \tag{2.147}$$

Proof. We let a and b be arbitrary and let Z an arbitrary set containing a and b (using the fact that such a set exists in view of the Axiom of Pairing).

Concerning a), we let $\varphi(y)$ be the formula

$$y = a \vee y = b.$$

Consequently, the Axiom of Specification yields the true existential sentence

$$\exists \{a,b\} \, \forall y \, (y \in \{a,b\} \Leftrightarrow [y \in Z \wedge \varphi(y)]),$$

so that the proof of the existential part of the uniquely existential sentence (2.145) is complete. We may prove the uniqueness part by means of the

Equality Criterion for sets. Now, the proof of the uniquely existential sentence (2.145) is complete, so that a) is true.

Concerning b), we let $\{a, b\}$ be the unique set satisfying

$$\forall y\,(y \in \{a,b\} \Leftrightarrow [y \in Z \wedge (y = a \vee y = b)]). \tag{2.148}$$

Next, we observe the truth of the equation $a = a$, so that the disjunction $a = a \vee a = b$ is also true. Furthermore, we see that $a \in Z$ holds, recalling that we initially assumed Z to be a set which contains a. Thus, the conjunction

$$a \in Z \wedge (a = a \vee a = b)$$

is true, which then implies $a \in \{a,b\}$ with (2.148). This shows that there exists an element in $\{a,b\}$, so that (2.146) is clearly true.

Concerning c), we let y be arbitrary and observe in light of (2.145) that $\{a,b\}$ and y satisfy

$$y \in \{a,b\} \Leftrightarrow [y \in Z \wedge (y = a \vee y = b)]. \tag{2.149}$$

Noting that $\{a,b\}$ is nonempty according to b), we prove the first part ('$\Rightarrow$') of the equivalence in (2.147) directly, assuming $y \in \{a,b\}$ to be true. This assumption implies with (2.149) in particular the disjunction $y = a \vee y = b$, which proves already the first part of the equivalence in (2.147). To prove the second part ('$\Leftarrow$'), we now assume the preceding disjunction to be true, which evidently implies $y \in Z$ irrespective of whether $y = a$ or $y = b$ holds, because Z was assumed to contain both a and b. Thus, the conjunction in (2.149) is true, which in turn implies $y \in \{a,b\}$. This proves the second part of the equivalence in (2.147), and as y was arbitrary, we therefore conclude that the set $\{a,b\}$ satisfies the universal sentence (2.147).

As a, b and Z were arbitrary, we finally conclude that the theorem is true. □

Exercise 2.14. Prove the uniqueness part of the uniquely existential sentence (2.145).

(Hint: Proceed in analogy to the corresponding proof for the intersection of a pair.)

Note 2.13. The preceding exercise should convince us sufficiently of the fact that the proof of the uniqueness part always follows exactly the same line of arguments, irrespective of the specific form of the formula $\varphi(y)$ and irrespective of the underlying set from which an element y with that definite property $\varphi(y)$ is 'picked out'. We may indeed use the proof of the uniqueness part in the proof of Theorem 2.15 as a template and simply replace the set

A and the formula '$y \in B$' by another set and another formula. Therefore, whenever we use a given set and a given definite property to establish the unique existence of a new set, we will in the following simply state the uniquely existential sentence without proving it in detail.

Definition 2.14 (Pair). For any a and b we call the set $\{a, b\}$ consisting of a and b, in the sense that

$$\forall y\,(y \in \{a, b\} \Leftrightarrow [y = a \lor y = b]),$$

the *pair* formed by a and b. We symbolize this set also by

$$\{y : y = a \lor y = b\}. \tag{2.150}$$

The following fundamental property of a pair is readily obtained from its definition.

Exercise 2.15. Verify the following sentence.

$$\forall a, b\,(a, b \in \{a, b\}). \tag{2.151}$$

(Hint: Recall the proof of Theorem 2.48b) in connection with (2.151).)

Evidently, whenever a happens to be equal to b, the corresponding pair $\{a, b\}$ can be written as $\{a, a\}$ or $\{b, b\}$, which are sets with a single element.

Notation 2.2 (Singleton). For any a we write

$$\{a\} = \{a, a\} \tag{2.152}$$

and call this pair formed by a and a the *singleton* formed by a.

Corollary 2.49. *Any a is element of the singleton formed by a, that is,*

$$\forall a\,(a \in \{a\}). \tag{2.153}$$

Proof. Letting a be arbitrary, we have $a \in \{a, a\}$ due to (2.151), so that substitution based on (2.152) yields $a \in \{a\}$. Since a is arbitrary, this is then true for all a. □

The concepts of a pair and of a singleton allow us to define two more 'natural numbers'.

Definition 2.15 (One). We call the set

$$1 = \{0\} \tag{2.154}$$

the (*number*) *one*.

Note 2.14. Due to (2.49), the set 1 can also be written in terms of the empty set, i.e.

$$1 = \{0\} = \{\emptyset\}. \tag{2.155}$$

Definition 2.16 (Two). We call the set

$$2 = \{0, 1\} \tag{2.156}$$

the (*number*) *two*.

Note 2.15. Because of (2.49 and (2.154), we can also express the set 2 solely by using $\emptyset$, that is,

$$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}. \tag{2.157}$$

To summarize, we begin with the empty set $0 = \{\ \}$, then we include 0 in that set to form the singleton $1 = \{0\}$, and then we include 1 to the preceding set to form the pair $2 = \{0, 1\}$.

Before constructing further numbers, we establish a few facts for pairs and singletons, which we will apply in subsequent chapters.

2.5.1. Basic laws for pairs

We begin with the observation that the intersection of two sets is not only colloquially but also technically identical with the intersection of a pair.

Proposition 2.50. *The intersection of two sets A and B is identical with the intersection of the pair formed by A and B, that is,*

$$\forall A, B\,(A \cap B = \bigcap\{A, B\}). \tag{2.158}$$

Proof. We let A and B be arbitrary sets and apply the Equality Criterion for sets to prove the equation $A \cap B = \bigcap\{A, B\}$. To do this, we verify the equivalent

$$\forall y\,(y \in A \cap B \Leftrightarrow y \in \bigcap\{A, B\}). \tag{2.159}$$

Letting y be arbitrary, we now prove the first part ('$\Rightarrow$') of the equivalence directly by assuming $y \in A \cap B$ to be true. Consequently, $y \in A$ and $y \in B$ are both true by definition of the intersection of two sets. To show that this implies $y \in \bigcap\{A, B\}$, we now apply the definition of the intersection of a set system and demonstrate the truth of the equivalent universal sentence

$$\forall Y\,(Y \in \{A, B\} \Rightarrow y \in Y). \tag{2.160}$$

We take an arbitrary $Y \in \{A, B\}$, which gives $Y = A \vee Y = B$ with the definition of a pair, so that we may now prove the desired consequent

$y \in Y$ by cases. If the first part $Y = A$ of this disjunction holds, then the previously established $y \in A$ gives $y \in Y$ via substitution. Similarly, if the second part $Y = B$ of the preceding disjunction holds, then the previously obtained $y \in B$ yields $y \in Y$, too. We thus showed that $Y \in \{A, B\}$ implies $y \in Y$ in any case, and since Y is arbitrary, we may therefore conclude that the universal sentence (2.160) is true. Thus, the equivalent $y \in \bigcap\{A, B\}$ also holds, so that the proof of the first part of the equivalence in (2.159) is complete.

To prove the second part ('$\Leftarrow$'), we let y be arbitrary and assume $y \in \bigcap\{A, B\}$ to be true, so that y satisfies the universal sentence (2.160) by definition of the intersection of a set system. Next, we verify that this implies $y \in A \cap B$, or equivalently $y \in A \wedge y \in B$ (applying the definition of the intersection of two sets). Let us now recall the truth of $A \in \{A, B\}$ and $B \in \{A, B\}$ in light of (2.151), where the former implies $y \in A$ and the latter $y \in B$ with (2.160). Thus, the conjunction $y \in A \wedge y \in B$ is true, and therefore the equivalent $y \in A \cap B$ also holds, which completes the proof of the equivalence in (2.159). Since y is arbitrary, we may therefore conclude that the universal sentence (2.159), and thus the equivalent equation in (2.158), is true. Then, as A and B were also arbitrary, it follows from this that the proposed universal sentence (2.158) holds, as claimed. □

Next, we give a formal expression to the evident fact that two pairs are the same whenever their element are the same. The following exercise serves as a preparation for this task.

Exercise 2.16. Show for any a and any b that the pair formed by a and b is identical with the pair formed by b and a, that is,

$$\forall a, b\,(\{a, b\} = \{b, a\}). \tag{2.161}$$

(Hint: Use (2.147) and (1.37).)

Theorem 2.51 (Equality Criterion for pairs). *The following sentence is true.*

$$\forall a, b, a', b'\,(\{a, b\} = \{a', b'\} \Leftrightarrow [(a = a' \wedge b = b') \vee (a = b' \wedge b = a')]). \tag{2.162}$$

Proof. We begin the proof by letting a, b, a' and b' be arbitrary.

We prove the first part ('$\Rightarrow$') of the stated equivalence directly, assuming the equation $\{a, b\} = \{a', b'\}$ to be true, so that the universal sentence

$$\forall y\,(y \in \{a, b\} \Leftrightarrow y \in \{a', b'\}) \tag{2.163}$$

follows to be also true in view of the Equality Criterion for sets. Since we have

$$\begin{aligned} a &\in \{a, b\}, \\ b &\in \{a, b\}, \\ a' &\in \{a', b'\}, \\ b' &\in \{a', b'\} \end{aligned}$$

according to (2.151), it follows with the preceding universal sentence that

$$\begin{aligned} a &\in \{a', b'\}, \\ b &\in \{a', b'\}, \\ a' &\in \{a, b\}, \\ b' &\in \{a, b\} \end{aligned}$$

also hold. These four sentences in turn imply with the definition of a pair the respective disjunctions

$$\begin{aligned} a = a' &\vee a = b', \\ b = a' &\vee b = b', \\ a' = a &\vee a' = b, \\ b' = a &\vee b' = b, \end{aligned}$$

so that the conjunction

$$([a = a' \vee a = b'] \wedge [b = a' \vee b = b']) \wedge ([a' = a \vee a' = b] \wedge [b' = a \vee b' = b])$$

is true, which we may also write as

$$([a = a' \vee a = b'] \wedge [b = a' \vee b = b']) \wedge ([a = a' \vee b = a'] \wedge [a = b' \vee b = b']).$$

Because of the Commutative Law for sentences (1.37), this yields

$$([a = b' \vee a = a'] \wedge [b = a' \vee b = b']) \wedge ([b = a' \vee a = a'] \wedge [a = b' \vee b = b']).$$

Next, we apply the Associative Law for sentences (1.39) twice to rearrange the brackets, so that we obtain first

$$[a = b' \vee a = a'] \wedge ([b = a' \vee b = b'] \wedge ([b = a' \vee a = a'] \wedge [a = b' \vee b = b']))$$

and then

$$[a = b' \vee a = a'] \wedge (([b = a' \vee b = b'] \wedge [b = a' \vee a = a']) \wedge [a = b' \vee b = b']).$$

We are now in a position to apply the Distributive Law for sentences (1.45), which gives

$$[a = b' \ \vee \ a = a'] \ \wedge \ ((b = a' \ \vee \ [b = b' \ \wedge \ a = a']) \ \wedge \ [a = b' \ \vee \ b = b']).$$

Applying now first the Commutative Law for sentences (1.36), we obtain

$$[a = b' \ \vee \ a = a'] \ \wedge \ ([a = b' \ \vee \ b = b'] \ \wedge \ (b = a' \ \vee \ [b = b' \ \wedge \ a = a'])),$$

and then with the Associative Law for sentences (1.39)

$$([a = b' \ \vee \ a = a'] \ \wedge \ [a = b' \ \vee \ b = b']) \ \wedge \ (b = a' \ \vee \ [b = b' \ \wedge \ a = a']).$$

Subsequently, the Distributive Law for sentences (1.45) gives

$$(a = b' \vee [a = a' \wedge b = b']) \wedge (b = a' \vee [b = b' \wedge a = a']),$$

and another application of the Commutative Law for sentences (1.37) results in

$$([a = a' \ \wedge \ b = b'] \ \vee \ a = b') \ \wedge \ ([a = a' \ \wedge \ b = b'] \ \vee \ b = a').$$

Finally, we arrive at the disjunction in (2.162) by using the Distributive Law for sentences (1.45) once again, so that the proof of the first part of the equivalence in (2.162) is complete.

To prove the second part ('$\Leftarrow$'), we now assume the disjunction

$$(a = a' \ \wedge \ b = b') \ \vee \ (a = b' \ \wedge \ b = a')$$

to be true. On the one hand, if $a = a' \ \wedge \ b = b'$ holds, then we immediately obtain $\{a, b\} = \{a', b'\}$ via substitution. On the other hand, if $a = b' \wedge b = a'$ holds, then we obtain

$$\{a, b\} = \{b', a'\} = \{a', b'\}$$

by applying substitution and (2.161), and therefore again $\{a, b\} = \{a', b'\}$. Thus, the second part of the equivalence in (2.162) also holds.

As a, b, a' and b' are arbitrary, we therefore conclude that the theorem is true. $\square$

For later reference, we also state and prove the following rather obvious

Proposition 2.52. *For any elements a and b of a set X it is true that the pair formed by a and b is included in X, that is,*

$$\forall X, a, b\,(a, b \in X \Rightarrow \{a, b\} \subseteq X). \qquad (2.164)$$

Proof. We let X, a and b be arbitrary and assume $a, b \in X$ to be true. To prove the inclusion $\{a, b\} \subseteq X$, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall y\,(y \in \{a, b\} \Rightarrow y \in X). \tag{2.165}$$

For this purpose, we let $y \in \{a, b\}$ be arbitrary, which implies by definition of a pair the disjunction $y = a \lor y = b$, which we now use to prove the sentence $y \in X$ by cases. In case $y = a$ holds, then the assumed $a \in X$ gives the desired $y \in X$ via substitution. Similarly, if $y = b$ holds, then the initial assumption $b \in X$ yields $y \in X$, as desired. As y is arbitrary, we therefore conclude that (2.165) holds, so that $\{a, b\} \subseteq X$ follows to be true by definition of a subset. This in turn proves the implication in (2.164), and since a and b are arbitrary, we may then further conclude that the universal sentence (2.164) also holds. Finally, because X was also arbitrary, it then follows that the proposed sentence is true. □

2.5.2. Basic laws for singletons

Exercise 2.17. Show for any a, any a' and any b' that the singleton formed by a is identical with the pair formed by a' and b' iff a is identical both with a and with b', that is,

$$\forall a, a', b'\,(\{a\} = \{a', b'\} \Leftrightarrow [a = a' \land a = b']). \tag{2.166}$$

(Hint: Use (2.152), (2.162), (1.36), and (1.34)).

Exercise 2.18. Show for any a and any b that the pair formed by a and b is identical with the singleton formed by a iff a is identical with b, that is,

$$\forall a, b\,(\{a, b\} = \{a\} \Leftrightarrow a = b). \tag{2.167}$$

(Hint: Use (2.152), (2.162), (1.36), and (1.34)).

Theorem 2.53 (Equality Criterion for singletons). *The singleton formed by an a is identical with the singleton formed by an a' iff a and a' are identical, that is,*

$$\forall a, a'\,(\{a\} = \{a'\} \Leftrightarrow a = a'). \tag{2.168}$$

Proof. Letting a and a' be arbitrary, we obtain the true equivalences

$$\begin{aligned} \{a\} = \{a'\} &\Leftrightarrow \{a\} = \{a', a'\} \\ &\Leftrightarrow a = a' \land a = a' \\ &\Leftrightarrow a = a' \end{aligned}$$

by applying substitution based on the notation for singletons, then (2.166), and finally the Idempotent Law (1.33). Consequently, the equivalence in (2.168) is true, and since a and a' were arbitrary, it follows that the proposed universal sentence holds. $\square$

Proposition 2.54. *For any y and a it is true that y and a are identical iff y is an element of the singleton formed by a, that is,*

$$\forall y, a\,(y = a \Leftrightarrow y \in \{a\}). \tag{2.169}$$

Proof. We let y and a be arbitrary and observe the truth of the equivalences

$$\begin{aligned} y = a \ &\Leftrightarrow\ y = a \lor y = a \\ &\Leftrightarrow\ y \in \{a, a\} \\ &\Leftrightarrow\ y \in \{a\} \end{aligned}$$

in view of the Idempotent Law for the disjunction, the definition of a pair and the notation for a singleton. Then, the equivalence in (2.169) follows to be true, and since y and a were arbitrary, we therefore conclude that the proposed universal sentence holds. $\square$

Proposition 2.55. *It is true that the intersection of the singleton formed by any set is identical with that set, that is,*

$$\forall X\,(\bigcap\{X\} = X). \tag{2.170}$$

Proof. We let X be an arbitrary set and prove the equation by means of the Equality Criterion for sets, i.e. by verifying

$$\forall y\,(y \in \bigcap\{X\} \Leftrightarrow y \in X). \tag{2.171}$$

We take an arbitrary y and assume first $y \in \bigcap\{X\}$ to be true, which implies the truth of

$$\forall A\,(A \in \{X\} \Rightarrow y \in A). \tag{2.172}$$

with the definition of the intersection of a set system. Then, since $X \in \{X\}$ is also true according to (2.153), it follows with the preceding universal sentence that $y \in X$ holds, as desired. Thus, the first part ('$\Rightarrow$') of the equivalence in (2.171) holds. To establish the second part ('$\Leftarrow$'), we assume now $y \in X$ to be true and prove the universal sentence (2.172), which will imply the desired consequent $y \in \bigcap\{X\}$ by definition of the intersection of a set system. Letting A be arbitrary and assuming $A \in \{X\}$ to be true, we obtain the true equation $A = X$ with (2.169), so that the assumed $y \in X$ yields $y \in A$ via substitution. Thus, the implication in (2.172) is true, and because A is arbitrary, we may therefore conclude that the universal sentence (2.172) holds. Consequently, $y \in \bigcap\{X\}$ also holds, which finding completes the proof of the equivalence in (2.171). Since y is also arbitrary, we may further conclude that the universal sentence (2.171) holds, which in turn implies $\bigcap\{X\} = X$ (with the Equality Criterion for Sets). Finally, as the set X was arbitrary, we may infer from the truth of this equation the truth of the proposition. $\square$

Proposition 2.56. *A set X and the singleton formed by an a are disjoint if a is not element of X, that is,*

$$\forall a, X \, (a \notin X \Rightarrow X \cap \{a\} = \emptyset). \tag{2.173}$$

Proof. We let a and X be arbitrary and prove the implication by contradiction, assuming $a \notin X$ and $X \cap \{a\} \neq \emptyset$ to be both true. The latter implies with (2.42) that there exists an element in $X \cap \{a\}$, say $\bar{y}$. By definition of the intersection of a pair, it now follows that $\bar{y} \in X$ and $\bar{y} \in \{a\}$ are both true. The latter finding then further implies $\bar{y} = a$ with (2.169), so that the former former gives $a \in X$ via substitution. Since we assumed $a \notin X$ to be also true, we obtained a contradiction, so that the proof of the implication in (2.173) is complete. Since a and X were arbitrary, we may therefore conclude that the proposed universal sentence holds. $\square$

Exercise 2.19. Verify that the singletons formed by an y and by an a are disjoint if y and a are distinct, that is,

$$\forall y, a \, (y \neq a \Rightarrow \{y\} \cap \{a\} = \emptyset). \tag{2.174}$$

(Hint: You may proceed in analogy to the proof of Proposition 2.173.)

Proposition 2.57. *For any a and any b the intersection of the singleton formed by a and the pair formed by a and b is identical with the singleton formed by a, that is,*

$$\forall a, b \, (\{a\} \cap \{a, b\} = \{a\}). \tag{2.175}$$

Proof. We let a and b be arbitrary and apply then the Equality Criterion for sets to verify the equation in (2.175). For this purpose, we prove the universal sentence

$$\forall y \, (y \in \{a\} \cap \{a, b\} \Leftrightarrow y \in \{a\}). \tag{2.176}$$

Letting y be arbitrary, we obtain the true equivalences

$$\begin{aligned} y \in \{a\} \cap \{a, b\} \; &\Leftrightarrow \; y \in \{a\} \wedge y \in \{a, b\} \\ &\Leftrightarrow \; y = a \wedge (y = a \vee y = b) \end{aligned}$$

by applying the definition of the intersection of two sets and then (2.169) together with the definition of a pair. We may therefore evidently write the equivalence in (2.176) to be proven equivalently as

$$[y = a \land (y = a \lor y = b)] \Leftrightarrow y = a. \tag{2.177}$$

We now prove the first part ('⇒') of this equivalence directly, assuming that $y = a \land (y = a \lor y = b)$ holds. This conjunction implies in particular the desired $y = a$, which proves already the implication '⇒'. To prove the second part ('⇐'), we now assume $y = a$ to be true. Then, the disjunction $y = a \lor y = b$ is also true, so that the desired conjunction in (2.177) holds, completing the proof of the equivalence. Consequently, the equivalent equivalence in (2.176) is also true, and since y is arbitrary, we may now conclude that the universal sentence (2.176) holds. Then, the equation in (2.175) follows to be true with the Equality Criterion for sets, and as a and b were also arbitrary, we may further conclude that the proposed universal sentence (2.175) holds, as claimed. □

Proposition 2.58. *The difference of a genuine pair and the singleton formed by one of the pair's elements is the singleton formed by the other element of the pair, i.e.*

$$\forall a, b\,(a \neq b \Rightarrow \{a, b\} \setminus \{a\} = \{b\}). \tag{2.178}$$

Proof. We let a and b be distinct elements of the universe. To prove that the sets $\{a, b\} \setminus \{a\}$ and $\{b\}$ are identical, we apply the Equality Criterion for sets. For this purpose, we let y be arbitrary and then obtain the true equivalences

$$\begin{aligned}
y \in \{a, b\} \setminus \{a\} &\Leftrightarrow y \in \{a, b\} \land y \notin \{a\} \\
&\Leftrightarrow (y = a \lor y = b) \land y \neq a \\
&\Leftrightarrow (y \neq A \land y = a) \lor (y \neq a \land y = b) \\
&\Leftrightarrow y \neq A \land y = b \\
&\Leftrightarrow y \in \{b\} \land y \notin \{a\} \\
&\Leftrightarrow y \in \{b\} \setminus \{a\} \\
&\Leftrightarrow y \in \{b\}
\end{aligned}$$

using the definition of a set difference, the definition of a pair together with (2.169), the Commutative Law for sentences (1.36) together with the Distributive Law for sentences (1.44), (1.15) with the fact that $y \neq a \land y = a$ constitutes a contradiction according to (1.11), again (1.36) as well as (2.169), again the definition of a set difference, and finally (2.107) in connection with the fact that the assumption $a \neq b$ implies $\{a\} \cap \{b\} = \emptyset$ due to (2.174). As y is arbitrary, we therefore conclude that the equation in (2.178) holds. Since a and b were also arbitrary, it then follows that the proposition holds, as claimed. □

Proposition 2.59. *For any set A it is true that an y is not element of the difference of A and the singleton formed by y, that is,*

$$\forall A, y\,(y \notin A \setminus \{y\}). \tag{2.179}$$

Proof. We let A and y be arbitrary and prove $y \notin A \setminus \{y\}$ by contradiction. For this purpose, we assume $y \in A \setminus \{y\}$ to be true, which implies the conjunction $y \in A \wedge y \notin \{y\}$ by definition of a set difference. Then, this conjunction implies in particular $y \notin \{y\}$, which evidently contradicts the fact that $y \in \{y\}$ is true according to (2.153). This completes the proof by contradiction of $y \notin A \setminus \{y\}$ is true; since A and y were arbitrary, we may therefore conclude that (2.179) holds. □

Exercise 2.20. Show that every singleton has a unique element, that is,

$$\forall a\, \exists! y\,(y \in \{a\}). \tag{2.180}$$

(Hint: Apply Method 1.17, using (2.153) and (2.169).)

Theorem 2.60 (Characterization of singletons). *It is true that a set X is the singleton formed by constant a if, and only if, X is nonempty and any element of X is identical with a, that is,*

$$\forall X, a\,(X = \{a\} \Leftrightarrow [X \neq \emptyset \wedge \forall x\,(x \in X \Rightarrow x = a)]). \tag{2.181}$$

Proof. Letting X and a be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $X = \{a\}$ to be true. Let us now observe in light of (2.180) that there exists a unique constant y such that $y \in \{a\}$ holds. Consequently, $[X =]\ \{a\} \neq \emptyset$ is true according to (2.42), proving the first part of the conjunction in (2.181). Next, we let x be arbitrary and assume $x \in X$ to be true. Then, substitution yields $x \in \{a\}$, and this finding implies $x = a$ with (2.169). Because x is arbitrary, we may therefore conclude that the second part of the desired conjunction also holds, completing the proof of the first part of the equivalence in (2.181). We now prove the second part ('$\Leftarrow$') by contraposition, for which purpose we assume $X \neq \{a\}$ to be true. We may write the desired consequent

$$\neg[X \neq \emptyset \wedge \forall x\,(x \in X \Rightarrow x = a)]$$

equivalently as

$$X = \emptyset \vee \exists x\,(x \in X \wedge x \neq a), \tag{2.182}$$

using De Morgan's Law for the conjunction in connection with the Double Negation Law and the Negation Law for universal implications. We now prove this disjunction by cases, based on the fact that the disjunction $X = \emptyset \vee X \neq \emptyset$ is true because of the Law of the Excluded Middle. The first case $X = \emptyset$ immediately implies the truth of the disjunction (2.182) to be proven. Let us observe in the second case $X \neq \emptyset$ that the assumed $X \neq \{a\}$ implies the truth of the existential sentence

$$\exists y\,([y \in X \,\wedge\, y \notin \{a\}] \,\vee\, [y \in \{a\} \,\wedge\, y \notin X]).$$

according to (2.23). Thus, there is a particular constant $\bar{y}$ such that

$$[\bar{y} \in X \,\wedge\, \bar{y} \notin \{a\}] \,\vee\, [\bar{y} \in \{a\} \,\wedge\, \bar{y} \notin X]$$

holds. We may now use this disjunction to prove the existential sentence in (2.182) by cases. On the one hand, if $\bar{y} \in X \,\wedge\, \bar{y} \notin \{a\}$ is true, then the second part of this conjunction yields $\bar{y} \neq a$ with (2.169). Together with the first part of the preceding conjunction, this shows that there exists indeed a constant x satisfying both $x \in X$ and $x \neq a$. On the other hand, if $\bar{y} \in \{a\} \,\wedge\, \bar{y} \notin X$ is true, then the first part of this conjunction gives $\bar{y} = a$ with (2.169), so that the second part of the conjunction implies $a \notin X$ via substitution. Since the current case assumption $X \neq \emptyset$ implies the existence of a particular element $\bar{x} \in X$, we may now infer from the two findings $\bar{x} \in X$ and $a \notin X$ the truth of $\bar{x} \neq a$ by applying (2.4). Having found the constant $\bar{x}$ satisfying both $\bar{x} \in X$ and $\bar{x} \neq a$, we have that the existential sentence in (2.182) holds again. Thus, the proof of the disjunction (2.182) is complete, and this establishes the desired consequent of the proof by contraposition. Therefore, the equivalence in (2.181) holds, and since X and a were arbitrary, we may then further conclude that the proposed universal sentence is true. $\square$

Exercise 2.21. Show that a set X is neither empty nor a singleton iff X contains two distinct elements, that is,

$$\forall X\,([X \neq \emptyset \wedge \forall a\,(X \neq \{a\})] \Leftrightarrow \exists x\,\exists y\,(x \in X \,\wedge\, y \in X \,\wedge\, x \neq y)). \quad (2.183)$$

(Hint: Concerning '$\Rightarrow$', apply Method 1.10 with (1.54), (1.81), (1.90), (1.51), (1.53), (1.35), and consider then the two cases $X = \emptyset$ and $X \neq \emptyset$ in connection with the Characterization of singletons.)

Exercise 2.22. Show that a constant y is contained in a set A iff the singleton formed by y is included in A, that is,

$$\forall A, y\,(y \in A \Leftrightarrow \{y\} \subseteq A). \quad (2.184)$$

(Hint: Use Method 1.6 and Method 1.11 in connection with (2.169), (2.9) and (2.153).)

Corollary 2.61. *For any a and any b the singletons formed by a and b are subsets of the pair formed by a and b, that is,*

$$\forall a, b\,(\{a\} \subseteq \{a,b\} \,\wedge\, \{b\} \subseteq \{a,b\}). \tag{2.185}$$

Proof. We let a and b be arbitrary and observe the truth of $a \in \{a,b\}$ and of $b \in \{a,b\}$ in light of 2.151. These findings in turn imply, respectively, $\{a\} \subseteq \{a,b\}$ and $\{b\} \subseteq \{a,b\}$ with (2.184), so that the conjunction in (2.185) holds. Since a and b were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Proposition 2.62. *Any nonempty subset A of a singleton $\{y\}$ equals that singleton, that is,*

$$\forall A, y\,([A \subseteq \{y\} \,\wedge\, A \neq \emptyset] \Rightarrow A = \{y\}). \tag{2.186}$$

Proof. We let A and y be arbitrary and prove the implication directly. For this purpose, we assume both $A \subseteq \{y\}$ and $A \neq \emptyset$ to be true. Due to the definition of the empty set in connection with (2.41), we have that $A \neq \emptyset$ implies that there exists an element of A, say $\bar{y}$. Then, $\bar{y} \in A$ implies $\bar{y} \in \{y\}$ with the assumption $A \subseteq \{y\}$ and the definition of a subset. In view of (2.169), we now see that $\bar{y} \in \{y\}$ implies $\bar{y} = y$, so that $\bar{y} \in A$ implies $y \in A$ with substitution. The latter then further implies $\{y\} \subseteq A$ with (2.184). Finally, the conjunction of this and the assumption $A \subseteq \{y\}$ implies the desired $\{y\} = A$ with the Axiom of Extension. This equation proves the implication in (2.186), and since A and y were arbitrary, we therefore conclude that the proposition holds. □

Exercise 2.23. Show that any proper subset of a singleton is empty, i.e.

$$\forall A, y\,(A \subset \{y\} \Rightarrow A = \emptyset). \tag{2.187}$$

(Hint: Use Method 1.11, (2.25), and (2.186).)

Proposition 2.63. *If a set A is a subset of a set C and if y is not an element of A, then A is a subset of the difference of C and the singleton formed by y, that is,*

$$\forall A, C, y\,([A \subseteq C \,\wedge\, y \notin A] \Rightarrow A \subseteq C \setminus \{y\}). \tag{2.188}$$

Proof. We let A, C and y be arbitrary and prove the implication directly. For this purpose, we assume that $A \subseteq C$ and $y \notin A$ are true and show that this implies the inclusion $A \subseteq C \setminus \{y\}$, which is equivalent to

$$\forall x\,(x \in A \Rightarrow x \in C \setminus \{y\}) \tag{2.189}$$

by definition of a subset. To prove this universal sentence, we let x be arbitrary and consider the two cases $A = \emptyset$ and $A \neq \emptyset$. In the first case, $x \in A$ is false by definition of the empty set, and therefore the implication is true, having a false antecedent. In the second case $A \neq \emptyset$, we prove the implication by contradiction, assuming that both $x \in A$ and $\neg x \in C \setminus \{y\}$ are true. The latter means $\neg(x \in C \wedge x \notin \{y\})$ by definition of a set difference, and therefore $x \notin C \vee x \in \{y\}$ with De Morgan's Law (1.51) and the Double Negation Law. Thus, at least one part of this disjunction is true. If $x \notin C$ holds, then this contradicts the fact that the assumption $x \in A$ implies $x \in C$ with the initial assumption $A \subseteq C$ (and the definition of a subset). If $x \in \{y\}$ holds, then also $x = y$ due to (2.169), so that $x \in A$ now implies $y \in A$, which contradicts the initial assumption $y \notin A$. This proves the universal sentence (2.189), that is, $A \subseteq C \setminus \{y\}$, and thus the implication in (2.188). Since A, C and y were arbitrary, we therefore conclude that the proposition is true. □

Exercise 2.24. Show that if a set B is included in a set A, then the difference of B and a singleton $\{y\}$ is included in the difference of A and $\{y\}$, that is,

$$\forall A, B, y\,(B \subseteq A \Rightarrow B \setminus \{y\} \subseteq A \setminus \{y\}). \tag{2.190}$$

(Hint: Use Method 1.6 alongside Definition 2.2 and Definition 2.12.)

Proposition 2.64. *It is true that, if a set B is a proper subset of a set A, then the difference of B and the singleton $\{y\}$ formed by an element y of B is a proper subset of the difference of A and the singleton $\{y\}$, that is,*

$$\forall A, B, y\,([B \subset A \wedge y \in B] \Rightarrow B \setminus \{y\} \subset A \setminus \{y\}). \tag{2.191}$$

Proof. We let A, B and y be arbitrary and prove the implication directly, assuming $B \subset A$ and $y \in B$ to be true. To show that this implies $B \setminus \{y\} \subset A \setminus \{y\}$, we prove the equivalent (applying the definition of a proper subset)

$$B \setminus \{y\} \subseteq A \setminus \{y\} \;\wedge\; B \setminus \{y\} \neq A \setminus \{y\}. \tag{2.192}$$

To prove the first part of this conjunction, we observe that the assumption $B \subset A$ implies (by definition of a proper subset) that $B \subseteq A$ and $B \neq A$ are both true; here, $B \subseteq A$ already implies the desired $B \setminus \{y\} \subseteq A \setminus \{y\}$ with (2.190). To prove the second part of the conjunction (2.192), we notice that $B \neq A$ implies with (2.23) that there is an element, say $\bar{y}$, such that

$$(\bar{y} \in B \;\wedge\; \bar{y} \notin A) \;\vee\; (\bar{y} \in A \;\wedge\; \bar{y} \notin B). \tag{2.193}$$

holds. Let us also observe that the assumption $y \in B$ implies $y \in A$ with the previously obtained $B \subseteq A$ and the definition of a subset, so that $y \in B$

and $y \in A$ are both true. We now use the true disjunction (2.193) to carry out a proof by cases of the sentence $y \neq \bar{y}$. We assume the first part $\bar{y} \in B \wedge \bar{y} \notin A$ of the disjunction to be true, which implies in particular $\bar{y} \notin A$, and carry out a proof by contradiction. To do this, we assume $y = \bar{y}$, so that $y \in A$ implies $\bar{y} \in A$. Thus, $\bar{y} \notin A$ and $\bar{y} \in A$ are both true, so that we have a contradiction according to (1.11). We now assume the second part $\bar{y} \in A \wedge \bar{y} \notin B$ of the disjunction (2.193) to be true, which implies in particular $\bar{y} \notin B$, and carry out another proof by contradiction. For this purpose, we again assume $y = \bar{y}$, so that $y \in B$ implies $\bar{y} \in B$. Thus, $\bar{y} \notin B$ and $\bar{y} \in B$ are both true, so that we have a contradictioin also in the second case, which completes the proof of $y \neq \bar{y}$. Now, this inequality evidently implies $\bar{y} \notin \{y\}$ with (2.169). Thus, the conjunction of this and (2.193) is true, which reads

$$\bar{y} \notin \{y\} \wedge [(\bar{y} \in B \wedge \bar{y} \notin A) \vee (\bar{y} \in A \wedge \bar{y} \notin B)].$$

With the Distributive Law for sentences (1.44), this yields

$$[\bar{y} \notin \{y\} \wedge \bar{y} \in B \wedge \neg \bar{y} \in A] \vee [\bar{y} \notin \{y\} \wedge \bar{y} \in A \wedge \neg \bar{y} \in B]. \tag{2.194}$$

As $\bar{y} \notin \{y\}$ is true, it follows that $\neg \bar{y} \notin \{y\}$ is a contradiction, so that $\neg \bar{y} \in A$ is equivalent to $\neg \bar{y} \in A \vee \neg \bar{y} \notin \{y\}$ and $\neg \bar{y} \in B$ equivalent to $\neg \bar{y} \in B \vee \neg \bar{y} \notin \{y\}$ because of (1.15). We may therefore rewrite (2.194) equivalently as

$$\begin{aligned}&[\bar{y} \notin \{y\} \wedge \bar{y} \in B \wedge (\neg \bar{y} \in A \vee \neg \bar{y} \notin \{y\})] \\ &\quad \vee [\bar{y} \notin \{y\} \wedge \bar{y} \in A \wedge (\neg \bar{y} \in B \vee \neg \bar{y} \notin \{y\})],\end{aligned}$$

which is also equivalent to

$$\begin{aligned}&[\bar{y} \in B \wedge \bar{y} \notin \{y\} \wedge \neg(\bar{y} \in A \wedge \bar{y} \notin \{y\})] \\ &\quad \vee [\bar{y} \in A \wedge \bar{y} \notin \{y\} \wedge \neg(\bar{y} \in B \wedge \bar{y} \notin \{y\})]\end{aligned}$$

because of the Commutative Law for sentences (1.36) and De Morgan's Law for sentences (1.51). Then, the definition of a set difference gives

$$[\bar{y} \in B \setminus \{y\} \wedge \bar{y} \notin A \setminus \{y\}] \vee [\bar{y} \in A \setminus \{y\} \wedge \bar{y} \notin B \setminus \{y\}].$$

This proves the existential sentence

$$\exists x\,([x \in B \setminus \{y\} \wedge x \notin A \setminus \{y\}] \vee [x \in A \setminus \{y\} \wedge x \notin B \setminus \{y\}]),$$

which implies the desired $B \setminus \{y\} \neq A \setminus \{y\}$ with (2.23). Thus, the proof of the conjunction (2.192) is complete, so that $B \setminus \{y\}$ is indeed a proper subset of $A \setminus \{y\}$. Since A, B and y were arbitrary, we therefore conclude that the proposed universal sentence (2.64) is true. □

2.6. Unions of Sets

To express the next number 3 as a set according to this mechanism, we would should 'add' 2 to the set $\{0, 1\}$ to form a 'triplet'. To enable such an operation is the purpose of the following axiom.

Axiom 2.4 (Axiom of union). For any set system $\mathcal{S}$ there exists a set V which contains all the elements of all the sets in $\mathcal{S}$, that is,

$$\forall \mathcal{S}\, \exists V\, \forall A\, \forall y\, ([y \in A \,\wedge\, A \in \mathcal{S}] \Rightarrow y \in V). \tag{2.195}$$

As for the pair, we will rule out for the sake of uniqueness any possible elements of V which are not in any of the sets of the given set system.

Theorem 2.65. *The following sentences are true for any set system $\mathcal{S}$ and any set V containing all the elements of all the sets in $\mathcal{S}$.*

a) There exists a unique set $\bigcup \mathcal{S}$ such that an element y is in $\bigcup \mathcal{S}$ iff y is in V and moreover in some set A in $\mathcal{S}$, that is,

$$\exists! \bigcup \mathcal{S}\, \forall y\, (y \in \bigcup \mathcal{S} \Leftrightarrow [y \in V \,\wedge\, \exists A\, (A \in \mathcal{S} \,\wedge\, y \in A)]). \tag{2.196}$$

b) The set $\bigcup \mathcal{S}$ satisfies also

$$\forall y\, (y \in \bigcup \mathcal{S} \Leftrightarrow \exists A\, (A \in \mathcal{S} \,\wedge\, y \in A)). \tag{2.197}$$

Exercise 2.25. Prove Theorem 2.65.

(Hint: Proceed in analogy to the proof of Theorem 2.28.)

Definition 2.17 (Union of a set system). For any set system $\mathcal{S}$ we call the set $\bigcup \mathcal{S}$ consisting of all the elements in all of the sets in $\mathcal{S}$, in the sense that

$$\forall y\, (y \in \bigcup \mathcal{S} \Leftrightarrow \exists A\, (A \in \mathcal{S} \,\wedge\, y \in A)),$$

the *union* of $\mathcal{S}$. This set is also symbolized by

$$\bigcup_{A \in \mathcal{S}} A \qquad \text{or} \qquad \{y : \exists A\, (A \in \mathcal{S} \,\wedge\, y \in A)\}. \tag{2.198}$$

Proposition 2.66. *The union of the singleton formed by any set is identical with that set, that is,*

$$\forall A\, (\bigcup \{A\} = A). \tag{2.199}$$

Proof. Letting $\bar{A}$ be an arbitrary set, we prove $\bigcup\{\bar{A}\} = \bar{A}$ by applying the Equality Criterion for sets, i.e. by verifying the equivalent universal sentence

$$\forall y\,(y \in \bigcup\{\bar{A}\} \Leftrightarrow y \in \bar{A}). \tag{2.200}$$

To do this, we take an arbitrary y and assume first $y \in \bigcup\{\bar{A}\}$ to be true. By definition of the union of a set system, there exists then a set, say $\bar{\bar{A}}$, such that $\bar{\bar{A}} \in \{\bar{A}\}$ and $y \in \bar{\bar{A}}$ hold. The former implies $\bar{\bar{A}} = \bar{A}$ with (2.169), so that we obtain via substitution $y \in \bar{A}$, as desired. Assuming now conversely $y \in \bar{A}$ to be true, which gives $\bar{A} \in \{\bar{A}\}$ with (2.153), we thus see that the existential sentence $\exists A\,(A \in \{\bar{A}\} \wedge y \in \{\bar{A}\})$ holds. This in turn implies the desired $y \in \bigcup\{\bar{A}\}$ with the definition of the union of a set system. Since y is arbitrary, we may therefore conclude that the universal sentence (2.200) is true, which yields then the equality $\bigcup\{\bar{A}\} = \bar{A}$. As $\bar{A}$ was arbitrary, we may now infer from the preceding equation the truth of the proposed universal sentence. □

Proposition 2.67. *For any set system $\mathcal{S}$ it is true that all of its sets are included in the union of $\mathcal{S}$, that is,*

$$\forall \mathcal{S}, A\,(A \in \mathcal{S} \Rightarrow A \subseteq \bigcup \mathcal{S}). \tag{2.201}$$

Proof. We let $\mathcal{S}$ be an arbitrary set system, A an arbitrary set in $\mathcal{S}$, and we prove $A \subseteq \bigcup \mathcal{S}$ by verifying

$$\forall y\,(y \in A \Rightarrow y \in \bigcup \mathcal{S}). \tag{2.202}$$

To do this, we let y be arbitrary and assume that $y \in A$ is true. The conjunction of this assumption and the initially assumed $A \in \mathcal{S}$ proves the existential sentence

$$\exists A\,(A \in \mathcal{S} \wedge y \in A),$$

which in turn implies $y \in \bigcup \mathcal{S}$ because of (2.197), and which thus proves the implication in (2.202). Since y is arbitrary, we therefore conclude that the universal sentence (2.202) holds, which means $A \subseteq \bigcup \mathcal{S}$ by definition of a subset. As A and $\mathcal{S}$ were also arbitrary, we finally conclude that the proposition holds, as claimed. □

Proposition 2.68. *It is true that removing the empty set from any set (system) $\mathcal{S}$ does not alter its union, that is,*

$$\forall \mathcal{S}\,(\bigcup[\mathcal{S} \setminus \{\emptyset\}] = \bigcup \mathcal{S}). \tag{2.203}$$

Proof. We take an arbitrary any set (system) $\mathcal{S}$ and apply the Equality Criterion for sets to establish the equation. For this purpose, we prove the universal sentence

$$\forall y\,(y \in \bigcup[\mathcal{S} \setminus \{\emptyset\}] \Leftrightarrow y \in \bigcup \mathcal{S}), \tag{2.204}$$

letting y be arbitrary. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $y \in \bigcup[\mathcal{S} \setminus \{\emptyset\}]$, so that there exists by definition of the union of a set system a set, say $\bar{A}$, satisfying $\bar{A} \in \mathcal{S} \setminus \{\emptyset\}$ and $y \in \bar{A}$. Because of the definition of a set difference, $\bar{A} \in \mathcal{S}$ is then true in particular. In conjunction with $y \in \bar{A}$, this implies $y \in \bigcup \mathcal{S}$ with the definition of the union of a set system, as desired.

Regarding the second part ('$\Leftarrow$') of the equivalence, we assume now $y \in \bigcup \mathcal{S}$, so that there evidently exists a particular set $\bar{B} \in \mathcal{S}$ with $y \in \bar{B}$. The latter clearly shows that $\bar{B}$ is nonempty, so that $\bar{B} \notin \{\emptyset\}$ follows to be true with (2.169). In conjunction with $\bar{B} \in \mathcal{S}$, this negation implies evidently $\bar{B} \in \mathcal{S} \setminus \{\emptyset\}$, which further implies – together with $y \in \bar{B}$ – the desired consequent $y \in \bigcup[\mathcal{S} \setminus \{\emptyset\}]$.

Having thus completed the proof of the equivalence, we may now infer from this the truth of (2.204), since y was arbitrary. Consequently, the equation in (2.203) holds as well, in which $\mathcal{S}$ is arbitrary, so that the proposed universal sentence follows to be true. □

Exercise 2.26. Prove that

a) the union of the empty set system $\emptyset$ is itself empty, that is,

$$\bigcup \emptyset = \emptyset. \tag{2.205}$$

b) the union of the singleton formed by the empty set is empty, that is,

$$\bigcup \{\emptyset\} = \emptyset. \tag{2.206}$$

(Hint: Apply (2.169).)

c) the non-emptiness of the union of a set system implies the non-emptiness of the set system, that is,

$$\forall \mathcal{S}\,(\bigcup \mathcal{S} \neq \emptyset \Rightarrow \mathcal{S} \neq \emptyset). \tag{2.207}$$

Definition 2.18 (Partition). We say for any set X that a system $\mathcal{S}$ of subsets of X is a *partition* of X iff $\mathcal{S}$ is a system of pairwise disjoint sets whose union is identical with X, i.e. iff

$$\forall A, B\,([A, B \in \mathcal{S} \,\wedge\, A \neq B] \Rightarrow A \cap B = \emptyset) \,\wedge\, \bigcup \mathcal{S} = X. \tag{2.208}$$

Definition 2.19 (Union of two sets, union of a pair). For any sets A and B we call the set

$$A \cup B = \bigcup \{A, B\} \tag{2.209}$$

the *union* of A and B.

Proposition 2.69. *The union of two sets A and B satisfies*

$$\forall y \, (y \in A \cup B \Leftrightarrow y \in A \,\vee\, y \in B). \tag{2.210}$$

Proof. Letting A and B be arbitrary sets, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $y \in A \cup B$ to be true. This in turn implies $y \in \bigcup\{A, B\}$ with (2.209), and then moreover with (2.197) that there exists a set, say $\bar{Y}$, such that $\bar{Y} \in \{A, B\}$ and $y \in \bar{Y}$ hold. Here, $\bar{Y} \in \{A, B\}$ implies the true disjunction $\bar{Y} = A \,\vee\, \bar{Y} = B$ with (2.147), which we may now use to prove the desired disjunction $y \in A \,\vee\, y \in B$ by cases. On the one hand, if $\bar{Y} = A$ holds, then $y \in \bar{Y}$ gives $y \in A$, and the disjunction $y \in A \,\vee\, y \in B$ is then also true. On the other hand, if $\bar{Y} = B$ is true, then $y \in \bar{Y}$ yields $y \in B$, and the disjunction $y \in A \vee y \in B$ holds again. Thus, the proof of the first part of the equivalence in (2.210) is complete.

Next, we prove the second part ('$\Leftarrow$') directly, assuming that the disjunction $y \in A \,\vee\, y \in B$ is true, which we use to prove the existential sentence

$$\exists Y \, (Y \in \{A, B\} \,\wedge\, y \in Y) \tag{2.211}$$

by cases. Let us first observe that $A, B \in \{A, B\}$ holds according to (2.151). Now, if the first part $y \in A$ of the preceding disjunction is true, then this implies together with $A \in \{A, B\}$ the truth of the existential sentence (2.211). Similarly, the second case $y \in B$ yields with $B \in \{A, B\}$ again (2.211), completing the proof by cases. Then, $y \in \bigcup\{A, B\}$ follows to be true with (2.197), and we therefore obtain with (2.209) $y \in A \cup B$, as desired, completing the proof of the second part of the equivalence in (2.210) and thus the proof of that equivalence. As y is arbitrary, we may therefore conclude that the universal sentence (2.210) is true, and since A and B were also arbitrary, it then follows that the proposition holds, as claimed. □

Notation 2.3. We symbolize the union $A \cup B$ of two sets A and B also by

$$\{y : y \in A \,\vee\, y \in B\}$$

Exercise 2.27. Show that the union of any set and any singleton is nonempty, that is,

$$\forall X, a \, (X \cup \{a\} \neq \emptyset). \tag{2.212}$$

(Hint: Use (2.153), (2.210) and (2.42).)

The concept of union also allows us to define some further *numbers*.

Definition 2.20 (Three). We call the set

$$3 = 2 \cup \{2\} \tag{2.213}$$

the (*number*) *three*.

2.6.1. Basic laws

Theorem 2.70 (Commutative Law for the union of two sets). *The union of two sets A and B is identical with the union of B and A, that is,*

$$\forall A, B\, (A \cup B = B \cup A). \tag{2.214}$$

Exercise 2.28. Prove the Commutative Law for the union of two sets.

(Hint: Proceed in analogy to the proof of the Commutative Law for the intersection of two sets.)

Theorem 2.71 (Idempotent Law for the union of two sets). *The pairwise union is idempotent in the sense that*

$$\forall A\, (A \cup A = A). \tag{2.215}$$

Exercise 2.29. Prove the Idempotent Law for the union of two sets.

(Hint: Proceed in analogy to the proof of the Idempotent Law for the intersection of two sets.)

Proposition 2.72. *The union of any set and the empty set as well as the union of the empty set and any set is the set itself, that is,*

$$\forall A\, (A \cup \emptyset = A \,\wedge\, \emptyset \cup A = A). \tag{2.216}$$

Proof. We let A be arbitrary and prove the first part of the stated conjunction by verifying

$$\forall y\, (y \in A \cup \emptyset \Leftrightarrow y \in A). \tag{2.217}$$

For this purpose, we let y be arbitrary and observe the truth of the equivalences

$$\begin{aligned} y \in A \cup \emptyset \;&\Leftrightarrow\; y \in A \,\vee\, y \in \emptyset \\ &\Leftrightarrow\; y \in A, \end{aligned}$$

using (2.210) and then (1.15) in connection with the fact that $y \in \emptyset$ is a contradiction by definition of the empty set. Thus, the equivalence $y \in$

$A \cup \emptyset \Leftrightarrow y \in A$ follows to be true, and since y is arbitrary, we may therefore conclude that (2.217) holds. This universal sentence in turn implies the desired equation $A \cup \emptyset = A$ with the Equality Criterion for sets.

Regarding the second part of the conjunction in (2.216), we notice that the equation

$$\emptyset \cup A = A \cup \emptyset \;(= A)$$

holds due to the Commutative Law for the union of two sets, which yields $\emptyset \cap A = A$. As the set A was arbitrary, we may now conclude that the proposed sentence (2.216) holds. □

Corollary 2.73. *The union of the empty set and the empty set is also empty, that is,*

$$\emptyset \cup \emptyset = \emptyset. \tag{2.218}$$

The Associative Law for sentences regarding the disjunction translates naturally to the

Theorem 2.74 (Associative Law for the union of two sets). *The union of two sets is associative in the sense that*

$$\forall A, B, C\, (A \cup [B \cup C] = [A \cup B] \cup C). \tag{2.219}$$

Exercise 2.30. Prove the Associative Law for the union of two sets.

(Hint: Proceed similarly as in the proof of the Associative Law for the intersection of two sets.)

Theorem 2.75 (Distributive Law for the union & the intersection of two sets). *The following laws are true.*

a) The union of two sets is distributive over the intersection of two sets in the sense that

$$\forall A, B, C\, (A \cup [B \cap C] = [A \cup B] \cap [A \cup C]). \tag{2.220}$$

b) The intersection of two sets is distributive over the union of two sets in the sense that

$$\forall A, B, C\, (A \cap [B \cup C] = [A \cap B] \cup [A \cap C]). \tag{2.221}$$

Proof. Concerning a), we let A, B and C be arbitrary sets and verify first the truth of

$$\forall y\, (y \in A \cup [B \cap C] \Leftrightarrow y \in [A \cup B] \cap [A \cup C]). \tag{2.222}$$

Letting y be arbitrary, we obtain the true equivalences

$$\begin{aligned}
y \in A \cup [B \cap C] &\Leftrightarrow (y \in A \lor y \in B \cap C) \\
&\Leftrightarrow (y \in A \lor [y \in B \land y \in C]) \\
&\Leftrightarrow ([y \in A \lor y \in B] \land [y \in A \lor y \in C]) \\
&\Leftrightarrow (y \in A \cup B \land y \in A \cup C) \\
&\Leftrightarrow (y \in [A \cup B] \cap [A \cup C])
\end{aligned}$$

by applying (2.210), (2.57), (1.45), again (2.210), and finally again (2.57). Thus, the equivalence in (2.222) holds, and since y is arbitrary, we may therefore conclude that (2.222) is true. It then follows from this with the Equality Criterion for sets that the equation in (2.220) also holds. As A, B and C were arbitrary, we may finally conclude that the Distributive Law a) is true. □

Exercise 2.31. Verify the Distributive Law (2.221).

Proposition 2.76. *For disjoint sets B and C we may either first 'subtract' B from a set A and then form the union with C, or equivalently form first the union of A and C and then 'subtract' B, that is,*

$$\forall A, B, C\,(B \cap C = \emptyset \Rightarrow [A \setminus B] \cup C = [A \cup C] \setminus B). \tag{2.223}$$

Proof. We let A, B and C be arbitrary sets and assume B and C to be disjoint, which assumption implies $C \setminus B = C$ with (2.107). Then, to prove the equation in (2.223), we apply the Equality Criterion for sets and verify accordingly the equivalent universal sentence

$$\forall y\,(y \in [A \setminus B] \cup C \Leftrightarrow y \in [A \cup C] \setminus B). \tag{2.224}$$

Letting y be arbitrary, we then obtain the true equivalences

$$\begin{aligned}
y \in (A \setminus B) \cup C &\Leftrightarrow y \in (A \setminus B) \cup (C \setminus B) \\
&\Leftrightarrow y \in A \setminus B \lor y \in C \setminus B \\
&\Leftrightarrow (y \in A \land y \notin B) \lor (y \in C \land y \notin B) \\
&\Leftrightarrow (y \notin B \land y \in A) \lor (y \notin B \land y \in C) \\
&\Leftrightarrow y \notin B \land (y \in A \lor y \in C) \\
&\Leftrightarrow y \in A \cup C \land y \notin B \\
&\Leftrightarrow y \in (A \cup C) \setminus B
\end{aligned}$$

using the previously established equation $C \setminus B = C$, the definition of the union of a pair, the definition of a set difference, the Commutative Law for sentences (1.36), the Distributive Law for sentences (1.44), then again the Commutative Law for sentences (1.36), the definition of the union of a pair, and the definition of a set difference. As y is arbitrary, it therefore follows that the universal sentence (2.224) holds, so that the sets $[A \setminus B] \cup C$ and $[A \cup C] \setminus B$ are identical. Since A, B and C were also arbitrary, we then conclude that the proposition (2.223) is true. □

Exercise 2.32. Show that 'removing' first a set B from a set A and then a set C from that set difference yields the same set as removing the union of B and C from A, that is,

$$\forall A, B, C\, (A \setminus [B \cup C] = [A \setminus B] \setminus C). \tag{2.225}$$

Proposition 2.77. *The union of two singletons is a pair, that is,*

$$\forall a, b\, (\{a\} \cup \{b\} = \{a, b\}). \tag{2.226}$$

Proof. We let a and b be arbitrary and establish first

$$\forall y\, (y \in \{a\} \cup \{b\} \Leftrightarrow y \in \{a, b\}), \tag{2.227}$$

letting y be arbitrary and observing the truth of

$$\begin{aligned} y \in \{a\} \cup \{b\} &\Leftrightarrow y \in \{a\} \lor y \in \{b\} \\ &\Leftrightarrow y = a \lor y = b \\ &\Leftrightarrow y \in \{a, b\} \end{aligned}$$

in light of (2.210), (2.169), and (2.147). Since y is arbitrary, we may therefore conclude that (2.227) holds, and as a and b were also arbitrary, the proposition follows to be true. □

Note 2.16. Recalling the definition of the number $2 = \{0, 1\}$ we also have according to (2.226)

$$2 = \{0\} \cup \{1\}. \tag{2.228}$$

Definition 2.21 (Triple). For any a, b and c we call the union of the pair formed by a, b and the singleton formed by c the *triple* formed by a, b and c, symbolically

$$\{a, b, c\} = \{a, b\} \cup \{c\}. \tag{2.229}$$

Exercise 2.33. Show for any a, b and c that an element y is in the triple formed by a, b and c iff y is identical with a, or with b, or with c, that is,

$$\forall a, b, c, y\, (y \in \{a, b, c\} \Leftrightarrow [y = a \lor y = b \lor y = c]). \tag{2.230}$$

(Hint: Apply (2.229), (2.210), (2.147), and (2.169).)

Exercise 2.34. Verify for any a that the triple formed by a, a and a is identical with the singleton formed by a, that is,

$$\forall a\,(\{a,a,a\} = \{a\}). \tag{2.231}$$

(Hint: Apply Exercise 2.33, Theorem 1.9, and Proposition 2.54.)

Exercise 2.35. Show for any a, any b and any c that the triple formed by a, b and c is identical with the triple formed by b, c and a, that is,

$$\forall a,b,c\,(\{a,b,c\} = \{b,c,a\}). \tag{2.232}$$

(Hint: Use (2.230) and (1.37).)

Corollary 2.78. *For any a, any b and any c it is true that a, b and c are elements of the triple formed by these constants, that is,*

$$\forall a,b,c\,(a,b,c \in \{a,b,c\}). \tag{2.233}$$

Proof. Letting a, b and c be arbitrary and $\{a,b,c\}$ the triple formed by a, b and c, we observe the truth of the equations $a = a$, $b = b$ and $c = c$. Then, the three disjunctions

$$\begin{aligned} &a = a \,\vee\, a = b \,\vee\, a = c \\ &b = a \,\vee\, b = b \,\vee\, b = c \\ &c = a \,\vee\, c = b \,\vee\, c = c \end{aligned}$$

are also true, which in turn imply $a \in \{a,b,c\}$, $b \in \{a,b,c\}$ and $c \in \{a,b,c\}$, respectively, with (2.230). Since a, b and c were arbitrary, we may therefore conclude that (2.233) holds, as claimed. □

Exercise 2.36. Show for any elements a, b and c of a set X that the triple formed by a, b and c is included in X, that is,

$$\forall a,b,c\,(a,b,c \in X \Rightarrow \{a,b,c\} \subseteq X). \tag{2.234}$$

(Hint: Proceed in analogy to the proof of Proposition 2.52.)

Note 2.17. Recalling the definition of the number $3 = 2 \cup \{2\}$, where $2 = \{0,1\}$, we obtain with the definition of a triple

$$3 = \{0,1,2\}. \tag{2.235}$$

Thus, 0 is empty, 1 is a singleton, 2 a pair and 3 a triple. Using (2.49), (2.155) and (2.157), we may also write

$$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}. \tag{2.236}$$

Note 2.18. Because of the Commutative Law for the union of two sets, the union of a singleton $\{a\}$ and a pair $\{b, c\}$ also yields a triple.

Proposition 2.79. *For any a, b and c it is true that both the pair formed by a, b and the singleton formed by c are subsets of the triple formed by a, b, c, that is,*

$$\forall a, b, c\,(\{a, b\} \subseteq \{a, b, c\} \,\wedge\, \{c\} \subseteq \{a, b, c\}). \tag{2.237}$$

Proof. We let a, b and c be arbitrary, and we prove the first stated inclusion by verifying the equivalent (applying the definition of a subset)

$$\forall y\,(y \in \{a, b\} \Rightarrow y \in \{a, b, c\}). \tag{2.238}$$

For this purpose, we let y be arbitrary and assume $y \in \{a, b\}$ to be true. Then, the disjunction $y \in \{a, b\} \vee y \in \{c\}$ is also true, which in turn implies $y \in \{a, b\} \cup \{c\}$ with (2.210), and therefore $y \in \{a, b, c\}$ by definition of a triple. As y is arbitrary, we may therefore conclude that the universal sentence (2.238) is true, so that $\{a, b\} \subseteq \{a, b, c\}$ follows to be true by definition of a subset. Thus, the proof of the first part of the conjunction in (2.237) is complete.

To prove the second part $\{c\} \subseteq \{a, b, c\}$, we verify the equivalent

$$\forall y\,(y \in \{c\} \Rightarrow y \in \{a, b, c\}). \tag{2.239}$$

To do this, we let y be arbitrary and assume $y \in \{c\}$, so that the disjunction $y \in \{a, b\} \vee y \in \{c\}$ is again true. Using the same arguments as before, we obtain the desired $y \in \{a, b, c\}$, and since y is arbitrary, it follows that (2.239) holds. Consequently, $\{c\} \subseteq \{a, b, c\}$ is true (by definition of a subset), completing the proof of the stated conjunction. Because a, b and c were arbitrary, we may finally conclude that the proposed universal sentence is true. □

We now see that a pair may be viewed as a special case of a triple.

Proposition 2.80. *For any a and any b the union of the singleton formed by a and the pair formed by a and b is identical with the pair formed by a and b, that is,*

$$\forall a, b\,(\{a\} \cup \{a, b\} = \{a, b\}). \tag{2.240}$$

Proof. We let a and b be arbitrary and apply then the Equality Criterion for sets to verify the equation in (2.240). For this purpose, we prove the universal sentence

$$\forall y\,(y \in \{a\} \cup \{a, b\} \Leftrightarrow y \in \{a, b\}). \tag{2.241}$$

Letting y be arbitrary, we obtain the true equivalences

$$\begin{aligned} y \in \{a\} \cup \{a,b\} &\Leftrightarrow y \in \{a\} \lor y \in \{a,b\} \\ &\Leftrightarrow y = a \lor (y = a \lor y = b) \\ &\Leftrightarrow (y = a \lor y = a) \lor y = b \\ &\Leftrightarrow y = a \lor y = b \\ &\Leftrightarrow y \in \{a\} \lor y \in \{b\} \\ &\Leftrightarrow y \in \{a\} \cup \{b\} \\ &\Leftrightarrow y \in \{a,b\} \end{aligned}$$

by applying the definition of the union of two sets, then (2.169) together with the definition of a pair, the Associative Law for the disjunction, the Idempotent Law for the disjunction, again (2.169), again the definition of the union of two sets, and finally (2.226). As y is arbitrary, we may now conclude that the universal sentence (2.241) holds, and the equation in (2.240) follows then to be true with the Equality Criterion for sets. Since a and b were arbitrary, too, we may finally conclude that the proposed universal sentence (2.240) is true. □

Definition 2.22 (Quadruple). For any a, b, c and d we call the union of the triple formed by a, b, c and the singleton formed by d the *quadruple* formed by a, b, c and d, symbolically

$$\{a,b,c,d\} = \{a,b,c\} \cup \{d\}. \tag{2.242}$$

Exercise 2.37. Verify for any a, b, c, d that an element y is in the triple formed by a, b, c and d iff y is identical with a, or with b, or with c, or with d, that is,

$$\forall a,b,c,d,y\,(y \in \{a,b,c,d\} \Leftrightarrow [(y = a \lor y = b \lor y = c) \lor y = d]). \tag{2.243}$$

(Hint: Proceed in analogy to Exercise 2.33.)

Exercise 2.38. Show for any a, b, c, d that a, b, c and d are elements of the quadruple formed by these constants, that is,

$$\forall a,b,c,d\,(a,b,c,d \in \{a,b,c,d\}). \tag{2.244}$$

(Hint: Apply a proof similarly to the one for Corollary 2.78.)

Exercise 2.39. Define the number 4 in analogy to the number 3 and express then 4 as a quadruple in terms of the empty set.

(Hint: Use (2.235), (2.236), and (2.242).)

In the same manner we could systematically define 5 as $4 \cup \{4\}$ and so on. However, as there would clearly always be one further number to be defined (6, then 7, then 8, etc.), we could not define in this way a set consisting of these numbers in their entirety without using unlimited space and time. To achieve this, we will have to introduce another set-theoretical axiom, which we postpone until Section 2.7.

2.6.2. Basic relationships of unions with subsets

Exercise 2.40. Show that the union of two sets includes both of the sets, that is,

$$\forall A, B\,(A \subseteq A \cup B \,\wedge\, B \subseteq A \cup B). \tag{2.245}$$

(Hint: Apply (2.151), (2.201), and (2.209).)

Corollary 2.81. *The intersection of any two sets is included in the union of these sets, that is,*

$$\forall A, B\,(A \cap B \,\subseteq\, A \cup B). \tag{2.246}$$

Proof. Letting A and B be arbitrary, we obtain on the one hand $A \cap B \subseteq A$ with (2.74), and on the other hand $A \subseteq A \cup B$ with (2.245). The conjunction of these two inclusions then implies the desired $A \cap B \subseteq A \cup B$ with (2.13), and since A and B were arbitrary, we therefore conclude that the proposed universal sentence (2.246) is true. □

Proposition 2.82. *The union of a set with a subset yields the former set, that is,*

$$\forall A, B\,(A \subseteq B \Rightarrow A \cup B = B). \tag{2.247}$$

Proof. We let A and B be arbitrary sets, and we prove the implication directly, assuming A to be a subset of B, so that

$$\forall y\,(y \in A \Rightarrow y \in B) \tag{2.248}$$

holds. Next, we prove the equation $A \cup B = B$ by applying the Equality Criterion for sets, i.e. by verifying

$$\forall y\,(y \in A \cup B \Leftrightarrow y \in B). \tag{2.249}$$

To do this, we let y be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $y \in A \cup B$ to hold. This means by definition of the union of two sets that the disjunction $y \in A \,\vee\, y \in B$ is true. Let us now carry out a proof by cases of the sentence $y \in B$ based on that disjunction. On the one hand, if $y \in A$ holds, then the desired consequent

$y \in B$ follows to be true with (2.248). On the other hand, if the second part $y \in B$ of the preceding disjunction holds, then we already arrived at the desired consequent, and the proof of the first part of the equivalence in (2.249) is complete.

To prove the second part ('$\Leftarrow$'), we now assume that $y \in B$ is true. Then, the disjunction $y \in A \vee y \in B$ is also true (irrespective of the truth value of $y \in A$), so that $y \in A \cup B$ holds by definition of the union of two sets. This completes the proof of the equivalence in (2.249), and since y is arbitrary, we may therefore conclude that (2.249) holds. This universal sentence then further implies $A \cup B = B$ with the Equality Criterion for sets, so that the proof of the implication in (2.247) is also complete. As A and B were arbitrary, we finally conclude that the proposition holds. □

Since any set is a subset of itself according to (2.10), we immediately obtain the following result with the preceding proposition.

Corollary 2.83. *Any set is identical to the union of the set with itself, i.e.*

$$\forall A\,(A \cup A = A). \tag{2.250}$$

Proposition 2.84. *The following sentences are true.*

a) If a set A is a subset of a set B, then the union of A with a set C is included in the union of B with C, that is,

$$\forall A, B, C\,(A \subseteq B \Rightarrow A \cup C \subseteq B \cup C). \tag{2.251}$$

b) Two sets A and B are subsets of a set C iff the union of A and B is a subset of C, that is,

$$\forall A, B, C\,([A \subseteq C \,\wedge\, B \subseteq C] \Leftrightarrow A \cup B \subseteq C). \tag{2.252}$$

Proof. Concerning a), we let A, B, C be arbitrary and prove the implication directly, assuming that $A \subseteq B$, or equivalently (using the definition of a subset) that

$$\forall y(y \in A \Rightarrow y \in B) \tag{2.253}$$

holds. To prove $A \cup C \subseteq B \cup C$, we demonstrate the truth of the equivalent

$$\forall y\,(y \in A \cup C \Rightarrow y \in B \cup C). \tag{2.254}$$

For this purpose, we let y be arbitrary and assume $y \in A \cup C$, which means $y \in A \vee y \in C$ by definition of the union of a pair. We use this true disjunction to prove the sentence $y \in B \cup C$ by cases. In the first case that $y \in A$ holds, it follows with (2.253) that $y \in B$ is true, which further implies the truth of the disjunction $y \in B \lor y \in C$ (no matter if $y \in C$ is true or false), and therefore the truth of $y \in B \cup C$ (using again the definition of the union of a pair). In the second case that $y \in C$ is true, the disjunction $y \in B \lor y \in C$ is evidently true as well (no matter if $y \in B$ is true or false), which yields again $y \in B \cup C$. We thus showed that $y \in A \cup C$ implies $y \in B \cup C$, and since y is arbitrary, we may therefore conclude that (2.254) is true. This proves $A \cup C \subseteq B \cup C$ and thus the implication in (2.251). As A, B and C were arbitrary, we finally conclude that a) holds.

Concerning b), we let A, B, C again be arbitrary, so that we obtain the equivalences

$$\begin{aligned} A \subseteq C \land B \subseteq C &\Leftrightarrow \forall y\,(y \in A \Rightarrow y \in C) \land \forall y\,(y \in B \Rightarrow y \in C) \\ &\Leftrightarrow \forall y\,([y \in A \Rightarrow y \in C] \land [y \in B \Rightarrow y \in C]) \\ &\Leftrightarrow \forall y\,([y \in A \lor y \in B] \Rightarrow y \in C) \\ &\Leftrightarrow \forall y\,([y \in A \cup B] \Rightarrow y \in C) \\ &\Leftrightarrow A \cup B \subseteq C \end{aligned}$$

using the definition of a subset, the Distributive Law for quantification (1.74), the Distributive Law for sentences (1.50), the definition of the union of a pair, and finally again the definition of a subset. Since A, B and C are arbitrary, we therefore conclude that b) is also true. □

Proposition 2.85. *If a set B is not included in a set A, then A is a proper subset of the union of A and B, that is,*

$$\forall A, B\,(\neg B \subseteq A \Rightarrow A \subset A \cup B). \tag{2.255}$$

Proof. We let A and B be arbitrary sets and prove the stated implication directly, assuming $\neg B \subseteq A$ to be true. According to Note 2.12, this assumption implies that there exists an element in the set difference $B \setminus A$, say $\bar{y}$. It then follows with with the definition of a subset that $\bar{y} \in B$ and $\bar{y} \notin A$ are true. Here, $\bar{y} \in B$ further implies the truth of the disjunction $\bar{y} \in A \lor \bar{y} \in B$ (even though $\bar{y} \in A$ is false), so that $\bar{y} \in A \cup B$ holds by definition of the union of two sets. Thus, the conjunction of $\bar{y} \in A \cup B$ and $\bar{y} \notin A$ is true, and then the disjunction

$$(\bar{y} \in A \land \bar{y} \notin A \cup B) \lor (\bar{y} \in A \cup B \land \bar{y} \notin A)$$

also holds, which proves the existential sentence

$$\exists y\,([y \in A \land y \notin A \cup B] \lor [y \in A \cup B \land y \notin A]).$$

Therefore, we obtain with (2.23) the inequality $A \neq A \cup B$. Let us now observe that $A \subseteq A \cup B$ is also true in view of (2.40). Then, the conjunction of $A \subseteq A \cup B$ and $A \neq A \cup B$ yields $A \subset A \cup B$ with the definition of a proper subset, which proves the implication in (2.255). Since A and B are arbitrary, we therefore conclude that the proposition is true. □

Corollary 2.86. *If a set B is not included in a set A, then A is a proper subset of the union of A and B, that is,*

$$\forall A, B\,([B \neq \emptyset \,\wedge\, B \cap A = \emptyset] \Rightarrow A \subset A \cup B). \tag{2.256}$$

Proof. Letting A and B be arbitrary sets such that $B \neq \emptyset$ and $B \cap A = \emptyset$ hold, we obtain the negation $\neg B \subseteq A$ with (2.86), which then implies the desired $A \subset A \cup B$ with (2.255). Since A and B are arbitrary, we therefore conclude that the corollary holds, as claimed. □

Let us now inspect some relationships of the union with complements.

Proposition 2.87. *The union of a subset A of a set X and the complement of A with respect to X is identical with X, that is,*

$$\forall X, A\,(A \subseteq X \Rightarrow A \cup A^c = X). \tag{2.257}$$

Proof. We let X and A be arbitrary sets, and we prove the implication directly, assuming $A \subseteq X$, so that $A \cup X = X$ follows to be true with (2.247). Next, we verify the equation $A \cup A^c = X$ by proving

$$\forall y\,(y \in A \cup A^c \Leftrightarrow y \in X). \tag{2.258}$$

We let y be arbitrary and obtain the true equivalences

$$\begin{aligned} y \in A \cup A^c &\Leftrightarrow y \in A \,\vee\, y \in A^c \\ &\Leftrightarrow y \in A \,\vee\, (y \in X \setminus A) \\ &\Leftrightarrow y \in A \,\vee\, (y \in X \,\wedge\, y \notin A) \\ &\Leftrightarrow (y \in A \,\vee\, y \in X) \,\wedge\, (y \in A \,\vee\, y \notin A) \\ &\Leftrightarrow y \in A \,\vee\, y \in X \\ &\Leftrightarrow y \in A \cup X \\ &\Leftrightarrow y \in X, \end{aligned}$$

using the definition of the union of two sets, the notation for a complement, the definition of a set difference, the Distributive Law for sentences (1.45), then (1.13) in connection with the fact that the disjunction $y \in A \,\vee\, y \notin A$ is a tautology according to (1.10), again the definition of the union of two sets, and finally the previously established equation $A \cup X = X$. These equivalences evidently yield the equivalence in (2.258), and since y is arbitrary, we therefore conclude that the (2.258) holds. This universal sentence in turn implies the desired equation $A \cup A^c = X$ with the Equality Criterion for sets, so that the proof of the implication in (2.257) is complete. As A was an arbitrary set, we may finally conclude that the proposed sentence is true. □

Theorem 2.88 (De Morgan's Laws for the intersection & the union two sets). *The following sentences are true (where the complement is formed with respect to X).*

a) **De Morgan's Law for the intersection of two sets:**

$$\forall X, A, B\,(A, B \subseteq X \Rightarrow [A \cap B]^c = A^c \cup B^c). \tag{2.259}$$

b) **De Morgan's Law for the union of two sets:**

$$\forall X, A, B\,(A, B \subseteq X \Rightarrow (A \cup B)^c = A^c \cap B^c). \tag{2.260}$$

Proof. Concerning a), we let X be an arbitrary set and A and B arbitrary subsets of X. Next, we verify

$$\forall y\,(y \in [A \cap B]^c \Leftrightarrow y \in A^c \cup B^c), \tag{2.261}$$

letting y be arbitrary. Let us now observe the truth of the equivalences

$$\begin{aligned} y \in [A \cap B]^c &\Leftrightarrow y \in X \setminus [A \cap C] \\ &\Leftrightarrow y \in X \wedge \neg y \in A \cap C \\ &\Leftrightarrow y \in X \wedge \neg(y \in A \wedge y \in C) \\ &\Leftrightarrow y \in X \wedge (y \notin A \vee y \notin C) \\ &\Leftrightarrow (y \in X \wedge y \notin A) \vee (y \in X \wedge y \notin C) \\ &\Leftrightarrow y \in X \setminus A \vee y \in X \setminus C \\ &\Leftrightarrow y \in A^c \cup B^c \end{aligned}$$

in light of the Notation for complements, the definition of a set difference, the definition of the intersection of two sets, De Morgan's Law for sentences (1.51), the Distributive Law for sentences (1.44), the definition of a set difference, and the notation for complements. As y is arbitrary, we may therefore conclude that the universal sentence (2.261) holds, which then yields the equation in (2.259) with the Equality Criterion for sets. Since A, B and C were arbitrary, we may further conclude that a) holds. □

Exercise 2.41. Prove De Morgan's Law (2.260).

(Hint: Proceed similarly as in the proof of (2.259) and apply (1.33), (1.39) as well as (1.36) instead of the Distributive Law for sentences (1.44).)

The union of two disjoint sets has two useful direct relationships with the set difference.

Proposition 2.89. *If C is the union of two disjoint sets A and B, then A is obtained by 'subtracting' B from the union C, that is,*

$$\forall A, B, C\,([A \cup B = C \,\wedge\, A \cap B = \emptyset] \Rightarrow A = C \setminus B). \tag{2.262}$$

Proof. We let A, B and Cbe arbitrary such that $A \cup B = C$ and $A \cap B = \emptyset$ hold. Here, the latter equation implies $A \setminus B = A$ with (2.107). Next, we establish for an arbitrary y the equivalences

$$\begin{aligned} y \in C \setminus B \;&\Leftrightarrow\; (y \in C \,\wedge\, y \notin B) \\ &\Leftrightarrow\; (y \in A \cup B \,\wedge\, y \notin B) \\ &\Leftrightarrow\; ([y \in A \,\vee\, y \in B] \,\wedge\, y \notin B) \\ &\Leftrightarrow\; (y \notin B \,\wedge\, [y \in A \,\vee\, y \in B]) \\ &\Leftrightarrow\; ([y \notin B \,\wedge\, y \in A] \,\vee\, [y \notin B \,\wedge\, y \in B]) \\ &\Leftrightarrow\; ([y \in A \,\wedge\, y \notin B] \,\vee\, [y \in B \,\wedge\, y \notin B]) \\ &\Leftrightarrow\; ([y \in A \setminus B] \,\vee\, [y \in B \setminus B]) \\ &\Leftrightarrow\; (y \in A \setminus B \,\vee\, y \in \emptyset) \\ &\Leftrightarrow\; (y \in A \,\vee\, y \in \emptyset) \\ &\Leftrightarrow\; (y \in A \cup \emptyset) \\ &\Leftrightarrow\; y \in A \end{aligned}$$

by applying the definition of a set difference, substitution based on the assumed equation $A \cup B = C$, the definition of the union of two sets, the Commutative Law for sentences (1.36), the Distributive Law for sentences (1.44), again (1.36), again the definition of a set difference, then (2.104), substitution based on the previously established equation $A \setminus B = A$, again the definition of the union of two sets, and finally (2.216). As y is arbitrary, we see in light of the Equality Criterion for sets that $C \setminus B = A$ holds. Then, since A, B and C were also arbitrary, it follows that (2.262) holds as claimed. □

Lemma 2.90. *If A is a subset of B, then subtracting A from B and subsequently adjoining A to that difference yields again B, that is,*

$$\forall A, B\,(A \subseteq B \Rightarrow [B \setminus A] \cup A = B). \tag{2.263}$$

Proof. We let A and B be arbitrary sets and assume $A \subseteq B$, so that on the one hand

$$\forall y\,(y \in A \Rightarrow y \in B)$$

holds by definition of a subset, and on the other hand $A \cup B = A$ due to (2.247). We may now verify the truth of the equivalences (letting y be arbitrary)

$$\begin{aligned}
y \in (B \setminus A) \cup A &\Leftrightarrow y \in B \setminus A \lor y \in A \\
&\Leftrightarrow (y \in B \land y \notin A) \lor y \in A \\
&\Leftrightarrow y \in A \lor (y \in B \land y \notin A) \\
&\Leftrightarrow (y \in A \lor y \in B) \land (y \in A \lor y \notin A) \\
&\Leftrightarrow y \in A \cup B \land (y \in A \lor y \notin A) \\
&\Leftrightarrow y \in A \cup B \\
&\Leftrightarrow y \in B
\end{aligned}$$

using the definition of the union of two sets, the definition of a set difference, the Commutative Law for sentences (1.37), the Distributive Law for sentences (1.45), again the definition of the union of two sets, then (1.13) with the fact that the disjunction $y \in A \lor y \notin A$ is a tautology as in (1.10), and finally substitution based on the previously obtained equation $A \cup B = A$. Since y is arbitrary, it now follows with the Equality Criterion for sets that $(B \setminus A) \cup A = B$ is true. As A and B are also arbitrary, we may conclude that the proposed universal sentence holds. □

Proposition 2.91. *Show that the union of any sets A and B can be written as the union of two disjoint sets, that is,*

$$\forall A, B \, \exists C \, (A \cup B = A \cup C \land A \cap C = \emptyset). \tag{2.264}$$

Proof. We let A and B be arbitrary sets and observe for the set $C = B \setminus A$ that $A \cap C = \emptyset$ holds with Lemma 2.35. Furthermore, we obtain for an arbitrary y the true equivalences

$$\begin{aligned}
y \in A \cup C &\Leftrightarrow y \in A \cup (B \setminus A) \\
&\Leftrightarrow y \in A \lor (y \in B \land y \notin A) \\
&\Leftrightarrow (y \in A \lor y \in B) \land (y \in A \lor y \notin A) \\
&\Leftrightarrow y \in A \lor y \in B \\
&\Leftrightarrow y \in A \cup B
\end{aligned}$$

by applying substitution, the definition of the union of two sets together with the definition of a set difference, the Distributive Law for sentences (1.45), then (1.13) with the fact that the disjunction $y \in A \lor y \notin A$ is a tautology according to (1.10), and finally again the definition of the union of two sets. Since y is arbitrary, the equation in $A \cup B = A \cup C$ follows to be true with the Equality Criterion for sets. We thus proved the conjunction in (2.264) for $C = B \setminus A$, and thus the existential sentence. As A and B were arbitrary, we therefore conclude that the proposed sentence is true. $\square$

Note 2.19. The proof of the preceding proposition reveals the truth of the universal sentence

$$\forall A, B\,(A \cup B = A \cup [B \setminus A]). \tag{2.265}$$

2.6.3. Symmetric differences

The following set-theoretical operation, which is defined in terms of the union and the set difference, will be applied later on in the context of 'rings of sets'.

Definition 2.23 (Symmetric difference/Boolean sum). For any sets A and B we call the set

$$A \Delta B = (A \setminus B) \cup (B \setminus A). \tag{2.266}$$

the *symmetric difference* or the *Boolean sum* of A and B.

In case that sets A and B are included in a set X, (2.138) allows for the following equivalent expression of the symmetric difference.

Corollary 2.92. *For any sets A, B and X with $A \subseteq X$ and $B \subseteq X$, it is true that*

$$A \Delta B = (A \cap B^c) \cup (B \cap A^c). \tag{2.267}$$

Recalling that the union of two sets is commutative, we immediately obtain the

Theorem 2.93 (Commutative Law for the symmetric difference). *The symmetric difference of a set A and a set B is identical with the symmetric difference of B and A, that is,*

$$\forall A, B\,(A \Delta B = B \Delta A). \tag{2.268}$$

Proof. Letting A and B be arbitrary sets, we obtain with (2.266) and the Commutative Law for the union of two sets the true equations

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) = B \Delta A,$$

so that the proposed universal sentence (2.268) follows to be true since A and B were arbitrary. $\square$

The following basic laws for the symmetric difference are readily established.

Exercise 2.42. Show that

a) a set A is equal to its Boolean sum with $\emptyset$, that is,

$$\forall A\,(\emptyset \Delta A = A \,\wedge\, A \Delta \emptyset = A). \tag{2.269}$$

(Hint: Use (2.266), (2.105), (2.102), (2.216), and (2.268).)

b) the symmetric difference of a set A and itself is empty, that is,

$$\forall A\,(A \Delta A = \emptyset), \tag{2.270}$$

(Hint: Apply (2.104) and (2.218).)

c) the Boolean sum of a subset A of a set X and X is equal to the complement of A with respect to X, that is,

$$\forall X, A\,(A \subseteq X \Rightarrow A \Delta X = A^c). \tag{2.271}$$

(Hint: Apply first (2.118) and then (2.266) together with (2.216) as well as (2.131).)

d) the pairwise intersection is distributive over the symmetric difference in the sense that

$$\forall A, B, C\,(A \cap [B \Delta C] = [A \cap B]\,\Delta\,[A \cap C]). \tag{2.272}$$

(Hint: Use (2.266), (2.220) and (2.101).)

Proposition 2.94. *The symmetric difference of two sets A, B and the intersection of A, B are disjoint sets, that is,*

$$\forall A, B\,([A \Delta B] \cap [A \cap B] = \emptyset). \tag{2.273}$$

Proof. Letting A and B be arbitrary sets, we now prove the equation $[A \Delta B] \cap [A \cap B] = \emptyset$ by means of the Equality Criterion for sets. For this purpose, we verify the universal sentence

$$\forall y\,(y \in [A \Delta B] \cap [A \cap B] \Leftrightarrow y \in \emptyset). \tag{2.274}$$

We let y be arbitrary and obtain then the true equivalences

$$\begin{aligned}
&y \in (A\Delta B) \cap (A \cap B) \\
&\quad \Leftrightarrow\ y \in A\Delta B \wedge y \in A \cap B \\
&\quad \Leftrightarrow\ y \in (A \setminus B) \cup (B \setminus A) \wedge (y \in A \wedge y \in B) \\
&\quad \Leftrightarrow\ (y \in A \setminus B \vee y \in B \setminus A) \wedge (y \in A \wedge y \in B) \\
&\quad \Leftrightarrow\ (y \in A \wedge y \in B) \wedge (y \in A \setminus B \vee y \in B \setminus A) \\
&\quad \Leftrightarrow\ y \in A \wedge [y \in B \wedge (y \in A \setminus B \vee y \in B \setminus A)] \\
&\quad \Leftrightarrow\ y \in A \wedge [(y \in B \wedge y \in A \setminus B) \vee (y \in B \wedge y \in B \setminus A)] \\
&\quad \Leftrightarrow\ y \in A \wedge [(y \in B \cap [A \setminus B]) \vee (y \in B \wedge y \in B \wedge y \notin A)] \\
&\quad \Leftrightarrow\ y \in A \wedge [y \in \emptyset \vee (y \in B \wedge y \notin A)] \\
&\quad \Leftrightarrow\ y \in A \wedge [y \in \emptyset \vee y \in B \setminus A] \\
&\quad \Leftrightarrow\ y \in A \wedge y \in \emptyset \cup (B \setminus A) \\
&\quad \Leftrightarrow\ y \in A \wedge y \in B \setminus A \\
&\quad \Leftrightarrow\ y \in A \cap (B \setminus A) \\
&\quad \Leftrightarrow\ y \in \emptyset
\end{aligned}$$

by using the definition of the intersection of two sets, the definition of the symmetric difference together with the definition of the intersection of two sets, the definition of the union of two sets, the Commutative Law for the conjunction, the Associative Law for the conjunction, the Distributive Law for sentences (1.44), the definition of the intersection of two sets together with the definition of a set difference, (2.111) together with the Idempotent Law for the conjunction, again the definition of a set difference, the definition of the union of two sets, (2.216), the definition of the intersection of two sets, and finally again (2.111). Since y is arbitrary, we may therefore conclude that (2.274) holds, so that the Equality Criterion for sets yields the desired equation in (2.273). As A and B were also arbitrary, we finally conclude that the proposition is true. □

Proposition 2.95. *The Boolean sum of two sets A, B is the set difference of the union of A, B and the intersection of A, B, that is,*

$$\forall A, B\,(A\Delta B = [A \cup B] \setminus [A \cap B]). \tag{2.275}$$

Proof. We let A and B be arbitrary sets. We then obtain for an arbitrary

y the true equivalences

$$
\begin{aligned}
&y \in (A \cup B) \setminus (A \cap B) \\
&\quad \Leftrightarrow \; y \in A \cup B \,\wedge\, \neg y \in A \cap B \\
&\quad \Leftrightarrow \; (y \in A \,\vee\, y \in B) \,\wedge\, \neg(y \in A \,\wedge\, y \in B) \\
&\quad \Leftrightarrow \; (y \in A \,\vee\, y \in B) \,\wedge\, (y \notin A \,\vee\, y \notin B) \\
&\quad \Leftrightarrow \; [(y \in A \,\vee\, y \in B) \,\wedge\, y \notin A] \,\vee\, [(y \in A \,\vee\, y \in B) \,\wedge\, y \notin B] \\
&\quad \Leftrightarrow \; [y \notin A \,\wedge\, (y \in A \,\vee\, y \in B)] \,\vee\, [y \notin B \,\wedge\, (y \in A \,\vee\, y \in B)] \\
&\quad \Leftrightarrow \; [(y \notin A \,\wedge\, y \in A) \,\vee\, (y \notin A \,\wedge\, y \in B)] \\
&\quad \; \vee\, [(y \notin B \,\wedge\, y \in A) \,\vee\, (y \notin B \,\wedge\, y \in B)] \\
&\quad \Leftrightarrow \; [(y \in A \,\wedge\, y \notin A) \,\vee\, (y \in B \,\wedge\, y \notin A)] \\
&\quad \; \vee\, [(y \in A \,\wedge\, y \notin B) \,\vee\, (y \in B \,\wedge\, y \notin B)] \\
&\quad \Leftrightarrow \; [y \in A \setminus A \,\vee\, y \in B \setminus A] \,\vee\, [y \in A \setminus B \,\vee\, y \in B \setminus B] \\
&\quad \Leftrightarrow \; [y \in \emptyset \,\vee\, y \in B \setminus A] \,\vee\, [y \in A \setminus B \,\vee\, y \in \emptyset] \\
&\quad \Leftrightarrow \; y \in \emptyset \cup (B \setminus A) \,\vee\, y \in (A \setminus B) \cup \emptyset \\
&\quad \Leftrightarrow \; y \in B \setminus A \,\vee\, y \in A \setminus B \\
&\quad \Leftrightarrow \; y \in A \setminus B \,\vee\, y \in B \setminus A \\
&\quad \Leftrightarrow \; y \in (A \setminus B) \cup (B \setminus A) \\
&\quad \Leftrightarrow \; y \in A \Delta B.
\end{aligned}
$$

applying the definition of a set difference, the definition of the union of two sets together with the definition of the intersection of two sets, De Morgan's Law for sentences (1.51), the Distributive Law for sentences (1.44), the Commutative Law for the conjunction, the Distributive Law (1.44), the Commutative Law for the conjunction, the definition of a set difference, then (2.104), the definition of the union of two sets, (2.216), the Commutative Law for the conjunction, the definition of the union of two sets, and finally the definition of a symmetric difference. As y is arbitrary, we therefore obtain with the Equality Criterion for sets the equation in (2.275). Since A and B were arbitrary, we then further conclude that the proposed sentence is true. □

Corollary 2.96. *The union of two sets A, B is the union of the symmetric difference of A, B and the intersection of A, B, that is,*

$$\forall A, B\,(A \cup B = [A \Delta B] \,\cup\, [A \cap B]). \tag{2.276}$$

Proof. Letting A and B be arbitrary, we first observe the truth of $A \cap B \subseteq$

$A \cup B$ in light of (2.246), which inclusion then implies

$$\begin{aligned} A \cup B &= ([A \cup B] \setminus [A \cap B]) \,\cup\, [A \cap B] \\ &= [A \Delta B] \,\cup\, [A \cap B]. \end{aligned}$$

with (2.263) and (2.275). Since A and B were arbitrary, we therefore conclude that the proposed universal sentence is true. □

Proposition 2.97. *The union of two sets A, B is identical with the symmetric difference of the Boolean sum of A, B and the intersection of A, B, that is,*

$$\forall A, B\, (A \cup B = [A \Delta B]\, \Delta\, [A \cap B]). \tag{2.277}$$

Proof. Let us take arbitrary sets A and B and then notice that $A \Delta B$ and $A \cap B$ are disjoint according to (2.273), so that

$$[A \Delta B] \setminus [A \cap B] = A \Delta B \tag{2.278}$$

$$[A \cap B] \setminus [A \Delta B] = A \cap B \tag{2.279}$$

follow to be true with (2.107). We then obtain the true equations

$$\begin{aligned} [A \Delta B]\, \Delta\, [A \cap B] &= ([A \Delta B] \setminus [A \cap B]) \,\cup\, ([A \cap B] \setminus [A \Delta B]) \\ &= [A \Delta B] \,\cup\, [A \cap B] \\ &= A \cup B \end{aligned}$$

by applying the definition of a symmetric difference, then substitutions based on (2.278) – (2.279), and finally (2.276). Because A and B were arbitrary sets, the corollary therefore follows to be true. □

Corollary 2.98. *The intersection of two sets A, B is the set difference of the union of A, B and the symmetrical difference of A, B, that is,*

$$\forall A, B\, (A \cap B = [A \cup B] \setminus [A \Delta B]). \tag{2.280}$$

Proof. Letting A and B be arbitrary, we see that the equation

$$[A \cap B] \,\cup\, [A \Delta B] = A \cup B$$

holds in view of (2.276) and the Commutative Law for the union of two sets. Observing now that $A \cap B$ and $A \Delta B$ are disjoint according to Proposition 2.273 (using the Commutative Law for the intersection of two sets), we may apply (2.262) to obtain the equation in (2.280). We may then conclude that this equation holds for any A and any B. □

Proposition 2.99. *The set difference of two sets A, B is the symmetric difference of A and the intersection of A, B, that is,*

$$\forall A, B\,(A \setminus B = A\,\Delta\,[A \cap B]). \tag{2.281}$$

Proof. We let A, B and y be arbitrary and observe the truth of the equivalences

$$\begin{aligned}
& y \in A\Delta[A \cap B] \\
& \quad \Leftrightarrow\ y \in [A \setminus (A \cap B)] \cup [(A \cap B) \setminus A] \\
& \quad \Leftrightarrow\ y \in A \setminus (A \cap B)\ \vee\ y \in (A \cap B) \setminus A \\
& \quad \Leftrightarrow\ (y \in A\ \wedge\ \neg y \in A \cap B)\ \vee\ (y \in A \cap B\ \wedge\ y \notin A) \\
& \quad \Leftrightarrow\ (y \in A\ \wedge\ \neg[y \in A\ \wedge y \in B)\ \vee\ ([y \in A\ \wedge\ y \in B]\ \wedge\ y \notin A) \\
& \quad \Leftrightarrow\ (y \in A\ \wedge\ [y \notin A\ \vee\ y \notin B])\ \vee\ ([y \in B\ \wedge\ y \in A]\ \wedge\ y \notin A) \\
& \quad \Leftrightarrow\ ([y \in A\ \wedge\ y \notin A]\ \vee\ [y \in A\ \wedge\ y \notin B])\ \vee\ (y \in B\ \wedge\ [y \in A\ \wedge\ y \notin A]) \\
& \quad \Leftrightarrow\ y \in A\ \wedge\ y \notin B \\
& \quad \Leftrightarrow\ y \in A \setminus B,
\end{aligned}$$

in light of the definition of the symmetric difference, the definition of the union of two sets, the definition of a set difference, the definition of the intersection of two sets, De Morgan's Law for sentences (1.51) together with the Commutative Law for the conjunction, the Distributive Law for sentences (1.44) together with the Associative Law for the conjunction, twice (1.15) with the fact that $y \in A \wedge y \notin A$ and then evidently also $y \in B \wedge [y \in A \wedge y \notin A]$ are contradictions, and finally again the definition of a set difference. Then, the equation $A\Delta(A \cap B) = A \setminus B$ follows to be true with the Equality Criterion for sets, because y is arbitrary. Since A and B were also arbitrary, we may therefore conclude that (2.281) is true. □

In the remainder of this section on the symmetric difference, we introduce some useful laws that involve complements. We begin with the proof that any symmetric difference may be written as the intersection of two unions.

Proposition 2.100. *For any set X and any subsets $A \subseteq X$, $B \subseteq X$ it is true that*

$$A\Delta B = (A \cup B)\ \cap\ (A^c \cup B^c). \tag{2.282}$$

Proof. We let A, B and X be arbitrary and assume $A \subseteq X$ as well as $B \subseteq X$ to be true. Let us first observe that these inclusions imply $A^c \subseteq X$ and $B^c \subseteq X$, respectively, with (2.137); furthermore, the inclusions $A \subseteq X$ and $B \subseteq X$ imply $A \cup B \subseteq X$ with (2.252), and for the same reason the previously established inclusions $A^c \subseteq X$ and $B^c \subseteq X$ imply $A^c \cup B^c \subseteq X$. It then follows from the inclusions $A \cup B \subseteq X$ and $A^c \cup B^c \subseteq X$ that

$$(A \cup B) \cap X = A \cup B, \tag{2.283}$$
$$(A^c \cup B^c) \cap X = A^c \cup B^c \tag{2.284}$$

hold, according to (2.77). We then obtain the equations

$$\begin{aligned}
A\Delta B &= (A \cap B^c) \,\cup\, (B \cap A^c) \\
&= [(A \cap B^c) \cup B] \,\cap\, [(A \cap B^c) \cup A^c] \\
&= [B \cup (A \cap B^c)] \,\cap\, [A^c \cup (A \cap B^c)] \\
&= [(B \cup A) \cap (B \cup B^c)] \,\cap\, [(A^c \cup A) \cap (A^c \cup B^c)] \\
&= [(A \cup B) \cap (B \cup B^c)] \,\cap\, [(A \cup A^c) \cap (A^c \cup B^c)] \\
&= [(A \cup B) \cap X] \,\cap\, [X \cap (A^c \cup B^c)] \\
&= [(A \cup B) \cap X] \,\cap\, [(A^c \cup B^c) \cap X] \\
&= (A \cup B) \,\cap\, (A^c \cup B^c)
\end{aligned}$$

by using (2.267), the Distributive Law for the union of two sets, the Commutative Law for the union of two sets, again the Distributive Law for the union of two sets, again the Commutative Law for the union of two sets, (2.257), the Commutative Law for the intersection of two sets, and finally substitutions based on the previously established equations (2.283) – (2.284). Since A, B and X are arbitrary, we may therefore conclude that proposition is true. □

Exercise 2.43. Show for any set X and any subsets $A \subseteq X$, $B \subseteq X$ that

$$(A\Delta B)^c = (A^c \cup B) \,\cap\, (A \cup B^c). \tag{2.285}$$

(Hint: Apply Corollary 2.92, Theorem 2.88 and (2.136).)

The next result prepares us for the task of establishing the associativity of the symmetric difference.

Exercise 2.44. Show for any set X and any subsets $A \subseteq X$, $B \subseteq X$ that

$$(A\Delta B)\Delta C = (A \cup B \cup C) \cap (A^c \cup B^c \cup C) \cap (A^c \cup B \cup C^c) \cap (A \cup B^c \cup C^c). \tag{2.286}$$

(Hint: Apply (2.282), (2.285), (2.214), (2.220), (2.219), and (2.72).)

Theorem 2.101 (Associative Law for the symmetric difference). *For any set X, the Boolean sum is associative in the sense that*

$$\forall A, B, C\, ([A \subseteq X \,\wedge\, B \subseteq X \,\wedge\, C \subseteq X] \Rightarrow A\Delta[B\Delta C] = [A\Delta B]\Delta C). \tag{2.287}$$

Proof. We let A, B, C and X be arbitrary sets and assume $A \subseteq X$, $B \subseteq X$ as well as $C \subseteq X$ to be true. We then obtain

$$\begin{aligned}
A\Delta(B\Delta C) &= (B\Delta C)\Delta A \\
&= (B \cup C \cup A) \cap (B^c \cup C^c \cup A) \cap (B^c \cup C \cup A^c) \cap (B \cup C^c \cup A^c) \\
&= (A \cup B \cup C) \cap (A \cup B^c \cup C^c) \cap (A^c \cup B^c \cup C) \cap (A^c \cup B \cup C^c) \\
&= (A \cup B \cup C) \cap (A^c \cup B^c \cup C) \cap (A^c \cup B \cup C^c) \cap (A \cup B^c \cup C^c) \\
&= (A\Delta B)\Delta C
\end{aligned}$$

with the Commutative Law for the symmetric difference, (2.286), the Commutative Law for union of two sets and the Commutative Law for the intersection of two sets in connection with the Associative Law for the intersection of two sets (which allows to omit brackets), and again with (2.286). As A, B, C and X were arbitrary sets, we may infer from the truth of these equations the truth of the stated theorem. $\square$

2.7. The Set of Natural Numbers $\mathbb{N}$

In this section, we finalize the construction of the natural numbers.

Axiom 2.5 (Axiom of infinity). There exists a set (system) I which contains 0 and for any of its elements n also the union of n and the singleton formed by n, that is,

$$\exists I\,(0 \in I \,\wedge\, \forall n\,(n \in I \Rightarrow n \cup \{n\} \in I)). \tag{2.288}$$

The existence of such a set gives rise to the following definition and terminology.

Definition 2.24 (Inductive set/successor set, initial element, successor, predecessor). We say that a set I is an *inductive set* or a *successor set* iff

$$0 \in I \,\wedge\, \forall n\,(n \in I \Rightarrow n \cup \{n\} \in I). \tag{2.289}$$

Here, we call 0 the *initial element* of I, and for any $n \in I$ we call the union

$$n^+ = n \cup \{n\} \tag{2.290}$$

the *successor* of n. We then also say that n is a *predecessor* of n^+.

Note 2.20. We speak of 'the' successor of x because the union $n \cup \{n\}$ is a uniquely specified set.

Proposition 2.102. *For any inductive set I it is true that*

$$0^+ = 1 \quad (\in I), \tag{2.291}$$
$$1^+ = 2 \quad (\in I). \tag{2.292}$$

Proof. Letting I be an arbitrary inductive set, we obtain on the one hand the true equations

$$0^+ = 0 \cup \{0\} = \emptyset \cup \{0\} = \{0\} = 1,$$

using (2.290), (2.49), (2.216), and (2.154). Since $0 \in I$ holds according to the first part of the conjunction in (2.289), it follows with the second part that $(1 =)\ 0 \cup \{0\} \in I$, i.e. that $1 \in I$. On the other hand, we obtain

$$1^+ = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\} = 2,$$

applying (2.290), (2.154), (2.226), and (2.156). Now, the previously established $1 \in I$ implies with the second part of the conjunction in (2.289) that $(2 =)\ 1 \cup \{1\} \in I$, i.e. that $2 \in I$. Because I was arbitrary, we therefore conclude that the proposition is true. $\square$

Combining these findings with the definition of 3 as $2 \cup \{2\}$, we immediately obtain the following result.

Corollary 2.103. *For any inductive set I it is true that*

$$2^+ = 3 \quad (\in I). \tag{2.293}$$

Theorem 2.104. *The following sentences are true.*

a) For any inductive set J there exists a unique set $\mathbb{N}$ such that an element n is in $\mathbb{N}$ iff n is element of J and moreover element of all inductive sets, i.e.

$$\begin{aligned} &\exists! \mathbb{N}\, \forall n\, (n \in \mathbb{N} \\ &\quad \Leftrightarrow [n \in J \wedge \forall I\, ([0 \in I \wedge \forall m\, (m \in I \Rightarrow m^+ \in I)] \Rightarrow n \in I)]). \end{aligned} \tag{2.294}$$

b) The set $\mathbb{N}$ is not empty, that is,

$$\mathbb{N} \neq \emptyset. \tag{2.295}$$

c) The set $\mathbb{N}$ satisfies also

$$\forall n\, (n \in \mathbb{N} \Leftrightarrow \forall I\, ([0 \in I \wedge \forall m\, (m \in I \Rightarrow m^+ \in I)] \Rightarrow n \in I)). \tag{2.296}$$

d) Furthermore, the set $\mathbb{N}$ is inductive, that is,

$$0 \in \mathbb{N} \wedge \forall n\, (n \in \mathbb{N} \Rightarrow n^+ \in \mathbb{N}). \tag{2.297}$$

Proof. Concerning a), we let J be an arbitrary inductive set, using the fact that such a set exists due to the Axiom of Infinity, and we let $\varphi(I)$ be the definite property of I that I is an inductive set, i.e.

$$0 \in I \wedge \forall m\, (m \in I \Rightarrow m^+ \in I)$$

according to (2.289). Due to the Axiom of Specification

$$\exists \mathbb{N}\, \forall n\, (n \in \mathbb{N} \Leftrightarrow [n \in J \wedge \forall I\, (\varphi(I) \Rightarrow n \in I)]) \tag{2.298}$$

is then true, so that we established thusly the existential part of the uniquely existential sentence to be proven. The uniqueness part follows with the Equality Criterion for sets (in analogy to the proof for the intersection of two sets).

Concerning b), we notice first that the unique set $\mathbb{N}$ satisfies, in view of its specification (2.294),

$$\forall n\,(n \in \mathbb{N} \Leftrightarrow [n \in J \,\wedge\, \forall I\,([0 \in I \,\wedge\, \forall m\,(m \in I \Rightarrow m^+ \in I)] \Rightarrow n \in I)]). \tag{2.299}$$

Next, we observe that $0 \in J$ holds by definition of an inductive set. Moreover, letting I be an arbitrary set and assuming $\varphi(I)$ to be true (i.e., assuming I to be inductive), we also see that $0 \in I$ is true; since I is arbitrary, we may therefore conclude that the universal sentence $\forall I\,(\varphi(I) \Rightarrow 0 \in I)$ holds. The conjunction of the previously established $0 \in J$ and the preceding universal sentence now implies $0 \in \mathbb{N}$ due to (2.299). Thus, there exists an element in $\mathbb{N}$, so that the set $\mathbb{N}$ is clearly nonempty.

Concerning c), we take an arbitrary n and prove the first part ('$\Rightarrow$') of the equivalence in (2.296) directly, assuming $n \in \mathbb{N}$ to be true. This assumption implies with (2.299) in particular

$$\forall I\,([0 \in I \,\wedge\, \forall m\,(m \in I \Rightarrow m^+ \in I)] \Rightarrow n \in I), \tag{2.300}$$

proving already the first part of the equivalence. To prove the second part ('$\Leftarrow$'), we now assume (2.300) to be true. Since J was assumed to be an inductive set, it satisfies

$$0 \in J \,\wedge\, \forall m\,(m \in J \Rightarrow m^+ \in J),$$

so that $n \in J$ is implied. Together with (2.300), this further implies $n \in \mathbb{N}$, completing the proof of the equivalence in (2.296). As n was arbitrary, we therefore conclude that c) is also true.

Concerning d), we recall first from the proof of b) that

$$\emptyset \in \mathbb{N}. \tag{2.301}$$

Secondly, to verify

$$\forall n\,(n \in \mathbb{N} \Rightarrow n^+ \in \mathbb{N}), \tag{2.302}$$

we let n be an arbitrary element of $\mathbb{N}$; according to (2.296), $n \in \mathbb{N}$ then implies (2.300) with (2.296). We may prove the universal sentence

$$\forall I\,([0 \in I \,\wedge\, \forall m\,(m \in I \Rightarrow m^+ \in I)] \Rightarrow n^+ \in I). \tag{2.303}$$

Indeed, letting I be an arbitrary inductive set, it follows with (2.300) that $n \in I$ holds, which in turn implies $n^+ \in I$ by definition of an inductive set. As I was arbitrary, we may therefore conclude that (2.303) is true. Consequently, $n^+ \in \mathbb{N}$ follows also to be true in view of (2.296). Since

n was arbitrary, it follows then that the universal sentence (2.302) holds, which shows – together with (2.301) – that $\mathbb{N}$ is inductive.

In the proofs of a) – d) the set J was arbitrary, so that the theorem is in fact true for any such set J. □

We thus completed von Neumann's construction of the set of natural numbers.

Definition 2.25 ((Von Neumann's) set of natural numbers, natural number). We call the set $\mathbb{N}$, which we also symbolize by

$$\{0, 1, 2, \ldots\}, \tag{2.304}$$

the *set of natural numbers*. Thus, we call the elements $n = 0, 1, 2, \ldots$ of $\mathbb{N}$ *natural numbers*.

Note 2.21. We may informally view the set $\mathbb{N}$, due to its specification (2.294), as the intersection of all inductive sets. As this intersection $\mathbb{N}$ would then be a subset of every inductive set due to (2.92), $\mathbb{N}$ could be viewed as the 'smallest' inductive set.

Proposition 2.105. *The following sentences are true.*

a) Any natural number is an element of its successor, that is,

$$\forall n\,(n \in \mathbb{N} \Rightarrow n \in n^+). \tag{2.305}$$

b) Any natural number is a subset of its successor, that is,

$$\forall n\,(n \in \mathbb{N} \Rightarrow n \subseteq n^+). \tag{2.306}$$

Proof. Concerning a), we let n be arbitrary and assume $n \in \mathbb{N}$. Let us first recall that $n^+ = n \cup \{n\}$ holds by definition of a successor. Then, the basic fact $n = n$ implies $n \in \{n\}$ with (2.169), which further implies the disjunction $n \in n \vee n \in \{n\}$ and therefore $n \in n \cup \{n\}$ by definition of the union of a pair. This means $n \in n^+$, and since n is arbitrary, we therefore conclude that a) is true.

Concerning b), we again let n be arbitrary and assume $n \in \mathbb{N}$. By definition of the union of a pair, we have $n \cup \{n\} = \bigcup\{n, \{n\}\}$. Clearly, $n \in \{n, \{n\}\}$ holds by definition of a pair. It then follows with Proposition 2.67 that $n \subseteq \bigcup\{n, \{n\}\}$ is true, which means $n \subseteq n \cup \{n\}$ and thus $n \subseteq n^+$. As n was arbitrary, the therefore conclude that b) also holds, as claimed. □

It is occasionally useful to exclude the number 0 from the set $\mathbb{N}$.

Definition 2.26 (Set of positive natural numbers, positive natural number). We call

$$\mathbb{N}_+ = \{1, 2, \ldots\} = \mathbb{N} \setminus \{0\} \tag{2.307}$$

the *set of positive natural numbers* and the elements of $\mathbb{N}_+$ *positive natural numbers*.

Exercise 2.45. Show that

a) the set of positive natural numbers is a subset of the set of natural numbers, i.e.

$$\mathbb{N}_+ \subseteq \mathbb{N}. \tag{2.308}$$

(Hint: Apply (2.125) to the set difference in (2.307).)

b) the set of natural numbers is the union of the singleton formed by 0 and the set of positive natural numbers, that is,

$$\mathbb{N} = \{0\} \cup \mathbb{N}_+. \tag{2.309}$$

(Hint: Use (2.301) together (2.49) and (2.184), then apply (2.263) and (2.214).)

c) a natural number is 0 or a positive natural number, that is,

$$\forall n \, (n \in \mathbb{N} \Leftrightarrow [n = 0 \, \vee \, n \in \mathbb{N}_+]). \tag{2.310}$$

(Hint: Use (2.18) to (2.309), then Definition 2.19, and finally (2.169).)

We conclude this section with a widely used proof technique based on the inductive nature of the natural numbers.

Theorem 2.106 (Principle of Mathematical Induction). *Any inductive subset M of $\mathbb{N}$ is identical with $\mathbb{N}$, that is,*

$$\forall M \, ([M \subseteq \mathbb{N} \, \wedge \, M \text{ is an inductive set}] \Rightarrow M = \mathbb{N}). \tag{2.311}$$

Proof. To prove the universal sentence we let M be an arbitrary set. Then, to proof the implication directly, we assume that M is inductive and a subset of $\mathbb{N}$. To prove $M = \mathbb{N}$, we show that $M \subseteq \mathbb{N} \wedge \mathbb{N} \subseteq M$; here, we only need to show that $\mathbb{N} \subseteq M$ since the first part of the conjunction is true by assumption. In light of (2.8), we must then show that $\forall y \, (y \in \mathbb{N} \Rightarrow y \in M)$. To prove this universal sentence, we let y be arbitrary. Then, to prove the involved implication directly, we assume $y \in \mathbb{N}$; due to the definition of $\mathbb{N}$, y is an element of every inductive set, thus in particular of M, so that $y \in M$ is true. This proves $\forall y \, (y \in \mathbb{N} \Rightarrow y \in M)$ since y was arbitrary, which means that $\mathbb{N} \subseteq M$ is also true. Consequently, $M = \mathbb{N}$ holds, which completes the proof of the theorem since M is arbitrary. □

We can now summarize the rationale of the proof method we intend to describe.

Proposition 2.107. *Let $\varphi(n)$ be an arbitrary formula and n a free variable with values in the set of natural numbers $\mathbb{N}$. Then, the following sentences are true.*

a) There exists a unique set

$$X = \{n : n \in \mathbb{N} \wedge \varphi(n)\}$$

which contains precisely all the natural numbers n for which $\varphi(n)$ is true, and this set X is a subset of $\mathbb{N}$.

b) If this set X is inductive, then $\varphi(n)$ is true for all natural numbers, that is,

$$X \text{ is an inductive set } \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow \varphi(n)). \tag{2.312}$$

c) If

$$\varphi(0) \tag{2.313}$$

and

$$\forall n\,(n \in \mathbb{N} \Rightarrow [\varphi(n) \Rightarrow \varphi(n^+)]) \tag{2.314}$$

are true, then the set X specified in a) is an inductive set.

Proof. Concerning a), we may apply the Axiom of Specification in connection with the Equality Criterion for sets to obtain the true uniquely existential sentence

$$\exists! X\,\forall n\,(n \in X \Leftrightarrow [n \in \mathbb{N} \wedge \varphi(n)]). \tag{2.315}$$

Then, the set X satisfies

$$\forall n\,(n \in X \Leftrightarrow [n \in \mathbb{N} \wedge \varphi(n)]). \tag{2.316}$$

To prove $X \subseteq \mathbb{N}$, we verify (applying the definition of a subset) the equivalent

$$\forall n\,(n \in X \Rightarrow n \in \mathbb{N}). \tag{2.317}$$

Fot this purpose, we let n be arbitrary and assume $n \in X$, which then implies $n \in \mathbb{N} \wedge \varphi(n)$ with (2.316). By definition of a conjunction, we have in particular that $n \in \mathbb{N}$ is true. This proves the implication in (2.317), and since n is arbitrary, we therefore conclude that (2.317) holds. We thus proved $X \subseteq \mathbb{N}$.

Concerning b), we prove the implication directly, assuming that X is an inductive set. To prove the consequent, which is a universal sentence, we let n be arbitrary and assume $n \in \mathbb{N}$. Next, we notice that the conjunction of the previously established fact $X \subseteq \mathbb{N}$ and the assumption that X is inductive implies $X = \mathbb{N}$ with the Principle of Mathematical Induction (2.311). Thus, $n \in \mathbb{N}$ implies $n \in X$, which further implies the conjunction $n \in \mathbb{N} \wedge \varphi(n)$ due to (2.316), and therefore in particular $\varphi(n)$. As n is arbitrary, we therefore conclude that the universal sentence to proven holds, which proves the stated implication.

Concerning c), we assume that (2.313) and (2.314) hold. To show that this implies that X is inductive, we verify that X satisfies (2.289), i.e.

$$0 \in X \wedge \forall n\, (n \in X \Rightarrow n^+ \in X). \tag{2.318}$$

To do this, we carry out a proof of a conjunction. Regarding the first part of the conjunction, we observe that the conjunction of the fact $0 \in \mathbb{N}$ and the assumption $\varphi(0)$ implies $\emptyset \in X$ with (2.316). Regarding the second part of the conjunction, which is a universal sentence, we let n be arbitrary, then assume $n \in X$, and show that this implies $n^+ \in X$. Here, $n \in X$ implies with (2.316) that $n \in \mathbb{N}$ and $\varphi(n)$ are both true. Because of the assumption (2.314), $n \in \mathbb{N}$ implies $\varphi(n) \Rightarrow \varphi(n^+)$. Since $\varphi(n)$ is true, it follows that $\varphi(n^+)$ is also true. Since $\mathbb{N}$ is inductive, $n \in \mathbb{N}$ furthermore implies $n^+ \in \mathbb{N}$, so that the conjunction $n^+ \in \mathbb{N} \wedge \varphi(n^+)$ is true. This conjunction then implies $n^+ \in X$ with (2.316). As n was arbitrary, we therefore conclude that the second part of the conjunction (2.318) is true. Thus, X is an inductive set, which proves the implication c). □

Corollary 2.108. *The following implication holds for an arbitrary formula $\varphi(n)$. If (2.313) and (2.314) are true, then $\varphi(n)$ is true for all $n \in \mathbb{N}$.*

Proof. Letting $\varphi(n)$ be an arbitrary formula, the truth of the conjunction of the implications c) and b) in Proposition 2.107 implies the stated implication with the Law of the Hypothetical Syllogism. □

Thus, a proof along these lines can be considered to be a proof of a universal sentence where we seek to show that a given formula is true for all natural numbers.

Method 2.2 (Proof by mathematical induction (for $\mathbb{N}$)). To prove that a given formula $\varphi(n)$ holds for all natural numbers, we show that the so-called *base case* (2.313) and *induction step* (2.314) are true, where $\varphi(n)$ is called the *induction assumption*.

We will apply this method when we could otherwise not prove the given formula via a standard proof of a universal sentence based by simply 'letting n be arbitrary'.

Proposition 2.109. *0 is an element of the successor of any natural number, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow 0 \in n^+). \tag{2.319}$$

Proof. We apply a proof by mathematical induction, where $\varphi(n)$ is given by $0 \in n^+$. To prove the base case $\varphi(0)$, i.e. $0 \in 0^+$, we observe that $0 \in 0^+$ holds in view of (2.305). To prove the inducion step, i.e.

$$\forall n\,(n \in \mathbb{N} \Rightarrow [0 \in n^+ \Rightarrow 0 \in (n^+)^+]),$$

we let n be arbitrary in $\mathbb{N}$, make the induction assumption $0 \in n^+$, and show that this implies $0 \in (n^+)^+$. Now, we obtain the equations

$$(n^+)^+ = n^+ \cup \{n^+\} = \bigcup\{n^+, \{n^+\}\}$$

using the definition of the successor and the definition of the union of a pair. Since $0 \in n^+$ holds (by the induction assumption) and evidently also $n^+ \in \{n^+, \{n^+\}\}$ (by definition of a pair), it follows that $0 \in \bigcup\{n^+, \{n^+\}\}$ is true by definition of the union of a set system, which yields the desired $0 \in (n^+)^+$ with the previously established equations. Since n is arbitrary, we therefore conclude that the induction step holds, so that the proof by mathematical induction is complete. □

Proposition 2.110. *All natural numbers are subsets of* $\mathbb{N}$*, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow n \subseteq \mathbb{N}). \tag{2.320}$$

Proof. Concerning the base case $(n = 0)$, we observe that $\emptyset$ is a subset of $\mathbb{N}$ due to (2.43) and that $0 = \emptyset$ holds by definition of 0; thus, $0 \subseteq \mathbb{N}$. Concerning the induction step, we let n be arbitrary in $\mathbb{N}$, make the induction assumption $n \subseteq \mathbb{N}$, and prove $n^+ \subseteq \mathbb{N}$, or equivalently $n \cup \{n\} \subseteq \mathbb{N}$ (using the definition (2.290) of a successor). Let us now observe that the fact $n \in \mathbb{N}$ implies $\{n\} \subseteq \mathbb{N}$ with (2.184). Then, the conjunction of the induction assumption $n \subseteq \mathbb{N}$ and $\{n\} \subseteq \mathbb{N}$ implies $n \cup \{n\} \subseteq \mathbb{N}$ with (2.252). Thus, $n^+ \subseteq \mathbb{N}$ holds, and since n was arbitrary, the induction step follows to be true. This completes the proof by mathematical induction. □

Chapter 3.

Ordered Sets

3.1. Ordered Pairs and Cartesian Products of Two Sets

The concepts of this section will prepare us for the subsequent tasks of establishing certain 'relations' between given objects and algebraic entities.

3.1.1. Ordered pairs, ordered triples and ordered quadruples

We begin with the observation that a given pair $\{A, B\}$ is 'unordered' in the sense that we may list its elements A and B in any 'order', because $\{A, B\}$ and $\{B, A\}$ are identical sets according to Exercise 2.16. To impose an order structure on two given sets A and B, we may – instead of forming directly the pair of A and B – form a certain pair of pairs.

Definition 3.1 (Ordered pair, coordinate). For any A and B we call

$$(A, B) = \{\{A\}, \{A, B\}\} \tag{3.1}$$

the *ordered pair* formed by A and B. Here, we call A the *first* and B the *second coordinate* of (A, B). We also say that a set Z is an *ordered pair* iff there exist sets A and B such that Z constitutes the ordered pair formed by A and B, that is, iff

$$\exists A, B\,(\,Z = (A, B)\,). \tag{3.2}$$

Note 3.1. Consisting of a singleton and a pair, an ordered pair is thus a set system. The order of given sets A and B within the ordered pair (A, B) is retrievable via the characteristic that the first coordinate A is element of both the singleton and the pair within (A, B), and that the second coordinate B arises only in one of these sets.

This ordering property is also reflected by the following condition that two ordered pairs must satisfy in order to be identical, which is qualitatively different from the equality condition with respect to unordered pairs.

Theorem 3.1 (Equality Criterion for ordered pairs). *Two ordered pairs (A,B) and (A',B') are identical iff the first coordinates and the second coordinates are identical, that is,*

$$\forall A,B,A',B'\,((A,B)=(A',B')\Leftrightarrow[A=A'\,\wedge\,B=B']). \tag{3.3}$$

Proof. We let A, B, A' and B' be arbitrary sets and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $(A,B)=(A',B')$ holds. Since the ordered pairs (A,B) and (A',B') are set systems, the intersections and the unions of these ordered pairs are uniquely specified, and we obtain on the one hand

$$\begin{aligned}
\bigcap(A,B) &= \bigcap\{\{A\},\{A,B\}\}\\
&= \{A\}\cap\{A,B\}\\
&= \{A\},\\
\bigcap(A',B') &= \bigcap\{\{A'\},\{A',B'\}\}\\
&= \{A'\}\cap\{A',B'\}\\
&= \{A'\}
\end{aligned}$$

using (3.1), (2.209), and (2.175). On the other hand, we obtain

$$\begin{aligned}
\bigcup(A,B) &= \bigcup\{\{A\},\{A,B\}\}\\
&= \{A\}\cup\{A,B\}\\
&= \{A,B\},\\
\bigcup(A',B') &= \bigcup\{\{A'\},\{A',B'\}\}\\
&= \{A'\}\cup\{A',B'\}\\
&= \{A',B'\}
\end{aligned}$$

by applying (3.1), (2.158), and (2.240). Furthermore, substitution yields then the equations

$$\begin{aligned}
(\{A\}=)\quad \bigcap(A,B) &= \bigcap(A',B') \quad (=\{A'\})\\
(\{A,B\}=)\quad \bigcup(A,B) &= \bigcup(A',B') \quad (=\{A',B'\})
\end{aligned}$$

so that the equations

$$\{A\} = \{A'\}$$
$$\{A, B\} = \{A', B'\}$$

hold. Because of the Equality Criterion for singletons and the Equality Criterion for pairs, the conjunction

$$A = A' \wedge [(A = A' \wedge B = B') \vee (A = B' \wedge B = A')]$$

follows to be true. Therefore, the equation $A = A'$ and the disjunction

$$(A = A' \wedge B = B') \vee (A = B' \wedge B = A')$$

are both true. On the one hand, if the first part $A = A' \wedge B = B'$ of this disjunction holds, then we may infer from this conjunction that $B = B'$ holds besides $A = A'$. On the other hand, if the second part $A = B' \wedge B = A'$ of the preceding disjunction holds, we may infer from the second part of this conjunction and the previously established equation $A = A'$ that $B = A' = A$ is true, so that first part of the preceding conjunction can be written as $(A =)\ B = B'$. Consequently, we have in any case that $A = A'$ and $B = B'$ are true, and thus the conjunction in (3.3) holds, as desired.

To prove the second part ('$\Leftarrow$') of the equivalence in (3.3), we now assume $A = A'$ and $B = B'$ to be both true. Then, substitution based on these equations immediately yields the desired $(A, B) = (A', B')$, which proves already the implication. As A, B, A' and B' were arbitrary, we may therefore conclude that the proposed universal sentence (3.3) is true. □

Theorem 3.2 (Characterization of a uniquely existential sentence with two variables). *Any uniquely existential sentence involving two variables can be equivalently expressed as a uniquely existential sentence with respect to a single variable, in the sense that*

$$\exists! X, Y\, (\varphi(X, Y)) \;\Leftrightarrow\; \exists! Z\, (\exists X, Y\, (Z = (X, Y) \wedge \varphi(X, Y))). \tag{3.4}$$

Proof. Letting $\varphi(X, Y)$ be an arbitrary given formula, we establish the first part ('$\Leftarrow$') of the equivalence directly, assuming $\exists! X, Y\, (\varphi(X, Y))$ to be true. According to Notation 1.4, this implies

$$\begin{aligned} &\exists X, Y\, (\varphi(X, Y)) \\ &\quad \wedge \forall X, Y, X', Y'\, ([\varphi(X, Y) \wedge \varphi(X', Y')] \Rightarrow [X = X' \wedge Y = Y']), \end{aligned} \tag{3.5}$$

whereas the desired consequent is equivalent to

$$\begin{aligned}\exists Z\,(\exists X,Y\,(Z=(X,Y)\,\wedge\,\varphi(X,Y)))\\ \wedge\,\forall Z,Z'\,([\exists X,Y\,(Z=(X,Y)\,\wedge\,\varphi(X,Y))\\ \wedge\,\exists X,Y\,(Z'=(X',Y')\,\wedge\,\varphi(X',Y'))]\Rightarrow Z=Z'),\end{aligned}\tag{3.6}$$

in view of the Criterion for unique existence. Due to the assumed antecedent, there exist constants, say $\bar{X}$ and $\bar{Y}$, such that $\varphi(\bar{X},\bar{Y})$ is satisfied. Forming now the ordered pair $\bar{Z}=(\bar{X},\bar{Y})$, we see that the existential sentence

$$\exists X,Y\,(\bar{Z}=(X,Y)\,\wedge\,\varphi(X,Y))$$

is true, which in turn demonstrates the truth of the existential sentence

$$\exists Z\,(\exists X,Y\,(Z=(X,Y)\,\wedge\,\varphi(X,Y))),$$

proving the existential part of (3.6). Concerning the uniqueness part, we let Z and Z' be arbitrary such that the two existential sentences

$$\begin{aligned}&\exists X,Y\,(Z=(X,Y)\,\wedge\,\varphi(X,Y))\\ &\exists X,Y\,(Z'=(X',Y')\,\wedge\,\varphi(X',Y'))\end{aligned}$$

are satisfied. Thus, there are particular constants $\bar{X}$, $\bar{Y}$, $\bar{X}'$ and $\bar{Y}'$ satisfying $Z=(\bar{X},\bar{Y})$, $\varphi(\bar{X},\bar{Y})$, $Z'=(\bar{X}',\bar{Y}')$ and $\varphi(\bar{X}',\bar{Y}')$. Here, the truth of $\varphi(\bar{X},\bar{Y})$ and $\varphi(\bar{X}',\bar{Y}')$ implies the truth of the equations $\bar{X}=\bar{X}'$ and $\bar{Y}=\bar{Y}'$ with (3.5). Consequently, the Equality Criterion for ordered pairs yields $(\bar{X},\bar{Y})=(\bar{X}',\bar{Y}')$, so that $Z=Z'$ follows to be true via substitutions. Because Z and Z' are arbitrary, we may therefore conclude that the second part of the conjunction in (3.6), which proves the uniqueness part and thus the first part of the equivalence in (3.4).

Regarding the second implication '$\Leftarrow$', we assume now that the uniquely existential sentence with respect to Z is true, so that the equivalent conjunction (3.6) holds as well. We note that the consequent to be proven is equivalent to the conjunction (3.5) as shown by Notation 1.4. The first part of the assumed conjunction gives us particular constants $\bar{Z}$, $\bar{X}$ and $\bar{Y}$ satisfying both $\bar{Z}=(\bar{X},\bar{Y})$ and $\varphi(\bar{X},\bar{Y})$. Thus, the existential sentence $\exists X,Y\,(\varphi(X,Y))$ is true, so that the existential part in (3.5) holds. To prove the uniqueness part, we take arbitrary $\bar{X}$, $\bar{Y}$, $\bar{X}'$ and $\bar{Y}'$, assuming $\varphi(\bar{X},\bar{Y})$ and $\varphi(\bar{X}',\bar{Y}')$ to be true. Defining now the ordered pairs $\bar{Z}=(\bar{X},\bar{Y})$ and $\bar{Z}'=(\bar{X}',\bar{Y}')$, we see that the conjunctions

$$\begin{aligned}\bar{Z}&=(\bar{X},\bar{Y})\,\wedge\,\varphi(\bar{X},\bar{Y})\\ \bar{Z}'&=(\bar{X}',\bar{Y}')\,\wedge\,\varphi(\bar{X}',\bar{Y}')\end{aligned}$$

hold, which in turn give rise to the true existential sentences

$$\exists X, Y\, (\bar{Z} = (X, Y) \wedge \varphi(X, Y)),$$
$$\exists X, Y\, (\bar{Z}' = (X', Y') \wedge \varphi(X', Y')).$$

Due to the second part of (3.6), the conjunction of the preceding existential sentences implies $\bar{Z} = \bar{Z}'$. Then, substitutions give us the equation $(\bar{X}, \bar{Y}) = (\bar{X}', \bar{Y}')$, which further implies $\bar{X} = \bar{X}'$ and $\bar{Y} = \bar{Y}'$ with the Equality Criterion for ordered pairs. Since $\bar{X}$, $\bar{Y}$, $\bar{X}'$ and $\bar{Y}'$ are arbitrary, the second part of the conjunction (3.5) follows therefore to be true, which finding completes the proof of the uniqueness part and thus the proof of the second part of the equivalence in (3.4). □

The following definition demonstrates the principle that is applied to extend the concept of an ordered pair to sets with more than two coordinates.

Notation 3.1 (Ordered triple, ordered quadruple). We call

(1) for any sets A, B, C the ordered pair formed by the ordered pair (A, B) and C, symbolically

$$(A, B, C) = ((A, B), C), \tag{3.7}$$

the *ordered triple* formed by A, B and C. We then refer to A, B, C as the *first*, *second* and *third coordinate* of (A, B, C).

(2) for any sets A, B, C, D the ordered pair formed by the ordered triple (A, B, C) and D, symbolically

$$(A, B, C, D) = ((A, B, C), D) = (((A, B), C), D), \tag{3.8}$$

the *ordered quadruple* formed by A, B, C and D. Here, we say that A, B, C, D are the *first*, *second*, *third* and *fourth coordinate* of (A, B, C, D).

3.1.2. Power sets and Cartesian products of two sets

Before we can define particular sets of ordered pairs, it is necessary to introduce the concept of a *set of subsets* of a given set.

Axiom 3.1 (Axiom of powers). For any set A there exists a set (system) $\mathcal{P}$ which contains every subset of A, that is,

$$\forall A\, \exists \mathcal{P}\, \forall Y\, (Y \subseteq A \Rightarrow Y \in \mathcal{P}). \tag{3.9}$$

In order to establish an unambiguous definition of a set with that property, we now specify those elements in $\mathcal{P}$ which are truly subsets of A.

Proposition 3.3. *The following sentences are true for any set A and any set system $\mathcal{P}$ containing all subsets of A.*

a) There exists a unique set (system) $\mathcal{P}(A)$ such that a set Y is in $\mathcal{P}(A)$ iff Y is in $\mathcal{P}$ and moreover a subset of A, that is,

$$\exists! \mathcal{P}(A)\, \forall Y\, (Y \in \mathcal{P}(A) \Leftrightarrow [Y \in \mathcal{P} \wedge Y \subseteq A]). \tag{3.10}$$

b) The set $\mathcal{P}(A)$ is nonempty, that is,

$$\mathcal{P}(A) \neq \emptyset. \tag{3.11}$$

c) The set $\mathcal{P}(A)$ satisfies

$$\forall Y\, (Y \in \mathcal{P}(A) \Leftrightarrow Y \subseteq A). \tag{3.12}$$

Proof. We let A be an arbitrary set and $\mathcal{P}$ an arbitrary set system containing all subsets of A.

Concerning a), we define $\varphi(Y)$ to be the formula

$$Y \subseteq A$$

and apply the Axiom of Specification to obtain the true existential sentence

$$\exists \mathcal{P}(A)\, \forall Y\, (Y \in \mathcal{P}(A) \Leftrightarrow [Y \in \mathcal{P} \wedge \varphi(Y)]). \tag{3.13}$$

Then, $\mathcal{P}(A)$ exists uniquely in view of the Equality Criterion for sets.

Concerning b), we see in light of (2.43) that $\emptyset \subseteq A$ holds, which then implies also $\emptyset \in \mathcal{P}$ since $\mathcal{P}$ contains every subset of A. Thus, the conjunction $\emptyset \in \mathcal{P} \wedge \emptyset \subseteq A$ holds, so that $\emptyset \in \mathcal{P}(A)$ follows to be true with (3.10). This proves the existence of an element in $\mathcal{P}(A)$, which yields (3.11) according to (2.42).

Concerning c), to show that $\mathcal{P}(A)$ satisfies also (3.12), we let $\bar{Y}$ be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $\bar{Y} \in \mathcal{P}(A)$ holds. Then, this assumption implies with (3.10) in particular $\bar{Y} \subseteq A$, as desired. To prove the second part ('$\Leftarrow$') of the equivalence, we now assume $\bar{Y} \subseteq A$ to be true, so that $\bar{Y} \in \mathcal{P}$ holds in view of the specification of $\mathcal{P}$. Thus, the conjunction $\bar{Y} \in \mathcal{P} \wedge \bar{Y} \subseteq A$ is true, which in turn implies the desired $\bar{Y} \in \mathcal{P}(A)$ with (3.10). As $\bar{Y}$ is arbitrary, we may therefore conclude that the universal sentence (3.12) holds. Since A was also arbitrary, we then further conclude that the proposition is true. $\square$

Definition 3.2 (Power set). For any set A we call the set $\mathcal{P}(A)$ consisting of all the subsets of A in the sense of

$$\forall Y \, (Y \in \mathcal{P}(A) \Leftrightarrow Y \subseteq A).$$

the *power set* of A. This set is also symbolized by

$$\{Y : Y \subseteq A\}. \tag{3.14}$$

Corollary 3.4. *The power set of a set A contains $\emptyset$ and A, that is,*

$$\forall A \, (\emptyset, A \in \mathcal{P}(A)). \tag{3.15}$$

Proof. Letting A be an arbitrary set, we recall from the proof of Proposition 3.3b) that $\emptyset \in \mathcal{P}(A)$ holds. Furthermore, $A \subseteq A$ is true according to (2.10), and therefore $A \in \mathcal{P}(A)$ follows to be true with (3.12). Since A is arbitrary, we therefore conclude that the proposed universal sentence is true. $\square$

Exercise 3.1. Show for any sets A and B that the power set of A is included in the power set of B of A is included in B, that is,

$$\forall A, B \, (A \subseteq B \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)). \tag{3.16}$$

(Hint: Apply the definition of a subset in connection with (2.13).)

Exercise 3.2. Show that the power set of the empty set is the singleton formed by the empty set, that is,

$$\mathcal{P}(\emptyset) = \{\emptyset\}. \tag{3.17}$$

(Hint: Apply Method 2.1 in connection with (3.12), (2.46) and (2.169).)

Proposition 3.5. *It is true for any set X and any nonempty subset $\mathcal{S}$ of the power set of X that the intersection of the set system $\mathcal{S}$ is an element of the power set of X, that is,*

$$\forall X, \mathcal{S} \, ([\mathcal{S} \subseteq \mathcal{P}(X) \wedge \mathcal{S} \neq \emptyset] \Rightarrow \bigcap \mathcal{S} \in \mathcal{P}(X)). \tag{3.18}$$

Proof. We let X and $\mathcal{S}$ be arbitrary sets, assuming that $\mathcal{S}$ is included in the power set $\mathcal{P}(X)$ and assuming that $\mathcal{S} \neq \emptyset$ holds. Let us observe here that the former assumption implies by definition of a subset

$$\forall A\,(A \in \mathcal{S} \Rightarrow A \in \mathcal{P}(X)). \tag{3.19}$$

Because the desired consequent $\bigcap \mathcal{S} \in \mathcal{P}(X)$ is equivalent to $\bigcap \mathcal{S} \subseteq X$ by definition of a power set, we may apply again the definition of a subset and prove the equivalent universal sentence

$$\forall y\,(y \in \bigcap \mathcal{S} \Rightarrow y \in X). \tag{3.20}$$

Letting $\bar{y}$ be arbitrary and assuming $\bar{y} \in \bigcap \mathcal{S}$ to be true, we obtain now by definition of the intersection of a set system the true universal sentence

$$\forall A\,(A \in \mathcal{S} \Rightarrow \bar{y} \in A). \tag{3.21}$$

Observing now that the initial assumption $\mathcal{S} \neq \emptyset$ implies the existence of a particular element $\bar{A} \in \mathcal{S}$, we obtain on the one hand in view of (3.19) $\bar{A} \in \mathcal{P}(X)$ and therefore $\bar{A} \subseteq X$ (by definition of a power set). On the other hand, $\bar{A} \in \mathcal{S}$ implies with (3.21) $\bar{y} \in \bar{A}$, which in turn implies evidently $\bar{y} \in X$ by virtue of the preceding inclusion. As $\bar{y}$ is arbitrary, we may therefore conclude that the universal sentence (3.20) is true, giving the inclusion $\bigcap \mathcal{S} \subseteq X$ and then also $\bigcap \mathcal{S} \in \mathcal{P}(X)$, as desired. Because X and $\mathcal{S}$ are arbitrary, we may now further conclude that the proposed universal sentence (3.18) holds as well. □

Exercise 3.3. Prove for any set X and any subset $\mathcal{S}$ of the power set of X that the union of the set system $\mathcal{S}$ is in the power set of X, that is,

$$\forall X, \mathcal{S}\,(\mathcal{S} \subseteq \mathcal{P}(X) \Rightarrow \bigcup \mathcal{S} \in \mathcal{P}(X)). \tag{3.22}$$

Definition 3.3 (Covering). We say for any set X and any set $\mathcal{K}$ that $\mathcal{K}$ is a *covering* of X (alternatively, that $\mathcal{K}$ *covers* X) iff

1. $\mathcal{K}$ consists of subsets of X, that is,

   $$\mathcal{K} \subseteq \mathcal{P}(X) \tag{3.23}$$

   and

2. the union of $\mathcal{K}$ equals X, that is,

   $$\bigcup \mathcal{K} = X. \tag{3.24}$$

Corollary 3.6. *It is true for any set X and any subset A of X that the pair formed by A and its complement covers X.*

Proof. Letting X and A be arbitrary sets and assuming $A \subseteq X$ to hold, we obtain $A^c \subseteq X$ with (2.137). Thus, A and A^c are by definition of a power set elements of $\mathcal{P}(X)$, which finding allows us to establish the inclusion $\{A, A^c\} \subseteq \mathcal{P}(X)$ via the definition of a subset. Indeed, letting $Y \in \{A, A^c\}$ be arbitrary, we obtain the disjunction $Y = A \vee Y = A^c$ with the definition of a pair; since $A \in \{A, A^c\}$ and $A^c \in \{A, A^c\}$ are evidently both true, we immediately obtain the desired consequent $Y \in \mathcal{P}(X)$ by means of a proof by cases in connection with corresponding substitutions (which follows then to be true for any Y). Thus, the pair $\{A, A^c\}$ consists of subsets of X, for which we furthermore obtain

$$\bigcup\{A, A^c\} = A \cup A^c = X$$

using the notation for the union of two sets and (2.257). Thus, $\{A, A^c\}$ is by definition a covering of X, where, X and A are arbitrary, so that the proposed universal sentence follows to be true. □

Note 3.2. The proof of the preceding corollary shows that the complement of any element of the power set of a set is also element of that power set, that is,

$$\forall X, A\,(A \in \mathcal{P}(X) \Rightarrow A^c \in \mathcal{P}(X)). \tag{3.25}$$

Given two sets A and B, we can use the concept of a power set to specify the following set of ordered pairs.

Theorem 3.7. *The following sentences are true for any sets A and B.*

a) There exists a unique set (system) $A \times B$ such that a set Y is in $A \times B$ iff Y is in the power set $\mathcal{P}(\mathcal{P}(A \cup B))$ and if there are elements $a \in A$ and $b \in B$ such that Y is the ordered pair formed by a and b, that is,

$$\begin{aligned} &\exists! A \times B\, \forall Y\,(Y \in A \times B \\ &\quad \Leftrightarrow [Y \in \mathcal{P}(\mathcal{P}(A \cup B)) \wedge \exists a, b\,(a \in A \wedge b \in B \wedge (a, b) = Y)]). \end{aligned} \tag{3.26}$$

b) The set $A \times B$ is empty iff A or B is empty, that is,

$$(A = \emptyset \vee B = \emptyset) \Leftrightarrow A \times B = \emptyset. \tag{3.27}$$

c) The set $A \times B$ satisfies

$$\forall a, b\,((a, b) \in A \times B \Leftrightarrow [a \in A \wedge b \in B]). \tag{3.28}$$

Proof. We let A, B be arbitrary sets.

Concerning a), we verify first that any ordered pair $(a, b) = \{\{a\}, \{a, b\}\}$ with $a \in A$ and $b \in B$ is in the power set of the power set of the union of A and B, that is,

$$\forall a, b\,([a \in A \,\wedge\, b \in B] \Rightarrow (a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))). \tag{3.29}$$

Letting a and b be arbitrary, we prove the implication directly by assuming $a \in A$ and $b \in B$ to be both true, from which it evidently follows by definition of the union of a pair that $a, b \in A \cup B$ holds. On the one hand, $a \in A \cup B$ implies

$$\{a\} \subseteq A \cup B \tag{3.30}$$

with (2.184). On the other hand, $a, b \in A \cup B$ implies

$$\{a, b\} \subseteq A \cup B \tag{3.31}$$

with (2.164). Since $A \cup B$ uniquely specifies the power set $\mathcal{P}(A \cup B)$, the two inclusions (3.30) and (3.31) yield

$$\{a\}, \{a, b\} \in \mathcal{P}(A \cup B),$$

and therefore

$$[(a, b) =] \quad \{\{a\}, \{a, b\}\} \subseteq \mathcal{P}(A \cup B),$$

applying again (2.164). The set $\mathcal{P}(A \cup B)$ uniquely specifies the power set $\mathcal{P}(\mathcal{P}(A \cup B))$, so that the preceding inclusion $(a, b) \subseteq \mathcal{P}(A \cup B)$ implies

$$(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B)),$$

which proves the implication in (3.29). Since a and b were arbitrary, we may therefore conclude that the universal sentence (3.29) is true.

Next, we use the Axiom of Specification in connection with the Equality Criterion of sets to obtain the true uniquely existential sentence (3.26). Thus, the set $A \times B$ satisfies

$$\forall Y\,(Y \in A \times B \Leftrightarrow [Y \in \mathcal{P}(\mathcal{P}(A \cup B)) \,\wedge\, \exists a, b\,(a \in A \wedge b \in B \wedge (a, b) = Y)]). \tag{3.32}$$

Concerning b), to prove the first part ('$\Rightarrow$') of the equivalence, we assume

$$A = \emptyset \,\vee\, B = \emptyset \tag{3.33}$$

and show that this implies $A \times B = \emptyset$, i.e. that

$$\forall Y\,(Y \notin A \times B) \tag{3.34}$$

holds (using the definition of the empty set). To prove this universal sentence, we let Y be arbitrary. Next, we demonstrate the truth of the negation

$$\neg\exists a, b\,(a \in A \,\wedge\, b \in B \,\wedge\, (a,b) = Y), \tag{3.35}$$

which we may write equivalently as

$$\neg\exists a, b\,(Y = (a,b) \,\wedge\, [a \in A \,\wedge\, b \in B])$$

by applying (1.36) as well as (1.39), and then also as

$$\forall a, b\,(Y = (a,b) \Rightarrow [a \notin A \,\vee\, b \notin B]), \tag{3.36}$$

applying the Quantifier Negation Law (1.54), then (1.81), and subsequently De Morgan's Law for sentences (1.51). We let a and b be arbitrary such that $Y = (a,b)$ holds. Now, if the first part $A = \emptyset$ of the assumed disjunction (3.33) is true, then $a \notin A\,(= \emptyset)$ follows by definition of the empty set. Consequently, the disjunction $a \notin A \vee b \notin B$ also holds, irrespective of the truth value of $b \notin B$. Similarly, if the second part $B = \emptyset$ of the disjunction (3.33) is true, then we obtain $b \notin B$, and the disjunction $a \notin A \vee b \notin B$ is again true (irrespective of the truth value of $a \notin A$). This proves the implication in (3.36), and since a and b are arbitrary, we may therefore conclude that (3.36) holds, and thus equivalently that (3.35) is true. Thus, the existential sentence in (3.32) is false, and therefore the conjunction in (3.32) is also false, which then gives $\neg Y \in A \times B$ by applying the Law of Contraposition to the equivalence in (3.32). As Y is arbitrary, we may then further conclude that (3.34) holds, which gives $A \times B = \emptyset$ with the definition of the empty set. Thus, the proof of the first part of the equivalence in (3.27) is complete.

We prove the second part ('$\Leftarrow$') by contradiction, assuming $A \times B = \emptyset$ and the negation $\neg[A = \emptyset \vee B = \emptyset]$ to be true. We may write the latter equivalently as $A \neq \emptyset \wedge B \neq \emptyset$ by applying De Morgan's Law for sentences (1.52). This conjunction implies in particular $A \neq \emptyset$, so that there exists an element in A, say $\bar{a}$, because of (2.42). Similarly, the preceding conjunction implies in particular also $B \neq \emptyset$, so that there exists also an element in B, say $\bar{b}$. Then, the ordered pair $\bar{Y} = (\bar{a}, \bar{b})$ is uniquely specified, and it follows with (3.29) that $(\bar{a}, \bar{b}) \in \mathcal{P}(\mathcal{P}(A \cup B))$. Moreover, we showed that there exist elements a and b with $a \in A$, $b \in B$, and with $(a,b) = \bar{Y}$. Consequently, $\bar{Y} \in A \times B$ holds according to (3.32). Now, due to the assumption $A \times B = \emptyset$, we have that $\bar{Y} \notin A \times B$ is also true (by definition of the empty set), so that we obtained the contradiction $\bar{Y} \in A \times B \wedge \bar{Y} \notin A \times B$ according to (1.11). Thus, the proof of the second part of the equivalence in (3.27) is also complete.

Concerning c), we now prove that $A \times B$ satisfies (3.28), letting a and b be arbitrary and assuming $(a, b) \in A \times B$ to be true. This implies with (3.32) in particular that there exist elements, say $\bar{a}$ and $\bar{b}$, such that $\bar{a} \in A$, $\bar{b} \in B$ and $(\bar{a}, \bar{b}) = (a, b)$ hold. This equation now gives $\bar{a} = a$ and $\bar{b} = b$ with (3.3), so that $\bar{a} \in A$ and $\bar{b} \in B$ yield $a \in A$ and $b \in B$ via substitution. This proves the conjunction in (3.28), so that the proof of the first part of the equivalence is complete.

To prove the second part ('$\Leftarrow$'), we now assume $a \in A$ and $b \in B$ to be true, so that we obtain $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$ with (3.29). Moreover, we then see that the existential sentence in (3.32) also holds, so that (a, b) follows to be an element of $\mathcal{X}$. This completes the proof of the equivalence in (3.28), and since a and b were arbitrary, we may therefore conclude that the universal sentence (3.28) is true.

Then, as A and B were also arbitrary, we may finally conclude that the proposition holds, as claimed. □

Definition 3.4 (Cartesian product of two sets). For any sets A and B we call the set $A \times B$ consisting of all the ordered pairs with first coordinate in A and second coordinate in B in the sense of

$$\forall a, b \, ((a, b) \in A \times B \Leftrightarrow [a \in A \, \wedge \, b \in B]).$$

the *Cartesian product* of A and B, and we symbolize this set also by

$$\{(a, b) : a \in A \, \wedge \, b \in B\}. \tag{3.37}$$

Exercise 3.4. Show for any sets A and B that the Cartesian product $A \times B$ satisfies

$$\forall Y \, (Y \in A \times B \Leftrightarrow \exists a, b \, (a \in A \, \wedge \, b \in B \wedge (a, b) = Y)). \tag{3.38}$$

(Hint: Proceed in analogy to the proof of Theorem 3.7b), using the findings obtained in the proof of Theorem 3.7c).)

Exercise 3.5. The Cartesian product of a singleton $\{a\}$ and a singleton $\{b\}$ is identical with the singleton formed by the ordered pair whose first coordinate is a and whose second coordinate is b, that is,

$$\forall a, b \, (\{a\} \times \{b\} = \{(a, b)\}). \tag{3.39}$$

(Hint: Apply Method 2.1 in connection with (3.38) and (2.169).)

3.1.3. Basic laws for the Cartesian product of two sets

Proposition 3.8. *For any sets A, B, X and Y it is true that, if A is included in X and B is included in Y, then the Cartesian product of A and B is included in the Cartesian product of X and Y, i.e.*

$$(A \subseteq X \wedge B \subseteq Y) \Rightarrow A \times B \subseteq X \times Y. \tag{3.40}$$

Proof. We let A, B, X and Y be arbitrary sets and prove implication directly, assuming that $A \subseteq X$ and $B \subseteq Y$ are both true. To show that this implies $A \times B \subseteq X \times Y$, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall z\, (z \in A \times B \Rightarrow z \in X \times Y). \tag{3.41}$$

Letting $\bar{z}$ be arbitrary and assuming $\bar{z} \in A \times B$ to be true, it then follows by definition of the Cartesian product of two sets that there exist elements, say $\bar{a}$ and $\bar{b}$, such that $\bar{a} \in A$, $\bar{b} \in B$ and $(\bar{a}, \bar{b}) = \bar{z}$ hold. On the one hand, as the initial assumption $A \subseteq X$ means (by definition of a subset) that the implication $a \in A \Rightarrow a \in X$ is true for any a, we see that the previously obtained $\bar{a} \in A$ then implies $\bar{a} \in X$. On the other hand, the initial assumption $B \subseteq Y$ means (again by definition of a subset) that the implication $b \in B \Rightarrow b \in Y$ holds for all b, so that the previously established $\bar{b} \in B$ implies $\bar{b} \in Y$. We thus showed that there exist a and b such that $a \in X$, $b \in Y$ and $(a, b) = \bar{z}$ are true, with the consequence that $\bar{z} \in X \times Y$ holds by definition of the Cartesian product of two sets. Since $\bar{z}$ is arbitrary, we may therefore conclude that (3.41) is true, which yields $A \times B \subseteq X \times Y$ with the definition of a subset, proving the implication (3.40). Since A, B, X and Y were arbitrary, the proposed universal sentence then follows to be true. □

Proposition 3.9. *For any nonempty sets A, B, X and Y it is true that, if the Cartesian product of A and B is included in the Cartesian product of X and Y, then A is included in X and B is included in Y, i.e.*

$$A \times B \subseteq X \times Y \Rightarrow (A \subseteq X \wedge B \subseteq Y). \tag{3.42}$$

Proof. We let A, B, X and Y be arbitrary sets such that $A \neq \emptyset$, $B \neq \emptyset$, $X \neq \emptyset$ and $Y \neq \emptyset$ hold. Consequently, we obtain the nonempty Cartesian products $A \times B \neq \emptyset$ and $X \times Y \neq \emptyset$ with (3.27) and the Law of Contraposition. Now, to prove the implication directly, we assume $A \times B \subseteq X \times Y$ to be true, so that (3.41) holds by definition of a subset. To prove $A \subseteq X \wedge B \subseteq Y$, we apply the definition of a subset again and verify the equivalent conjunction of the universal sentences

$$\forall a\, (a \in A \Rightarrow a \in X) \tag{3.43}$$

and

$$\forall b\,(b \in B \Rightarrow b \in Y). \tag{3.44}$$

To prove (3.43), we let $\bar{a} \in A$ be arbitrary. Since we initially assumed $B \neq \emptyset$, it follows with (2.42) that there exists an element in B, say $\bar{b}$. Then, the ordered pair $\bar{z} = (\bar{a}, \bar{b})$ exists also. We thus showed that there exist a and b satisfying $a \in A$, $b \in B$ and $(a, b) = \bar{z}$, so that $\bar{z}$ is in $A \times B$ by definition of the Cartesian product of two sets. Here, $\bar{z} \in A \times B$ implies with the assumed $A \times B \subseteq X \times Y$ and with the definition of a subset that $\bar{z} \in X \times Y$ holds. Consequently, there exist (again by definition of the Cartesian product of two sets) elements, say $\bar{x}$ and $\bar{y}$, with $\bar{x} \in X$, $\bar{y} \in Y$ and

$$(\bar{x}, \bar{y}) = \bar{z} \quad [= (\bar{a}, \bar{b})].$$

These equations yield $(\bar{x}, \bar{y}) = (\bar{a}, \bar{b})$ and therefore in particular $\bar{x} = \bar{a}$ with (3.3). In view of this equation, we now see that the previously found $\bar{x} \in X$ implies $\bar{a} \in X$. Since $\bar{a}$ is arbitrary, we may therefore conclude that the universal sentence (3.43) and thus the equivalent $A \subseteq X$ is true.

Regarding (3.43), we take an arbitrary $\bar{\bar{b}} \in B$ and recall the initial assumption $A \neq \emptyset$, so that there is an element in A, say $\bar{\bar{a}}$, according to (2.42). The ordered pair $\bar{\bar{z}} = (\bar{\bar{a}}, \bar{\bar{b}})$ is then also specified. These findings show that there are a and b with $a \in A$, $b \in B$ and $(a, b) = \bar{\bar{z}}$, with the consequence that $\bar{\bar{z}} \in A \times B$ (by definition of the Cartesian product of two sets). This further implies $\bar{z} \in X \times Y$ with the assumed $A \times B \subseteq X \times Y$ (and with the definition of a subset). Therefore, there are (once again by definition of the Cartesian product of two sets) elements, say $\bar{\bar{x}}$ and $\bar{\bar{y}}$, such that $\bar{\bar{x}} \in X$, $\bar{\bar{y}} \in Y$ and

$$(\bar{\bar{x}}, \bar{\bar{y}}) = \bar{\bar{z}} \quad [= (\bar{\bar{a}}, \bar{\bar{b}})].$$

These equations give $(\bar{\bar{x}}, \bar{\bar{y}}) = (\bar{\bar{a}}, \bar{\bar{b}})$ and thus in particular $\bar{\bar{y}} = \bar{\bar{b}}$ because of (3.3), so that the previously established $\bar{\bar{y}} \in Y$ gives $\bar{\bar{b}} \in Y$. As $\bar{\bar{b}}$ was arbitrary, it then follows that the universal sentence (3.43) and thus the equivalent $B \subseteq Y$ also holds.

Since $A \subseteq X$ and $B \subseteq Y$ are both true, the proof of the implication in (3.42) is complete. As A, B, X and Y were arbitrary, the proposition follows then to be true. □

Corollary 3.10. *For any nonempty sets A, B, X and Y it is true that A is included in X and B included in Y iff the Cartesian product of A and B is included in the Cartesian product of X and Y, i.e.*

$$(A \subseteq X \,\wedge\, B \subseteq Y) \Leftrightarrow A \times B \subseteq X \times Y. \tag{3.45}$$

Proof. Letting A, B, X and Y be arbitrary sets and assuming further $A \neq \emptyset$, $B \neq \emptyset$, $X \neq \emptyset$ and $Y \neq \emptyset$ to be true, we see that the first part ('$\Rightarrow$') of the equivalence holds with (3.40) and moreover that the second part ('$\Leftarrow$') holds with (3.42). Since A, B, X and Y are arbitrary, we may therefore conclude that the corollary is true. □

Theorem 3.11 (Equality Criterion for Cartesian products of two sets). *The following equivalence holds for any nonempty sets A, B, X, Y.*

$$(A = X \wedge B = Y) \Leftrightarrow A \times B = X \times Y. \tag{3.46}$$

Proof. We let $A \neq \emptyset$, $B \neq \emptyset$, $X \neq \emptyset$ and $Y \neq \emptyset$ be arbitrary sets. To prove the first part ('$\Rightarrow$') of the equivalence, we assume that $A = X$ and $B = Y$ hold. Applying now (2.22), the former equation $A = X$ implies the truth of $A \subseteq X$ and of $X \subseteq A$, and the latter equation $B = Y$ implies $B \subseteq Y$ and $Y \subseteq B$. Therefore, the two conjunctions $A \subseteq X \wedge B \subseteq Y$ and $X \subseteq A \wedge Y \subseteq B$ are also true, which subsequently imply with (3.45) the truth of $A \times B \subseteq X \times Y$ and of $X \times Y \subseteq A \times B$, respectively. The conjunction of these two inclusions then further implies the desired equality $A \times B = X \times Y$ with (2.22), which proves the first part of the equivalence in (3.46).

To prove the second part ('$\Leftarrow$'), we now assume $A \times B = X \times Y$ to be true, which implies the conjunction of $A \times B \subseteq X \times Y$ and $X \times Y \subseteq A \times B$ with (2.22). In view of (3.45), the first part $A \times B \subseteq X \times Y$ implies the truth of $A \subseteq X$ and of $B \subseteq Y$, and the second part $X \times Y \subseteq A \times B$ the truth of $X \subseteq A$ and of $Y \subseteq B$. Consequently, the conjunctions $A \subseteq X \wedge X \subseteq A$ and $B \subseteq Y \wedge Y \subseteq B$ are both true as well, which yield the two respective equations $A = X$ and $B = Y$ with (2.22), proving the desired conjunction in (3.46).

Thus, the proof of the equivalence (3.46) is complete, and since A, B, X and Y were arbitrary, the theorem follows then to be true. □

Exercise 3.6. Show that the following equivalence holds for any nonempty sets A, B, X and Y.

$$(A \neq X \vee B \neq Y) \Leftrightarrow A \times B \neq X \times Y. \tag{3.47}$$

Lemma 3.12 (Elementary characterization of the complement of a Cartesian product). *For any set X, any subsets A and B of X and any elements a and b of X it is true that the ordered pair formed by a and b is in the complement of $A \times B$ with respect to $X \times X$ if, and only if, a is not an element of A or b is not an element of B, that is,*

$$\begin{aligned} &\forall X, A, B, a, b\,([A \subseteq X \wedge B \subseteq X \wedge a \in X \wedge b \in X] \\ &\quad \Rightarrow [(a,b) \in (A \times B)^c \Leftrightarrow a \notin A \vee b \notin B]). \end{aligned} \tag{3.48}$$

Proof. We let X, A, B, a and b be arbitrary and prove the implication directly, assuming $A \subseteq X$, $B \subseteq X$, $a \in X$ and $b \in X$ to be true. Let us first observe that $A \subseteq X$ and $B \subseteq X$ imply $A \times B \subseteq X \times X$ with (3.40), so that the complement of $A \times B$ with respect to $X \times X$ is defined.

We now prove the first part ('⇒') of the equivalence in (3.48) directly, assuming $(a, b) \in (A \times B)^c$ to be true. This assumption implies by definition of a complement that $(a, b) \in (X \times X) \setminus (A \times B)$ is true, so that the conjunction of $(a, b) \in (X \times X)$ and $(a, b) \notin (A \times B)$ holds, according to the definition of a set difference. Here, the latter implies with (3.28) and the Law of Contraposition $\neg(a \in A \wedge b \in B)$, and therefore the desired disjunction $a \notin A \vee b \notin B$ with De Morgan's Law for sentences (1.51). Thus, the proof of the first part of the equivalence in (3.48) is complete.

To prove the second part ('⇐'), we now assume the disjunction $a \notin A \vee b \notin B$ to be true, which implies the negation $\neg(a \in A \wedge b \in B)$ with De Morgan's Law for sentences (1.51). This negation in turn implies $(a, b) \notin A \times B$ with (3.28) and the Law of Contraposition. Furthermore, introducing the denotation $Y = (a, b)$ and recalling that $a \in X$ and $b \in X$ hold, we see that there exist a and b such that $a \in X$, $b \in X$ and $(a, b) = Y$ are true, so that $Y \in X \times X$ follows to be true with (3.38). Thus, $(a, b) \in X \times X$ and $(a, b) \notin A \times B$ both hold, which conjunction gives $(a, b) \in (X \times X) \setminus (A \times B)$ with the definition of a set difference. Consequently, (a, b) is an element of the complement of $A \times B$ with respect to $X \times X$, so that the proof of the second part of the equivalence in (3.48) is also complete. This in turn proves the implication in (3.48), and since X, A, B, a and b were arbitrary, we may therefore conclude that the proposed universal sentence (3.48) is true. □

Theorem 3.13 (Characterization of the complement of a Cartesian product). *For any set X and any subsets A and B of X it is true that the complement of $A \times B$ with respect to $X \times X$ is identical with the union of the Cartesian products $A^c \times B^c$, $A^c \times B$ and $A \times B^c$ (where the complements are taken with respect to X), i.e.*

$$(A \times B)^c = (A^c \times B^c) \cup (A^c \times B) \cup (A \times B^c). \tag{3.49}$$

Proof. We let X be an arbitrary set and A and B arbitrary subsets of X. Thus, the complements of A and B with respect to X are defined. Furthermore, since $A \subseteq X$ and $B \subseteq X$ imply $A \times B \subseteq X \times X$ with Proposition 3.8, the complement of $A \times B$ with respect to $X \times X$ is also defined. We now consider the two exhaustive cases $(A \times B)^c = \emptyset$ and $(A \times B)^c \neq \emptyset$. The first case $(A \times B)^c = \emptyset$ implies $(X \times X) \setminus (A \times B) = \emptyset$ by definition of a complement, which equation in turn implies $X \times X \subseteq A \times B$ with (2.113) and the Law of Contraposition. Together with the previously established inclusion $A \times B \subseteq X \times X$, this further implies $A \times B = X \times X$ with the Axiom of Extension, and then also the conjunction $A = X \wedge B = X$ with the Identity Criterion for Cartesian products of two sets. We now obtain

$$A^c = X \setminus A \;=\; X \setminus X \;=\; \emptyset,$$
$$B^c = X \setminus B \;=\; X \setminus X \;=\; \emptyset,$$

using the definition of a complement, then substitution based on the equations of the preceding conjunction, and (2.104). Thus, $A^c = B^c = \emptyset$, so that the equations $A^c \times B^c = \emptyset$, $A^c \times B = \emptyset$ and $A \times B^c = \emptyset$ evidently follow to be true with (3.27). Then, we obtain the equations

$$(A^c \times B^c) \cup (A^c \times B) \cup (A \times B^c) = \emptyset \cup \emptyset \cup \emptyset = \emptyset \quad [= (A \times B)^c]$$

by applying substitutions and (2.216), recalling the current case assumption. These equations show that (3.49) holds in the first case.

The second case $(A \times B)^c \neq \emptyset$ shows in light of (2.42) that there exists an element in $(X \times X) \setminus (A \times B)$, say $\bar{z}$. Thus, $\bar{z} \in (X \times X) \setminus (A \times B)$ holds by definition of a complement, and therefore $\bar{z} \in X \times X$ and $\bar{z} \notin A \times B$ are true by definition of a set difference. The former implies by definition of the Cartesian product of two sets that are elements, say $\bar{a}$ and $\bar{b}$, satisfying $\bar{a} \in X$, $\bar{b} \in X$ and $(\bar{a}, \bar{b}) = \bar{z}$. Thus, $(\bar{a}, \bar{b}) \in (A \times B)^c$ holds, which then gives the disjunction $\bar{a} \notin A \vee \bar{b} \notin B$ with Lemma 3.12. Let us now observe that there are three possible cases in which this disjunction is true according to its defining truth table:

1. $\bar{a} \notin A$ is true and $\bar{b} \notin B$ is true,
2. $\bar{a} \notin A$ is true and $\bar{b} \notin B$ is false,
3. $\bar{a} \notin A$ is false and $\bar{b} \notin B$ is true.

In the first case, the conjunctions $\bar{a} \in X \wedge \bar{a} \notin A$ and $\bar{b} \in X \wedge \bar{b} \notin B$ hold, so that $\bar{a} \in A^c$ and $\bar{b} \in B^c$ follow to be true by definition of a set difference and of a complement. Therefore, $(\bar{a}, \bar{b}) \in A^c \times B^c$ follows to be true with (3.28). We thus showed that there exists an element in $A^c \times B^c$, so that $A^c \times B^c \neq \emptyset$ holds according to (2.42). Moreover, the disjunction

$$(\bar{a}, \bar{b}) \in A^c \times B^c \vee (\bar{a}, \bar{b}) \in A^c \times B \vee (\bar{a}, \bar{b}) \in A \times B^c \tag{3.50}$$

is then also true.

In the second case, the conjunction $\bar{a} \in X \wedge \bar{a} \notin A$ and $\bar{b} \in B$ are now evidently true, with the consequence that $\bar{a} \in A^c$ and $\bar{b} \in B$ hold simultaneously. We therefore obtain $(\bar{a}, \bar{b}) \in A^c \times B$ due to (3.28), which shows on the one hand that there is an element in $A^c \times B$, so that $A^c \times B \neq \emptyset$; on the other hand, the disjunction (3.50) is then again true.

Similarly, the third case clearly yields $\bar{a} \in A$ and the conjunction $\bar{b} \in X \wedge \bar{b} \notin B$, so that $\bar{a} \in A$ and $\bar{b} \in B^c$ are both true. Consequently, $(\bar{a}, \bar{b}) \in A \times B^c$ holds, and we now see that there exists an element also in $A \times B^c$, so that $A \times B^c \neq \emptyset$. Clearly, the disjunction (3.50) holds then also in this case.

Since (3.50) holds in any case, this means by definition of the union of two sets that

$$(\bar{a}, \bar{b}) \in A^c \times B^c \cup A \times B^c \cup A \times B^c. \tag{3.51}$$

We thus showed that there exists an element in the preceding union, i.e. that the case assumption $(A \times B)^c \neq \emptyset$ implies $A^c \times B^c \cup A \times B^c \cup A \times B^c \neq \emptyset$.

Based on these findings, we now prove the equation (3.49) by verifying

$$\forall Y \, (Y \in (A \times B)^c \Leftrightarrow Y \in (A^c \times B^c) \cup (A^c \times B) \cup (A \times B^c)). \tag{3.52}$$

To do this, we let Y be arbitrary, and we may prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $Y \in (A \times B)^c$ to be true. Therefore, obtain $Y \in X \times X$ and $Y \notin A \times B$ follow to be true with the definitions of a complement and of a set difference. Here, $Y \in X \times X$ implies with (3.38) that there exist elements, say $\bar{a}$ and $\bar{b}$, such that $\bar{a} \in X$, $\bar{b} \in X$ and $(\bar{a}, \bar{b}) = Y$ hold. We thus have $(\bar{a}, \bar{b}) \in (A \times B)^c$, which in turn implies $\bar{a} \notin X \vee \bar{b} \notin X$ with (3.48). Then, the three aforementioned cases 1. – 3. evidently apply here for $Y = (\bar{a}, \bar{b})$, and we may follow the same line of reasoning as before to infer the truth of (3.51) for that ordered pair. Thus,

$$Y \in A^c \times B^c \cup A^c \times B \cup A \times B^c. \tag{3.53}$$

is true, completing the proof of the first part of the equivalence in (3.52). To prove the second part ('$\Leftarrow$'), we may now assume (3.53) to be true, which then yields the multiple disjunction

$$Y \in A^c \times B^c \vee Y \in A^c \times B \vee Y \in A \times B^c. \tag{3.54}$$

If the first part $Y \in A^c \times B^c$ of this disjunction is true, then there exist in view of (3.38) elements, say $\bar{a}$ and $\bar{b}$, with $\bar{a} \in A^c$, $\bar{b} \in B^c$ and $(\bar{a}, \bar{b}) = Y$. Here, $\bar{a} \in A^c$ $(= X \setminus A)$ gives $\bar{a} \in X$ and $\bar{a} \notin A$, and similarly $\bar{b} \in B^c$ $(= X \setminus B)$ yields $\bar{b} \in X$ and $\bar{b} \notin B$ (applying again the definitions of a complement and of a set difference). Thus, $A \subseteq X$, $B \subseteq X$, $\bar{a}, \bar{b} \in X$ and the disjunction $\bar{a} \notin A \vee \bar{b} \notin B$ hold, and therefore $(\bar{a}, \bar{b}) \in (A \times B)^c$ follows to be true with (3.48).

If the second part $Y \in A^c \times B$ of the disjunction (3.54) holds, then there are elements, say $\bar{a}'$ and $\bar{b}'$, such that $\bar{a}' \in A^c$, $\bar{b}' \in B$ and $(\bar{a}', \bar{b}') = Y$, where $\bar{a}' \in A^c$ $(= X \setminus A)$ implies the truth of $\bar{a}' \in X$ and $\bar{a}' \notin A$. Consequently, the disjunction $\bar{a}' \notin A \vee \bar{b}' \notin B$ also holds. Furthermore, since we initially assumed $B \subseteq X$, we have that $\bar{b}' \in B$ implies $\bar{b}' \in X$. Then, the simultaneous truth of $A \subseteq X$, $B \subseteq X$, $\bar{a}', \bar{b}' \in X$ and of $\bar{a}' \notin A \vee \bar{b}' \notin B$ implies $(\bar{a}', \bar{b}') \in (A \times B)^c$ again with (3.48).

Similarly, if the third part $Y \in A \times B^c$ of the disjunction (3.54) is true, then there now exist elements, say $\bar{a}''$ and $\bar{b}''$, satisfying $\bar{a}'' \in A$, $\bar{b}'' \in B^c$ and $(\bar{a}'', \bar{b}'') = Y$, where $\bar{b}'' \in B^c$ $(= X \setminus B)$ implies the truth of $\bar{b}'' \in X$ and $\bar{b}'' \notin B$. Then, the disjunction $\bar{a}'' \notin A \vee \bar{b}'' \notin B$ is also true. Moreover, as we initially assumed $A \subseteq X$, we see that $\bar{a}'' \in A$ implies $\bar{a}'' \in X$. Together with $A \subseteq X$, $B \subseteq X$, $\bar{b}'' \in X$ and the preceding disjunction, this gives $(\bar{a}'', \bar{b}'') \in (A \times B)^c$ due to (3.48).

We thus obtain $Y \in (A \times B)^c$ no matter which part of the disjunction (3.54) is true, which proves also the second part of the equivalence in (3.52). As Y is arbitrary, we may therefore conclude that the universal sentence (3.52) holds, which in turn implies the equation (3.49) because of (2.18). This also completes the proof by cases, and since X, A and B were arbitrary sets, we may finally conclude that the theorem is true. □

Proposition 3.14. *If two sets C and D are disjoint, then the Cartesian product of a set A with C and the Cartesian product of a set B with D are also disjoint, that is,*

$$\forall A, B, C, D \, (C \cap D = \emptyset \Rightarrow [A \times C] \cap [B \times D] = \emptyset). \tag{3.55}$$

Proof. We let A, B, C and D be arbitrary sets and prove then the implication by contradiction, assuming $C \cap D = \emptyset$ and $[A \times C] \cap [B \times D] \neq \emptyset$ to be both true. The preceding inequality implies now with (2.42) that there exists an element in $[A \times C] \cap [B \times D]$, say $\bar{y}$. Then, $\bar{y} \in [A \times C] \cap [B \times D]$ implies $\bar{y} \in A \times C$ and $\bar{y} \in B \times D$ by definition of the intersection of two sets. On the one hand, $\bar{y} \in A \times C$ implies by definition of the Cartesian product of two sets that there exist elements, say $\bar{a}$ and $\bar{c}$, with $\bar{a} \in A$, $\bar{c} \in C$ and $(\bar{a}, \bar{c}) = \bar{y}$. On the other hand, $\bar{y} \in B \times D$ implies (again with the Cartesian product of two sets) that there are elements, say $\bar{b}$ and $\bar{d}$, such that $\bar{b} \in B$, $\bar{d} \in D$ and $(\bar{b}, \bar{d}) = \bar{y}$ hold. We thus obtained two expressions for $\bar{y}$, which we may combine via substitution to obtain $[\bar{y} =]$ $(\bar{a}, \bar{c}) = (\bar{b}, \bar{d})$, which further implies $\bar{a} = \bar{b}$ and $\bar{b} = \bar{d}$ with (3.3). In view of the latter equation, we see that the previously established $\bar{c} \in C$ gives $\bar{d} \in C$. Since $\bar{d} \in D$ is also true, it follows (by definition of the intersection of two sets) that $\bar{d} \in C \cap D$ holds. This shows that there exists an element in $C \cap D$, which existential sentence in turn implies $C \cap D \neq \emptyset$ with (2.42), in contradiction to the assumed $C \cap D = \emptyset$. This completes the proof of the implication in (3.55), and since the sets A, B, C, D were arbitrary, the proposed universal sentence finally follows to be true. □

Proposition 3.15. *The Cartesian product is distributive over the union of a pair in the sense that*

$$\forall A, B, C\,(A \times [B \cup C] = [A \times B] \cup [A \times C]). \tag{3.56}$$

Proof. We let A, B and C be arbitrary and apply the Equality Criterion for sets to prove the stated equation. To do this, we verify the equivalent

$$\forall Z\,(Z \in A \times [B \cup C] \Leftrightarrow Z \in [A \times B] \cup [A \times C]), \tag{3.57}$$

letting Z be arbitrary. To prove the first part ('$\Rightarrow$') of the equivalence, we assume $Z \in A \times (B \cup C)$ to hold. This assumption implies with (3.38) that there exist an element of A, say $\bar{a}$, and an element of $B \cup C$, say $\bar{x}$, such that $(\bar{a}, \bar{x}) = Z$ holds. Here, $\bar{x} \in B \cup C$ in turn implies the disjunction

$$\bar{x} \in B \vee \bar{x} \in C \tag{3.58}$$

by definition of the union of two sets. In case the first part of this disjunction is true, the conjunction of $\bar{a} \in A$, $\bar{x} \in B$ and $(\bar{a}, \bar{x}) = Z$ holds, so that $Z \in A \times B$ follows to be true with (3.38). Then, the disjunction $Z \in A \times B \vee Z \in A \times C$ is also true, so that

$$Z \in [A \times B] \cup [A \times C] \tag{3.59}$$

holds by definition of the union of two sets. In case the second part $\bar{x} \in C$ of the disjunction (3.58) is true, we now see that the conjunction of $\bar{a} \in A$, $\bar{x} \in C$ and $(\bar{a}, \bar{x}) = Z$ is true, and we therefore obtain $Z \in A \times C$ again with (3.38). Consequently, the disjunction $Z \in A \times B \vee Z \in A \times C$ is true again, and thus the equivalent (3.59) holds, too. This completes the proof of the first part of the equivalence in (3.57).

We now prove the second part ('$\Leftarrow$') directly, assuming $Z \in [A \times B] \cup [A \times C]$ to be true. This implies (by definition of the union of two sets) that the disjunction

$$Z \in A \times B \vee Z \in A \times C \tag{3.60}$$

holds. On the one hand, if $Z \in A \times B$, then there are elements, say $\bar{a}$ and $\bar{b}$, with $\bar{a} \in A$, $\bar{b} \in B$ and $(\bar{a}, \bar{b}) = Z$. Since $\bar{b} \in B$ is true, the disjunction $\bar{b} \in B \vee \bar{b} \in C$ also holds, so that $\bar{b} \in B \cup C$ holds (by definition of the union of two sets). Thus, there exist elements a and b such that $a \in A$, $b \in B \cup C$ and $(a, b) = Z$, and therefore

$$Z \in A \times [B \cup C] \tag{3.61}$$

follows to be true with (3.38). On the other hand, if the second part $Z \in A \times C$ of the disjunction (3.60) holds, then there exist elements, say $\bar{a}$ and $\bar{c}$, with $\bar{a} \in A$, $\bar{c} \in C$ and $(\bar{a}, \bar{c}) = Z$. Here, $\bar{c} \in C$ implies the truth of the disjunction $\bar{c} \in B \lor \bar{c} \in C$, which evidently means that $\bar{c} \in B \cup C$. These findings show that there are elements a and c with $a \in A$, $c \in B \cup C$ and $(a, c) = Z$, with the consequence that (3.61) holds as before. This proves the second part of the equivalence in (3.57), which is thus true. Since Z is arbitrary, we may therefore conclude that the universal sentence (3.57) holds, which in turn implies the equation in (3.56) with the Equality Criterion for sets. As A, B and C were also arbitrary, we may finally conclude that the proposition is true. □

Exercise 3.7. Show that the Cartesian product is distributive over the union also in the sense that

$$\forall A, B, C\,([A \cup B] \times C = [A \times C] \cup [B \times C]). \tag{3.62}$$

(Hint: Proceed similarly as in the proof of Proposition 3.15.)

Proposition 3.16. *The following universal sentence is true.*

$$\forall A, B, C, D\,([A \cap C] \times [B \cap D] = [A \times B] \cap [C \times D]). \tag{3.63}$$

Proof. We let A, B, C and D be arbitrary sets and verify

$$\forall Z\,(Z \in [A \cap C] \times [B \cap D] \Leftrightarrow Z \in [A \times B] \cap [C \times D]), \tag{3.64}$$

letting Z be arbitrary. We now prove the first part ('$\Rightarrow$') of the equivalence and assume $Z \in [A \cap C] \times [B \cap D]$, so that there exist due to (3.38) an element in $A \cap C$, say $\bar{x}$, and an element in $B \cap D$, say $\bar{y}$, satisfying $(\bar{x}, \bar{y}) = Z$. On the one hand, $\bar{x} \in A \cap C$ yields $\bar{x} \in A$ and $\bar{x} \in C$, and on the other hand, $\bar{y} \in B \cap D$ gives $\bar{y} \in B$ and $\bar{y} \in D$, using the definition of the intersection of two sets. As $\bar{x} \in A$, $\bar{y} \in B$ and $(\bar{x}, \bar{y}) = Z$ are thus true, it follows with (3.38) that $Z \in A \times B$ holds. Similarly, the conjunction of $\bar{x} \in C$, $\bar{y} \in D$ and $(\bar{x}, \bar{y}) = Z$ is also true, so that $Z \in C \times D$. Thus, the conjunction of $Z \in A \times B$ and $Z \in C \times D$ holds, which yields the desired $Z \in [A \times B] \cap [C \times D]$ with the definition of the intersection of two sets.

To prove the second part ('$\Leftarrow$') of the equivalence in (3.64), we assume $Z \in [A \times B] \cap [C \times D]$ to be true, which means (by definition of the intersection of two sets) that $Z \in A \times B$ and $Z \in C \times D$ both hold. The former implies with (3.38) that there are elements of A and B, say $\bar{a}$ and $\bar{b}$, such that $(\bar{a}, \bar{b}) = Z$; similarly, the latter implies the existence of elements of C and D, say of $\bar{c}$ and of $\bar{d}$, with $(\bar{c}, \bar{d}) = Z$. Combining now the two equations for Z, we obtain $(\bar{a}, \bar{b}) = (\bar{c}, \bar{d})$ and therefore $\bar{a} = \bar{c}$ as well as $\bar{b} = \bar{d}$, according to the Equality Criterion for ordered pairs. With these equations, the previously established $\bar{c} \in C$ implies $\bar{a} \in C$, and the previously obtained $\bar{d} \in D$ gives $\bar{b} \in D$. Recalling the truth of $\bar{a} \in A$ and $\bar{b} \in B$, we thus have the true conjunctions $\bar{a} \in A \wedge \bar{a} \in C$ and $\bar{b} \in B \wedge \bar{b} \in D$, which further imply $\bar{a} \in A \cap C$ and $\bar{b} \in B \cap D$, respectively (applying again the definition of the intersection of two sets). Recalling now the equation $(\bar{a}, \bar{b}) = Z$, we now see that there are elements a and b such that $a \in A \cap C$, $b \in B \cap D$ and $(a, b) = Z$ are simultaneously true, with the consequence that $Z \in [A \cap C] \times [B \cap D]$ holds, according to (3.38).

This completes the proof of the equivalence in (3.64), and since Z is arbitrary, we may therefore conclude that the universal sentence (3.64) is true. It then follows from this with the Equality Criterion for sets that the equation in (3.63) holds. Finally, as A, B, C and D were arbitrary sets, we may conclude that the proposed universal sentence (3.63) is true. □

Corollary 3.17. *The Cartesian product is distributive over the intersection in the sense that*

$$\forall A, B, C\, (A \times [B \cap C] = [A \times B] \cap [A \times C]), \tag{3.65}$$
$$\forall A, B, C\, ([A \cap B] \times C = [A \times C] \cap [B \times C]). \tag{3.66}$$

Proof. We first observe that the conjunction of the universal sentences (3.65) and (3.66) is equivalent to

$$\forall A, B, C\, (A \times [B \cap C] = [A \times B] \cap [A \times C] \wedge [A \cap B] \times C = [A \times C] \cap [B \times C]) \tag{3.67}$$

because of the Distributive Law for quantification (1.74). Now, letting A, B and C be arbitrary sets, we obtain the true equations

$$A \times (B \cap C) = (A \cap A) \times (B \cap C) \;=\; [A \times B] \cap (A \times C)$$
$$(A \cap B) \times C = (A \cap B) \times (C \cap C) \;=\; (A \times C) \cap (B \times C)$$

by using (2.60) and (3.64). Since A, B and C were arbitrary, we may therefore conclude that the universal sentence (3.67), and thus the equivalent conjunction of the two proposed universal sentences, is true. □

Proposition 3.18. *The following equations hold for any set X and any subsets A_1, A_2, B_1, B_2 of X.*

$$[A_1 \times (B_1 \setminus B_2)] \cap [(A_1 \setminus A_2) \times (B_1 \cap B_2)] = \emptyset. \tag{3.68}$$
$$[A_1 \times (B_1 \setminus B_2)] \cup [(A_1 \setminus A_2) \times (B_1 \cap B_2)] = (A_1 \times B_1) \setminus (A_2 \times B_2). \tag{3.69}$$

Proof. Letting X, A_1, A_2, B_1 and B_2 be arbitrary sets such that $A_1 \subseteq X$, $A_2 \subseteq X$, $B_1 \subseteq X$ and $B_2 \subseteq X$ are true, we prove first the universal sentence

$$\forall y\,(y \notin [A_1 \times (B_1 \setminus B_2)] \cap [(A_1 \setminus A_2) \times (B_1 \cap B_2)]) \tag{3.70}$$

which will imply the stated equation (3.68) by definition of the empty set. For this purpose, we take an arbitrary y, and we prove then the negation by contradiction, assuming that the negation of that negation holds, so that

$$y \in [A_1 \times (B_1 \setminus B_2)] \cap [(A_1 \setminus A_2) \times (B_1 \cap B_2)]$$

follows to be true with the Double Negation Law. This implies with the definition of the intersection of two sets

$$y \in A_1 \times (B_1 \setminus B_2) \,\wedge\, y \in (A_1 \setminus A_2) \times (B_1 \cap B_2), \tag{3.71}$$

so that $y \in A_1 \times (B_1 \setminus B_2)$ is especially true. According to the definition of the Cartesian product of two sets, there exist then particular elements $\bar{a} \in A_1$ and $\bar{b} \in B_1 \setminus B_2$ with $(\bar{a}, \bar{b}) = y$. Let us observe here that $\bar{b} \in B_1 \setminus B_2$ implies in particular $\bar{b} \notin B_2$ with the definition of a set difference. Because the second part of the conjunction in (3.71) is also true, the preceding equation for y yields via substitution

$$(\bar{a}, \bar{b}) \in (A_1 \setminus A_2) \times (B_1 \cap B_2),$$

so that $\bar{a} \in A_1 \setminus A_2$ and $\bar{b} \in B_1 \cap B_2$ evidently follow to be both true. Clearly, the latter implies in particular $\bar{b} \in B_2$, in contradiction to the previously established negation $\bar{b} \notin B_2$. Thus, the proof of the negation in (3.70) is complete, and as y is arbitrary, we may therefore conclude that the universal sentence (3.70) holds, which gives then the equation (3.68) by definition of the empty set.

To establish the second proposed equation (3.69), we apply the Equality Criterion for sets, letting y be arbitrary. We now obtain the true equivalences

$$
\begin{aligned}
& y \in (A_1 \times B_1) \setminus (A_2 \times B_2) \\
& \Leftrightarrow y \in (A_1 \times B_1) \cap (A_2 \times B_2)^c \\
& \Leftrightarrow y \in (A_1 \times B_1) \cap [[(A_2^c \times B_2^c) \cup (A_2^c \times B_2)] \cup (A_2 \times B_2^c)] \\
& \Leftrightarrow y \in [(A_1 \times B_1) \cap [(A_2^c \times B_2^c) \cup (A_2^c \times B_2)]] \\
& \qquad\qquad \cup [(A_1 \times B_1) \cap (A_2 \times B_2^c)] \\
& \Leftrightarrow y \in [(A_1 \times B_1) \cap (A_2^c \times B_2^c)] \cup [(A_1 \times B_1) \cap (A_2^c \times B_2)] \\
& \qquad\qquad \cup [(A_1 \times B_1) \cap (A_2 \times B_2^c)] \\
& \Leftrightarrow y \in [(A_1 \cap A_2^c) \times (B_1 \cap B_2^c)] \cup [(A_1 \cap A_2^c) \times (B_1 \cap B_2)] \\
& \qquad\qquad \cup [(A_1 \cap A_2) \times (B_1 \cap B_2^c)] \\
& \Leftrightarrow y \in [(A_1 \cap A_2^c) \times (B_1 \cap B_2^c)] \cup [(A_1 \cap A_2) \times (B_1 \cap B_2^c)] \\
& \qquad\qquad \cup [(A_1 \cap A_2^c) \times (B_1 \cap B_2)] \\
& \Leftrightarrow y \in [[(A_1 \cap A_2^c) \cup (A_1 \cap A_2)] \times (B_1 \cap B_2^c)] \\
& \qquad\qquad \cup [(A_1 \cap A_2^c) \times (B_1 \cap B_2)] \\
& \Leftrightarrow y \in [[A_1 \cap (A_2^c \cup A_2)] \times (B_1 \cap B_2^c)] \cup [(A_1 \cap A_2^c) \times (B_1 \cap B_2)] \\
& \Leftrightarrow y \in ([A_1 \cap X] \times (B_1 \setminus B_2)) \cup [(A_1 \setminus A_2) \times (B_1 \cap B_2)] \\
& \Leftrightarrow y \in [A_1 \times (B_1 \setminus B_2)] \cup [(A_1 \setminus A_2) \times (B_1 \cap B_2)]
\end{aligned}
$$

by applying (2.138) where the complement is formed with respect to $X \times X$, then the Characterization of the complement of a Cartesian product (where the complements are now taken with respect to X), the Distributive Law for the intersection of two sets, again the Distributive Law for the intersection of two sets, subsequently (3.63) to each of the three Cartesian products, the Commutative Law for the union of two sets, then (3.62), once again the Distributive Law for the intersection of two sets, then (2.257) together with (2.138), and finally (2.77) in connection with the initial assumption of $A_1 \subseteq X$.

Since y is arbitrary, we may therefore infer from the preceding equivalences the truth of the equation (3.69). As the sets X, A_1, A_2, B_1, and B_2 were arbitrary, we may finally conclude that the proposition holds. □

3.2. Binary Relations

The concept of a 'binary relation' is fundamental to subsequently introduced concepts such as 'orderings', 'equivalence relations', and 'functions'.

Definition 3.5 (Binary relation). We say that a set R is a *binary relation* iff R is a set of ordered pairs, i.e. iff

$$\forall Y\,(Y \in R \Rightarrow \exists a, b\,(Y = (a, b))). \tag{3.72}$$

Proposition 3.19. *Any subset of the Cartesian product of any two sets is a binary relation, that is,*

$$\forall A, B, R\,(R \subseteq A \times B \Rightarrow R \text{ is a binary relation}). \tag{3.73}$$

Proof. Letting A, B and R be arbitrary sets and assuming the inclusion $R \subseteq A \times B$ to be true, we need to show that R satisfies (3.72). To do this, we take an arbitrary set Y and assume that $Y \in R$ is true. This assumption implies now with the preceding inclusion $Y \in A \times B$ by definition of a subset. According to the definition of the Cartesian product of two sets, there are then constants, say $\bar{a}$ and $\bar{b}$, such that $\bar{a} \in A$, $\bar{b} \in B$ and $(\bar{a}, \bar{b}) = Y$ hold. Thus, the desired existential sentence $\exists a, b\,(Y = (a, b))$ is clearly true, and since Y is arbitrary, we may therefore conclude that R satisfies indeed (3.72). This means that R is a binary relation, and as the sets A, B and Z were arbitrary, we may infer from this the truth of the proposed universal sentence. □

Note 3.3. We then say for any set X that R is a binary relation <u>on</u> X iff R is a subset of the Cartesian product $X \times X$, i.e. iff

$$R \subseteq X \times X. \tag{3.74}$$

Notation 3.2. Instead of $(a, b) \in R$ we also write

$$a\,R\,b. \tag{3.75}$$

In the following, we will usually abbreviate "for any set R such that R is a binary relation" by "for any binary relation".

Let us inspect two simple examples for a binary relation.

Proposition 3.20. *The empty set $\emptyset$ is a binary relation.*

Proof. We verify (3.72) for $R = \emptyset$, i.e.

$$\forall Y\,(Y \in \emptyset \Rightarrow \exists a, b\,(Y = (a, b))). \tag{3.76}$$

For this purpose, we let Y be arbitrary and observe that the antecedent $Y \in \emptyset$ is false by definition of the empty set, so that the implication itself is true. As Y is arbitrary, we may therefore conclude that the universal sentence (3.76) is true, so that $\emptyset$ is a binary relation by definition. $\square$

Exercise 3.8. Show that any binary relation on the empty set is itself empty.
(Hint: Use (3.74), (3.27) and (2.46).)

Proposition 3.21. *For any x and any y it is true that the singleton $\{(x,y)\}$ is a binary relation on $\{x,y\}$.*

Proof. We let x and y be arbitrary and verify

$$\forall Y\,(Y \in \{(x,y)\} \Rightarrow \exists a,b\,(Y = (a,b))). \tag{3.77}$$

For this purpose, we take an arbitrary Y and assume $Y \in \{(x,y)\}$, which implies $Y = (x,y)$ with (2.169). Thus, the existential sentence to be proven is clearly true, and since Y is arbitrary, we may therefore conclude that the universal sentence (3.77) holds. Thus, $\{(x,y)\}$ is a binary relation by definition.

Let us now observe that $x \in \{x,y\}$ and $y \in \{x,y\}$ are true according to (2.151), so that we obtain $\{x\} \subseteq \{x,y\}$ $\{y\} \subseteq \{x,y\}$ with (2.184). These two inclusion now give $\{x\} \times \{y\} \subseteq \{x,y\} \times \{x,y\}$ with Corollary 3.10, where $\{x\} \times \{y\} = \{(x,y)\}$ holds according to (3.39). Applying now substitution based on this equation, the preceding inclusion yields $\{(x,y)\} \subseteq \{x,y\} \times \{x,y\}$, so that $\{(x,y)\}$ is a binary relation on $\{x,y\}$.

As x and y were arbitrary, we may finally conclude that the proposed sentence is true. $\square$

Since ordered triples are structurally ordered pairs for which the first coordinates are themselves ordered pairs, we may view the following kind of relation (which we will consider in greater detail in the context of binary operations) as a special case of a binary relation.

Definition 3.6 (Ternary relation). We say that a set R is a *ternary relation* iff R is a set of ordered triples, that is, iff

$$\forall Y\,(Y \in R \Rightarrow \exists a,b,c\,(Y = (a,b,c))). \tag{3.78}$$

Moreover, we say for any set X that R is a ternary relation <u>on</u> X iff R is a subset of the Cartesian product $(X \times X) \times X$, i.e. iff

$$R \subseteq (X \times X) \times X. \tag{3.79}$$

Exercise 3.9. Verify that the $\emptyset$ is a ternary relation.

(Hint: Use the same arguments as in the proof of Proposition 3.20.)

Exercise 3.10. Show for any x, y and z that the singleton $\{(x, y, z)\}$ is a ternary relation.

(Hint: Proceed in analogy to the proof of Proposition 3.21.)

The next theorem allows us to extract from a given binary relation all the ordered pairs whose first coordinate is element of a given set.

Theorem 3.22. *For any binary relation R and any set A there exists a unique set $R \restriction A$ such that an element Z is in $R \restriction A$ iff Z is in R and moreover if Z is some ordered pair with first coordinate in A, i.e. such that*

$$\forall Z\,(Z \in R \restriction A \Leftrightarrow [Z \in R \,\wedge\, \exists a, b\,(a \in A \,\wedge\, (a, b) = Z)]). \tag{3.80}$$

holds. Then, the set $R \restriction A$ satisfies also

$$\forall a, b\,((a, b) \in R \restriction A \Leftrightarrow [(a, b) \in R \,\wedge\, a \in A]). \tag{3.81}$$

Proof. Letting R be an arbitrary binary relation and A an arbitrary set, we may apply the Axiom of Specification to obtain the true existential sentence

$$\exists R \restriction A\, \forall Z\,(Z \in R \restriction A \Leftrightarrow [Z \in R \,\wedge\, \exists a, b\,(a \in A \,\wedge\, (a, b) = Z)]).$$

It then follows with the Equality Criterion for sets that $R \restriction A$ exists uniquely, that is,

$$\exists! R \restriction A\, \forall Z\,(Z \in R \restriction A \Leftrightarrow [Z \in R \,\wedge\, \exists a, b\,(a \in A \,\wedge\, (a, b) = Z)]). \tag{3.82}$$

We now take arbitrary $\bar{a}$ and $\bar{b}$ and assume first $(\bar{a}, \bar{b}) \in R \restriction A$ to be true, which implies with (3.80) that $(\bar{a}, \bar{b}) \in R$ holds and moreover that there exist constants, say $\bar{\bar{a}}$ and $\bar{\bar{b}}$, such that $\bar{\bar{a}} \in A$ and $(\bar{\bar{a}}, \bar{\bar{b}}) = (\bar{a}, \bar{b})$ are true. The latter equation yields with the Equality Criterion for ordered pairs in particular $\bar{\bar{a}} = \bar{a}$, so that $\bar{\bar{a}} \in A$ gives $\bar{a} \in A$ via substitution. Thus, the conjunction $(\bar{a}, \bar{b}) \in R \wedge \bar{a} \in A$ is true, which proves the first part ('$\Rightarrow$') of the equivalence in (3.81).

To prove the second part ('$\Leftarrow$'), we now assume the conjunction of $(\bar{a}, \bar{b}) \in R$ and $\bar{a} \in A$ to be true. Since the equation $(\bar{a}, \bar{b}) = (\bar{a}, \bar{b})$ also holds, the existential sentence $\exists a, b\,(a \in A \wedge (a, b) = (\bar{a}, \bar{b}))$ is true. Consequently, the conjunction

$$(\bar{a}, \bar{b}) \in R \,\wedge\, \exists a, b\,(a \in A \,\wedge\, (a, b) = (\bar{a}, \bar{b}))$$

holds, which in turn implies $(\bar{a}, \bar{b}) \in R \restriction A$ with (3.80). Thus, the proof of the equivalence in (3.81) is complete, and because a and b are arbitrary, we may infer from this the truth of the universal sentence (3.81).

Since R and A were initially arbitrary, we may therefore conclude that the theorem is true. $\square$

Definition 3.7 (Restriction). For any binary relation R and any set A we call the set $R \upharpoonright A$, also symbolized by

$$\{(a,b) : (a,b) \in R \wedge a \in A\}, \tag{3.83}$$

the *restriction* of R to A.

Corollary 3.23. *For any binary relation R and any set A, it is true that the restriction of R to A is a binary relation included in R.*

Proof. We let R be an arbitrary binary relation and A an arbitrary set. To prove $R \upharpoonright A \subseteq R$, i.e. (applying the definition of a subset)

$$\forall Z\,(Z \in R \upharpoonright A \Rightarrow Z \in R),$$

we let Z be arbitrary and assume $Z \in R \upharpoonright A$ to be true, which assumption implies in particular the desired $Z \in R$ with (3.80). Since Z is arbitrary, the preceding universal sentence follows then to be true, so that $R \upharpoonright A \subseteq R$ holds by definition of a subset. Therefore, the restriction $R \upharpoonright A$ consists of ordered pairs in R and constitutes thus a binary relation. As R and A were also arbitrary, we may finally conclude that the corollary is true. □

Exercise 3.11. Show that the restriction of any binary relation on the empty set to any set is itself empty, that is,

$$\forall R, A\,(R \text{ is a binary relation on } \emptyset \;\Rightarrow\; R \upharpoonright A = \emptyset). \tag{3.84}$$

(Hint: Apply Exercise 3.8, Corollary 3.23, and Proposition 2.13.)

Exercise 3.12. Show that the restriction of any binary relation R to the empty set is itself empty, that is,

$$\forall R\,(R \text{ is a binary relation} \;\Rightarrow\; R \upharpoonright \emptyset = \emptyset). \tag{3.85}$$

(Hint: Use (2.39), Method 1.12, and (3.80).)

Proposition 3.24. *For any set X and any binary relation R on X, the restriction of R to X is identical with R, that is,*

$$\forall X, R\,(R \subseteq X \times X \Rightarrow R \upharpoonright X = R). \tag{3.86}$$

Proof. We let X be an arbitrary set and R an arbitrary binary relation on X, i.e. we let R be an arbitrary subset of the Cartesian production $X \times X$. To prove the equation $R \upharpoonright X = R$, we verify that $R \upharpoonright X \subseteq R$ and $R \subseteq R \upharpoonright X$ are both true (which will then imply the desired equation with the Axiom of Extension). Clearly, the first part $R \upharpoonright X \subseteq R$ of this conjunction is true because of Corollary 3.23. To prove the second part $R \subseteq R \restriction X$, we apply the definition of a subset and verify

$$\forall Z\,(Z \in R \Rightarrow Z \in R \restriction X). \tag{3.87}$$

To do this, we let Z be arbitrary and assume $Z \in R$ to be true, which implies on the one hand $Z \in X \times X$ with the assumed inclusion $R \subseteq X \times X$ (applying again the definition of a subset). On the other hand, $Z \in R$ implies by definition of a binary relation that there exist elements, say $\bar{a}$ and $\bar{b}$, such that $Z = (\bar{a}, \bar{b})$ holds. With this, $Z \in R$ implies $(\bar{a}, \bar{b}) \in R$, and $Z \in X \times X$ implies $(\bar{a}, \bar{b}) \in X \times X$. It then follows with the definition of a Cartesian product that there are elements, say $\bar{a}'$ and $\bar{b}'$, with $\bar{a}' \in X$, $\bar{b}' \in X$ and $(\bar{a}', \bar{b}') = (\bar{a}, \bar{b})$. The latter equation implies in particular $\bar{a}' = \bar{a}$ with the Equality Criterion for ordered pairs, so that $\bar{a}' \in X$ gives $\bar{a} \in X$ after substitution. In view of these findings, the sentence

$$(\bar{a}, \bar{b}) \in R \wedge \exists a, b\,(a \in X \wedge (a, b) = (\bar{a}, \bar{b}))$$

is evidently true, so that $(\bar{a}, \bar{b}) \in R \restriction X$ follows to be true with the definition of a restriction. This finally implies the desired $Z \in R \restriction X$ with the previously established equation $Z = (\bar{a}, \bar{b})$. Since Z is arbitrary, we may therefore conclude that (3.87) holds, which yields $R \subseteq R \restriction X$ by definition of a subset. The conjunction of this inclusion and the already proved $R \restriction X \subseteq R$ then gives the equation $R \restriction X = R$ with the Axiom of Extension. As X and R were arbitrary, it follows that the proposition is true. □

Exercise 3.13. Show that any subset R of the Cartesian product of any two sets X and Y is identical with the restriction of R to X, that is,

$$\forall X, R\,(R \subseteq X \times Y \Rightarrow R \restriction X = R). \tag{3.88}$$

(Hint: Proceed in analogy to the proof of Proposition 3.24, recalling Proposition 3.19.)

Proposition 3.25. *For any binary relation R it is true that, if a set A is included in a set B, then the restriction of R to A is included in the restriction of R to B, that is,*

$$\forall R, A, B\,(R \text{ is a binary relation } \Rightarrow [A \subseteq B \Rightarrow R \restriction A \subseteq R \restriction B]). \tag{3.89}$$

Proof. We let R, A and B be arbitrary sets, assume that R is a binary relation, and then we assume furthermore that $A \subseteq B$ holds, that is (by definition of a subset)

$$\forall y\,(y \in A \Rightarrow y \in B). \tag{3.90}$$

To verify $R \upharpoonright A \subseteq R \upharpoonright B$, we prove the equivalent

$$\forall Z\,(Z \in R \upharpoonright A \Rightarrow Z \in R \upharpoonright B). \tag{3.91}$$

To do this, we let Z be arbitrary and assume that $Z \in R \upharpoonright A$ holds, which implies by definition of a restriction $Z \in R$ and moreover that there exist elements, say $\bar{a}$ and $\bar{b}$, with $\bar{a} \in A$ and $(\bar{a}, \bar{b}) = Z$. Here, $\bar{a} \in A$ implies $\bar{a} \in B$ with (3.90); thus, $\bar{a} \in B$ and $(\bar{a}, \bar{b}) = Z$ are both true, which shows that the existential sentence

$$\exists a, b\,(a \in B \,\wedge\, (a, b) = Z)$$

holds. The conjunction of this and the previously established $Z \in R$ then implies the desired $Z \in R \upharpoonright B$ (with the definition of a restriction). Since Z is arbitrary, we may therefore conclude that (3.91) holds, so that $R \upharpoonright A \subseteq R \upharpoonright B$ is indeed true. As R, A and B were also arbitrary, it then follows that the proposed universal sentence (3.89) holds. □

We now establish two characteristic sets which collect, respectively, the first and the second coordinates of a binary relation.

Theorem 3.26. *The following sentences hold for any binary relation R.*

a) There exists a unique set $\mathrm{dom}(R)$ *such that an element a is in* $\mathrm{dom}(R)$ *iff a is in the union of the union of R and moreover if there is an element b such that the ordered pair formed by a and b is in R, i.e.*

$$\exists!\mathrm{dom}(R)\ \forall a\,(a \in \mathrm{dom}(R) \Leftrightarrow [a \in \bigcup\bigcup R \wedge \exists b\,((a, b) \in R)]). \tag{3.92}$$

b) The set $\mathrm{dom}(R)$ *satisfies also*

$$\forall a\,(a \in \mathrm{dom}(R) \Leftrightarrow \exists b\,((a, b) \in R)). \tag{3.93}$$

Proof. Concerning a), we let R be an arbitrary binary relation and use then the Axiom of Specification to obtain the true existential sentence

$$\exists \mathrm{dom}(R)\ \forall a\,(a \in \mathrm{dom}(R) \Leftrightarrow [a \in \bigcup\bigcup R \,\wedge\, \exists b\,((a, b) \in R)]).$$

Then, we may apply the Equality Criterion for sets to establish the unique existence of $\mathrm{dom}(R)$. Thus, the set $\mathrm{dom}(R)$ satisfies

$$\forall a\,(a \in \mathrm{dom}(R) \Leftrightarrow [a \in \bigcup\bigcup R \,\wedge\, \exists b\,((a, b) \in R)]). \tag{3.94}$$

Concerning b), we let a be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $a \in \mathrm{dom}(R)$ to be true, so that the existential sentence $\exists b\,((a,b)\in R)$ to be proven follows in particular to be true with (3.94).

To prove the second part ('$\Leftarrow$') of the equivalence in (3.93), we assume that there is an element, say $\bar{b}$, such that $(a,\bar{b})\in R$ is true. We may write the latter also as $\{\{a\},\{a,\bar{b}\}\}$, using the notation for an ordered pair. Then, $\bigcup R$ evidently contains by definition of the union of a set system the elements of $\{\{a\},\{a,\bar{b}\}\}$, namely $\{a\}$ and $\{a,\bar{b}\}$. Thus, we have in particular $\{a\}\in\bigcup R$, so that $\bigcup\bigcup R$ contains (again by definition of the union of a set system) the element of $\{a\}$, namely a. We now arrived at the desired $a\in\bigcup\bigcup R$, which implies together with the assumed existential sentence $\exists b\,((a,b)\in R)$ because of (3.94) that $a\in\mathrm{dom}(R)$ holds. This completes the proof of the implication '$\Leftarrow$' and thus the proof of the equivalence in (3.93). Since a is arbitrary, we therefore conclude that the universal sentence (3.93) is true.

As R was arbitrary in the proofs of a) and b), we may therefore conclude that the theorem is true. □

Definition 3.8 (Domain). For any binary relation R we call the set $\mathrm{dom}(R)$ consisting of the first coordinates of the ordered pairs in R in the sense that

$$\forall a\,(a\in\mathrm{dom}(R)\Leftrightarrow\exists b\,((a,b)\in R))$$

the *domain* of R, and we symbolize this set also by

$$\{a:\exists b\,((a,b)\in R)\}. \tag{3.95}$$

Corollary 3.27. *For any binary relation R the domain of R is element of the power set of the union of the union of R, that is,*

$$\mathrm{dom}(R)\in\mathcal{P}(\bigcup\bigcup R). \tag{3.96}$$

Proof. Letting R be arbitrary and assuming R to be a binary relation, we see in light of (3.94) that $a\in\mathrm{dom}(R)$ implies in particular $a\in\bigcup\bigcup R$ for any a, so that

$$\mathrm{dom}(R)\subseteq\bigcup\bigcup R \tag{3.97}$$

holds by definition of a subset. Then, (3.96) follows to be true by definition of a power set. Since R was arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Proposition 3.28. *For any set X and any binary relation R on X, it is true that the domain of R is included in X, that is,*

$$\forall X,R\,(R\subseteq X\times X\Rightarrow\mathrm{dom}(R)\subseteq X). \tag{3.98}$$

Proof. We let X be an arbitrary set and R an arbitrary binary relation on X, i.e. an arbitrary subset of $X \times X$. To prove the inclusion $\mathrm{dom}(R) \subseteq X$, we verify

$$\forall a\,(a \in \mathrm{dom}(R) \Rightarrow a \in X). \tag{3.99}$$

To verify this universal sentence, we let $\bar{a}$ be arbitrary and assume $\bar{a} \in \mathrm{dom}(R)$ to be true. By definition of a domain, this assumption implies that there exists an element, say $\bar{b}$, such that $(\bar{a}, \bar{b}) \in R$ holds. This further implies $(\bar{a}, \bar{b}) \in X \times X$ with the assumed $R \subseteq X \times X$, applying the definition of a subset. It then follows with the definition of a Cartesian product in particular that $\bar{a} \in X$ is true, which proves the implication in (3.99). Since a is arbitrary, we may therefore conclude that (3.99) holds, which yields $\mathrm{dom}(R) \subseteq X$ with the definition of a subset. As X and R were also arbitrary, it follows that the proposed universal sentence (3.98) is true. □

Proposition 3.29. *It is true for any a and any b that the domain of the binary relation $\{(a, b)\}$ is identical with the singleton formed by a, that is,*

$$\forall a, b\,(\mathrm{dom}(\{(a, b)\}) = \{a\}). \tag{3.100}$$

Proof. We let $\bar{a}$ and $\bar{b}$ be arbitrary and prove the stated equation by means of the Equality Criterion for sets, by verifying

$$\forall x\,(x \in \mathrm{dom}(\{(\bar{a}, \bar{b})\}) \Leftrightarrow x \in \{\bar{a}\}). \tag{3.101}$$

We take an arbitrary $\bar{x}$, and we assume first $\bar{x} \in \mathrm{dom}(\{(\bar{a}, \bar{b})\})$ to be true. By definition of a domain, there is then a particular $\bar{\bar{b}}$ such that $(\bar{x}, \bar{\bar{b}}) \in \{(\bar{a}, \bar{b})\}$ holds. Consequently, we obtain $(\bar{x}, \bar{\bar{b}}) = (\bar{a}, \bar{b})$ with (2.169), which equation yields especially $\bar{x} = \bar{a}$ with the Equality Criterion for ordered pairs, and this equation implies the desired $\bar{x} \in \{\bar{a}\}$ with (2.169).

We now assume $\bar{x} \in \{\bar{a}\}$ to be true, which gives $\bar{x} = \bar{a}$ again with (2.169). As $(\bar{a}, \bar{b}) \in \{(\bar{a}, \bar{b})\}$ holds according to (2.153), we obtain $(\bar{x}, \bar{b}) \in \{(\bar{a}, \bar{b})\}$ via substitution, which shows that the existential sentence $\exists b((\bar{x}, b) \in \{(\bar{a}, \bar{b})\})$ is true. Consequently, we obtain $\bar{x} \in \mathrm{dom}(\{(\bar{a}, \bar{b})\})$ by definition of a domain, as desired. Since $\bar{x}$ is arbitrary, we may therefore conclude that the universal sentence (3.101) is true, so that the equation in (3.100) holds indeed. Then, as $\bar{a}$ and $\bar{b}$ are also arbitrary, the proposed sentence follows to be true. □

Proposition 3.30. *If the domain of a binary relation R and a set A are not disjoint, then the restriction of R to A is nonempty, i.e.*

$$\forall R, A\,([R \text{ is a binary relation } \wedge \mathrm{dom}(R) \cap A \neq \emptyset] \Rightarrow R \upharpoonright A \neq \emptyset). \tag{3.102}$$

Proof. We let R and A be arbitrary sets such that R is a binary relation and such that $\mathrm{dom}(R) \cap A \neq \emptyset$ holds. Consequently, there exists an element in $\mathrm{dom}(R) \cap A$, say $\bar{a}$ (according to Proposition 2.11). It then follows with the definition of the intersection of two sets that $\bar{a} \in \mathrm{dom}(R)$ and $\bar{a} \in A$ are both true. The former finding implies by definition of a domain that there exists an element, say $\bar{b}$, such that $(\bar{a}, \bar{b}) \in R$ holds. Thus, there are a and b satisfying $a \in A$ and $(\bar{a}, \bar{b}) = (a, b)$, which existential sentence implies – together with $(\bar{a}, \bar{b}) \in R$ – that $(\bar{a}, \bar{b}) \in R \restriction A$ holds, using (3.80). Thus, there exists an element in $R \restriction A$, so that this restriction is evidently nonempty, proving the implication in (3.102). Since R and A were arbitrary sets, we may therefore conclude that the proposed universal sentence is true. □

Exercise 3.14. Show for any set X and any binary relation R with domain X that the restriction of R to X is identical with R, that is,

$$\forall X, R\,([R \text{ is a binary relation } \wedge \mathrm{dom}(R) = X] \Rightarrow R \restriction X = R). \quad (3.103)$$

(Hint: Proceed similarly to the proof of Proposition 3.24, applying now the definition of a domain instead of the definition of a Cartesian product.)

Exercise 3.15. Show that

a) the domain of the restriction of any binary relation to any set is included in that set, that is,

$$\forall R, A\,(R \text{ is a binary relation } \Rightarrow \mathrm{dom}(R \restriction A) \subseteq A). \quad (3.104)$$

b) the domain of the restriction of any binary relation to any subset of the domain of the binary relation includes the subset, that is,

$$\forall R, A\,([R \text{ is a binary relation } \wedge A \subseteq \mathrm{dom}(R)] \Rightarrow A \subseteq \mathrm{dom}(R \restriction A)). \quad (3.105)$$

Corollary 3.31. *It is true that the domain of the restriction of any binary relation to any subset of its domain includes the subset, that is,*

$$\forall R, A\,([R \text{ is a binary relation } \wedge A \subseteq \mathrm{dom}(R)] \Rightarrow \mathrm{dom}(R \restriction A) = A). \quad (3.106)$$

Proof. We let R and A be arbitrary sets, assuming R to be a binary relation and assuming now in addition $A \subseteq \mathrm{dom}(R)$ to hold. On the one hand, the assumption that R is a binary relation implies with Exercise 3.15a) the inclusion $\mathrm{dom}(R \restriction A) \subseteq A$. On the other hand, the assumed antecedent implies with Exercise 3.15b) the inclusion $A \subseteq \mathrm{dom}(R \restriction A)$. These two inclusions gives us then with the Axiom of Extension the desired equation $\mathrm{dom}(R \upharpoonright A) = A$. Since R and A were arbitrary, the corollary follows therefore to be true. □

In analogy to the domain, we may obtain the set of all second coordinates of a binary relation.

Theorem 3.32. *The following sentences hold for any binary relation R.*

a) *There exists a unique set* $\mathrm{ran}(R)$ *such that an element b is in* $\mathrm{ran}(R)$ *iff b is in the union of the union of R and moreover if there is an element a such that the ordered pair formed by a and b is in R, i.e.*

$$\exists!\mathrm{ran}(R)\, \forall b\, (b \in \mathrm{ran}(R) \Leftrightarrow [b \in \bigcup \bigcup R \wedge \exists a\, ((a, b) \in R)]). \quad (3.107)$$

b) *The set* $\mathrm{ran}(R)$ *satisfies also*

$$\forall b\, (b \in \mathrm{ran}(R) \Leftrightarrow \exists a\, ((a, b) \in R)). \quad (3.108)$$

Exercise 3.16. Prove Theorem 3.32.
(Hint: Proceed in analogy to the proof of Theorem 3.26.)

Definition 3.9 (Range). For any binary relation R we call the set $\mathrm{ran}(R)$ consisting of the second coordinates of the ordered pairs in R in the sense that

$$\forall b\, (b \in \mathrm{ran}(R) \Leftrightarrow \exists a\, ((a, b) \in R))$$

the *range* of R, and we symbolize this set also by

$$\{b : \exists a\, ((a, b) \in R)\}. \quad (3.109)$$

Exercise 3.17. Show for any a and any b that the range of the binary relation $\{(a, b)\}$ is identical with the singleton formed by b, that is,

$$\forall a, b\, (\mathrm{ran}(\{(a, b)\}) = \{b\}). \quad (3.110)$$

Proposition 3.33. *Any binary relation R is included in the Cartesian product of its domain and its range, i.e.*

$$R \subseteq \mathrm{dom}(R) \times \mathrm{ran}(R). \quad (3.111)$$

Proof. We let R be an arbitrary set and assume that R is a binary relation. We now prove the stated inclusion by verifying the equivalent (using the definition of a subset)

$$\forall Y\, (Y \in R \Rightarrow Y \in \mathrm{dom}(R) \times \mathrm{ran}(R)). \quad (3.112)$$

We let Y be arbitrary and assume $Y \in R$ to be true. Then, by definition of a binary relation, there are elements, say $\bar{a}$ and $\bar{b}$, such that $Y = (\bar{a}, \bar{b})$ holds. Thus, $Y \in R$ gives $(\bar{a}, \bar{b}) \in R$ via substitution. This finding shows firstly that there exists an element b satisfying $(\bar{a}, b) \in R$, so that $\bar{a} \in \mathrm{dom}(R)$ follows to be true by definition of a domain; secondly, we see that there is an element a such that $(a, \bar{b}) \in R$ holds, with the consequence that $\bar{b} \in \mathrm{ran}(R)$ is true by definition of a range. Thus, the conjunction $\bar{a} \in \mathrm{dom}(R) \wedge \bar{b} \in \mathrm{ran}(R)$ holds, which in turn implies $[Y =]\ (\bar{a}, \bar{b}) \in \mathrm{dom}(R) \times \mathrm{ran}(R)$ with the definition of the Cartesian product of two sets. This completes the proof of the implication in (3.112), and since Y is arbitrary, we may therefore conclude that the universal sentence (3.112) holds. By definition of a subset, the inclusion (3.111) is then true. As R was initially arbitrary, we may finally conclude that the proposition holds. □

Proposition 3.34. *If a binary relation R is included in a binary relation S, then the range of R is included in the range of S, that is,*

$$\begin{aligned}&\forall R, S\,([R \text{ is a binary relation } \wedge\ S \text{ is a binary relation}]\\ &\quad \Rightarrow [R \subseteq S \Rightarrow \mathrm{ran}(R) \subseteq \mathrm{ran}(S)]).\end{aligned} \tag{3.113}$$

Proof. We let R and S be arbitrary binary relations and then assume $R \subseteq S$, that is (using the definition of a subset),

$$\forall z\,(z \in R \Rightarrow z \in S), \tag{3.114}$$

and show that this implies $\mathrm{ran}(R) \subseteq \mathrm{ran}(S)$, that is,

$$\forall y\,(y \in \mathrm{ran}(R) \Rightarrow y \in \mathrm{ran}(S)). \tag{3.115}$$

For this purpose, we let y be arbitrary. and assume $y \in \mathrm{ran}(R)$, which implies by definition of a range that there exists an element, say $\bar{x}$, such that $(\bar{x}, y) \in R$ holds. This further implies $(\bar{x}, y) \in S$ with (3.114), so that $y \in \mathrm{ran}(S)$ holds (by definition of a range). As y is arbitrary, we therefore conclude that (3.115) holds, so that $\mathrm{ran}(R) \subseteq \mathrm{ran}(S)$ is true. Since R and S were also arbitrary, we finally conclude that the proposed universal sentence holds, as claimed. □

Exercise 3.18. Show that, if a binary relation R is included in a binary relation S, then the domain of R is included in the domain of S, that is,

$$\begin{aligned}&\forall R, S\,([R \text{ is a binary relation } \wedge\ S \text{ is a binary relation}]\\ &\quad \Rightarrow [R \subseteq S \Rightarrow \mathrm{dom}(R) \subseteq \mathrm{dom}(S)]).\end{aligned} \tag{3.116}$$

Corollary 3.35. *For any binary relation R and any set A it is true that the domain and the range of the restriction of R to A are included in, respectively, the domain and the range of R, i.e.*

$$\forall A\,(\mathrm{dom}(R \upharpoonright A) \subseteq \mathrm{dom}(R) \wedge \mathrm{ran}(R \upharpoonright A) \subseteq \mathrm{ran}(R)). \tag{3.117}$$

Proof. Letting R be an arbitrary binary relation and A an arbitrary set, we see in light of Corollary 3.23 that $R \upharpoonright A$ is a binary relation included in the binary relation R, so that $\mathrm{dom}(R \upharpoonright A) \subseteq \mathrm{dom}(R)$ and $\mathrm{ran}(R \upharpoonright A) \subseteq \mathrm{ran}(R)$ follow to be true with Exercise 3.18 and Proposition 3.34, respectively. Since R and A were arbitrary, we therefore conclude that the proposed sentence holds, as claimed. □

Proposition 3.36. *For any binary relation R it is true that, if the domain of R is empty, then the range of R is also empty, that is,*

$$\forall R\,(R \text{ is a binary relation } \Rightarrow [\mathrm{dom}(R) = \emptyset \Rightarrow \mathrm{ran}(R) = \emptyset]). \tag{3.118}$$

Proof. We let R be an arbitrary binary relation and prove the implication by contradiction, assuming $\mathrm{dom}(R) = \emptyset$ and $\mathrm{ran}(R) \neq \emptyset$ to be both true. As the latter assumption implies with (2.42) that there exists an element in $\mathrm{ran}(R)$, say $\bar{b}$. This further implies by definition of a range that there exists an element, say $\bar{a}$, such that $(\bar{a}, \bar{b}) \in R$ holds. This shows that the existential sentence $\exists b\,((\bar{a}, b) \in R)$ is true, which then implies $\bar{a} \in \mathrm{dom}(R)$ by definition of a range. With this finding, the previously made assumption $\mathrm{dom}(R) = \emptyset$ implies $\bar{a} \in \emptyset$, so that the existential sentence $\exists y\,(y \in \emptyset)$ holds. This however contradicts (2.41), so that the proof of the implication (3.118) is complete. As R was arbitrary, we therefore conclude that the proposition is true. □

Exercise 3.19. Prove for any binary relation R that the domain of R is empty iff the range of R is empty, that is,

$$\forall R\,(R \text{ is a binary relation } \Rightarrow [\mathrm{dom}(R) = \emptyset \Leftrightarrow \mathrm{ran}(R) = \emptyset]). \tag{3.119}$$

(Hint: Prove the second part ('$\Leftarrow$') of the equivalence according to the proof of the first part ('$\Rightarrow$') in (3.118), then apply Definition 1.5.)

Exercise 3.20. Verify the following sentences for any binary relation R.

$$R = \emptyset \;\Leftrightarrow\; \mathrm{dom}(R) = \emptyset, \tag{3.120}$$

$$R = \emptyset \;\Leftrightarrow\; \mathrm{ran}(R) = \emptyset. \tag{3.121}$$

(Hint: Regarding (3.120), apply Method 1.11 in connection with (2.42) and the Definitions 3.8 & 3.5. Regarding (3.121), apply (3.119).)

It is also possible to combine two given binary relations in the following sense.

Theorem 3.37. *The following sentences are true for any binary relations Q and R.*

a) There exists a unique set $R \circ Q$ which contains precisely every ordered pair in the Cartesian product of the domain of Q and the range of R such that its first coordinate coincides with the first coordinate of some ordered pair in Q and its second coordinate coincides with the second coordinate of some ordered pair in R, where the second coordinate of the ordered pair in Q and the first coordinate of the ordered pair in R are identical, that is,

$$\exists! R \circ Q \, \forall Y \, (Y \in R \circ Q \Leftrightarrow [Y \in \mathrm{dom}(Q) \times \mathrm{ran}(R) \\ \wedge \, \exists a, b, c \, ((a,b) \in Q \, \wedge \, (b,c) \in R \, \wedge \, (a,c) = Y)]). \tag{3.122}$$

b) The set $R \circ Q$ satisfies

$$R \circ Q \neq \emptyset \Leftrightarrow \exists a, b, c \, ((a,b) \in Q \, \wedge \, (b,c) \in R). \tag{3.123}$$

c) The set $R \circ Q$ satisfies also

$$\forall Y \, (Y \in R \circ Q \Leftrightarrow \exists a, b, c \, ((a,b) \in Q \, \wedge \, (b,c) \in R \, \wedge \, (a,c) = Y)). \tag{3.124}$$

Proof. Concerning a), letting Q and R be arbitrary binary relations, we we obtain with the Axiom of Specification

$$\exists R \circ Q \, \forall Y \, (Y \in R \circ Q \Leftrightarrow [Y \in \mathrm{dom}(Q) \times \mathrm{ran}(R) \\ \wedge \, \exists a, b, c \, ((a,b) \in Q \, \wedge \, (b,c) \in R \, \wedge \, (a,c) = Y)]).$$

The fact that $R \circ Q$ is unique follows by means of the Equality Criterion for sets in the usual way. Thus, the set $R \circ Q$ satisfies

$$\forall Y \, (Y \in R \circ Q \Leftrightarrow [Y \in \mathrm{dom}(Q) \times \mathrm{ran}(R) \\ \wedge \, \exists a, b, c \, ((a,b) \in Q \, \wedge \, (b,c) \in R \, \wedge \, (a,c) = Y)]). \tag{3.125}$$

Concerning b), we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $R \circ Q \neq \emptyset$ to hold, so that there evidently exists an element in $R \circ Q$, say $\bar{Y}$. Then, $\bar{Y} \in R \circ Q$ implies with (3.125) in particular that there are elements, say $\bar{a}$, $\bar{b}$ and $\bar{c}$, satisfying the (multiple) conjunction of $(\bar{a}, \bar{b}) \in Q$, $(\bar{b}, \bar{c}) \in R$ and $(\bar{a}, \bar{c}) = \bar{Y}$. The first two parts of this multiple conjunction show that the existential sentence in (3.123) holds, so that the proof of the first part of the equivalence (3.123) is complete.

To prove the second part ('$\Leftarrow$'), we now assume that Q and R are such that there exist elements, say $\bar{\bar{a}}$, $\bar{\bar{b}}$ and $\bar{\bar{c}}$, with $(\bar{\bar{a}}, \bar{\bar{b}}) \in Q$ and $(\bar{\bar{b}}, \bar{\bar{c}}) \in R$. Then, the ordered pair $\bar{\bar{Y}} = (\bar{\bar{a}}, \bar{\bar{c}})$ is uniquely specified, so that the existential sentence

$$\exists a, b, c\,((a,b) \in Q \;\wedge\; (b,c) \in R \;\wedge\; (a,c) = \bar{Y}) \tag{3.126}$$

holds. Furthermore, $(\bar{\bar{a}}, \bar{\bar{b}}) \in Q$ shows that there is an element b with $(\bar{\bar{a}}, b) \in Q$, so that $\bar{\bar{a}} \in \mathrm{dom}(Q)$ holds by definition of a domain. Moreover, $(\bar{\bar{b}}, \bar{\bar{c}}) \in R$ shows that there is an element b with $(b, \bar{\bar{c}}) \in R$, and therefore $\bar{\bar{c}} \in \mathrm{ran}(R)$ follows to be true by definition of a range. The conjunction of $\bar{\bar{a}} \in \mathrm{dom}(Q)$ and $\bar{\bar{c}} \in \mathrm{ran}(R)$ then implies

$$[(\bar{\bar{a}}, \bar{\bar{c}}) =] \quad \bar{Y} \in \mathrm{dom}(Q) \times \mathrm{ran}(R) \tag{3.127}$$

with the definition of the Cartesian product of two sets. Now, the conjunction of (3.127) and (3.126) further implies with (3.125) $\bar{Y} \in R \circ Q$, which shows that there exists an element in $R \circ Q$. Thus, the set $R \circ Q$ is clearly nonempty, and the proof of the equivalence (3.123) is therefore complete.

As Q and R were arbitrary, we may conclude that a) and b) are true universally. $\square$

Exercise 3.21. Prove Part c) of Theorem 3.37.

(Hint: Proceed similarly as in the proof of Theorem 3.7.)

Definition 3.10 (Composition). For any binary relations Q and R we call the set $R \circ Q$, characterized by

$$\forall Y\,(Y \in R \circ Q \Leftrightarrow \exists a, b, c\,((a,b) \in Q \;\wedge\; (b,c) \in R \;\wedge\; (a,c) = Y)),$$

the *composition* of R and Q.

Proposition 3.38. *The composition of a binary relations Q and a binary relation R is empty if Q or R is empty, that is,*

$$[Q = \emptyset \;\vee\; R = \emptyset] \;\Rightarrow\; R \circ Q = \emptyset. \tag{3.128}$$

Proof. We let Q and R be arbitrary binary relations, and we prove the implication by contraposition, assuming $R \circ Q \neq \emptyset$ to be true. According to (2.42), the composition then has some element, say $\bar{Y} \in R \circ Q$. Due to (3.124), there also exist constant, say $\bar{a}, \bar{b}, \bar{c}$, such that $(\bar{a}, \bar{b}) \in Q$ and $(\bar{b}, \bar{c}) \in R$. Consequently, the sets Q and R both are nonempty. The truth of the conjunction $Q \neq \emptyset \wedge R \neq \emptyset$ implies now the truth of the negation $\neg(Q = \emptyset \vee R = \emptyset)$ by virtue of De Morgan's Law for the disjunction. This finding completes the proof of the implication, and since Q and R were initially arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Corollary 3.39. *The composition of a binary relation R and the empty set yields the empty set, that is,*

$$R \circ \emptyset = \emptyset \;\wedge\; \emptyset \circ R = \emptyset. \tag{3.129}$$

Proof. Letting R be an arbitrary binary relation, the evident truth of $\emptyset = \emptyset$ implies the truth of the disjunctions $\emptyset = \emptyset \vee R = \emptyset$ and $R = \emptyset \vee \emptyset = \emptyset$, which in turn imply $R \circ \emptyset = \emptyset$ and $\emptyset \circ R = \emptyset$ with (3.128). As R was arbitrary, we therefore conclude that the corollary holds. □

Proposition 3.40. *For any binary relations Q and R such that the range of Q is included in the domain of R, it is true that the domain of the composition $R \circ Q$ is identical with the domain of Q, that is,*

$$\begin{aligned}&\forall Q, R\,([Q, R \text{ are binary relations} \wedge \operatorname{ran}(Q) \subseteq \operatorname{dom}(R)]\\ &\quad \Rightarrow \operatorname{dom}(R \circ Q) = \operatorname{dom}(Q)).\end{aligned} \tag{3.130}$$

Proof. We let Q and R be arbitrary binary relations with $\operatorname{ran}(Q) \subseteq \operatorname{dom}(R)$, so that

$$\forall b\,(b \in \operatorname{ran}(Q) \Rightarrow b \in \operatorname{dom}(R)). \tag{3.131}$$

holds by definition of a subset. To prove $\operatorname{dom}(R \circ Q) = \operatorname{dom}(Q)$, we verify

$$\forall a\,(a \in \operatorname{dom}(R \circ Q) \Leftrightarrow a \in \operatorname{dom}(Q)), \tag{3.132}$$

letting $\bar{a}$ be arbitrary. To show that the first implication ($\Rightarrow$) holds, we assume $\bar{a} \in \operatorname{dom}(R \circ Q)$. Then, by definition of a domain, there exists an element, say $\bar{c}$, such that $(\bar{a}, \bar{c}) \in R \circ Q$ holds. This implies with the definition of a composition in particular $(\bar{a}, \bar{c}) \in \operatorname{dom}(Q) \times \operatorname{ran}(R)$, which further implies $\bar{a} \in \operatorname{dom}(Q)$ by definition of the Cartesian product of two sets, as desired.

We now show that the second implication ($\Leftarrow$) also holds, which we do directly by assuming $\bar{a} \in \operatorname{dom}(Q)$ to be true. Then, by definition of a domain, there exists an element, say $\bar{b}$, such that $(\bar{a}, \bar{b}) \in Q$ is true. This in turn implies $\bar{b} \in \operatorname{ran}(Q)$ by definition of a range and then $\bar{b} \in \operatorname{dom}(R)$ with (3.131). By definition of a domain, there exists an element, say $\bar{c}$, such that $(\bar{b}, \bar{c}) \in R$ holds. Let us now define the ordered pair $Y = (\bar{a}, \bar{c})$ Then, $(\bar{a}, \bar{b}) \in Q$, $(\bar{b}, \bar{c}) \in R$ and $(\bar{a}, \bar{c}) = Y$ show that there exist a, b, c such that

$(a,b) \in Q$, $(b,c) \in R$ and $(a,c) = Y$ hold. This existential sentence implies $Y \in R \circ Q$ with (3.124), and therefore $(\bar{a}, \bar{c}) \in R \circ Q$. Applying now the definition of a domain, we obtain the desired $\bar{a} \in \mathrm{dom}(R \circ Q)$.

Thus, the proof of the equivalence in (3.132) is complete. Since $\bar{a}$ is arbitrary, we therefore conclude that (3.132) holds, so that the sets $\mathrm{dom}(R \circ Q)$ and $\mathrm{dom}(Q)$ follow to be identical with the Equality Criterion for sets. As Q and R were also arbitrary, we further conclude that the proposition (3.130) is true. □

Exercise 3.22. Show for any binary relations Q and R such that the range of Q is included in the domain of R that the range of the composition $R \circ Q$ is included in the range of R, that is,

$$\begin{aligned} &\forall Q, R\,([Q, R \text{ are binary relations} \wedge \mathrm{ran}(Q) \subseteq \mathrm{dom}(R)] \\ &\quad \Rightarrow \mathrm{ran}(R \circ Q) \subseteq \mathrm{ran}(R)). \end{aligned} \tag{3.133}$$

(Hint: Proceed in analogy to the proof of the first part of the equivalence in (3.132), using the definition of a range instead of the definition of a domain.)

We also obtain a new binary relation by interchanging the coordinates of the ordered pairs in a given binary relation.

Theorem 3.41. *The following sentences hold for any binary relation R.*

a) There exists a unique set R^{-1} such that an element Y is in R^{-1} iff Y is in the Cartesian product of the range of R and the domain of R, and moreover if there is an ordered pair in R whose first coordinate is the second coordinate of Y and whose second coordinate is the first coordinate of Y, that is,

$$\begin{aligned} &\exists! R^{-1}\, \forall Y\, (Y \in R^{-1} \\ &\quad \Leftrightarrow [Y \in \mathrm{ran}(R) \times \mathrm{dom}(R) \wedge \exists a, b\,((a,b) \in R \wedge (b,a) = Y])). \end{aligned} \tag{3.134}$$

b) The set R^{-1} is nonempty iff R is nonempty, that is,

$$R^{-1} \neq \emptyset \Leftrightarrow R \neq \emptyset. \tag{3.135}$$

c) The set R^{-1} satisfies also

$$\forall Y\, (Y \in R^{-1} \Leftrightarrow \exists a, b\,((a,b) \in R \wedge (b,a) = Y)). \tag{3.136}$$

Proof. We let R be an arbitrary binary relation. Concerning a), we may use the Axiom of Specification to obtain the true existential sentence

$$\exists R^{-1}\,\forall Y\,(Y \in R^{-1} \\ \Leftrightarrow [Y \in \mathrm{ran}(R) \times \mathrm{dom}(R) \,\wedge\, \exists a,b\,((a,b) \in R \,\wedge\, (b,a) = Y])),$$

and we may apply then the Equality Criterion for sets to establish the unique existence of R^{-1} in the usual way. This set thusly satisfies

$$\forall Y\,(Y \in R^{-1} \Leftrightarrow [Y \in \mathrm{ran}(R) \times \mathrm{dom}(R) \,\wedge\, \exists a,b\,((a,b) \in R \,\wedge\, (b,a) = Y])). \tag{3.137}$$

Concerning b), we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $R^{-1} \neq \emptyset$ holds. Clearly, there exists then an element in R^{-1}, say $\bar{Y}$. Consequently, $\bar{Y} \in R^{-1}$ implies with (3.137) in particular that there are elements, say $\bar{a}$ and $\bar{b}$, such that $(\bar{a},\bar{b}) \in R$ and $(\bar{b},\bar{a}) = \bar{Y}$ hold. Here, $(\bar{a},\bar{b}) \in R$ clearly shows that there exists an element in R, so that R is nonempty. Thus, the first part of the equivalence in (3.135) is true.

To prove the second part, we now assume R to be nonempty, so that there is an element in R, say $\bar{Z}$. This implies with the definition of a binary relation that there exist elements, say $\bar{\bar{a}}$ and $\bar{\bar{b}}$, with $\bar{Z} = (\bar{\bar{a}},\bar{\bar{b}})$. Then, $\bar{Z} \in R$ yields $(\bar{\bar{a}},\bar{\bar{b}}) \in R$. Thus, there is, on the one hand, an element b with $(\bar{\bar{a}},b) \in R)$, so that $\bar{\bar{a}} \in \mathrm{dom}(R)$ holds by definition of a domain. On the other hand, we see that there is an element a such that $(a,\bar{\bar{b}}) \in R$ holds, which implies $\bar{\bar{b}} \in \mathrm{ran}(R)$ by definition of a range. The previous two findings give with (3.28)

$$(\bar{\bar{b}},\bar{\bar{a}}) \in \mathrm{ran}(R) \times \mathrm{dom}(R). \tag{3.138}$$

Denoting $Y = (\bar{\bar{b}},\bar{\bar{a}})$, we thus showed that $Y \in \mathrm{ran}(R) \times \mathrm{dom}(R)$ holds and that there are elements a,b with $(a,b) \in R$ as well as $(b,a) = Y$. Consequently, Y is an element of R^{-1} because of (3.137), and therefore the set R^{-1} is nonempty. Thus, the proof of b) is complete.

Concerning c), we let Y be arbitrary, and we prove the first part ('$\Rightarrow$') of the equivalence in (3.136) directly, assuming $Y \in R^{-1}$ to be true. Then, this gives in particular the existential sentence in (3.136) to be proven. To prove the second part ('$\Leftarrow$') directly, we assume the truth of the existential sentence

$$\exists a,b\,((a,b) \in R \,\wedge\, (b,a) = Y), \tag{3.139}$$

i.e., that there exist elements, say $\bar{a}$ and $\bar{b}$ such that $(\bar{a},\bar{b}) \in R$ and $(\bar{b},\bar{a}) = Y$ hold. Here, $(\bar{a},\bar{b}) \in R$ implies on the one hand $\bar{a} \in \mathrm{dom}(R)$ by definition of a domain, and on the other hand $\bar{b} \in \mathrm{ran}(R)$ by definition of a range. Consequently,

$$[(\bar{b},\bar{a}) =] \quad Y \in \mathrm{ran}(R) \times \mathrm{dom}(R)$$

follows to be true with (3.28). Together with the assumed existential sentence, this further implies with (3.137) $Y \in R^{-1}$, as desired. This completes the proof of the equivalence, and since Y is arbitrary, we may therefore conclude that c) holds, too.

Finally, as R was arbitrary, the theorem follows to be true. □

Definition 3.11 (Inverse of a binary relation). For any binary relation R we call the set R^{-1} containing precisely any ordered pair whose first coordinate is the second coordinate and whose second coordinate is the first coordinate of some ordered pair in R, in the sense that

$$\forall Y\,(Y \in R^{-1} \Leftrightarrow \exists a, b\,((a,b) \in R \wedge (b,a) = Y)),$$

the *inverse* of R. We symbolize this set also by

$$\{(b,a) : (a,b) \in R\}. \tag{3.140}$$

Corollary 3.42. *For any binary relation R it is true that the inverse of R is a subset of the Cartesian product of the range and the domain of R, i.e.*

$$R^{-1} \subseteq \mathrm{ran}(R) \times \mathrm{dom}(R). \tag{3.141}$$

Proof. Letting R be an arbitrary binary relation, we see in light of (3.137) that $Y \in R^{-1}$ implies in particular $Y \in \mathrm{ran}(R) \times \mathrm{dom}(R)$ for any Y, so that (3.141) holds by definition of a subset. As R is arbitrary, this inclusion is then true for any R. □

Proposition 3.43. *For any binary relation R, it is true that the ordered pair formed by some b and some a is element of the inverse of R iff the ordered pair formed by a and b is in R, i.e.*

$$\forall a, b\,((b,a) \in R^{-1} \Leftrightarrow (a,b) \in R). \tag{3.142}$$

Proof. We let R be an arbitrary set, assume that R is a binary relation, and let $\bar{a}$ as well as $\bar{b}$ be arbitrary. To prove the first part ('⇒') of the equivalence, we assume $(\bar{b}, \bar{a}) \in R^{-1}$ to be true, which implies with the definition of the inverse of a binary relation that there exist elements, say $\bar{\bar{a}}$ and $\bar{\bar{b}}$, such that $(\bar{\bar{a}}, \bar{\bar{b}}) \in R$ and $(\bar{\bar{b}}, \bar{\bar{a}}) = (\bar{b}, \bar{a})$ are true. This equation implies with the Equality Criterion for ordered pairs that the equations $\bar{\bar{b}} = \bar{b}$ and $\bar{\bar{a}} = \bar{a}$ hold, so that the previously established $(\bar{\bar{a}}, \bar{\bar{b}}) \in R$ yields $(\bar{a}, \bar{b}) \in R$ via substitution. Thus, the first part of the equivalence in (3.142) is true. To prove the second part ('⇐'), we now assume $(\bar{a}, \bar{b}) \in R$ to be true. Since the ordered pair $(\bar{b}, \bar{a})$ is also specified by $\bar{a}$ and $\bar{b}$, we thus see that there are elements a and b such that the conjunction $(a,b) \in R \wedge (b,a) = (\bar{b}, \bar{a})$ holds.

This existential sentence then implies (with the definition of the inverse of a binary relation) that $(\bar{b}, \bar{a}) \in R^{-1}$ is true, which completes the proof of the equivalence. Since $\bar{a}$ and $\bar{b}$ were arbitrary, we may therefore conclude that the universal sentence (3.142) holds. As R was initially also arbitrary, we may then infer from this the truth of the proposed sentence. □

Theorem 3.44 (Characterization of the domain of the inverse of a binary relation). *The domain of the inverse of a binary relation R is identical with the domain of R, that is,*

$$\forall R\,(R \text{ is a binary relation } \Rightarrow \operatorname{dom}(R^{-1}) = \operatorname{ran}(R)). \tag{3.143}$$

Proof. We let R be arbitrary and assume that R is a binary relation. Now, we apply the Equality Criterion for sets to prove the proposed equation, by verifying equivalently

$$\forall b\,(b \in \operatorname{dom}(R^{-1}) \Leftrightarrow b \in \operatorname{ran}(R)). \tag{3.144}$$

We let $\bar{b}$ be arbitrary and assume first $\bar{b} \in \operatorname{dom}(R^{-1})$ to be true. By definition of a domain, there then exists an element, say $\bar{a}$, such that $(\bar{b}, \bar{a}) \in R^{-1}$ holds. This implies with Proposition 3.43 the truth of $(\bar{a}, \bar{b}) \in R$, which shows that there is an element a satisfying $(a, \bar{b}) \in R$, so that $\bar{b} \in \operatorname{ran}(R)$ follows to be true by definition of a range. Conversely, we now assume $\bar{b} \in \operatorname{ran}(R)$ to be true, so that there exists (by definition of a range) an element, say $\bar{a}$, with $(\bar{a}, \bar{b}) \in R$. Then, Proposition 3.43 gives $(\bar{b}, \bar{a}) \in R^{-1}$, which shows that there is an element a which satisfies $(\bar{b}, a) \in R^{-1}$, so that $\bar{b} \in \operatorname{dom}(R^{-1})$ holds (by definition of a domain). Thus, the equivalence in (3.144) holds, and since $\bar{b}$ is arbitrary, we may therefore conclude that the (3.144) is true. This universal sentence in turn implies the equation $\operatorname{dom}(R^{-1}) = \operatorname{ran}(R)$ (with the Equality Criterion for sets), and as R was arbitrary, we may then further conclude that the proposed universal sentence (3.143) is true. □

3.3. Ordered Sets

3.3.1. Basic characteristics of partially, totally and linearly ordered sets

In this section we will investigate binary relations which satisfy certain combinations of the definite properties listed in the subsequent definition.

Definition 3.12. For any set X we say that a binary relation R on X is

(1) *reflexive* iff
$$\forall a\,(a \in X \Rightarrow a\,R\,a). \tag{3.145}$$

(2) *irreflexive* iff
$$\forall a\,(a \in X \Rightarrow \neg a\,R\,a), \tag{3.146}$$

(3) *symmetric* iff
$$\forall a, b\,(a, b \in X \Rightarrow [a\,R\,b \Rightarrow b\,R\,a]). \tag{3.147}$$

(4) *antisymmetric* iff
$$\forall a, b\,(a, b \in X \Rightarrow [(a\,R\,b \,\wedge\, b\,R\,a) \Rightarrow a = b]). \tag{3.148}$$

(5) *transitive* iff
$$\forall a, b, c\,(a, b, c \in X \Rightarrow [(a\,R\,b \,\wedge\, b\,R\,c) \Rightarrow a\,R\,c]). \tag{3.149}$$

(6) *total* iff
$$\forall a, b\,(a, b \in X \Rightarrow [a\,R\,b \,\vee\, b\,R\,a]). \tag{3.150}$$

(7) *connex* iff
$$\forall a, b\,(a, b \in X \Rightarrow [a\,R\,b \,\vee\, b\,R\,a \,\vee\, a = b]), \tag{3.151}$$

(8) *comparable* or *trichotomous* iff
$$\begin{aligned}\forall a, b\,(a, b \in X \Rightarrow [&(a\,R\,b \,\vee\, b\,R\,a \,\vee\, a = b) \\ &\wedge \neg(a\,R\,b \,\wedge\, b\,R\,a) \,\wedge\, \neg(a\,R\,b \,\wedge\, a = b) \,\wedge\, \neg(b\,R\,a \,\wedge\, a = b)])\end{aligned} \tag{3.152}$$

Corollary 3.45. *The empty binary relation $R = \emptyset$ on $X = \emptyset$ is reflexive, irreflexive, symmetric, antisymmetric, transitive, total, connex, comparable.*

Proof. In view of Exercise 3.8, any binary relation on $X = \emptyset$ is itself the empty set. Furthermore, letting a, b and c be arbitrary, we see that the antecedents $a \in X$ regarding Property (1) and (2), the antecedents $a, b \in X$ with respect to the Properties (3), (4), (6), (7) and (8), and the antecedent $a, b, c \in X$ concerning Property (5) are all false by definition of the empty set. Therefore, the corresponding implications are all true, so that the Properties (1) – (8) follow to be true for $R = \emptyset$ on $X = \emptyset$. $\square$

Corollary 3.46. *The empty binary relation $R = \emptyset$ on a nonempty set X is irreflexive (and not reflexive), symmetric, antisymmetric, transitive, and not total.*

Proof. We let X be an arbitrary nonempty set, $R = \emptyset$ be the empty binary relation on X, and we let in the following a, b and c be arbitrary elements of X. Since R is empty, we have that $(a, a) \in R$, $(a, b) \in R$, $(b, a) \in R$ and $(b, c) \in R$ are all false; thus, $a \, R \, a$, $a \, R \, b$, $b \, R \, a$ and $b \, R \, c$ are also false.

Since $a \in X$ implies the false $a \, R \, a$, we see that R is not reflexive according to (3.145). Then, $\neg a \, R \, a$ is true, so that R is indeed irreflexive according to (3.146). Next, since $a \, R \, b$ is false, the implication in (3.147) based on this antecedent is true, and therefore R is symmetric. Moreover, as the conjunction of $a \, R \, b$ and $b \, R \, a$ is false, the implication in (3.148) based on this false antecedent is true, so that R follows to be also antisymmetric. Similarly, because the conjunction of $a \, R \, b$ and $b \, R \, c$ is false, the implication in (3.149) with this false antecedent holds, and R follows therefore to be transitive. Furthermore, the previously mentioned fact that $a \, R \, b$ and $b \, R \, a$ are both false implies that the disjunction $a \, R \, b \vee b \, R \, a$ in (3.150) is also false, so that R is not total. $\square$

Exercise 3.23. Show that any binary relation R on any singleton $\{x\}$ is connex.

(Hint: Establish the disjunction in (3.151) by using (2.169).)

Exercise 3.24. Show for any x that the empty binary relation $R = \emptyset$ on the singleton $\{x\}$ is comparable.

(Hint: Use Exercise 3.23.)

Proposition 3.47. *For any x it is true that the singleton $R = \{(x, x)\}$ is a reflexive, symmetric, antisymmetric, transitive, total and connex binary relation on the singleton $X = \{x\}$.*

Proof. We let x be arbitrary and observe in light of Proposition 3.21 and the definition of a singleton that $R = \{(x, x)\}$ is a binary relation $\{x, x\} =$

$\{x\}$. Next, we verify the claimed properties of R and let for this purpose $a, b, c \in \{x\}$ be arbitrary, which assumption implies the equations

$$a = x,\ b = x,\ c = x \tag{3.153}$$

with (2.169). The latter argument also gives $(x, x) \in R\ [= \{(x, x)\}]$, so that substitutions based on the equations (3.153) yield

$$(a, a), (a, b), (a, c), (b, a), (b, c) \in R,$$

which we may also write as

$$a\,R\,a,\ a\,R\,b,\ a\,R\,c,\ b\,R\,a,\ b\,R\,c. \tag{3.154}$$

Here, the truth of $a\,R\,a$ shows that $R = \{(x, x)\}$ satisfies the implication in (3.145), and therefore we may conclude that R is reflexive (since a is arbitrary). In addition, we notice that the equations (3.153) imply

$$a = b. \tag{3.155}$$

Let us now observe that the truth of (3.154) and of (3.155) immediately causes the implications

$$\begin{aligned} a\,R\,b &\Rightarrow b\,R\,a, \\ (a\,R\,b \wedge b\,R\,a) &\Rightarrow a = b, \\ (a\,R\,b \wedge b\,R\,c) &\Rightarrow a\,R\,c, \end{aligned}$$

in (3.147) – (3.149) to be true themselves; since a, b and c are arbitrary, we may therefore conclude that R is symmetric, antisymmetric, and transitive. Clearly, the truth of (3.154) causes also the truth of the disjunction $a\,R\,b \vee b\,R\,a$ in (3.150), from which we may infer that R is total (since a and b are arbitrary). In view of Exercise 3.23, R is also connex. As x was initially arbitrary, we may finally conclude that the proposed universal sentence holds. □

Proposition 3.48. *For any x and y it is true that the singleton $R = \{(x, y)\}$ is a transitive binary relation on the pair $\{x, y\}$.*

Proof. Letting x and y be arbitrary, we have due to Proposition 3.21 that $R = \{(x, y)\}$ is a binary relation on $\{x, y\}$. To prove that R is transitive, we let a, b and c be arbitrary, assume $a, b, c \in \{x, y\}$ to be true, and assume furthermore that $a\,R\,b$ and $b\,R\,c$ are both true. Since we may write the latter assumptions also as $(a, b) \in \{(x, y)\}$ and $(b, c) \in \{(x, y)\}$, we obtain the equations $(a, b) = (x, y)$ and $(b, c) = (x, y)$ with (2.169). Then, combining these equations clearly gives $(a, b) = (b, c)$ and therefore $b = c$ with the Equality Criterion for ordered pairs. With this equation, the assumed sentence $a\,R\,b$ yields $a\,R\,c$ via substitution. Since a, b and c are arbitrary, we may therefore conclude that R satisfies (3.149), so that $R = \{(x, y)\}$ is a transitive binary relation on the pair $\{x, y\}$. As x and y were also arbitrary, we may now further conclude that the proposition is true. $\square$

Theorem 3.49 (Characterization of comparability). *For any set X and any binary relation R on X it is true that R is comparable iff precisely one of the three sentences $a\,R\,b$, $b\,R\,a$ and $a = b$ is true for any elements a and b in X, i.e. iff*

$$\begin{aligned}\forall a, b\,(a, b \in X \Rightarrow [(a\,R\,b \,\wedge\, [\neg b\,R\,a] \,\wedge\, a \neq b) \\ \vee ([\neg a\,R\,b] \,\wedge\, b\,R\,a \,\wedge\, a \neq b) \\ \vee ([\neg a\,R\,b] \,\wedge\, [\neg b\,R\,a] \,\wedge\, a = b)]).\end{aligned} \tag{3.156}$$

Proof. Let us take arbitrary sets X and R, and let us assume that R is a binary relation on X. To prove the first part ('$\Rightarrow$') of the stated equivalence, we further assume that R is comparable, so that R satisfies the universal sentence (3.152). To show that this implies (3.156), we let a and b be arbitrary, and we assume $a, b \in X$ to hold. This assumption implies with (3.152) on the one hand the truth in particular of

$$\neg(a\,R\,b \,\wedge\, b\,R\,a), \tag{3.157}$$
$$\neg(a\,R\,b \,\wedge\, a = b), \tag{3.158}$$
$$\neg(b\,R\,a \,\wedge\, a = b), \tag{3.159}$$

which negations in turn imply with De Morgan's Law for sentences (1.51)

$$\neg a\,R\,b \,\vee\, \neg b\,R\,a, \tag{3.160}$$
$$\neg a\,R\,b \,\vee\, a \neq b, \tag{3.161}$$
$$\neg b\,R\,a \,\vee\, a \neq b. \tag{3.162}$$

On the other hand, we obtain from (3.152) the true disjunction

$$a\,R\,b \,\vee\, b\,R\,a \,\vee\, a = b. \tag{3.163}$$

In case the first part $a\,R\,b$ is true, we see immediately that the first part of the multiple conjunction

$$a\,R\,b \,\wedge\, [\neg b\,R\,a] \,\wedge\, a \neq b \tag{3.164}$$

in (3.156) holds. Furthermore, since the case assumption implies that $\neg a\,R\,b$ is false, it follows from the truth of the disjunctions (3.160) and (3.161) that their respective second parts $\neg b\,R\,a$ and $a \neq b$ are true, so that the second and third part of the multiple conjunction (3.164) also hold. Then, the truth of (3.164) clearly implies the truth of the multiple disjunction in (3.156) to be proven.

In case the second part $b\,R\,a$ of the disjunction (3.163) is true, this means that the second part of the multiple conjunction

$$[\neg a\,R\,b] \wedge b\,R\,a \wedge a \neq b \tag{3.165}$$

holds. Since $\neg b\,R\,a$ is now false, we also obtain from the true disjunctions (3.160) and (3.162) the true negations $\neg a\,R\,b$ and $a \neq b$, respectively, so that the multiple conjunction (3.165) holds. Then, the multiple disjunction in (3.156) is again true.

In the final case that the third part $a = b$ in (3.163) is true, we now see that the third part of the conjunction

$$[\neg a\,R\,b] \wedge [\neg b\,R\,a] \wedge a = b \tag{3.166}$$

holds. Moreover, the falsehood of $a \neq b$ yields in view of the true disjunctions (3.161) and (3.162) the true negations $\neg a\,R\,b$ and $\neg b\,R\,a$. Thus, (3.166) holds, so that the multiple disjunction in (3.156) is true also in this case. Since we showed that the consequent of the implication in (3.156) holds in any case, and since a and b are arbitrary, we may infer from this finding the truth of the universal sentence (3.156), completing the proof of the first part of the proposed equivalence.

To prove the second part ('$\Leftarrow$'), we now assume (3.156) to be true and show that R satisfies (3.152). Letting a and b be arbitrary and assuming $a, b \in X$ to be true, it now follows with the preceding assumption that the multiple disjunction

$$\begin{aligned} &(a\,R\,b \wedge [\neg b\,R\,a] \wedge a \neq b) \\ \vee\, &([\neg a\,R\,b] \wedge b\,R\,a \wedge a \neq b) \\ \vee\, &([\neg a\,R\,b] \wedge [\neg b\,R\,a] \wedge a = b) \end{aligned} \tag{3.167}$$

is true. If its first part (3.167) holds, then $a\,R\,b$ is in particular true, and therefore the multiple disjunction (3.163) is also satisfied by R. Moreover, (3.167) implies in particular $\neg b\,R\,a$ as well as $a \neq b$, so that the disjunctions (3.160) – (3.162) are clearly all satisfied by R. These further imply (3.157) – (3.159) with De Morgan's Law for sentences (1.51); together with the previously established (3.163), this shows that the multiple conjunction in (3.152) is thus satisfied by R in the first case.

Next, if the second part of the disjunction (3.167) holds, then $b\,R\,a$ is now in particular true, with the consequence that the multiple disjunction

(3.163) is again satisfied by R. Since we also obtain the true negations $\neg a\,R\,b$ and $a \neq b$, the disjunctions (3.160) – (3.162) are evidently all true, as in the first case. Consequently, they give again the true negations (3.157) – (3.159), besides the true (3.163). Thus, the multiple conjunction in (3.152) is satisfied also in the second case.

Finally, if the third part in (3.167) holds, then we have in particular the true equation $a = b$, and the disjunction (3.163) holds then as well. In addition, (3.167) yields the true negations $\neg a\,R\,b$ and $\neg b\,R\,a$, and we see once again that the disjunctions (3.160) – (3.162) are true. In analogy to the first and the second case, we may therefore establish the truth of the multiple conjunction in (3.152), which thus holds in any case.

Because a and b are arbitrary, we may therefore conclude that R satisfies (3.152), so that R is comparable by definition. Thus, the proof of the proposed equivalence is complete, and since X and R were initially arbitrary sets, we may now further conclude that the stated theorem is true. □

Proposition 3.50. *For any x and y satisfying $x \neq y$ it is true that the singleton $R = \{(x, y)\}$ is a comparable binary relation on $\{x, y\}$.*

Proof. We take arbitrary x and y such that $x \neq y$ holds and recall from Proposition 3.50 that $R = \{(x, y)\}$ is a (transitive) binary relation on $\{x, y\}$. It will be useful for us to establish first the following four sentences:

$$x\,R\,y, \tag{3.168}$$

$$\neg x\,R\,x, \tag{3.169}$$

$$\neg y\,R\,y, \tag{3.170}$$

$$\neg y\,R\,x. \tag{3.171}$$

Because $(x, y) \in \{(x, y)\}$ $[= R]$ holds according to (2.153), we obtain $(x, y) \in R$, which we may also write as (3.168). We may prove the remaining three sentences by contradiction. Assuming $x\,R\,x$ to be true, we may write this as $(x, x) \in \{(x, y)\}$ $[= R]$, which yields $(x, x) = (x, y)$ with (2.169) and therefore especially $x = y$ with the Equality Criterion for ordered pairs, in contradiction to the assumption $x \neq y$. Similarly, the assumption $y\,R\,y$ may be stated as $(y, y) \in \{(x, y)\}$ $[= R]$, which gives $(y, y) = (x, y)$ (again with (2.169)) and consequently in particular $y = x$ (using again the Equality Criterion for ordered pairs), so that we obtain the same contradiction as before. Finally, the assumption $y\,R\,x$ means $(y, x) \in \{(x, y)\}$ $[= R]$, which implies $(y, x) = (x, y)$ and then $x = y$ (using the same arguments as before), once again in contradiction to the true negation $x \neq y$.

We now apply the Characterization of comparability, letting a and b be arbitrary and assuming $a, b \in \{x, y\}$ to be true. Here, $a \in \{x, y\}$ implies by definition of a pair the true disjunction $a = x \vee a = y$, which we use to prove the multiple disjunction in (3.156) for $R = \{(x, y)\}$ by cases. In the first case of $a = x$, we observe that $b \in \{x, y\}$ yields the the true disjunction $b = x \vee b = y$, which we employ for a proof of the desired sentence by cases within the current first case. On the one hand, if $b = x$ holds (besides the current case assumption $a = x$), we obtain via substitution $a = b$ and from (3.169) the $\neg a\, R\, b$ as well as $\neg b\, R\, a$, so that the multiple conjunction

$$\neg a\, R\, b \wedge \neg b\, R\, a \wedge a = b \tag{3.172}$$

holds. Then, the multiple disjunction in (3.156) is evidently also true. On the other hand, if $b = y$ holds (alongside $a = x$), then the initial assumption $x \neq y$ yields $a \neq b$ via substitution, the sentence (3.168) gives $a\, R\, b$, and (3.171) implies $\neg b\, R\, a$. Thus, we now have the true multiple conjunction

$$a\, R\, b \wedge \neg b\, R\, a \wedge a \neq b,$$

so that the multiple disjunction in (3.156) follows again to be true. We now consider the second case $a = y$ and then again the two (sub)cases $b = x$ and $b = y$. On the one hand, if $b = x$ is true, the assumed $x \neq y$ gives $a \neq b$, and the sentences (3.168) and (3.171) imply $b\, R\, a$ and $\neg a\, R\, b$, respectively, so that we obtain the truth of

$$\neg a\, R\, b \wedge b\, R\, a \wedge a \neq b,$$

and therefore the truth of the desired multiple disjunction. On the other hand, if $b = y$ is true, it follows with the current case assumption $a = y$ that $a = b$ holds; furthermore, (3.170) gives $\neg a\, R\, b$ as well as $b\, R\, a$, so that (3.172) is true again. Thus, $R = \{(x, y)\}$ satisfies the multiple disjunction in (3.156) in any case, and since a and b were arbitrary, we may therefore conclude that this binary relation is indeed comparable. □

It will be also useful for to know the following link between comparability, irreflexivity and connexity.

Lemma 3.51. *For any set X and any transitive binary relation R on X, it is true that R is irreflexive and connex iff R is comparable.*

Proof. We let X be an arbitrary set and R an arbitrary transitive binary relation on X. We now prove the first part ('⇒') of the stated equivalence by contradiction, assuming that R is irreflexive and connex, i.e.

$$\forall a\, (a \in X \Rightarrow \neg a\, R\, a) \wedge \forall a, b\, (a, b \in X \Rightarrow [a\, R\, b \vee b\, R\, a \vee a = b]) \tag{3.173}$$

and assuming that R is not comparable, i.e.

$$\neg\forall a,b\,(a,b\in X\Rightarrow[(a\,R\,b\,\vee\,b\,R\,a\,\vee\,a=b)\\ \wedge\,\neg(a\,R\,b\,\wedge\,b\,R\,a)\,\wedge\,\neg(a\,R\,b\,\wedge\,a=b)\,\wedge\,\neg(b\,R\,a\,\wedge\,a=b)]). \tag{3.174}$$

Here, we may write the latter negation equivalently as

$$\exists a,b\,(a,b\in X\wedge\neg[(a\,R\,b\,\vee\,b\,R\,\vee\,a=b)\\ \wedge\,\neg(a\,R\,b\,\wedge\,b\,R\,a)\,\wedge\,\neg(a\,R\,b\,\wedge\,a=b)\,\wedge\,\neg(b\,R\,a\,\wedge\,a=b)])$$

by applying the Quantifier Negation Law (1.53) and (1.82) in connection with the Double Negation Law. Thus, there are elements in X, say $\bar{a}$ and $\bar{b}$, with

$$\neg[(\bar{a}\,R\,\bar{b}\,\vee\,\bar{b}\,R\,\bar{a}\,\vee\,\bar{a}=\bar{b})\\ \wedge\,\neg(\bar{a}\,R\,\bar{b}\,\wedge\,\bar{b}\,R\,\bar{a})\,\wedge\,\neg(\bar{a}\,R\,\bar{b}\,\wedge\,\bar{a}=\bar{b})\,\wedge\,\neg(\bar{b}\,R\,\bar{a}\,\wedge\,\bar{a}=\bar{b})]$$

Let us now apply De Morgan's Law for the conjunction successively to obtain first

$$\neg(\bar{a}\,R\,\bar{b}\,\vee\,\bar{b}\,R\,\bar{a}\,\vee\,\bar{a}=\bar{b})\\ \vee\,\neg[\neg(\bar{a}\,R\,\bar{b}\,\wedge\,\bar{b}\,R\,\bar{a})\,\wedge\,\neg(\bar{a}\,R\,\bar{b}\,\wedge\,\bar{a}=\bar{b})\,\wedge\,\neg(\bar{b}\,R\,\bar{a}\,\wedge\,\bar{a}=\bar{b})],$$

then in connection with the Double Negation Law and the Associative Law for the conjunction as well as for the disjunction

$$\neg(\bar{a}\,R\,\bar{b}\,\vee\,\bar{b}\,R\,\bar{a}\,\vee\,\bar{a}=\bar{b})\,\vee\,(\bar{a}\,R\,\bar{b}\,\wedge\,\bar{b}\,R\,\bar{a})\\ \vee\,\neg[\neg(\bar{a}\,R\,\bar{b}\,\wedge\,\bar{a}=\bar{b})\,\wedge\,\neg(\bar{b}\,R\,\bar{a}\,\wedge\,\bar{a}=\bar{b})],$$

and moreover

$$\neg(\bar{a}\,R\,\bar{b}\vee\bar{b}\,R\,\bar{a}\vee\bar{a}=\bar{b})\vee(\bar{a}\,R\,\bar{b}\wedge\bar{b}\,R\,\bar{a})\vee(\bar{a}\,R\,\bar{b}\wedge\bar{a}=\bar{b})\vee(\bar{b}\,R\,\bar{a}\wedge\bar{a}=\bar{b}). \tag{3.175}$$

If the first part of this multiple disjunction is true, then we obtain a contradiction since $\bar{a},\bar{b}\in X$ implies with the second part of the assumed conjunction (3.173), i.e. with the assumed connexity of R, the truth of $\bar{a}\,R\,\bar{b}\,\vee\,\bar{b}\,R\,\bar{a}\,\vee\,\bar{a}=\bar{b}$. Next, if the second part $\bar{a}\,R\,\bar{b}\,\wedge\,\bar{b}\,R\,\bar{a}$ of the multiple disjunction (3.175) holds, then this yields $\bar{a}\,R\,\bar{a}$ with the initially assumed transitivity (3.149) of R. Also, if the third part $\bar{a}\,R\,\bar{b}\,\wedge\,\bar{a}=\bar{b}$ or the fourth part $\bar{b}\,R\,\bar{a}\,\wedge\,\bar{a}=\bar{b}$ is true, then the equation $\bar{a}=\bar{b}$ holds in particular, so that substitution in the true $\bar{a}\,R\,\bar{b}$ and $\bar{b}\,R\,\bar{a}$ yields $\bar{a}\,R\,\bar{a}$ in any case. Since $\bar{a}\in X$ implies also $\neg\bar{a}\,R\,\bar{a}$ with the first part of the assumed conjunction (3.173), i.e. with the assumed irreflexivity of R, we have a contradiction also in case the second, third or fourth part of the disjunction (3.175) is true. Thus, the proof of the implication '$\Rightarrow$' via contradiction is complete.

We now prove the second part ('$\Leftarrow$') of the proposed equivalence also by contradiction, assuming

$$\begin{aligned} &\forall a, b\,(a, b \in X \Rightarrow [(a\,R\,b\ \vee\ b\,R\,a\ \vee\ a = b) \\ &\quad \wedge\ \neg(a\,R\,b\ \wedge\ b\,R\,a)\ \wedge\ \neg(a\,R\,b\ \wedge\ a = b)\ \wedge\ \neg(b\,R\,a\ \wedge\ a = b\,)]). \end{aligned} \tag{3.176}$$

and the negation of (3.173) to be both true. We may write this negation also as

$$\neg\forall a\,(a \in X \Rightarrow \neg a\,R\,a)\ \vee\ \neg\forall a, b\,(a, b \in X \Rightarrow [a\,R\,b\ \vee\ b\,R\,a\ \vee\ a = b])$$

by means of De Morgan's Law for the conjunction. Using now the Quantifier Negation Law (1.53) as well as (1.82) in connection with the Double Negation Law, we obtain equivalently

$$\exists a\,(a \in X\ \wedge\ a\,R\,a)\ \vee\ \exists a, b\,(a, b \in X\ \wedge\ \neg[a\,R\,b\ \vee\ b\,R\,a\ \vee\ a = b]). \tag{3.177}$$

In case the first part of this disjunction is true, there exists an element in X, say $\bar{a}$, with $\bar{a}\,R\,\bar{a}$; since $\bar{a}, \bar{a} \in X$ implies with (3.176) in particular $\neg(\bar{a}\,R\,\bar{a}\ \wedge\ \bar{a}\,R\,\bar{a})$ and therefore $\neg\bar{a}\,R\,\bar{a}\ \vee\ \neg\bar{a}\,R\,\bar{a}$ with De Morgan's Law for the conjunction, so that $\neg\bar{a}\,R\,\bar{a}$ follows to be true with the Idempotent Law for the disjunction, we arrived at a contradiction with the previously found $\bar{a}\,R\,\bar{a}$. Next, in case the second part of the disjunction (3.177) holds, there are elements in X, say $\bar{a}$ and $\bar{b}$, satisfying the negation $\neg[\bar{a}\,R\,\bar{b} \vee \bar{b}\,R\,\bar{a} \vee \bar{a} = \bar{b}]$; as $\bar{a}, \bar{b} \in X$ implies with (3.176) in particular $\bar{a}\,R\,\bar{b}\ \vee\ \bar{b}\,R\,\bar{a}\ \vee\ \bar{a} = \bar{b}$, we again have a contradiction. This completes the proof of the implication '$\Leftarrow$' and thus the proof of the proposed equivalence. Since X and R were arbitrary, we may therefore conclude that the lemma holds. □

Let us apply this lemma in connection with the Propositions 3.48 & 3.50.

Corollary 3.52. *It is true for any x and y with $x \neq y$ that the singleton $R = \{(x, y)\}$ is an irreflexive, connex binary relation on $\{x, y\}$.*

In the following, we will consider standard types of binary relations which satisfy certain combinations of the properties listed in Definition 3.12.

Definition 3.13 (Equivalence relation, equivalent elements). We say for any set X that a binary relation $\sim$ on X is an *equivalence relation* on X iff $\sim$ is

1. reflexive,

2. symmetric, and
3. transitive.

We then say for any set X and any equivalence relation $\sim$ on X that two elements x_1 and x_2 of X are *equivalent* under $\sim$ iff

$$x_1 \sim x_2. \tag{3.178}$$

Corollary 3.45 and Proposition 3.47 give immediately the following basic examples for an equivalence relation.

Corollary 3.53. *The empty binary relation $\sim = \emptyset$ on $X = \emptyset$ is an equivalence relation.*

Corollary 3.54. *For any x it is true that the binary relation $\sim = \{(x,x)\}$ on $\{x\}$ is an equivalence relation.*

Proposition 3.55. *It is true for any set X, any equivalence relation $\sim$ on X and any element x of X that there exists a unique set $[x]_\sim$ consisting of all the elements of X which are equivalent to x, in the sense that*

$$\forall y\, (y \in [x]_\sim \Leftrightarrow y \sim x). \tag{3.179}$$

Proof. We let X and $\sim$ be arbitrary sets and assume $\sim$ to be an equivalence relation on X. Concerning a), letting x be arbitrary and assuming $x \in X$ to be true, we see in light of the Axiom of Specification and the Equality Criterion for sets that there exists a unique set $[x]_\sim$ satisfying

$$\forall y\, (y \in [x]_\sim \Leftrightarrow [y \in X \wedge y \sim x]). \tag{3.180}$$

In order to prove that the unique set $[x]_\sim$ satisfies (3.179), we take an arbitrary y. On the one hand, the assumption $y \in [x]_\sim$ implies especially $y \sim x$ with (3.180), so that the first part ('$\Rightarrow$') of the equivalence in (3.179) holds. To establish the second part ('$\Leftarrow$'), we assume now $y \sim x$ to be true, which we may write also as $(y,x) \in \sim$. Since $\sim$ is an equivalence relation on X, the inclusion $\sim \subseteq X \times X$ holds, so that the preceding finding implies $(y,x) \in X \times X$ by definition of a subset. We thus have in particular $y \in X$ by definition of the Cartesian product of two sets, and this implies in conjunction with $y \sim x$ because of (3.180) that $y \in [x]_\sim$ is true. Since y is arbitrary, we may therefore conclude that $[x]_\sim$ satisfies indeed the universal sentence (3.179), and as x, X and $\sim$ were also arbitrary, we may then further conclude that the proposed universal sentence holds. $\square$

Definition 3.14 (Equivalence class, representative). We call for any set X, for any equivalence relation $\sim$ on X and for any element $x \in X$ the set

$$[x]_\sim = [x]_{\sim_X} \tag{3.181}$$

consisting of all the elements of X that are equivalent to x the *equivalence class* of x with respect to $\sim$. Furthermore, we call any element of an equivalence class a *representative* of that equivalence class.

Corollary 3.56. *It is true for any set X and any equivalence relation $\sim$ on X that the equivalence class of any element of X is included in X, i.e.*

$$\forall x\,(x \in X \Rightarrow [x]_\sim \subseteq X). \tag{3.182}$$

Proof. Letting X, $\sim$ and x be arbitrary such that $\sim$ is an equivalence relation on X and such that x is an element of X, we observe in light of (3.180) $y \in [x]_\sim$ implies especially $y \in X$ for an arbitrary y, so that the inclusion $[x]_\sim \subseteq X$ follows to be true by definition of a subset. Because X, $\sim$ and x are also arbitrary, we may therefore conclude that the proposed sentence holds. □

Corollary 3.57. *It is true for any set X and any equivalence relation $\sim$ on X that any element of X is contained in its own equivalence class, i.e.*

$$\forall x\,(x \in X \Rightarrow x \in [x]_\sim). \tag{3.183}$$

Proof. We take arbitrary sets X, $\sim$ and x, assuming $\sim$ to be an equivalence relation on X and assuming x to be an element of X. We therefore obtain with Property 1 of an equivalence relation (i.e., with the reflexivity of $\sim$) the true sentence $x \sim x$, which implies $x \in [x]_\sim$ with (3.179). As the sets X, $\sim$ and x were initially arbitrary, we may infer from this finding the truth of the proposed universal sentence. □

Note 3.4. Corollary 3.57 shows that every equivalence class $[x]_\sim$ contains x and thus some element of X, so that we have

$$\forall x\,(x \in X \Rightarrow [x]_\sim \neq \emptyset) \tag{3.184}$$

according to (2.42).

Theorem 3.58 (Equality Criterion for equivalence classes). *It is true for any equivalence relation $\sim$ on any set X that the equivalence classes of two elements x and y in X are identical iff x and y are equivalent, i.e.*

$$\forall x, y\,(x, y \in X \Rightarrow ([x]_\sim = [y]_\sim \Leftrightarrow x \sim y)). \tag{3.185}$$

Proof. We let X and $\sim$ be arbitrary sets, we assume $\sim$ to be an equivalence relation on X, and we assume $\bar{x}, \bar{y} \in X$ to be true. We establish the first part ('$\Rightarrow$') of the equivalence in (3.185) by contraposition, assuming the negation $\neg \bar{x} \sim \bar{y}$ to be true. This assumption implies then with (3.179) and the Law of Contraposition that $\bar{x} \notin [\bar{y}]_\sim$ holds. Since $\bar{x} \in [\bar{x}]_\sim$ is also true according to (3.183), we obtain the true disjunction $\bar{x} \in [\bar{x}]_\sim \wedge \bar{x} \notin [\bar{y}]_\sim$. Let us observe that the existential sentence

$$\exists x\,([x \in [\bar{x}]_\sim \wedge x \notin [\bar{y}]_\sim] \vee [x \in [\bar{y}]_\sim \wedge x \notin [\bar{x}]_\sim])$$

is then also true, and that this sentence implies $[\bar{x}]_\sim \neq [\bar{y}]_\sim$ due to (2.23). We thus completed the proof of the implication '$\Rightarrow$' by contraposition.

We establish the second implication '$\Leftarrow$' directly, assuming $\bar{x} \sim \bar{y}$ to be true. Next, we apply the Equality Criterion for sets to prove the equation $[\bar{x}]_\sim = [\bar{x}]_\sim$, by verifying the universal sentence

$$\forall z\,(z \in [\bar{x}]_\sim \Leftrightarrow z \in [\bar{y}]_\sim). \tag{3.186}$$

For this purpose, we let z be arbitrary and assume first $z \in [\bar{x}]_\sim$ to be true. Thus, z is equivalent to $\bar{x}$, i.e. $z \sim \bar{x}$ holds according to (3.179). In conjunction with the assumed $\bar{x} \sim \bar{y}$, this implies then $z \sim \bar{y}$ with Property 3 of an equivalence relation (i.e., with the transitivity of $\sim$). This finding gives us then the desired consequent $z \in [\bar{y}]_\sim$ by applying again (3.179). We now conversely assume $z \in [\bar{y}]_\sim$ to be true, so that $z \sim \bar{y}$ holds evidently. Let us observe now that the assumed $\bar{x} \sim \bar{y}$ implies $\bar{y} \sim \bar{x}$ with Property 2 of an equivalence relation (i.e., with the symmetry of $\sim$). Then, the simultaneous truth of $z \sim \bar{y}$ and $\bar{y} \sim \bar{x}$ implies the truth of $z \sim \bar{x}$ because of the transitivity of $\sim$, with the evident consequence that $z \in [\bar{x}]_\sim$ holds, as desired. We thus completed the proof of the equivalence in (3.186), in which z is arbitrary, so that the universal sentence (3.186) follows to be true. Consequently, the equation $[\bar{x}]_\sim = [\bar{x}]_\sim$ holds indeed, and this finding completes the proof of the second part ('$\Leftarrow$') of the equivalence in (3.185).

Since X, $\sim$, $\bar{a}$ and $\bar{b}$ were initially arbitrary, we may therefore conclude that the stated theorem holds. □

Exercise 3.25. Show for any equivalence relation $\sim$ on any set X that the equivalence classes of two elements x and y in X are not disjoint iff x and y are equivalent, i.e.

$$\forall x, y\,(x, y \in X \Rightarrow ([x]_\sim \cap [y]_\sim \neq \emptyset \Leftrightarrow x \sim y)). \tag{3.187}$$

(Hint: Prove the implications directly, applying (2.42), (2.57) and some of the arguments of the proof of the Equality Criterion for equivalence classes.)

Combining the equivalences in (3.185) and (3.187) gives us the following alternative characterization of equality and non-disjointness of two equivalence classes.

Corollary 3.59. *It is true for any equivalence relation $\sim$ on any set X that the equivalence classes of two elements in X are identical iff they are not disjoint, i.e.*

$$\forall x, y\,(x, y \in X \Rightarrow ([x]_\sim = [y]_\sim \Leftrightarrow [x]_\sim \cap [y]_\sim \neq \emptyset)). \tag{3.188}$$

Note 3.5. The preceding Corollary 3.59 shows that two equivalence classes are either identical or disjoint, because equality and disjointness cannot occur concurrently.

Proposition 3.60. *It is true for any equivalence relation $\sim$ on any set X that there exists a unique set (system) $X/\sim$ consisting of all the equivalence classes $[x]_\sim$ with $x \in X$, in the sense that*

$$\forall Z\,(Z \in X/\sim\ \Leftrightarrow \exists x\,(x \in X \ \wedge\ [x]_\sim = Z)). \tag{3.189}$$

Proof. Letting X and $\sim$ be arbitrary sets such that $\sim$ is an equivalence relation of X, we obtain by means of the Axiom of Specification and the Equality Criterion for sets a unique set $X/\sim$ whose elements satisfy

$$\forall Z\,(Z \in X/\sim\ \Leftrightarrow [Z \in \mathcal{P}(X) \ \wedge\ \exists x\,(x \in X \ \wedge\ [x]_\sim = Z)]). \tag{3.190}$$

Letting Z be arbitrary, we note on the one hand that $Z \in X/\sim$ yields in particular the existential sentence in (3.189) in view of (3.190). On the other hand, assuming that there exists a constant, say $\bar{x}$, with $\bar{x} \in X$ and $[\bar{x}]_\sim = Z$, we see that (3.182) gives the inclusion $[Z =]\ [\bar{x}]_\sim \subseteq X$. We therefore have $Z \in \mathcal{P}(X)$ by definition of a power set, which finding implies in conjunction with the assumed existential sentence that $Z \in X/\sim$ is true, according to (3.190). We thus completed the proof of the equivalence in (3.189), where Z was arbitrary, so that the universal sentence (3.189) holds, too. Since X and $\sim$ are also arbitrary, the stated proposition follows therefore to be true. □

Definition 3.15 (Quotient set). We say for any set X and any equivalence relation $\sim$ on X that the set

$$X/\sim \tag{3.191}$$

consisting of all equivalence classes with respect to $\sim$ is the *quotient set* of X with respect to $\sim$.

Theorem 3.61 (Partitioning Property of quotient sets). *It is true that the quotient set $X/\sim$ of any set X with respect to any equivalence relation $\sim$ on X constitutes a partition of X, i.e.*

$$\forall A, B\,([A, B \in X/\sim \;\wedge A \neq B] \Rightarrow A \cap B = \emptyset) \;\wedge\; \bigcup X/\sim\, = X. \tag{3.192}$$

Proof. We take arbitrary sets X and $\sim$ such that $\sim$ is an equivalence relation on X. Then, the quotient set $X/\sim$ is defined and constitutes a subset of $\mathcal{P}(X)$ in view of (3.190), noting that $Z \in X/\sim$ implies in particular $Z \in \mathcal{P}(X)$ for any Z. Thus, $X/\sim$ is by definition of a power set a system of subsets of X.

We now show that all sets in this set system are disjoint, letting $A, B \in X/\sim$ be arbitrary such that $A \neq B$ holds. Due to (3.189), the preceding assumption $A \in X/\sim$ implies that there exists a constant, say $\bar{x}$, satisfying $\bar{x} \in X$ and $[\bar{x}]_\sim = A$. Similarly, the assumed $B \in X/\sim$ implies the existence of a particular constant $\bar{y} \in X$ with $[\bar{y}]_\sim = B$. Then, the other assumption $A \neq B$ yields $[\bar{x}]_\sim \neq [\bar{y}]_\sim$ via substitution, and this finding further implies $\neg[\bar{x}]_\sim \cap [\bar{y}]_\sim \neq \emptyset$ with (3.188) and the Law of Contraposition. Consequently, we obtain $[\bar{x}]_\sim \cap [\bar{y}]_\sim = \emptyset$ with the Double Negation Law, so that substitutions gives us the desired equation $A \cap B = \emptyset$, proving the implication in (3.192). Since A and B are arbitrary, we may therefore conclude that the universal sentence in (3.192) is true, which means that the sets in $X/\sim$ are indeed pairwise disjoint.

Next, we verify that the union of the set system $X/\sim$ is identical with X. To do this, we apply the Equality Criterion for sets and prove accordingly the universal sentence

$$\forall c\,(c \in \bigcup X/\sim\, \Leftrightarrow c \in X), \tag{3.193}$$

letting c be arbitrary. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $c \in \bigcup X/\sim$ to be true, which means by definition of the union of a set system that $c \in \bar{A}$ holds for some particular set $\bar{A} \in X/\sim$. According to (3.189) there exists then a particular element $\bar{x} \in X$ such that $[\bar{x}]_\sim = \bar{A}$. We therefore obtain $[\bar{x}]_\sim \subseteq X$ by means of (3.182), as well as $c \in [\bar{x}]_\sim$ via substitution. Consequently, the definition of a subset yields $c \in X$, as desired. Regarding the second part ('$\Leftarrow$') of the equivalence, we assume conversely $c \in X$ to be true, which gives us $c \in [c]_\sim$ with (3.183), and the simultaneous truth of $c \in X$ and of $[c]_\sim = [c]_\sim$ demonstrates that there exists a constant x satisfying both $x \in X$ and $[x]_\sim = [c]_\sim$. This existential sentence implies now with (3.189) that $[c]_\sim \in X/\sim$ is true. In conjunction with the previously established $c \in [c]_\sim$, this shows in light of the definition of the union of a set system that there exists a set $A \in X/\sim$ that contains c, with the desired consequence that $c \in \bigcup X/\sim$ holds. We thus completed the proof of the equivalence in (3.193), and as c was arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (3.193), and therefore the truth of the equation $\bigcup X/\sim = X$ in (3.192).

We thus proved both parts of the conjunction (3.192), in which X and $\sim$ were arbitrary, so that the stated theorem is indeed true. □

The following kind of binary relation will be used frequently throughout this exposition.

Definition 3.16 (Reflexive partial ordering, partially ordered set / poset). For any set X we say that a binary relation R on X is a *reflexive partial ordering* of X iff R is

1. reflexive,
2. antisymmetric, and
3. transitive.

We then call the ordered pair (X, R) a *partially ordered set* or a *poset*.

Corollary 3.62. *The empty binary relation $R = \emptyset$ on $X = \emptyset$ is a reflexive partial ordering of $\emptyset$, and*

$$(\emptyset, \emptyset) \tag{3.194}$$

is a partially ordered set.

Corollary 3.63. *For any x it is true that the binary relation $R = \{(x, x)\}$ on $\{x\}$ is a reflexive partial ordering of $\{x\}$, and*

$$(\{x\}, \{(x, x)\}) \tag{3.195}$$

is a partially ordered set.

Theorem 3.64 (Reflexive partial ordering of inclusion ($\subseteq$)). *It is true for any set system $\mathcal{K}$ that there exists a unique set $\subseteq_{\mathcal{K}}$ such that an element Y is in $\subseteq_{\mathcal{K}}$ iff there exists an ordered pair (A, B) in the Cartesian product of $\mathcal{K}$ with itself such that A is included in B, i.e. iff*

$$\forall Y\,(Y \in \subseteq_{\mathcal{K}} \Leftrightarrow [Y \in \mathcal{K} \times \mathcal{K} \wedge \exists A, B\,(A \subseteq B \wedge (A, B) = Y)]). \tag{3.196}$$

Then, the set $\subseteq_{\mathcal{K}}$ is a reflexive partial ordering of $\mathcal{K}$ and satisfies

$$\forall A, B\,(A, B \in \mathcal{K} \Rightarrow [A \subseteq_{\mathcal{K}} B \Leftrightarrow A \subseteq B]). \tag{3.197}$$

Proof. Letting $\mathcal{K}$ be an arbitrary set system, we may evidently apply the Axiom of Specification in connection with the Equality Criterion for sets to establish the unique existence sentence of a set $\subseteq_{\mathcal{K}}$ satisfying (3.196) holds.

Next, we verify that $\subseteq_{\mathcal{K}}$ is a binary relation on $\mathcal{K}$, i.e. $\subseteq_{\mathcal{K}} \subseteq \mathcal{K} \times \mathcal{K}$. For this purpose, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall Y\,(Y \in \subseteq_{\mathcal{K}} \Rightarrow Y \in \mathcal{K} \times \mathcal{K}), \tag{3.198}$$

letting Y be arbitrary and assuming $Y \in \subseteq_{\mathcal{K}}$ to be true. This assumption implies with (3.196) in particular $Y \in \mathcal{K} \times \mathcal{K}$, completing the proof of the implication in (3.198). As Y is arbitrary, we may therefore conclude that the universal sentence (3.198) holds, which means $\subseteq_{\mathcal{K}} \subseteq \mathcal{K} \times \mathcal{K}$ (by definition of a subset). Thus, $\subseteq_{\mathcal{K}}$ is indeed a binary relation on $\mathcal{K}$, by definition.

To demonstrate that the universal sentence (3.197) is true for the binary relation $\subseteq_{\mathcal{K}}$, we take arbitrary sets $\bar{A}$ and $\bar{B}$, and we assume $\bar{A}, \bar{B} \in \mathcal{K}$ to be true. To establish now the first part ('$\Rightarrow$') of the equivalence in (3.197), we assume $\bar{A} \subseteq_{\mathcal{K}} \bar{B}$, which assumption we may write also as $(\bar{A}, \bar{B}) \in \subseteq_{\mathcal{K}}$. This in turn implies with (3.196) especially that there constants, say $\bar{\bar{A}}$ and $\bar{\bar{B}}$, such that $\bar{\bar{A}} \subseteq \bar{\bar{B}}$ and $(\bar{\bar{A}}, \bar{\bar{B}}) = (\bar{A}, \bar{B})$ are both true. Then, the latter equation gives with the Equality Criterion for ordered pairs $\bar{\bar{A}} = \bar{A}$ as well as $\bar{\bar{B}} = \bar{B}$, so that the former inclusion yields via substitutions $\bar{A} \subseteq \bar{B}$, as desired. Thus, the first part of the equivalence to be proven holds.

Regarding the second part ('$\Leftarrow$'), we now assume $\bar{A} \subseteq \bar{B}$ to be true, which assumption shows in light of the obviously true equation $(\bar{A}, \bar{B}) = (\bar{A}, \bar{B})$ that the existential sentence

$$\exists A, B\,(A \subseteq B \,\wedge\, (A, B) = (\bar{A}, \bar{B}))$$

holds. Furthermore, the initial assumption $\bar{A}, \bar{B} \in \mathcal{K}$ implies $(\bar{A}, \bar{B}) \in \mathcal{K} \times \mathcal{K}$ by definition of the Cartesian product of two sets, and this finding implies then together with the preceding existential sentence the truth of $(\bar{A}, \bar{B}) \in \subseteq_{\mathcal{K}}$ with (3.196). We already established $\subseteq_{\mathcal{K}}$ as a binary relation, so that we may write this also as $\bar{A} \subseteq_{\mathcal{K}} \bar{B}$, completing the proof of the equivalence in (3.197). Since A and B were arbitrary, we may therefore conclude that the proposed universal sentence (3.197) is indeed true.

It now remains for us to verify that $\subseteq_{\mathcal{K}}$ is a reflexive partial ordering of $\subseteq_{\mathcal{K}}$. We first prove that $\subseteq_{\mathcal{K}}$ is reflexive and verify accordingly

$$\forall A\,(A \in \mathcal{K} \Rightarrow A \subseteq_{\mathcal{K}} A). \tag{3.199}$$

We let A be an arbitrary set, we assume $A \in \mathcal{K}$ to be true, and we observe that $A \subseteq A$ holds according to (2.10). Because of (3.197), we then obtain immediately the desired consequent $A \subseteq_{\mathcal{K}} A$, and since A was arbitrary, we may therefore conclude that the universal sentence (3.199) holds, which means that $\subseteq_{\mathcal{K}}$ is a reflexive binary relation on $\mathcal{K}$, by definition.

Concerning the antisymmetry of $\subseteq_{\mathcal{K}}$, we need to verify

$$\forall A, B\,(A, B \in \mathcal{K} \Rightarrow [(A \subseteq_{\mathcal{K}} B \,\wedge\, B \subseteq_{\mathcal{K}} A) \Rightarrow A = B]). \tag{3.200}$$

To do this, we let A and B be arbitrary sets, assume $A, B \in \mathcal{K}$ to be true, and assume in addition that $A \subseteq_{\mathcal{K}} B$ and $B \subseteq_{\mathcal{K}} A$ both hold. The latter two assumptions immediately imply $A \subseteq B$ and $B \subseteq A$, respectively, in view of (3.197). Consequently, the conjunction $A \subseteq B \,\wedge\, B \subseteq A$ is true, which implies then the true equation $A = B$ with the Axiom of Extension. Thus, the proofs of the implications in (3.200) are complete, and as A and B were arbitrary, we may therefore conclude that the universal sentence (3.200) holds, so that $\subseteq_{\mathcal{K}}$ is indeed an antisymmetric binary relation on $\mathcal{K}$.

Finally, we demonstrate that $\subseteq_{\mathcal{K}}$ is transitive, i.e. that $\subseteq_{\mathcal{K}}$ satisfies the universal sentence

$$\forall A, B, C\,(A, B, C \in \mathcal{K} \Rightarrow [(A \subseteq_{\mathcal{K}} B \,\wedge\, B \subseteq_{\mathcal{K}} C) \Rightarrow A \subseteq_{\mathcal{K}} C]). \tag{3.201}$$

Letting A, B, C be arbitrary, assuming $A, B, C \in \mathcal{K}$ to be true, and assuming also $A \subseteq_{\mathcal{K}} B$ as well as $B \subseteq_{\mathcal{K}} C$ to hold, we obtain the inclusions $A \subseteq B$ and $B \subseteq C$ with (3.197). Thus, the conjunction $A \subseteq B \,\wedge\, B \subseteq C$ is true, which in turn implies $A \subseteq C$ with (2.13). The latter inclusion now gives the desired $A \subseteq_{\mathcal{K}} C$ again with (3.197), and since A, B and C are arbitrary, the universal sentence (3.201) follows also to be true. Thus, $\subseteq_{\mathcal{K}}$ is a transitive binary relation by definition, and because $\subseteq_{\mathcal{K}}$ is reflexive and antisymmetric as well, it is a reflexive binary relation by definition.

Since $\mathcal{K}$ was initially arbitrary, we may now finally conclude that the stated theorem is true. □

The properties of a reflexively-partially ordered set also hold for any 'subset' of such an ordered set.

Theorem 3.65 (Reflexive partial ordering of subsets). *The following sentences are true for any set X, for any reflexive partial ordering R of X, and for any subset $A \subseteq X$.*

a) There exists a unique set R_A consisting of all the ordered pairs in R whose components are elements of A, that is,

$$\exists! R_A\, \forall Z\,(Z \in R_A \Leftrightarrow [Z \in R \,\wedge\, \exists x, y\,(x, y \in A \,\wedge\, (x, y) = Z)]). \tag{3.202}$$

b) Then, R_A is a binary relation on X, which is included in R, i.e.

$$R_A \subseteq R, \tag{3.203}$$

and which satisfies

$$\forall x, y\,(x, y \in A \Rightarrow [x\,R_A\,y \Leftrightarrow x\,R\,y]). \tag{3.204}$$

c) Furthermore, R_A is a reflexive partial ordering of A.

Proof. We let X, R and A be arbitrary sets, we assume that R is a reflexive partial ordering of X, and we assume that A is a subset of X.

Concerning a), we may evidently apply the Axiom of Specification together with the Equality Criterion for sets to establish the stated uniquely existential sentence. Thus, the set R_A satisfies

$$\forall Z\,(Z \in R_A \Leftrightarrow [Z \in R \wedge \exists x, y\,(x, y \in A \wedge (x, y) = Z)]). \tag{3.205}$$

Concerning b), we observe in light of (3.205) that $Z \in R_A$ implies in particular $Z \in R$ for any Z, so that the inclusion (3.203) holds by definition of a subset. Furthermore, the reflexive partial ordering R of X is by definition a binary relation on X, so that $R \subseteq X \times X$ also holds. The previous two inclusions then give the inclusion $R_A \subseteq X \times X$ with (2.13), which means that R_A is a binary relation on X. We now take arbitrary constants $\bar{x}$ and $\bar{y}$, and we assume $\bar{x}, \bar{y} \in A$ to be true. To prove the first part ('$\Rightarrow$') of the equivalence in (3.204), we assume now also that $\bar{x}\,R_A\,\bar{y}$ holds. We may write this assumption also as $(\bar{x}, \bar{y}) \in R_A$, which in turn implies $(\bar{x}, \bar{y}) \in R$ with (3.203) by definition of a subset. Since R is a binary relation, we may write the latter also as $\bar{x}\,R\,\bar{y}$, so that the proof of the first part of the equivalence is complete. To prove the second part ('$\Leftarrow$'), we now assume conversely $\bar{x}\,R\,\bar{y}$ to be true, that is, $(\bar{x}, \bar{y}) \in R$. The initial assumption $\bar{x}, \bar{y} \in A$ clearly shows that there exist constants x and y such that $x, y \in A$ and $(x, y) = (\bar{x}, \bar{y})$ are both true. Then, $(\bar{x}, \bar{y}) \in R$ implies together with this existential sentence that $(\bar{x}, \bar{y}) \in R_A$ holds, according to (3.205), which we may write also as $\bar{x}\,R_A\,\bar{y}$ (recalling that R_A is a binary relation). Thus, the second part of the equivalence is also true, and since $\bar{x}, \bar{y}$ were arbitrary, we may therefore conclude that the universal sentence (3.204) holds.

Concerning c), we first verify that R_A is a binary relation on A, i.e. that the inclusion $R_A \subseteq A \times A$ holds. For this purpose, we take an arbitrary set Z and assume $Z \in R_A$ to be true. This assumption implies now with (3.205) in particular that there exist constants, say $\bar{x}$ and $\bar{y}$, such that $\bar{x}, \bar{y} \in A$ and $(\bar{x}, \bar{y}) = Z$ are both true. The former yields $(\bar{x}, \bar{y}) \in A \times A$ with the definition of the Cartesian product of two sets, so that substitution based on the latter equation gives $Z \in A \times A$. Since Z is arbitrary, we may infer from this finding that $Z \in R_A$ implies $Z \in A \times A$ for any set Z, so that $R_A \subseteq A \times A$ is indeed true (by definition of a subset). Thus, R_A is a binary relation on A by definition.

Let us check that R_A is reflexive, letting x be arbitrary and assuming $x \in A$ to be true. Then, the initially assumed inclusion $A \subseteq X$ evidently yields $x \in X$, and this implies $x\,R\,x$ since we assumed R to be a reflexive partial ordering of X. Because of (3.204), the previous finding further implies $x\,R_A\,x$. As x was arbitrary, we may therefore infer from this the reflexivity of R_A.

Regarding the antisymmetry, we take arbitrary $x, y \in A$ such that $x\,R_A\,y$ and $y\,R_A\,x$ are both true. Clearly, we then obtain $x, y \in X$ and consequently $x\,R\,y$ as well as $y\,R\,x$. As the reflexive partial ordering R is antisymmetric, we get the desired equation $x = y$, and since x and y are arbitrary, we may therefore conclude that R_A is indeed antisymmetric.

Finally, we demonstrate that R_A is transitive, by letting $x, y, z \in A$ be arbitrary such that $x\,R_A\,y$ and $y\,R_A\,z$ hold. Evidently, $x, y, z \in X$ is then also true, with the consequence that $x\,R\,y$ and $y\,R\,z$ both hold. Since the reflexive partial ordering R is transitive, we obtain $x\,R\,z$ and therefore $x\,R_A\,z$, as desired. Here, x, y and z were arbitrary, so that R_A follows to be transitive as well.

We thus proved that the binary relation R_A on A is a reflexive partial ordering of A. Because the sets X, R and A were initially arbitrary, we may now finally conclude that the stated theorem is true. □

We continue now with another basic kind of binary ordering relation, which we will encounter mainly in the context of number systems. The main idea is to replace the previously considered reflexivity property by irreflexivity.

Definition 3.17 (Irreflexive partial ordering, less than). For any set X we say that a binary relation $<$ on X is an *irreflexive partial ordering* of X iff $<$ is

1. irreflexive and
2. transitive.

We then call the ordered pair

$$(X, <) \tag{3.206}$$

also a *partially ordered set* or a *poset*. Furthermore, we say for any elements a, b of X that a is *less than* b iff $a < b$.

Corollary 3.66. *For any set X the empty binary relation $R = \emptyset$ on X is an irreflexive partial ordering of X, and*

$$(X, \emptyset) \tag{3.207}$$

is a partially ordered set.

Proposition 3.67. *For any set X and any irreflexive partial ordering $<$ of X it is true that if an element a of X is less than an element b of X, then a and b are different, i.e.*

$$\forall a, b\,(a, b \in X \Rightarrow [a < b \Rightarrow a \neq b]). \tag{3.208}$$

Proof. We let X be an arbitrary set, $<$ an arbitrary irreflexive partial ordering of X, let a and b be arbitrary, and assume further $a, b \in X$ to hold. Next, we prove the implication $a < b \Rightarrow a \neq b$ by contradiction, assuming $a < b$ and $\neg a \neq b$ to be true. The latter means $\neg(\neg a = b)$ and therefore implies $a = b$ with the Double Negation Law. With this equation, the assumed $a < b$ becomes $a < a$ after substitution. Since $\neg a < a$ is also true in view of the irreflexivity of $<$, we thus obtained a contradiction according to (1.11), which proves the implications in (3.208). Since X, $<$, a and b were arbitrary, the proposed universal sentence follows to be true. □

Exercise 3.26. Verify the following universal sentence for any set X and any irreflexive partial ordering $<$ of X.

$$\forall a, b\,(a, b \in X \Rightarrow \neg(a < b \,\wedge\, b < a)). \tag{3.209}$$

(Hint: Apply Method 1.11 in connection with Definition 3.17.)

We now state a similar version of Reflexive partial orderings of subsets in the context of irreflexive partial orderings.

Theorem 3.68 (Irreflexive partial ordering of subsets). *The following sentences are true for any set X, for any irreflexive partial ordering $<_X$ of X, and for any subset $A \subseteq X$.*

a) *There exists a unique set $<_A$ consisting of all the ordered pairs in $<_X$ whose components are elements of A, that is,*

$$\exists! <_A \forall Z\,(Z \in <_A \Leftrightarrow [Z \in <_X \,\wedge \exists x, y\,(x, y \in A \,\wedge\, (x, y) = Z)]). \tag{3.210}$$

b) *Then, $<_A$ is a binary relation on X, which is included in R, i.e.*

$$<_A \subseteq <_X, \tag{3.211}$$

and which satisfies

$$\forall x, y\,(x, y \in A \Rightarrow [x <_A y \Leftrightarrow x <_X y]). \tag{3.212}$$

c) Furthermore, $<_A$ is an irreflexive partial ordering of A.

Exercise 3.27. Prove Theorem 3.68.

(Hint: Proceed in analogy to the proof of Theorem 3.65; use the Law of Contraposition.)

Special forms of reflexive and irreflexive partial orderings are the 'total' and the 'linear' ordering, as defined in the following.

Definition 3.18 (Total ordering, totally ordered set/chain). For any set X we say that a reflexive partial ordering $\leq$ of X is a *total ordering* of X iff $\leq$ is total. We then call $(X, \leq)$ a *totally ordered set* or a *chain*.

Corollary 3.69. *The empty binary relation $R = \emptyset$ on $\emptyset$ is a total ordering of $\emptyset$, and*

$$(\emptyset, \emptyset) \tag{3.213}$$

is a totally ordered set.

To obtain another simple example, we recall Corollary 3.63 and Proposition 3.47.

Corollary 3.70. *For any x it is true that the binary relation $R = \{(x,x)\}$ on $\{x\}$ is a total ordering of $\{x\}$, and*

$$(\{x\}, \{(x,x)\}) \tag{3.214}$$

is a totally ordered set.

Theorem 3.71 (Total ordering of subsets). *For any totally ordered set $(X, \leq)$ and any subset $A \subseteq X$, it is true that the set $(A, \leq_A)$ is also totally ordered.*

Proof. We let X, $\leq$ and A be arbitrary sets such that $\leq$ is a total ordering of X and such that A is included in X. Thus, $\leq$ is a reflexive partial ordering of X, so that there exists the unique reflexive partial ordering $\leq_A$ of A according to Theorem 3.65. It thus remains for us to show that $\leq_A$ is total, i.e. that $\leq_A$ satisfies

$$\forall a, b\,(a, b \in A \Rightarrow [a \leq_A b \,\vee\, b \leq_A a]). \tag{3.215}$$

To do this, we let a and b be arbitrary and assume $a, b \in A$ to be true. Then, the assumption $A \subseteq X$ yields (by definition of a subset) $a, b \in X$ and therefore $a \leq b \,\vee\, b \leq a$ due to the totality of $\leq$. Based on this true disjunction, we now prove the disjunction in (3.215) by cases. In the first case $a \leq b$, we obtain with (3.204) from the assumed $a, b \in A$ the true inequality $a \leq_A b$. Then, the disjunction to be proven is evidently also true. Similarly, the second case $b \leq a$ yields $b \leq_A a$ and therefore the truth of the disjunction in (3.215), completing the proof by cases. Since a and b are arbitrary, we may therefore conclude that the universal sentence (3.215) is true, which means that the binary relation $\leq_A$ is indeed total. Thus, $\leq_A$ is a total ordering of A, and as the sets X, $\leq$ and A were initially arbitrary, we may infer from this finding the truth of the stated theorem. □

Definition 3.19 (Linear ordering, linearly ordered set). For any set X we say that an irreflexive partial ordering $<$ on X is a *linear ordering* of X iff $<$ is connex. We then call $(X, <)$ a *linearly ordered set*.

Note 3.6. For future reference, any linear ordering $<$ of a set X is

1. irreflexive, i.e. satisfies
$$\forall a\,(a \in X \Rightarrow \neg a < a), \tag{3.216}$$
2. connex, i.e. satisfies
$$\forall a, b\,(a, b \in X \Rightarrow [a < b \,\vee\, b < a \,\vee\, a = b]), \tag{3.217}$$
and
3. transitive, i.e. satisfies
$$\forall a, b, c\,(a, b, c \in X \Rightarrow [(a < b \,\wedge\, b < c) \Rightarrow a < c]). \tag{3.218}$$

Corollary 3.72. *The empty binary relation $R = \emptyset$ on $\emptyset$ is a linear ordering of $\emptyset$, and*
$$(\emptyset, \emptyset) \tag{3.219}$$
is a linearly ordered set.

Combining Corollary 3.66 with the finding of Exercise 3.24 yields the following nontrivial example for a linearly ordered set.

Corollary 3.73. *For any x the empty binary relation $R = \emptyset$ on $\{x\}$ is a linear ordering of $\{x\}$, and*
$$(\{x\}, \emptyset) \tag{3.220}$$
is a linearly ordered set.

Another simple example is immediately derived from Proposition 3.48 and Corollary 3.52.

Corollary 3.74. *For any x and y satisfying $x \neq y$ it is true that the singleton $R = \{(x,y)\}$ is a linear ordering of $\{x,y\}$, and*

$$(\{x,y\},\{(x,y)\}) \tag{3.221}$$

is a linearly ordered set.

It is possible replace Property 1 and Property 2 of a linearly ordered set by the equivalent property of comparability.

Theorem 3.75 (Characterization of linearly ordered sets). *For any set X and any binary relation $<$ on X, it is true that $<$ is a linear ordering of X iff*

2. $<$ is comparable, that is,*

$$\begin{aligned}\forall a,b\,(a,b \in X \Rightarrow [(a < b \,\vee\, b < a \,\vee\, a = b) \\ \wedge\, \neg(a < b \,\wedge\, b < a) \,\wedge\, \neg(a < b \,\wedge\, a = b) \wedge \neg(b < a \,\wedge\, a = b\,)]),\end{aligned} \tag{3.222}$$

and if

3. $<$ is transitive, that is,

$$\forall a,b,c\,(a,b,c \in X \Rightarrow [(a < b \,\wedge\, b < c) \Rightarrow a < c]). \tag{3.223}$$

Proof. We let X be an arbitrary set and $<$ an arbitrary binary relation. On the one hand, if the ordered pair $(X,<)$ is linearly ordered, then $<$ is by definition irreflexive, connex, and transitive. It then follows with Lemma 3.51 from the irreflexivity and the connexity that $<$ is also comparable. Thus, Property 2* and Property 3 hold, so that the proof of the first part of the proposed equivalence is complete. On the other hand, if $<$ is comparable and transitive, then it follows with Lemma 3.51 from the comparability of $<$ that $<$ is also irreflexive and connex. Together with the transitivity of $<$, this shows that $<$ is a linear ordering of X, which finding completes the proof of the equivalence. Since X and $<$ were arbitrary, the theorem follows then to be true. □

Note 3.7. Theorem 3.49 shows for any linearly ordered set $(X,<)$ in light of Property 2* that precisely one of the three sentences $a < b$, $b < a$ and $a = b$ is true for any elements a,b in X, that is,

$$\begin{aligned}&(a < b \,\wedge\, [\neg b < a] \,\wedge\, a \neq b)) \\ &\vee\, ([\neg a < b] \,\wedge\, b < a \,\wedge\, a \neq b) \\ &\vee\, ([\neg a < b] \,\wedge\, [\neg b < a] \,\wedge\, a = b).\end{aligned} \tag{3.224}$$

Theorem 3.76 (Linear ordering of subsets). *For any linearly ordered set $(X, <)$ and any subset $A \subseteq X$ defining the irreflexive partial ordering $<_A$ according to Theorem 3.68, it is true that $(A, <_A)$ is linearly ordered.*

Exercise 3.28. Prove Theorem 3.76.
(Hint: Apply a proof similar to that of Theorem 3.71.)

The notion of comparability by means of $<$ suggests the following terminology.

Definition 3.20 (Greater than, strictly between, intermediate value, less than or equal to, greater than or equal to, between). For any set X, any irreflexive partial ordering $<$ of X and any elements $a, b, c \in X$, we say that

(1) b is *greater than* a, symbolically

$$b > a. \tag{3.225}$$

if, and only if, $a < b$.

(2) b is *strictly between* or an *intermediate value* of a and c, symbolically

$$a < b < c, \tag{3.226}$$

if, and only if $a < b$ and $b < c$.

(3) a is *less than or equal to* b, symbolically

$$a \leq b \tag{3.227}$$

if, and only if $a < b$ or $a = b$.

(4) b is *greater than or equal to* a, symbolically

$$b \geq a \tag{3.228}$$

if, and only if $a \leq b$.

(5) b is *between* a and c, symbolically

$$a \leq b \leq c, \tag{3.229}$$

if, and only if $a \leq b$ and $b \leq c$.

The following immediate consequences of the definition of $\leq$ and the properties of $<$ will be applied frequently.

Theorem 3.77 (Negation Formula for $\leq$ & for $<$). *The following laws are true for any linearly ordered set $(X, <)$.*

a) **Negation Formula for $\leq$:**

$$\forall a, b\,(a, b \in X \Rightarrow [\neg a \leq b \Leftrightarrow a > b]). \tag{3.230}$$

b) **Negation Formula for $<$:**

$$\forall a, b\,(a, b \in X \Rightarrow [\neg a < b \Leftrightarrow a \geq b]). \tag{3.231}$$

Proof. We let X and $<$ be arbitrary sets and assume further that the ordered pair $(X, <)$ is linearly ordered. Concerning a), we let a and b be arbitrary and assume that $a, b \in X$ holds. Let us now observe the truth of the equivalences

$$\begin{aligned} \neg a \leq b &\Leftrightarrow \neg(a < b \,\vee\, a = b) \\ &\Leftrightarrow \neg a < b \,\wedge\, a \neq b, \end{aligned}$$

using Definition 3.20(3) and De Morgan's Law for the disjunction. Therefore, the assumption $\neg a \leq b$ implies that $a < b$ and $a = b$ are both false. Since the linear ordering $<$ of X is connex, the disjunction of $a < b$, $b < a$ and $a = b$ is true, so that the second part $b < a$ holds. This gives the desired consequent $a > b$ of the first part of the equivalence with Definition 3.20(1). To prove the second part, we now assume $a > b$ to be true, so that $b < a$ holds. As the linear ordering $<$ is comparable, it follows with the Characterization of comparability that precisely one of the sentences $a < b$, $b < a$ and $a = b$ is true. Therefore, $a < b$ and $a = b$ are false, which means that the conjunction of $\neg a < b$ and $a \neq b$ is true, so that $\neg a \leq b$ follows to be true with the previously established equivalences. Thus, the second part of the equivalence in (3.230) also holds. Since a and b are arbitrary, we may now conclude that the universal sentence (3.230) is true, so that the proof of a) is complete. As X and $<$ are also arbitrary, we may further conclude that a) holds for any X and any $<$. □

Exercise 3.29. Prove the Negation Formula for $<$.

(Hint: Use similar arguments as in the proof of the Negation Formula for $\leq$.)

The following formulas will be applied frequently.

Theorem 3.78 (Transitivity Formula for $<$ and $\leq$ & for $\leq$ and $<$). *The following laws are true for any partially ordered set $(X, <)$.*

a) **Transitivity Formula for $<$ and $\leq$**:

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [(a < b \,\wedge\, b \leq c) \Rightarrow a < c]). \tag{3.232}$$

b) **Transitivity Formula for $\leq$ and $<$**:

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [(a \leq b \,\wedge\, b < c) \Rightarrow a < c]). \tag{3.233}$$

Proof. We let X and $<$ be arbitrary sets and assume that the ordered pair $(X, <)$ is partially ordered. Concerning a), we take arbitrary a, b and c and assume $a, b, c \in X$ to be true. We now assume furthermore that $a < b$ and $b \leq c$ both hold, where the latter means that the disjunction $b < c \vee b = c$ is true, according to Definition 3.20(3). In case the first part $b < c$ of this disjunction holds, we obtain the true conjunction $a < b \wedge b < c$, which in turn implies the desired $a < c$ with the transitivity of the partial ordering $<$ of X. In case the second part $b = c$ of the preceding disjunction holds, then the previously assumed $a < b$ yields via substitution again $a < c$, as desired. Thus, the two implications in (3.232) are true, and since a, b, c are arbitrary, the universal sentence (3.232) follows to be also true. As X and $<$ were also arbitrary, we may now further conclude that a) holds for any X and any $<$. □

Exercise 3.30. Prove the Transitivity Formula for $\leq$ and $<$.

Note 3.8. We will apply a similar notation as in (3.229) to write, for instance, $a < b \leq c$ instead of $a < b \wedge b \leq c$,

Proposition 3.79. *The following implication holds for any set X, any irreflexive partial ordering $<$ of X, and any $a, b \in X$.*

$$[\neg(a < b \,\wedge\, b < a) \,\wedge\, a \neq b] \;\Rightarrow\; \neg(a \leq b \,\wedge\, b \leq a). \tag{3.234}$$

Proof. We let X be an arbitrary set, $<$ an arbitrary irreflexive partial ordering of X, and a, b arbitrary elements. We now prove the implication by contradiction, assuming the negation $\neg(a < b \wedge b < a)$, the inequality $a \neq b$ and the double negation $\neg\neg(a \leq b \wedge b \leq a)$ to be true. The latter implies $a \leq b \wedge b \leq a$ with (1.35), which conjunction we may also write as

$$(a < b \,\vee\, a = b) \,\wedge\, (b < a \,\vee\, b = a),$$

using Definition 3.20(3). Here, $a = b$ and $b = a$ are both false in view of the assumed $a \neq b$, so that the preceding conjunction implies the truth of $a < b$ and $b < a$. Thus, $a < b \wedge b < a$ is true, and $\neg(a < b \wedge b < a)$ also holds due to Exercise 3.26, so that we obtained a contradiction. This proves the implication (3.234), and since X, $<$, a and b were arbitrary, we therefore conclude that the proposition is true. □

Proposition 3.80. *The following equivalence is true for any set X, any irreflexive partial ordering $<$ of X, and any elements $a, b \in X$.*

$$a < b \;\Leftrightarrow\; (a \leq b \,\wedge\, a \neq b) \tag{3.235}$$

Proof. We let X be an arbitrary set, $<$ an arbitrary irreflexive partial ordering of X, and a and b arbitrary elements of X.

To prove the first part ('$\Rightarrow$') of the stated equivalence, we assume $a < b$. On the one hand, the disjunction $a < b \vee a = b$ is then evidently also true (irrespective of the truth value of $a = b$), which further implies $a \leq b$ with Definition 3.20(3). On the other hand, the assumed $a < b$ implies $a \neq b$ with Proposition 3.67, which shows together with the previously established $a \leq b$ that the right-hand side of the equivalence (3.235) is true.

To prove the second part ('$\Leftarrow$') of that equivalence, we assume the conjunction

$$a \leq b \,\wedge\, a \neq b$$

to be true. Because of Definition 3.20(3) and the Commutative Law for sentences (1.36), we may write this sentence equivalently as

$$a \neq b \,\wedge\, (a < b \,\vee\, a = b).$$

Then, an application of the Distributive Law for Sentences (1.44) gives

$$(a \neq b \,\wedge\, a < b) \,\vee\, (a \neq b \,\wedge\, a = b).$$

Here, $a \neq b \wedge a = b$ is evidently a contradiction and therefore false, so that the first part $a \neq b \wedge a < b$ of the preceding disjunction is true. This conjunction then implies in particular $a < b$, so that the left-hand side of the equivalence (3.235) is true. Thus, the proof of the equivalence is complete, and since X, $<$, a and b were arbitrary, we finally conclude that the proposition holds, as claimed. □

Definition 3.21 (Densely ordered set). We say that a linearly ordered set $(X, <)$ is *densely ordered* iff

1. X is neither empty nor a singleton, that is,

$$X \neq \emptyset \,\wedge\, \forall a\,(X \neq \{a\}), \tag{3.236}$$

   and

2. for any two elements x and y in X with $x < y$ there exists an element of X which is strictly between x and y, that is,

$$\forall x, y\,([x, y \in X \,\wedge\, x < y] \Rightarrow \exists z\,(z \in X \,\wedge\, x < z < y)). \tag{3.237}$$

3.3.2. Induced partial orderings

Any given irreflexive partial ordering may be used to define a corresponding reflexive partial ordering of the same set.

Theorem 3.81 (Characterization of induced reflexive partial orderings). *The following sentences hold for any set X and any irreflexive partial ordering $<_X$ of X.*

a) There exists a unique set $\leq_X$ such that an element Z is in $\leq_X$ iff Z is in $X \times X$ and moreover if there is an ordered pair (a,b) identical with Z where a is less than b or a is equal to b, that is,

$$\exists!\leq_X \forall Z\,(Z \in \leq_X \\ \Leftrightarrow [Z \in X \times X \wedge \exists a,b\,([a <_X b \vee a = b] \wedge (a,b) = Z)]).$$

b) The set $\leq_X$ is a binary relation on X satisfying

$$\forall a,b\,(a,b \in X \Rightarrow [a \leq_X b \Leftrightarrow (a <_X b \vee a = b)]) \tag{3.238}$$

and furthermore

$$\forall a,b\,(a,b \in X \Rightarrow [a <_X b \;\Leftrightarrow\; (a \leq_X b \wedge a \neq b)]). \tag{3.239}$$

c) Moreover, the binary relation $\leq_X$ is a reflexive partial ordering of X.

Proof. Letting X be an arbitrary set and $<_X$ an arbitrary irreflexive partial ordering of X, we evidently obtain the uniquely existential sentence a) with the Axiom of Specification in connection with the Equality Criterion for sets. Thus, the set $\leq_X$ satisfies

$$\forall Z\,(Z \in \leq_X \Leftrightarrow [Z \in X \times X \wedge \exists a,b\,([a <_X b \vee a = b] \wedge (a,b) = Z)]). \tag{3.240}$$

Concerning b), letting Z be arbitrary, we see that the assumption $Z \in \leq_X$ implies with (3.240) in particular $Z \in X \times X$, so that $\leq_X \subseteq X \times X$ follows to be true by definition of a subset. Thus, $\leq_X$ is a binary relation on X by definition.

Next, regarding (3.238), we let $\bar{a}$ and $\bar{b}$ be arbitrary and assume $\bar{a},\bar{b} \in X$. To prove the first part ('$\Rightarrow$') of the equivalence, we further assume $\bar{a} \leq_X \bar{b}$, which we may write also as $(\bar{a},\bar{b}) \in \leq_X$. This implies with (3.240) in particular that there exist elements, say $\bar{\bar{a}}$ and $\bar{\bar{b}}$, satisfying the disjunction $\bar{\bar{a}} <_X \bar{\bar{b}} \vee \bar{\bar{a}} = \bar{\bar{b}}$ and $(\bar{\bar{a}},\bar{\bar{b}}) = (\bar{a},\bar{b})$. The latter equation now gives $\bar{\bar{a}} = \bar{a}$ and $\bar{\bar{b}} = \bar{b}$ with the Equality Criterion for ordered pairs, so that the preceding disjunction becomes $\bar{a} <_X \bar{b} \vee \bar{a} = \bar{b}$ after carrying out substitutions based on these equations. We thus proved the first part of the equivalence in (3.238).

To prove the second part ('$\Leftarrow$'), we now assume the disjunction $\bar{a} <_X \bar{b} \vee \bar{a} = \bar{b}$ to be true, so that the existential sentence

$$\exists a, b\,([a <_X b \vee a = b] \wedge (a,b) = (\bar{a},\bar{b}))$$

evidently holds. Furthermore, the initially assumed $\bar{a}, \bar{b} \in X$ implies $(\bar{a},\bar{b}) \in X \times X$ by definition of the Cartesian product of two sets. Together with the preceding existential sentence, this further implies $(\bar{a},\bar{b}) \in \leq_X$ according to (3.240), which finding we may then write also as $\bar{a} \leq_X \bar{b}$. This completes the proof of the equivalence and thus the proof of the implication in (3.238). Since $\bar{a}$ and $\bar{b}$ are arbitrary, we may therefore conclude that the universal sentence (3.238) holds.

Then, Proposition 3.80 applies also to the binary relation $\leq_X$, so that (3.235) holds as well.

Concerning c), we first verify that $\leq_X$ is reflexive, i.e. that $\leq_X$ satisfies

$$\forall a\,(a \in X \Rightarrow a \leq_X a). \tag{3.241}$$

For this purpose, we let a be arbitrary and assume $a \in X$. Because $a = a$ is obviously true, the disjunction $a <_X a \vee a = a$ holds then as well, so that $a \leq_X a$ follows to be true with (3.238). Since a is arbitrary, we may therefore conclude that the universal sentence (3.241) holds, which means indeed that $\leq_X$ is a reflexive binary relation on X.

To establish the antisymmetry of $\leq_X$, we verify accordingly

$$\forall a, b\,(a, b \in X \Rightarrow [(a \leq_X b \wedge b \leq_X a) \Rightarrow a = b]), \tag{3.242}$$

letting a and b be arbitrary in X, and assuming moreover $a \leq_X b$ and $b \leq_X a$ to hold. These assumptions imply with (3.238) the truth of the disjunctions $a <_X b \vee a = b$ and $b <_X a \vee b = a$. Thus, the conjunction of these two disjunctions holds, which we may write equivalently as

$$(a = b \vee a <_X b) \wedge (a = b \vee b <_X a),$$

applying the Commutative Law for the disjunction. We now obtain with the Distributive Law for sentences (1.45) the true disjunction

$$a = b \vee (a <_X b \wedge b <_X a). \tag{3.243}$$

Since $<_X$ is an irreflexive partial ordering, we may apply Exercise 3.26 to infer from $a, b \in X$ that the negation $\neg(a <_X b \wedge b <_X a)$ is true. Thus, the second part of the true disjunction (3.243) is false, and therefore the first part $a = b$ is true. We thus arrived at the desired consequent of the implication in (3.242) to be proven. The truth of this implication in turn established the truth of the implication based on the antecedent $a, b \in X$, and since a and b were arbitrary, we may therefore conclude that (3.242) is true, i.e. that $\leq_X$ is antisymmetric.

Finally, we prove the transitivity of $\leq_X$ by verifying the universal sentence

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [(a \leq_X b \wedge b \leq_X c) \Rightarrow a \leq_X c]). \tag{3.244}$$

Letting a, b, c be arbitrary in X, we further assume that $a \leq_X b$ and $b \leq_X c$ hold, so that we obtain the true conjunction

$$(a <_X b \vee a = b) \wedge (b <_X c \vee b = c)$$

by applying (3.238). Then, the Distributive Law for sentences (1.44) yields the true disjunction

$$[(a <_X b \vee a = b) \wedge b <_X c] \vee [(a <_X b \vee a = b) \wedge b = c]. \tag{3.245}$$

In case the first part of this disjunction is true, we have that the disjunction $a <_X b \vee a = b$ and $b <_X c$ are both true. On the one hand, if $a <_X b$ holds, the conjunction $a <_X b \wedge b <_X c$ is evidently true, which further implies $a <_X c$ with the transitivity if $<_X$. Then, the disjunction $a <_X c \vee a = c$ is also true (irrespective of the truth value of $a = c$), so that we obtain the desired $a \leq_X c$ with (3.238). On the other hand, if $a = b$ holds, then the previously established $b <_X c$ yields again $a <_X c$ (via substitution), so that the disjunction $a <_X c \vee a = c$ and therefore $a \leq_X c$ follow again to be true.

In case the second part of the disjunction (3.245) is true, we now see that the disjunction $a <_X b \vee a = b$ and $b = c$ are both true. On the one hand, if $a <_X b$ holds, then substitution based on the preceding equation yields $a <_X c$ and therefore the true disjunction $a <_X c \vee a = c$, so that the desired $a \leq_X c$ follows to true as shown before. On the other hand, if $a = b$ holds, then we obtain in view of $b = c$ the true equation $a = c$, so that the disjunction $a <_X c \vee a = c$ holds again (irrespective of the truth value of $a <_X c$). Thus, we obtain once again $a \leq_X c$, completing the proof of the implications in (3.244). As a, b and c are arbitrary, we may then infer from this that (3.244) holds, so that $\leq_X$ is also transitive.

We thus demonstrated that the binary relation $\leq_X$ on X is reflexive, antisymmetric and transitive, so that $\leq_X$ is a reflexive partial ordering of X, by definition. In the proofs of a) – c) the sets X and $<_X$ were arbitrary, so that the theorem holds, as claimed. $\square$

Note 3.9. The binary relation $\leq_X$ specified within the preceding theorem is thus compatible with Definition 3.20 of '$\leq$'.

Definition 3.22 (Induced reflexive partial ordering). For any set X and any irreflexive partial ordering $<_X$ of X we call the binary relation $\leq_X$ on X which satisfies

$$\forall a, b\,(a, b \in X \Rightarrow [a \leq_X b \Leftrightarrow (a <_X b \vee a = b)]) \tag{3.246}$$

the *reflexive partial ordering induced by* $<_X$.

Recalling that $\{(x, x)\}$ is a reflexive partial of ordering of a singleton $\{x\}$ according to Corollary 3.63 and that $\emptyset$ is an irreflexive partial ordering of $\{x\}$ due to Corollary 3.66, we now see that the preceding definition provides a direct link between the two partial orderings.

Proposition 3.82. *For any x it is true that the reflexive partial ordering $R = \{(x, x)\}$ of $X = \{x\}$ is identical with the reflexive partial ordering ($\leq_X$) induced by the irreflexive partial ordering $<_X = \emptyset$ of $\{x\}$.*

Proof. We let x be arbitrary and apply the Equality Criterion for sets to establish the identity of $R = \{(x, x)\}$ and the reflexive partial ordering $\leq_X$ induced by the irreflexive partial ordering $<_X = \emptyset$ of $X = \{x\}$. For this purpose, we verify

$$\forall Z\,(Z \in R \Leftrightarrow Z \in \leq_X), \tag{3.247}$$

letting Z be arbitrary. Let us first observe that $x \in X$ $[= \{x\}]$ holds according to (2.153). Now, to prove the first part ('$\Rightarrow$') of the equivalence, we assume $Z \in R$ $[= \{(x, x)\}]$ to be true, which implies $Z = (x, x)$ with (2.169). Then, the basic fact $x = x$ implies the truth of the disjunction $x <_X x \vee x = x$, so that (3.246) gives $x \leq_X x$. We may write this also as $[(x, x) =]\, Z \in \leq_X$, which finding completes the proof of the first part of the equivalence in (3.247).

To prove the second part ('$\Leftarrow$'), we now assume $Z \in \leq_X$ to be true. Since the (induced) reflexive partial ordering $\leq_X$ is by definition a binary relation on X, the equation $(\bar{a}, \bar{b}) = Z$ holds for particular constants $\bar{a}$ and $\bar{b}$, and $\leq_X$ is included in $X \times X$, i.e. included in $\{x\} \times \{x\}$. Consequently, $[Z =]\ (\bar{a}, \bar{b}) \in \leq_X$ implies $(\bar{a}, \bar{b}) \in \{x\} \times \{x\}$ with the definition of a subset. Since $\{x\} \times \{x\} = \{(x, x)\}$ holds due to (3.39), we therefore obtain $(\bar{a}, \bar{b}) \in \{(x, x)\}$ $[= R]$, so that the second part of the equivalence also holds. Since Z is arbitrary, we may now further conclude that (3.247) is true, so that the sets $\{(x, x)\}$ and $\leq_X$ are indeed equal. As x was also arbitrary, we may finally conclude that the proposition holds, as claimed. □

Theorem 3.83 (Induced total ordering). *For any set X and any linear ordering $<_X$ of X it is true that the induced reflexive partial ordering $\leq_X$ is a total ordering of X.*

Proof. Letting X and $<_X$ be arbitrary sets and assume that $<_X$ is a linear ordering of X. Let us first observe that $<_X$ is then more generally an irreflexive partial ordering of X, which induces the reflexive partial ordering $\leq_X$ of X. To show that $\leq_X$ is in the given situation also total, we now prove the universal sentence

$$\forall a, b\,(a, b \in X \Rightarrow [a \leq_X b \,\vee\, b \leq_X a]). \tag{3.248}$$

We let a and b be arbitrary and assume $a, b \in X$ to be true. Since the linear ordering $<_X$ is in particular connex, the multiple disjunction $a <_X b \vee b <_X a \vee a = b$ is true. If $a <_X b$ holds, then the disjunction $a <_X b \vee a = b$ is also true, which in turn implies $a \leq_X b$ with (3.238). Then, the desired disjunction $a \leq_X b \vee b \leq_X a$ also holds. Similarly, if the second part $b <_X a$ of the preceding multiple disjunction holds, then we obtain the true disjunction $b <_X a \vee b = a$, and therefore $b \leq_X a$ again with (3.238). Finally, if $a = b$ holds, then the previously considered disjunction $a <_X b \vee a = b$ is again true, with the consequence that $a \leq_X$ and then also $a \leq_X b \vee b \leq_X a$ hold, as shown earlier. Thus, the proof of the implication in (3.248) is complete, and since a and b are arbitrary, we may therefore conclude that the stated universal sentence is true. This means that the (induced) reflexive partial ordering $\leq_X$ is total and therefore, by definition, a total ordering of X. As X and $<_X$ were arbitrary, the theorem follows then to be true. □

Definition 3.23 (Induced total ordering). For any set X and any linear ordering $<_X$ of X we call the induced reflexive partial ordering $\leq_X$, which is total according to Theorem 3.83, the *total ordering induced by* $<_X$.

Proposition 3.84. *For any x and any y satisfying $x \neq y$ it is true that the total ordering $\leq_X$ induced by the linear ordering $<_X = \{(x, y)\}$ of the pair $X = \{x, y\}$ is identical with the triple $\{(x, x), (x, y), (y, y)\}$.*

Proof. We take arbitrary x and y and assume $x \neq y$ to be true, so that $<_X = \{(x, y)\}$ is indeed a linear ordering of $X = \{x, y\}$ due to Corollary 3.74. To prove that the triple $\{(x, x), (x, y), (y, y)\}$ is identical with the total ordering $\leq_X$ induced by $<_X$, we apply the Equality Criterion for sets and demonstrate accordingly the truth of

$$\forall Z\,(Z \in \{(x, x), (x, y), (y, y)\} \Leftrightarrow Z \in\, \leq_X). \tag{3.249}$$

To do this, we let Z be arbitrary and assume first $Z \in \{(x,x),(x,y),(y,y)\}$ to be true, so that

$$Z = (x,x) \lor Z = (x,y) \lor Z = (y,y) \tag{3.250}$$

holds according to (2.230). We now use this true disjunction to prove the sentence $Z \in \leq_X$ by cases. In the first case $Z = (x,x)$, we observe that the basic fact $x = x$ gives the true disjunction $x <_X x \lor x = x$. Then, since $x \in \{x,y\}$ is true in view of (2.151), the preceding disjunction implies $x \leq_X x$ with (3.238), which we may write also as $(x,x) \in \leq_X$. This in turn gives the desired consequent $Z \in \leq_X$ by means of substitution based on the current case assumption $Z = (x,x)$. In the second case $Z = (x,y)$, we notice that $(x,y) \in \{(x,y)\}$ $[= <_X]$ holds according to (2.153), so that we obtain $(x,y) \in <_X$, which we may write as $x <_X y$. Then, the disjunction $x <_X y \lor x = y$ is true as well, and since $x,y \in \{x,y\}$ holds with (2.151), this disjunction implies $x \leq_X y$ with (3.238). This means that $[Z =]$ $(x,y) \in \leq_X$ is true, so that $Z \in \leq_X$ holds also in the second case. Finally, in the third case $Z = (y,y)$, we see that $y = y$ yields the true disjunction $y <_X y \lor y = y$ where $y \in \{x,y\}$ holds (as mentioned within the proof for the second case), so that $y \leq_X y$ follows to be true with (3.238). Thus, we have $[Z =]$ $(y,y) \in \leq_X$, which shows that $Z \in \leq_X$ is true once again, so that the proof of that sentence by cases is complete. Therefore, the first part ('$\Rightarrow$') of the equivalence in (3.249) holds.

To prove the second part ('$\Leftarrow$'), we now assume conversely $Z \in \leq_X$ to be true. Because the induced total ordering $\leq_X$ is a binary relation on $X = \{x,y\}$, it is included in the Cartesian product $\{x,y\} \times \{x,y\}$, so that $Z \in \leq_X$ implies $Z \in \{x,y\} \times \{x,y\}$ with the definition of a subset. Moreover, there exist by definition of a binary relation two constants, say $\bar{a}$ and $\bar{b}$, such that $(\bar{a},\bar{b}) = Z$ holds. Then, $Z \in \{x,y\} \times \{x,y\}$ gives $(\bar{a},\bar{b}) \in \{x,y\} \times \{x,y\}$ via substitution, so that we obtain $\bar{a} \in \{x,y\}$ and $\bar{b} \in \{x,y\}$ with the definition of the Cartesian product of two sets. Furthermore, $Z \in \leq_X$ yields with the preceding equation $(\bar{a},\bar{b}) \in \leq_X$, which we may also write as $\bar{a} \leq_X \bar{b}$. Consequently, this inequality implies the disjunction $\bar{a} <_X \bar{b} \lor \bar{a} = \bar{b}$ with (3.238), which we may now use to prove the sentence $Z \in \{(x,x),(x,y),(y,y)\}$ by cases. In the first case that $\bar{a} <_X \bar{b}$ is true, we thus have $(\bar{a},\bar{b}) \in <_X$ $[= \{(x,y)\}]$, so that we obtain $[Z =]$ $(\bar{a},\bar{b}) = (x,y)$ with (2.169). Because of $Z = (x,y)$, the multiple disjunction (3.250) is then also true, which in turn implies the desired $Z \in \{(x,x),(x,y),(y,y)\}$ with (2.230). In the second case that $\bar{a} = \bar{b}$ is true, we see that $Z = (\bar{a},\bar{b})$ yields $Z = (\bar{a},\bar{a})$ by means substitution, where $\bar{a} \in \{x,y\}$ holds, as mentioned earlier. Therefore, the disjunction $\bar{a} = x \lor \bar{a} = y$ is true by definition of a pair, which we may now use to prove (3.250). by cases. On the one hand, if $\bar{a} = x$ holds, then the previously established $Z = (\bar{a},\bar{a})$ gives $Z = (x,x)$, and the disjunction (3.250) is then also true. On the other hand, if $\bar{a} = y$ holds, then $Z = (\bar{a}, \bar{a})$ yields $Z = (y, y)$, so that (3.250) is again true. Thus, the proof of that multiple disjunction by cases is complete, and this disjunction implies $Z \in \{(x,x),(x,y),(y,y)\}$ as in the first case. The truth of this sentence in turn proves the second part of the equivalence in (3.249).

As Z is arbitrary, we may therefore conclude that (3.249) is true, and we may now infer from this universal sentence the truth of the equality $\{(x,x),(x,y),(y,y)\} = \leq_X$. This shows that the total ordering $\leq_X$ induced by the linear ordering $<_X$ is indeed identical with $\{(x,x),(x,y),(y,y)\}$. Since x and y were initially arbitrary, we may finally conclude that the proposition is true. □

Corollary 3.85. *For any x and any y the ordered pair*

$$(\{x,y\}, \{(x,x),(x,y),(y,y)\}) \tag{3.251}$$

is a totally ordered set.

Proof. We let x and y be arbitrary and observe that the Law of the Excluded Middle yields the true disjunction $x = y \lor x \neq y$, which we now use to prove by cases that (3.251) is a totally ordered set. In the first case when $x = y$ is true, we obtain on the one hand the equation $\{x,y\} = \{x\}$ with (2.167). On the other hand, we evidently have that the two conjunctions $x = x \land y = x$ and $y = x \land y = x$ are true, which imply with the Equality Criterion for ordered pairs $(x,y) = (x,x)$ and $(y,y) = (x,x)$, respectively; these equations then imply together with (2.231)

$$\{(x,x),(x,y),(y,y)\} = \{(x,x),(x,x),(x,x)\} = \{(x,x)\}.$$

These findings show that the ordered pair (3.251) reduces in the current first case to the totally ordered set $(\{x\}, \{(x,x)\})$ established in Corollary 3.70. Regarding the other case $x \neq y$, we have that (3.251) is a totally ordered set because of Proposition 3.84. Since x and y were initially arbitrary, we may therefore conclude that the corollary is true. □

We may also begin with a reflexive partial ordering and induce a corresponding irreflexive partial ordering.

Theorem 3.86 (Characterization of induced irreflexive partial orderings). *The following sentences are true for any set X and any reflexive partial ordering $\leq_X$ of X.*

a) There exists a unique set $<_X$ such that an element Z is in $<_X$ iff Z is in $X \times X$ and moreover if there is an ordered pair (a,b) identical *with Z where a is less than or equal to b and simultaneously unequal to b, that is,*

$$\exists! <_X \forall Z \, (Z \in <_X \\ \Leftrightarrow [Z \in X \times X \, \wedge \, \exists a, b \, ([a \leq_X b \, \wedge \, a \neq b] \, \wedge \, (a, b) = Z)]).$$

b) *The set $<_X$ is a binary relation on X satisfying*

$$\forall a, b \, (a, b \in X \Rightarrow [a <_X b \; \Leftrightarrow \; (a \leq_X b \, \wedge \, a \neq b)]) \tag{3.252}$$

and moreover

$$\forall a, b \, (a, b \in X \Rightarrow [a \leq_X b \Leftrightarrow (a <_X b \, \vee \, a = b)]). \tag{3.253}$$

c) *The binary relation $<_X$ is an irreflexive partial ordering of X.*

Proof. We let X be an arbitrary set and $\leq_X$ an arbitrary reflexive partial ordering of X. We now evidently obtain the uniquely existential sentence a) with the Axiom of Specification in connection with the Equality Criterion for sets. Thus, the set $<_X$ satisfies

$$\forall Z \, (Z \in <_X \Leftrightarrow [Z \in X \times X \wedge \exists a, b \, ([a \leq_X b \wedge a \neq b] \wedge (a, b) = Z)]). \tag{3.254}$$

Concerning b), letting Z be arbitrary, we see that the assumption $Z \in <_X$ implies with (3.254) in particular $Z \in X \times X$, so that $<_X \subseteq X \times X$ follows to be true by definition of a subset. Thus, $<_X$ is a binary relation on X by definition.

Next, regarding (3.252), we let $\bar{a}$ and $\bar{b}$ be arbitrary and assume $\bar{a}, \bar{b} \in X$. To prove the first part ('$\Rightarrow$') of the equivalence, we further assume $\bar{a} <_X \bar{b}$, which we may write also as $(\bar{a}, \bar{b}) \in <_X$. This implies with (3.254) in particular that there exist elements, say $\bar{\bar{a}}$ and $\bar{\bar{b}}$, satisfying the conjunction $\bar{\bar{a}} \leq_X \bar{\bar{b}} \wedge \bar{\bar{a}} \neq \bar{\bar{b}}$ and $(\bar{\bar{a}}, \bar{\bar{b}}) = (\bar{a}, \bar{b})$. The latter equation now gives $\bar{\bar{a}} = \bar{a}$ and $\bar{\bar{b}} = \bar{b}$ with the Equality Criterion for ordered pairs, so that the preceding conjunction becomes $\bar{a} \leq_X \bar{b} \wedge \bar{a} \neq \bar{b}$ after carrying out substitutions based on these equations. We thus proved the first part of the equivalence in (3.252).

To prove the second part ('$\Leftarrow$'), we now assume the conjunction $\bar{a} \leq_X \bar{b} \wedge \bar{a} \neq \bar{b}$ to be true, so that the existential sentence

$$\exists a, b \, ([a \leq_X b \, \wedge \, a \neq b] \, \wedge \, (a, b) = (\bar{a}, \bar{b}))$$

evidently holds. Furthermore, the initially assumed $\bar{a}, \bar{b} \in X$ implies $(\bar{a}, \bar{b}) \in X \times X$ by definition of the Cartesian product of two sets. Together with the preceding existential sentence, this further implies $(\bar{a}, \bar{b}) \in <_X$ according to (3.254), which finding we may then write also as $\bar{a} <_X \bar{b}$. This completes the proof of the equivalence and thus the proof of the implication in (3.252). Since $\bar{a}$ and $\bar{b}$ are arbitrary, we may therefore conclude that the universal sentence (3.252) holds.

Regarding (3.253), we now let $\bar{a}$ and $\bar{b}$ be arbitrary and assume $\bar{a}, \bar{b} \in X$ to be true. To prove the first part ('$\Rightarrow$') of the equivalence by contradiction, we further assume $\bar{a} \leq_X \bar{b}$ and the negation $\neg(\bar{a} <_X \bar{b} \vee \bar{a} = \bar{b})$ to be true. The latter yields with De Morgan's Law for the disjunction the truth of $\neg \bar{a} <_X \bar{b}$ and of $\bar{a} \neq \bar{b}$. Here, we may write the former negation also as $\neg(\bar{a}, \bar{b}) \in <_X$, which negation in turn implies with (3.254) in connection with the Law of Contraposition and De Morgan's Law for the conjunction that $(\bar{a}, \bar{b}) \notin X \times X$ or

$$\neg \exists a, b \, (a \leq_X b \wedge a \neq b \wedge (a, b) = (\bar{a}, \bar{b}))$$

holds. Since the initial assumption $\bar{a}, \bar{b} \in X$ implies the truth of $(\bar{a}, \bar{b}) \in X \times X$ with the definition of the Cartesian product of two sets, the first part $(\bar{a}, \bar{b}) \notin X \times X$ of the preceding disjunction is false, so that the second part of that disjunction, i.e. the preceding negated existential sentence, is true. We may write the latter equivalently as

$$\forall a, b \, ((a, b) = (\bar{a}, \bar{b}) \Rightarrow \neg[a \leq_X b \wedge a \neq b])$$

by using the Quantifier Negation Law (1.54) in connection with the Commutative Law for the conjunction and (1.81). We may now infer from this for $\bar{a}$ and $\bar{b}$ that $\neg[\bar{a} \leq_X \bar{b} \wedge \bar{a} \neq \bar{b}]$ is true, in contradiction to the previously assumed/established truth of $\bar{a} \leq_X \bar{b}$ and $\bar{a} \neq \bar{b}$. Thus, the proof of the first part of the equivalence in (3.253) via contradiction is complete.

To prove the second part ('$\Leftarrow$'), we assume the disjunction $\bar{a} <_X \bar{b} \vee \bar{a} = \bar{b}$ to be true. In case of $\bar{a} <_X \bar{b}$, it follows with (3.254) in particular that there are elements, say $\bar{\bar{a}}$ and $\bar{\bar{b}}$, satisfying $\bar{\bar{a}} \leq_X \bar{\bar{b}}$, $\bar{\bar{a}} \neq \bar{\bar{b}}$ and $(\bar{\bar{a}}, \bar{\bar{b}}) = (\bar{a}, \bar{b})$. This equation gives with the Equality Criterion for ordered pairs $\bar{\bar{a}} = \bar{a}$ and $\bar{\bar{b}} = \bar{b}$, so that the previous finding $\bar{\bar{a}} \leq_X \bar{\bar{b}}$ yields the desired $\bar{a} \leq_X \bar{b}$ by substitution. In case the second part $\bar{a} = \bar{b}$ of the assumed disjunction holds, we see that the reflexivity of $\leq_X$ gives first $a \leq_X a$ and then again the desired $a \leq_X b$ via substitution based on the preceding equation. This completes the proof of the equivalence in (3.253), and since $\bar{a}$ and $\bar{b}$ were arbitrary, we may therefore conclude that the universal sentence (3.253) is true. As X and $\leq_X$ were initially arbitrary, this sentence is then true for any X and any $\leq_X$. $\square$

Exercise 3.31. Prove Theorem 3.86c).

Definition 3.24 (Induced irreflexive partial ordering). For any set X and any reflexive partial ordering $\leq_X$ of X we call the binary relation $<_X$ on X that satisfies

$$\forall a, b\,(a, b \in X \Rightarrow [a <_X b \Leftrightarrow (a \leq_X b \wedge a \neq b)]) \tag{3.255}$$

the *reflexive partial ordering induced by* $\leq_X$.

Example 3.1. The irreflexive partial ordering $<_{\mathcal{X}}$ induced by the reflexive partial ordering of inclusion '$\subseteq$' on a set system $\mathcal{X}$ satisfies

$$\begin{aligned}&\forall a, b\,(a, b \in X \Rightarrow [a <_X b \;\Leftrightarrow\; (a \subseteq b \wedge a \neq b)])\\ &\forall a, b\,(a, b \in X \Rightarrow [a \subseteq b \Leftrightarrow (a <_X b \vee a = b)])\end{aligned}$$

according to (3.252) and (3.253). Because of the definition of a proper subset and due to (2.26), these universal sentences evidently remain true if we replace $<_X$ by $\subset$. Thus, $\subset$ represents the irreflexive partial ordering of proper inclusion on a set system $\mathcal{X}$.

Proposition 3.87. *For any x it is true that the irreflexive partial ordering $R = \emptyset$ of $X = \{x\}$ is identical with the irreflexive partial ordering ($<_X$) induced by the reflexive partial ordering $\leq_X = \{(x, x)\}$ of $\{x\}$.*

Proof. We let x be arbitrary and apply the definition of the empty set to establish the identity of $R = \emptyset$ and of the irreflexive partial ordering $<_X$ induced by the reflexive partial ordering $\leq_X = \{(x, x)\}$ of $X = \{x\}$, by verifying

$$\forall Z\,(Z \notin\, <_X). \tag{3.256}$$

To do this, we let Z be arbitrary and show that assuming $Z \in\, <_X$ to be true implies a contradiction. As the (induced) irreflexive partial ordering $<_X$ is (by definition) a binary relation on X, we have $(\bar{a}, \bar{b}) = Z$ for particular constants $\bar{a}$ and $\bar{b}$, and $<_X$ is included in $X \times X$, that is, included in $\{x\} \times \{x\}$. With this, $[Z =]\ (\bar{a}, \bar{b}) \in\, <_X$ yields $(\bar{a}, \bar{b}) \in \{x\} \times \{x\}$ by definition of a subset, and therefore with the definition of the Cartesian product of two sets $\bar{a}, \bar{b} \in \{x\}$. The latter in turn implies $\bar{a} = x$ and $\bar{b} = x$ with (2.169), so that $(\bar{a}, \bar{b}) \in\, <_X$ gives $(x, x) \in\, <_X$ via substitution, which we may now write also as $x <_X x$. This inequality contradicts the fact that $\neg x <_X x$ is true because $<_X$ is irreflexive, and this finding completes the proof of $Z \notin\, <_X$ by contradiction. Since Z is arbitrary, we may therefore conclude that (3.256) holds, so that $<_X = \emptyset$ follows to be true (by definition of the empty set). Then, as x was also arbitrary, we may further conclude that the proposed universal sentence is true. □

Note 3.10. Proposition 3.82 and Proposition 3.87 show that the reflexive partial ordering $\{(x,x)\}$ of $\{x\}$ and the irreflexive partial ordering $\emptyset$ of $\{x\}$ induce each other.

The preceding note and the properties of an induced reflexive/irreflexive partial ordering suggest that there are pairs of partial orderings, consisting of a reflexive and an irreflexive partial ordering which correspond to each other in the sense that each of them uniquely determines the other one.

Proposition 3.88. *For any set X and any reflexive partial ordering $\leq_X$ of X, it is true that there exists a unique irreflexive partial ordering of X which induces $\leq_X$, and this inducing irreflexive partial ordering is identical with the irreflexive partial ordering $<_X$ of X induced by $\leq_X$.*

Proof. We let X and $\leq_X$ be arbitrary, assume that $\leq_X$ is a reflexive partial ordering of X, and denote by $<_X$ the irreflexive partial ordering of X induced by $\leq_X$. In a first step, we apply the Equality Criterion for sets to prove that the reflexive partial ordering $\leq'_X$ induced by $<_X$ is identical with the given $\leq_X$. For this purpose, we verify

$$\forall Z\,(Z \in\, \leq_X \Leftrightarrow\; Z \in\, \leq'_X). \tag{3.257}$$

We let Z be arbitrary and assume first $Z \in\, \leq_X$ to be true. As a partial ordering of X, we have that $\leq_X$ is a binary relation such that $\leq_X \subseteq X \times X$ holds. By definition of a binary relation, there are then constants, say $\bar{a}$ and $\bar{b}$, satisfying $(\bar{a},\bar{b}) = Z$. Therefore, $Z \in\, \leq_X$ yields by substitution $(\bar{a},\bar{b}) \in\, \leq_X$, which in turn gives with the preceding inclusion $(\bar{a},\bar{b}) \in X \times X$ (by definition of a subset). This further implies $\bar{a},\bar{b} \in X$ with the definition of the Cartesian product of two sets, and we may write $(\bar{a},\bar{b}) \in\, \leq_X$ as $\bar{a} \leq_X \bar{b}$. Since $<_X$ is induced by $\leq_X$, these findings imply in view of (3.253) the disjunction $\bar{a} <_X \bar{b} \vee \bar{a} = \bar{b}$, which in turn implies $\bar{a} \leq'_X \bar{b}$ because of (3.238) in connection with the fact that $\leq'_X$ is induced by $<_X$. Writing this inequality now as $[Z =]\ (\bar{a},\bar{b}) \in\, \leq'_X$, we see that the desired consequent $Z \in\, \leq'_X$ is true, so that the proof of the first part ('$\Rightarrow$') of the equivalence in (3.257) is complete.

To prove the second part ('$\Leftarrow$'), we now conversely assume $Z \in\, \leq'_X$ to be true. In analogy to the proof of the first part, there exist particular constants $\bar{a}$ and $\bar{b}$ such that $(\bar{a},\bar{b}) = Z$ holds. We thus have $(\bar{a},\bar{b}) \in\, \leq'_X$, which implies $(\bar{a},\bar{b}) \in X \times X$ since the partial ordering $\leq'_X$ is included in $X \times X$. We therefore obtain $\bar{a},\bar{b} \in X$ (by definition of the Cartesian product of two sets), and we may write for $(\bar{a},\bar{b}) \in\, \leq'_X$ also $\bar{a} \leq'_X \bar{b}$. Because $\leq'_X$ is induced by $<_X$, (3.238) gives $\bar{a} <_X \bar{b} \vee \bar{a} = \bar{b}$, which further implies $\bar{a} \leq_X \bar{b}$ with (3.253) – recalling that $<_X$ is induced by $\leq_X$. Since we may write the preceding inequality also as $[Z =]\ (\bar{a},\bar{b}) \in\, \leq_X$, the consequent $Z \in\, \leq_X$ of the implication '$\Leftarrow$' holds, as desired. Thus, the proof of the equivalence in (3.257) is complete, and since Z was arbitrary, we may therefore conclude that (3.257) is true.

Consequently, the reflexive partial orderings $\leq_X$ and $\leq'_X$ are indeed identical. Based on this equality, we may now apply substitution to the assumption that $<_X$ induces $\leq'_X$ to find that $<_X$ induces $\leq_X$. To show that there exists a unique irreflexive partial ordering of X which induces $\leq_X$, it therefore remains for us to demonstrate that $<_X$ is the only irreflexive partial ordering of X which induces $\leq_X$. To do this, we establish

$$\forall <'_X \, (<'_X \text{ is an irreflexive partial ordering of } X \text{ inducing } \leq_X \Rightarrow <_X = <'_X), \tag{3.258}$$

according to Method 1.18. We let $<'_X$ be an arbitrary set, assume that $<'_X$ is an irreflexive partial ordering of X such that $\leq_X$ is the reflexive partial ordering induced by $<'_X$, and show that $<_X$ and $<'_X$ follow to be identical sets. For this purpose, we verify

$$\forall Z \, (Z \in <_X \Leftrightarrow \; Z \in <'_X), \tag{3.259}$$

letting Z be arbitrary. To establish the first part ('$\Rightarrow$') of the equivalence, we may evidently follow the same line of arguments as in the proof of the first part of the equivalence in (3.257) to infer from the assumption $Z \in <_X$ the truth of $\bar{a} <_X \bar{b}$ for two particular elements $\bar{a}, \bar{b} \in X$ with $Z = (\bar{a}, \bar{b})$. Since $\leq_X$ is induced by $<_X$, we then obtain with (3.239) the true conjunction $\bar{a} \leq_X \bar{b} \wedge \bar{a} \neq \bar{b}$, which in turn implies $\bar{a} <'_X \bar{b}$ with (3.239) and the assumption that $\leq_X$ is also induced by $<'_X$. The preceding inequality shows that $[(\bar{a}, \bar{b}) =] \; Z \in <'_X$ holds, as desired.

We may establish the second part ('$\Leftarrow$') of the equivalence by applying similar arguments as in the proof of the second part of the equivalence in (3.257). Assuming $Z \in <'_X$ to be true, where $<'_X \subseteq X \times X$ holds, we obtain $\bar{a} <'_X \bar{b}$ holds for some particular constants $\bar{a}, \bar{b} \in X$ with $Z = (\bar{a}, \bar{b})$. As $\leq_X$ is induced by $<'_X$, this yields $\bar{a} \leq_X \bar{b} \wedge \bar{a} \neq \bar{b}$ with (3.239), which conjunction in turn gives $\bar{a} <_X \bar{b}$ again with (3.239) – recalling that $\leq_X$ is also induced by $<_X$. This inequality may be written as $[(\bar{a}, \bar{b}) =] \; Z \in <_X$, so that the second part of the equivalence also holds. As Z was arbitrary, we may therefore conclude that (3.259) is true, which completes the proof of the proposed uniquely existential sentence. We thus also showed that the irreflexive partial ordering $<_X$ (induced by $\leq_X$) induces itself $\leq_X$. $\square$

Exercise 3.32. Prove for any set X and any irreflexive partial ordering $<_X$ of X that there exists a unique reflexive partial ordering of X which induces $<_X$, and show furthermore that this inducing reflexive partial ordering is identical with the reflexive partial ordering $\leq_X$ of X induced by $<_X$.

(Hint: Proceed in analogy to the proof of Proposition 3.88.)

Theorem 3.89 (Induced linear ordering). *For any set X and any total ordering $\leq_X$ of X it is true that the induced irreflexive binary relation $<_X$ is a linear ordering of X.*

Proof. We let X and $\leq_X$ be arbitrary and assume that $\leq_X$ is a total ordering of X. Since $\leq_X$ is then a reflexive partial ordering of X, it induces the irreflexive partial ordering $<_X$. We now prove

$$\forall a, b\,(a, b \in X \Rightarrow [a <_X b \,\vee\, b <_X a \,\vee\, a = b]), \tag{3.260}$$

letting $\bar{a}$ and $\bar{b}$ be arbitrary in X. Thus, the ordered pairs $\bar{U} = (\bar{a}, \bar{b})$ and $\bar{V} = (\bar{b}, \bar{a})$ exist, where the first equation shows – together with $\bar{a} \in X$ and $\bar{b} \in X$ – that there are elements a and b with $a \in X$, $b \in X$ and $(a, b) = \bar{U}$, so that $\bar{U} \in X \times X$ follows to be true by definition of the Cartesian product of two sets. Similarly, the second equation equation shows – jointly with $\bar{b} \in X$ and $\bar{a} \in X$ – that there are elements b and a such that $b \in X$, $a \in X$ and $(b, a) = \bar{V}$ hold, with the consequence that $\bar{V} \in X \times X$. Furthermore, since $\leq_X$ is total, the disjunction $\bar{a} \leq_X \bar{b} \,\vee\, \bar{b} \leq_X \bar{a}$ is true.

In case that $\bar{a} \leq_X \bar{b}$ holds, we observe that the disjunction $\bar{a} \neq \bar{b} \vee \bar{a} = \bar{b}$ is also true, according to (1.10). On the one hand, if $\bar{a} \neq \bar{b}$ holds (alongside the current case assumption $\bar{a} \leq_X \bar{b}$), then we see that there exist a and b such that $a \leq_X b$, $a \neq b$ and $(a, b) = \bar{U}$ hold. Together with the previously established $\bar{U} \in X \times X$, this existential sentence implies $\bar{U} \in <_X$ with (3.254) and therefore $\bar{a} <_X \bar{b}$. Then, the disjunction

$$\bar{a} <_X \bar{b} \,\vee\, \bar{b} <_X \bar{a} \,\vee\, \bar{a} = \bar{b} \tag{3.261}$$

is evidently also true (irrespective of the truth values of $\bar{b} <_X \bar{a}$ and $\bar{a} = \bar{b}$). On the other hand, if $\bar{a} = \bar{b}$ holds, then the disjunction (3.261) follows immediately to be true (irrespective of the truth values of $\bar{a} <_X \bar{b}$ and $\bar{b} <_X \bar{a}$).

In case that $\bar{b} \leq_X \bar{a}$ is true, we may apply exactly the same arguments to infer from this the truth of the equivalence (3.261). To begin with, we notice that now $\bar{b} \neq \bar{a}$ or $\bar{b} = \bar{a}$ holds. On the one hand, if $\bar{b} \neq \bar{a}$ holds (alongside the current case assumption $\bar{b} \leq_X \bar{a}$), then there are evidently b and a satisfying $b \leq_X a$, $b \neq a$ and $(b, a) = \bar{V}$. Together with the previously established $\bar{V} \in X \times X$, this existential sentence yields $\bar{V} \in <_X$ and therefore $\bar{b} <_X \bar{a}$, proving the disjunction (3.261) irrespective of the truth values of $\bar{a} <_X \bar{b}$ and $\bar{a} = \bar{b}$. On the other hand, if $\bar{b} = \bar{a}$ holds, then evidently also $\bar{a} = \bar{b}$, and the disjunction (3.261) holds again.

We thus proved that $\bar{a}, \bar{b} \in X$ implies (3.261) in any case, and since $\bar{a}, \bar{b}$ are arbitrary, we may therefore conclude that the universal sentence

(3.260) is true. Thus, the irreflexive partial ordering $<_X$ is connex, too, and therefore a linear ordering of X by definition. Since X and $\leq_X$ were arbitrary, it follows that the theorem holds, as claimed. □

Exercise 3.33. Simplify the proof of Theorem 3.89 by using (3.252) and (3.253).

Definition 3.25 (Induced linear ordering). For any set X and any total ordering $\leq_X$ of X we call the induced irreflexive partial ordering $<_X$ (which is linear due to Theorem 3.89) the *linear ordering induced by* $\leq_X$.

Notation 3.3. In view of Exercise 3.32, we will use expressions such as

- "any partially ordered set $(X, <_X)$"
- "any linearly ordered set $(X, <_X)$"
- "any partially ordered set $(X, \leq_X)$"
- "any totally ordered set $(X, \leq_X)$"

in the sense that $<_X$ and $\leq_X$ are any sets such that, respectively,

- $<_X$ is an irreflexive partial and $\leq_X$ a reflexive partial ordering of X
- $<_X$ is a linear ordering and $\leq_X$ a total ordering of X
- $\leq_X$ is a reflexive partial and $<_X$ an irreflexive partial ordering of X
- $\leq_X$ is a total ordering and $<_X$ a linear ordering of X

where $<_X$ and $\leq_X$ induce each other.

3.3.3. Extremal elements

Aside from comparing elements of a partially/totally/linearly ordered set only relatively, it is in certain cases possible to also compare in an absolute sense with respect to certain 'extreme' elements.

Definition 3.26 (Maximal & minimal element). For any partially ordered set $(X, <_X)$ and any nonempty subset A of X,

(1) we say that an element $m \in A$ is a *maximal element* of A (with respect to $<_X$) iff no element of A is greater than m, i.e. iff

$$\forall x\, (x \in A \Rightarrow \neg m <_X x). \tag{3.262}$$

(2) we say that an element $m \in A$ is a *minimal* of A (with respect to $<_X$) iff no element of A is less than m, i.e. iff

$$\forall x\,(x \in A \Rightarrow \neg x <_X m). \tag{3.263}$$

Note 3.11. Due to (1.81), a maximal element m of A is equivalently characterized by

$$\neg\exists x\,(x \in A \wedge m <_X x), \tag{3.264}$$

and a minimal element m of A by

$$\neg\exists x\,(x \in A \wedge x <_X m). \tag{3.265}$$

It will also be useful to have a concept for a 'largest' element which need not necessarily lie in the considered set A.

Definition 3.27 (Upper & lower bound, bounded-from-above & bounded from below set). For any partially ordered set $(X, \leq_X)$ and any subset A of X,

(1) we say that an element $u \in X$ is an *upper bound* for A (with respect to $\leq_X$) iff every element of A is less than or equal to u, i.e. iff

$$\forall x\,(x \in A \Rightarrow x \leq_X u). \tag{3.266}$$

Furthermore, we say that A is *bounded from above* (with respect to $\leq_X$) iff there exists an upper bound for A (with respect to $\leq_X$), i.e. iff

$$\exists u\,(u \in X \wedge \forall x\,(x \in A \Rightarrow x \leq_X u)). \tag{3.267}$$

(2) we say that an element $a \in X$ is a *lower bound* for A (with respect to $\leq_X$) iff any element of A is greater than or equal to a, i.e. iff

$$\forall x\,(x \in A \Rightarrow a \leq_X x). \tag{3.268}$$

Moreover, we say that A is *bounded from below* (with respect to $\leq_X$) iff there exists a lower bound for A (with respect to $\leq_X$), i.e. iff

$$\exists a\,(a \in X \wedge \forall x\,(x \in A \Rightarrow a \leq_X x)). \tag{3.269}$$

(3) we say that A is *bounded* (with respect to $\leq_X$) iff A is both bounded from above and bounded from below (with respect to $\leq_X$).

Proposition 3.90. *For any partially ordered set $(X, \leq_X)$ it is true that any element of X is an upper bound for the empty set, i.e.*

$$\forall x\,(x \in X \Rightarrow \forall y\,(y \in \emptyset \Rightarrow y \leq_X x)). \tag{3.270}$$

Proof. We let X and $\leq_X$ be arbitrary and assume that $\leq_X$ is a reflexive partial ordering of X. Next, we let x be arbitrary and assume $x \in X$ to be true. Then, we take an arbitrary y and observe that $y \in \emptyset$ is false by definition of the empty set. Thus, the implication in (3.270) based on the false antecedent $y \in \emptyset$ is itself true. Since y is arbitrary, we may therefore conclude that the stated universal sentence with respect to y is true, which in turn proves the implication based on the antecedent $x \in X$. Because x is also arbitrary, we may now further conclude that the universal sentence (3.270) holds. Finally, as X and $\leq_X$ were initially arbitrary, the proposed sentence follows then to be true. □

Exercise 3.34. Verify for any partially ordered set $(X, \leq_X)$ that every element of X is a lower bound for the empty set, i.e.

$$\forall x\,(x \in X \Rightarrow \forall y\,(y \in \emptyset \Rightarrow x \leq_X y)). \tag{3.271}$$

Proposition 3.91. *For any partially ordered set $(X, \leq_X)$ and any element $y \in X$, it is true that the singleton formed by y is bounded from above by y.*

Proof. We let X, $\leq_X$ and y be arbitrary, assume that $\leq_X$ is a reflexive partial ordering of X, and assume moreover that $y \in X$ is true. Therefore, the singleton $\{y\}$ is a subset of X according to (2.184). To show that y is an upper bound for $\{y\}$, we now verify

$$\forall x\,(x \in \{y\} \Rightarrow x \leq_X y). \tag{3.272}$$

To do this, we let x be arbitrary and assume $x \in \{y\}$ to be true, which implies $x = y$ with (2.169). As we initially assumed $\leq_X$ to be reflexive, $x \leq_X x$ is true, so that substitution based on the preceding equation gives $x \leq_X y$, which proves the implication in (3.272). Since x is arbitrary, we may therefore conclude that the universal sentence (3.272) is true, so that y is by definition an upper bound for $\{y\}$. As X, $\leq_X$ and y were arbitrary, the proposed universal sentence follows then to be true. □

Exercise 3.35. Show for any partially ordered set $(X, \leq_X)$ and any $y \in X$ that the singleton formed by y is bounded from below by y.

Theorem 3.92 (Characterization of upper & lower bounds for pairs). *For any partially ordered set $(X, \leq_X)$ and any y, z it is true that*

a) an element $u \in X$ is an upper bound for the pair $\{y, z\}$ iff

$$y \leq_X u \wedge z \leq_X u. \tag{3.273}$$

b) an element $a \in X$ is a lower bound for the pair $\{y, z\}$ iff

$$a \leq_X y \wedge a \leq_X z. \tag{3.274}$$

Proof. We let X, $\leq_X$, y and z be arbitrary, assume $\leq_X$ to be a reflexive partial ordering of X, and assume $y, z \in X$ to be true, so that $\{y, z\} \subseteq X$ holds according to (2.164). Concerning a), we let u be arbitrary and assume that $u \in X$ holds. Next, we assume that u is an upper bound for $\{y, z\}$, so that

$$\forall x \, (x \in \{y, z\} \Rightarrow x \leq_X u) \tag{3.275}$$

holds. According to (2.151), $y \in \{y, z\}$ and $z \in \{y, z\}$ are both true, so that (3.275) gives $y \leq_X u$ and $z \leq_X u$, and therefore the conjunction (3.273) holds.

Conversely, we now assume this conjunction $y \leq_X u \wedge z \leq_X u$ to be true, and we now show that this implies the truth of the universal sentence (3.275). Letting x be arbitrary and assuming $x \in \{y, z\}$ to be true, it follows by definition of a pair that the disjunction $x = y \vee x = z$ holds. If $x = y$ is true, then the assumed $y \leq_X u$ yields $x \leq_X u$ via substitution, and if $x = z$ holds, then the assumed $z \leq_X u$ gives $x \leq_X u$ again. Therefore, the implication in (3.275) is true, and as x is arbitrary, we may infer from this that (3.275) holds. Consequently, u is by definition an upper bound for $\{y, z\}$, and the proof of the stated equivalence is complete. Since u is arbitrary, we may then further conclude that a) is true. This is then true for any X, any $\leq_X$, and any a, b. □

Exercise 3.36. Prove Theorem 3.92b).

The preceding characterization extends in a natural way to triples.

Theorem 3.93 (Characterization of upper & lower bounds for triples). *For any partially ordered set $(X, \leq_X)$ and any $w, y, z \in X$ it is true that*

a) an element $u \in X$ is an upper bound for the triple $\{w, y, z\}$ iff

$$w \leq_X u \wedge y \leq_X u \wedge z \leq_X u. \tag{3.276}$$

b) an element $a \in X$ is a lower bound for the pair $\{y, z\}$ iff

$$a \leq_X w \wedge a \leq_X y \wedge a \leq_X z. \tag{3.277}$$

Exercise 3.37. Prove Theorem 3.93.

(Hint: Proceed in analogy to the proof of Theorem 3.92.)

Proposition 3.94. *For any partially ordered set $(X, \leq_X)$ and any bounded-from-above subset A of X (with respect to $\leq_X$) it is true that any subset B of A is itself bounded from above (with respect to $\leq_X$), having the same upper bound as A.*

Proof. We let X and $\leq_X$ be arbitrary sets such that $\leq_X$ is a reflexive partial ordering of X, let A an arbitrary bounded-from-above subset of X, and let B an arbitrary subset of A. Since A is bounded from above, there exists an element of X, say $\bar{u}$, such that

$$\forall x\,(x \in A \Rightarrow x \leq_X \bar{u}). \tag{3.278}$$

We now show that $\bar{u}$ is also an upper bound for B, i.e.

$$\forall x\,(x \in B \Rightarrow x \leq_X \bar{u}). \tag{3.279}$$

For this purpose, we let x be arbitrary in B, which implies $x \in A$ due to the assumption $B \subseteq A$ (applying the definition of a subset). Then, $x \in A$ implies in turn $x \leq_X \bar{u}$ with (3.278), which proves the implication in (3.279). As x is arbitrary, we therefore conclude that the universal sentence (3.279) holds, so that the existential sentence

$$\exists u\,(u \in X \;\wedge\; \forall x\,(x \in B \Rightarrow x \leq_X u))$$

is true. Thus, the set B is bounded from above by definition. Since X, $\leq_X$, A and B were also arbitrary, it follows that the proposition holds, as claimed. □

Exercise 3.38. Show for any partially ordered set $(X, \leq_X)$ and any bounded-from-below subset A of X that any subset B of A is itself bounded from below, having the same lower bound as A.

We now consider the particular subset $X \subseteq X$ in place of A.

Proposition 3.95. *For any partially ordered set $(X, \leq_X)$ and any subset A of X, it is true that, if there exists an upper bound for A in A, then it exists uniquely, i.e.*

$$\exists u\,(u \in A \wedge \forall x\,(x \in A \Rightarrow x \leq_X u)) \Rightarrow \exists! u\,(u \in A \wedge \forall x\,(x \in A \Rightarrow x \leq_X u)). \tag{3.280}$$

Proof. We let X, $\leq_X$ and A be arbitrary sets, assume that $\leq_X$ is a reflexive partial ordering of X and that A is a subset of X, and we further assume that there exists an upper bound for A in A. Thus, the existential part of the uniquely existential sentence to be proven holds already by assumption. To establish the truth of the uniqueness part, we verify

$$\forall u, u' \, ([u \in A \, \wedge \, \forall x \, (x \in A \Rightarrow x \leq_X u) \, \wedge \, u' \in A \, \wedge \, \forall x \, (x \in A \Rightarrow x \leq_X u')] \\ \Rightarrow u = u'). \tag{3.281}$$

For this purpose, we let u and u' be arbitrary, assume $u, u' \in A$, and assume further that the universal sentences

$$\forall x \, (x \in A \Rightarrow x \leq_X u) \tag{3.282}$$

$$\forall x \, (x \in A \Rightarrow x \leq_X u') \tag{3.283}$$

hold. Then, the assumed $u' \in A$ implies $u' \leq_X u$ with (3.282), and similarly the assumed $u \in A$ implies $u \leq_X u'$ with (3.283). As the reflexive partial ordering $\leq$ is by definition antisymmetric, the conjunction $u' \leq_X u \wedge u \leq_X u'$ implies then the desired $u = u'$, so that the proof of the implication in (3.281) is complete. Since u and u' are arbitrary, we may therefore conclude that (3.281) and thus the uniqueness part of the uniquely existential sentence to be proven also hold. This completes the proof of the implication (3.280); as X, $\leq_X$ and A were arbitrary, we may therefore conclude that the proposition is true. □

Exercise 3.39. Show for any partially ordered set $(X, \leq_X)$ and any $A \subseteq X$ that, if there exists a lower bound for A, then it exists uniquely, i.e.

$$\exists a \, (a \in A \wedge \forall x \, (x \in A \Rightarrow a \leq_X x)) \Rightarrow \exists! a \, (a \in A \wedge \forall x \, (x \in A \Rightarrow a \leq_X x)). \tag{3.284}$$

Definition 3.28 (Maximum/greatest/top & minimum/least/bottom element). For any partially ordered set $(X, \leq_X)$ and any subset A of X,

(1) we say that an element $u \in X$ is the *maximum element* or the *greatest element* or the *top element* (with respect to $\leq_X$), symbolically

$$u = \overset{\leq_X}{\max} A = \max A, \tag{3.285}$$

iff u is an upper bound for A (with respect to $\leq_X$) contained in A, i.e. iff

$$\forall x \, (x \in A \Rightarrow x \leq_X u) \, \wedge \, u \in A. \tag{3.286}$$

(2) we say that an element $a \in X$ is the *minimum element* or the *least element* or the *bottom element* (with respect to $\leq_X$), symbolically

$$a = \overset{\leq_X}{\min} A = \min A, \tag{3.287}$$

iff a is a lower bound for A (with respect to $\leq_X$) contained in A, i.e. iff

$$\forall x\,(x \in A \Rightarrow a \leq_X x) \wedge a \in A. \tag{3.288}$$

Corollary 3.96. *For any partially ordered set* $(X, \leq_X)$ *and any element* y *in* X *it is true that* y *is the maximum and the minimum of the singleton formed by* y*, i.e.*

$$\forall y\,(y \in X \Rightarrow y = \max\{y\} = \min\{y\}). \tag{3.289}$$

Proof. We let X, $\leq_X$ and y be arbitrary, assume $\leq_X$ to be a reflexive partial ordering of X, and moreover we assume $y \in X$ to be true. Then, y is both an upper and a lower bound for $\{y\}$ according to Proposition 3.91 and Exercise 3.35. Since $y \in \{y\}$ is evidently true as well, it follows that y is the maximum and also the minimum of $\{y\}$. As y is arbitrary, we may therefore conclude that (3.289) holds, and since X and $\leq_X$ were also arbitrary, it follows that the corollary is true. □

Maximal and maximum elements coincide in case of total/linear orderings.

Proposition 3.97. *For any linearly ordered set* $(X, <_X)$ *and any subset* A *of* X*, it is true that an element* $m \in A$ *is a maximal element of* A *iff* m *is the maximum element of* A*.*

Proof. We let X, $<_X$ and $\leq_X$ be arbitrary sets such that $<_X$ is a linear and $\leq_X$ a total ordering of X inducing each other. Moreover, we let A be an arbitrary subset of X and m an arbitrary element of A. To prove the first implication '⇒', we assume that m is a maximal element of A so that

$$\forall x\,(x \in A \Rightarrow \neg m <_X x) \tag{3.290}$$

holds. To show that m is then the maximum elements, i.e. an upper bound for A (in A), we verify

$$\forall x\,(x \in A \Rightarrow x \leq_X m). \tag{3.291}$$

To do this, we let x be arbitrary and assume $x \in A$, which implies $\neg m <_X x$ with (3.290). The latter further implies $m \geq_X x$ with the Negation Formula for $<$, so that the desired consequent $x \leq_X m$ in (3.291) is true. Since x is arbitrary, we may therefore conclude that the universal sentence (3.291) holds, which means that m is indeed the maximum element of A.

Conversely, to prove the second implication '⇐', we now assume that m is the maximum element of X, so that the universal sentence (3.291) is by definition true. To prove that m follows to be also a maximal element of A, we demonstrate the truth of (3.290). For this purpose, we let $x \in A$ be arbitrary, so that $x \leq_X m$ and therefore $m \geq_X x$ is true. The latter implies now $\neg m <_X x$ with the Negation Formula for $<$, so that the the implication in (3.290) holds. As x was arbitrary, we may infer from this the truth of the universal sentence (3.290). Thus, m is by definition a maximal element of A. Since X and $<_X/\leq_X$ were arbitrary, we may finally conclude that the proposition is true. $\square$

Exercise 3.40. Show for any linearly ordered set $(X, <_X)$ and any subset A of X that an element m of A is a minimal element of A iff m is the minimum element of A.

A special case of a linearly ordered set is one for which all subsets have a least element.

Definition 3.29 (Well-ordering, well-ordered set). For any totally ordered set $(X, \leq_X)$ we say that $\leq_X$ is a *well-ordering* of X iff every nonempty subset of X has a least element, that is, iff

$$\forall A\,([A \subseteq X \;\wedge\; A \neq \emptyset] \Rightarrow \exists m\,(m \in A \;\wedge\; \forall x\,(x \in A \Rightarrow m \leq_X x))). \quad (3.292)$$

We then call $(X, \leq_X)$ a *well-ordered set*.

The notions of a 'least upper bound' and of a 'greatest lower bound' will be of great practical relevance. define these concepts after the following preparatory exercise.

Exercise 3.41. Verify the following sentences for any partially ordered set $(X, \leq_X)$ and any subset A of X.

a) There exists a unique set U consisting of all the upper bounds for A (with respect to $\leq_X$), in the sense that

$$\forall u\,(u \in U \Leftrightarrow \forall x\,(x \in A \Rightarrow x \leq_X u)). \quad (3.293)$$

b) There exists a unique set L consisting of all the lower bounds for A (with respect to $\leq_X$), in the sense that

$$\forall a\,(a \in L \Leftrightarrow \forall x\,(x \in A \Rightarrow a \leq_X x)). \quad (3.294)$$

(Hint: Apply Axiom 2.2 in connection with Theorem 2.6, recalling that any upper/lower bound for A is by definition an element of X.)

Definition 3.30 (Supremum/least upper bound, infimum/greatest lower bound). For any partially ordered set $(X, \leq_X)$, for any subset A of X, and

(1) for any S, we say that S is the *supremum* or the *least upper bound* of A (with respect to $\leq_X$), symbolically

$$S = \overset{\leq}{\sup} A = \sup A, \tag{3.295}$$

iff S is the least element of the set U consisting of all upper bounds for A (with respect to $\leq_X$).

(2) for any I, we say that I is the *infimum* or the *greatest lower bound* of A (with respect to $\leq_X$), symbolically

$$I = \overset{\leq}{\inf} A = \inf A, \tag{3.296}$$

iff I is the greatest element of the set L consisting of all lower bounds for A (with respect to $\leq_X$).

Theorem 3.98 (Characterization of suprema & infima). *For any partially ordered set $(X, \leq_X)$ and any subset A of X, it is true*

a) for any S that S is the supremum of A (with respect to $\leq_X$) iff S is an upper bound for A and moreover if S is less than or equal to any upper bound S' for A, i.e. iff

$$\forall x\,(x \in A \Rightarrow x \leq_X S) \wedge \forall S'\,(\forall x\,(x \in A \Rightarrow x \leq_X S') \Rightarrow S \leq_X S'). \tag{3.297}$$

b) for any I that I is the infimum of A (with respect to $\leq_X$) iff I is a lower bound for A and moreover if I is greater than or equal to any lower bound I' for A, i.e. iff

$$\forall x\,(x \in A \Rightarrow I \leq_X x) \wedge \forall I'\,(\forall x\,(x \in A \Rightarrow I' \leq_X x) \Rightarrow I' \leq_X I). \tag{3.298}$$

Proof. We let X be an arbitrary set, $\leq_X$ an arbitrary reflexive partial ordering of X, and A an arbitrary subset of X. Concerning a), we let S be arbitrary and prove the first part ('$\Rightarrow$') of the proposed equivalence directly, assuming that S is the supremum of A, so that S is by definition the least element of the set U consisting of all upper bounds for A. By definition of a least element, we thus have that S is a lower bound for U contained in U, i.e.

$$\forall S'\,(S' \in U \Rightarrow S \leq_X S') \wedge S \in U. \tag{3.299}$$

Here, $S \in U$ implies with (3.293) that S is an upper bound for A, i.e.

$$\forall x \, (x \in A \Rightarrow x \leq_X S), \tag{3.300}$$

so that the first part of the conjunction (3.297) to be proven holds (for the given S). We now verify the second part

$$\forall S' \, (\forall x \, (x \in A \Rightarrow x \leq_X S') \Rightarrow S \leq_X S'). \tag{3.301}$$

To do this, we let $\bar{S}'$ be arbitrary and assume

$$\forall x \, (x \in A \Rightarrow x \leq_X \bar{S}') \tag{3.302}$$

to be true, which conjunction implies $\bar{S}' \in U$ with (3.293) and then $S \leq_X \bar{S}'$ with the first part of the conjunction (3.299). This proves the implication in (3.301), and since $\bar{S}'$ is arbitrary, we may therefore conclude that the universal sentence (3.301) is true. Thus, the proof of the conjunction 3.297) (for the given S) is complete, so that the first part of the proposed equivalence holds.

To prove the second part ('$\Leftarrow$'), we now assume that S satisfies (3.297) and show that S is the supremum of A, i.e. that S is the least element of the set U, i.e. that S satisfies (3.299). Regarding the first part of that conjunction, we let $\bar{S}'$ be arbitrary and assume that $\bar{S}' \in U$ holds, which assumption implies (3.302) with the specification of U, i.e. that $\bar{S}'$ is an upper bound for A. This further implies with the second part of the assumed conjunction (3.297) that $S \leq_X \bar{S}'$ holds. Therefore, $\bar{S}'$ satisfies indeed the first part of the conjunction (3.299). Furthermore, the first part of the assumed conjunction (3.297) implies $S \in U$ by definition of U, so that the second part of the conjunction (3.299) also holds. This means that S is the supremum of A, completing the proof of the stated equivalence. Since S is arbitrary, we may therefore conclude that a) holds.

As X, $\leq_X$ and A were also arbitrary, we may further conclude that a) is true for any X, and $\leq_X$ and any A. □

Exercise 3.42. Prove Part b) of Theorem 3.98.

Note 3.12. If S is the supremum of a subset A of a set X, then we have $S \in X$ since upper bounds of $A \subseteq X$ are by definition elements of X. Similarly, if I is the infimum of $A \subseteq X$, then we have $I \in X$ because lower bounds of $A \subseteq X$ are by definition elements of X.

We will frequently encounter sets that are 'closed' under the formation of suprema and infima in the following sense.

Definition 3.31 (Lattice, complete lattice). For any nonempty set X we say that

(1) a partially ordered set $(X, \leq_X)$ is a *lattice* iff every pair $\{x, y\}$ of elements in X has both a supremum and an infimum with respect to $\leq_X$, i.e. iff

$$\forall x, y\, (x, y \in X \Rightarrow \exists S, I\, (S, I \in X \,\wedge\, S = \overset{\leq_X}{\sup}\{x, y\} \,\wedge\, I = \overset{\leq_X}{\inf}\{x, y\})). \tag{3.303}$$

(2) a partially ordered set $(X, \leq_X)$ is a *complete lattice* iff every subset A of X has both a supremum and an infimum with respect to $\leq_X$, i.e. iff

$$\forall A\, (A \subseteq X \Rightarrow \exists S, I\, (S, I \in X \,\wedge\, S = \overset{\leq_X}{\sup} A \,\wedge\, I = \overset{\leq_X}{\inf} A)). \tag{3.304}$$

Corollary 3.99. *It is true that any complete lattice $(X, \leq_X)$ is a lattice.*

Proof. Letting X and $\leq_X$ be arbitrary sets, assuming X to be nonempty and $\leq_X$ to be a reflexive partial ordering of X such that the ordered pair $(X, \leq_X)$ is a complete lattice, we take arbitrary constants $x, y \in X$, so that $\{x, y\} \subseteq X$ holds according to (2.164). Since $(X, \leq_X)$ is a complete lattice, there exist then elements of X, say $\bar{S}$ and $\bar{I}$, which are respectively the supremum $\bar{S} = \sup^{\leq_X}\{x, y\}$ and the infimum $\bar{S} = \sup^{\leq_X}\{x, y\}$ of the pair. As x and y are arbitrary, we may therefore conclude that the $(X, \leq_X)$ is a lattice, by definition. Because X was initially arbitrary, we may then infer from this finding the truth of the stated universal sentence. $\square$

To shed some light on the lattice concept, we develop in the following a first simple example of a complete lattice.

Proposition 3.100. *For any set X, it is true that*

a) the intersection of any subset $\mathcal{K}$ of the power set of X is identical with the infimum of $\mathcal{K}$, i.e.

$$\forall \mathcal{K}\, (\mathcal{K} \subseteq \mathcal{P}(X) \Rightarrow \bigcap \mathcal{K} = \inf \mathcal{K}), \tag{3.305}$$

b) the union of any subset $\mathcal{K}$ of the power set of X is identical with the supremum of $\mathcal{K}$, i.e.

$$\forall \mathcal{K}\, (\mathcal{K} \subseteq \mathcal{P}(X) \Rightarrow \bigcup \mathcal{K} = \sup \mathcal{K}), \tag{3.306}$$

where the infimum and the supremum are taken with respect to the reflexive partial ordering of inclusion of $\mathcal{P}(X)$.

Proof. We let X be an arbitrary set. Concerning a), we let $\mathcal{K}$ also be an arbitrary set and assume moreover $\mathcal{K} \subseteq \mathcal{P}(X)$ to be true. Since the universal sentence $\forall A\,(A \in \mathcal{K} \Rightarrow \bigcap \mathcal{K} \subseteq A)$ holds in view of (2.92), we evidently obtain with the specification of a reflexive partial ordering of inclusion

$$\forall A\,(A \in \mathcal{K} \Rightarrow \bigcap \mathcal{K} \subseteq_{\mathcal{P}(\Omega)} A), \tag{3.307}$$

so that $\bigcap \mathcal{K}$ is a lower bound for $\mathcal{K}$ (with respect to $\subseteq_{\mathcal{P}(\Omega)}$). According to the Characterization of the infimum, it therefore remains for us to prove

$$\forall L\,(\forall A\,(A \in \mathcal{K} \Rightarrow L \subseteq_{\mathcal{P}(\Omega)} A) \Rightarrow L \subseteq_{\mathcal{P}(\Omega)} \bigcap \mathcal{K}) \tag{3.308}$$

To do this, we take an arbitrary set L, assume

$$\forall A\,(A \in \mathcal{K} \Rightarrow L \subseteq_{\mathcal{P}(\Omega)} A) \tag{3.309}$$

(i.e., that L is a lower bound for $\mathcal{K}$ in $\mathcal{P}(X)$), and we show that $L \subseteq_{\mathcal{P}(\Omega)} \bigcap \mathcal{K}$ follows to be true, which we may write equivalently as $L \subseteq \bigcap \mathcal{K}$. For this purpose, we verify (according to the definition of a subset)

$$\forall y\,(y \in L \Rightarrow y \in \bigcap \mathcal{K}), \tag{3.310}$$

letting $\bar{y}$ be arbitrary and assuming $\bar{y} \in L$ to be true. To establish the desired consequent $\bar{y} \in \bigcap \mathcal{K}$, we apply the definition of the intersection of a set system and prove the universal sentence

$$\forall A\,(A \in \mathcal{K} \Rightarrow \bar{y} \in A). \tag{3.311}$$

We take an arbitrary set $\bar{A}$ and assume that $\bar{A} \in \mathcal{K}$ holds, which implies $L \subseteq_{\mathcal{P}(\Omega)} \bar{A}$ with (3.309), which we may write as $L \subseteq \bar{A}$ and therefore (by definition of a subset)

$$\forall y\,(y \in L \Rightarrow y \in \bar{A}).$$

With this, the previously made assumption $\bar{y} \in L$ yields $\bar{y} \in \bar{A}$, proving the implication in (3.311). Since $\bar{A}$ is arbitrary, we may therefore conclude that (3.311) is true, which universal sentence then implies the truth of $\bar{y} \in \bigcap \mathcal{K}$ (by definition of the intersection of a set system). This finding in turn proves the implication in (3.310), and as $\bar{y}$ was arbitrary, we may infer from this the truth of the universal sentence (3.310), which then gives $L \subseteq \bigcap \mathcal{K}$ (by definition of a subset), and equivalently $L \subseteq_{\mathcal{P}(\Omega)} \bigcap \mathcal{K}$. Thus, the proof of the implication in (3.308) is complete, and because L was arbitrary, we may now further conclude that the universal sentence (3.308) is true as well. The conjunction of (3.307) and (3.308) finally shows (in light of the Characterization of the infimum) that $\bigcap \mathcal{K}$ is the greatest lower bound for $\mathcal{K}$ in $\mathcal{P}(X)$. Since X was initially an arbitrary set, Part a) of the proposition follows to be true. □

Exercise 3.43. Establish Part b) of Proposition 3.100.

(Hint: Simply apply the definition of the union of a set system in place of the definition of the intersection of a set system.)

Recalling that the power set of any set X is nonempty in view of (3.15) and observing in light of the preceding proposition and exercise that the infimum and the supremum of any subset of a power set $\mathcal{P}(X)$ exist, we may immediately apply the definition of a complete lattice to power sets that are partially ordered by inclusion.

Corollary 3.101. *The ordered pair $(\mathcal{P}(X), \subseteq_{\mathcal{P}(X)})$ formed by the power set of any set X and the reflexive partial ordering of inclusion on $\mathcal{P}(X)$ is a complete lattice.*

Proposition 3.102. *For any set X, any subset $\mathcal{Y} \subseteq \mathcal{P}(X)$, any subset $\mathcal{K} \subseteq \mathcal{Y}$ and any set $I \in \mathcal{Y}$, it is true that I is the intersection of $\mathcal{K}$ iff I is the infimum of $\mathcal{K}$ with respect to $\subseteq_{\mathcal{Y}}$, i.e.*

$$I = \bigcap \mathcal{K} \;\Leftrightarrow\; I = \overset{\subseteq_{\mathcal{Y}}}{\inf} \mathcal{K}. \tag{3.312}$$

Proof. We let X, $\mathcal{Y}$, $\mathcal{K}$ and I be arbitrary sets and assume $\mathcal{Y} \subseteq \mathcal{P}(X)$, $\mathcal{K} \subseteq \mathcal{Y}$ as well as $I \in \mathcal{Y}$ to be true. Let us observe here that the assumed inclusions give $\mathcal{K} \subseteq \mathcal{P}(X)$ with (2.13), which implies with Proposition 3.100a) the equation

$$\bigcap \mathcal{K} = \overset{\subseteq_{\mathcal{P}(X)}}{\inf} \mathcal{K}. \tag{3.313}$$

To prove now the first part ('⇒') of the equivalence (3.312), we assume $I = \bigcap \mathcal{K}$ to be true, so that (3.313) gives

$$I = \overset{\subseteq_{\mathcal{P}(X)}}{\inf} \mathcal{K}, \tag{3.314}$$

which we may write also as

$$\forall A\,(A \in \mathcal{K} \Rightarrow I \subseteq_{\mathcal{P}(X)} A) \wedge \forall I'\,(\forall A\,(A \in \mathcal{K} \Rightarrow I' \subseteq_{\mathcal{P}(X)} A) \Rightarrow I' \subseteq_{\mathcal{P}(X)} I), \tag{3.315}$$

according to the Characterization of the infimum. For the same reason, the desired consequent

$$I = \overset{\subseteq_{\mathcal{Y}}}{\inf} \mathcal{K} \tag{3.316}$$

can be written equivalently as

$$\forall A\,(A \in \mathcal{K} \Rightarrow I \subseteq_{\mathcal{Y}} A) \wedge \forall I'\,(\forall A\,(A \in \mathcal{K} \Rightarrow I' \subseteq_{\mathcal{Y}} A) \Rightarrow I' \subseteq_{\mathcal{Y}} I). \tag{3.317}$$

To show that the first part

$$\forall A\,(A \in \mathcal{K} \Rightarrow I' \subseteq_{\mathcal{Y}} A) \tag{3.318}$$

of this conjunction holds, we take an arbitrary set A and assume $A \in \mathcal{K}$ to be true, which implies with the first part of the conjunction (3.315) that $I \subseteq_{\mathcal{P}(X)} A$. Due to the specification of the reflexive partial ordering of inclusion, we may write this first as $I \subseteq A$, and then also as $I \subseteq_{\mathcal{Y}} A$. Since A is arbitrary, we may therefore conclude that the first part of the conjunction (3.317) is true. To establish the second part, we take an arbitrary set I', assume the universal sentence

$$\forall A\,(A \in \mathcal{K} \Rightarrow I' \subseteq_{\mathcal{Y}} A) \tag{3.319}$$

to be true, and show that $I' \subseteq_{\mathcal{Y}} I$ is implied. To do this, we first prove the universal sentence

$$\forall A\,(A \in \mathcal{K} \Rightarrow I' \subseteq_{\mathcal{P}(X)} A), \tag{3.320}$$

letting A be arbitrary and assuming $A \in \mathcal{K}$ to hold, so that (3.319) yields $I' \subseteq_{\mathcal{Y}} A$. As before, we may apply here the specification of the reflexive partial ordering of inclusion to obtain write this first as $I' \subseteq A$, and then as $I' \subseteq_{\mathcal{P}(X)} A$. Because A is arbitrary, we may infer from this finding the truth of the universal sentence (3.320), which in turn implies $I' \subseteq_{\mathcal{P}(X)} I$ with the second part of the conjunction (3.315). Clearly, we may write this first as $I' \subseteq I$ and then as $I' \subseteq_{\mathcal{Y}} I$. As the set I' was arbitrary, we may therefore conclude that the second part of the conjunction (3.317) also holds, so that this true conjunction gives the desired consequent (3.316) of the first part of the equivalence (3.312) to be proven.

Regarding the second part ('$\Leftarrow$'), we now assume (3.316) to be true, so that the equivalent conjunction (3.317) also holds. Let us establish (3.314), by verifying the equivalent conjunction (3.315). To prove the first part

$$\forall A\,(A \in \mathcal{K} \Rightarrow I \subseteq_{\mathcal{P}(X)} A) \tag{3.321}$$

of this conjunction, we take an arbitrary set A and assume $A \in \mathcal{K}$ to be true. This assumption implies $I \subseteq_{\mathcal{Y}} A$ with the first part of the conjunction

(3.317), and this may evidently written first as the inclusion $I \subseteq A$, and then also as $I \subseteq_{\mathcal{P}(X)} A$. Since A is arbitrary, we may therefore conclude that (3.321) holds indeed. To prove the second part of the conjunction (3.315), we let I' be arbitrary, assume (3.320), and show that $I' \subseteq_{\mathcal{P}(X)} I$ follows to be true. Let us first prove that the universal sentence (3.319) is true. Indeed, letting A be arbitrary and assuming $A \in \mathcal{K}$, the assumed (3.320) gives $I' \subseteq_{\mathcal{P}(X)} A$, which we may write as $I' \subseteq A$ and then as $I' \subseteq_{\mathcal{Y}} A$. As A was arbitrary, we may therefore conclude that (3.319) is indeed true. This universal sentence implies now with the second part of the conjunction (3.317) that $I' \subseteq_{\mathcal{Y}} I$ is true. Clearly, this may be written as $I' \subseteq_{\mathcal{P}(X)} I$, which finding proves the second part of the conjunction (3.315) because I' was arbitrary. This conjunction gives now (3.314), which in turn implies $I = \bigcap \mathcal{K}$ (3.313).

Thus, the proof of the equivalence (3.312) is complete, and since X, $\mathcal{Y}$, $\mathcal{K}$ and I were initially arbitrary sets, we may now finally conclude that the proposition holds, as claimed. □

Exercise 3.44. Show for any set X, any subset $\mathcal{Y} \subseteq \mathcal{P}(X)$, any subset $\mathcal{K} \subseteq \mathcal{Y}$ and any set $S \in \mathcal{Y}$ that S is the union of $\mathcal{K}$ iff S is the supremum of $\mathcal{K}$ with respect to $\subseteq_{\mathcal{Y}}$, i.e.

$$S = \bigcup \mathcal{K} \;\Leftrightarrow\; S = \overset{\subseteq_{\mathcal{Y}}}{\sup} \mathcal{K}. \tag{3.322}$$

We now establish a useful property of complete lattices.

Corollary 3.103. *For any complete lattice $(X, \leq_X)$ and any nonempty subset A of X, it is true that the infimum of A is less than or equal to the supremum of A, i.e.*

$$\forall A\,([A \subseteq X \;\wedge\; A \neq \emptyset] \Rightarrow \inf A \leq_X \sup A). \tag{3.323}$$

Proof. We let X and $\leq_X$ be arbitrary sets such that $(X, \leq_X)$ is a complete lattice and let A be an arbitrary nonempty subset of X. It then follows with (2.42) that there exists an element in A, say $\bar{x}$. Since $\inf A$ is a lower bound for A, we have on the one hand $\inf A \leq_X \bar{x}$, and since $\sup A$ is an upper bound for A, we have on the other hand $\bar{x} \leq_X \sup A$. The conjunction of these two inequalities then implies the desired $\inf A \leq_X \sup A$ with the transitivity of the reflexive partial ordering $\leq_X$ of X. Since A is arbitrary, we therefore conclude that (3.323) holds. Then, as $(X, <_X)$ was also arbitrary, we may further conclude that the proposed sentence is true. □

Proposition 3.104. *For any partially ordered set $(X, \leq_X)$ it is true that the supremum of the pair formed by two elements a, b in X is less than or* *equal to an element c in X iff both elements a, b of the pair are less than or equal to c, i.e.*

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [\sup\{a, b\} \leq_X c \Leftrightarrow (a \leq_X c \wedge b \leq_X c)]). \tag{3.324}$$

Proof. We let X and $\leq_X$ be arbitrary sets, assume that $\leq_X$ is a reflexive partial ordering of X, let then a, b, c be arbitrary, and assume $a, b, c \in X$ to be true. To prove the first part ('$\Rightarrow$') of the stated equivalence, we assume that $\sup\{a, b\} \leq_X c$ holds. Since the supremum $\sup\{a, b\}$ is an upper bound for $\{a, b\}$ due to Theorem 3.98a), we obtain with Theorem 3.92a)

$$\begin{aligned} a &\leq_X \sup\{a, b\} \quad (\leq_X c), \\ b &\leq_X \sup\{a, b\} \quad (\leq_X c), \end{aligned}$$

which yield with the transitivity of the partial ordering $\leq_X$ the desired inequalities $a \leq_X c$ and $b \leq_X c$.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume $a \leq_X c$ and $b \leq_X c$. Thus, c is evidently an upper bound for $\{a, b\}$. As $\sup\{a, b\}$ is the supremum of $\{a, b\}$, it follows with (3.297) that $\sup\{a, b\} \leq_X c$ holds, completing the proof of the equivalence. Since X, $\leq_X$, a, b and c were arbitrary, we may therefore conclude that the proposition is true. □

Exercise 3.45. Show for any partially ordered set $(X, \leq_X)$ that the infimum of the pair formed by two elements a, b in X is greater than or equal to an element c in X iff both elements a, b of the pair are greater than or equal to c, i.e.

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [c \leq_X \inf\{a, b\} \Leftrightarrow (c \leq_X a \wedge c \leq_X b)]). \tag{3.325}$$

(Hint: Proceed in analogy to the proof of Proposition 3.104.)

Theorem 3.105. *For any partially ordered set $(X, \leq_X)$ and any subset A of X, it is true that*

a) *m is the maximum of A iff m is both the supremum of A and an element of A, i.e.*

$$\forall m\,(m = \max A \Leftrightarrow [m = \sup A \wedge m \in A]). \tag{3.326}$$

b) *m is the minimum of A iff m is both the infimum of A and an element of A, i.e.*

$$\forall m\,(m = \min A \Leftrightarrow [m = \inf A \wedge m \in A]). \tag{3.327}$$

Proof. We let X and $\leq_X$ be arbitrary sets, assume $\leq_X$ to be a reflexive partial ordering of X, let A be an arbitrary set such that $A \subseteq X$ holds, and let then m also be arbitrary.

Concerning a) and regarding the first part ('$\Rightarrow$') of the stated equivalence, we assume that $m = \max A$ holds, so that m is, by definition of a maximum, both an upper bound for A and an element of A. Thus, $m \in A$ holds, so that it remains to verify $m = \sup A$. For this purpose, we use the Characterization of the supremum. Since we already found that m is an upper bound for A, we only need to establish

$$\forall S' \, (\forall x \, (x \in A \Rightarrow x \leq_X S') \Rightarrow m \leq_X S'). \tag{3.328}$$

We let S' be arbitrary and assume

$$\forall x \, (x \in A \Rightarrow x \leq_X S').$$

With this, the previously obtained $m \in A$ implies the desired consequent $m \leq_X S'$ in (3.328). Since S' is arbitrary, we may therefore conclude that the universal sentence (3.328) is true, so that $m = \sup A$ is indeed true, besides $m \in A$.

Regarding the second part ('$\Leftarrow$') of the equivalence, we now assume $m = \sup A$ and $m \in A$ to be true. Since m is then (according to the Characterization of the supremum) an upper bound for A, and since this upper bound m for A is an element of A, it follows by definition of a maximum that $m = \max A$ is true. Thus, the proof of the equivalence is complete, and as m is arbitrary, the universal sentence (3.326) follows then to be true.

Since X, $\leq_X$ and A were also arbitrary, we may then conclude that a) holds for any X, any $\leq_X$ and any A. □

Exercise 3.46. Prove Theorem 3.105b).

The following result is then an immediate consequence of Corollary 3.96.

Corollary 3.106. *For any partially ordered set $(X, \leq_X)$ and any element y in X, it is true that y is the supremum and the infimum of the singleton formed by y, i.e.*

$$\forall y \, (y \in X \Rightarrow y = \sup\{y\} = \inf\{y\}). \tag{3.329}$$

Proposition 3.107. *For any partially ordered set $(X, \leq_X)$ and any subsets A, B of X for which the suprema of A and B exist, it is true that the supremum of A is less than or equal to the supremum of B if A is included in B, i.e.*

$$A \subseteq B \Rightarrow \sup A \leq_X \sup B. \tag{3.330}$$

Proof. We let X, $\leq_X$, A and B be arbitrary sets, assume that $\leq_X$ is a reflexive partial ordering of X, and assume also that $\sup A$ and $\sup B$ exist. To prove the stated implication directly, we assume $A \subseteq B$, which is equivalent to $\forall x\,(x \in A \Rightarrow x \in B)$ by definition of a subset. Next, we show that $\sup B$ is an upper bound for A, i.e. $\forall x\,(x \in A \Rightarrow x \leq_X \sup B)$. For this purpose, we let $x \in A$ be arbitrary and demonstrate that this implies $x \leq_X \sup B$. Let us note that $x \in A$ implies $x \in B$ in view of the assumption $A \subseteq B$. Then, as $\sup B$ is an upper bound for B, we see that $x \in B$ implies the desired $x \leq_X \sup B$. Since x is arbitrary, we therefore conclude that $\sup B$ is an upper bound for A. Consequently, as $\sup A$ is the least upper of A, we obtain $\sup A \leq_X \sup B$, which completes the proof of the implication (3.330). As X, $\leq_X$, A and B were arbitrary, we may now conclude that the proposition is true. □

Exercise 3.47. Show for any partially ordered set $(X, \leq_X)$ and any subsets A, B of X for which the infima of A and B exist that the infimum of B is less than or equal to the infimum of A if A is included in B, i.e.

$$A \subseteq B \Rightarrow \inf B \leq_X \inf A. \tag{3.331}$$

Theorem 3.108 (Nested determination of the supremum & the infimum). *The following equations hold for any partially ordered set $(X, \leq_X)$ and any $a, b, c \in X$ such that the suprema/infima of $\{a, b, c\}$ and $\{a, b\}$ exist.*

$$\sup\{a, b, c\} = \sup\{\sup\{a, b\}, c\}, \tag{3.332}$$

$$\inf\{a, b, c\} = \inf\{\inf\{a, b\}, c\}. \tag{3.333}$$

Proof. We let X, $\leq_X$, a, b and c be arbitrary, assume that $\leq_X$ is a reflexive partial ordering of X, assume that a, b, c are in X, and assume concerning (3.332) also that $S = \sup\{a, b, c\}$ as well as $\sup\{a, b\}$ exist (in X). To show that the constant S is the supremum of $\{\sup\{a, b\}, c\}$, as claimed, we apply the Characterization of the supremum. To show that S is an upper bound for $\{\sup\{a, b\}, c\}$, we first note that both $\{a, b\}$ and $\{c\}$ are subsets of $\{a, b, c\}$ according to (2.237), so that we obtain $\sup\{a, b\} \leq_X S$ and $\sup\{c\} \leq_X S$ with Proposition 3.107. Since Corollary 3.106 gives $\sup\{c\} = c$, the preceding conjunction gives $\sup\{a, b\} \leq_X S \wedge c \leq_X S$, which in turn implies with the Characterization of upper bounds for pairs that S is an upper bound for $\{\sup\{a, b\}, c\}$.

Next, we verify (according to the Characterization of the supremum) that S is less than or equal to any upper bound S' for $\sup\{\{a, b\}, c\}$, i.e.

$$\forall S'\,(\forall x\,(x \in \sup\{\{a, b\}, c\} \Rightarrow x \leq_X S') \Rightarrow S \leq_X S') \tag{3.334}$$

For this purpose, we let S' be arbitrary and assume the universal sentence in (3.334) to be true, i.e. we assume S' to be an upper bound for $\sup\{\{a,b\},c\}$, so that the conjunction $\sup\{a,b\} \leq_X S' \wedge c \leq_X S'$ holds (according to the Characterization of upper bounds for pairs), and we demonstrate that this implies $S \leq_X S'$. Now, as $\{a\}$ and $\{b\}$ are subsets of $\{a,b\}$ due to (2.185), we obtain

$$\begin{aligned} (a =) \quad \sup\{a\} &\leq_X \sup\{a,b\} \quad (\leq_X S'), \\ (b =) \quad \sup\{b\} &\leq_X \sup\{a,b\} \quad (\leq_X S'), \end{aligned}$$

using again Corollary 3.106, Proposition 3.107 (as well as the previously established inequality $\sup\{a,b\} \leq_X S'$). Then, the transitivity of the partial ordering $\leq_X$ gives $a \leq_X S'$ and $b \leq_X S'$, alongside the previously obtained $c \leq_X S'$. These findings imply with Theorem 3.93a) that S' is an upper bound for the triple $\{a,b,c\}$, that is,

$$\forall x\,(x \in \{a,b,c\} \Rightarrow x \leq_X S'). \tag{3.335}$$

Since we defined $S = \sup\{a,b,c\}$ to be the supremum of $\{a,b,c\}$, it follows now from (3.335) with the Characterization of the supremum that $S \leq_X S'$ is true, which completes the proof of the implication in (3.334). Because S' is arbitrary, we may infer from this the truth of the universal sentence (3.334), which completes the proof that S is the supremum of $\{\sup\{a,b\},c\}$. Since X, $\leq_X$, a, b and c were arbitrary, we may finally conclude that the first part of the theorem holds. □

Exercise 3.48. Verify the second equation of Theorem 3.108.

Proposition 3.109. *For any lattice $(X, \leq_X)$ it is true that, if an element a of X is less than or equal to an element b of X, then the supremum of the pair formed by a and b is identical with b, and the infimum of that pair equals a, that is,*

$$\forall a,b\,(a,b \in X \Rightarrow [a \leq_X b \Rightarrow (\sup\{a,b\} = b \wedge \inf\{a,b\} = a)]). \tag{3.336}$$

Proof. We let X and $\leq_X$ be arbitrary sets and assume that $(X, \leq_X)$ is a lattice. Next, we let a and b be arbitrary and assume $a,b \in X$ to be true. Let us now observe that $a,b \in X$ implies $a \leq_X a$ as well as $b \leq_X b$ since $\leq_X$ is reflexive. Next, we assume that $a \leq_X b$ holds. Consequently, the conjunctions $a \leq_X b \wedge b \leq_X b$ and $a \leq_X a \wedge a \leq_X b$ are true. It then follows with the Characterization of upper & lower bounds for a pair that b is an upper bound and that a is a lower bound for $\{a,b\}$. Since $a,b \in \{a,b\}$ is also true according to Exercise 2.15, we see that $b = \max\{a,b\}$ and $a = \min\{a,b\}$ hold by definition of a greatest and of a least element. These equations in turn imply, respectively, $b = \sup\{a, b\}$ and $a = \inf\{a, b\}$ with Theorem 3.105. Thus, the proof of the conjunction in (3.336) is complete, so that the implications follow to be true. Since a and b are arbitrary, we may therefore conclude that the proposition holds. □

Proposition 3.110. *For any lattice $(X, \leq_X)$ it is true that the infimum of the pair formed by two elements of X is less than or equal to the supremum of that pair, i.e.*

$$\forall a, b\,(a, b \in X \Rightarrow \inf\{a, b\} \leq_X \sup\{a, b\}). \tag{3.337}$$

Proof. Letting X and $\leq_X$ be arbitrary such that $(X, \leq_X)$ is a lattice and letting a and b be arbitrary such that $a, b \in X$ is true, we see in light of the Characterization of the supremum & the infimum that $\sup\{a, b\}$ is an upper bound and $\inf\{a, b\}$ a lower bound for $\{a, b\}$, so that the universal sentences

$$\forall x\,(x \in \{a, b\} \Rightarrow x \leq_X \sup\{a, b\})$$
$$\forall x\,(x \in \{a, b\} \Rightarrow \inf\{a, b\} \leq_X x)$$

are both true. Then, the evident fact that $a \in \{a, b\}$ is true implies on the one hand $a \leq_X \sup\{a, b\}$, and on the other hand $\inf\{a, b\} \leq_X a$. Since the partial ordering $\leq_X$ is transitive, it follows from these two findings that $\inf\{a, b\} \leq_X \sup\{a, b\}$ holds, so that the implication in (3.337) is true. Then, since X, $\leq_X$, a and b are arbitrary, it follows that the proposed universal sentence holds, as claimed. □

Exercise 3.49. Show for any complete lattice $(X, \leq_X)$ that the infimum of a nonempty subset A of X is less than or equal to the supremum of that subset, i.e.

$$\forall A\,([A \subseteq X \wedge A \neq \emptyset] \Rightarrow \inf A \leq_X \sup A). \tag{3.338}$$

(Hint: Proceed in analogy to the proof of Proposition 3.110, applying now (2.42).)

In case of a linear ordering, the definition of the supremum can be shown to be equivalent to the following often applied formulation: If we take an element S' of X which is less than the supremum S of A, then there exists an element x in A which is greater than S', so that S' cannot be an upper bound for A, *a fortiori* not the supremum of A. The infimum is characterized analogously.

Theorem 3.111 (Supremum & Infimum Criterion). *For any linearly ordered set $(X, <_X)$ and any nonempty subset A of X, it is true that* *a) an element S in X is the supremum of A iff*

$$\begin{aligned}&\forall x\,(x \in A \Rightarrow x \leq_X S)\\&\quad\wedge\ \forall S'\,([S' \in X \ \wedge\ S' <_X S] \Rightarrow \exists x\,(x \in A \ \wedge\ S' <_X x)).\end{aligned} \tag{3.339}$$

b) an element I in X is the infimum of A iff

$$\begin{aligned}&\forall x\,(x \in A \Rightarrow I \leq_X x)\\&\quad\wedge\ \forall I'\,([I' \in X \ \wedge\ I <_X I'] \Rightarrow \exists x\,(x \in A \ \wedge\ x <_X I')).\end{aligned} \tag{3.340}$$

Proof. We let X, $<_X$ and $\leq_X$ be arbitrary sets such that $<_X$ is a linear ordering and $\leq_X$ a total ordering of X inducing each other,and we let A be an arbitrary nonempty subset of X.

Concerning a), we let S be arbitrary and assume $S \in X$ to be true. To prove the first part ('$\Rightarrow$') of the stated equivalence, we assume (3.339) and show that this implies (3.297). Clearly, the first part of the assumed conjunction (3.339) is then true and therefore immediately implies the truth of the identical first part of the conjunction (3.297), which characterizes the supremum of A. Next, we demonstrate that the second part

$$\forall S'\,([S' \in X \ \wedge\ S' <_X S] \Rightarrow \exists x\,(x \in A \ \wedge\ S' <_X x)) \tag{3.341}$$

of the assumed conjunction (3.339) also implies the truth of the second part

$$\forall S'\,([S' \in X \ \wedge\ \forall x\,(x \in A \Rightarrow x \leq_X S')] \Rightarrow S \leq_X S') \tag{3.342}$$

of the conjunction (3.297). To prove the latter universal sentence, we let $\bar{S}'$ be arbitrary, and we prove the implication

$$[\bar{S}' \in X \ \wedge\ \forall x\,(x \in A \Rightarrow x \leq_X \bar{S}')] \Rightarrow S \leq_X \bar{S}' \tag{3.343}$$

by contradiction, assuming $\bar{S}' \in X$, the universal sentence

$$\forall x\,(x \in A \Rightarrow x \leq_X \bar{S}') \tag{3.344}$$

and $\neg S \leq_X \bar{S}'$ to be true. Since S and $\bar{S}'$ are thus elements of X, we have that $\neg S \leq_X \bar{S}'$ implies $S >_X \bar{S}'$ with the negation Formula for $\leq$. Thus, the conjunction of $\bar{S}' \in X$ and $\bar{S}' <_X S$ is true, which then implies with the initial assumption $A \neq \emptyset$ and with (3.341) that there exists an element in A, say $\bar{x}$, such that $\bar{S}' <_X \bar{x}$ holds. As we initially assumed $A \subseteq X$, we then see that $\bar{x} \in A$ implies $\bar{x} \in X$ by definition of a subset. Thus, $\bar{S}'$ and $\bar{x}$ are both in X, so that the preceding inequality $\bar{S}' <_X \bar{x}$ implies $\neg\bar{x} \leq_X \bar{S}'$ with the Negation Formula for $\leq$. Furthermore, $\bar{x} \in A$ implies $\bar{x} \leq_X \bar{S}'$

with (3.344), which evidently contradicts the preceding negation, so that the proof of the implication (3.343) is complete. Because $\bar{S}'$ is arbitrary, we may therefore conclude that the universal sentence (3.342) is true, which completes the proof of the conjunction (3.297) and thus the proof of the first part of the proposed equivalence.

To prove the second part ('$\Leftarrow$'), we now assume (3.297) to be true and establish the truth of (3.339). Then, the first part of the conjunction (3.297) implies the first part of the conjunction (3.339), because both are identical. It now remains for us to verify that the second part (3.342) of the assumed conjunction implies the truth of the second part (3.341) of the conjunction to be proven. Letting $\bar{S}'$ be arbitrary, we now prove the implication in (3.341) directly, assuming $\bar{S}' \in X$ and $\bar{S}' <_X S$ to hold. Thus, $\bar{S}'$ and S are both elements of X, and therefore the assumed inequality $\bar{S}' <_X S$ implies $\neg S \leq_X \bar{S}'$ with the Negation Formula for $\leq$. Let us now observe that the implication in (3.342) holds for $\bar{S}'$, so that we may apply the Law of Contraposition to infer from $\neg S \leq_X \bar{S}'$ the truth of the negation

$$\neg[\bar{S}' \in X \wedge \forall x\,(x \in A \Rightarrow x \leq_X \bar{S}')].$$

It then follows with De Morgan's Law (1.51) that $\bar{S}' \notin X$ or

$$\neg\forall x\,(x \in A \Rightarrow x \leq_X \bar{S}')$$

holds. Here, $\bar{S}' \notin X$ is false as we assumed $\bar{S}' \in X$ to be true. Consequently, the preceding negated universal sentence is true, which we may then write equivalently as

$$\exists x\,(x \in A \wedge \neg x \leq_X \bar{S}'),$$

applying the Double Negation Law and (1.82). Thus, there exists an element in A, say $\bar{x}$, such that $\neg\bar{x} \leq_X \bar{S}'$ holds. Recalling the initial assumption $A \subseteq X$, we also see that $\bar{x} \in A$ implies $\bar{x} \in A$ with the definition of a subset. Thus, $\bar{x}$ and $\bar{S}'$ are both in X, and therefore the preceding negation $\neg\bar{x} \leq_X \bar{S}'$ implies $\bar{S}' <_X \bar{x}$ with the Negation Formula for $\leq$. This proves the existence of an element x satisfying the conjunction $x \in A \wedge \bar{S}' <_X x$, which in turn proves the implication in (3.341). Since $\bar{S}'$ is arbitrary, the universal sentence (3.341) then follows to be true, so that the second part of the conjunction (3.339) to be proven also holds. Thus, the proof of the second part of the proposed equivalence is complete, and therefore the equivalence is itself true. As X, $<_X$, $\leq_X$, A and S were arbitrary, we may now conclude that Part a) of the theorem is true. $\square$

Exercise 3.50. Prove the Infimum Criterion.

Theorem 3.112 (Equality Criterion for suprema & infima). *It is true for any linearly ordered set* $(X, <_X)$ *and any sets* $A, B \subseteq X$ *with* $A \subseteq B$ *such that*

a) the suprema $\sup A$ *and* $\sup B$ *exist that these suprema are identical if, for any element* x *in* B*, there exists an element* y *in* A *which is greater than or equal to* x*, i.e.,*

$$\forall x\,(x \in B \Rightarrow \exists y\,(y \in A \wedge y \geq_X x)) \Rightarrow \sup A = \sup B. \tag{3.345}$$

b) the infima $\inf A$ *and* $\inf B$ *exist that these infima are identical if, for any element* x *in* B*, there exists an element* y *in* A *which is less than or equal to* x*, i.e.,*

$$\forall x\,(x \in B \Rightarrow \exists y\,(y \in A \wedge y \leq_X x)) \Rightarrow \inf A = \inf B. \tag{3.346}$$

Proof. We let X, $\leq_X$, A and B be arbitrary sets such that $(X, \leq_X)$ is linearly ordered, such that the inclusions $A \subseteq X$, $B \subseteq X$ and $A \subseteq B$ hold, and such that $\sup A$ and $\sup B$ exist. We note that these assumptions imply $\sup A \leq_X \sup B$ with Proposition 3.107. We now prove the implication (3.345) by contradiction, assuming its antecedent and the negation $\neg \sup A = \sup B$ to be true. Since the linear ordering $<_X$ is connex, this implies the truth of the disjunction $\sup A <_X \sup B \vee \sup B <_X \sup A$. As the previously found inequality $\sup A \leq_X \sup B$ implies $\neg \sup B <_X \sup A$, the first part $\sup A <_X \sup B$ of the disjunction must be true. Noting that $S' = \sup A$ is an element of X, it follows from $S' <_X \sup B$ with the Supremum Criterion that there exists a particular element $\bar{x} \in B$ which is greater than S', i.e., which satisfies $[\sup A =]\ S' <_X \bar{x}$. Due to the assumed antecedent, it follows from $\bar{x} \in B$ that there exists a particular element $\bar{y} \in A$ which satisfies $\bar{y} \geq_X \bar{x}$. Since the supremum $\sup A$ is by definition an upper bound for A, $\bar{y} \in A$ implies $[\bar{x} \leq_X]\ \bar{y} \leq_X \sup A$. These inequalities in turn give us $\bar{x} \leq_X \sup A$ with the transitivity of the induced total ordering $\leq_X$. In conjunction with the previously found inequality $\sup A <_X \bar{x}$, this further implies $\sup A <_X \sup A$ with the Transitivity Formula for $<$ and $\leq$. This finding contradicts the true negation $\neg \sup A <_X \sup A$, which holds due to the irreflexivity of the linear ordering $<_X$. We thus completed the proof of the implication (3.345).

The second implication (3.346) can be proved similarly. Since X, $\leq_X$, A and B were initially arbitrary, we therefore conclude that the theorem is true. □

Exercise 3.51. Establish the Equality Criterion for infima.

Proposition 3.113. *The following sentences are true for any totally ordered set $(X, \leq_X)$ and any $a, b \in X$.*

a) *$\{a, b\}$ is bounded from above by a or by b.*

b) *The maximum of $\{a, b\}$ is identical with a or with b, that is,*

$$\max\{a, b\} = a \lor \max\{a, b\} = b. \tag{3.347}$$

c) *The supremum of $\{a, b\}$ is identical with a or with b, and moreover identical with the maximum of $\{a, b\}$, that is,*

$$(\sup\{a, b\} = a \lor \sup\{a, b\} = b) \land \sup\{a, b\} = \max\{a, b\}. \tag{3.348}$$

Proof. We let X, $\leq_X$, a and b be arbitrary, assume that $(X, \leq_X)$ is totally ordered, and assume furthermore that and $a, b \in X$ is true.

Concerning a), we note that $a \leq_X a$ and $b \leq_X b$ are true since the total ordering $\leq_X$ is reflexive. Furthermore, the disjunction $a \leq_X b \lor b \leq_X a$ holds because $\leq_X$ is total. Now, in case $a \leq_X b$ is true, the conjunction $a \leq_X b \land b \leq_X b$ also holds, so that b is an upper bound for $\{a, b\}$ in view of Theorem 3.92a). Similarly, in the other case that $b \leq_X a$ is true, the conjunction $a \leq_X a \land b \leq_X a$ also holds, so that a is now an upper bound for $\{a, b\}$. We thus showed that $\{a, b\}$ is bounded from above by a or by b.

Concerning b), let us first observe that a and b are clearly elements of $\{a, b\}$. Now, in case a is an upper bound for A, it follows together with $a \in A$ that a is the maximum of A by definition. Similarly, if b is an upper bound for A, then this implies with $b \in A$ that b is the maximum of A.

Concerning c), in case a is the maximum of $\{a, b\}$, it follows with Theorem 3.105a) that a is also the supremum of A. Similarly, in the other case that b is the maximum of $\{a, b\}$, this implies that b is the supremum of $\{a, b\}$.

As X, $\leq_X$, a and b were arbitrary, we may therefore conclude that the proposition is true. □

Exercise 3.52. The following sentences are true for any totally ordered set $(X, \leq_X)$ and any $a, b \in X$.

a) $\{a, b\}$ is bounded from below by a or by b.

b) The minimum of $\{a, b\}$ is identical with a or with b, that is,

$$\min\{a, b\} = a \lor \min\{a, b\} = b. \tag{3.349}$$

c) The infimum of $\{a,b\}$ is identical with a or with b, and moreover identical with the minimum of $\{a,b\}$, that is,

$$(\inf\{a,b\} = a \vee \inf\{a,b\} = b) \wedge \inf\{a,b\} = \min\{a,b\}. \tag{3.350}$$

Corollary 3.114. *Any totally ordered set $(X, \leq_X)$ with $X \neq \emptyset$ is a lattice.*

Proof. Letting X and $\leq_X$ be arbitrary sets, assuming $X \neq \emptyset$, and assuming $\leq_X$ to be a total ordering of X, we see in light of Proposition 3.113c) and Exercise 3.52c) that both the supremum and the infimum exist for any pair $\{a,b\}$. This is then true for any X and any $\leq_X$. $\square$

The preceding corollary allows us to obtain the following simple examples of a lattice.

Corollary 3.115. *For any x the ordered pair $(\{x\}, \{(x,x)\})$ is a lattice.*

Proof. Letting x be arbitrary, we have that due to Corollary 3.70 that $(\{x\}, \{(x,x)\})$ is a totally ordered set. Now, since $x \in \{x\}$ holds according to (2.153), we see that there exists an element in $\{x\}$, so that $\{x\} \neq \emptyset$ follows to be true with (2.42). Consequently, $(\{x\}, \{(x,x)\})$ is a lattice in view of Corollary 3.114. Since x was arbitrary, we may therefore conclude that the proposed universal sentence is true. $\square$

Corollary 3.116. *For any x and any y satisfying $x \neq y$ the ordered pair $(\{x,y\}, \{(x,x),(x,y),(y,y)\})$ is a lattice.*

Proof. We take arbitrary x and y, assume $x \neq y$ to be true, and recall from Corollary 3.85 that $(\{x,y\}, \{(x,x),(x,y),(y,y)\})$ is a totally ordered set, where $\{x,y\}$ is evidently nonempty since $x, y \in \{x,y\}$ holds according to (2.151). Thus, Corollary 3.114 shows that $(\{x,y\}, \{(x,x),(x,y),(y,y)\})$ is a lattice, which is then true for any x and any y. $\square$

3.3.4. Intervals

We end the current section with the introduction of another basic type of set that arises frequently in the context of ordered sets.

Proposition 3.117. *The following sentences are true for any partially ordered set* $(X, \leq_X)$ *and for any elements* $a, b \in X$.

a) There exists a unique set $[a,b]$ *such that an element* x *is in* $[a,b]$ *iff (1)* x *is in* X*, (2)* x *is greater than or equal to* a *and (3)* x *is less than or equal to* b*, that is,*

$$\exists![a,b]\, \forall x\, (x \in [a,b] \Leftrightarrow [x \in X \,\wedge\, (x \geq_X a \,\wedge\, x \leq_X b)]). \tag{3.351}$$

b) The set $[a,b]$ *is a subset of* X*, that is,*

$$[a,b] \;\subseteq\; X. \tag{3.352}$$

c) Furthermore, an element x *is in* $[a,b]$ *if, and only if,* x *is between* a *and* b*, that is,*

$$\forall x\, (x \in [a,b] \Leftrightarrow a \leq_X x \leq_X b). \tag{3.353}$$

a is a lower bound and b an upper bound for $[a,b]$ with respect to $\leq_X$.

d) Moreover, the set $[a,b]$ *is nonempty iff* a *is less than or equal to* b*, that is,*

$$[a,b] \neq \emptyset \;\Leftrightarrow a \leq_X b. \tag{3.354}$$

e) There exists a unique set $\{[a,b] : a, b \in X\}$ *consisting of all the sets* $[a,b]$ *in* $\mathcal{P}(X)$ *with* $a, b \in X$*, and this set satisfies*

$$\forall Z\, (Z \in \{[a,b] : a,b \in X\} \Leftrightarrow \exists a, b\, (a, b \in X \,\wedge\, [a,b] = Z)). \tag{3.355}$$

f) The set $\{[a,b] : a,b \in X\}$ *is included in the power set of* X*, that is,*

$$\{[a,b] : a,b \in X\} \;\subseteq\; \mathcal{P}(X). \tag{3.356}$$

Proof. We let X, $\leq_X$, a and b be arbitrary, assume $\leq_X$ to be a reflexive partial ordering of X, and assume $a, b \in X$ to be true.

Concerning a), we may evidently apply the Axiom of Specification together with the Equality Criterion for sets to establish the uniquely existential sentence (3.351). Thus, the set $[a,b]$ satisfies

$$\forall x\, (x \in [a,b] \Leftrightarrow [x \in X \,\wedge\, (x \geq_X a \,\wedge\, x \leq_X b)]). \tag{3.357}$$

Concerning b), we notice in light of (3.357) that $x \in [a, b]$ implies in particular $x \in X$ for any X, so that the inclusion (3.352) follows to be true by definition of a subset.

Concerning c), we take an arbitrary x and assume first $x \in [a, b]$ to be true. This assumption implies with (3.357) in particular the truth of $x \geq_X a \wedge x \leq_X b$, which conjunction we may write also as $a \leq_X x \leq_X b$. We thus proved the first part ('⇒') of the equivalence in (3.353). To establish the second part ('⇐'), we now assume $a \leq_X x \leq_X b$ to be true, which conjunction we may write first as $x \geq_X a \wedge x \leq_X b$, and then also $(a, x) \in \leq_X \wedge (x, b) \in \leq_X$, recalling that $\leq_X$ is by definition a binary relation on X. Thus, the inclusion $\leq_X \subseteq X \times X$ holds, so that the two parts of the preceding conjunction imply $(a, x), (x, b) \in X \times X$ by definition of a subset, and therefore especially $x \in X$ with the definition of the Cartesian product of two sets. We thus showed that $x \in X$ and $x \geq_X a \wedge x \leq_X b$ are both true, so that $x \in [a, b]$ follows to be also true with (3.357), completing the proof of the equivalence in (3.353). Because x is arbitrary, we may therefore conclude that the universal sentence (3.353) is true. Then, since $x \in [a, b]$ evidently implies in particular $a \leq_X x$ for any x, we have that a is a lower bound for $[a, b]$, by definition. Similarly, $x \in [a, b]$ implies in particular $x \leq_X b$ for any x, so that b is by definition an upper bound for $[a, b]$.

Concerning d), we prove the first part ('⇒') of the equivalence directly, assuming $[a, b] \neq \emptyset$ to hold. Then, there exists a particular element $\bar{x} \in [a, b]$, according to (2.42), so that $a \leq_X \bar{x} \leq_X b$ follows to be true with (3.353). As the reflexive partial ordering $\leq_X$ is transitive by definition, the preceding inequalities imply $a \leq_X b$, as desired. Next, we prove the second part ('⇐') of the equivalence by assuming $a \leq_X b$ to be true. Since $\leq_X$ is reflexive, the inequality $a \leq_X a$ also holds, so that we obtained the two inequalities $a \leq_X a \leq_X b$. These now give $a \in [a, b]$ due to (3.353), which finding clearly shows that $[a, b]$ is not empty. Thus, the proof of the equivalence is complete. Concerning e), we can use again the Axiom of Specification in connection with the Equality Criterion for sets to prove the unique existence of a set $\{[a, b] : a, b \in X\}$ such that

$$\forall Z \, (Z \in \{[a, b] : a, b \in X\} \Leftrightarrow [Z \in \mathcal{P}(X) \wedge \exists a, b \, (a, b \in X \wedge [a, b] = Z)]). \tag{3.358}$$

Next, we take an arbitrary set Z, assuming first $Z \in \{[a, b] : a, b \in X\}$. Consequently, we obtain with (3.358) in particular the desired existential sentence in (3.355). Let us now conversely assume that there exist constants a and b satisfying $a, b \in X \wedge [a, b] = Z$, say $\bar{a}$ and $\bar{b}$. According to b), we thus have $Z \subseteq X$ and therefore $Z \in \mathcal{P}(X)$ by definition of a power set. Together with the assumed existential sentence, this gives with (3.358) the desired consequent $Z \in \{[a,b] : a,b \in X\}$, so that the proof of the equivalence in (3.355) is complete. As Z is arbitrary, we may infer from the truth of that equivalence the truth of the universal sentence (3.355).

Finally, concerning f), we see in light of (3.358) that $Z \in \{[a,b] : a,b \in X\}$ implies in particular $Z \in \mathcal{P}(X)$ for any Z, so that the stated inclusion follows to be true by definition of a subset.

Since X, $\leq_X$, a and b were initially arbitrary, we may therefore conclude that the proposition holds. □

Definition 3.32 (Closed interval, set of closed intervals). For any partially ordered set $(X, \leq_X)$ and any elements $a, b \in X$, we call the unique set

$$[a,b]_X = [a,b] \tag{3.359}$$

consisting of all the elements in X between a and b in the sense of

$$\forall x\, (x \in [a,b]_X \Leftrightarrow a \leq_X x \leq_X b)$$

the *closed interval* from a to b (with respect to $\leq_X$). Furthermore, we call

$$\{[a,b] : a,b \in X\} \tag{3.360}$$

the *set of closed intervals* in X (with respect to $\leq_X$).

Corollary 3.118. *For any partially ordered set $(X, \leq_X)$ and for any elements $a, b \in X$ with $a \leq_X b$, it is true that a is the minimum/infimum and b the maximum/supremum of $[a,b]$, that is,*

$$\min\,[a,b] = a = \inf\,[a,b], \tag{3.361}$$

$$\max\,[a,b] = b = \sup\,[a,b]. \tag{3.362}$$

Proof. Letting X, $\leq_X$, a and b be arbitrary such that $\leq_X$ is a reflexive partial ordering of X, such that $a, b \in X$ and such that $a \leq_X b$ holds, we observe in light of Proposition 3.117c) that a is a lower and b an upper bound for $[a,b]$. Then, because $\leq_X$ is reflexive and because of the assumption $a \leq_X b$, the inequalities

$$a \leq_X a \leq_X b$$

$$a \leq_X b \leq_X b$$

hold, so that $a, b \in [a,b]$ follows to be true again with Proposition 3.117c). Consequently, we obtain the first equation in (3.361) and in (3.362) with the definition of a minimum & of a maximum. In addition, the respective second equations follow to be true with Theorem 3.105. As X, $\leq_X$, a and b were arbitrary, we may therefore conclude that the corollary is true. □

Proposition 3.119. *For any partially ordered set $(X, \leq_X)$, where the reflexive partial ordering $\leq_X$ induces the irreflexive partial ordering $<_X$ of X, and for any element $a \in X$, it is true that the closed interval from a to a is identical with the singleton formed by a, i.e.*

$$[a, a] = \{a\}. \tag{3.363}$$

Proof. Letting X, $\leq_X$ and a be arbitrary, we assume that $(X, \leq_X)$ is partially ordered and that a is in X. Let us apply the Equality Criterion for sets to establish $[a, a] = \{a\}$. We thus need to prove the universal sentence

$$\forall x\, (x \in [a, a] \Leftrightarrow x \in \{a\}). \tag{3.364}$$

Letting x be arbitrary, we first assume $x \in [a, a]$, so that $a \leq_X x \wedge x \leq_X a$ holds by definition of a closed interval. Because the reflexive partial ordering $\leq_X$ is antisymmetric, the preceding conjunction implies $x = a$, which equation in turn implies the desired consequent $x \in \{a\}$ with (2.169). To establish the second part of the equivalence, we now assume conversely $x \in \{a\}$ to be true, so that $x = a$ holds according to (2.169). Then, letting $<_X$ be the irreflexive partial ordering of X induced by $\leq_X$, the two disjunctions

$$a <_X x \vee a = x$$
$$x <_X a \vee x = a$$

are evidently also both true, which now imply $a \leq_X x$ as well as $x \leq_X a$ by definition of an induced irreflexive partial ordering. Thus, $x \in [a, a]$ holds by definition of a closed interval, which finding completes the proof of the equivalence. As x was arbitrary, we may therefore conclude that the universal sentence (3.364) is true, which in turn implies the truth of the equation (3.363). Since X, $\leq_X$ and a were initially arbitrary, we may infer from this the truth of the stated proposition. □

Theorem 3.120 (Generation of lattices based on closed intervals). *For any lattice $(X, \leq_X)$ and any elements $a, b \in X$ with $a \leq_X b$, it is true that the ordered pair $([a, b], \leq_{[a,b]})$ is a also a lattice.*

Proof. We take arbitrary sets X and $\leq_X$ such that $(X, \leq_X)$ is a lattice, and we take arbitrary elements $a, b \in X$ satisfying $a \leq_X b$, so that the closed interval $[a, b]$ is a nonempty subset of X due to Proposition 3.117b). According to the Reflexive partial ordering of subsets, there exists then the unique reflexive partial ordering $\leq_{[a,b]}$ of the closed interval $[a, b]$, which satisfies

$$\forall x, y\, (x, y \in [a, b] \Rightarrow [a \leq_{[a,b]} b \Leftrightarrow a \leq_X b]). \tag{3.365}$$

To prove that the partially ordered set $([a,b], \leq_{[a,b]})$ is a lattice, we observe first that the initial assumption $a \leq_X b$ implies $[a,b] \neq \emptyset$ with Proposition 3.117d). Next, we verify the universal sentence

$$\forall x, y\, (x, y \in [a,b] \Rightarrow \exists S, I\, (S, I \in [a,b] \wedge S = \sup^{\leq_{[a,b]}}\{x,y\} \wedge I = \inf^{\leq_{[a,b]}}\{x,y\})). \tag{3.366}$$

To do this, we take arbitrary constants x and y, and we assume $x, y \in [a,b]$ to be true. Consequently, $x, y \in X$ holds due to the mentioned inclusion $[a,b] \subseteq X$, so that there are constants, say $\bar{S}$ and $\bar{I}$, satisfying $\bar{S}, \bar{I} \in X$, $\bar{S} = \sup^{\leq_X}\{x,y\}$, and $\bar{I} = \inf^{\leq_X}\{x,y\}$, because $(X, \leq_X)$ is a lattice. Thus, $\bar{S}$ is an upper and $\bar{I}$ a lower bound for $\{x,y\}$ (with respect to $\leq_X$). Therefore, the Characterization of upper & lower bounds for pairs yields, respectively,

$$x \leq_X \bar{S} \wedge y \leq_X \bar{S} \tag{3.367}$$

and

$$\bar{I} \leq_X x \wedge \bar{I} \leq_X y. \tag{3.368}$$

Furthermore, the inequalities

$$a \leq_X x \wedge x \leq_X b \tag{3.369}$$

as well as

$$a \leq_X y \wedge y \leq_X b \tag{3.370}$$

are true according to Proposition 3.117c); thus, the conjunction

$$x \leq_X b \wedge y \leq_X b$$

and the conjunction

$$a \leq_X x \wedge a \leq_X y$$

are true, which show in light of the Characterization of upper & lower bounds for pairs that b is an upper and a a lower bound for $\{x,y\}$ (with respect to $\leq_X$). Since $\bar{S}$ is the least upper and $\bar{I}$ the greatest lower bound for $\{x,y\}$ (with respect to $\leq_X$), we have

$$\bar{S} \leq_X b \tag{3.371}$$

and

$$a \leq_X \bar{I}. \tag{3.372}$$

Let us now combine the first inequality in (3.369), the first one in (3.367) and the inequality in (3.371) to

$$a \leq_X x \leq_X \bar{S} \leq_X b,$$

and similarly the inequality in (3.372), the first one in (3.368) and the second one in (3.369) to

$$a \leq_X \bar{I} \leq_X x \leq_X b.$$

As the reflexive partial ordering $\leq_X$ is transitive, we therefore obtain

$$a \leq_X \bar{S} \leq_X b$$

as well as

$$a \leq_X \bar{I} \leq_X b,$$

consequently $\bar{S}, \bar{I} \in [a, b]$ with (3.353), as required by the desired existential sentence in (3.366). In addition, because x, y, $\bar{S}$ and $\bar{I}$ are thus all elements in $[a, b]$, we may write (3.367) – (3.368) equivalently as

$$x \leq_{[a,b]} \bar{S} \wedge y \leq_{[a,b]} \bar{S}$$

and

$$\bar{I} \leq_{[a,b]} x \wedge \bar{I} \leq_{[a,b]} y.$$

respectively, by applying (3.365). These inequalities clearly show that $\bar{S}$ is an upper and $\bar{I}$ a lower bound for the pair $\{x, y\}$ with respect to $\leq_{[a,b]}$. To prove that $\bar{S}$ is the least upper and $\bar{I}$ the greatest lower bound for $\{x, y\}$, we apply the Characterization of the supremum & infimum, letting S' and I' be arbitrary constants such that S' is an upper and I' a lower bound for $\{x, y\}$ with respect to $\leq_{[a,b]}$, so that the inequalities

$$x \leq_{[a,b]} S' \wedge y \leq_{[a,b]} S'$$

and

$$I' \leq_{[a,b]} x \wedge I' \leq_{[a,b]} y.$$

are evidently true. Thus, x, y, S' and I' are all elements in $[a, b]$, so that we may write the preceding inequalities also as

$$x \leq_X S' \wedge y \leq_X S'$$

and

$$I' \leq_X x \wedge I' \leq_X y.$$

These inequalities show now that S' is an upper and I' a lower bound for $\{x, y\}$ also with respect to $\leq_X$. Then, as $\bar{S}$ is the least upper and $\bar{I}$ the greatest lower bound for $\{x, y\}$ with respect to $\leq_X$ (whose existence we established previously), we obtain

$$\bar{S} \leq_X S'$$

and

$$I' \leq_X \bar{I}$$

with the Characterization of the supremum & infimum. We already mentioned before that $\bar{S}$, S', $\bar{I}$ and I' are elements in $[a,b]$, which allows us to write these two inequalities equivalently as

$$\bar{S} \leq_{[a,b]} S'$$

and

$$I' \leq_{[a,b]} \bar{I}.$$

Since S' and I' were arbitrary, we may infer from these findings that the upper bound $\bar{S}$ for $\{x,y\}$ is indeed the least one and that the lower bound $\bar{I}$ for $\{x,y\}$ is the greatest one for $\{x,y\}$ (with respect to $\leq_{[a,b]}$). Thus, the proof of the existential sentence in (3.366) is complete, and as x and y were arbitrary, we may therefore conclude that the universal sentence (3.366) is true. This means that the partially ordered set $([a,b], \leq_{[a,b]})$ is a lattice, by definition. Because X, $\leq_X$, a and b were initially also arbitrary, we may now finally conclude that the stated theorem holds. □

Note 3.13. For any lattice $([a,b], \leq_{[a,b]})$ generated by a lattice $(X, \leq_X)$, we have the underlying assumption $a \leq_X b$, so that a is the bottom and b the top of $[a,b]$ according to Corollary 3.118.

Exercise 3.53. Prove the following sentences for any partially ordered set $(X, \leq_X)$ and for any elements $a, b \in X$.

a) There exist unique sets (a,b), $(a,b]$ and $[a,b)$ such that

$$\forall x\,(x \in (a,b) \Leftrightarrow [x \in X \,\wedge\, (x >_X a \,\wedge\, x <_X b)]), \tag{3.373}$$
$$\forall x\,(x \in (a,b] \Leftrightarrow [x \in X \,\wedge\, (x >_X a \,\wedge\, x \leq_X b)]), \tag{3.374}$$
$$\forall x\,(x \in [a,b) \Leftrightarrow [x \in X \,\wedge\, (x \geq_X a \,\wedge\, x <_X b)]). \tag{3.375}$$

b) The set (a,b), $(a,b]$ and $[a,b)$ are subsets of X, that is,

$$(a,b) \subseteq X, \tag{3.376}$$
$$(a,b] \subseteq X, \tag{3.377}$$
$$[a,b) \subseteq X. \tag{3.378}$$

c) Furthermore, the sets (a,b), $(a,b]$ and $[a,b)$ satisfy

$$\forall x\,(x \in (a,b) \Leftrightarrow a <_X x <_X b), \tag{3.379}$$
$$\forall x\,(x \in (a,b] \Leftrightarrow a <_X x \leq_X b), \tag{3.380}$$
$$\forall x\,(x \in [a,b) \Leftrightarrow a \leq_X x <_X b). \tag{3.381}$$

a is a lower bound and b an upper bound for all of these sets (with respect to $\leq_X$).

d) The sets (a,b), $(a,b]$ and $[a,b)$ are empty if a is not less than b, i.e.

$$\neg a <_X b \Leftrightarrow (a,b) = \emptyset, \tag{3.382}$$
$$\neg a <_X b \Leftrightarrow (a,b] = \emptyset, \tag{3.383}$$
$$\neg a <_X b \Leftrightarrow [a,b) = \emptyset. \tag{3.384}$$

e) Establish the unique sets $\{(a,b) : a,b \in X\}$, $\{(a,b] : a,b \in X\}$ and $\{[a,b) : a,b \in X\}$ that satisfy

$$\forall Z\,(Z \in \{(a,b) : a,b \in X\} \Leftrightarrow \exists a,b\,(a,b \in X \,\wedge\, (a,b) = Z)), \tag{3.385}$$
$$\forall Z\,(Z \in \{(a,b] : a,b \in X\} \Leftrightarrow \exists a,b\,(a,b \in X \,\wedge\, (a,b] = Z)), \tag{3.386}$$
$$\forall Z\,(Z \in \{[a,b) : a,b \in X\} \Leftrightarrow \exists a,b\,(a,b \in X \,\wedge\, [a,b) = Z)). \tag{3.387}$$

f) The sets $\{(a,b) : a,b \in X\}$, $\{(a,b] : a,b \in X\}$ and $\{[a,b) : a,b \in X\}$ are all included in the power set of X, that is,

$$\{(a,b) : a,b \in X\} \subseteq \mathcal{P}(X), \tag{3.388}$$
$$\{(a,b] : a,b \in X\} \subseteq \mathcal{P}(X), \tag{3.389}$$
$$\{[a,b) : a,b \in X\} \subseteq \mathcal{P}(X). \tag{3.390}$$

(Hint: Proceed similarly as in the proof of Proposition 3.117.)

Definition 3.33 (Open interval, set of open intervals, left-open and right-closed interval, set of left-open and right-closed intervals, left-closed and right-open interval, set of left-closed and right-open intervals). For any partially ordered set $(X, \leq_X)$ and any elements $a,b \in X$,

(1) we call the set
$$(a,b)_X = (a,b) \tag{3.391}$$
consisting of all the elements in X strictly between a and b in the sense of
$$\forall x\,(x \in (a,b)_X \Leftrightarrow a <_X x <_X b)$$
the *open interval* from a to b (with respect to $\leq_X$). Furthermore, we call
$$\{(a,b) : a,b \in X\} \tag{3.392}$$
the *set of open intervals* in X (with respect to $\leq_X$).

(2) we call the set

$$(a,b]_X = (a,b] \tag{3.393}$$

consisting of all the elements in X satisfying

$$\forall x\,(x \in (a,b]_X \Leftrightarrow a <_X x \leq_X b)$$

the *left-open and right-closed interval* from a to b (with respect to $\leq_X$). Moreover, we call

$$\{(a,b] : a,b \in X\} \tag{3.394}$$

the *set of left-open and right-closed intervals* in X (with respect to $\leq_X$).

(3) we call the set

$$[a,b)_X = [a,b) \tag{3.395}$$

consisting of all the elements in X satisfying

$$\forall x\,(x \in [a,b)_X \Leftrightarrow a \leq_X x <_X b)$$

the *left-open and right-closed interval* from a to b (with respect to $\leq_X$). In addition, we call

$$\{[a,b) : a,b \in X\} \tag{3.396}$$

the *set of left-closed and right-open intervals* in X (with respect to $\leq_X$).

Note 3.14. We may define open intervals also more directly from a given partially ordered set $(X, <_X)$ where $<_X$ is irreflexive, instead of using the irreflexive partial ordering induced by $\leq_X$.

Corollary 3.121. *For any partially ordered set $(X, \leq_X)$ with $X \neq \emptyset$ it is true that the sets of open, of left-open and right-closed, and of left-closed and right-open intervals contain the empty set, i.e.*

$$X \neq \emptyset \;\Rightarrow\; \emptyset \in \{(a,b) : a,b \in X\}, \tag{3.397}$$
$$X \neq \emptyset \;\Rightarrow\; \emptyset \in \{(a,b] : a,b \in X\}, \tag{3.398}$$
$$X \neq \emptyset \;\Rightarrow\; \emptyset \in \{[a,b) : a,b \in X\}. \tag{3.399}$$

Proof. Letting X and $\leq_X$ be arbitrary such that $(X, \leq_X)$ is a partially ordered set and assuming $X \neq \emptyset$ to be true, there evidently exists an element in X, say $\bar{a}$. Since the induced partial ordering $<_X$ is irreflexive, the negation $\neg \bar{a} <_X \bar{a}$ holds then, so that $(\bar{a}, \bar{a}) = \emptyset$, $(\bar{a}, \bar{a}] = \emptyset$ and $[\bar{a}, \bar{a}) = \emptyset$ follow to be true with (3.382) – (3.384). Thus, the existential sentences

$$\exists a, b\,(a, b \in X \;\wedge\; (a, b) = \emptyset),$$
$$\exists a, b\,(a, b \in X \;\wedge\; (a, b] = \emptyset),$$
$$\exists a, b\,(a, b \in X \;\wedge\; [a, b) = \emptyset)$$

hold, which turn implies the desired consequents (3.397) – (3.399) with (3.385) – (3.387). Since X and $\leq_X$ were initially arbitrary sets, we may therefore conclude that the corollary holds indeed. □

Proposition 3.122. *It is true for any linearly ordered set $(X, <_X)$ that the singleton formed by an element a of X and the open interval from a to an element b of X are disjoint, i.e.*

$$\forall a, b\,(a, b \in X \Rightarrow \{a\} \cap (a, b) = \emptyset). \tag{3.400}$$

Proof. We take arbitrary sets X and $<_X$, assuming that $(X, <_X)$ is linearly ordered, and take arbitrary a and b, assuming $a, b \in X$ to hold. We may now establish the equation $\{a\} \cap (a, b) = \emptyset$ via the definition of the empty set, i.e. by proving the universal sentence

$$\forall x\,(x \notin \{a\} \cap (a, b)). \tag{3.401}$$

We let x be arbitrary, and we prove the sentence $x \notin \{a\} \cap (a, b)$ by contradiction, assuming its negation to be true. Then, the Double Negation Law gives us the true sentence $x \in \{a\} \cap (a, b)$, so that $x \in \{a\}$ and $x \in (a, b)$ follow to be both true by definition of the intersection of two sets. Here, the former yields $x = a$ with (2.169), and the latter $a <_X x$ as well as $x <_X b$ by definition of an open interval in X; we thus found $a = x$ and $a <_X x$ to be both true. Since the linear ordering $<_X$ is characterized by comparability, it is however false that $a = x$ and $a <_X x$ are simultaneously true, so that we arrived at a contradiction, and the proof of $x \notin \{a\} \cap (a, b)$ is complete. As x was arbitrary, we may therefore conclude that the universal sentence (3.401) is true, with the consequence that $\{a\} \cap (a, b) = \emptyset$. Here, a and b were arbitrary, so that the universal sentence (3.400) follows now to be also true. Finally, since X and $<_X$ were initially arbitrary as well, we may further conclude that the proposed universal sentence holds, as claimed. □

Proposition 3.123. *It is true for any partially ordered set $(X, <_X)$ that the left-closed and right-open interval from an element a to an element b can be written as the union of the open interval from a to b and the singleton formed by a if a is less than b (with respect to $<_X$), i.e.*

$$\forall a, b\,(\,a <_X b \;\Rightarrow\; [a, b) = (a, b) \cup \{a\}\,). \tag{3.402}$$

Proof. We take arbitrary sets X, $<_X$, a and b, assuming $(X, <_X)$ to be partially ordered and assuming $a <_X b$ to be true. Let us observe here that the assumed inequality can be written as $(a, b) \in <_X$; since $<_X$ is a binary relation on X, it is included in $X \times X$, so that $(a, b) \in <_X$ implies $(a, b) \in X \times X$ with the definition of a subset, and subsequently $a, b \in X$ with the definition of the Cartesian product of two sets. Thus, $[a, b)$ and (a, b) represent indeed intervals in X. We now prove the stated equation via an application of the Equality Criterion for sets, proving the equivalent universal sentence

$$\forall x\,(\,x \in [a, b) \;\Leftrightarrow\; x \in (a, b) \cup \{a\}\,). \tag{3.403}$$

We take now an arbitrary x, assuming first $x \in [a, b)$ to hold. By definition of a left-closed and right-open interval, this assumption gives us the two inequalities $a \leq_X x$ and $x <_X b$, where the former implies the disjunction $a <_X x \vee a = x$ by definition of an induced reflexive partial ordering. Let us use this disjunction to prove the disjunction $x \in (a, b) \vee x \in \{a\}$ by cases. The first case $a <_X x$ implies together with the previously established $x <_X b$, by definition of an open interval, $x \in (a, b)$, so that the disjunction to be proven also holds. In the second case $a = x$, we may write equivalently $x = a$, which implies then $x \in \{a\}$ with (2.169), so that the disjunction $x \in (a, b) \vee x \in \{a\}$ follows to be true again. Having thus completed the proof by cases, we may infer from the truth of that disjunction the truth of the desired consequent $x \in (a, b) \cup \{a\}$ by means of the definition of the union of two sets, and therefore the truth of the first part ('$\Rightarrow$') of the equivalence in (3.403).

Regarding the second part ('$\Leftarrow$'), we assume conversely $x \in (a, b) \cup \{a\}$ to be true, giving the true disjunction $x \in (a, b) \vee x \in \{a\}$ (using again definition of the union of two sets). We may use this disjunction to prove $x \in [a, b)$ by cases. On the one hand, if $x \in (a, b)$ is true, so that the inequalities $a <_X x$ and $x <_X b$ hold (by definition of an open interval), we now see that the former inequality yields the true disjunction $a <_X x \vee a = x$. This disjunction in turn gives $a \leq_X x$ (by definition of an induced reflexive partial ordering), which further implies – in connection with the previously obtained $x <_X b$, the desired consequent $x \in [a, b)$ (by definition of a left-closed and right-open interval). On the other hand, if $x \in \{a\}$ is true, so that $x = a$ holds according to (2.169), we obtain from the initial assumption $a <_X b$ via substitution $x <_X b$. Furthermore, writing the preceding equation as $a = x$, the disjunction $a <_X x \vee a = x$ is then true again, with the previously inferred consequence that $a \leq_X x$ holds. The conjunction of the two findings $a \leq_X x$ and $x <_X b$ gives us now the desired consequent $x \in [a, b)$ also in the second case, so that the proof of the second part of the equivalence in (3.403) is complete. Because

x is arbitrary, the universal sentence (3.403) follows to be true, which in turn implies the truth of the equation in (3.402). As the sets X, $\leq_X$, a and b were all arbitrary, we may therefore conclude that the proposition is true. □

Exercise 3.54. Show for any partially ordered set $(X, \leq_X)$ that

a) the left-open and right-closed interval from an element a to an element b can be written as the union of the open interval from a to b and the singleton formed by b if a is less than b (with respect to $\leq_X$), i.e.

$$\forall a, b\,(\, a <_X b \Rightarrow (a, b] = (a, b) \cup \{b\}\,). \tag{3.404}$$

b) the closed interval from an element a to an element b can be written as the union of the left-closed, right-open interval from a to b and the singleton formed by b if a is less than or equal to b (with respect to $\leq_X$), i.e.

$$\forall a, b\,(\, a \leq_X b \Rightarrow [a, b] = [a, b) \cup \{b\}\,). \tag{3.405}$$

(Hint: Proceed in analogy to the proof of Proposition 3.123.)

Corollary 3.124. *It is true for any linearly ordered set $(X, <_X)$ that the singleton formed by an element a can be written as the difference of the left-closed, right-open interval from a to an element b and the open interval from a to b if a is less than b (with respect to $<_X$), i.e.*

$$\forall a, b\,(\, a <_X b \Rightarrow \{a\} = [a, b) \setminus (a, b)\,). \tag{3.406}$$

Proof. Letting X, $<_X$, a and b be arbitrary, assuming $(X, <_X)$ to be linearly ordered and assuming $a <_X b$ to hold (where $a, b \in X$ is then true), we obtain for the union C of $\{a\}$ and (a, b)

$$C = \{a\} \cup (a, b) = (a, b) \cup \{a\} = [a, b)$$

with the Commutative Law for the union of two sets and with (3.402). Furthermore, we obtain

$$\{a\} \cap (a, b) = \emptyset$$

with (3.400). We may therefore apply (2.262) to infer from these findings the truth of

$$\{a\} = C \setminus (a, b) = [a, b) \setminus (a, b),$$

which equations yield the desired consequent of the implication in (3.406). Because X, $<_X$, a and b were initially arbitrary, the proposed universal sentence follows then to be true. □

In the following, we explore some set-theoretical interactions of, respectively, two open and two left-closed and right-open intervals.

Proposition 3.125. *The following sentences are true for any partially ordered set* $(X, \leq_X)$ *and any elements* $a_1, a_2, b_1, b_2 \in X$*.*

a) Concerning the inclusion of two open intervals,

$$(a_2 \leq_X a_1 \,\wedge\, b_1 \leq_X b_2) \Rightarrow (a_1, b_1) \subseteq (a_2, b_2). \tag{3.407}$$

b) Concerning the intersection of two open intervals,

$$(a_1, b_1) \cap (a_2, b_2) = (a_1, b_2) \cap (a_2, b_1). \tag{3.408}$$

c) Concerning the disjointness of two open intervals,

$$(\neg a_2 <_X b_1 \,\vee\, \neg a_1 <_X b_2) \Rightarrow (a_1, b_1) \cap (a_2, b_2) = \emptyset. \tag{3.409}$$

Proof. Letting X, $\leq_X$, a_1, a_2, b_1 and b_2 be arbitrary such that $(X, \leq_X)$ is a partially ordered set and such that $a_1, a_2, b_1, b_2 \in X$ holds, we assume concerning a) the antecedent $a_2 \leq_X a_1 \,\wedge\, b_1 \leq_X b_2$ to be true. To show that this implies the inclusion $(a_1, b_1) \subseteq (a_2, b_2)$, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall x \, (x \in (a_1, b_1) \Rightarrow x \in (a_2, b_2)).$$

For this purpose, we let $x \in (a_1, b_1)$ be arbitrary and show that this implies $x \in (a_2, b_2)$. The assumption $x \in (a_1, b_1)$ means by definition of an open interval that $a_1 <_X x$ and $x <_X b_1$ are true. On the one hand, the two assumed inequalities $a_2 \leq_X a_1$ and $a_1 <_X x$ imply $a_2 <_X x$ with the Transitivity Formula for $\leq$ and $<$. On the other hand, the other two assumed inequalities $x <_X b_1$ and $b_1 \leq_X b_2$ imply $x < b_2$ with the Transitivity Formula for $<$ and $\leq$. Then, since $a_2 <_X x$ and $x <_X b_2$ are simultaneously true, we obtain $x \in (a_2, b_2)$ by definition of an open interval, as desired. As x was arbitrary, we may therefore infer from this finding the truth of the inclusion $(a_1, b_1) \subseteq (a_2, b_2)$, proving (3.407).

Concerning b), we obtain for an arbitrary x the true equivalences

$$\begin{aligned}
x \in (a_1, b_1) \cap (a_2, b_2) \;&\Leftrightarrow\; x \in (a_1, b_1) \,\wedge\, x \in (a_2, b_2) \\
&\Leftrightarrow\; (a_1 <_X x \,\wedge\, x <_X b_1) \,\wedge\, (a_2 <_X x \,\wedge\, x <_X b_2) \\
&\Leftrightarrow\; (a_1 <_X x \,\wedge\, x <_X b_2) \,\wedge\, (a_2 <_X x \,\wedge\, x <_X b_1) \\
&\Leftrightarrow\; x \in (a_1, b_2) \,\wedge\, x \in (a_2, b_1) \\
&\Leftrightarrow\; x \in (a_1, b_2) \cap (a_2, b_1)
\end{aligned}$$

by applying the definition of the intersection of two sets, the definition of an open interval, the Associative and Commutative Law for the conjunction, again the definition of an open interval, and finally again the definition of the intersection of two sets. As x is arbitrary, we may infer from the truth of these equivalences the truth of the equation (3.408) with the Equality Criterion for sets.

For the proof of c) we assume $\neg a_2 <_X b_1 \vee \neg a_1 <_X b_2$ and prove the desired consequent $(a_1, b_1) \cap (a_2, b_2) = \emptyset$ by cases. The first case $\neg a_2 <_X b_1$ implies $(a_2, b_1) = \emptyset$ with (3.382), so that we obtain

$$(a_1, b_1) \cap (a_2, b_2) = (a_1, b_2) \cap (a_2, b_1) = (a_1, b_2) \cap \emptyset = \emptyset$$

by applying b), substitution and (2.62). Using exactly the same arguments, the second case $\neg a_1 <_X b_2$ gives $(a_1, b_2) = \emptyset$ and then

$$(a_1, b_1) \cap (a_2, b_2) = (a_1, b_2) \cap (a_2, b_1) = \emptyset \cap (a_2, b_1) = \emptyset,$$

completing the proof by cases.

Because X, $\leq_X$, a_1, a_2, b_1 and b_2 were initially also arbitrary, we may then further conclude that the proposition is true. □

Exercise 3.55. Establish the following sentences for any partially ordered set $(X, \leq_X)$ and any elements $a_1, a_2, b_1, b_2 \in X$.

a) Concerning the inclusion of two left-closed and right-open intervals,

$$(a_2 \leq_X a_1 \wedge b_1 \leq_X b_2) \Rightarrow [a_1, b_1) \subseteq [a_2, b_2). \tag{3.410}$$

(Hint: Apply the Transitivity Formula for $<$ and $\leq$.)

b) Concerning the intersection of two left-closed and right-open intervals,

$$[a_1, b_1) \cap [a_2, b_2) = [a_1, b_2) \cap [a_2, b_1). \tag{3.411}$$

(Hint: Use the Equality Criterion for sets in connection with Theorem 1.12 & 1.11.)

c) Concerning the disjointness of two left-closed and right-open intervals,

$$(\neg a_2 <_X b_1 \vee \neg a_1 <_X b_2) \Rightarrow [a_1, b_1) \cap [a_2, b_2) = \emptyset. \tag{3.412}$$

(Hint: Use b) and (2.62).)

We now establish a rule for left-closed and right-open intervals that requires the partial ordering to be linear/total.

Proposition 3.126. *The following universal sentence holds for any linearly ordered set* $(X, <_X)$.

$$\forall a, b, c\,([a, b, c \in X \;\wedge\; a \leq_X b \leq_X c] \Rightarrow [a, b) \cup [b, c) = [a, c)). \qquad (3.413)$$

Proof. We let X, $\leq_X$, a, b and c be arbitrary assuming that $(X, \leq_X)$ is partially ordered, that $a, b, c \in X$ holds, and assuming moreover the inequalities $a \leq_X b \leq_X c$ to be true. We now prove the stated equation by applying the Equality Criterion for sets, by verifying the universal sentence

$$\forall x\,(x \in [a, b) \cup [b, c) \Leftrightarrow x \in [a, c)). \qquad (3.414)$$

We let x be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $x \in [a, b) \cup [b, c)$ to be true. Therefore, the disjunction $x \in [a, b) \vee x \in [b, c)$ is true by definition of the union of two sets. We now use this true disjunction to prove the desired consequent $x \in [a, c)$ by cases. In case $x \in [a, b)$ holds, we have $a \leq_X x$ and $x <_X b$ according to the definition of a left-closed and right-open interval. The latter implies now with the initial assumption $b \leq_X c$ because of the Transitivity Formula for $<$ and $\leq$ that $x <_X c$ holds. Together with the previously established $a \leq_X x$, this in turn implies $x \in [a, c)$, as desired. In the other case that $x \in [b, c)$ holds, i.e. $b \leq_X x$ and $x <_X c$, the former implies with the initial assumption $a \leq_X b$ and the transitivity of the induced total ordering $\leq_X$ that $a \leq_X x$ is true. Thus, $a \leq_X x$ and $x <_X c$ are both true, which means again $x \in [a, c)$, completing the proof by cases, and then also the proof of the first part of the equivalence.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume $x \in [a, c)$ to be true, i.e. $a \leq_X x$ and $x <_X c$. Noting that the disjunction $x <_X b \vee \neg x <_X b$ is true according to the Law of the Excluded Middle, we now prove $x \in [a, b) \vee x \in [b, c)$ by cases. The first case $x <_X b$ implies together with $a \leq_X x$ that $x \in [a, b)$ holds, which further implies the truth of the disjunction $x \in [a, b) \vee x \in [b, c)$. The other case $\neg x <_X b$ gives with the Negation Formula for $<$ the inequality $b \leq_X x$ and therefore – in view of $x <_X c$ – the truth of $x \in [b, c)$. Then, the disjunction $x \in [a, b) \vee x \in [b, c)$ is true again, which thus holds in any case. Consequently, we obtain by definition of the union of two sets $x \in [a, b) \cup [b, c)$, which finding completes the proof of the equivalence (3.414).

As x is arbitrary, we may infer from the truth of that equivalence the truth of the equation in (3.413). Since a, b, c and also X and $<_X$ were arbitrary, we may therefore conclude that the proposition is true. □

The following sentence shows that an open interval may, under certain conditions, be constructed via the intersection of the other two other types of intervals considered in Definition 3.33.

Proposition 3.127. *Show for any partially ordered set $(X, <_X)$ and any elements $a_1, b_1, a_2, b_2 \in X$ satisfying $a_1 \leq_X a_2$ as well as $b_1 \leq_X b_2$ that the intersection of the left-closed and right-open interval from a_1 to b_1 and the left-open and right-closed interval from a_2 to b_2 is the open interval from a_2 to b_1, i.e.*

$$[a_1 \leq_X a_2 \wedge b_1 \leq_X b_2] \Rightarrow [a_1, b_1) \cap (a_2, b_2] = (a_2, b_1)\,). \tag{3.415}$$

Proof. We let X, $<_X$, a_1, b_1, a_2 and b_2 be arbitrary such that $(X, <_X)$ is partially ordered, such that $a_1, b_1, a_2, b_2 \in X$ is true, and such that the inequalities $a_1 \leq_X a_2 \wedge b_1 \leq_X b_2$ hold. We now apply the Equality Criterion for sets to establish the stated equation. To do this, we prove the universal sentence

$$\forall x\,(\, x \in [a_1, b_1) \cap (a_2, b_2] \Leftrightarrow x \in (a_2, b_1)\,), \tag{3.416}$$

letting x be arbitrary and assuming first $x \in [a_1, b_1) \cap (a_2, b_2]$ to be true. Therefore, we obtain with the definition of the intersection of two sets $x \in [a_1, b_1)$ and $x \in (a_2, b_2]$, and then with the definition of a left-closed and right-open as well as with the definition of a left-open and right-closed interval the inequalities $a_1 \leq_X x$, $x <_X b_1$, $a_2 <_X x$ and $x \leq_X b_2$. We thus have $a_2 <_X x <_X b_1$, so that $x \in (a_2, b_1)$ follows to be true by definition of an open interval, as desired. We now conversely assume $x \in (a_2, b_1)$ to be true, so that $a_2 <_X x$ and $x <_X b_1$ hold (by definition of an open interval). These two inequalities clearly imply the truth of the two disjunctions $a_2 <_X x \vee a_2 = x$ and $x <_X b_1 \vee x = b_1$ and therefore the truth of the two inequalities $a_2 \leq_X x$ and $x \leq_X b_1$ (using the definition of an induced reflexive partial ordering). Here, the conjunction of the assumed inequality $a_1 \leq_X a_2$ and $a_2 \leq_X x$ implies $a_1 \leq_X x$ with the transitivity of the induced reflexive partial ordering $\leq_X$. For the same reason, the conjunction of $x \leq_X b_1$ the other assumed inequality $b_1 \leq_X b_2$ gives $x \leq_X b_2$. The previous findings evidently allow us to form the conjunctions

$$(a_1 \leq_X x \wedge x <_X b_1) \wedge (a_2 <_X x \wedge x \leq_X b_2),$$

with the consequence that $x \in [a_1, b_1)$ and $x \in (a_2, b_2]$ are both true. Therefore, x is in the intersection of these two intervals, so that the second part of the equivalence in (3.416) also holds. As x is arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (3.416) and consequently the truth of the equation $[a_1, b_1) \cap (a_2, b_2] = (a_2, b_1)$. This proves the implication (3.415), and since X, $<_X$, a_1, b_1, a_2 and b_2 were initially arbitrary, we may then further conclude that the proposed universal sentence holds. □

In the following, we establish four more basic types of intervals, which will be shown to have diverse relationships to the previously introduced intervals.

Exercise 3.56. Prove the following sentences for any partially ordered set $(X, \leq_X)$ and for any elements $a, b \in X$.

a) There exist unique sets $(a, +\infty)$, $[a, +\infty)$, $(-\infty, b)$ and $(-\infty, b]$ such that

$$\forall x\,(x \in (a, +\infty) \Leftrightarrow [x \in X \,\wedge\, x >_X a]), \tag{3.417}$$
$$\forall x\,(x \in [a, +\infty) \Leftrightarrow [x \in X \,\wedge\, x \geq_X a]), \tag{3.418}$$
$$\forall x\,(x \in (-\infty, b) \Leftrightarrow [x \in X \,\wedge\, x <_X b]), \tag{3.419}$$
$$\forall x\,(x \in (-\infty, b] \Leftrightarrow [x \in X \,\wedge\, x \leq_X b]). \tag{3.420}$$

b) The sets $(a, +\infty)$, $[a, +\infty)$, $(-\infty, b)$ and $(-\infty, b]$ are included in X, i.e.

$$(a, +\infty) \subseteq X, \tag{3.421}$$
$$[a, +\infty) \subseteq X, \tag{3.422}$$
$$(-\infty, b) \subseteq X, \tag{3.423}$$
$$(-\infty, b] \subseteq X. \tag{3.424}$$

c) Furthermore, the sets $(a, +\infty)$, $[a, +\infty)$, $(-\infty, b)$ and $(-\infty, b]$ satisfy

$$\forall x\,(x \in (a, +\infty) \Leftrightarrow a <_X x), \tag{3.425}$$
$$\forall x\,(x \in [a, +\infty) \Leftrightarrow a \leq_X x), \tag{3.426}$$
$$\forall x\,(x \in (-\infty, b) \Leftrightarrow x <_X b), \tag{3.427}$$
$$\forall x\,(x \in (-\infty, b] \Leftrightarrow x \leq_X b). \tag{3.428}$$

d) The sets $[a, +\infty)$ and $(-\infty, b]$ are nonempty, i.e.

$$[a, +\infty) \neq \emptyset, \tag{3.429}$$
$$(-\infty, b] \neq \emptyset. \tag{3.430}$$

(Hint: Use (2.42) and the reflexivity of $\leq_X$.)

e) Establish the unique sets $\{(a, +\infty) : a \in X\}$, $\{[a, +\infty) : a \in X\}$, $\{(-\infty, b) : b \in X\}$ and $\{(-\infty, b] : b \in X\}$ that satisfy

$$\forall Z\,(Z \in \{(a, +\infty) : a \in X\} \Leftrightarrow \exists a\,(a \in X \,\wedge\, (a, +\infty) = Z)), \tag{3.431}$$
$$\forall Z\,(Z \in \{[a, +\infty) : a \in X\} \Leftrightarrow \exists a\,(a \in X \,\wedge\, [a, +\infty) = Z)), \tag{3.432}$$
$$\forall Z\,(Z \in \{(-\infty, b) : b \in X\} \Leftrightarrow \exists b\,(b \in X \,\wedge\, (-\infty, b) = Z)), \tag{3.433}$$
$$\forall Z\,(Z \in \{(-\infty, b] : b \in X\} \Leftrightarrow \exists b\,(b \in X \,\wedge\, (-\infty, b] = Z)). \tag{3.434}$$

f) All of the sets in e) are included in the power set of X, that is,

$$\{(a,+\infty) : a \in X\} \subseteq \mathcal{P}(X), \tag{3.435}$$
$$\{[a,+\infty) : a \in X\} \subseteq \mathcal{P}(X), \tag{3.436}$$
$$\{(-\infty,b) : b \in X\} \subseteq \mathcal{P}(X), \tag{3.437}$$
$$\{(-\infty,b] : b \in X\} \subseteq \mathcal{P}(X). \tag{3.438}$$

Definition 3.34 ((Set of) open and unbounded & left-closed and right-unbounded & open and left-unbounded interval & left-unbounded and right-closed intervals). For any partially ordered set $(X,\leq_X)$ and any elements $a,b \in X$,

(1) we call the set
$$(a,+\infty)_X = (a,+\infty) \tag{3.439}$$
consisting of all the elements in X that are greater than a the *open and right-unbounded interval* beginning in a (with respect to $\leq_X$). Furthermore, we call
$$\{(a,+\infty) : a \in X\} \tag{3.440}$$
the *set of open and right-unbounded intervals* in X (w.r.t. $\leq_X$).

(2) we call the set
$$[a,+\infty)_X = [a,+\infty) \tag{3.441}$$
consisting of all the elements in X that are greater than or equal to a the *left-closed and right-unbounded interval* beginning in a (w.r.t. $\leq_X$). Moreover, we call
$$\{[a,+\infty) : a \in X\} \tag{3.442}$$
the *set of left-closed and right-unbounded intervals* in X (w.r.t. $\leq_X$).

(3) we call the set
$$(-\infty,b)_X = (-\infty,b) \tag{3.443}$$
consisting of all the elements in X that are less than b the *open and left-unbounded interval* up to b (w.r.t. $\leq_X$). In addition, we call
$$\{(-\infty,b) : b \in X\} \tag{3.444}$$
the *set of open and left-unbounded intervals* in X (w.r.t. $\leq_X$).

(4) we call the set

$$(-\infty, b]_X = (-\infty, b] \tag{3.445}$$

consisting of all the elements in X that are less than or equal to b the *left-unbounded and right-closed interval* up to b (w.r.t. $\leq_X$), and

$$\{(-\infty, b] : b \in X\} \tag{3.446}$$

the *set of left-unbounded and right-closed intervals* in X (w.r.t. $\leq_X$).

Proposition 3.128. *It is true for any linearly ordered set $(X, <_X)$ that the open and left-unbounded interval ending in any element b of X is the complement (with respect to X) of the left-closed and right-unbounded interval beginning in b, that is,*

$$\forall b\,(b \in X \Rightarrow (-\infty, b) = [b, +\infty)^c). \tag{3.447}$$

Proof. We let X, $<_X$ and b be arbitrary, assuming $<_X$ to be a linear ordering of X (inducing thus the total ordering $\leq_X$ of X) and assuming b to be contained in X. Let us now apply the Equality Criterion for sets to prove $(-\infty, b) = [b, +\infty)^c$. We then need to establish the truth of the universal sentence

$$\forall x\,(x \in (-\infty, b) \Leftrightarrow x \in [b, +\infty)^c). \tag{3.448}$$

For this purpose, we let x be arbitrary and observe the truth of the equivalences

$$\begin{aligned} x \in (-\infty, b) &\Leftrightarrow x \in X \wedge x <_X b \\ &\Leftrightarrow x \in X \wedge \neg b \leq_X x \\ &\Leftrightarrow x \in X \wedge \neg x \in [b, +\infty) \\ &\Leftrightarrow x \in X \setminus [b, +\infty) \\ &\Leftrightarrow x \in [b, +\infty)^c \end{aligned}$$

in light of (3.419), the Negation Formula for $\leq$, (3.426), the definition of a set difference and the definition of a complement (with respect to X) in connection with (3.422). Since x is arbitrary, we may infer from the truth of the resulting equivalence $x \in (-\infty, b) \Leftrightarrow x \in [b, +\infty)^c$ the truth of (3.448), and therefore the truth of the equation (3.447). Because b, X and $<_X$ were arbitrary, the proposition follows then to be true. □

Corollary 3.129. *It is true for any linearly ordered set $(X, <_X)$ that the left-closed and right-unbounded interval beginning in any element a of X is the complement of the open and left-unbounded interval ending in a, i.e.*

$$\forall a\,(a \in X \Rightarrow [a, +\infty) = (-\infty, a)^c). \tag{3.449}$$

Proof. Letting $(X, <_X)$ be an arbitrary linearly ordered set and a an arbitrary element of X, we obtain

$$[a, +\infty) = ([a, +\infty)^c)^c = (-\infty, a)^c$$

by applying (2.136) and substitution based on (3.447). Then, the resulting equation $[a, +\infty) = (-\infty, a)^c$ is universally true since a and $(X, <_X)$ were arbitrary. $\square$

Exercise 3.57. Show for any linearly ordered set $(X, <_X)$ that

a) the left-unbounded and right-closed interval ending in any element b of X is the complement of the open and right-unbounded interval beginning in b, i.e.

$$\forall b\,(b \in X \Rightarrow (-\infty, b] = (b, +\infty)^c). \tag{3.450}$$

b) the open and right-unbounded interval beginning in any element a of X is the complement of the left-unbounded right-closed interval ending in a, i.e.

$$\forall a\,(a \in X \Rightarrow (a, +\infty) = (-\infty, a]^c). \tag{3.451}$$

Proposition 3.130. *It is true for any partially ordered set $(X, \leq_X)$ that the left-closed, right-unbounded interval beginning in an element a of X can be written as the union of the open, right-unbounded interval beginning in a and the singleton formed by a, i.e.*

$$\forall a\,(a \in X \;\Rightarrow\; [a, +\infty) = (a, +\infty) \cup \{a\}). \tag{3.452}$$

Proof. We take arbitrary X, $\leq_X$ and a such that $(X, \leq_X)$ is a partially ordered set and such that a is an element of X. Letting then also x be arbitrary, we obtain the true equivalences

$$\begin{aligned} x \in [a, +\infty) \;&\Leftrightarrow\; a \leq_X x \\ &\Leftrightarrow\; a <_X x \,\vee\, a = x \\ &\Leftrightarrow\; a <_X x \,\vee\, x = a \\ &\Leftrightarrow\; x \in (a, +\infty) \,\vee\, x \in \{a\} \\ &\Leftrightarrow\; x \in (a, +\infty) \,\cup\, \{a\} \end{aligned}$$

by applying the definition of a left-closed and right-unbounded interval, the definition of an induced irreflexive partial ordering, (1.96), the definition of an open, right-unbounded interval together with (2.169), and finally the definition of a union of two sets. Because x is arbitrary, we may now infer from these equivalences by means of the Equality Criterion for sets the truth of the desired equation in (3.452). Since a, X and $\leq_X$ were arbitrary, we may therefore conclude that the proposed universal sentence holds. $\square$

Exercise 3.58. Prove for any partially ordered set $(X, \leq_X)$ that the left-unbounded, right-closed interval ending in an element b of X can be written as the union of the open, left-unbounded interval ending in b and the singleton formed by b, i.e.

$$\forall b\,(b \in X \;\Rightarrow\; (-\infty, b] = (-\infty, b) \cup \{b\}). \tag{3.453}$$

Proposition 3.131. *The following implication is true for any partially ordered set* $(X, \leq_X)$ *and any elements* $a_1, b_1, a_2 \in X$.

$$a_2 \leq_X a_1 \;\Rightarrow\; [a_1, b_1) \subseteq [a_2, +\infty). \tag{3.454}$$

Proof. We let X and $\leq_X$ be arbitrary such that $(X, \leq_X)$ is partially ordered, and we let a_1, b_1 and a_2 also be arbitrary in X. To prove the implication, we assume $a_2 \leq_X a_1$ and establish the desired inclusion $[a_1, b_1) \subseteq [a_2, +\infty)$ by applying the definition of a subset, i.e. by verifying

$$\forall x\,(x \in [a_1, b_1) \Rightarrow x \in [a_2, +\infty)). \tag{3.455}$$

To do this, we let x be arbitrary and assume $x \in [a_1, b_1)$ to be true, so that the inequalities $a_1 \leq_X x$ and $x <_X b_1$ are true by definition of a left-closed and right-open interval. We now see that the assumption $a_2 \leq_X a_1$ and the previously obtained $a_1 \leq_X x$ imply $a_2 \leq_X x$ with the transitivity of the partial ordering $\leq_X$. Consequently, $x \in [a_2, +\infty)$ holds by definition of a left-closed and right-unbounded interval, which was to be proven. Since x is arbitrary, we may therefore conclude that the universal sentence (3.455) holds, which then implies the truth of the inclusion $[a_1, b_1) \subseteq [a_2, +\infty)$, proving the implication (3.454). As X, $\leq_X$, a_1, b_1 and a_2 were arbitrary, we may now further conclude that the proposed universal sentence is true. $\square$

Exercise 3.59. Verify the following implication for any partially ordered set $(X, \leq_X)$ and any elements $a_1, b_1, b_2 \in X$.

$$b_1 \leq_X b_2 \;\Rightarrow\; [a_1, b_1) \subseteq (-\infty, b_2). \tag{3.456}$$

(Hint: Proceed similarly as in the proof of Proposition 3.131, using now the Transitivity Formula for $<$ and $\leq$.)

Proposition 3.132. *It is true for any linearly ordered set $(X, <_X)$ that the open and left-unbounded interval up to an $a \in X$ is included in the open and left-unbounded interval up to a $b \in X$ iff a is less than or equal to b, that is,*

$$\forall a, b\,(a, b \in X \Rightarrow [(-\infty, a) \subseteq (-\infty, b) \Leftrightarrow a \leq_X b]). \tag{3.457}$$

Proof. Letting $(X, <_X)$ be an arbitrary linearly ordered set and a, b arbitrary constants in X, we prove the first part ('$\Rightarrow$') of the equivalence by contradiction, assuming the inclusion $(-\infty, a) \subseteq (-\infty, b)$ and the negation $\neg a \leq_X b$. By definition of a subset, we thus have

$$\forall x\,(x \in (-\infty, a) \Rightarrow x \in (-\infty, b)), \tag{3.458}$$

and the negation implies $b <_X a$ with the Negation Formula for $\leq$. The latter inequality implies $b \in (-\infty, a)$ by definition of an open and left-unbounded interval, so that $b \in (-\infty, b)$ follows to be true with (3.458). Consequently, $b <_X b$ holds by definition of an open and left-unbounded interval, in contradiction to the fact that $\neg b <_X b$ is true since the linear ordering $<_X$ is irreflexive by definition. Thus, the proof of the first implication is complete. To establish the second part ('$\Leftarrow$') of the equivalence, we assume $a \leq_X b$ to be true, and we show that the inclusion $(-\infty, a) \subseteq (-\infty, b)$ also holds. For this purpose, we prove the universal sentence (3.458), letting $x \in (-\infty, a)$ be arbitrary. Since this implies $x <_X a$ $[\leq_X b]$ with the definition of an open and left-unbounded interval, we obtain with the Transitivity Formula for $<$ and $\leq$ the true inequality $x <_X b$, which evidently implies $x \in (-\infty, b)$, as desired. As x was arbitrary, we may conclude that the universal sentence (3.458) is true, so that the inclusion $(-\infty, a) \subseteq (-\infty, b)$ holds indeed. Thus, the proof of the equivalence in (3.457) is complete, and since a and b were arbitrary, the universal sentence (3.457) follows to be true, too. Initially, $(X, <_X)$ was arbitrary, so that we may finally conclude that the stated proposition is true. □

Proposition 3.133. *It is true for any partially ordered set $(X, <_X)$ that the open, left-unbounded interval up to an element $b \in X$ and the open, right-unbounded interval beginning at an element $a \in X$ are disjoint if a is not less than b, i.e.*

$$\forall a, b\,([a, b \in X \wedge \neg a <_X b] \;\Rightarrow\; (-\infty, b) \cap (a, +\infty) = \emptyset). \tag{3.459}$$

Proof. We take arbitrary X, $\leq_X$, a and b, assume that $(X, <_X)$ is a partially ordered set, and we assume $a, b \in X$ as well as $\neg a <_X b$ to be true. Next, we establish the universal sentence

$$\forall y\,(y \notin (-\infty, b) \cap (a, +\infty)), \tag{3.460}$$

letting y be arbitrary. Let us now prove $y \notin (-\infty, b) \cap (a, +\infty)$ by contradiction, assuming its negation to be true. Then, $y \in (-\infty, b) \cap (a, +\infty)$ follows to be true with the Double Negation Law, and the definition of the intersection of two sets gives $y \in (-\infty, b)$ as well as $y \in (a, +\infty)$. By definition of open and left-/right-unbounded intervals, we therefore have the inequalities $y <_X b$ and $a <_X y$. Consequently, the transitivity of the partial ordering $<_X$ yields $a <_X b$, in evident contradiction to the the assumed negation $\neg a <_X b$. This finding completes the proof of $y \notin (-\infty, b) \cap (a, +\infty)$ by contradiction, and since y is arbitrary, we may therefore conclude that the universal sentence (3.460) is true. We then obtain $(-\infty, b) \cap (a, +\infty) = \emptyset$ with the definition of the empty set, proving the implication in (3.459). As X, $\leq_X$, a and b were initially arbitrary, we may now finally conclude that the proposition holds, as claimed. □

Proposition 3.134. *It is true for any partially ordered set* $(X, <_X)$ *that the open interval from an element* $a \in X$ *to an element* $b \in X$ *can be written as the intersection of the open, left-unbounded interval ending in* b *and the open, right-unbounded interval beginning in* a*, i.e.*

$$(a, b) = (-\infty, b) \cap (a, +\infty). \tag{3.461}$$

Proof. Letting X, $\leq_X$, a and b be arbitrary, assuming $(X, <_X)$ to be partially ordered, assuming furthermore a and b to be elements of X, and letting now also y be arbitrary, we obtain the true equivalences

$$\begin{aligned} y \in (a, b) \;&\Leftrightarrow\; a <_X y <_X b \\ &\Leftrightarrow\; y <_X b \,\wedge\, a <_X y \\ &\Leftrightarrow\; y \in (-\infty, b) \,\wedge\, y \in (a, +\infty) \\ &\Leftrightarrow\; y \in (-\infty, b) \cap (a, +\infty) \end{aligned}$$

by applying the definition of an open interval, the Commutative Law for the conjunction, the definition of an open and left-unbounded interval together with the definition of an open and right-unbounded interval, and finally the definition of the intersection of two sets. Since y is arbitrary, we may then infer from the preceding equivalences the truth of the equation (3.461) because of the Equality Criterion for sets. Because X, $\leq_X$, a and b are also arbitrary, the proposition follows to true. □

Proposition 3.135. *It is true for any for any linearly ordered set* $(X, <_X)$ *and for any elements* $a, b \in X$ *that the complement of the closed interval from* a *to* b *(with respect to* X*) can be written as the union of the open, left-unbounded interval ending in* a *and the open, right-unbounded interval beginning in* b*, i.e.*

$$[a, b]^c = (-\infty, a) \cup (b, +\infty). \tag{3.462}$$

Proof. We take arbitrary X, $<_X$, a and b, assume that $(X, <_X)$ is a linearly ordered set, and we assume that $a, b \in X$ holds. Letting now y be arbitrary, we obtain the equivalences

$$\begin{aligned} y \in [a,b]^c &\Leftrightarrow y \in X \setminus [a,b] \\ &\Leftrightarrow y \in X \wedge \neg y \in [a,b] \\ &\Leftrightarrow y \in X \wedge \neg(a \leq_X y \wedge y \leq_X b) \\ &\Leftrightarrow y \in X \wedge (\neg a \leq_X y \vee \neg y \leq_X b) \\ &\Leftrightarrow y \in X \wedge (y <_X a \vee b <_X y) \\ &\Leftrightarrow (y \in X \wedge y <_X a) \vee (y \in X \wedge b <_X y) \\ &\Leftrightarrow y \in (-\infty, a) \vee y \in (b, +\infty) \\ &\Leftrightarrow y \in (-\infty, a) \cup (b, +\infty) \end{aligned}$$

using the definition of a complement based on the inclusion (3.352), the definition of a set difference, the definition of a closed interval (in X), De Morgan's Law for the conjunction, the Negation Formulas for $\leq$ and $<$, the Distributive Law for the conjunction, the definitions of an open, left-/right-unbounded interval (in X), and the definition of the union of two sets. Since y is arbitrary, we may infer from the truth of these equivalences the truth of the equation (3.462). Because X, $<_X$, a and b were initially also arbitrary, we may therefore conclude that the proposed universal sentence is true. $\square$

Exercise 3.60. Show that the following equation holds for any for any linearly ordered set $(X, <_X)$ and for any elements $a, b \in X$.

$$[a,b)^c = (-\infty, a) \cup [b, +\infty) \tag{3.463}$$
$$(a,b]^c = (-\infty, a] \cup (b, +\infty) \tag{3.464}$$

(Hint: Proceed in analogy to the proof of Proposition 3.135.).

Exercise 3.61. Prove the following equation for any partially ordered set $(X, \leq_X)$ and any elements $a_1, a_2, b_1, b_2 \in X$.

$$[a_1, b_1) \setminus [a_2, b_2) = (\, [a_1, a_2) \cap (-\infty, b_1)\,) \cup (\, [b_2, b_1) \cap [a_1, +\infty)\,)\,. \tag{3.465}$$

(Hint: Use the Equality Criterion for sets in connection with (2.138), (3.378), Definition 2.8, (3.463), Definition 2.19, (1.44), (1.39) and (1.36).)

Proposition 3.136. *It is true for any linearly ordered set $(X, <_X)$ that the left-closed and right-open interval from an element a of X to an element b of X can be written as the difference of the open, left-unbounded interval ending in b and the open, left-unbounded interval ending in a, i.e.*

$$\forall a, b\,(\, a, b \in X \Rightarrow [a,b) = (-\infty, b) \setminus (-\infty, a)\,). \tag{3.466}$$

Proof. Letting X, $<_X$, a and b be arbitrary such that $(X, <_X)$ is a linearly ordered set and such that a and b are elements in X, we may apply the Equality Criterion for sets to establish $[a, b) = (-\infty, b) \setminus (-\infty, a)$, by proving the corresponding universal sentence

$$\forall x\,(\,x \in [a, b) \Leftrightarrow x \in (-\infty, b) \setminus (-\infty, a)\,). \tag{3.467}$$

We let now also x be arbitrary, so that we obtain the true equivalences

$$\begin{aligned} x \in [a, b) &\Leftrightarrow a \leq_X x \wedge x <_X b \\ &\Leftrightarrow x <_X b \wedge a \leq_X x \\ &\Leftrightarrow x <_X b \wedge \neg x <_X a \\ &\Leftrightarrow x \in (-\infty, b) \wedge \neg x \in (-\infty, a) \\ &\Leftrightarrow x \in (-\infty, b) \setminus (-\infty, a) \end{aligned}$$

by applying the definition of a left-closed and right-open interval, the Commutative Law for the conjunction, the Negation Formula for $<$, the definition of an open and left-unbounded interval, and finally the definition of a set difference. Since x is arbitrary, we may infer from the resulting equivalence $x \in [a, b) \Leftrightarrow x \in (-\infty, b) \setminus (-\infty, a)$ the truth of the equation $[a, b) = (-\infty, b) \setminus (-\infty, a)$. Here, X, $<_X$, a and b are also arbitrary, so that the proposed universal sentence follows now to be true. □

Exercise 3.62. Prove for any linearly ordered set $(X, <_X)$ that the left-closed and right-open interval from an a in X to a b in X can be written as the difference of the left-closed, right-unbounded interval beginning in a and the left-closed, right-unbounded interval beginning in b, i.e.

$$\forall a, b\,(\,a, b \in X \;\Rightarrow\; [a, b) = [a, +\infty) \setminus [b, +\infty)\,). \tag{3.468}$$

Exercise 3.63. Verify for any linearly ordered set $(X, <_X)$ that the left-open and right-closed interval from an element a of X to an element b of X can be written as

a) the difference of the open, right-unbounded interval beginning in a and the open, right-unbounded interval beginning in a, i.e.

$$\forall a, b\,(\,a, b \in X \;\Rightarrow\; (a, b] = (a, +\infty) \setminus (b, +\infty)\,). \tag{3.469}$$

b) the difference of the left-unbounded, right-closed interval ending in b and the left-unbounded, right-closed interval ending in a, i.e.

$$\forall a, b\,(\,a, b \in X \;\Rightarrow\; (a, b] = (-\infty, b] \setminus (-\infty, a]\,). \tag{3.470}$$

(Hint: Proceed as in the proof of Proposition 3.136.)

Proposition 3.137. *For any linearly ordered set $(X, <_X)$ such that X is not bounded from above and for any element $a \in X$, it is true that the left-closed and right-unbounded interval $[a, +\infty)$ is not bounded from above.*

Proof. Letting X, $<_X$ and a be arbitrary such that $(X, <_X)$ is a linearly ordered without an upper bound and such that a is an element of X, we establish the negation

$$\neg \exists u \, (u \in X \ \wedge \ \forall x \, (x \in [a, +\infty) \Rightarrow x \leq_X u)).$$

First, we apply the Negation Law for existential conjunctions and write the preceding the negation as the equivalent universal sentence

$$\forall u \, (u \in X \Rightarrow \neg \forall x \, (x \in [a, +\infty) \Rightarrow x \leq_X u)). \tag{3.471}$$

We now take an arbitrary $\bar{u}$, and we prove the implication by contradiction, assuming $\bar{u} \in X$ and the negation of the negated universal sentence to be true. The latter assumption yields then with the Double Negation Law the true universal sentence

$$\forall x \, (x \in [a, +\infty) \Rightarrow x \leq_X \bar{u}). \tag{3.472}$$

Let us observe here the truth of $a \leq_X a$ in light of the reflexivity of the induced reflexive partial ordering $\leq_X$. This inequality gives us $a \in [a, +\infty)$ by definition of a left-closed and right-unbounded interval in X, and this yields $a \leq_X \bar{u}$ with the preceding universal sentence. Because we assumed that an upper bound for X does not exist, we also have the true negation

$$\neg \exists u \, (u \in X \ \wedge \ \forall x \, (x \in X \Rightarrow x \leq_X u)),$$

which we can evidently write in the equivalent form

$$\forall u \, (u \in X \Rightarrow \neg \forall x \, (x \in X \Rightarrow x \leq_X u)).$$

Therefore, the assumed $\bar{u} \in X$ implies $\neg \forall x \, (x \in X \Rightarrow x \leq_X \bar{u})$, which in turn implies (according to the Negation Law for universal implications) that there exists a constant, say $\bar{x}$, satisfying $\bar{x} \in X$ and $\neg \bar{x} \leq_X \bar{u}$. The latter gives us $\bar{u} <_X \bar{x}$ with the Negation Formula for $\leq$, and this implies in conjunction with the previously obtained $a \leq_X \bar{u}$ the inequality $a <_X \bar{x}$ by means of the Transitivity Formula for $\leq$ and $<$. Then, the disjunction $a <_X \bar{x} \vee a = \bar{x}$ also holds, so that $a \leq_X \bar{x}$ follows to be true by definition of an induced reflexive partial ordering. Consequently, we find $\bar{x} \in [a, +\infty)$ with the definition of a left-closed and right-unbounded interval, which further implies $\bar{x} \leq_X \bar{u}$ with (3.472), and subsequently also $\neg \bar{u} <_X \bar{x}$ with the Negation Formula for $<$. As $\bar{u} <_X \bar{x}$ also holds, we arrived at a contradiction, which completes the proof of the implication in (3.471). Since $\bar{u}$ is arbitrary, we therefore conclude that the universal sentence (3.471) is true, which proves the equivalent assertion that an upper bound for $[a, +\infty)$ does not exist. Because X, $<_X$ and a were initially arbitrary, we now further conclude that the proposed universal sentence holds. □

Exercise 3.64. Show for any linearly ordered set $(X, <_X)$ such that X is not bounded from below and for any element $b \in X$ that the left-unbounded and right-closed interval $(-\infty, b]$ is not bounded from below.

Proposition 3.138. *For any linearly ordered set $(X, <_X)$ such that the maximum of X does not exist and for any element $a \in X$, it is true that*

a) there exists an element in the open, right-unbounded interval beginning in a which is greater than a, i.e.

$$\exists x\, (x \in (a, +\infty) \wedge x >_X a). \tag{3.473}$$

b) the open, right-unbounded interval beginning in a is nonempty, i.e.

$$(a, +\infty) \neq \emptyset. \tag{3.474}$$

Proof. We take arbitrary sets X and $<_X$, for which we assume that $<_X$ is a linear ordering of X (inducing the total ordering $\leq_X$ of X) and that the maximum of X does not exist. The latter assumption can be stated as the negation $\neg \exists m\, (m \in X \wedge m = \max X)$, noting that X is a subset of itself according to (2.10). This negation implies with the Negation Law for existential conjunctions $\forall m\, (m \in X \Rightarrow \neg m = \max X)$. We now take an arbitrary set a, and we assume a to be an element of X, so that the preceding universal sentence yields the true negation $\neg a = \max X$. According to the definition of a maximum, this implies

$$\neg[\forall x\, (x \in X \Rightarrow x \leq_X a) \wedge a \in X],$$

so that the disjunction

$$\exists x\, (x \in X \wedge \neg x \leq_X a) \vee a \notin X,$$

follows to be true with De Morgan's Law for the conjunction and the Negation Law for universal implications. Since we assumed $a \in X$ to be true, the second part of the preceding disjunction is false, so that its first part must be true. This means that there exists a particular element $\bar{x} \in X$ with

$\neg\bar{x} \leq_X a$. The latter implies then with the Negation Formula for $\leq$ the truth of $\bar{x} >_X a$, which in turn gives $\bar{x} \in (a, +\infty)$ by definition of an open and right-unbounded interval in X. We thus proved the existential sentence (3.473), and as $\bar{x} \in (a, +\infty)$ clearly shows that the interval $(a, +\infty)$ is nonempty, we established the truth also of (3.429). Because the sets X, $<_X$ and a were arbitrary, we can now infer from the truth of a) and b) the truth of the proposition. □

Exercise 3.65. Show for any linearly ordered set $(X, <_X)$ such that the minimum of X does not exist and for any element $b \in X$ that

a) there exists an element in the open, left-unbounded interval ending in b which is less than b, i.e.

$$\exists x\,(x \in (-\infty, b) \,\wedge\, x <_X b). \tag{3.475}$$

b) the open and left-unbounded interval ending in b is nonempty, i.e.

$$(-\infty, b) \neq \emptyset. \tag{3.476}$$

Proposition 3.139. *For any partially ordered set $(X, \leq_X)$ such that the maximum of X exists and for any element $a \in X$, it is true that*

a) the open and right-unbounded interval beginning in a is identical with the left-open and right-closed interval from a to $\max X$, i.e.

$$(a, +\infty) = (a, \max X]. \tag{3.477}$$

b) the left-closed and right-unbounded interval beginning in a is identical with the closed interval from a to $\max X$, i.e.

$$[a, +\infty) = [a, \max X]. \tag{3.478}$$

Proof. Letting X, $\leq_X$ and a be arbitrary such that $(X, \leq_X)$ is partially ordered, such that the maximum of X exists and such that $a \in X$ holds, we may prove a) via the Equality Criterion for sets. For this purpose, we verify the universal sentence

$$\forall x\,(x \in (a, +\infty) \Leftrightarrow x \in (a, \max X]), \tag{3.479}$$

letting x be arbitrary. On the one hand, the assumption $x \in (a, +\infty)$ implies $x \in X$ and $a <_X x$ with (3.417). Since the maximum of X is by definition an upper bound for X (contained in X), we have that $x \in X$ implies $x \leq_X \max X$, so that the inequalities $a <_X x \leq_X \max X$ hold.

According to the definition of a left-open and right-closed interval, this finding in turn implies $x \in (a, \max X]$, which is the desired consequent of the first part ('$\Rightarrow$') of the equivalence in (3.479). On the other hand, the converse assumption $x \in (a, \max X]$ implies (by definition of a left-open and right-closed interval) the truth of the inequalities $a <_X x \leq_X \max X$, so that $a <_X x$ is especially true. Consequently, we obtain $x \in (a, +\infty)$ by definition of an open and right-unbounded interval, proving the second part ('$\Leftarrow$') of the equivalence. Since x is arbitrary, we may therefore conclude that (3.479) holds, and this universal sentence implies now the truth of (3.477), and thus the truth of a). Then, Part b) can be proved in analogy to Part a). As X, $\leq_X$ and a were initially arbitrary, we may infer from the truth of a) and b) the truth of the proposed universal sentence. □

Exercise 3.66. Prove Part b) of Proposition 3.139 in analogy to Part a). Show then for any partially ordered set $(X, \leq_X)$ such that the minimum of X exists and for any element $b \in X$ that

c) the open and left-unbounded interval ending in b is identical with the left-closed and right-open interval from $\min X$ to b, i.e.

$$(-\infty, b) = [\min X, b). \tag{3.480}$$

d) the left-unbounded and right-closed interval ending in b is identical with the closed interval from $\min X$ to b, i.e.

$$(-\infty, b] = [\min X, b]. \tag{3.481}$$

Note 3.15. As a consequence of the preceding Proposition 3.139 and Exercise 3.66, we may rewrite the previously established sentences concerning left- or right-unbounded intervals in 'closed forms' by using the minimum or the maximum of the underlying set X if they exist.

The next results follow immediately from Proposition 3.128, Corollary 3.129 and Exercise 3.57.

Corollary 3.140. *The following sentences are true for any linearly ordered set* $(X, <_X)$ *for which* $\min X$ *and* $\max X$ *both exist.*

$$\forall b \, (b \in X \Rightarrow [\min X, b) = [b, \max X]^c), \tag{3.482}$$

$$\forall a \, (a \in X \Rightarrow [a, \max X] = [\min X, a)^c), \tag{3.483}$$

$$\forall b \, (b \in X \Rightarrow [\min X, b] = (b, \max X]^c), \tag{3.484}$$

$$\forall a \, (a \in X \Rightarrow (a, \max X] = [\min X, a]^c). \tag{3.485}$$

The next result is obtained from Proposition 3.135

Corollary 3.141. *For any set X and any linear ordering $<_X$ such that the minimum and the maximum of X exist both, and for any elements $a, b \in X$, it is true that the complement of the closed interval from a to b (with respect to X) can be written as the union of the left-closed, right-open interval from the minimum of X to a and the left-open, right-closed interval from b to the maximum of X, i.e.*

$$[a, b]^c = [\min X, a) \cup (b, \max X]. \tag{3.486}$$

We now similarly specialize the findings of Proposition 3.136, Exercise 3.62 and Exercise 3.63.

Corollary 3.142. *The following sentences are true for any linearly ordered set $(X, <_X)$ for which $\min X$ and $\max X$ both exist.*

$$\forall a, b\, (a, b \in X \;\Rightarrow\; [a, b) = [\min X, b) \setminus [\min X, a)), \tag{3.487}$$
$$\forall a, b\, (a, b \in X \;\Rightarrow\; [a, b) = [a, \max X] \setminus [b, \max X]), \tag{3.488}$$
$$\forall a, b\, (a, b \in X \;\Rightarrow\; (a, b] = (a, \max X] \setminus (b, \max X]), \tag{3.489}$$
$$\forall a, b\, (a, b \in X \;\Rightarrow\; (a, b] = [\min X, b] \setminus [\min X, a]). \tag{3.490}$$

In what follows we intend to elaborate some common aspects of intervals, for which task it will be useful to subsume the previously introduced interval types.

Definition 3.35 (Interval). We say for any partially ordered set $(X, <_X)$ that a set I is an *interval in X* iff

$$\begin{aligned} & I \in \{[a, b] : a, b \in X\} \;\vee\; I \in \{(a, b) : a, b \in X\} \\ \vee\, & I \in \{(a, b] : a, b \in X\} \;\vee\; I \in \{[a, b) : a, b \in X\} \\ \vee\, & I \in \{(a, +\infty) : a \in X\} \vee I \in \{[a, +\infty) : a \in X\} \\ \vee\, & I \in \{(-\infty, b) : b \in X\} \vee I \in \{(-\infty, b] : b \in X\} \\ \vee\, & I = \emptyset \\ \vee\, & I = X. \end{aligned}$$

We now use open intervals to define a new type of subset of a partially ordered set.

Definition 3.36 (Convex set (with respect to a partial ordering)). For any partially ordered set $(X, <_X)$, we say that a set A is *convex* in X with respect to $<_X$ iff

1. A is a subset of X, i.e.
$$A \subseteq X, \tag{3.491}$$
and

2. the open interval in X from any element a in A to any element b in A is included in A, i.e.

$$\forall a, b\,(a, b \in A \Rightarrow (a, b)_X \subseteq A). \tag{3.492}$$

Exercise 3.67. Show for any partially ordered set $(X, <_X)$ that a subset A of X is convex in X with respect to $<_X$ iff

$$\forall a, b\,(a, b \in A \Rightarrow \forall x\,(a <_X x <_X b \Rightarrow x \in A)). \tag{3.493}$$

Proposition 3.143. *It is true for any partially ordered set $(X, <_X)$ and any elements $x, y \in X$ that the intervals*

$$[x, y],\ (x, y),\ (x, y],\ [x, y),\ (x, +\infty),\ [x, +\infty),\ (-\infty, y),\ (-\infty, y],\ \emptyset,\ X$$

are all convex in X with respect to $<_X$.

Proof. We let in the following X, $<_X$, x and y be arbitrary such that $(X, <_X)$ is a partially ordered set and such that x and y are elements of X. Then, $\leq_X$ denotes the induced reflexive partial ordering of X.

Regarding the interval $[x, y]$ in X, we observe first that the inclusion $[x, y] \subseteq X$ is true according to (3.352), so that the set $[x, y]$ satisfies Property 1 of a convex set in X with respect to $<_X$. To establish Property 2, we verify

$$\forall a, b\,(a, b \in [x, y] \Rightarrow (a, b) \subseteq [x, y]), \tag{3.494}$$

taking arbitrary constants a and b, assuming $a, b \in [x, y]$ to be true, and showing that the inclusion $(a, b) \subseteq [x, y]$ also holds. For this purpose, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall z\,(z \in (a, b) \Rightarrow z \in [x, y]), \tag{3.495}$$

letting z be arbitrary and assuming $z \in (a, b)$ to be true. This assumption gives us now, by definition of an open interval in X, the inequalities $a <_X z$ and $z <_X b$. Furthermore, the previously made assumption $a, b \in [x, y]$ yields, by definition of a closed interval in X, the inequalities $x \leq_X a \leq_X y$ and $x \leq_X b \leq_X y$. Combining now $x \leq_X a$ with $a <_X z$, we obtain $x <_X z$ with the Transitivity Formula for $\leq$ and $<$; similarly, $z <_X b$ in connection with $b \leq_X y$ yields $z <_X y$ with the Transitivity Formula for $<$ and $\leq$. Then, the disjunctions $x <_X z \vee x = z$ and $z <_X y \vee z = y$ also hold, so that the definition of an induced reflexive partial ordering gives $x \leq_X z$ and $z \leq_X y$. Consequently, the definition of a closed interval in X yields the desired consequent $z \in [x, y]$ of the implication in (3.495). Here, z was arbitrary, so that the universal sentence (3.495) follows to be true, which in turn implies the desired inclusion $(a, b) \subseteq [x, y]$. Thus, the proof of the implication in (3.494) is also complete. Since a and b are arbitrary, we may now infer from the truth of that implication the truth of the universal sentence (3.494). This means that the interval $[x, y]$ in X satisfies also Property 2 of a convex set in X with respect to $<_X$, which then holds universally because x and y are arbitrary.

The proofs for the remaining intervals are carried out similarly.

As X and $<_X$ were arbitrary, the proposition follows then to be true. □

Exercise 3.68. Complete the proof of Proposition 3.143.

(Hint: Use the transitivity of $<_X$.)

Proposition 3.144. *The following implications are true for any linearly ordered set* $(X, <_X)$*, any convex set* A *in* X *w.r.t.* $<_X$ *and any* $a \in X$*.*

$$a \in A \;\Rightarrow\; A \cap (a, +\infty)_X = (a, +\infty)_A, \tag{3.496}$$

$$a \notin A \;\Rightarrow\; [A \cap (a, +\infty)_X = A \,\vee\, A \cap (a, +\infty)_X = \emptyset]. \tag{3.497}$$

Proof. We take arbitrary X, $<_X$, A and $\bar{a}$, assuming $(X, <_X)$ to be a linearly ordered set, assuming A to be a convex set in X with respect to $<_X$, and assuming $\bar{a}$ to be an element of X. Thus, the open and right-unbounded interval $(\bar{a}, +\infty)_X$ is defined, and the convex set A satisfies by definition $A \subseteq X$ as well as (3.492).

To prove the first implication (3.496) directly, we assume $\bar{a} \in A$ to be true, and we establish the consequent by means of the Equality Criterion for sets, i.e. by verifying the equivalent universal sentence

$$\forall y \, (y \in A \cap (\bar{a}, +\infty)_X \Leftrightarrow y \in (\bar{a}, +\infty)_A). \tag{3.498}$$

For this purpose, we let y be arbitrary and assume first $y \in A \cap (\bar{a}, +\infty)_X$ to hold, so that $y \in A$ and $y \in (\bar{a}, +\infty)_X$ follow to be true by definition of the intersection of two sets. The latter implies $\bar{a} <_X y$ with the definition of an open and right-unbounded interval in X. Because of $\bar{a}, y \in A$, the preceding inequality further implies $\bar{a} <_A y$ according to the Irreflexive partial ordering of subsets, so that $y \in (\bar{a}, +\infty)_A$ follows to be true by definition of an open and right-open interval in A. Thus, the first part ('$\Rightarrow$') of the equivalence in (3.498) holds.

Regarding the second part ('$\Leftarrow$'), we now assume $y \in (\bar{a}, +\infty)_A$ to be true, which yields $y \in A$ as well as $\bar{a} <_A y$, according to (3.417). Due to $\bar{a}, y \in A$, that inequality in turn implies $\bar{a} <_X y$ (with the Irreflexive partial ordering of subsets), and then evidently $y \in (\bar{a}, +\infty)_X$, which is the desired consequent of the second part of the equivalence to be proven. Therefore, that implication is true, and as y was arbitrary, we may now infer from that equivalence the truth of the universal sentence (3.498), and then also the truth of the equation $A \cap (\bar{a}, +\infty)_X = (\bar{a}, +\infty)_A$. This finding in turn proves the implication (3.496).

We prove the second implication (3.497) also directly, assuming $\bar{a} \notin A$ to be true. Let us first verify the assertion that $\bar{a}$ is a lower or an upper bound for A with respect to the induced total ordering $\leq_X$, i.e. that the disjunction

$$\forall y\,(y \in A \Rightarrow \bar{a} \leq_X y) \,\vee\, \forall y\,(y \in A \Rightarrow y \leq_X \bar{a}) \tag{3.499}$$

holds. To do this, we apply a proof by contradiction, assuming the negation of that disjunction to be true, so that De Morgan's Law for the disjunction gives us the true sentence

$$\neg\forall y\,(y \in A \Rightarrow \bar{a} \leq_X y) \,\wedge\, \neg\forall y\,(y \in A \Rightarrow y \leq_X \bar{a}).$$

We therefore obtain with the Negation Law for universal implications

$$\exists y\,(y \in A \,\wedge\, \neg\bar{a} \leq_X y) \,\wedge\, \exists y\,(y \in A \,\wedge\, \neg y \leq_X \bar{a}).$$

Thus, there are particular constants $y_1, y_2 \in A$ with $\neg\bar{a} \leq_X y_1$ as well as $\neg y_2 \leq_X \bar{a}$. These inequalities imply with the Negation Formula for $\leq$ the inequalities $y_1 <_X \bar{a}$ and $\bar{a} <_X y_2$, respectively, so that $\bar{a} \in (y_1, y_2)$ follows to be true by definition of an open interval in X. Recalling now the assumption that A is convex in X with respect to $<_X$, it follows from $y_1, y_2 \in A$ that the inclusion $(y_1, y_2) \subseteq A$ holds. With this, $\bar{a} \in (y_1, y_2)$ implies $\bar{a} \in A$ by definition of a subset, which finding contradicts the initial assumption $\bar{a} \notin A$, so that the proof of the disjunction (3.499) is now complete. In the next step, we use this true disjunction to prove the desired disjunction in (3.497) by cases.

In the first case that $\bar{a}$ is a lower bound for A with respect to $\leq_X$, we may establish the equation $A \cap (\bar{a}, +\infty)_X = A$ by means of the Equality Criterion for sets, i.e. by proving equivalently

$$\forall y\,(y \in A \cap (\bar{a}, +\infty)_X \Leftrightarrow y \in A). \tag{3.500}$$

Letting y be arbitrary and assuming first $y \in A \cap (\bar{a}, +\infty)_X$ to be true, the definition of the intersection of two sets gives us especially the desired consequent $y \in A$. Assuming conversely $y \in A$ to be true, we obtain $\bar{a} \leq_X y$ with the currently assumed first part of the disjunction (3.499); here, the truth of $y \in A$ and of $\bar{a} \notin A$ implies $y \neq \bar{a}$ with (2.4), and the conjunction of $\bar{a} \leq_X y$ and this inequality $\bar{a} \neq y$ implies then $\bar{a} <_X y$ with the definition of an induced reflexive partial ordering. We obtain then $y \in$

$(\bar{a}, +\infty)_X$ by definition of an open and right-unbounded interval in X. In conjunction with the assumed $y \in A$, this further implies $y \in A \cap (\bar{a}, +\infty)_X$ (by definition of the intersection of two sets), as desired. Because y is arbitrary, we may therefore conclude that (3.500) is true, which universal sentence implies then $A \cap (\bar{a}, +\infty)_X = A$; thus, the disjunction in (3.497) holds as well.

In the second case that $\bar{a}$ is an upper bound for A with respect to $\leq_X$, we may establish $A \cap (\bar{a}, +\infty)_X = \emptyset$ via the definition of the empty set, i.e. by demonstrating the truth of

$$\forall y \, (y \notin A \cap (\bar{a}, +\infty)_X). \tag{3.501}$$

We take an arbitrary y, and we prove $y \notin A \cap (\bar{a}, +\infty)_X$ by contradiction, assuming the negation of that sentence to be true. Then, $y \in A \cap (\bar{a}, +\infty)_X$ holds in view of the Double Negation Law, with the evident consequence that $y \in A$ and $y \in (\bar{a}, +\infty)_X$ are both true. Here, the latter finding gives us the inequality $\bar{a} <_X y$, which in turn implies the truth of $\neg y \leq_X \bar{a}$. However, since we currently assume the second part of the disjunction (3.499) to hold, it follows from $y \in A$ that $y \leq_X \bar{a}$ is also true, so that we arrived at a contradiction. Thus, the proof of $y \notin A \cap (\bar{a}, +\infty)_X$ is complete, and as y was arbitrary, we may infer from the preceding negation the truth of (3.501). This universal sentence gives us $A \cap (\bar{a}, +\infty)_X = \emptyset$ and then also the disjunction in (3.497), which thus holds in any case.

Having now completed the proofs of the implications (3.496) and (3.497), we may furthermore conclude that the proposed universal sentence is true, since X, $<_X$, A and $\bar{a}$ were initially arbitrary. □

Exercise 3.69. Prove the following implications for any linearly ordered set $(X, <_X)$, any convex set A in X with respect to $<_X$ and any $b \in X$.

$$b \in A \;\Rightarrow\; A \cap (-\infty, b)_X = (-\infty, b)_A, \tag{3.502}$$

$$b \notin A \;\Rightarrow\; [A \cap (-\infty, b)_X = A \,\vee\, A \cap (-\infty, b)_X = \emptyset]. \tag{3.503}$$

(Hint: Proceed in analogy to the proof of Proposition 3.144.)

3.4. Functions

Definition 3.37 (Function/mapping, function value, argument, codomain). We say that a set f is a *function* or a *mapping* iff

1. f is a binary relation and
2. ordered pairs in f with identical first coordinates also have identical second coordinates, that is,

$$\forall x, y, y' \left([(x, y) \in f \wedge (x, y') \in f] \Rightarrow y = y' \right). \tag{3.504}$$

For any $(x, y) \in f$, we call y the *function value* at (the *argument*) x.

Furthermore, we say for any sets X and Y that a function f is a function from X to Y, symbolically

$$f : X \to Y \tag{3.505}$$

if, and only if,

1. the domain of f is identical with X, that is,

$$\mathrm{dom}(f) = X, \tag{3.506}$$

   and
2. the range of f is a subset of Y, that is,

$$\mathrm{ran}(f) \subseteq Y. \tag{3.507}$$

Here, we call Y the *codomain* of f.

Notation 3.4. Instead of $(x, y) \in f$ we will usually write

$$y = f(x) \tag{3.508}$$

or

$$x \mapsto y. \tag{3.509}$$

We now bring out more clearly the fact a function value is uniquely determined by the argument.

Theorem 3.145 (Function Criterion). *For any sets X and Y and any binary relation $f \subseteq X \times Y$, it is true that f is a function with domain X and codomain Y iff, for any element x of X, there exists a unique element y of Y such that the ordered pair (x, y) is an element of f, i.e. it is true that*

$$f : X \to Y \Leftrightarrow \forall x \left(x \in X \Rightarrow \exists! y \left(y \in Y \wedge (x, y) \in f \right) \right). \tag{3.510}$$

Proof. We let X and Y be arbitrary sets and f an arbitrary binary relation.

To prove the first part ('$\Rightarrow$') of the equivalence, we assume that f is a function from X to Y, i.e. that $\mathrm{dom}(f) = X$ and $\mathrm{ran}(f) \subseteq Y$ hold and that f satisfies (3.504). Then, to demonstrate the truth of the right-hand side, we let x be arbitrary and show that this implies $\exists! y\,(y \in Y \,\wedge\, (x,y) \in f)$, or equivalently (using the Criterion for unique existence)

$$\begin{aligned}&\exists y\,(y \in Y \,\wedge\, (x,y) \in f)\\ &\quad \wedge\, \forall y, y'\,([y \in y \,\wedge\, (x,y) \in f \,\wedge\, y' \in Y \,\wedge\, (x,y') \in f] \Rightarrow y = y').\end{aligned} \quad (3.511)$$

Since X is the domain of f, it follows by definition that $x \in X$ implies that there exists an element, say $\bar{y}$, with $(x,\bar{y}) \in f$. Thus, $\bar{y}$ is an element of the range of f, and therefore $\bar{y} \in Y$ holds with the definitions of a codomain and of a subset. This proves the first part of the conjunction (3.511). To prove the second part, we let y and y' be arbitrary and assume $y \in Y$, $(x,y) \in f$, $y' \in Y$, and $(x,y') \in f$. As this multiple conjunction implies in particular $(x,y) \in f \wedge (x,y') \in f$, it follows with the assumed (3.504) that $y = y'$ is true. As y and y' are arbitrary, we therefore conclude that that the second part of the conjunction (3.511) holds, which proves this conjunction, and thus the uniquely existential sentence in (3.510). Since x was also arbitrary, we further conclude that the right-hand side of the equivalence (3.510) is true.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume the right-hand side. To prove $f : X \to Y$, we first verify $\mathrm{dom}(f) = X$, i.e.

$$\forall x\,(x \in X \Leftrightarrow x \in \mathrm{dom}(f)). \quad (3.512)$$

For this purpose, we let x be arbitrary and first assume $x \in X$. With the assumed right-hand side of (3.510), this implies that there exists the unique y with $y \in Y$ and $(x,y) \in f$. Therefore, x is an element of the domain of f. We now assume conversely $x \in \mathrm{dom}(f)$, so that there exists (by definition of a domain) an element, say $\bar{y}$, such that $(x,\bar{y}) \in f$. The latter implies with the initial assumption $f \subseteq X \times Y$ that $(x,\bar{y}) \in X \times Y$, so that $x \in X$ holds by definition of a Cartesian product. As x was arbitrary, we therefore conclude that (3.512) is true, which means $\mathrm{dom}(f) = X$. Next, we verify $\mathrm{ran}(f) \subseteq Y$, which means

$$\forall y\,(y \in \mathrm{ran}(f) \Rightarrow y \in Y)). \quad (3.513)$$

by definition of a subset. To do this, we let y be an arbitrary element of the range of f. Therefore, there exists an element, say $\bar{x}$, with $(\bar{x},y) \in f$, where $\bar{x}$ is then evidently an element of the domain X. Then, $\bar{x} \in X$ implies with the assumed right-hand side of (3.510) that there exists the unique element

y' in Y with $(\bar{x}, y') \in f$. Therefore, $(\bar{x}, y) = (\bar{x}, y')$, so that $y' = y$ follows with (3.3). Then, $y' \in Y$ implies $y \in Y$. As y is arbitrary, we therefore conclude that (3.513) holds as well.

We now show f satisfies also (3.504). We let x, y and y' be arbitrary and assume $(x, y) \in f \wedge (x, y') \in f$. Thus, x is an element of the domain X and y, y' are elements of the range of f, thus of Y. Then, $x \in X$ implies with the assumption that the uniquely existential sentence $\exists! y\,(y \in Y \ \wedge\ (x, y) \in f)$ or equivalently (3.511) holds. The latter conjunction implies in particular its second part, from which it then follows with the previously established $y, y' \in Y$ and with the assumption $(x, y), (x, y') \in f$ that $y = y'$ holds. Since x, y and y' are arbitrary, it follows that f satisfies (3.504). We thus showed that f is a function from X to Y, which completes the proof of the proposed equivalence. As X, Y and f were also arbitrary, it follows that the theorem holds, as claimed. $\square$

The following method summarizes the application of the Function Criterion.

Method 3.1 (Verification that a binary relation is a function). To verify that a given binary relation $f \subseteq X \times Y$ is a function from X to Y, we may show for any element x of X that there exists a unique element y in Y such that the ordered pair formed by x and y is in f.

Proposition 3.146. *Any function $f : X \to Y$ is included in the Cartesian product of X and Y, that is,*

$$\forall X, Y, f\,(f : X \to Y \ \Rightarrow\ f \subseteq X \times Y). \tag{3.514}$$

Proof. We let X, Y and f be arbitrary sets and assume that f is a function from X to Y. Thus, f is a binary relation with $\mathrm{dom}(f) = X$ and $\mathrm{ran}(f) \subseteq Y$. Since $X \subseteq X$ holds according to (2.10), the conjunction $X \subseteq X \wedge \mathrm{ran}(f) \subseteq Y$ is true, which further implies

$$X \times \mathrm{ran}(f) \subseteq X \times Y \tag{3.515}$$

with (3.40). Furthermore, we have $f \subseteq \mathrm{dom}(f) \times \mathrm{ran}(f)$ because of (3.111), and therefore

$$f \subseteq X \times \mathrm{ran}(f). \tag{3.516}$$

The conjunction of (3.516) and (3.515) now yields the desired inclusion $f \subseteq X \times Y$ with (2.13), and since X, Y and f were arbitrary, we may therefore conclude that the proposed universal sentence is true. $\square$

Corollary 3.147. *For any X, Y, f, x and y it is true that, if f is a function from X to Y and if y is the value of f at x, then x is in X and y is in Y, that is,*

$$\forall X, Y, f, x, y\,([f : X \to Y \,\wedge\, y = f(x)] \Rightarrow [x \in X \,\wedge\, y \in Y]). \tag{3.517}$$

Proof. Letting X, Y, f, x and y be arbitrary and assuming $f : X \to Y$ as well as $y = f(x)$ to be true, we observe that the former assumption implies $f \subseteq X \times Y$ with (3.514), and that the latter assumption may be written as as $(x, y) \in f$ in view of Notation 3.4, which in turn implies $(x, y) \in X \times Y$ with the preceding inclusion and the definition of a subset. It then follows by definition of the Cartesian product of two sets that the desired conjunction $x \in X \,\wedge\, y \in Y$ is true. As X, Y, f, x and y were arbitrary, we may therefore conclude that (3.517) holds. □

Exercise 3.70. Verify the following sentences.

a) Any function f with domain X is a function from X to the range of f, that is,

$$\forall X, f\,([f \text{ is a function} \wedge \operatorname{dom}(f) = X] \;\Rightarrow\; f : X \to \operatorname{ran}(f)). \tag{3.518}$$

(Hint: Apply (2.10).)

b) Any function f from a set X to a subset Y of a set Z is also a function from X to Z, that is,

$$\forall X, Y, f, Z\,([f : X \to Y \,\wedge\, Y \subseteq Z] \Rightarrow\; f : X \to Z). \tag{3.519}$$

(hint: Use (2.13).)

Definition 3.38 (Transformation). We say for any sets X, Y and any function $f : X \to Y$ that f is a *transformation* (on X) iff $X = Y$.

Note 3.16. The domain and codomain of a transformation f coincide, which we may therefore write as

$$f : X \to X. \tag{3.520}$$

Our next task is to construct a set which consists of all functions having a fixed, common domain and codomain.

Theorem 3.148. *The following sentences are true for any sets X and Y.*

a) There exists a unique set Y^X such that an element f is in Y^X iff f is in the power set of the Cartesian product of X and Y and moreover if f is a function from X to Y, that is,

$$\exists! Y^X\; \forall f\,(f \in Y^X \Leftrightarrow [f \in \mathcal{P}(X \times Y) \,\wedge\, f : X \to Y]).$$

b) The set Y^X satisfies also

$$\forall f\,(f \in Y^X \Leftrightarrow f : X \to Y). \tag{3.521}$$

Proof. We let X and Y be arbitrary sets. Concerning a), we use the Axiom of Specification in connection with the Equality Criterion for sets to obtain the true existential sentence

$$\exists! Y^X\, \forall f\,(f \in Y^X \Leftrightarrow [f \in \mathcal{P}(X \times Y) \wedge \forall x, y, y'\,([(x,y) \in f \wedge (x,y') \in f] \\ \Rightarrow y = y') \wedge \mathrm{dom}(f) = X \wedge \mathrm{ran}(f) \subseteq Y]),$$

which clearly proves a). Thus, the set Y^X satisfies

$$\forall f\,(f \in Y^X \Leftrightarrow [f \in \mathcal{P}(X \times Y) \wedge f : X \to Y]). \tag{3.522}$$

Concerning b), we let f be arbitrary and assume first $f \in Y^X$, which implies in particular $f : X \Rightarrow Y$ in view of the preceding universal sentence. Conversely, we now assume $f : X \Rightarrow Y$ to be true, so that $f \subseteq X \times Y$ holds according to (3.514). This inclusion in turn implies $f \in \mathcal{P}(X \times Y)$ with the definition of a power set. Thus, the conjunction in (3.522) is true, which then implies $f \in Y^X$, completing the proof of the equivalence in (3.521). Since f is arbitrary, we may therefore conclude that the universal sentence (3.521) holds.

As X and Y were arbitrary, we may further conclude that the theorem is true. □

Definition 3.39 (Set of functions, set of transformations). For any sets X and Y we call the set (system)

$$Y^X \tag{3.523}$$

consisting of all functions from X to Y in the sense of

$$\forall f\,(f \in Y^X \Leftrightarrow f : X \to Y)$$

the *set of functions* from X to Y. In particular, we call

$$X^X \tag{3.524}$$

the *set of transformations* on X.

We conclude this introductory section on functions with a first basic, set-theoretical property.

Definition 3.40 (Bounded-from-above & bounded-from-below function, bounded & unbounded function). For any sets X and Y, any partial ordering $\leq$ of Y and any function $f : X \to Y$, we say that

(1) an element a in Y is a lower bound for f iff a is a lower bound for the range of f.

(2) f is *bounded from below* iff $\mathrm{ran}(f)$ is a bounded-from-below set.

(3) an element u in Y is an upper bound for f iff u is an upper bound for the range of f.

(4) f is *bounded from above* iff $\mathrm{ran}(f)$ is a bounded-from-above set.

(5) f is *bounded* iff $\mathrm{ran}(f)$ is a bounded set.

(6) f is *unbounded* iff f is not bounded.

3.4.1. Basic laws for functions

Proposition 3.149. *For any function f it is true that two distinct function values cannot have identical arguments, i.e.*

$$\forall x, x' \, (x, x' \in \mathrm{dom}(f) \Rightarrow [f(x) \neq f(x') \Rightarrow x \neq x']). \tag{3.525}$$

Proof. We let f be arbitrary and assume that f is a function. Next, we let $\bar{x}$ and $\bar{x}'$ be arbitrary and assume these to be elements of the domain of f. Observing that the function f is a binary relation, we now see that $\bar{x}, \bar{x}' \in \mathrm{dom}(f)$ implies by definition of a domain that there exist corresponding elements, say $\bar{y}$ and $\bar{y}'$, such that $(\bar{x}, \bar{y}), (\bar{x}', \bar{y}') \in f$ holds. In view of Notation 3.4, we thus have $\bar{y} = f(\bar{x})$ and $\bar{y}' = f(\bar{x}')$. We may now prove the implication $f(\bar{x}) \neq f(\bar{x}') \Rightarrow \bar{x} \neq \bar{x}'$ by contraposition, assuming $\neg \bar{x} \neq \bar{x}'$ to be true, so that $\bar{x} = \bar{x}'$ follows to be true with the Double Negation Law. Consequently, the previously established $(\bar{x}', \bar{y}') \in f$ gives $(\bar{x}, \bar{y}') \in f$ via substitution based on the preceding equation, so that the conjunction $(\bar{x}, \bar{y}) \in f \wedge (\bar{x}, \bar{y}') \in f$ is true. According to Property 2 of a function, this conjunction further implies $\bar{y} = \bar{y}'$, and therefore substitution yields $f(\bar{x}) = f(\bar{x}')$. This completes the proof by contraposition, so that the implications in (3.525) are true. Since $\bar{x}$ and $\bar{x}'$ are arbitrary, we may therefore conclude that the universal sentence (3.525) holds; as f was initially arbitrary, we may further conclude that the proposition is true. □

An application of the Law of Contraposition immediately yields an alternative representation of Property 2 of a function.

Corollary 3.150. *For any function f it is true that identical arguments imply identical corresponding functions, i.e.*

$$\forall x, x' \,(x, x' \in \mathrm{dom}(f) \Rightarrow [x = x' \Rightarrow f(x) = f(x')]). \tag{3.526}$$

Proposition 3.151. *There exists a unique function from $\emptyset$ to any set Y, that is,*

$$\forall Y \,\exists! f \,(f : \emptyset \to Y). \tag{3.527}$$

This function is the empty set $\emptyset$.

Proof. We let Y be an arbitrary set and prove the existential part first, recalling that $\emptyset$ is a binary relation due to Proposition 3.20; thus $\emptyset$ satisfies Property 1 of a function. We now demonstrate that $\emptyset$ satisfies also Property 2, that is, (3.504). For this purpose, we let x, y, y' be arbitrary and prove the implication

$$[(x, y) \in \emptyset \,\wedge\, (x, y') \in \emptyset] \Rightarrow y = y'.$$

Since $(x, y) \in \emptyset$ and $(x, y') \in \emptyset$ are both false by definition of the empty set, the conjunction and thus the antecedent is false. Therefore, the implication is true (no matter if $y = y'$ is true or false). As x, y and y' are arbitrary, we therefore conclude that (3.504) is true for $f = \emptyset$. This completes the proof that $\emptyset$ is a function. Let us also recall that $\mathrm{dom}(\emptyset) = \emptyset$ holds with (3.120) and that $\mathrm{ran}(\emptyset) = \emptyset$ holds with (3.121). Since $\emptyset \subseteq Y$ is true due to (2.43), we see that $\mathrm{ran}(\emptyset) \subseteq Y$ holds. We thus showed that $\emptyset$ is a function from $\emptyset$ to Y, which evidently proves the existential part.

To prove the uniqueness part, we let f and f' be arbitrary such that $f : \emptyset \to Y$ and $f' : \emptyset \to Y$ holds. Thus, f and f' are binary relations with $\mathrm{dom}(f) = \emptyset$ and $\mathrm{dom}(f') = \emptyset$. It then follows that $f = \emptyset$ and $f' = \emptyset$ because of (3.120), which equations then give the desired $f = f'$. This completes the proof of the uniquely existential sentence in (3.527), and since Y was arbitrary, we therefore conclude that the proposition is true. □

Proposition 3.152. *There is no function with nonempty domain and empty codomain, that is,*

$$\forall X \,(X \neq \emptyset \Rightarrow \neg\exists f \,(f : X \to \emptyset)). \tag{3.528}$$

Proof. We let X be an arbitrary set and prove the implication by contradiction. For this purpose, we assume $X \neq \emptyset$ and that there exists a function from X to $\emptyset$, say $\bar{f}$. Here, $X \neq \emptyset$ implies with (2.42) that there exists an element in X, say $\bar{x} \in X$. By Property 1 of a function, $\bar{f}$ is a binary relation (with domain X). Then, by definition of a domain, $\bar{x} \in X$ implies that there exists an element, say $\bar{y}$, such that $(\bar{x}, \bar{y}) \in \bar{f}$ holds. This implies by definition of a range that $\bar{y} \in \operatorname{ran}(\bar{f})$ is true, which shows that $\operatorname{ran}(\bar{f}) \neq \emptyset$. By definition of a codomain, $\operatorname{ran}(\bar{f}) \subseteq \emptyset$ is also true, which further implies $\operatorname{ran}(\bar{f}) = \emptyset$ with (2.46). The conjunction of this and the previously established $\operatorname{ran}(\bar{f}) \neq \emptyset$ constitutes a contradiction according to (1.11), so that the proof of the implication in (3.528) is complete. As X was arbitrary, we therefore conclude that the proposed universal sentence holds. □

Let us now apply the Function Criterion in a simple setting.

Proposition 3.153. *The Cartesian product of a set X and a singleton $\{y\}$ is a function from X to the singleton, that is,*

$$\forall X, y\,(X \times \{y\} \in \{y\}^X). \tag{3.529}$$

Proof. We let X and $\bar{y}$ be arbitrary and observe that $X \times \{\bar{y}\}$ is a binary relation (included in itself). We may therefore apply Method 3.1 to prove that this binary relation is a function from X to $\{\bar{y}\}$. To do this, we verify

$$\forall x\,(x \in X \Rightarrow \exists! y\,(y \in \{\bar{y}\} \,\wedge\, (x, y) \in X \times \{\bar{y}\})), \tag{3.530}$$

letting x be arbitrary and assuming $x \in X$ to be true. To establish the existential part, we observe the evident truth of $\bar{y} \in \{\bar{y}\}$, so that the conjunction $x \in X \,\wedge\, \bar{y} \in \{\bar{y}\}$ holds, which in turn implies $(x, \bar{y}) \in X \times \{\bar{y}\}$ with the definition of the Cartesian product of two sets. We thus showed that there exists an y with $y \in \{\bar{y}\}$ and $(x, y) \in X \times \{\bar{y}\}$. To establish the uniqueness part, we verify

$$\forall y, y'\,([y \in \{\bar{y}\} \wedge (\bar{x}, y) \in X \times \{\bar{y}\} \wedge y' \in \{\bar{y}\} \wedge (\bar{x}, y') \in X \times \{\bar{y}\}] \Rightarrow y = y'). \tag{3.531}$$

Letting y and y' be arbitrary and assuming $y, y' \in \{\bar{y}\}$ as well as

$$(\bar{x}, y), (\bar{x}, y') \in X \times \{\bar{y}\}$$

to be true, we now obtain from the former assumption with (2.169) the equations $y = \bar{y} = y'$, which give the desired consequent $y = y'$ of the implication in (3.531). Since y and y' are arbitrary, we may therefore conclude that the universal sentence (3.531) holds, so that the proof of the uniquely existential sentence in (3.530) is complete. Because $\bar{x}$ is also arbitrary, we may further conclude that the universal sentence (3.530) holds. It then follows from this with the Function Criterion that $X \times \{\bar{y}\} : X \to \{y\}$. Consequently, $X \times \{\bar{y}\}$ is an element of the set of functions $\{\bar{y}\}^X$. As X and $\bar{y}$ were initially arbitrary, we may now finally conclude that the proposed universal sentence (3.529) is true. □

Note 3.17. Every element of the domain X of a function $X \times \{y\}$ is associated with the unique element $y \in \{y\}$, which fact motivates the following definition.

Definition 3.41 (Constant function). For any X and any c we call the function $X \times \{c\}$, which we also symbolize by

$$g_c : X \to \{c\}, \tag{3.532}$$

the *constant function* on X with value c.

Notation 3.5. In case that Y is a codomain of a constant function $g_c : X \to \{c\}$, we also write

$$g_c : X \to Y. \tag{3.533}$$

Corollary 3.154. *The function values of a constant function $g_c : X \to \{c\}$ are identical with c for any element of the domain, i.e.*

$$\forall x\, (x \in X \Rightarrow g_c(x) = c). \tag{3.534}$$

Proof. Letting X, c and g_c be arbitrary sets, assuming $g_c : X \to \{c\}$ to be true, letting x also be arbitrary, and assuming moreover $x \in X$ to hold, it follows with the Function Criterion that there exists a unique element $y \in \{c\}$ with $(x, y) \in g_c$, i.e. with $y = g_c(x)$. As $y \in \{c\}$ implies $y = c$ with (2.169), we obtain the desired consequent $g_c(x) = c$ via substitution. Since x is arbitrary, we may therefore conclude that the universal sentence (3.534) is true. Because X, c and g_c were initially also arbitrary, the proposed sentence follows then to be true. □

Proposition 3.155. *The set of functions from a set X to a nonempty set Y is nonempty, that is,*

$$\forall X, Y\, (Y \neq \emptyset \Rightarrow Y^X \neq \emptyset). \tag{3.535}$$

Proof. We take arbitrary sets X and Y and prove the implication by cases, assuming first $Y \neq \emptyset$ and $X = \emptyset$ to be true. Then, we see in light of (3.527) that $\emptyset$ is a function from $X = \emptyset$ to Y, so that $\emptyset \in Y^X$ holds. Thus, there exists an element in Y^X, and therefore $Y^X \neq \emptyset$ follows to be true with (2.42). In the second case, we assume $Y \neq \emptyset$ and $X \neq \emptyset$ to be true. The former assumption implies with (2.42) that there exists an element in Y, say $\bar{y}$. With this element, we may form first the singleton $\{\bar{y}\}$ and then the Cartesian product $X \times \{\bar{y}\}$, which is by definition the constant function $g_{\bar{y}} : X \to \{\bar{y}\}$. Let us now observe that $\mathrm{ran}(g_{\bar{y}}) \subseteq \{\bar{y}\}$ holds by definition of a codomain, and moreover that $\bar{y} \in Y$ implies $\{\bar{y}\} \subseteq Y$ with (2.184). Consequently, we obtain from these two inclusions $\mathrm{ran}(g_{\bar{y}}) \subseteq Y$

with the transitivity of $\subseteq$, so that Y is also a codomain of $g_{\bar{y}}$. Thus, we have $g_{\bar{y}} : X \to Y$ and therefore $g_{\bar{y}} \in Y^X$, which clearly shows that $Y^X \neq \emptyset$ holds also in the second case. As X and Y were initially arbitrary sets, we may now infer from these findings the truth of the proposed universal sentence (3.535). □

The fact that the Cartesian product of two singletons constitutes a singleton formed by the ordered pair whose coordinates are the elements of the singletons (see Exercise 3.5) yields the following special case of a constant function.

Corollary 3.156. *The singleton formed by a pair (x, y) is a function from the singleton formed by x to the singleton formed by y, that is,*

$$\forall x, y\, (\{(x, y)\} \in \{y\}^{\{x\}}). \tag{3.536}$$

We now apply the Function Criterion to establish the following standard function.

Theorem 3.157. *The following sentences are true for any Y and any subset X of Y.*

a) There exists a unique set (system) j such that an element Z is in j iff Z is in $X \times Y$ and moreover if there is an element x such that Z is the ordered pair formed by x and x, that is,

$$\exists! j\, \forall Z\, (Z \in j \Leftrightarrow [Z \in X \times Y \,\wedge\, \exists x\, (Z = (x, x))]). \tag{3.537}$$

b) The set j is a function from X to Y, and this function satisfies

$$\forall x\, (x \in X \Rightarrow j(x) = x). \tag{3.538}$$

Proof. We let X and Y be arbitrary sets and assume that X is a subset of Y. Then, we may evidently apply the Axiom of Specification in connection with the Equality Criterion for sets to obtain the true uniquely existential sentence (3.537), which proves a). Thus, the set j satisfies

$$\forall Z\, (Z \in j \Leftrightarrow [Z \in X \times Y \,\wedge\, \exists x\, (Z = (x, x))]). \tag{3.539}$$

Concerning b), let us first observe that $Z \in j$ implies in particular $Z \in X \times Y$ for any Z, so that $j \subseteq X \times Y$ follows to be true by definition of a subset. We may now apply Method 3.1 to verify that this binary relation j is a function from X to Y. For this purpose, we prove

$$\forall x\, (x \in X \Rightarrow \exists! y\, (y \in Y \,\wedge\, (x, y) \in j)), \tag{3.540}$$

letting $\bar{x}$ be arbitrary and assuming $\bar{x} \in X$ to be true. To establish the uniquely existential sentence in (3.540), we first prove the existential part. To do this, we observe that $\bar{x} \in X$ implies $\bar{x} \in Y$ with the initial assumption $X \subseteq Y$ and the definition of a subset. Thus, the conjunction $\bar{x} \in X \wedge \bar{x} \in Y$ holds, so that $(\bar{x}, \bar{x}) \in X \times Y$ follows to be true by definition of the Cartesian product of two sets. This finding shows that there exists an x such that $(\bar{x}, \bar{x}) = (x, x)$ holds. Together with $(\bar{x}, \bar{x}) \in X \times Y$, this existential sentence implies with (3.539) $(\bar{x}, \bar{x}) \in j$; together with the previously established $\bar{x} \in Y$, this shows that there exists an y such that $y \in Y$ and $(\bar{x}, y) \in j$ are true, proving the existential part of the uniquely existential sentence in (3.540).

Regarding the uniqueness part, we verify

$$\forall y, y' \left([y \in Y \;\wedge\; (\bar{x}, y) \in j \;\wedge\; y' \in Y \;\wedge\; (\bar{x}, y') \in j] \Rightarrow y = y' \right). \tag{3.541}$$

We let $\bar{y}$ and $\bar{y}'$ be arbitrary and assume $\bar{y}, \bar{y}' \in Y$ and $(\bar{x}, \bar{y}), (\bar{x}, \bar{y}') \in j$ to hold. On the one hand, $(\bar{x}, \bar{y}) \in j$ implies with (3.539) in particular that there exists an element, say $\bar{\bar{x}}$, such that $(\bar{x}, \bar{y}) = (\bar{\bar{x}}, \bar{\bar{x}})$ holds. This equation yields with the Equality Criterion for ordered pairs $\bar{x} = \bar{\bar{x}}$ and $\bar{y} = \bar{\bar{x}}$, which equations in turn imply (via substitution)

$$\bar{x} = \bar{y}. \tag{3.542}$$

On the other hand, $(\bar{x}, \bar{y}') \in j$ implies with (3.539) in particular that there is an element, say $\bar{\bar{x}}'$, with $(\bar{x}, \bar{y}') = (\bar{\bar{x}}', \bar{\bar{x}}')$. Therefore, the Equality Criterion for ordered pairs gives now $\bar{x} = \bar{\bar{x}}'$ as well as $\bar{y}' = \bar{\bar{x}}'$. Combining these two equations, we now obtain

$$\bar{x} = \bar{y}'.$$

Applying now substitution to this equation based on (3.542), we arrive at the desired $\bar{y} = \bar{y}'$, so that the proof of the implication in (3.541) is complete. As $\bar{y}$ and $\bar{y}'$ were arbitrary, we may therefore conclude that the uniqueness part (3.541) of the uniquely existential sentence in (3.540) holds as well. This in turn proves the implication in (3.540), and since $\bar{x}$ was arbitrary, we may now further conclude that the universal sentence (3.540) is true. This implies then $j : X \to Y$ with the Function Criterion (3.510).

It now remains for us to verify (3.538). Letting $\bar{x} \in X$ be arbitrary, we already showed earlier that this implies $(\bar{x}, \bar{x}) \in j$, which we may now write as $\bar{x} = j(\bar{x})$ by applying Notation 3.4. Thus, the implication in (3.538) is clearly true, and since $\bar{x}$ is arbitrary, we may finally conclude that j satisfies indeed (3.538). As X and Y were arbitrary in the proofs of a) and b), the theorem follows to be true. □

Definition 3.42 (Inclusion function/inclusion map). For any set Y and any $X \subseteq Y$ we call the function $j : X \to Y$ which maps every element of its domain into itself in the sense of

$$\forall x\,(x \in X \Rightarrow j(x) = x).$$

the *inclusion function* or the *inclusion map* from X to Y.

As there exists a vast variety of functions it is crucial to have a method at one's disposal to check whether two given functions are actually identical.

Theorem 3.158 (Equality Criterion for functions). *Two functions f and g with common domain X are equal iff their function values at any common element of their domain are identical, that is,*

$$\begin{aligned}&\forall X, f, g\,(f, g \text{ are functions with domain } X\\ &\quad \Rightarrow [f = g \Leftrightarrow \forall x\,(x \in X \Rightarrow f(x) = g(x))]).\end{aligned} \tag{3.543}$$

Proof. We let X, f and g be arbitrary sets and assume that f and g are functions with domain X. We first observe that the equivalence

$$f = g \;\Leftrightarrow\; \forall z\,(z \in f \Leftrightarrow z \in g) \tag{3.544}$$

holds because of the Equality Criterion for sets. Now, to prove the first part ('$\Rightarrow$') of the equivalence in (3.543), we assume $f = g$, or equivalently the right-hand side

$$\forall z\,(z \in f \Leftrightarrow z \in g) \tag{3.545}$$

of the equivalence (3.544), and then let $x \in X$ be arbitrary. As f is a function with domain X, there exists a unique function value $y = f(x)$, which means $(x, y) \in f$. Defining $z = (x, y)$, it follows with the assumption (3.545) that $z \in g$, that is, $(x, y) \in g$, which means $y = g(x)$ since g is a function on X; consequently, $y = f(x) = g(x)$. As x is arbitrary, the right-hand side of the equivalence in (3.543) is true.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume

$$\forall x\,(x \in X \Rightarrow f(x) = g(x)) \tag{3.546}$$

and then let z be arbitrary. To prove the first part ('$\Rightarrow$') of the equivalence $z \in f \Leftrightarrow z \in g$, we first assume $z \in f$, which means that there exists an element of the domain X of f, say $\bar{x}$, and an element of the range of f, say $\bar{y}$, such that $z = (\bar{x}, \bar{y})$ and $\bar{y} = f(\bar{x})$. Due to the assumption (3.546), $\bar{x} \in X$ implies $(\bar{y} =)\ f(\bar{x}) = g(\bar{x})$, that is, $\bar{y} = g(\bar{x})$, which means $(z =)\ (\bar{x}, \bar{y}) \in g$, that is, $z \in g$. We thus showed that $z \in f$ implies $z \in g$. To prove the second part ('$\Leftarrow$') of the equivalence $z \in f \Leftrightarrow z \in g$, we now assume $z \in g$, so that $z = (\bar{x}, \bar{y})$ and $\bar{y} = g(\bar{x})$ hold for particular $\bar{x} \in X$ and $\bar{y}$. Because of the assumption (3.546), $\bar{x} \in X$ implies $(\bar{y} =)\ g(\bar{x}) = f(\bar{x})$, i.e. $\bar{y} = f(\bar{x})$, which means $(z =)\ (\bar{x}, \bar{y}) \in f$, i.e. $z \in f$. We thus verified that $z \in g$ implies $z \in f$, which completes the proof of the equivalence $z \in f \Leftrightarrow z \in g$. Since z is arbitrary, we therefore conclude that the right-hand side of the equivalence (3.544) holds, and therefore $f = g$, i.e. the left-hand side of the equivalence in (3.543).

As X, f and g are arbitrary we then conclude that the universal sentence (3.543) is true. □

Method 3.2 (Verification that two given functions are equal). To verify that two given functions f and g with the same domain X are identical, we may show that $f(x) = g(x)$ holds for every element x of the domain X of f and g.

We demonstrate the preceding method for a simple situation.

Proposition 3.159. *For any x and any function f whose domain consists of x, it is true that $f = \{(x, f(x))\}$.*

Proof. We take arbitrary sets $\bar{x}$ and f where we assume f to be a function with $\mathrm{dom}(f) = \{\bar{x}\}$. Evidently then, the function value $f(\bar{x})$ exists uniquely, so that we may form the singleton $g = \{(\bar{x}, f(\bar{x}))\}$, which is an element of $\{f(\bar{x})\}^{\{\bar{x}\}}$ due to (3.536) and consequently a function with domain $\{\bar{x}\}$ according to (3.521). We now establish $f = g$ by means of the Equality Criterion for functions. To do this, we verify

$$\forall x\,(x \in \{\bar{x}\} \Rightarrow f(x) = g(x)), \tag{3.547}$$

letting x be arbitrary and assuming $x \in \{\bar{x}\}$ to be true, which assumption implies $x = \bar{x}$ with (2.169). Then, substitution yields $f(x) = f(\bar{x})$. Furthermore, $(\bar{x}, f(\bar{x})) \in g\ [= \{(\bar{x}, f(\bar{x}))\}]$ holds according to (2.153), which we may write also as $f(\bar{x}) = g(\bar{x})$, because g is a function. Thus, substitution gives $f(x) = g(\bar{x}) = g(x)$ and therefore the equation $f(x) = g(x)$. As x was arbitrary, we may infer from this finding the truth of the universal sentence (3.547), which in turn implies $f = g\ [= \{(\bar{x}, f(\bar{x}))\}]$, proving the proposed equation. Since $\bar{x}$ and f were also arbitrary, we may finally conclude that the proposition is true. □

The following set-theoretical axiom will be useful for defining functions in a more direct way than via the Axiom of Specification.

Axiom 3.2 (Axiom of Replacement). If, for any element x of a set X, there exists a unique y such that a given formula $\varphi(x, y)$ becomes a true sentence, then there exists a set Y such that, for any element x in X, there is an element y in Y for which $\varphi(x,y)$ becomes true, that is,

$$\forall x\,(x \in X \Rightarrow \exists! y\,(\varphi(x,y))) \Rightarrow \exists Y\,\forall x\,(x \in X \Rightarrow \exists y\,(y \in Y \,\wedge\, \varphi(x,y))). \tag{3.548}$$

Theorem 3.160 (Function definition by replacement). *For an arbitrary formula $\varphi(x,y)$ and any set X such that*

$$\forall x\,(x \in X \Rightarrow \exists! y\,(\varphi(x,y))) \tag{3.549}$$

holds, there exists a unique function f with domain X satisfying

$$\forall x\,(x \in X \Rightarrow \varphi(x, f(x))). \tag{3.550}$$

Proof. We let X be an arbitrary set satisfying (3.549) and prove first the existential part. With the Axiom of Replacement, the assumption (3.549) implies that there exists a set, say $\bar{Y}$, such that

$$\forall x\,(x \in X \Rightarrow \exists y\,(y \in \bar{Y} \,\wedge\, \varphi(x,y))) \tag{3.551}$$

holds. We then obtain with the Axiom of Specification and the Equality Criterion for sets

$$\exists! f\,\forall Z\,(Z \in f \Leftrightarrow [Z \in X \times \bar{Y} \,\wedge\, \exists x, y\,(\varphi(x,y) \,\wedge\, (x,y) = Z)]),$$

so that the set f satisfies

$$\forall Z\,(Z \in f \Leftrightarrow [Z \in X \times \bar{Y} \,\wedge\, \exists x, y\,(\varphi(x,y) \,\wedge\, (x,y) = Z)]). \tag{3.552}$$

Since $Z \in f$ implies in particular $Z \in X \times \bar{Y}$ for any Z, we obtain $f \subseteq X \times \bar{Y}$ with the definition of a subset, so that f is a binary relation. We now verify

$$\forall x\,(x \in X \Rightarrow \exists! y\,(y \in \bar{Y} \,\wedge\, (x,y) \in f)), \tag{3.553}$$

which will prove that f is a function from X to $\bar{Y}$, according to the Function Criterion (3.510). For this purpose, we let $\bar{x}$ be arbitrary in X and prove the uniquely existential sentence in (3.553), which is equivalent to the conjunction

$$\begin{aligned}&\exists y\,(y \in \bar{Y} \,\wedge\, (\bar{x},y) \in f)\\ &\quad \wedge\, \forall y, y'\,([y \in \bar{Y} \,\wedge\, (\bar{x},y) \in f \,\wedge\, y' \in \bar{Y} \,\wedge\, (\bar{x},y') \in f] \Rightarrow y = y')\end{aligned} \tag{3.554}$$

because of the Criterion for unique existence. Regarding the existential part, we first observe that $\bar{x} \in X$ implies with (3.551) that there exists an element of $\bar{Y}$, say $\bar{y}$, such that $\varphi(\bar{x},\bar{y})$ is true. Then, the conjunction of $\varphi(\bar{x},\bar{y})$ and $(\bar{x},\bar{y})=(\bar{x},\bar{y})$ is clearly true as well, so that the existential sentence

$$\exists x,y\,(\varphi(x,y)\ \wedge\ (x,y)=(x,y)) \tag{3.555}$$

holds. Because of $\bar{x}\in X$ and $\bar{y}\in\bar{Y}$, we have $(\bar{x},\bar{y})\in X\times\bar{Y}$ by definition of the Cartesian product. The conjunction of this and (3.555) then implies with (3.552) that $(\bar{x},\bar{y})\in f$. The conjunction of this and $\bar{y}\in\bar{Y}$ proves the existential part of (3.554). We now prove the uniqueness part. To do this, we let $\bar{y}$ and $\bar{y}'$ be arbitrary and assume $\bar{y}\in\bar{Y}$, $(\bar{x},\bar{y})\in f$, $\bar{y}'\in\bar{Y}$, and $(\bar{x},\bar{y}')\in f$. The second and fourth part of this conjunction imply with (3.552) in particular the two existential sentences

$$\exists x,y\,(\varphi(x,y)\ \wedge\ (x,y)=(\bar{x},\bar{y}))$$
$$\exists x,y\,(\varphi(x,y)\ \wedge\ (x,y)=(\bar{x},\bar{y}')).$$

Thus, there exist x and y, say x' and y' with $\varphi(x',y')$ and $(x',y')=(\bar{x},\bar{y})$, and there exist x and y, say x^* and y^* with $\varphi(x^*,y^*)$ and $(x^*,y^*)=(\bar{x},\bar{y}')$. Because of the Equality Criterion for ordered pairs, the two preceding equations imply $x'=\bar{x}$, $y'=\bar{y}$, $x^*=\bar{x}$, and $y^*=\bar{y}'$, so that substitution yields $\varphi(\bar{x},\bar{y})$ and $\varphi(\bar{x},\bar{y}')$. Since $\bar{x}\in X$ implies with (3.549) that there exists a unique y such that $\varphi(\bar{x},y)$ holds, we see that $y=\bar{y}=\bar{y}'$. As $\bar{y}$ and $\bar{y}'$ are arbitrary, we therefore conclude that uniqueness part and thus (3.554) is true. Since $\bar{x}$ was also arbitrary, we then further conclude that (3.553) holds, which implies with the Function Criterion that f is a function with domain X and codomain $\bar{Y}$.

It remains for us to show that f satisfies (3.550). Letting $\bar{x}$ be arbitrary in X, it follows with (3.553) that there exists a unique $\bar{y}\in\bar{Y}$ satisfying $(\bar{x},\bar{y})\in f$ (which we may write as $\bar{y}=f(\bar{x})$ with Notation 3.4). This implies with (3.552) in particular

$$\exists x,y\,(\varphi(x,y)\ \wedge\ (x,y)=(\bar{x},\bar{y})).$$

Thus, there are x and y, say x' and y', such that $\varphi(x',y')$ and $(x',y')=(\bar{x},\bar{y})$ are true. The latter equation implies $x'=\bar{x}$ and $y'=\bar{y}$ with 3.3, so that substitution gives us $\varphi(\bar{x},\bar{y})$. With the previously introduced notation $\bar{y}=f(\bar{x})$, we then obtain $\varphi(\bar{x},f(\bar{x})$. As $\bar{x}$ is arbitrary, we therefore conclude that f indeed satisfies 3.550).

We now prove the uniqueness part, letting f and f' be arbitrary functions with domain X such that they both satisfy (3.550). To show that this implies $f=f'$, we apply Method 3.2. To do this, we let x be an arbitrary element of X and show that this implies $f(x)=f'(x)$. Now, as f and f' satisfy (3.550), we see that $x\in X$ implies that $\varphi(x,f(x))$ and $\varphi(x,f'(x))$

are both true. Furthermore, $x \in X$ implies $\exists! y (\varphi(x, y))$, and therefore in particular (using the Criterion for unique existence)

$$\forall y, y' \, ([\varphi(x, y) \, \wedge \, \varphi(x, y')] \Rightarrow y = y').$$

With this, the conjunction of $\varphi(x, f(x))$ and $\varphi(x, f'(x))$ implies the desired $f(x) = f'(x)$; as x is arbitrary, we therefore conclude that $f = f'$ holds. Since f and f' were arbitrary, we then conclude that the uniqueness part of the proposed sentence holds as well, which completes the proof of the stated uniquely existential sentence.

Since X was also arbitrary, we finally conclude that the theorem is true. □

Method 3.3 (Function definition by replacement). To establish for a given set X and a given formula $\varphi(x, y)$ the unique existence of a function f with domain X such that f satisfies $\varphi(x, f(x))$ for any element x of the domain X, we may establish for any $x \in X$ the unique existence of an y satisfying $\varphi(x, y)$.

We demonstrate the application this method to define the following special function.

Theorem 3.161 (Existence of identity functions). *For any set X there exists a unique function* id_X *with domain X such that every element x of X is mapped to itself, i.e. such that*

$$\forall x \, (x \in X \Rightarrow \mathrm{id}_X(x) = x) \tag{3.556}$$

holds. Furthermore, the range of id_X *is identical with X.*

Proof. We let X be an arbitrary set and $\varphi(x, y)$ the formula $y = x$. Let us first verify the truth of

$$\forall x \, (x \in X \Rightarrow \exists! y \, (y = x)). \tag{3.557}$$

To do this, we let x be arbitrary, assume $x \in X$ to be true, and note that the uniquely existential sentence is true due to (1.109). Now, as x is arbitrary, it follows that (3.557) holds indeed. Then, it follows with Theorem 3.160 that there exists a unique function id_X with domain X satisfying

$$\forall x \, (x \in X \Rightarrow \varphi(x, \mathrm{id}_X(x))),$$

which is evidently equivalent to (3.556) for the given formula $\varphi(x, y)$.

We now prove $X = \mathrm{ran}(\mathrm{id}_X)$ by verifying the equivalent (using the Equality Criterion for sets)

$$\forall x \, (x \in X \Leftrightarrow x \in \mathrm{ran}(\mathrm{id}_X)). \tag{3.558}$$

For this purpose, we let x be arbitrary and assume first $x \in X$, which implies $\mathrm{id}_X(x) = x$ with (3.556), which we may write as $(x, x) \in \mathrm{id}_X$ according to Notation 3.4. Thus, there exists an a with $(a, x) \in \mathrm{id}_X$, which implies $x \in \mathrm{ran}(\mathrm{id}_X)$ with the definition of a range. This proves the first part ('$\Rightarrow$') of the equivalence in (3.558). We now assume $x \in \mathrm{ran}(\mathrm{id}_X)$ to be true, which implies that there exists a particular $\bar{x}$ with $(\bar{x}, x) \in \mathrm{id}_X$. Thus, there exists a b with $(\bar{x}, b) \in \mathrm{id}_X$, so that $\bar{x} \in \mathrm{dom}(\mathrm{id}_X)$ $[= X]$ by definition of a domain. Then, $\bar{x} \in X$ implies $\mathrm{id}_X(\bar{x}) = \bar{x}$ with (3.556). As the previously established $(\bar{x}, x) \in \mathrm{id}_X$ can be written as $\mathrm{id}_X(\bar{x}) = x$, we obtain from the previous two equations $x = \bar{x}$. With the latter equation, the previous finding $\bar{x} \in X$ implies $x \in X$, which proves the second part ('$\Leftarrow$') of the equivalence. Since x is arbitrary, we therefore conclude that (3.558) is true, so that $X = \mathrm{ran}(\mathrm{id}_X)$ holds, as claimed.

As X was also arbitrary, we finally conclude that the theorem is true. □

Definition 3.43 (Identity function). For any set X we call the function id_X which maps every element of its domain to itself in the sense of

$$\forall x\,(x \in X \Rightarrow \mathrm{id}_X(x) = x)$$

the *identity function* on X. We symbolize this function also by

$$\{(x, x) : x \in X\} \tag{3.559}$$

We may apply the Equality Criterion for functions to show that the identity function and the inclusion function on the same domain are set-theoretically identical.

Corollary 3.162. *For any sets X and Y it is true that the identity function on X and the inclusion map from X to Y are identical functions.*

Proof. Letting X and Y be arbitrary sets and observing that id_X and $j : X \to Y$ share the same domain X, we may now apply Method 3.2 to prove the stated equation. To do this, we verify

$$\forall x\,(x \in X \Rightarrow \mathrm{id}_X(x) = j(x)). \tag{3.560}$$

We let x be arbitrary and assume that $x \in X$ holds. We then obtain $\mathrm{id}_X(x) = x$ by definition of an identity function and $j(x) = x$ by definition of an inclusion map. Then, substitution yields $\mathrm{id}_X(x) = j(x)$, which proves the implication in (3.560). As x is arbitrary, we may therefore conclude that the universal sentence (3.560) holds, which then implies $\mathrm{id}_X = j$ with the Equality Criterion for functions (3.543). Since X and Y were arbitrary, we may then further conclude that the proposed universal sentence is true. □

Note 3.18. The identity function on a set X and the inclusion function from X to X are identical functions with identical codomains.

Proposition 3.163. *It is true for any sets X_1 and X_2 that there exists a unique function π_1 with domain $X_1 \times X_2$ such that*

$$\forall Z\,(Z \in X_1 \times X_2 \Rightarrow \exists x_1, x_2\,(Z = (x_1, x_2) \wedge \pi_1(Z) = x_1)) \tag{3.561}$$

holds. Moreover, X_1 is a codomain of this function.

Proof. Letting X_1 and X_2 be arbitrary sets, we apply Function definition by replacement and prove accordingly the universal sentence

$$\forall Z\,(Z \in X_1 \times X_2 \Rightarrow \exists! y\,(\exists x_1, x_2\,(Z = (x_1, x_2) \wedge y = x_1))). \tag{3.562}$$

For this purpose, we take an arbitrary set Z, and we assume that $Z \in X_1 \times X_2$ is true. By definition of the Cartesian product of two sets, there are then particular elements $\bar{x}_1 \in X_1$ and $\bar{x}_2 \in X_2$ with $(\bar{x}_1, \bar{x}_2) = Z$. Denoting here $\bar{x}_1$ by $\bar{y}$, we thus have the true existential sentence

$$\exists x_1, x_2\,(Z = (x_1, x_2) \wedge \bar{y} = x_1),$$

which demonstrates also the truth of the existential part of the uniquely existential sentence in (3.562). Regarding the uniqueness part, we apply Method 1.18 and take an arbitrary set y' satisfying the existential sentence

$$\exists x_1, x_2\,(Z = (x_1, x_2) \wedge y' = x_1).$$

This means that there are constants, say $\bar{x}_1'$ and $\bar{x}_2'$, for which $Z = (\bar{x}_1', \bar{x}_2')$ and $y' = \bar{x}_1'$ are satisfied. Recalling the truth of $(\bar{x}_1, \bar{x}_2) = Z$, we therefore obtain $\bar{x}_1' = \bar{x}_1$ and $\bar{x}_2' = \bar{x}_2$ with the Equality Criterion for ordered pairs. Then, substitution into the previously established $\bar{y} = \bar{x}_1$ gives us $\bar{y} = \bar{x}_1'$ $[= y']$. Since y' was arbitrary, we may infer from the resulting equation $\bar{y} = y'$ the truth of the uniqueness part of the uniquely existential sentence in (3.562). As Z was also arbitrary, we may further conclude that the universal sentence (3.562) holds, so that there exists indeed a unique function π_1 such that (3.561). To show that X_1 is a codomain of π_1, which means by definition that the inclusion $\mathrm{ran}(\pi_1) \subseteq X_1$ is true, we apply the definition of a subset and verify

$$\forall y\,(y \in \mathrm{ran}(\pi_1) \Rightarrow y \in X_1). \tag{3.563}$$

For this purpose, we let y be arbitrary, where we assume $y \in \mathrm{ran}(\pi_1)$ to be true. According to the definition of a range, there exists then a constant, say $\bar{Z}$, for which $(\bar{Z}, y) \in \pi_1$ is satisfied. Since π_1 is a function, we can apply function notation and write $y = \pi_1(\bar{Z})$; on the other hand, the definition of a domain yields $\bar{Z} \in X_1 \times X_2$. These findings imply now with the definition of the function π_1 in (3.561) that there exist particular constants $\bar{\bar{x}}_1, \bar{\bar{x}}_2$ such that $\bar{Z} = (\bar{\bar{x}}_1, \bar{\bar{x}}_2)$ and $\pi_1(\bar{Z}) = \bar{\bar{x}}_1$. Thus, substitution yields $(\bar{\bar{x}}_1, \bar{\bar{x}}_2) \in X_1 \times X_2$, with the consequence that

$$[y = \pi_1(\bar{Z}) =] \quad \bar{\bar{x}}_1 \in X_1$$

is true, by definition of the Cartesian product of two sets. The resulting $y \in X_1$ proves the implication in (3.563), in which y is arbitrary, so that the universal sentence (3.563) turns out to be true. Thus, the range of π_1 is included in X_1, and this means that X_1 is indeed a codomain of π_1. Because X_1 and X_2 were initially arbitrary sets, the proposition follows now to be true. □

Exercise 3.71. Show for any sets X_1 and X_2 that there exists a unique function π_2 with domain $X_1 \times X_2$ such that

$$\forall Z\,(Z \in X_1 \times X_2 \Rightarrow \exists x_1, x_2\,(Z = (x_1, x_2) \;\wedge\; \pi_2(Z) = x_2)) \tag{3.564}$$

holds. Verify then that X_2 is a codomain of this function.

Definition 3.44 (Projection function on the Cartesian product of two sets). We call for any sets X_1 and X_2 the functions $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$, respectively, the *projection function* from $X_1 \times X_2$ to X_1 and from $X_1 \times X_2$ to X_2.

3.4.2. Restrictions and compositions of functions

Note 3.19. Since any function $f : X \to Y$ is a subset of $X \times Y$ according to (3.514), we see in light of (3.88) that the restriction of the function f to its domain X is the function f itself, that is,

$$\forall X, Y, f\,(f : X \to Y \;\Rightarrow\; f \restriction X = f). \tag{3.565}$$

Proposition 3.164. *For any function $f : X \to Y$ and any subset A of X the restriction $f \restriction A$ is a function from A to Y, that is,*

$$\forall X, Y, f, A\,([f \in Y^X \;\wedge\; A \subseteq X] \Rightarrow f \restriction A \in Y^A). \tag{3.566}$$

Proof. We let X, Y, f and A be arbitrary sets, assume that f is a function from X to Y, and assume furthermore that A is a subset of X. By definition of a restriction, $f \restriction A$ is a set of ordered pairs and therefore a binary relation. Thus, $f \restriction A$ satisfies Property 1 of a function. Regarding Property 2, we verify that $f \restriction A$ satisfies (3.504). For this purpose, we let x, y, y' be arbitrary, assume that $(x, y), (x, y') \in f \restriction A$ holds, and show that this implies $y = y'$. We see from (3.80) that $(x, y) \in f \restriction A$ implies in particular $(x, y) \in f$, and similarly that $(x, y') \in f \restriction A$ implies $(x, y') \in f$. As a function, f satisfies (3.504), so that the conjunction of $(x, y) \in f$ and $(x, y') \in f$ implies the desired $y = y'$. Thus, since y was arbitrary, we conclude that $f \restriction A$ satisfies also Property 2 of a function.

Next, we observe that f constitutes a binary relation with domain X. Then, the initial assumption $A \subseteq X$ [$= \mathrm{dom}(f)$] implies $\mathrm{dom}(f \restriction A) = A$ because of (3.106).

Finally, we show that $\mathrm{ran}(f \restriction A) \subseteq Y$ is also true, by verifying

$$\forall y \, (y \in \mathrm{ran}(f \restriction A) \Rightarrow y \in Y)$$

according to the definition of a subset. Letting $y \in \mathrm{ran}(f \restriction A)$ be arbitrary, it follows by definition of the range that there is an x, say $\bar{x}$, such that $(\bar{x}, y) \in f \restriction A$. The latter implies $(\bar{x}, y) \in f$ with (3.80), which shows that $y \in \mathrm{ran}(f)$. Since f satisfies (3.507), we have that $y \in \mathrm{ran}(f)$ implies the desired $y \in Y$ (by definition of a subset). Then, as y is arbitrary, we may conclude that $\mathrm{ran}(f \restriction A) \subseteq Y$ holds.

In summary, we thus proved that $f \restriction A : A \to Y$ holds, so that $f \restriction A$ is an element of Y^A. Since X, Y, f and A were arbitrary, the proposition follows then to be true. □

Corollary 3.165. *For any function $f : X \to Y$ and any subset A of X, it is true that the restriction of f to A takes the same value as f at any element of A, that is,*

$$\forall X, Y, f, A \, ([f \in Y^X \wedge A \subseteq X] \Rightarrow \forall x \, (x \in A \Rightarrow f \restriction A(x) = f(x))). \tag{3.567}$$

Proof. Letting X, Y, f and A be arbitrary, assuming $f : X \to Y$ and $A \subseteq X$ to be true, letting x be arbitrary in A, and letting the function value of $f \restriction A$ at x be $y = f \restriction A(x)$ (using the fact that this restriction is a function with domain A according to Proposition 3.164), we may write this equation as $(x, y) \in f \restriction A$. This in turn implies $(x, y) \in f$ with the definition of a restriction, which we may write also as $y = f(x)$. Then, substitution based on the previously established equation $y = f \restriction A(x)$ yields $f \restriction A(x) = f(x)$, as desired. Since x is arbitrary, we may therefore conclude that the universal sentence with respect to x is true. As X, Y, f and A were also arbitrary, we may further conclude that the proposed universal sentence holds. □

Proposition 3.166. *It is true for any set X, any partially ordered set $(Y, \leq_Y)$, any function $f : X \to Y$, any set $A \subseteq X$ and any element $a \in Y$ that a is a lower bound for $f \restriction A$ if a is a lower bound for f, i.e.*

$$\forall y\,(y \in \mathrm{ran}(f) \Rightarrow a \leq_Y y) \;\Rightarrow\; \forall y\,(y \in \mathrm{ran}(f \restriction A) \Rightarrow a \leq_Y y). \tag{3.568}$$

Proof. We let X, Y, $<_Y$, f, A and a be arbitrary, assume $(Y, <_Y)$ to be partially ordered, f to be a function from X to Y, A to be a subset of X and a to be an element of Y. To prove the stated implication directly, we also assume the universal sentence

$$\forall y\,(y \in \mathrm{ran}(f) \Rightarrow a \leq_Y y) \tag{3.569}$$

to be true, and we show that this implies

$$\forall y\,(y \in \mathrm{ran}(f \restriction A) \Rightarrow a \leq_Y y), \tag{3.570}$$

letting $\bar{y}$ be arbitrary and assuming $\bar{y} \in \mathrm{ran}(f \restriction A)$ to be true. Let us now observe that the range of the restriction $f \restriction A$ is included in the range of f according to Corollary 3.35, so that

$$\forall y\,(y \in \mathrm{ran}(f \restriction A) \Rightarrow y \in \mathrm{ran}(f))$$

follows to be true by definition of a subset. Therefore, the assumed $\bar{y} \in \mathrm{ran}(f \restriction A)$ implies $\bar{y} \in \mathrm{ran}(f)$, which in turn implies the desired $a \leq_Y y$ with (3.569). Thus, the proof of the implication in (3.570) is complete, and since y is arbitrary, we may infer from this the truth of the universal sentence (3.570). We thus completed also the proof of the implication (3.568), and because X, Y, $<_Y$, f, A and a were initially arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Exercise 3.72. Show for any set X, any partially ordered set $(Y, \leq_Y)$, any function $f : X \to Y$, any set $A \subseteq X$ and any element $u \in Y$ that u is an upper bound for $f \restriction A$ if u is an upper bound for f, i.e.

$$\forall y\,(y \in \mathrm{ran}(f) \Rightarrow y \leq_Y u) \;\Rightarrow\; \forall y\,(y \in \mathrm{ran}(f \restriction A) \Rightarrow y \leq_Y u). \tag{3.571}$$

(Hint: Apply Corollary 3.35.)

Proposition 3.167. *For any function $f : X \to Y$ and any element x of X, it is true that the restriction of f to the singleton formed by x is identical with the constant function on that singleton with the value of f at x, i.e.*

$$\forall X, Y, f, x\,([f \in Y^X \,\wedge\, x \in X] \Rightarrow f \restriction \{x\} = \{(x, f(x))\}). \tag{3.572}$$

Proof. We let X, Y, f and $\bar{x}$ be arbitrary, assume $f \in Y^X$, so that $f : X \to Y$ holds, and assume furthermore $\bar{x} \in X$ to be true. As the latter implies $\{\bar{x}\} \subseteq X$ with (2.184), we may apply Proposition 3.164 to infer from the assumptions that $f \restriction \{\bar{x}\}$ is a function with domain $\{\bar{x}\}$. In view of Corollary 3.156, the singleton $g = \{(\bar{x}, f(\bar{x}))\}$ is also a function with domain $\{\bar{x}\}$, where the function value $f(\bar{x})$ is indeed specified because $\bar{x}$ is in the domain X of f. Let us now apply the Equality Criterion for functions to establish the desired equation $f \restriction \{\bar{x}\} = \{(\bar{x}, f(\bar{x}))\}$, by verifying

$$\forall x \, (x \in \{\bar{x}\} \Rightarrow f \restriction \{\bar{x}\}(x) = g(x)). \tag{3.573}$$

Letting x be arbitrary and assuming $x \in \{\bar{x}\}$, it now follows from $f \in Y^X$, the previously established $\{\bar{x}\} \subseteq X$ and the preceding assumption with (3.165) that $f \restriction \{\bar{x}\}(x) = f(x)$ holds. Since $x \in \{\bar{x}\}$ implies also $x = \bar{x}$ with (2.169) and since $g = \{(\bar{x}, f(\bar{x}))\}$ implies evidently $(\bar{x}, f(\bar{x})) \in g$, which we may write as $f(\bar{x}) = g(\bar{x})$, we obtain via substitution

$$[f \restriction \{\bar{x}\}(x) =] \quad f(x) = f(\bar{x}) = g(\bar{x}) = g(x).$$

This proves the implication in (3.573), and since x was arbitrary, we may therefore conclude that the universal sentence (3.573) holds. Thus, the functions $f \restriction \{\bar{x}\}$ and $\{(\bar{x}, f(\bar{x}))\}$ are indeed identical, and as X, Y, f and $\bar{x}$ were initially also arbitrary, we may infer from this the truth of the proposition. □

Proposition 3.168. *For any function $f : X \to Y$ it is true that removing an element (x, y) from f yields the restriction of f to the set difference of X and the singleton formed by x, i.e.*

$$\forall x, y \, ((x, y) \in f \Rightarrow f \restriction X \setminus \{x\} = f \setminus \{(x, y)\}). \tag{3.574}$$

Proof. We let X, Y, f, x and y be arbitrary, assume that f is a function from X to Y, and moreover that $(x, y) \in f$ is true. To show that this implies $f \restriction X \setminus \{x\} = f \setminus \{(x, y)\}$, we verify the equivalent (applying the Equality Criterion for sets)

$$\forall z \, (z \in f \restriction X \setminus \{x\} \Leftrightarrow z \in f \setminus \{(x, y)\}). \tag{3.575}$$

Letting z be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $z \in f \restriction X \setminus \{x\}$. Then, by definition of a binary relation, there exist elements, say $\bar{x}$ and $\bar{y}$, with $(\bar{x}, \bar{y}) = z$, so that $(\bar{x}, \bar{y}) \in f \restriction X \setminus \{x\}$. Therefore, $\bar{x}$ is by definition an element of the domain $X \setminus \{x\}$ of the restriction $f \restriction X \setminus \{x\}$, which is a function in view of Proposition 3.164. It then follows from $\bar{x} \in X \setminus \{x\}$ that $\bar{x} \in X$ and $\bar{x} \notin \{x\}$ are both true, by definition of a set difference. The latter further implies $\bar{x} \neq x$ with (2.169), and therefore $(\bar{x}, \bar{y}) \neq (x, y)$ with the Equality Criterion for ordered pairs, so that we obtain $(\bar{x}, \bar{y}) \notin \{(x, y)\}$ again with (2.169). Now, $(\bar{x}, \bar{y}) \in f \upharpoonright X \setminus \{x\}$ also implies $(\bar{x}, \bar{y}) \in f$ by definition of a restriction. Thus, the conjunction of $(\bar{x}, \bar{y}) \in f$ and $(\bar{x}, \bar{y}) \notin \{(x, y)\}$ holds, so that the previously established equation $(\bar{x}, \bar{y}) = z$ gives $z \in f \setminus \{(x, y)\}$ (with the definition of a set difference). This proves the first part of the equivalence in (3.575).

To prove the second part ('$\Leftarrow$'), we now assume $z \in f \setminus \{(x, y)\}$ to be true and apply similar arguments as in the proof of the first part. Clearly then, $z \in f$ and $z \notin \{(x, y)\}$ both hold, and the latter implies $z \neq (x, y)$. Furthermore, $z \in f$ evidently implies that there exist elements, say $\bar{x}$ and $\bar{y}$, with $(\bar{x}, \bar{y}) = z$; thus, $(\bar{x}, \bar{y}) \in f$ as well as $(\bar{x}, \bar{y}) \neq (x, y)$. Let us now prove by contradiction that $\bar{x} \neq x$ is true. To do this, we assume $\neg \bar{x} \neq x$ to be true, which implies $\bar{x} = x$ with the Double Negation Law. With this equation, we obtain $(\bar{x}, \bar{y}) = (x, \bar{y})$, so that $(\bar{x}, \bar{y}) \in f$ implies $(x, \bar{y}) \in f$. Recalling that also $(x, y) \in f$, we have the true conjunction $(x, y) \in f \wedge (x, \bar{y}) \in f$, which implies $y = \bar{y}$ with the initial assumption that f is a function. This equation further implies $(x, \bar{y}) = (x, y)$; together with the equation $(\bar{x}, \bar{y}) = (x, \bar{y})$, this evidently yields $(\bar{x}, \bar{y}) = (x, y)$, which contradicts the previously established inequality $(\bar{x}, \bar{y}) \neq (x, y)$. This proves $\bar{x} \neq x$, so that $\bar{x} \notin \{x\}$. Moreover, the previously obtained $(\bar{x}, \bar{y}) \in f$ implies that $\bar{x}$ is an element of the domain X of f. Thus, the conjunction of $\bar{x} \in X$ and $\bar{x} \notin \{x\}$ holds, which means that $\bar{x} \in X \setminus \{x\}$. Then, the simultaneous truth of $(\bar{x}, \bar{y}) \in f$ and of $\bar{x} \in X \setminus \{x\}$ implies $(\bar{x}, \bar{y}) \in f \upharpoonright X \setminus \{x\}$ by definition of a restriction, which gives the desired $z \in f \upharpoonright X \setminus \{x\}$. This completes the proof of the equivalence in (3.575), and as z is arbitrary, we therefore conclude that the sets $f \upharpoonright X \setminus \{x\}$ and $f \setminus \{(x, y)\}$ are indeed identical. Since x, y, X, Y and f were also arbitrary, it follows that the proposition holds, as claimed. □

Proposition 3.169. *For any function $f : X \to Y$ and any elements $x_1, x_2 \in X$ it is true that the range of the restriction of f to the pair formed by x_1 and x_2 is identical with the pair formed by the corresponding function values $f(x_1)$ and $f(x_2)$, that is,*

$$\begin{aligned} &\forall X, Y, f, x_1, x_2\, ([f \in Y^X \wedge x_1, x_2 \in X] \\ &\quad \Rightarrow \operatorname{ran}(f \upharpoonright \{x_1, x_2\}) = \{f(x_1), f(x_2)\})). \end{aligned} \tag{3.576}$$

Proof. Letting X, Y, f, x_1 and x_2 be arbitrary, we assume $f \in Y^X$ and $x_1, x_2 \in X$ to be true, and we apply the Equality Criterion for sets to establish the stated equation. For this purpose, we verify

$$\forall y\,(y \in \operatorname{ran}(f \restriction \{x_1, x_2\}) \Leftrightarrow y \in \{f(x_1), f(x_2)\}). \tag{3.577}$$

We let y be arbitrary and we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $y \in \operatorname{ran}(f \restriction \{x_1, x_2\})$ to be true. Then, there exists by definition of a range a constant, say $\bar{x}$, with $(\bar{x}, y) \in f \restriction \{x_1, x_2\}$. Since the assumed $x_1, x_2 \in X$ implies $\{x_1, x_2\} \subseteq X$ with Proposition 2.52, we have that $f \restriction \{x_1, x_2\}$ is a function with domain $\{x_1, x_2\}$ and codomain Y, according to Proposition 3.164. Consequently, we obtain $\bar{x} \in \{x_1, x_2\}$ (by definition of a domain), and we may write $y = f \restriction \{x_1, x_2\}(\bar{x})$, which yields $y = f(\bar{x})$ with (3.567). Here, $\bar{x} \in \{x_1, x_2\}$ also implies $\bar{x} = x_1 \vee \bar{x} = x_2$ with the definition of a pair, which disjunction we now use to prove

$$y = f(x_1) \vee y = f(x_2) \tag{3.578}$$

by cases. On the one hand, if $\bar{x} = x_1$ holds, then the previous established $y = f(\bar{x})$ yields $y = f(x_1)$ via substitution, so that the disjunction (3.578) is true. On the other hand, if $\bar{x} = x_2$ holds, then $y = f(\bar{x})$ gives $y = f(x_2)$, and the disjunction (3.578) is true again. This completes the proof by cases, and we may now infer from the truth of the disjunction (3.578) the truth of $y \in \{f(x_1), f(x_2)\}$ because of the definition of a pair. Thus, the first part of the equivalence in (3.577) holds.

To prove the second part ('$\Leftarrow$'), we now assume $y \in \{f(x_1), f(x_2)\}$ to be true, so that the disjunction (3.578) holds (by definition of a pair). We now carry out a proof by cases based on this disjunction to establish the desired consequent. In the first case $y = f(x_1)$, we may write $(x_1, y) \in f$. Since $x_1 \in \{x_1, x_2\}$ holds according to (2.151), we then obtain $(x_1, y) \in f \restriction \{x_1, x_2\}$ with the definition of a restriction. Therefore, the definition of a range yields the desired $y \in \operatorname{ran}(f \restriction \{x_1, x_2\})$ in the first case. Similarly, in the second case $y = f(x_2)$, we may write $(x_2, y) \in f$, which implies with $x_2 \in \{x_1, x_2\}$ that $(x_2, y) \in f \restriction \{x_1, x_2\}$ holds. Consequently, $y \in \operatorname{ran}(f \restriction \{x_1, x_2\})$ follows to be true also in the second case. This completes the proof of the equivalence in (3.577), and since y was arbitrary, we may therefore conclude that the universal sentence (3.577) is true. Then, the equation in (3.576) follows to be true with the Equality Criterion for sets, which in turn proves the implication in (3.576). As X, Y, f, x_1 and x_2 were initially arbitrary, we may now finally conclude that the proposition holds. □

Corollary 3.170. *For any functions f and g it is true that the restrictions of f and g to the intersection of their domains are functions from* $\operatorname{dom}(f) \cap \operatorname{dom}(g)$ *to the union* $\operatorname{ran}(f) \cup \operatorname{ran}(g)$ *of their ranges.*

Proof. Letting f and g be arbitrary functions, we see on the one hand with (2.74) that the inclusions

$$\operatorname{dom}(f) \cap \operatorname{dom}(g) \subseteq \operatorname{dom}(f)$$
$$\operatorname{dom}(f) \cap \operatorname{dom}(g) \subseteq \operatorname{dom}(g)$$

are true. On the other hand, it follows with (2.201) that the inclusions

$$\operatorname{ran}(f) \subseteq \operatorname{ran}(f) \cup \operatorname{ran}(g)$$
$$\operatorname{ran}(g) \subseteq \operatorname{ran}(f) \cup \operatorname{ran}(g)$$

hold, so that we may take $\operatorname{ran}(f) \cup \operatorname{ran}(g)$ for the codomain of both f and g. It then follows with Proposition 3.164 that

$$f \upharpoonright (\operatorname{dom}(f) \cap \operatorname{dom}(g)) : \operatorname{dom}(f) \cap \operatorname{dom}(g) \to \operatorname{ran}(f) \cup \operatorname{ran}(g)$$
$$g \upharpoonright (\operatorname{dom}(f) \cap \operatorname{dom}(g)) : \operatorname{dom}(f) \cap \operatorname{dom}(g) \to \operatorname{ran}(f) \cup \operatorname{ran}(g)$$

This is then true for any functions f and g. □

The preceding result gives rise to the following definition.

Definition 3.45 (Compatible functions, compatible set of functions). We say that

(1) any two functions f and g are *compatible* iff their restrictions to the intersection of their domains are identical, i.e. iff

$$f \upharpoonright (\operatorname{dom}(f) \cap \operatorname{dom}(g)) = g \upharpoonright (\operatorname{dom}(f) \cap \operatorname{dom}(g)). \tag{3.579}$$

(2) a set $\mathcal{F}$ of functions is *compatible* iff any two functions f and g in $\mathcal{F}$ are compatible, i.e. iff

$$\forall f, g\,(f, g \in \mathcal{F} \Rightarrow f \text{ and } g \text{ are compatible}). \tag{3.580}$$

Note 3.20. In view of Theorem 3.158, the equation (3.579) means that the function values of the restrictions $f \upharpoonright (\operatorname{dom}(f) \cap \operatorname{dom}(g))$ and $g \upharpoonright (\operatorname{dom}(f) \cap \operatorname{dom}(g))$ at any $x \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$ are identical.

Corollary 3.171. *If two functions f and g are compatible, then g and f are also compatible.*

Proof. Letting f and g be arbitrary and assuming f and g to be compatible functions, we obtain the equations

$$\begin{aligned} g \upharpoonright (\operatorname{dom}(g) \cap \operatorname{dom}(f)) &= g \upharpoonright (\operatorname{dom}(f) \cap \operatorname{dom}(g)) \\ &= f \upharpoonright (\operatorname{dom}(f) \cap \operatorname{dom}(g)) \\ &= f \upharpoonright (\operatorname{dom}(g) \cap \operatorname{dom}(f)) \end{aligned}$$

by applying the Commutative Law for the intersection of two sets, the definition of compatible functions, and then again the Commutative Law for the intersection of two sets. It then follows from these equations that g and f are compatible, by definition. As f and g were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

We now consider a situation in which two given functions are automatically compatible.

Proposition 3.172. *If a function f is a subset of a function g, then f and g are compatible functions.*

Proof. We let f and g be arbitrary functions and assume $f \subseteq g$, i.e. (by definition of a subset)

$$\forall z\,(z \in f \Rightarrow z \in g). \tag{3.581}$$

Since the restrictions in (3.579) share the same domain $X = \mathrm{dom}(f) \cap \mathrm{dom}(g)$ according to Corollary 3.170, we may now apply Method 3.2 to verify the equality of these restrictions. Thus, we verify

$$\forall x\,(x \in X \Rightarrow f \restriction X(x) = g \restriction X(x). \tag{3.582}$$

To do this, we let x be arbitrary in X and define

$$y = f \restriction X(x), \tag{3.583}$$

so that $(x, y) \in f \restriction X$. It then follows by definition of a restriction that $(x, y) \in f$, which further implies $(x, y) \in g$ with (3.581). This and $x \in X$ now imply $(x, y) \in g \restriction X$ (again by definition of a restriction), and therefore

$$y = g \restriction X(x). \tag{3.584}$$

The two equations (3.583) and (3.584) show that the equation in (3.582) holds for an arbitrary $x \in X$, so that the universal sentence (3.582) is true. This proves (3.579), which means that f and g are compatible. As f and g were arbitrary, we therefore conclude that the proposition holds, as claimed. □

The fact that any set is a subset of itself according to Proposition 2.4 naturally translates into the observation that any function is compatible with itself.

Corollary 3.173. *For any function f it is true that f and f are compatible.*

Proposition 3.174. *Any pair of compatible functions is compatible.*

Proof. Letting $\bar{f}$ and $\bar{g}$ be arbitrary and assuming $\bar{f}$ and $\bar{g}$ to be compatible functions, we form the pair $\{\bar{f}, \bar{g}\}$ and verify

$$\forall f, g\,(f, g \in \{\bar{f}, \bar{g}\} \Rightarrow f \text{ and } g \text{ are compatible}). \tag{3.585}$$

We let f and g be arbitrary and assume $f \in \{\bar{f}, \bar{g}\}$ as well as $g \in \{\bar{f}, \bar{g}\}$ to be true. These assumptions imply with the definition of a pair

$$f = \bar{f} \vee f = \bar{g} \tag{3.586}$$

as well as

$$g = \bar{f} \vee g = \bar{g}. \tag{3.587}$$

In case the first part $f = \bar{f}$ of the first disjunction and then, on the one hand, also the first part $g = \bar{f}$ of the latter assumption are true, we obtain $f = g$ via substitution, so that f and g are compatible due to Corollary 3.173. On the other hand, if the second part $g = \bar{g}$ of the second disjunction holds (alongside $f = \bar{f}$), it again follows that f and g are compatible since we assumed $\bar{f}$ and $\bar{g}$ to be compatible.

In case the second part $f = \bar{g}$ of the disjunction (3.586) and then, on the one hand, also the first part $g = \bar{f}$ of the disjunction (3.587) hold, the assumed compatibility of $\bar{f}$ and $\bar{g}$ implies the compatibility of g and f. Therefore, f and g follow again to be compatible with Corollary 3.171. If, on the other hand, $g = \bar{g}$ holds, we obtain from the case assumption $f = \bar{g}$ via substitution $f = g$, with the already demonstrated consequence that f and g are compatible.

We may infer from these findings that the implication in (3.585) is true, and since f and g are arbitrary, we may further conclude that the universal sentence (3.585) holds. Thus, the pair $\{\bar{f}, \bar{g}\}$ is compatible by definition, and as $\bar{f}$ and $\bar{g}$ were arbitrary, we may finally conclude that the proposition is true. □

Exercise 3.73. Show that, if the domains of two functions f and g are disjoint, then f and g are compatible functions.

(Hint: Apply (2.43) in connection with Corollary 3.170 and Proposition 3.164, and then Proposition 3.151.)

Theorem 3.175 (Concatenation of functions). *The following sentences are true for any compatible set $\mathcal{F}$ of functions.*

a) $\bigcup \mathcal{F}$ *is a function.*

b) Furthermore, there exists a unique set (system) $\mathcal{D}$ consisting of the domains of all the functions in $\mathcal{F}$ in the sense that

$$\forall D\,(D \in \mathcal{D} \Leftrightarrow \exists f\,(f \in \mathcal{F} \wedge \operatorname{dom}(f) = D)), \tag{3.588}$$

and the union of this set system is the domain of the union of $\mathcal{F}$, i.e.

$$\mathrm{dom}(\bigcup \mathcal{F}) = \bigcup \mathcal{D}. \tag{3.589}$$

Proof. We let $\mathcal{F}$ be an arbitrary set and assume that $\mathcal{F}$ is a compatible set of functions.

Concerning a), let us first observe that every element of $\bigcup \mathcal{F}$ is element of some element of $\mathcal{F}$ (by definition of the union of a set system). Furthermore, every element of $\mathcal{F}$ is a function and therefore a set of ordered pairs (by definition of a function). Thus, any element of $\bigcup \mathcal{F}$ is an ordered pair, so that $\bigcup \mathcal{F}$ is a binary relation and consequently satisfies Property 1 of a function. Regarding Property 2, we prove

$$\forall x, y, y' \, ([(x,y) \in \bigcup \mathcal{F} \wedge (x,y') \in \bigcup \mathcal{F}] \Rightarrow y = y'). \tag{3.590}$$

For this purpose, we let x, y and y' be arbitrary, assume $(x,y) \in \bigcup \mathcal{F}$ as well as $(x,y') \in \bigcup \mathcal{F}$, and demonstrate that $y = y'$ holds. With our initial observations, we have that (x,y) is an element of some element f of $\mathcal{F}$, say $\bar{f}$, and we have that (x,y') is an element of some element g of $\mathcal{F}$, say $\bar{g}$. Then, on the one hand, as there exists an y with $(x,y) \in \bar{f}$, we have $x \in \mathrm{dom}(\bar{f})$ (by definition of a domain); on the other hand, as there is an y' with $(x,y') \in \bar{g}$, we have $x \in \mathrm{dom}(\bar{g})$, so that

$$x \in \mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}) \tag{3.591}$$

holds by definition of the intersection of pair. Then, as elements of $\mathcal{F}$, the functions $\bar{f}$ and $\bar{g}$ are compatible, so that

$$\bar{f} \upharpoonright (\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}))(x) = \bar{g} \upharpoonright (\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}))(x) \tag{3.592}$$

holds in view of Note 3.20. Now, $(x,y) \in \bar{f}$ and (3.591) imply

$$(x,y) \in \bar{f} \upharpoonright (\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}))$$

by definition of restriction; similarly, $(x,y') \in \bar{g}$ and (3.591) imply

$$(x,y') \in \bar{g} \upharpoonright (\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g})).$$

Thus, the two equations

$$\bar{f} \upharpoonright (\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}))(x) = y$$
$$\bar{g} \upharpoonright (\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}))(x) = y'$$

are true, so that (3.592) is equivalent to the desired $y = y'$. Since x, y and y' are arbitrary, we conclude that the binary relation $\bigcup \mathcal{F}$ is a function.

Concerning b), we may apply the Axiom of Specification and the Equality Criterion for sets to obtain the uniquely existential sentence

$$\exists! \mathcal{D} \forall D \, (D \in \mathcal{D} \Leftrightarrow [D \in \mathcal{P}(\bigcup\bigcup\bigcup \mathcal{F}) \wedge \exists f \, (f \in \mathcal{F} \wedge \mathrm{dom}(f) = D)]),$$

so that the set $\mathcal{D}$ satisfies

$$\forall D \, (D \in \mathcal{D} \Leftrightarrow [D \in \mathcal{P}(\bigcup\bigcup\bigcup \mathcal{F}) \wedge \exists f \, (f \in \mathcal{F} \wedge \mathrm{dom}(f) = D)]). \quad (3.593)$$

To show that $\mathcal{D}$ satisfies then also (3.588), we let D be arbitrary, and we prove the first part ('$\Rightarrow$') of the equivalence directly. The assumption $D \in \mathcal{D}$ implies then with (3.593) in particular the existential sentence in (3.588), so that the first part of the equivalence holds. To prove the second part ('$\Leftarrow$'), we now assume that existential sentence, i.e. we assume that there is an element of $\mathcal{F}$, say $\bar{f}$, with $\mathrm{dom}(\bar{f}) = D$. We now show that this implies the truth of $D \in \mathcal{P}(\bigcup\bigcup\bigcup \mathcal{F})$, by verifying first $D \subseteq \bigcup\bigcup\bigcup \mathcal{F}$. To do this, we apply the definition of a subset and prove the equivalent

$$\forall x \, (x \in D \Rightarrow x \in \bigcup\bigcup\bigcup \mathcal{F}). \quad (3.594)$$

Letting $\bar{x}$ be arbitrary and assuming $\bar{x} \in D$, so that $\bar{x} \in \mathrm{dom}(\bar{f})$, it follows with the definition of a domain that there exists an element, say $\bar{y}$, such that $(\bar{x}, \bar{y}) \in \bar{f}$ holds. Recalling $\bar{f} \in \mathcal{F}$, we thus see that there is an element $f \in \mathcal{F}$ with $(\bar{x}, \bar{y}) \in f$, so that

$$(\bar{x}, \bar{y}) \in \bigcup \mathcal{F} \quad (3.595)$$

follows to be true by definition of the union of a set system. According to the definition of an ordered pair, we have $(\bar{x}, \bar{y}) = \{\{\bar{x}\}, \{\bar{x}, \bar{y}\}\}$, and therefore $\{\bar{x}\} \in (\bar{x}, \bar{y})$ in view of (2.151). Together with (3.595), this shows that there exists a set $A \in \bigcup \mathcal{F}$ with $\{\bar{x}\} \in A$, so that we obtain (using again the definition of the union of a set system)

$$\{\bar{x}\} \in \bigcup\bigcup \mathcal{F}. \quad (3.596)$$

Since $\bar{x} \in \{\bar{x}\}$ is evidently also true, we see moreover that there is a set $A' \in \bigcup\bigcup \mathcal{F}$ with $\bar{x} \in A'$, and therefore we arrive at

$$\bar{x} \in \bigcup\bigcup\bigcup \mathcal{F}, \quad (3.597)$$

applying once more the definition of the union of a set system. We thus proved the implication in (3.594), and as $\bar{x}$ was arbitrary, we may infer from this the truth of (3.594). This universal sentence then gives (with the definition of a subset) $D \subseteq \bigcup\bigcup\bigcup\mathcal{F}$, which inclusion in turn yields $D \in \mathcal{P}(\bigcup\bigcup\bigcup\mathcal{F})$ (by definition of a power set). Together with the assumed existential sentence in (3.588), this further implies $D \in \mathcal{D}$ with (3.593), so that the proof of the implication ('$\Leftarrow$') in (3.588) is complete. Therefore, the equivalence in (3.588) is satisfied by D, and since D is arbitrary, we may now conclude that the universal sentence (3.588) is indeed true for $\mathcal{D}$.

Next, to prove the equation (3.589), we show that the equivalent

$$\forall x\,(x \in \mathrm{dom}(\bigcup\mathcal{F}) \Leftrightarrow x \in \bigcup\mathcal{D}) \tag{3.598}$$

holds (according to (2.18)). Letting x be arbitrary, we first assume $x \in \mathrm{dom}(\bigcup\mathcal{F})$. Then, there exists an element, say $\bar{y}$, with $(x,\bar{y}) \in \bigcup\mathcal{F}$. As noted before, the ordered pair $(x,\bar{y})$ is thus an element of some function f in $\mathcal{F}$, say of $\bar{f}$. Consequently, x is an element of the domain of $\bar{f}$. Because of $\bar{f} \in \mathcal{F}$ there exists an element of $\mathcal{D}$, say $\bar{D}$, with $\mathrm{dom}(\bar{f}) = \bar{D}$, so that $x \in \mathrm{dom}(\bar{f})$ implies $x \in \bar{D}$, Then, $\bar{D} \in \mathcal{D}$ and $x \in \bar{D}$ imply $x \in \bigcup\mathcal{D}$ (by definition of the union of a set system). This proves the first part of the equivalence in (3.598).

To prove the second part of that equivalence, we now assume $x \in \bigcup\mathcal{D}$, so that there exists an element of $\mathcal{D}$, say $\bar{D}$, with $x \in \bar{D}$. Since $\bar{D}$ is the domain of some function f in $\mathcal{F}$, say of $\bar{f}$, there exists an element, say $\bar{y}$, with $(x,\bar{y}) \in \bar{f}$. This and $\bar{f} \in \mathcal{F}$ imply $(x,\bar{y}) \in \bigcup\mathcal{F}$, so that x is an element of $\mathrm{dom}(\bigcup\mathcal{F})$.

This completes the proof of the equivalence, and since x is arbitrary, we conclude that (3.598) is true. Thus, the proposed equation (3.589) holds. As $\mathcal{F}$ was arbitrary, we finally conclude that the theorem holds, as claimed. □

Proposition 3.176. *It is true for any compatible functions $\bar{g}$ and $\bar{h}$ that their union constitutes a function with domain* $\mathrm{dom}(\bar{g}) \cup \mathrm{dom}(\bar{g})$.

Proof. We let $\bar{g}$ and $\bar{h}$ be arbitrary sets and assume these to be compatible functions, so that the pair $\{\bar{g},\bar{h}\}$ is a compatible set of functions by virtue of Proposition 3.174. This allows us to apply Concatenation of functions to define the new function $\bigcup\{\bar{g},\bar{h}\} = \bar{g} \cup \bar{h}$, and there exists a unique set $\mathcal{D}$ satisfying

$$\forall D\,(D \in \mathcal{D} \Leftrightarrow \exists f\,(f \in \{\bar{g},\bar{h}\} \,\wedge\, \mathrm{dom}(f) = D)) \tag{3.599}$$

and

$$\mathrm{dom}(\bar{g} \cup \bar{h}) = \bigcup\mathcal{D}. \tag{3.600}$$

We now prove by means of the Equality Criterion for sets that

$$\mathcal{D} = \{\mathrm{dom}(\bar{g}), \mathrm{dom}(\bar{g})\} \tag{3.601}$$

holds, by verifying accordingly the universal sentence

$$\forall D\,(D \in \mathcal{D} \Leftrightarrow D \in \{\mathrm{dom}(\bar{g}), \mathrm{dom}(\bar{g})\}). \tag{3.602}$$

We let D be an arbitrary set. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $D \in \mathcal{D}$ to be true, which implies with (3.599) that there exists a set, say $\bar{f}$, for which $\bar{f} \in \{\bar{g}, \bar{h}\}$ and $\mathrm{dom}(\bar{f}) = D$ are true. By definition of a pair, the disjunction $\bar{f} = \bar{g} \lor \bar{f} = \bar{h}$ holds then, too, which we now use to prove $D \in \{\mathrm{dom}(\bar{g}), \mathrm{dom}(\bar{g})\}$ by cases. Observing the truth of $\bar{g}, \bar{h} \in \{\bar{g}, \bar{h}\}$ in light of (2.151), we thus see that both the first case $\bar{f} = \bar{g}$ and the second case $\bar{f} = \bar{h}$ imply the desired consequent via substitution.

Concerning the second part ('$\Leftarrow$') of the equivalence in (3.602), we assume $D \in \{\mathrm{dom}(\bar{g}), \mathrm{dom}(\bar{g})\}$, so that the definition of a pair yields the true disjunction $D = \mathrm{dom}(\bar{g}) \lor D = \mathrm{dom}(\bar{h})$, so that we can prove $D \in \mathcal{D}$ by cases. Indeed, noting the truth of $\bar{g}, \bar{h} \in \{\bar{g}, \bar{h}\}$ in view of the previously mentioned (2.151), both the first case $D = \mathrm{dom}(\bar{g})$ and the second case $D = \mathrm{dom}(\bar{h})$ show that there exists an element $f \in \{\bar{g}, \bar{h}\}$ with $\mathrm{dom}(f) = D$, so that (3.599) gives the desired consequent in any case.

We thus completed the proof of the equivalence in (3.602), and as D was arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (3.602), consequently also the truth of (3.601). Combining this equation with the equation (3.600), we therefore obtain

$$\mathrm{dom}(\bar{g} \cup \bar{h}) = \bigcup\{\mathrm{dom}(\bar{g}), \mathrm{dom}(\bar{g})\} = \mathrm{dom}(\bar{g}) \cup \mathrm{dom}(\bar{g}). \tag{3.603}$$

Here, $\bar{g}$ and $\bar{h}$ were initially arbitrary sets, so that the proposed universal sentence follows to be true. □

Proposition 3.177. *For any set X, any function f with domain X and any ordered pair (x, y) with $x \notin X$ it is true that $f \cup \{(x, y)\}$ is a function with domain $X \cup \{x\}$.*

Proof. We let X, f, x and y be arbitrary, assume that f is a function with $\mathrm{dom}(f) = X$, and assume moreover that $x \notin X$ holds. The latter assumption implies with (2.173) that X and $\{x\}$ are disjoint sets. Furthermore, as the singleton $\{(x, y)\}$ is a function with domain $\{x\}$ according to Corollary 3.156, we have that f and $\{(x, y)\}$ are functions with disjoint domains. Consequently, f and $\{(x, y)\}$ are compatible functions according to Exercise 3.73. It now follows with Proposition 3.176 that the union $f \cup \{(x, y)\}$ is a function with domain $X \cup \{x\}$. As X, f, x and y were arbitrary, we may therefore conclude that the proposed universal sentence holds. □

We now study compositions of functions.

Proposition 3.178. *For any functions $f : X \to Y$ and $g : Y \to Z$ the composition of g and f is a function from X to Z, that is,*

$$\forall f, g\,([f \in Y^X \wedge g \in Z^Y] \Rightarrow g \circ f \in Z^X). \tag{3.604}$$

Proof. We let X, Y, f and g be arbitrary sets and assume $f : X \to Y$ as well as $g : Y \to Z$ to be true. Since the functions f and g are then by definition binary relations where the domain of f equals X, we obtain $\mathrm{dom}(g \circ f) = X$ with (3.130). Next, we prove that the binary relation $g \circ f$ is a function, by verifying

$$\forall x, z, z'\,([(x, z) \in g \circ f \wedge (x, z') \in g \circ f] \Rightarrow z = z'). \tag{3.605}$$

For this purpose, we let x, z and z' be arbitrary and assume $(x, z) \in g \circ f$ as well as $(x, z') \in g \circ f$. By definition of a composition, $(x, z) \in g \circ f$ implies that there exist elements, say $\bar{x}$, $\bar{y}$ and $\bar{z}$ with $(\bar{x}, \bar{y}) \in f$, $(\bar{y}, \bar{z}) \in g$ and $(\bar{x}, \bar{z}) = (x, z)$; the latter equation implies $\bar{x} = x$ and $\bar{z} = z$ with the Equality Criterion for ordered pairs, so that substitution yields $(x, \bar{y}) \in f$ and $(\bar{y}, z) \in g$. Similarly, $(x, z') \in g \circ f$ implies that there exist elements, say $\bar{a}$, $\bar{b}$ and $\bar{c}$ with $(\bar{a}, \bar{b}) \in f$, $(\bar{b}, \bar{c}) \in g$ and $(\bar{a}, \bar{c}) = (x, z')$; the latter equation gives (again with the Equality Criterion for ordered pairs) $\bar{a} = x$ and $\bar{c} = z'$, which equations yield $(x, \bar{b}) \in f$ and $(\bar{b}, z') \in g$. Now, since f is a function, the conjunction of the previously found $(x, \bar{y}) \in f$ and $(x, \bar{b}) \in f$ implies $\bar{y} = \bar{b}$. With this equation, $(\bar{b}, z') \in g$ becomes $(\bar{y}, z') \in g$; together with the previously obtained $(\bar{y}, z) \in g$, this implies $z = z'$, because g is also a function. Since x, z and z' are arbitrary, we may therefore conclude that (3.605) holds. Thus, the binary relation $g \circ f$ satisfies also Property 2 of a function.

Furthermore, it follows with (3.133) that $\mathrm{ran}(g \circ f) \subseteq \mathrm{ran}(g)$ holds. Observing now that the assumption $g : Y \to Z$ implies $\mathrm{ran}(g) \subseteq Z$ by definition of a codomain, we obtain from these two inclusions $\mathrm{ran}(g \circ f) \subseteq Z$ with the transitivity of $\subseteq$. Thus, Z is a codomain of $g \circ f$, which completes the proof that $g \circ f$ is a function from X to Z. □

Notation 3.6. When f and g are functions, we write

$$x \mapsto (g \circ f)(x) \quad \text{or} \quad x \mapsto g(f(x)). \tag{3.606}$$

The latter is justified since every $x \in \mathrm{dom}(f)$ is mapped by f into a unique $y \in \mathrm{dom}(g)$ due to (3.604), which y in turn is mapped by g into a unique $z \in \mathrm{ran}(g)$, which z coincides with $(g \circ f)(x)$ by definition of a composition.

Certain functions do not alter a given function when composed with. For the following definition, we apply (3.604) to a transformation $t : X \to X$ and a function $f : X \to Y$.

Definition 3.46 (Invariant (function)). For any sets X and Y, for any function $f : X \to Y$,

(1) and for any transformation t on X, we say that f is (*an*) *invariant under* t iff the composition of f and t is identical with f, i.e., iff

$$f \circ t = f. \tag{3.607}$$

(2) and for any subset $\mathcal{T}$ of the set X^X of transformations on X, we say that f is (*an*) *invariant under* $\mathcal{T}$ iff f is invariant under any of the transformations in G, i.e., iff

$$\forall t\, (t \in \mathcal{T} \Rightarrow f \circ t = f). \tag{3.608}$$

Note 3.21. In view of the Equality Criterion for functions and Notation 3.6, the definite property (3.607) of an invariant function (under a transformation t) can also be stated as

$$\forall x\, (x \in X \Rightarrow f(t(x)) = f(x)). \tag{3.609}$$

Corollary 3.179. *It is true for any sets X and Y that the empty function $\emptyset : \emptyset \to Y$ is invariant under any transformation $t : X \to X$ and under any subset $\mathcal{T}$ of the set X^X of transformations on X.*

Proof. Letting X and Y be arbitrary sets and $t : X \to X$ an arbitrary transformation on X, we observe in light of Corollary 3.39 that $\emptyset \circ t = \emptyset$ holds. Since t is arbitrary we may therefore conclude that $\emptyset : \emptyset \to Y$ is invariant under any $t : X \to X$. Letting now $\mathcal{T}$ be an arbitrary subset of X^X and t an arbitrary element of $\mathcal{T}$, we find $t \in X^X$ with the definition of a subset, so that $t : X \to X$. Thus, the previously established equation $\emptyset \circ t = \emptyset$ applies, and as t was arbitrary, we may therefore conclude that $\emptyset : \emptyset \to Y$ is invariant under $\mathcal{T}$. Since X and Y are also arbitrary, we may then further conclude that the stated universal sentence holds. □

Theorem 3.180 (Neutrality of identity functions under composition). *Any function f from a set X to a set Y is identical both with the composition of f and the identity function on X and with the composition of the identity function on Y and f, that is,*

$$\forall X, Y, f\, (f : X \to Y \Rightarrow [f \circ \mathrm{id}_X = f \wedge \mathrm{id}_Y \circ f = f]). \tag{3.610}$$

Exercise 3.74. Establish the neutrality of identity functions under composition.

(Hint: Apply Method 3.2 with Notation 3.6 and Definition 3.43.)

Note 3.22. The Neutrality of identity functions under composition and the fact that the identity function id_X is a function from X to X (see Theorem 3.161) and thus a transformation on X imply in particular that every function $f : X \to Y$ is invariant under id_X.

Exercise 3.75. Show for any set X, any constant c and any subset $\mathcal{T}$ of the set X^X of transformations on X that the constant function g_c on X with value c is an invariant under $\mathcal{T}$.

(Hint: Apply the Equality Criterion for functions in connection with Notation 3.6 and (3.534).)

A restriction of a function can be expressed as the composition of the function and the inclusion function on the restricting set.

Proposition 3.181. *For any function $f : X \to Y$ and any subset A of X, the restriction of f to A is the composition of f and the inclusion function j from A to X, that is,*

$$\forall X, Y, f, A\,([f : X \to Y \,\wedge\, A \subseteq X] \Rightarrow f \restriction A = f \circ j). \tag{3.611}$$

Proof. We let X and Y be arbitrary sets, f an arbitrary function from X to Y, A an arbitrary subset of X, and we let $j : A \to X$ be the inclusion function from the subset A to X. Then, the restriction $f \restriction A$ is a function with domain A due to Proposition 3.164, and the composition $f \circ j$ is a function also with domain A according to Proposition 3.178. Thus, the function $f \restriction A$ and $f \circ j$ share the same domain A, so that we may apply Method 3.2 based on the Equality Criterion for functions to establish the equation $f \restriction A = f \circ j$. To do this, we verify

$$\forall x\,(x \in A \Rightarrow (f \restriction A)(x) = (f \circ j)(x)), \tag{3.612}$$

letting x be arbitrary and assuming $x \in A$ to be true. We then obtain the true equations

$$(f \circ j)(x) = f(j(x)) = f(x) = (f \restriction A)(x)$$

with Notation 3.6, the definition of an inclusion function, and (3.567). Therefore, the equation $(f \restriction A)(x) = (f \circ j)(x)$ follows to be true, and since x is arbitrary, we may therefore conclude that the universal sentence (3.612) holds. Since X, Y, f and A were initially arbitrary we may then further conclude that (3.611) holds, as proposed. □

Theorem 3.182 (Associative Law for function composition). *The composition of functions is associative in the sense that*

$$\forall f, g, h\,([f \in Y^X \,\wedge\, g \in Z^Y \,\wedge\, h \in A^Z] \Rightarrow [h \circ g] \circ f = h \circ [g \circ f]). \tag{3.613}$$

Proof. We let X, Y, Z, A, f and g be arbitrary sets and assume that $f : X \to Y$, $g : Y \to Z$ and $h : Z \to A$. Let us first observe in light of Proposition 3.178 that

$$\begin{aligned} h \circ g &\in A^Y, \\ (h \circ g) \circ f &\in A^X, \\ g \circ f &\in Z^X, \\ h \circ (g \circ f) &\in A^X. \end{aligned}$$

Thus, $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are both functions with the same domain X and the same codomain A. We now verify by means of Method 3.2 that the functions themselves are identical. Letting x be an arbitrary element of the domain X, we observe in light of Notation 3.6 that

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = h(g(f(x))) = h((g \circ f)(x)) \\ &= (h \circ (g \circ f))(x). \end{aligned}$$

We thus conclude that the proposed equation (3.613) holds. As X, Y, Z, A, f, g and h were arbitrary, we conclude that the theorem is true. □

Proposition 3.183. *It is true for any sets X, Y and Z, any subset $\mathcal{T}$ of the set X^X of transformations of X, any invariant $f : X \to Y$ under $\mathcal{T}$ and any function $h : Y \to Z$ that the composition $h \circ f$ is also an invariant under $\mathcal{T}$.*

Proof. We take arbitrary sets X and Y, an arbitrary subset $\mathcal{T} \subseteq X^X$, an arbitrary invariant $f : X \to Y$ under $\mathcal{T}$, an arbitrary function $h : Y \to Z$, and an arbitrary element $t \in \mathcal{T}$. The latter implies with the definition of a subset that $t \in X^X$, so that $t : X \to X$. We then obtain the compositions $h \circ f : X \to Z$ and $(h \circ f) \circ t : X \to Z$ with Proposition 3.178. Moreover, we obtain the equations

$$(h \circ f) \circ t = h \circ (f \circ t) = h \circ f$$

by using the Associative Law for function composition and the assumption that f is invariant under $\mathcal{T}$, so that $f \circ t = f$ is true. Since t was arbitrary, we may therefore conclude that $h \circ f$ is indeed an invariant under $\mathcal{T}$, by definition. As X, Y, Z, $\mathcal{T}$, f and h were initially also arbitrary, we may further conclude that the stated universal sentence is true. □

3.4.3. Injections, surjections, and bijections

Definition 3.47 (Injection, one-to-one function). We say that a function $f : X \to Y$ is *one-to-one* or an *injection*, symbolically

$$f : X \hookrightarrow Y, \tag{3.614}$$

iff the identity of function values implies identity of arguments, i.e. iff

$$\forall x, x' \,([x, x' \in X \wedge f(x) = f(x')] \Rightarrow x = x'). \tag{3.615}$$

Corollary 3.184. *The singleton formed by a pair (x, y) is an injection from the singleton formed by x to the singleton formed by y, that is,*

$$\forall x, y\,(\{(x, y)\} : \{x\} \hookrightarrow \{y\}). \tag{3.616}$$

Proof. We let $\bar{x}$ and $\bar{y}$ be arbitrary and recall from Corollary 3.156 that the singleton $\{(\bar{x}, \bar{y})\}$ is a function from $\{\bar{x}\}$ to $\{\bar{y}\}$. We now verify that $f = \{(\bar{x}, \bar{y})\}$ satisfies

$$\forall x, x' \,([x, x' \in \{\bar{x}\} \wedge f(x) = f(x')] \Rightarrow x = x'). \tag{3.617}$$

We let x and x' be arbitrary, assume $x, x' \in \{\bar{x}\}$ as well as $f(x) = f(x')$ to be true, and observe that the former assumption implies the equations $x = \bar{x}$ and $x' = \bar{x}$ with (2.169), so that substitution gives already the desired consequent $x = x'$. As x and x' are arbitrary, we may therefore conclude that (3.617) holds, so that $\{(\bar{x}, \bar{y})\} : \{\bar{x}\} \hookrightarrow \{\bar{y}\}$ follows to be true by definition of an injection. Since $\bar{x}$ and $\bar{y}$ were also arbitrary, we may infer from this the truth of the proposed universal sentence. $\square$

Exercise 3.76. Show for any set Y that the function $\emptyset : \emptyset \to Y$ is an injection, i.e.

$$\emptyset : \emptyset \hookrightarrow Y, \tag{3.618}$$

and show that this is the only injection from $\emptyset$ to Y.

Theorem 3.185 (Injection Criterion). *For any set X it is true that a function f with domain X is an injection iff distinct elements of the domain X are associated with distinct function values, i.e. iff*

$$\forall x, x' \,([x, x' \in X \wedge x \neq x'] \Rightarrow f(x) \neq f(x')). \tag{3.619}$$

Exercise 3.77. Prove the Injection Criterion and show furthermore for any set X that a function f with domain X is not an injection iff there are distinct elements of the domain which are mapped to the same value, i.e. iff

$$\exists x, x' \,(x, x' \in X \wedge x \neq x' \wedge f(x) = f(x')). \tag{3.620}$$

(Hint: Concerning the proof of the Injection Criterion, apply (1.49) and (1.19).)

Combining the Injection Criterion with the fact that two distinct function values cannot have identical arguments for any kind of function according to Proposition 3.149, we immediately obtain the following result.

Corollary 3.186. *For any set X and any injection f with domain X it is true that two elements of the domain are different iff the corresponding function values are different, i.e.*

$$\forall x, x' \, (x, x' \in X \Rightarrow [x \neq x' \Leftrightarrow f(x) \neq f(x')]). \tag{3.621}$$

Note 3.23. As the codomain Y does not occur in the definite property (3.615), we see that whether a given function is one-to-one or not is independent of the choice for the codomain.

The following exercise may clarify this fact.

Exercise 3.78. Verify the following sentences for any injection $f : X \to Y$.

a) f is also a one-to-one function from X to its range, that is,

$$\forall X, Y, f \, (f : X \hookrightarrow Y \;\Rightarrow\; f : X \hookrightarrow \mathrm{ran}(f)). \tag{3.622}$$

(Hint: Use (3.518).)

b) If Y is subset of a set Z, then f is also a one-to-one function from X to Z, that is,

$$\forall X, Y, f, Z \, ([f : X \hookrightarrow Y \wedge Y \subseteq Z] \Rightarrow f : X \hookrightarrow Z). \tag{3.623}$$

(Hint: Use (3.519).)

Proposition 3.187. *The restriction of an injection to a subset of the domain is also an injection, that is,*

$$\forall X, Y, f, A \, ([f : X \hookrightarrow Y \wedge A \subseteq X] \Rightarrow f \restriction A : A \hookrightarrow Y). \tag{3.624}$$

Proof. We let $f : X \hookrightarrow Y$ be an arbitrary injection and A an arbitrary subset of X. We first note that $f \restriction A$ is a function with domain A and codomain Y due to Proposition 3.164. To verify that $f \restriction A$ satisfies (3.615), we let x, x' be arbitrary, assume that $x, x' \in A$ and $f \restriction A(x) = f \restriction A(x')$ are true, and show that this implies $x = x'$. Let us define $y = f \restriction A(x)$ $(= f \restriction A(x'))$. Now, $x, x' \in A$ implies $x, x' \in X$ due to $A \subseteq X$ and the definition of a subset. Therefore, the fact $(x, y), (x', y) \in f \restriction A$ implies $(x, y), (x', y) \in f$ (by definition of a restriction), which means that $f(x) = f(x')$. As an injection, f satisfies (3.615), so that the conjunction of $x, x' \in X$ and $f(x) = f(x')$ implies the desired $x = x'$. We thus proved that $f \restriction A$ is an injection (from A to Y). As f and A were arbitrary, we conclude that the proposition is true. $\square$

Proposition 3.188. *For any subset X of a set Y the inclusion function from X to Y is one-to-one, that is,*

$$\forall X, Y\,(X \subseteq Y \Rightarrow j : X \hookrightarrow Y). \tag{3.625}$$

Proof. We let Y be an arbitrary set, X an arbitrary subset of Y, and verify (3.615) for $f = j$. For this purpose, we let x, x' be arbitrary and assume that $x, x' \in X$ and $j(x) = j(x')$ are true. This implies $(x, y), (x', y) \in j$ with $y = j(x) = j(x')$. By definition of the inclusion function, $x = y$ and $x' = y$ are then true, so that we obtain the desired $x = x'$ with substitution. Since x and x' are arbitrary, we conclude that the inclusion function j is one-to-one. □

Note 3.18 applies immediately to the preceding finding.

Corollary 3.189. *The identity function on any set X is one-to-one, that is,*

$$\forall X\,(\mathrm{id}_X : X \hookrightarrow X). \tag{3.626}$$

Theorem 3.190 (Injectivity of the composition of two injections). *It is true that the composition of an injection g from a set Y to a set Z and an injection f from a set X to Y is itself an injection from X to Z, i.e.*

$$\forall X, Y, Z, f, g\,([f : X \hookrightarrow Y \,\wedge\, g : Y \hookrightarrow Z] \Rightarrow g \circ f : X \hookrightarrow Z). \tag{3.627}$$

Proof. We let X, Y and Z be arbitrary sets, f an arbitrary injection from X to Y, and g an arbitrary injection from Y to Z. Let us now observe in light of (3.604) that the composition $g \circ f$ is a function from X to Z. To prove that $g \circ f$ is an injection, we show according to (3.615) that

$$\forall x, x'\,([x, x' \in X \,\wedge\, (g \circ f)(x) = (g \circ f)(x')] \Rightarrow x = x') \tag{3.628}$$

holds. For this purpose, let $\bar{x}$ and $\bar{x}'$ be arbitrary elements of X such that $(g \circ f)(\bar{x}) = (g \circ f)(\bar{x}')$ is true, which equation we may write also as $g(f(\bar{x})) = g(f(\bar{x}'))$. Let us now use the denotations $\bar{y} = f(\bar{x})$ and $\bar{y}' = f(\bar{x}')$, so that the preceding equations give $g(\bar{y}) = g(\bar{y}')$ via substitution. As we may write the equations $\bar{y} = f(\bar{x})$ and $\bar{y}' = f(\bar{x}')$ also as $(\bar{x}, \bar{y}) \in f$ and $(\bar{x}', \bar{y}') \in f$, we see on the one hand that there exists an x with $(x, \bar{y}) \in f$, so that $\bar{y} \in \mathrm{ran}(f)$ follows to be true by definition of a range. On the other hand, we see that there is an x' with $(x', \bar{y}') \in f$, and therefore $\bar{y}' \in \mathrm{ran}(f)$ also holds (by definition of a range). As f was assumed to be a function with codomain Y, we have that $\mathrm{ran}(f) \subseteq Y$ is true. Consequently, the previously established $\bar{y}, \bar{y}' \in \mathrm{ran}(f)$ imply with the definition of a subset $\bar{y}, \bar{y}' \in Y$. Together with the previously obtained $g(\bar{y}) = g(\bar{y}')$, this implies then $\bar{y} = \bar{y}'$, because g was assumed to be an injection with domain Y. Then, substitution based on this equation yields

$$f(\bar{x}) = \bar{y} = \bar{y}' = f(\bar{x}')$$

and therefore $f(\bar{x}) = f(\bar{x}')$. Then, in view of $\bar{x}, \bar{x}' \in X$ and the assumption that f is an injection with domain X, this equation implies $\bar{x} = \bar{x}'$. Since $\bar{x}$ and $\bar{x}'$ are arbitrary, we may therefore conclude that (3.628) holds, which shows that the function $g \circ f : X \to Z$ is indeed an injection. As X, Y, Z, f and g were also arbitrary, we may then further conclude that the proposed universal sentence (3.627) is true. □

Another type of function is characterized by the property that the function values fill up the entire codomain.

Definition 3.48 (Surjection, onto function). We say that a function $f : X \to Y$ is a *surjection* from X to Y or a function from X *onto* Y, symbolically

$$f : X \twoheadrightarrow Y, \tag{3.629}$$

iff the range of f is identical with Y, that is, iff

$$\mathrm{ran}(f) = Y. \tag{3.630}$$

Corollary 3.191. *A function f is onto a set Y iff any element y is in Y precisely when y is function value at some x, that is,*

$$\forall Y, f\,(f \text{ is a function} \Rightarrow [f : \mathrm{dom}(f) \twoheadrightarrow Y \;\Leftrightarrow \forall y\,(y \in Y \Leftrightarrow \exists x\,(f(x) = y))]). \tag{3.631}$$

Proof. Letting Y and f be arbitrary and assuming f to be a function, we notice the truth of the equivalence

$$Y = \mathrm{ran}(f) \;\Leftrightarrow\; \forall y\,(y \in Y \Leftrightarrow \exists x\,(f(x) = y)) \tag{3.632}$$

in light of the definition of a range, applying Notation 3.4. With this, the equivalence in (3.631) follows to be true. Since Y and f are arbitrary, we may therefore conclude that the proposed universal sentence (3.631) holds. □

Exercise 3.79. Show that the function $\emptyset : \emptyset \to \emptyset$ is a surjection, i.e.

$$\emptyset : \emptyset \twoheadrightarrow \emptyset, \tag{3.633}$$

and show that this is the only surjection from $\emptyset$ to $\emptyset$.
(Hint: Apply Proposition 3.118.)

Theorem 3.192 (Surjection Criterion). *A function f with codomain Y is onto Y iff any element y in Y is function value at some x, that is,*

$$\forall Y, f\,(f : \mathrm{dom}(f) \to Y \\ \Rightarrow\ [f : \mathrm{dom}(f) \twoheadrightarrow Y \ \Leftrightarrow\ \forall y\,(y \in Y \Rightarrow \exists x\,(f(x) = y))]). \tag{3.634}$$

Proof. We let Y and f be arbitrary and assume $f : \mathrm{dom}(f) \to Y$ to be true. To prove the first part ('⇒') of the stated equivalence, we further assume $f : \mathrm{dom}(f) \twoheadrightarrow Y$, so that $\mathrm{ran}(f) = Y$ holds by definition of a surjection. This equation implies

$$\forall y\,(y \in \mathrm{ran}(f) \Leftrightarrow y \in Y) \tag{3.635}$$

with the Equality Criterion for sets. Now, to prove the desired consequent of '⇒'

$$\forall y\,(y \in Y \Rightarrow \exists x\,(f(x) = y)), \tag{3.636}$$

we let $\bar{y}$ be arbitrary and assume that $\bar{y} \in Y$ is true. This assumption implies with (3.635) $\bar{y} \in \mathrm{ran}(f)$, so that – by definition of a range – there exists an element, say $\bar{x}$, such that $(\bar{x}, \bar{y}) \in f$ holds. As we initially assumed f to be a function, we may write this also as $\bar{y} = f(\bar{x})$, which shows that the existential sentence in (3.636) is true. As $\bar{y}$ was arbitrary, we may therefore conclude that the universal sentence (3.636) holds, completing the proof of the first part of the equivalence in (3.634).

We now prove the second part ('⇒') of the equivalence, assuming the universal sentence (3.636) to be true. To show that this implies $f : \mathrm{dom}(f) \twoheadrightarrow Y$, we verify $\mathrm{ran}(f) = Y$. For this purpose, we establish $\mathrm{ran}(f) \subseteq Y$ and $Y \subseteq \mathrm{ran}(f)$, which will then imply the preceding equation with the Axiom of Extension. Let us here observe that the first inclusion $\mathrm{ran}(f) \subseteq Y$ is true because f is by assumption a function with codomain Y. To prove the second inclusion $Y \subseteq \mathrm{ran}(f)$, we apply the definition of a subset and verify the equivalent

$$\forall y\,(y \in Y \Leftrightarrow y \in \mathrm{ran}(f)). \tag{3.637}$$

Letting $\bar{y}$ be arbitrary and assuming $\bar{y} \in Y$ to hold, it follows with the assumed (3.636) that there is an element, say $\bar{x}$, with $f(\bar{x}) = \bar{y}$. We may write this also as $(\bar{x}, \bar{y}) \in f$, which shows that there exists an x such that $(x, \bar{y}) \in f$, and therefore $\bar{y} \in \mathrm{ran}(f)$ follows to be true by definition of a range. This proves the implication in (3.637), and since $\bar{y}$ is arbitrary, we may infer from this the truth of (3.637) and thus the truth of the equivalent $Y \subseteq \mathrm{ran}(f)$. This completes the proof of the conjunction $\mathrm{ran}(f) \subseteq Y \wedge Y \subseteq \mathrm{ran}(f)$, so that the equation $\mathrm{ran}(f) = Y$ holds according to the Axiom of Extension. We therefore obtain $f : X \twoheadrightarrow Y$ with the definition of a surjection, which completes the proof of the equivalence and thus the proof of the implication in (3.634). Since Y and f were arbitrary, we may now finally conclude that the proposed universal sentence is true. □

Proposition 3.193. *Any constant function with nonempty domain is a surjection, that is,*

$$\forall X, y\,(X \neq \emptyset \Rightarrow g_y : X \twoheadrightarrow \{y\}). \tag{3.638}$$

Proof. We take an arbitrary set $X \neq \emptyset$ and an arbitrary $\bar{y}$, so that $g_{\bar{y}} = X \times \{\bar{y}\}$ is a function from X to $\{\bar{y}\}$, by definition of a constant function. We now show that $g_{\bar{y}}$ satisfies

$$\forall y\,(y \in \{\bar{y}\} \Rightarrow \exists x\,(g_{\bar{y}}(x) = y)), \tag{3.639}$$

letting y be arbitrary and assuming $y \in \{\bar{y}\}$ to hold. Furthermore, as we assumed X to be nonempty, there exists an element in X, say $\bar{x}$. Thus, the conjunction $\bar{x} \in X \wedge y \in \{\bar{y}\}$ is true, which implies with the definition of the Cartesian product of two sets $(\bar{x}, y) \in X \times \{\bar{y}\}$ $[= g_{\bar{y}}]$, which we may write as $y = g_{\bar{y}}(\bar{x})$. Thus, there exists an x with $g_{\bar{y}}(x) = y$, which means that the existential sentence in (3.639) is true. As y is arbitrary, we may therefore conclude that the universal sentence (3.639) is true. Thus, $g_{\bar{y}}$ is a surjection according to the Surjection Criterion, and since X and $\bar{y}$ were arbitrary, we may therefore conclude that the proposed sentence holds. □

As singletons are nonempty, we then obtain immediately the following special case with Corollary 3.156.

Corollary 3.194. *The singleton formed by a pair (x, y) is a surjection from the singleton formed by x to the singleton formed by y, that is,*

$$\forall x, y\,(\{(x, y)\} : \{x\} \twoheadrightarrow \{y\}). \tag{3.640}$$

Note 3.24. The range of a function $\{(x, y)\}$ is identical with the singleton $\{y\}$.

Proposition 3.167 then yields the following equivalent representation of a surjection.

Corollary 3.195. *For any function $f : X \to Y$ and any element x of X, it is true that the restriction of f to the singleton $\{x\}$ is a surjection from $\{x\}$ to the singleton formed by the value of f at x, that is,*

$$\forall X, Y, f, x\,([f \in Y^X \wedge x \in X] \Rightarrow f \restriction \{x\} : \{x\} \twoheadrightarrow \{f(x)\}). \tag{3.641}$$

Note 3.25. The range of the restriction of a function $f : X \to Y$ to a singleton $\{x\}$ with $x \in X$ is given by the singleton $\{f(x)\}$, that is,

$$\forall X, Y, f, x\,([f \in Y^X \wedge x \in X] \Rightarrow \mathrm{ran}(f \restriction \{x\}) = \{f(x)\}). \tag{3.642}$$

Proposition 3.196. *The identity function on any set is a surjection, i.e.*

$$\forall X\,(\mathrm{id}_X : X \twoheadrightarrow X). \tag{3.643}$$

Proof. Letting X be an arbitrary set, we notice that id_X is by definition a function with codomain X. We now apply the Surjection Criterion to prove $\mathrm{id}_X : X \twoheadrightarrow X$, by verifying the equivalent

$$\forall y\,(y \in X \Rightarrow \exists x\,(\mathrm{id}_X(x) = y)). \tag{3.644}$$

For this purpose, we take an arbitrary y and assume $y \in X$ to be true. Since id_X is by definition a function with domain X, we see that $y \in X$ implies $\mathrm{id}_X(y) = y$ with the definition of an identity function. Thus, the existential sentence in (3.644) is true, and as y was arbitrary, we may therefore conclude that the universal sentence (3.644) holds. It then follows with the Surjection Criterion that id_X is a surjection from X to X. Because X was an arbitrary set, we may further conclude that the proposed universal sentence (3.643) is true. □

Proposition 3.197. *It is true for any nonempty sets X_1 and X_2 that the projection function from $X_1 \times X_2$ to X_1 is a surjection, i.e.*

$$\forall X_1, X_2\,([X_1 \neq \emptyset \wedge X_2 \neq \emptyset] \;\Rightarrow\; \pi_1 : X_1 \times X_2 \twoheadrightarrow X_1). \tag{3.645}$$

Proof. We take arbitrary sets X_1 and X_2, and we assume $X_1 \neq \emptyset$ as well as $X_2 \neq \emptyset$ to be true. To prove that the projection function $\pi_1 : X_1 \times X_2 \to X_1$ is a surjection with range X_1, we apply the Surjection Criterion and verify accordingly

$$\forall y\,(y \in X_1 \Rightarrow \exists z\,(\pi_1(z) = y)), \tag{3.646}$$

letting $y \in X_1$ be arbitrary (noting that X_1 is by assumption nonempty). Since we assumed X_2 to be also nonempty, there evidently exists an element in X_2, say $\bar{x}_2$. Then, the ordered pair $\bar{z} = (y, \bar{x}_2)$ is an element of $X_1 \times X_2$ by definition of the Cartesian product of two sets. Being thus in the domain of the projection function $\pi_1 : X_1 \times X_2 \to X_1$, we have that $\bar{z} = (y, \bar{x}_2)$ is associated with the value $\pi_1(\bar{z}) = \pi_1((y, \bar{x}_2)) = y$. This equation clearly demonstrates the truth of the existential sentence in (3.646), and as y was arbitrary, we may therefore conclude that the universal sentence (3.646) holds. Consequently, the projection function $\pi_1 : X_1 \times X_2 \to X_1$ is indeed a surjection (with range X_1), and since the sets X_1 and X_2 were initially arbitrary, we may now infer from this the truth of the proposition. □

Exercise 3.80. Show for any nonempty sets X_1 and X_2 that the projection function from $X_1 \times X_2$ to X_2 is a surjection, i.e.

$$\forall X_1, X_2\,([X_1 \neq \emptyset \wedge X_2 \neq \emptyset] \Rightarrow \pi_2 : X_1 \times X_2 \twoheadrightarrow X_2). \tag{3.647}$$

Exercise 3.81. Show for any x, any Y and any surjection f from the singleton formed by x to Y that Y is identical with the singleton formed by the function value at x, that is,

$$\forall x, Y, f\,(f : \{x\} \twoheadrightarrow Y \Rightarrow Y = \{f(x)\}). \tag{3.648}$$

(Hint: Apply Method 1.6 and use (2.18), (3.630), (3.108), (3.93), (2.169), and Notation 3.4.)

Proposition 3.198. *For any functions $f : X \to Y$ and $g : Y \to Z$ it is true that, if the composition of g and f is a surjection, then g is a surjection, that is,*

$$\forall X, Y, Z, f, g\,([f \in Y^X \wedge g \in Z^Y \wedge g \circ f : X \twoheadrightarrow Z] \Rightarrow g : Y \twoheadrightarrow Z). \tag{3.649}$$

Proof. We let X, Y, Z, f and g be arbitrary and prove the implication directly, assuming that f is a function from X to Y, that g is a function from Y to Z, and that the composition $g \circ f$, which is a function from X to Z due to Proposition 3.178 and the fact that the range of f is included in the codomain Y and thus in the domain of g, is a surjection. The latter assumption means that

$$\operatorname{ran}(g \circ f) = Z \tag{3.650}$$

holds. To show that the assumptions imply that $g : Y \to Z$ is a surjection from Y to Z, we prove $\operatorname{ran}(g) = Z$. For this purpose, we apply the Equality Criterion for sets and prove the equivalent

$$\forall y\,(y \in \operatorname{ran}(g) \Leftrightarrow y \in Z). \tag{3.651}$$

We let y be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $y \in \operatorname{ran}(g)$ holds. Since Z is the codomain of g, we have by definition that $\operatorname{ran}(g) \subseteq Z$ holds. With this and the definition of a subset, the assumption $y \in \operatorname{ran}(g)$ immediately implies the desired $y \in Z$. To prove the second part ('$\Leftarrow$') of the equivalence directly, we now assume $y \in Z$ to be true. This implies with the equation (3.650) that $y \in \operatorname{ran}(g \circ f)$ is true, so that there exists by definition of a range an element, say $\bar{x}$, such that $(\bar{x}, y) \in g \circ f$. This means $g(f(\bar{x})) = y$ with Notation 3.6 and then also $(f(\bar{x}), y) \in g$ with Notation 3.4. Consequently, $y \in \operatorname{ran}(g)$ holds (by definition of a range), which completes the proof of the equivalence. As y

is arbitrary, we therefore conclude that (3.651) is true, which proves the equation $\mathrm{ran}(g) = Z$. Thus, $g : Y \to Z$ is a surjection by definition. Since X, Y, Z, f and g were also arbitrary, it follows that the proposed universal sentence holds, as claimed. □

Theorem 3.199 (Surjectivity of the composition of two surjections). *The composition of a surjection g from a set Y to a set Z and a surjection f from a set X to Y is itself a surjection from X to Z, that is,*

$$\forall X, Y, Z, f, g\,([f : X \twoheadrightarrow Y \,\wedge\, g : Y \twoheadrightarrow Z] \Rightarrow g \circ f : X \twoheadrightarrow Z). \tag{3.652}$$

Proof. We let X, Y and Z be arbitrary sets, f an arbitrary surjection from X to Y, and g an arbitrary injection from Y to Z. Then, $g \circ f$ is a function from X to Z due to (3.604). To prove that $g \circ f$ is a surjection, we demonstrate the truth of $\mathrm{ran}(g \circ f) = Z$ according to (3.630). For this purpose, we first verify $Z \subseteq \mathrm{ran}(g \circ f)$, i.e. (applying the definition of a subset)

$$\forall z\,(z \in Z \Rightarrow z \in \mathrm{ran}(g \circ f)). \tag{3.653}$$

To do this, we take an arbitrary element $\bar{z}$ from Z and show that $\bar{z} \in \mathrm{ran}(g \circ f)$ holds. Now, as g is by assumption a surjection from Y to Z, it is true that $\mathrm{ran}(g) = Z$. Therefore, $\bar{z} \in Z$ implies $\bar{z} \in \mathrm{ran}(g)$, so that there exists by definition of a range an element, say $\bar{y}$, with $(\bar{y}, \bar{z}) \in g$. This in turn implies $\bar{y} \in \mathrm{dom}(g)$ $[= Y]$ by definition of a domain, so that $\bar{y} \in Y$ is true. Moreover, since f was assumed to be a surjection from X to Y, we have that $\mathrm{ran}(f) = Y$. Consequently, $\bar{y} \in Y$ gives $\bar{y} \in \mathrm{ran}(f)$, which implies (again by definition of a range) the existence of an element, say $\bar{x}$, such that $(\bar{x}, \bar{y}) \in f$. Then, the ordered pair $(\bar{x}, \bar{z})$ is specified, where $(\bar{x}, \bar{y}) \in f$ and $(\bar{y}, \bar{z}) \in g$ hold, so that the existential sentence

$$\exists x, y, z\,((x, y) \in f \,\wedge\, (y, z) \in g \,\wedge\, (x, z) = (\bar{x}, \bar{z})) \tag{3.654}$$

is true. This gives $(\bar{x}, \bar{z}) \in g \circ f$ with the definition of a composition. Applying now the definition of a range again, we obtain the desired $\bar{z} \in \mathrm{ran}(g \circ f)$. As $\bar{z}$ was arbitrary, it then follows from this that the universal sentence (3.653) is true, which means (by definition of a subset) that $Z \subseteq \mathrm{ran}(g \circ f)$ holds. Since $g \circ f$ is a function with codomain Z, as mentioned earlier, the inclusion $\mathrm{ran}(g \circ f) \subseteq Z$ is also true, so that the Axiom of Extension gives the equality $\mathrm{ran}(g \circ f) = Z$. Thus, the composition $g \circ f : X \to Z$ is a surjection, by definition. Since X, Y, Z, f and g were arbitrary, we may therefore finally conclude that the theorem is true. □

Proposition 3.200. *If the composition of two functions f and g is the identity function (on the domain of f), then f is an injection and g a surjection, that is,*

$$\forall X, Y, f, g\,([f : X \to Y \,\wedge\, g : Y \to X \,\wedge\, g \circ f = \mathrm{id}_X] \\ \Rightarrow [f : X \hookrightarrow Y \,\wedge\, g : Y \twoheadrightarrow X]). \tag{3.655}$$

Proof. We let $f : X \to Y$ and $g : Y \to X$ be arbitrary functions such that $g \circ f = \mathrm{id}_X$ is true. To show that f is an injection, we prove

$$\forall x, x'\,([x, x' \in X \,\wedge\, f(x) = f(x')] \Rightarrow x = x'). \tag{3.656}$$

For this purpose, we take two arbitrary elements x and x' from X such that $f(x) = f(x')$ holds. Then, we obtain

$$x = \mathrm{id}_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = \mathrm{id}_X(x') = x'$$

by applying the definition of the identity function on X, substitution based on the initial assumption $g \circ f = \mathrm{id}_X$, Notation 3.6, then substitution based on the assumption$f(x) = f(x')$, again Notation 3.6, again the initial assumption $g \circ f = \mathrm{id}_X$, and finally again the definition of the identity function on X. These equations yield $x = x'$, which proves the implication in (3.656), and since x and x' are arbitrary, we may therefore conclude that (3.656) is true. Thus, the function $f : X \to Y$ is an injection by definition, i.e. $f : X \hookrightarrow Y$.

To show that $g : Y \to X$ is a surjection, we apply the Surjection Criterion and verify accordingly

$$\forall x\,(x \in X \Rightarrow \exists y\,(g(y) = x)). \tag{3.657}$$

We let x be arbitrary and prove the implication directly, assuming $x \in X$ to hold. Since f is a function from X to Y, we see in light of the Function Criterion that $x \in X$ implies the existence of the unique value $\bar{y}$ satisfying $\bar{y} \in X$ and $(x, \bar{y}) \in f(x)$. Observing that we may write the latter as $f(x) = \bar{y}$ (since f is a function) and recalling the truth of the equations

$$x = \mathrm{id}_X(x) = (g \circ f)(x) = g(f(x)),$$

we therefore obtain after substitution $g(\bar{y}) = x$. We thus showed that there exists an y in Y with $g(y) = x$, which completes the proof of the implication in (3.657). Then, as x is arbitrary, the universal sentence (3.657) follows to be true, so that $g : Y \to X$ is indeed a surjection (according to the Surjection Criterion), i.e. $g : Y \twoheadrightarrow X$.

Since X, Y, f and g were arbitrary, we may therefore conclude that the proposition holds, as claimed. □

We now combine the concepts of injectivity and surjectivity.

Definition 3.49 (Bijection, one-to-one correspondence). We say that a function $f : X \to Y$ is a *bijection* or a *one-to-one correspondence*, symbolically

$$f : X \rightleftarrows Y, \tag{3.658}$$

if, and only if,

1. f is a one-to-one function (from X to Y), that is,

$$f : X \hookrightarrow Y, \tag{3.659}$$

   and

2. f is a surjection (from X to Y), that is,

$$f : X \twoheadrightarrow Y, \tag{3.660}$$

The following trivial example for a bijection results directly from Exercise 3.76 and Exercise 3.79.

Corollary 3.201. *The function $\emptyset : \emptyset \to \emptyset$ is a bijection, i.e.*

$$\emptyset : \emptyset \rightleftarrows \emptyset, \tag{3.661}$$

and this function is the only bijection from $\emptyset$ to $\emptyset$.

Similarly, the Corollaries 3.184 and 3.194 give the bijectivity of functions with a single value.

Corollary 3.202. *The singleton formed by a pair (x, y) is a bijection from the singleton formed by x to the singleton formed by y, that is,*

$$\forall x, y\, (\{(x, y)\} : \{x\} \rightleftarrows \{y\}). \tag{3.662}$$

Moreover, the conjunction of Corollary 3.189 and Proposition 3.196 yields the following useful one-to-one correspondence.

Corollary 3.203. *The identity function on any set is a bijection, that is,*

$$\forall X\, (\mathrm{id}_X : X \rightleftarrows X). \tag{3.663}$$

Corollary 3.204. *For any injection f from a set X to a set Y it is true that f is a bijection from X to the range of f, that is,*

$$\forall X, Y, f\, (f : X \hookrightarrow Y \;\Rightarrow\; f : X \rightleftarrows \mathrm{ran}(f)) \tag{3.664}$$

Proof. We let X, Y and f be arbitrary sets and assume that $f : X \hookrightarrow Y$. Therefore, $f : X \hookrightarrow \mathrm{ran}(f)$ follows to be true with (3.622). Since $f : X \twoheadrightarrow \mathrm{ran}(f)$ also holds by definition of a surjection, we obtain the desired result $f : X \rightleftarrows \mathrm{ran}(f)$ with the definition of a bijection. As X, Y and f are arbitrary, the proposed sentence follows then to be true. □

Proposition 3.205. *Removing any ordered pair from a bijection yields again a bijection, that is,*

$$\forall X, Y, f, x, y\,([f : X \rightleftarrows Y \wedge (x, y) \in f] \Rightarrow f \setminus \{(x, y)\} : X \setminus \{x\} \rightleftarrows Y \setminus \{y\}). \tag{3.665}$$

Proof. We let X and Y be arbitrary sets, f an arbitrary bijection from X to Y, and (x, y) an arbitrary ordered pair in f. The latter may be written as $y = f(x)$ and furthermore implies that $x \in X$ and $y \in \mathrm{ran}(f)$ hold by definitions of a domain and of a range, respectively. As a bijection, f is by definition both an injection (i.e., $f : X \hookrightarrow Y$) and a surjection (i.e., $f : X \twoheadrightarrow Y$). As $X \setminus \{x\}$ is a subset of X due to (2.125), it follows with Proposition 3.187 from $f : X \hookrightarrow Y$ that $f \restriction X \setminus \{x\}$ is an injection from $X \setminus \{x\}$ to Y. This restriction is identical with $f \setminus \{(x, y)\}$ because of Proposition 3.168. Now, the fact that $f : X \twoheadrightarrow Y$ also holds implies $Y = \mathrm{ran}(f)$ by definition of a surjection. We now verify that the range of (the injection) $f \setminus \{(x, y)\}$ equals $Y \setminus \{y\}$ (which will prove that this function is a bijection from $X \setminus \{x\}$ to $Y \setminus \{y\}$). We demonstrate the truth of that equality by proving the equivalent (applying (2.18))

$$\forall z\,(z \in \mathrm{ran}(f \setminus \{(x, y)\}) \Leftrightarrow z \in Y \setminus \{y\}). \tag{3.666}$$

Letting z be arbitrary, we prove the first part ('$\Rightarrow$') of the stated equivalence directly, assuming $z \in \mathrm{ran}(f \setminus \{(x, y)\})$. By definition of a range, there exists an element, say $\bar{x}$, such that $(\bar{x}, z) \in f \setminus \{(x, y)\}$. This implies by definition of a domain that $\bar{x} \in \mathrm{dom}(f \setminus \{(x, y)\})$ and therefore $\bar{x} \in \mathrm{dom}(f \restriction X \setminus \{x\})$, so that $\bar{x} \in X \setminus \{x\}$. Consequently, $\bar{x} \in X$ and $\bar{x} \notin \{x\}$ are both true by definition of a set difference. The latter implies $\bar{x} \neq x$ with (2.169); together with the previously obtained $\bar{x}, x \in X$, the preceding inequality implies $f(\bar{x}) \neq f(x)$ with the Injection Criterion, since f is an injection. As $(\bar{x}, z) \in f \setminus \{(x, y)\}$ also implies in particular $(\bar{x}, z) \in f$ (with the definition of a set difference), we see that $f(\bar{x}) = z$ holds; together with the initially assumed $f(x) = y$, the inequality $f(\bar{x}) \neq f(x)$ gives $z \neq y$, and therefore $z \notin \{y\}$ (again with (2.169)). As the range of the injection $f \setminus \{(x, y)\}$ is included in its codomain Y, it also follows from $z \in \mathrm{ran}(f \setminus \{(x, y)\})$ that $z \in Y$ is true (applying the definition of a subset). Thus, the conjunction $z \in Y \wedge z \notin \{y\}$ holds, which means $z \in Y \setminus \{y\}$ (by definition of a set difference), which proves the first part of the equivalence in (3.666).

We now prove the second part ('$\Leftarrow$') directly, assuming now that $z \in Y \setminus \{y\}$ holds. Clearly, this implies in particular $z \in Y$, (by definition of a set difference) so that y is an element of the range Y of the bijection $f : X \rightleftarrows Y$. Thus, by definition of a binary relation, there exists an element, say $\bar{x}$, with $(\bar{x}, z) \in f$. Then, $\bar{x} \in \mathrm{dom}(f)$ holds by definition of a domain, so that $\bar{x} \in X$. Clearly, the assumption $z \in Y \setminus \{y\}$ also implies $z \notin \{y\}$ and therefore $z \neq y$ with (2.169). Then, the fact that $z = f(\bar{x})$ and $y = f(x)$ are both true implies with the preceding inequality that $f(\bar{x}) \neq f(x)$, with the consequence that $\bar{x} \neq x$ holds due to (3.525). This further implies $\bar{x} \notin \{x\}$ with (2.169), so that the conjunction $\bar{x} \in X \wedge \bar{x} \notin \{x\}$ is true, which clearly means that $\bar{x} \in X \setminus \{x\}$. Thus, $\bar{x}$ is an element of the domain $X \setminus \{x\}$ of the injection $f \restriction X \setminus \{x\}$, thus of $\mathrm{ran}(f \setminus \{(x, y)\})$, completing the proof of the second part of the equivalence. As z is arbitrary, we therefore conclude that (3.666) is true, so that the range of the injection $f \setminus \{(x, y)\}$ from $X \setminus \{x\}$ to Y is indeed identical with $Y \setminus \{y\}$. This shows that this injection is also a surjection, and therefore a bijection. Since X, Y, f, x and y were all arbitrary, we therefore conclude that (3.665) holds. □

Proposition 3.206. *Adjoining to a bijection f an ordered pair whose first coordinate is not in the domain and whose second coordinate is not in the range of f yields again a bijection, that is,*

$$\forall X, Y, f\,(f : X \rightleftarrows Y \Rightarrow \forall x, y\,([x \notin X \;\wedge\; y \notin Y] \\ \Rightarrow f \cup \{(x, y)\} : X \cup \{x\} \rightleftarrows Y \cup \{y\})). \tag{3.667}$$

Proof. Letting X, Y and f be arbitrary sets and assuming $f : X \rightleftarrows Y$ to be true, we also take arbitrary $\bar{x}$ and $\bar{y}$ and assume furthermore that $\bar{x} \notin X$ and $\bar{y} \notin Y$ hold. Then, $F = f \cup \{(\bar{x}, \bar{y})\}$ is a function with domain $X \cup \{\bar{x}\}$ because of Proposition 3.177, and $g = \{(\bar{x}, \bar{y})\}$ is a surjection with domain $\{\bar{x}\}$ and range $\{\bar{y}\}$ due to Corollary 3.194. We now prove that F is a surjection by applying Corollary 3.191, i.e. by verifying

$$\forall y\,(y \in Y \cup \{\bar{y}\} \Leftrightarrow \exists x\,(F(x) = y)). \tag{3.668}$$

We take an arbitrary y and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $y \in Y \cup \{\bar{y}\}$ to be true. Consequently, the disjunction $y \in Y \,\vee\, y \in \{\bar{y}\}$ holds by definition of the union of two sets. On the one hand, if $y \in Y$ is true, then it follows with the surjectivity of the assumed bijection $f : X \rightleftarrows Y$ in connection with the Surjection Criterion that there exists an element, say $\bar{\bar{x}}$, such that $f(\bar{\bar{x}}) = y$ holds, which we may also write as $(\bar{\bar{x}}, y) \in f$. Since $F = f \cup \{(\bar{x}, \bar{y})\}$ implies $f \subseteq F$ with Proposition 2.67, it follows that $(\bar{\bar{x}}, y) \in F$ is true, which we may also write as $F(\bar{\bar{x}}) = y$. This proves the existential sentence in (3.668) in case of $y \in Y$.

On the other hand, if $y \in \{\bar{y}\}$ is true, then we obtain $y = \bar{y}$ with (2.169). Since $F = f \cup \{(\bar{x}, \bar{y})\}$ implies also $\{(\bar{x}, \bar{y})\} \subseteq F$, the evident fact $(\bar{x}, \bar{y}) \in \{(\bar{x}, \bar{y})\}$ gives $(\bar{x}, \bar{y}) \in F$ with the definition of a subset. Then, substitution based on the previously established equation $y = \bar{y}$ yields $(\bar{x}, y) \in F$, which we may also write as $F(\bar{x}) = y$. Thus, the existential sentence in (3.668) holds also in case of $y \in \{\bar{y}\}$.

To prove the second part ('$\Leftarrow$') of the equivalence in (3.668), we now assume that there exists an element, say $\bar{\bar{x}}$, satisfying $F(\bar{\bar{x}}) = y$, i.e. $(\bar{\bar{x}}, y) \in F$. Therefore, the disjunction $(\bar{\bar{x}}, y) \in f \,\vee\, (\bar{\bar{x}}, y) \in \{(\bar{x}, \bar{y})\}$ holds by definition of the union of two sets in connection with the definition of F. On the one hand, if $(\bar{\bar{x}}, y) \in f$ is true, then we obtain $y \in \mathrm{ran}(f)$ $[= Y]$ with the definition of a range and the surjectivity of f. Then, the disjunction $y \in Y \vee y \in \{\bar{y}\}$ is also true, which gives the desired consequent $y \in Y \cup \{\bar{y}\}$ (using again the definition of the union of two sets).

On the other hand, if $(\bar{\bar{x}}, y) \in \{(\bar{x}, \bar{y})\}$ holds, then we obtain $y \in \mathrm{ran}(g)$ $[= \{\bar{y}\}]$ with the definition of a range and the surjectivity of $g = \{(\bar{x}, \bar{y})\}$. Then, the disjunction $y \in Y \,\vee\, y \in \{\bar{y}\}$ is again true, with the already established consequence that $y \in Y \cup \{\bar{y}\}$ holds, as desired.

Thus, the proof of the equivalence is complete. As y is arbitrary, we may therefore conclude that the universal sentence (3.668) is true, which then implies $F : X \cup \{\bar{x}\} \twoheadrightarrow Y \cup \{\bar{y}\}$ with Corollary 3.191.

We now prove that F is also an injection by applying the Injection Criterion, i.e. by verifying

$$\forall x, x' \left([x, x' \in X \cup \{\bar{x}\} \,\wedge\, x \neq x'] \Rightarrow F(x) \neq F(x')\right). \tag{3.669}$$

We let x and x' be arbitrary and assume that both $x, x' \in X \cup \{\bar{x}\}$ and $x \neq x'$ are true. Introducing the denotations $y = F(x)$ and $y' = F(x')$, we may also write $(x, y) \in F$ and $(x', y') \in F$. Recalling $F = f \cup \{(\bar{x}, \bar{y})\}$, it now follows from $(x, y) \in F$ with the definition of the union of two sets that the disjunction

$$(x, y) \in f \,\vee\, (x, y) \in \{(\bar{x}, \bar{y})\} \tag{3.670}$$

is true. We assume first that $(x, y) \in f$ holds, i.e. $y = f(x)$, which gives $x \in X$ and $y \in Y$ with (3.517). Now, the previously obtained $(x', y') \in F$ yields the true disjunction

$$(x', y') \in f \,\vee\, (x', y') \in \{(\bar{x}, \bar{y})\}, \tag{3.671}$$

applying again the definition of the union of two sets. On the one hand, if the first part $(x', y') \in f$ of this disjunction holds, i.e. $y' = f(x')$, then we obtain $x' \in X$ and $y' \in Y$, using (3.517) again. Thus, the conjunction $x, x' \in X \wedge x \neq x'$ is true, which in turn implies $f(x) \neq f(x')$ with the Injection Criterion (recalling that we assumed f to be an injection with domain X). Now, substitution yields $y \neq y'$ and then $F(x) \neq F(x')$, so that the desired consequent in (3.669) holds in case of $(x, y) \in f$ and $(x', y') \in f$.

On the other hand, if the second part $(x', y') \in \{(\bar{x}, \bar{y})\}$ of the disjunction (3.671) is true (alongside $(x, y) \in f$), then we obtain $(x', y') = (\bar{x}, \bar{y})$ with (2.169) and moreover $x' = \bar{x}$ as well as $y' = \bar{y}$ with the Equality Criterion for ordered pairs. Consequently, the assumed $\bar{y} \notin Y$ yields $y' \notin Y$, so that the conjunction $y \in Y \wedge y' \notin Y$ holds, which in turn implies $y \neq y'$ with (2.4). Therefore, $F(x) \neq F(x')$ holds also in case of $(x, y) \in f$ and $(x', y') \in \{(\bar{x}, \bar{y})\}$.

We now assume that the second part $(x, y) \in \{(\bar{x}, \bar{y})\}$ of the disjunction (3.670) holds, so that $(x, y) = (\bar{x}, \bar{y})$ follows to be true with (2.169), which equation gives then $x = \bar{x}$ as well as $y = \bar{y}$ with the Equality Criterion for ordered pairs. Thus, the assumed $\bar{y} \notin Y$ gives now $y \notin Y$. Let us now observe that the disjunction (3.671) is again true. We now prove by contradiction that $(x', y') \in \{(\bar{x}, \bar{y})\}$ is false. Indeed, assuming $(x', y') \in \{(\bar{x}, \bar{y})\}$ to be true, we evidently obtain $(x', y') = (\bar{x}, \bar{y})$ and then $x' = \bar{x}$ (alongside $y' = \bar{y}$), so that substitution based on the previously established $x = \bar{x}$ yields $x = x'$, in contradiction to the initially assumed $x \neq x'$. Therefore, the first part $(x', y') \in f$ of the disjunction (3.671) is true, which we may write as $y' = f(x')$. Then, we obtain as before $x' \in X$ and $y' \in Y$, so that the conjunction $y' \in Y \wedge y \notin Y$ is true. This further implies $y' \neq y$ with (2.4), and therefore $F(x) \neq F(x')$ holds also in case of $(x, y) \in \{(\bar{x}, \bar{y})\}$.

This completes the proof of the implication in (3.669), and since x, x' are arbitrary, we may infer from this the truth of the universal sentence (3.669), and therefore the injectivity of F. Consequently, the injective surjection $F : X \cup \{\bar{x}\} \twoheadrightarrow Y \cup \{\bar{x}\}$ is a bijection, by definition. Since $\bar{x}$ and $\bar{y}$ are arbitrary, we may therefore conclude that the universal sentence in (3.667) with respect to x and y is true. Moreover, as X, Y and f were also arbitrary, we may finally conclude that the proposition holds, as claimed. □

Theorem 3.207 (Bijectivity of the composition of two bijections). *It is true that the composition of a bijection g from a set Y to a set Z and a bijection f from a set X to Y is itself a bijection from X to Z, that is,*

$$\forall X, Y, Z, f, g\,([f : X \rightleftarrows Y \wedge g : Y \rightleftarrows Z] \Rightarrow g \circ f : X \rightleftarrows Z). \tag{3.672}$$

Proof. Letting X, Y and Z be arbitrary sets, f an arbitrary bijection from X to Y and g an arbitrary bijection from Y to Z, it follows with the definition of a bijection on the one hand that f is both an injection and a surjection from X to Y; on the other hand, g is then both an injection and a surjection from Y to Z. Therefore, the conjunctions

$$f : X \hookrightarrow Y \wedge g : Y \hookrightarrow Z$$
$$f : X \twoheadrightarrow Y \wedge g : Y \twoheadrightarrow Z$$

are true. Here, the former implies $g \circ f : X \hookrightarrow Z$ with (3.627) and the latter $g \circ f : X \twoheadrightarrow Z$ with (3.652). These findings show that $g \circ f$ is both an injection and a surjection from X to Z, so that (by definition of a bijection) $g \circ f : X \rightleftarrows Z$ holds. This proves the implication in (3.672), and since X, Y, Z, f and g were arbitrary, the theorem follows then to be true. □

3.4.4. Inverses of functions

Lemma 3.208. *The inverse of a one-to-one correspondence $f : X \rightleftarrows Y$ is a function from Y to X, that is,*

$$\forall X, Y, f\,(f : X \leftrightarrows Y \;\Rightarrow f^{-1} : Y \to X). \tag{3.673}$$

Proof. We let X, Y and f be arbitrary sets such that f is a one-to-one correspondence from X to Y; thus, f is in particular an onto function with $\mathrm{dom}(f) = X$ and $\mathrm{ran}(f) = Y$. It then follows with Corollary 3.42 that $f^{-1} \subseteq Y \times X$ holds. We may therefore apply the Function Criterion to prove $f^{-1} : Y \to X$, by verifying

$$\forall y\,(y \in Y \Rightarrow \exists! x\,(x \in X \;\wedge\; (y, x) \in f^{-1})). \tag{3.674}$$

For this purpose, we let $\bar{y} \in Y$ be arbitrary and establish first the existential part of the uniquely existential sentence. Because of $\mathrm{ran}(f) = Y$, we see in light of the definition of a range that $\bar{y} \in Y$ implies that there exists an element, say $\bar{x}$, such that $(\bar{x}, \bar{y}) \in f$ holds. Therefore, $\bar{x} \in X$ $[= \mathrm{dom}(f)]$ follows to be true by definition of a domain. Thus, the conjunction $\bar{y} \in Y \wedge \bar{x} \in X$ holds, which in turn implies $(\bar{y}, \bar{x}) \in Y \times X$ with the definition of the Cartesian product of two sets. Moreover, we see that there are elements x and y satisfying $(x, y) \in f$ and $(y, x) = (\bar{y}, \bar{x})$, which existential sentence then implies $(\bar{y}, \bar{x}) \in f^{-1}$ with the definition of the inverse of a binary relation. Thus, $\bar{x} \in X$ and $(\bar{y}, \bar{x}) \in f^{-1}$ are both true, proving the existential part in (3.674). To establish the uniqueness part, we verify

$$\forall x, x'\,([x \in X \wedge (\bar{y}, x) \in f^{-1} \wedge x' \in X \wedge (\bar{y}, x') \in f^{-1}] \Rightarrow x = x'). \tag{3.675}$$

We let $\bar{x}$ and $\bar{x}'$ be arbitrary elements of X satisfying $(\bar{y}, \bar{x}) \in f^{-1}$ and $(\bar{y}, \bar{x}') \in f^{-1}$. On the one hand, $(\bar{y}, \bar{x}) \in f^{-1}$ implies with the definition of the inverse of a binary relation that there are elements, say $\bar{\bar{x}}$ and $\bar{\bar{y}}$, such that $(\bar{\bar{x}}, \bar{\bar{y}}) \in f$ and $(\bar{\bar{y}}, \bar{\bar{x}}) = (\bar{y}, \bar{x})$; the latter equation further implies $\bar{\bar{y}} = \bar{y}$ and $\bar{\bar{x}} = \bar{x}$ with the Equality Criterion for ordered pairs, and therefore $(\bar{x}, \bar{y}) \in f$.

On the other hand, $(\bar{y}, \bar{x}') \in f^{-1}$ implies (again by definition of the inverse of a binary relation) that there are elements, say $\bar{\bar{x}}'$ and $\bar{\bar{y}}'$, with $(\bar{\bar{x}}', \bar{\bar{y}}') \in f$ and $(\bar{\bar{y}}', \bar{\bar{x}}') = (\bar{y}, \bar{x}')$; this equation now yields $\bar{\bar{y}}' = \bar{y}$ and $\bar{\bar{x}}' = \bar{x}'$ (using again the Equality Criterion for ordered pairs), so that $(\bar{x}', \bar{y}) \in f$.

Thus, $(\bar{x}, \bar{y}) \in f$ and $(\bar{x}', \bar{y}) \in f$ are both true, which we may write in function notation as $\bar{y} = f(\bar{x})$ and $\bar{y} = f(\bar{x}')$. Clearly, we may combine these equations via substitution to obtain $f(\bar{x}) = f(\bar{x}')$, which then implies $\bar{x} = \bar{x}'$ with the initial assumption that f is an injection. As $\bar{x}$ and $\bar{x}'$ are arbitrary, we therefore conclude that the universal sentence (3.675) is true, which then completes the proof of the uniqueness part, and thus the proof of the uniquely existential sentence in (3.674). Since $\bar{y}$ was also arbitrary, we may now further conclude that (3.674) holds, which then implies the desired $f^{-1} : Y \to X$ with the Function Criterion. Because X, Y and f were arbitrary, we finally conclude that the lemma is indeed true. □

Exercise 3.82. Show that the inverse of a one-to-one function $f : X \hookrightarrow Y$ is a function from the range to the domain of f, that is,

$$\forall X, Y, f\, (f : X \hookrightarrow Y \;\Rightarrow f^{-1} : \operatorname{ran}(f) \to X). \tag{3.676}$$

(Hint: Proceed in analogy to the proof of Lemma 3.208.)

Proposition 3.209. *For any injection f from a set X to a set Y it is true that an element x is the value of the inverse f^{-1} at an element y iff y is the value of f at x, that is,*

$$\forall X, Y, f, x, y\, (f : X \hookrightarrow Y \;\Rightarrow [x = f^{-1}(y) \Leftrightarrow y = f(x)]). \tag{3.677}$$

Proof. We let X, Y, f, x and y be arbitrary and assume that f is an injection from X to Y, so that the inverse f^{-1} of f is a function from $\operatorname{ran}(f)$ to X according to Exercise 3.82.

To prove the first part ('⇒') of the equivalence in (3.677), we assume $x = f^{-1}(y)$, which equation we may also write as $(y, x) \in f^{-1}$. This implies with Proposition 3.43 $(x, y) \in f$, which we may also write as $y = f(x)$. Thus, the first part of the equivalence in (3.678) holds.

Regarding the second part ('⇐'), we now assume $y = f(x)$ to be true, so that $(x, y) \in f$ holds. This implies $(y, x) \in f^{-1}$ with Proposition 3.43, so that $x = f^{-1}(y)$ holds, as desired.

This completes the proof of the equivalence and thus the proof of the implication in (3.678). As X, Y, f, x and y were initially arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Definition 3.50 (Inverse function). For any one-to-one correspondence f we call f^{-1} the *inverse function* of f.

Theorem 3.210 (Characterization of the function values of an inverse function). *For any bijection f from a set X to a set Y it is true that an element x is the value of the inverse function f^{-1} at an element y iff y is the value of f at x, i.e.*

$$\forall X, Y, f, x, y\,(f : X \rightleftarrows Y \;\Rightarrow [x = f^{-1}(y) \Leftrightarrow y = f(x)]). \tag{3.678}$$

Exercise 3.83. Prove Theorem 3.210.

(Hint: Proceed in analogy to the proof of Proposition 3.209.)

Lemma 3.211. *For any bijection $f : X \rightleftarrows Y$ it is true that*

a) the composition $f^{-1} \circ f$ is the identity function on X, that is,

$$\forall X, Y, f\,(f : X \rightleftarrows Y \;\Rightarrow f^{-1} \circ f = \mathrm{id}_X). \tag{3.679}$$

b) the composition $f \circ f^{-1}$ is the identity function on Y, that is,

$$\forall X, Y, f\,(f : X \rightleftarrows Y \;\Rightarrow f \circ f^{-1} = \mathrm{id}_Y). \tag{3.680}$$

Proof. Concerning a), letting $f : X \rightleftarrows Y$ be an arbitrary bijection, we first observe that (3.673) gives the inverse function $f^{-1} : Y \to X$. Then, (3.604) yields the composition $f^{-1} \circ f : X \to X$. Next, we apply the Function Criterion to verify $f^{-1} \circ f = \mathrm{id}_X$, by proving

$$\forall x\,(x \in X \Rightarrow (f^{-1} \circ f)(x) = \mathrm{id}_X(x)) \tag{3.681}$$

We let $\bar{x}$ be an arbitrary element of X, so that $\mathrm{id}_X(\bar{x}) = \bar{x}$ holds by definition of an identity function. Furthermore, there exists in view of the Function Criterion the unique value $\bar{y} \in Y$ with $(\bar{x}, \bar{y}) \in f$, which we may write in function notation as $\bar{y} = f(\bar{x})$. These findings show that there exist x and y such that $(x, y) \in f$ and $(y, x) = (\bar{y}, \bar{x})$ hold, which existential sentence implies $(\bar{y}, \bar{x}) \in f^{-1}$ by definition of the inverse of a binary relation, which we may also write as $\bar{x} = f^{-1}(\bar{y})$ (recalling that f^{-1} is a function) and then also as $\bar{x} = f^{-1}(f(\bar{x}))$, using the previously established equation $\bar{y} = f(\bar{x})$. Applying now Notation 3.6, this gives

$$[\mathrm{id}_X(\bar{x}) =] \quad \bar{x} = (f^{-1} \circ f)(\bar{x}),$$

and therefore $(f^{-1} \circ f)(\bar{x}) = \mathrm{id}_X(\bar{x})$. Since $\bar{x}$ is arbitrary, we may therefore conclude that the universal sentence (3.681) is true, so that the proof of $f^{-1} \circ f = \mathrm{id}_X$ is complete. Because X, Y and f were arbitrary, Part a) of the lemma follows then to be true. □

Exercise 3.84. Prove Part b) of Lemma 3.211.

Exercise 3.85. Show that Part a) of Lemma 3.211 holds also for any injection $f : X \hookrightarrow Y$, that is,

$$\forall X, Y, f\,(f : X \hookrightarrow Y \;\Rightarrow f^{-1} \circ f = \mathrm{id}_X). \tag{3.682}$$

(Hint: Use (3.676) and proceed similarly as in the proof of Lemma 3.211a).)

Theorem 3.212 (Bijectivity of inverse functions). *The inverse function of a bijection is also a bijection, that is,*

$$\forall X, Y, f\,(f : X \rightleftarrows Y \;\Rightarrow\; f^{-1} : Y \rightleftarrows X). \tag{3.683}$$

Proof. We let $f : X \rightleftarrows Y$ be an arbitrary bijection. Then, Lemma 3.211 yields the true conjunction

$$f^{-1} \circ f = \mathrm{id}_X \;\wedge\; f \circ f^{-1} = \mathrm{id}_Y.$$

Let us now observe in light of Lemma 3.208 that f^{-1} is a function from Y to X. Then, the first part $f^{-1} \circ f = \mathrm{id}_X$ of the preceding conjunction implies – together with $f : X \rightleftarrows Y$ and $f^{-1} : Y \rightleftarrows X$ – that f^{-1} is a surjection from Y to X, according to Proposition 3.200. Similarly, the second part $f \circ f^{-1} = \mathrm{id}_Y$ implies that f^{-1} is an injection from Y to X, so that f^{-1} is by definition a bijection from Y to X. Since X, Y and f were arbitrary, we therefore conclude that the theorem holds. □

The Bijectivity of inverse functions leads us to define the following type of function.

Definition 3.51 (Invertible transformation). We call any bijection

$$t : X \rightleftarrows X \tag{3.684}$$

on any set X an *invertible transformation*.

Note 3.26. The identity function id_X on any set X is an invertible transformations, as shown by (3.663).

Note 3.27. We may evidently apply the Axiom of Specification and the Equality Criterion for sets to prove that there exists a unique set $\mathcal{T}(X)$ consisting precisely of all bijective transformations on X, in the sense that

$$\forall f\,(f \in \mathcal{T}(X) \Leftrightarrow [f \in X^X \;\wedge\; f : X \rightleftarrows X]). \tag{3.685}$$

Since $f \in \mathcal{T}(X)$ implies $f \in X^X$ for every f, we find that the set $\mathcal{T}(X)$ is a subset of the set X^X of transformations on X. Given any subset $T(X) \subseteq \mathcal{T}(X)$, we also find that $T(X)$ is a subset of X^X due to (2.13).

Definition 3.52 (Set of invertible transformations). For any set X, we call

$$\mathcal{T}(X) \tag{3.686}$$

the *set of invertible transformations* on X. In addition, we call any subset

$$T(X) \tag{3.687}$$

of $\mathcal{T}(X)$ $\underline{\text{a}}$ set of invertible transformations.

Note 3.28. As an extension of Note 3.27, we may, for any set X, also establish the unique existence of a set $\mathcal{T}_{fi}(X)$ consisting of all the invertible transformations on X under which f is invariant, in the sense that

$$\forall t\,(t \in \mathcal{T}_{fi}(X) \Leftrightarrow [t \in \mathcal{T}(X) \,\wedge\, f \circ t = f]). \tag{3.688}$$

Since $t \in \mathcal{T}_{fi}(X)$ implies $t \in \mathcal{T}(X)$ and then $t \in X^X$ for any f, we see that $\mathcal{T}_{fi}(X)$ is a subset both of the set $\mathcal{T}(X)$ of invertible transformations and of the set X^X of transformations on X. Given any subset $T_{fi}(X) \subseteq \mathcal{T}_{fi}(X)$, we also find the inclusions $T_{fi}(X) \subseteq \mathcal{T}(X)$ and $T_{fi}(X) \subseteq X^X$ by virtue of (2.13).

Definition 3.53 (Set of invertible transformations under which a function is invariant). For any function $f : X \to Y$, we call

$$\mathcal{T}_{fi}(X) \tag{3.689}$$

the *set of invertible transformations* (on X) *under which f is invariant.* Furthermore, we call any subset

$$T_{fi}(X) \tag{3.690}$$

of $\mathcal{T}_{fi}(X)$ $\underline{\text{a}}$ set of invertible transformations under which f is invariant.

Exercise 3.86. Show that any bijection f is identical with the inverse function of its inverse function, that is,

$$\forall X, Y, f\,(f : X \rightleftarrows Y \;\Rightarrow (f^{-1})^{-1} = f). \tag{3.691}$$

The following Proposition shows how to construct a surjection on the basis of a given injection.

Proposition 3.213. *For any injection $f : X \hookrightarrow Y$ and any element $x' \in X$ it is true that the union of the inverse function of f and the Cartesian product $(Y \setminus \operatorname{ran}(f)) \times \{x'\}$ is a surjection from Y to X, that is,*

$$\forall X, Y, f, x'\,([f : X \hookrightarrow Y \wedge x' \in X] \;\Rightarrow\; f^{-1} \cup [(Y \setminus \operatorname{ran}(f)) \times \{x'\}] : Y \twoheadrightarrow X). \tag{3.692}$$

Proof. We let X, Y, f and x' be arbitrary, assume that f is an injection from X to Y, and assume moreover that $x' \in X$ is true. It then follows with (3.676) that f^{-1} is a function from $\mathrm{ran}(f)$ to X, and the Cartesian product $g_{x'} = (Y \setminus \mathrm{ran}(f)) \times \{x'\}$ is the constant function from $Y \setminus \mathrm{ran}(f)$ to $\{x'\}$, by definition. Since the domain $\mathrm{ran}(f)$ of the function f^{-1} and the domain $Y \setminus \mathrm{ran}(f)$ of the constant function $g_{x'}$ are disjoint sets because of (2.111), it follows with Exercise 3.73 that f^{-1} and $g_{x'}$ are compatible functions. Consequently, the pair formed by these functions constitutes a compatible set of functions due to Proposition 3.174, so that the union (3.692) is a function with domain $\mathrm{ran}(f) \cup [Y \setminus \mathrm{ran}(f)]$ according to Theorem 3.175. As we initially assumed f to be a function with codomain Y, we have $\mathrm{ran}(f) \subseteq Y$, which implies with (2.263)

$$[Y \setminus \mathrm{ran}(f)] \cup \mathrm{ran}(f) = Y,$$

so that the domain of g is clearly Y. We now apply Corollary 3.191 to prove $g : Y \twoheadrightarrow X$. To do this, we verify

$$\forall x\,(x \in X \Leftrightarrow \exists y\,(g(y) = x)). \tag{3.693}$$

We let x be arbitrary and assume first $x \in X$ to be true. Since f is a function from X to Y, the function value $\bar{y} = f(x)$ exists uniquely in Y. Then, we obtain $x = f^{-1}(\bar{y})$ with Proposition 3.209, which equation we may also write as $(\bar{y}, x) \in f^{-1}$. Since $f^{-1} \subseteq g$ holds due to (3.692) and Proposition 2.67, it now follows that $(\bar{y}, x) \in g$ holds as well, which finding we may also write as $x = g(\bar{y})$, because g is a function. This equation shows that there exists an y with $g(y) = x$, so that the first part ('$\Rightarrow$') of the equivalence in (3.693) holds.

To establish the second part ('$\Leftarrow$'), we now assume that there is an element, say $\bar{y}$, satisfying $g(\bar{y}) = x$. Thus, $(\bar{y}, x) \in g$ holds, which implies with (3.692) and the definition of the union of two sets that the disjunction $(\bar{y}, x) \in f^{-1} \vee (\bar{y}, x) \in g_{x'}$ is true. On the one hand, if $(\bar{y}, x) \in f^{-1}$ holds, i.e. $x = f^{-1}(\bar{y})$, then we obtain $\bar{y} = f(x)$ with Proposition 3.209, i.e. $(x, \bar{y}) \in f$. Consequently, $x \in \mathrm{dom}(f)$ $[= X]$ follows to be true by definition of a domain, so that the desired $x \in X$ holds in the first case.

On the other hand, if the second part $(\bar{y}, x) \in g_{x'}$ of the preceding disjunction is true, we obtain on the one hand $\bar{y} \in \mathrm{dom}(g_{x'})$ with the definition of a domain, which shows that $g_{x'}$ is a constant function with nonempty domain. On the other hand, we obtain $x \in \mathrm{ran}(g_{x'})$ $[= \{x'\}]$ with the definition of a range and the fact that the nonempty constant function $g_{x'}$ is onto $\{x'\}$ according to Proposition 3.193. Then, $x \in \{x'\}$ implies $x = x'$ with (2.169), so that the initial assumption $x' \in X$ gives $x \in X$, as desired.

This completes the proof of the equivalence in (3.693), and since x is arbitrary, the universal sentence (3.693) follows then to be true. Consequently, g is a surjection from Y to X according to Corollary 3.191. As X, Y, f and x' were arbitrary, we may then infer from this the truth of the proposed universal sentence. □

To construct, conversely, an injection from a given surjection, we must be able to pick out unspecified elements from the sets of a given set system.

Axiom 3.3 (Axiom of Choice). It is true for any set system $\mathcal{K}$ consisting of nonempty sets that there exists a function f with domain $\mathcal{K}$ and codomain $\bigcup\mathcal{K}$ such that the function value at any set K in $\mathcal{K}$ is an element of K, that is,

$$\forall \mathcal{K}\,(\emptyset \notin \mathcal{K} \Rightarrow \exists f\,(f : \mathcal{K} \to \bigcup \mathcal{K} \,\wedge\, \forall K\,(K \in \mathcal{K} \Rightarrow f(K) \in K))). \quad (3.694)$$

The Axiom of Choice allows us to construct a function from any given binary relation in such a way that their domain coincide.

Proposition 3.214. *It is true for any binary relation that there exists a function included in R with equal domain, that is,*

$$\begin{aligned}&\forall R\,(R \text{ is a binary relation}\\ &\quad \Rightarrow\ \exists F\,(F \text{ is a function} \,\wedge\, F \subseteq R \,\wedge\, \mathrm{dom}(F) = \mathrm{dom}(R))).\end{aligned} \quad (3.695)$$

Proof. We take an arbitrary set R, and we assume R to be a binary relation. Let us verify now the universal sentence

$$\forall a\,(a \in \mathrm{dom}(R) \Rightarrow \exists! B\,(\forall b\,(b \in B \Leftrightarrow [b \in \mathrm{ran}(R) \,\wedge\, (a,b) \in R]))). \quad (3.696)$$

Letting a be arbitrary and assuming $a \in \mathrm{dom}(R)$ to be true, we can evidently prove the uniquely existential by means of the Axiom of Specification and the Equality Criterion for sets (in the usual way). According to Function definition by replacement, there exists then a unique function f with domain $\mathrm{dom}(R)$ such that

$$\forall a\,(a \in \mathrm{dom}(R) \Rightarrow \forall b\,(b \in f(a) \Leftrightarrow [b \in \mathrm{ran}(R) \,\wedge\, (a,b) \in R])). \quad (3.697)$$

Since the range of f is included in itself according to (2.10), we may take $\mathrm{ran}(f)$ as a codomain of f, so that $f : \mathrm{dom}(R) \to \mathrm{ran}(f)$. Next, we demonstrate the truth of $\emptyset \notin \mathrm{ran}(f)$, via the verification of the universal sentence

$$\forall B\,(B \in \mathrm{ran}(f) \Rightarrow B \neq \emptyset). \quad (3.698)$$

We let B be arbitrary, assuming $B \in \mathrm{ran}(f)$ to be true. The definitions of a range and of a domain imply then the existence of a particular constant

$\bar{a} \in \mathrm{dom}(f)$ [$= \mathrm{dom}(R)$] such that $(\bar{a}, B) \in f$. We thus have $\bar{a} \in \mathrm{dom}(R)$ and, using function notation, $B = f(\bar{a})$. The former implies (by definition of a domain) that there exists a particular constant $\bar{b}$ with $(\bar{a}, \bar{b}) \in R$. This further implies $\bar{b} \in \mathrm{ran}(R)$ by definition of a range, so that $\bar{b} \in B$ follows to be true with (3.697). This finding clearly demonstrates the truth of the desired consequent $B \neq \emptyset$ of the implication in (3.698), in which B is arbitrary, so that the universal sentence (3.698) follows to be true. Consequently, we obtain the suggested negation $\emptyset \notin \mathrm{ran}(f)$ with (2.5).

This negation allows us now to apply the Axiom of Choice and to infer the existence of a particular function $\bar{g} : \mathrm{ran}(f) \to \bigcup \mathrm{ran}(f)$ such that

$$\forall B\, (B \in \mathrm{ran}(f) \Rightarrow \bar{g}(B) \in B). \tag{3.699}$$

Then, the composition $\bar{F} = \bar{g} \circ f$ constitutes a function from $\mathrm{dom}(R)$ to $\bigcup \mathrm{ran}(f)$, according to Proposition 3.178.

We establish now the inclusion $\bar{F} \subseteq R$ by means of the definition of a subset. To do this, we take an arbitrary Z, and we assume that $Z \in \bar{F}$ is true. Since $\bar{F}$ is a binary relation, there are particular constants $\bar{a}, \bar{b}$ for which $(\bar{a}, \bar{b}) = Z$ holds. Thus, substitution yields $(\bar{a}, \bar{b}) \in \bar{F}$, which we may write in function notation as $\bar{b} = \bar{F}(\bar{a})$. Recalling that $\bar{F}$ denotes the composed function $\bar{g} \circ f$, we can also write this equation in the form $\bar{b} = \bar{g}(f(\bar{a}))$. We note that the previously found $(\bar{a}, \bar{b}) \in \bar{F}$ implies $\bar{a} \in \mathrm{dom}(R)$ [$= \mathrm{dom}(\bar{F})$] by definition of a domain. Here, $\bar{a}$ is also in the domain of the function f, and $f(\bar{a})$ is evidently an element of the range of f. Consequently, [$\bar{b} =$] $\bar{g}(f(\bar{a})) \in f(\bar{a})$ is implied according to (3.699). The resulting $\bar{b} \in f(\bar{a})$ implies in addition $(\bar{a}, \bar{b}) \in R$ with (3.697), so that $Z \in R$ follows to be true via substitution. Since Z was arbitrary in $\bar{F}$, we may now infer from this the truth of the inclusion $\bar{F} \subseteq R$.

We thus proved that there exists a function F with domain $\mathrm{dom}(F) = \mathrm{dom}(R)$ and satisfying the inclusion $F \subseteq R$. As R was initially arbitrary, we may therefore conclude that the proposition holds, as claimed. $\square$

The next theorem gives another typical application of the Axiom of Choice.

Theorem 3.215. *For any surjection g from a set Y to a set X there exists an injection f from X to Y, that is,*

$$\forall X, Y, g\, (g : Y \twoheadrightarrow X \;\Rightarrow\; \exists f\, (f : X \hookrightarrow Y)) \tag{3.700}$$

Proof. We let X, Y and g be arbitrary sets and assume that $g : Y \twoheadrightarrow X$ holds, so that $\mathrm{dom}(g) = Y$ and $\mathrm{ran}(g) = X$ are true by definition of a surjection. We now consider the two cases $g = \emptyset$ and $g \neq \emptyset$. In the first case, $g = \emptyset$ implies $Y = \mathrm{dom}(g) = \emptyset$ and $X = \mathrm{ran}(g) = \emptyset$ with Exercise 3.20 and the assumption that g is a surjection with range X. Then, because of $X = \emptyset$, we see in light of Proposition 3.151, there exists a (unique) function $f : X \to Y$, which is an injection according to Exercise 3.76. This proves the proposition for the first case.

In the other case that $g \neq \emptyset$ holds, we will use the inverse g^{-1} to construct an injection from X to Y. Let us first observe that $g \neq \emptyset$ implies $[Y =] \ \mathrm{dom}(g) \neq \emptyset$ and $[X =] \ \mathrm{ran}(g) \neq \emptyset$ (applying again Exercise 3.20 in connection with the surjectivity of g), and moreover $g^{-1} \neq \emptyset$ with Theorem 3.41b). It follows with Corollary 3.42 also that $g^{-1} \subseteq X \times Y$ holds, and moreover that $\mathrm{dom}(g^{-1}) = \mathrm{ran}(g)$ $[= X]$ is true because of Theorem 3.44. Next, we observe that g^{-1} is not necessarily a function, as there might exist ordered pairs (y_1, x), (y_2, x) in g, so that $(x, y_1), (x, y_2)$ are elements of g^{-1} with multiple values for one element x of the domain of g^{-1}. Using now the Axiom of Specification in connection with the Equality Criterion for sets, we may establish the unique existence of a set (system) $\mathcal{K}$ containing precisely any element K of $\mathcal{P}(g^{-1})$, i.e. any subset K of g^{-1}, such that K is the restriction of g^{-1} to some x in X, that is,

$$\exists! \mathcal{K} \forall K \, (K \in \mathcal{K} \Leftrightarrow [K \in \mathcal{P}(g^{-1}) \ \wedge \ \exists x \, (x \in X \ \wedge \ g^{-1} \restriction \{x\} = K)]).$$

Thus, the set $\mathcal{K}$ satisfies

$$\forall K \, (K \in \mathcal{K} \Leftrightarrow [K \in \mathcal{P}(g^{-1}) \ \wedge \ \exists x \, (x \in X \ \wedge \ g^{-1} \restriction \{x\} = K)]). \quad (3.701)$$

We now prove that $\emptyset \notin \mathcal{K}$ is true. For this purpose, we verify the equivalent

$$\forall K \, (K \in \mathcal{K} \Rightarrow K \neq \emptyset), \quad (3.702)$$

using Proposition 2.2. We let K be arbitrary and assume $K \in \mathcal{K}$ to be true, which implies with (3.701) in particular that there is an element, say $\bar{x}$, with $\bar{x} \in X$ and $g^{-1} \restriction \{\bar{x}\} = K$. Recalling that $X = \mathrm{dom}(g^{-1})$ holds and observing the evident truth of $\bar{x} \in \{\bar{x}\}$, we thus obtain the true conjunction $\bar{x} \in \mathrm{dom}(g^{-1}) \ \wedge \ \bar{x} \in \{\bar{x}\}$, which in turn yields with the definition of the intersection of two sets $\bar{x} \in \mathrm{dom}(g^{-1}) \cap \{\bar{x}\}$. Clearly, the intersection is thus nonempty, so that the restriction $K = g^{-1} \restriction \{\bar{x}\}$ is also nonempty according to (3.102). This proves the implication in (3.702), and since K was arbitrary, we may therefore infer from this the truth of the universal sentence (3.702), which then gives $\emptyset \notin \mathcal{K}$ (according to Proposition 2.2).

We may therefore apply the Axiom of Choice to infer from the preceding finding that there exists a function, say $\bar{F}$, such that $\bar{F} : \mathcal{K} \to \bigcup \mathcal{K}$ and

$$\forall K \, (K \in \mathcal{K} \Rightarrow \bar{F}(K) \in K) \quad (3.703)$$

hold. Since $\bigcup\mathcal{K}$ is codomain of $\bar{F}$, we see that $\mathrm{ran}(\bar{F}) \subseteq \bigcup\mathcal{K}$. Next, we prove that $\bigcup\mathcal{K} \subseteq X \times Y$ also holds. To do this, we apply the definition of a subset and verify the equivalent

$$\forall Z\,(Z \in \bigcup\mathcal{K} \Rightarrow Z \in X \times Y). \tag{3.704}$$

We let Z be arbitrary and assume $Z \in \bigcup\mathcal{K}$ to be true, so that there exists – by definition of the union of a set system – a set in $\mathcal{K}$, say $\bar{K}$, with $Z \in \bar{K}$. Here, $\bar{K} \in \mathcal{K}$ implies with (3.702) in particular $\bar{K} \in \mathcal{P}(g^{-1})$ and therefore $\bar{K} \subseteq g^{-1}$ by definition of a power set. Recalling now that the inclusion $g^{-1} \subseteq X \times Y$ also holds, we obtain with the transitivity of $\subseteq$ the inclusion $\bar{K} \subseteq X \times Y$. With this, the previously established $Z \in \bar{K}$ implies the desired consequent $Z \in X \times Y$ by definition of a subset. Since Z was arbitrary, we may therefore conclude that the universal sentence (3.704) is true, which in turn implies $\bigcup\mathcal{K} \subseteq X \times Y$ (using again the definition of a subset). Together with the previously found $\mathrm{ran}(\bar{F}) \subseteq \bigcup\mathcal{K}$, this finding further implies $\mathrm{ran}(\bar{F}) \subseteq X \times Y$ (again with the transitivity of $\subseteq$).

Consequently, we may apply the Function Criterion to verify that $\mathrm{ran}(\bar{F})$ is a function from X to Y. For this purpose, we prove

$$\forall x\,(x \in X \Rightarrow \exists! y\,(y \in Y \ \wedge\ (x, y) \in \mathrm{ran}(\bar{F}))). \tag{3.705}$$

We let $\bar{x} \in X$ be arbitrary and establish first the existential part

$$\exists y\,(y \in Y \ \wedge\ (\bar{x}, y) \in \mathrm{ran}(\bar{F})). \tag{3.706}$$

Defining the restriction $\bar{K} = g^{-1} \restriction \{\bar{x}\}$, we thus see that there exists an x which satisfies the conjunction $x \in X \wedge g^{-1} \restriction \{x\} = \bar{K}$. Moreover, because $g^{-1} \restriction \{\bar{x}\} \subseteq g^{-1}$ holds according to Corollary 3.23, we obtain $\bar{K} \subseteq g^{-1}$ and therefore $\bar{K} \in \mathcal{P}(g^{-1})$ by definition of a power set. Together with the preceding existential sentence, this further implies $\bar{K} \in \mathcal{K}$ with (3.702), so that $g^{-1} \restriction \{\bar{x}\} \in \mathcal{K}$ holds. This in turn implies $\bar{F}(\bar{K}) \in g^{-1} \restriction \{\bar{x}\}$ with (3.703), so that there exist – by definition of a restriction – elements, say $\bar{a}$ and $\bar{b}$, such that $\bar{a} \in \{\bar{x}\}$ and $(\bar{a}, \bar{b}) = \bar{F}(\bar{K})$. Here, we may write the equation also as $(\bar{K}, (\bar{a}, \bar{b})) \in \bar{F}$, so that $(\bar{a}, \bar{b}) \in \mathrm{ran}(\bar{F})$ holds by definition of a range. As the previously found $\bar{a} \in \{\bar{x}\}$ yields $\bar{a} = \bar{x}$ with (2.169), substitution yields $(\bar{x}, \bar{b}) \in \mathrm{ran}(\bar{F})$. Recalling now $\mathrm{ran}(\bar{F}) \subseteq X \times Y$, we then obtain with the definition of a subset $(\bar{x}, \bar{b}) \in X \times Y$, so that $\bar{b} \in Y$ follows in particular to be true with the definition of the Cartesian product of two sets. We thus showed that there exists an y which satisfies $y \in Y$ and $(\bar{x}, y) \in \mathrm{ran}(\bar{F})$, which proves the existential part (3.706). To establish the uniqueness part, we verify accordingly

$$\forall y, y'\,([y \in Y \ \wedge\ (\bar{x}, y) \in \mathrm{ran}(\bar{F}) \ \wedge\ y' \in Y \ \wedge\ (\bar{x}, y') \in \mathrm{ran}(\bar{F})] \Rightarrow y = y'). \tag{3.707}$$

Letting $\bar{y}$ and $\bar{y}'$ be arbitrary, we assume $\bar{y}, \bar{y}'$ and $(\bar{x}, \bar{y}), (\bar{x}, \bar{y}') \in \mathrm{ran}(\bar{F})$ to be true. The latter assumptions imply with the definition of a range that there exist elements, $\bar{K}$ and $\bar{K}'$, with

$$(\bar{K}, (\bar{x}, \bar{y})) \in \bar{F}, \tag{3.708}$$
$$(\bar{K}', (\bar{x}, \bar{y}')) \in \bar{F}. \tag{3.709}$$

On the one hand, $\bar{K}, \bar{K}' \in \mathrm{dom}(\bar{F})$ $[= \mathcal{K}]$ follows to be true by definition of a domain, and on the other hand we may write

$$(\bar{x}, \bar{y}) = \bar{F}(\bar{K}),$$
$$(\bar{x}, \bar{y}') = \bar{F}(\bar{K}').$$

Since $\bar{K}, \bar{K}' \in \mathcal{K}$ implies with (3.703) the truth of $\bar{F}(\bar{K}) \in \bar{K}$ and of $\bar{F}(\bar{K}') \in \bar{K}'$, substitution based on the preceding equations yields

$$(\bar{x}, \bar{y}) \in \bar{K}, \tag{3.710}$$
$$(\bar{x}, \bar{y}') \in \bar{K}'. \tag{3.711}$$

Furthermore, $\bar{K}, \bar{K}' \in \mathcal{K}$ implies with (3.702) in particular that there are elements, say $\bar{\bar{x}}$ and $\bar{\bar{x}}'$, such that $\bar{\bar{x}}, \bar{\bar{x}}' \in X$ and the equations

$$g^{-1} \upharpoonright \{\bar{\bar{x}}\} = \bar{K} \tag{3.712}$$
$$g^{-1} \upharpoonright \{\bar{\bar{x}}'\} = \bar{K}' \tag{3.713}$$

are true. Now, (3.710) and (3.711) give via substitution based on these equations

$$(\bar{x}, \bar{y}) \in g^{-1} \upharpoonright \{\bar{\bar{x}}\}$$
$$(\bar{x}, \bar{y}') \in g^{-1} \upharpoonright \{\bar{\bar{x}}'\}$$

By definition of a restriction, we therefore obtain in particular $\bar{x} \in \{\bar{\bar{x}}\}$ as well as $\bar{x} \in \{\bar{\bar{x}}'\}$, so that the equations $\bar{x} = \bar{\bar{x}}$ and $\bar{x} = \bar{\bar{x}}'$ follow to be true with (2.169). We may then apply substitutions to (3.712) and (3.713), that is,

$$\bar{K} = g^{-1} \upharpoonright \{\bar{\bar{x}}\} = g^{-1} \upharpoonright \{\bar{x}\} = g^{-1} \upharpoonright \{\bar{\bar{x}}'\} = \bar{K}',$$

which equations yield $\bar{K} = \bar{K}'$. Consequently, we may write (3.709) as $(\bar{K}, (\bar{x}, \bar{y}')) \in \bar{F}$, which implies together with (3.708) and the fact that $\bar{F}$ is a function $(\bar{x}, \bar{y}) = (\bar{x}, \bar{y}')$. This equation finally gives the desired $\bar{y} = \bar{y}'$ with the Equality Criterion for ordered pairs. As $\bar{y}$ and $\bar{y}'$ were arbitrary, we may therefore conclude that the universal sentence (3.707) is true, which establishes the uniqueness part and thus the truth of the uniquely existential sentence in (3.705). Because $\bar{x}$ was also arbitrary, the universal sentence (3.705) follows then also to be true, so that $\bar{f} = \mathrm{ran}(\bar{F})$ is a function from X to Y according to the Function Criterion.

It now only remains for us to verify that the function $\bar{f} : X \to Y$ is an injection. To do this, we prove

$$\forall x, x' \, ([x, x' \in X \, \wedge \, \bar{f}(x) = \bar{f}(x')] \Rightarrow x = x'). \tag{3.714}$$

We let x and x' be arbitrary elements in X and assume $\bar{f}(x) = \bar{f}(x')$ to be true. Denoting this function value by $y = \bar{f}(x)$ and $y = \bar{f}(x')$, we may then write these equations as $(x, y), (x', y) \in \mathrm{ran}(\bar{F})$. By definition of a range, there exist elements, say $\bar{K}$ and $\bar{K}'$, such that

$$(\bar{K}, (x, y)) \in \bar{F},$$
$$(\bar{K}', (x', y)) \in \bar{F}.$$

Then, we obtain on the one hand $\bar{K}, \bar{K}' \in \mathrm{dom}(\bar{F})$ $[= \mathcal{K}]$ (by definition of a domain) and on the other hand

$$(x, y) = \bar{F}(\bar{K}),$$
$$(x', y) = \bar{F}(\bar{K}').$$

As $\bar{K}, \bar{K}' \in \mathcal{K}$ implies $\bar{F}(\bar{K}) \in \bar{K}$ as well as $\bar{F}(\bar{K}') \in \bar{K}'$ with (3.703), we then obtain via substitution

$$(x, y) \in \bar{K}, \tag{3.715}$$
$$(x', y) \in \bar{K}'. \tag{3.716}$$

Moreover, $\bar{K}, \bar{K}' \in \mathcal{K}$ implies with (3.702) in particular $\bar{K}, \bar{K}' \in \mathcal{P}(g^{-1})$ and therefore $\bar{K} \subseteq g^{-1}$ as well as $\bar{K}' \subseteq g^{-1}$ with the definition of a power set. With these inclusions, (3.715) and (3.716) imply

$$(x, y) \in g^{-1},$$
$$(x', y) \in g^{-1},$$

and therefore $(y, x), (y, x') \in g$ with Proposition 3.43. Since g is a function, the preceding finding yields $x = x'$, completing the proof of the implication in (3.714). As x and x' are arbitrary, we may now infer from this the truth of the universal sentence (3.714), and thus the injectivity of the function $\bar{f} : X \to Y$. We thus showed that an injection $f : X \to Y$ exists, and since X, Y and g were initially arbitrary, we may finally conclude that the theorem is true. $\square$

3.4.5. Images and inverse images

Definition 3.54 ((Direct) image). For any function f from a set X to a set Y and for any subset A of X we call the range of the restriction of f to A, symbolically

$$f[A] = \mathrm{ran}(f \upharpoonright A), \tag{3.717}$$

the (*direct*) *image* of A under f.

Recalling that the restriction of a function to its domain is the function itself (see Note 3.19), we immediately obtain the following characterization of the range of a function as the image of its domain.

Corollary 3.216. *It is true that the image of the domain of/under any function is identical with the range of the function, that is,*

$$\forall X, Y, f\,(f : X \to Y \;\Rightarrow\; f[X] = \mathrm{ran}(f)). \tag{3.718}$$

The following proposition brings out more clearly how the image of a subset of a given function domain is determined.

Proposition 3.217. *It is true for any function f from a set X to a set Y and for any subset A of X that the value of f at any element of A is in the image of A under f, that is,*

$$\forall X, Y, f, A\,([f \in Y^X \wedge A \subseteq X] \Rightarrow \forall x\,(x \in A \Rightarrow f(x) \in f[A])). \tag{3.719}$$

Proof. We take arbitrary sets X, Y, f and A, we assume f to be an element of Y^X (i.e., to be a function from X to Y), and we assume the inclusion $A \subseteq X$ to be true. Thus, the image $f[A]$ of A under f is defined. Next, we take an arbitrary x and assume $x \in A$ to be true, so that $x \in X$ also holds by definition of a subset. Therefore, the function value $y = f(x)$ exists uniquely in Y, and we may write this equation also in the form $(x, y) \in f$. This implies in conjunction with the assumed $x \in A$ and the definition of a restriction $(x, y) \in f \upharpoonright A$, which shows in light of the definition of a range that $[f(x) =]\ y \in \mathrm{ran}(f \upharpoonright A)$ is true. This means according to the definition of an image that $f(x) \in f[A]$ holds, as desired. Here, x is arbitrary, so that the universal sentence with respect to x in (3.719) follows to be true. Since X, Y, f and A were initially also arbitrary, we may therefore conclude that the proposed universal sentence holds, as claimed. □

Corollary 3.218. *For any function f from a set X to a set Y and for any subset A of X it is true that the image of A under f is included in the range and the codomain of f, that is,*

$$\forall X, Y, f, A\,([f \in Y^X \wedge A \subseteq X] \Rightarrow [f[A] \subseteq \mathrm{ran}(f) \wedge f[A] \subseteq Y]). \tag{3.720}$$

Proof. Letting X and Y be arbitrary sets, f an arbitrary element of Y^X (so that $f : X \to Y$), and letting A be an arbitrary subset of X, we first observe in light of Corollary 3.23 that $f \upharpoonright A \subseteq f$ holds and that the restriction is itself a binary relation. These findings in turn imply with Proposition 3.34 that the inclusion $\mathrm{ran}(f \upharpoonright A) \subseteq \mathrm{ran}(f)$ is then true as well. Then, the definition of an image yields $f[A] \subseteq \mathrm{ran}(f)$. Since $\mathrm{ran}(f) \subseteq Y$ also holds by definition of a codomain, we obtain with the transitivity of $\subseteq$ the second desired inclusion $f[A] \subseteq Y$. As X, Y, f and A were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Note 3.29. In the particular case of a function $f : X \to Y$ and a singleton $A = \{x\}$ with $x \in X$, the corresponding image reads according to (3.642)

$$f[\{x\}] = \{f(x)\}. \tag{3.721}$$

Exercise 3.87. Show for function $f : X \to Y$ that the image of the empty set under f is empty, that is,

$$\forall X, Y, f\,(f \in Y^X \Rightarrow f[\emptyset] = \emptyset). \tag{3.722}$$

(Hint: Use (2.43), (3.717), (3.85) and (3.121).)

Exercise 3.88. Show that the image of any subset A of any set X under any function f from X to any set Y is nonempty iff the subset A is nonempty, that is,

$$\forall X, Y, f, A\,([f : X \to Y \,\wedge\, A \subseteq X] \Rightarrow [f[A] \neq \emptyset \Leftrightarrow A \neq \emptyset]). \tag{3.723}$$

(Hint: Use (2.42), (3.108), (3.81) and (3.719).)

Proposition 3.219. *For any functions $f : X \to Y$, $g : Y \to Z$ and any $A \subseteq X$ it is true that the image of the range of the restriction of f to A under g is identical with the range of the restriction of the composition of g and f to A, that is,*

$$\begin{aligned}&\forall X, Y, Z, f, g, A\,([f \in Y^X \,\wedge\, g \in Z^Y \,\wedge\, A \subseteq X]\\ &\quad \Rightarrow g[\mathrm{ran}(f \upharpoonright A)] = \mathrm{ran}([g \circ f] \upharpoonright A)).\end{aligned} \tag{3.724}$$

Proof. We let X, Y, Z, f, g and A be arbitrary sets, assume that f is a function from X to Y, furthermore that g is a function from Y to Z, and moreover that A is a subset of X. To prove the equation in (3.724), we apply the Equality Criterion for sets and verify accordingly

$$\forall z\,(z \in g[\mathrm{ran}(f \upharpoonright A)] \Leftrightarrow z \in \mathrm{ran}([g \circ f] \upharpoonright A)). \tag{3.725}$$

To do this, we let $\bar{z}$ be arbitrary and prove the first part ('⇒') of the equivalence directly, assuming

$$\bar{z} \in g[\operatorname{ran}(f \restriction A)] \tag{3.726}$$

to be true. By definition of an image, this assumption implies

$$\bar{z} \in \operatorname{ran}(g \restriction \operatorname{ran}(f \restriction A)) \tag{3.727}$$

and then, by definition of a range, that there exists an element, say $\bar{y}$, such that

$$(\bar{y}, \bar{z}) \in g \restriction \operatorname{ran}(f \restriction A). \tag{3.728}$$

This implies on, the one hand, in particular

$$(\bar{y}, \bar{z}) \in g \tag{3.729}$$

with the definition of a restriction and, on the other hand,

$$\bar{y} \in \operatorname{ran}(f \restriction A) \tag{3.730}$$

with the definition of a domain, since the assumption $g : Y \to Z$ implies with Proposition 3.164 that the restriction $g \restriction \operatorname{ran}(f \restriction A)$ is a function with domain $\operatorname{ran}(f \restriction A)$ (and codomain Z). Next, the sentence (3.730) implies (again by definition of a range) that there exists an element, say $\bar{x}$, with

$$(\bar{x}, \bar{y}) \in f \restriction A. \tag{3.731}$$

This implies on the one hand (by definition of a restriction)

$$(\bar{x}, \bar{y}) \in f, \tag{3.732}$$

and on the other hand (by definition of a domain)

$$\bar{x} \in A, \tag{3.733}$$

since the assumption $f : X \to Y$ implies (with Proposition 3.164) that the restriction $f \restriction A$ is a function with domain A. Now, the sentences (3.732) and (3.729) show that the existential sentence

$$\exists x, y, z\, ((x, y) \in f \,\wedge\, (y, z) \in g \,\wedge\, (x, z) = (\bar{x}, \bar{z})) \tag{3.734}$$

is true, which implies with the definition of a composition

$$(\bar{x}, \bar{z}) \in g \circ f. \tag{3.735}$$

The conjunction of (3.733) and (3.735) now gives with the definition of a restriction

$$(\bar{x}, \bar{z}) \in [g \circ f] \upharpoonright A, \tag{3.736}$$

and then finally (by definition of a range) the desired

$$\bar{z} \in \mathrm{ran}([g \circ f] \upharpoonright A). \tag{3.737}$$

Thus, the proof of the first part of the equivalence in (3.725) is complete.

To prove the second part ('$\Leftarrow$'), we apply the same arguments (plus one additional argument) in reversed order to obtain (3.726). Assuming now (3.737) to be true, there exists by definition of a range an element, say $\bar{x}$, such that (3.736) holds. This in turn implies (3.733) and (3.735) with the definition of a restriction. It now follows with the definition of a composition from (3.735) that (3.734) holds. Thus, there are elements, say $\bar{x}'$, $\bar{y}'$ and $\bar{z}'$, such that $(\bar{x}', \bar{y}') \in f$, $(\bar{y}', \bar{z}') \in g$ and $(\bar{x}', \bar{z}') = (\bar{x}, \bar{z})$ hold, because the ordered pair $(\bar{x}, \bar{z})$ if formed by the previously established elements $\bar{x}$ and $\bar{z}$. The preceding equation implies $\bar{x}' = \bar{x}$ as well as $\bar{z}' = \bar{z}$ with the Equality Criterion for ordered pairs. Performing substitution based on these equations, the previously obtained $(\bar{x}', \bar{y}') \in f$ and $(\bar{y}', \bar{z}') \in g$ imply (3.732) and (3.729), respectively. Since (3.733) and (3.732) are thus both true, it follows with the definition of a restriction that (3.731) also holds. This in turn implies (3.730) with the definition of a range. The conjunction of this and the previously found (3.729) now gives (3.728) with the definition of a restriction, then (3.727) with the definition of a range, and finally (3.726) with the definition of an image.

This completes the proof of the equivalence in (3.726), and since $\bar{z}$ is arbitrary, it follows that the universal sentence (3.726) is true. We may then infer from this that the sets $g[\mathrm{ran}(f \upharpoonright A)]$ and $\mathrm{ran}([g \circ f] \upharpoonright A)$ are identical (according to the Equality Criterion for sets). Since X, Y, Z, f, g and A were arbitrary, we may therefore conclude that the proposition is true. □

Since the composition $g \circ f$ of any functions $f : X \to Y$ and $g : Y \to Z$ constitutes a function from X to Z according to Proposition 3.178, we may immediately apply the definition of an image to the preceding proposition to obtain the following simpler statement.

Corollary 3.220. *It is true that*

$$\forall X, Y, Z, f, g, A\,([f \in Y^X \wedge g \in Z^Y \wedge A \subseteq X] \Rightarrow g[f[A]] = (g \circ f)[A]). \tag{3.738}$$

Exercise 3.89. Demonstrate the truth of the following universal sentence.

$$\forall X,Y,f,A\,([f\in Y^X \wedge g\in Z^Y \wedge A\subseteq X] \\ \Rightarrow \forall x\,(x\in A\Rightarrow g(f(x))\in g[f[A]])). \tag{3.739}$$

(Hint: Apply (3.719) and (3.720).)

In the particular case of $A=X$ we obtain the following characterization of the range of a composition.

Corollary 3.221. *For any functions $f:X\to Y$ and $g:Y\to Z$, the image of the range of f under g is identical with the range of the composition of g and f, that is,*

$$\forall X,Y,Z,f,g\,([f\in Y^X \wedge g\in Z^Y]\Rightarrow g[\mathrm{ran}(f)]=\mathrm{ran}(g\circ f)). \tag{3.740}$$

Proof. Letting $f:X\to Y$ and $g:Y\to Z$ be arbitrary functions, it follows with the fact $X\subseteq X$ (see Exercise 2.10) in view of (3.724) that

$$g[\mathrm{ran}(f\restriction X)]=\mathrm{ran}([g\circ f]\restriction X) \tag{3.741}$$

holds. Since $f\restriction X$ and $g\circ f$ are functions with domain X due to Proposition 3.164 and Proposition 3.178, it follows with (3.103) that $f\restriction X=f$ as well as $[g\circ f]\restriction X=g\circ f$, so that (3.741) implies via substitution the equation in (3.740), which then follows to be true for any sets X, Y, Z, f and g. □

Theorem 3.222. *The following sentences are true for any function $f:X\to Y$ and any subset $B\subseteq Y$.*

a) There exists a unique set $f^{-1}[B]$ consisting of all the elements x in X for which the function value $f(x)$ is in B.

b) The set $f^{-1}[B]$ satisfies also

$$\forall x\,(x\in f^{-1}[B]\Leftrightarrow f(x)\in B). \tag{3.742}$$

Proof. Letting X, Y, f and B be arbitrary sets such that $f:X\to Y$ and $B\subseteq Y$ hold, the Axiom of Specification yields in connection with the Equality Criterion the true uniquely existential sentence

$$\exists! f^{-1}[B]\,\forall x\,(x\in f^{-1}[B]\Leftrightarrow[x\in X \wedge f(x)\in B]),$$

which proves a). Thus, the set $f^{-1}[B]$ satisfies

$$\forall x\,(x\in f^{-1}[B]\Leftrightarrow[x\in X \wedge f(x)\in B]). \tag{3.743}$$

Concerning b), we let x be arbitrary and assume first $x \in f^{-1}[B]$, which implies with (3.743) in particular the desired $f(x) \in B$. Conversely, we now assume $f(x) \in B$ to be true, which expression makes only sense in case the function value $y = f(x)$ is specified. Since we may write this as $(x, y) \in f$, it follows by definition of a domain that $x \in \mathrm{dom}(f)$ $[= X]$ holds. Thus, the conjunction of $x \in X$ and $f(x) \in B$ is true, which in turn implies $x \in f^{-1}[B]$ with (3.743). As x was arbitrary, we may therefore conclude that the universal sentence (3.743) is true, so that the proof of b) is complete. Because X, Y, f and B were arbitrary, we may finally conclude that the theorem is true. $\square$

Note 3.30. Since $x \in f^{-1}[B]$ implies with (3.743) in particular $x \in X$ for any x, we obtain $f^{-1}[B] \subseteq X$ by definition of a subset, so that the set $f^{-1}[B]$ is included in the domain X of a function $f : X \to Y$ (for a subset B of the codomain Y).

Definition 3.55 (Inverse image/preimage). For any function $f : X \to Y$ and any subset $B \subseteq Y$ we call the set $f^{-1}[B]$ consisting of all elements x in X for which $f(x)$ is in B the *inverse image* or *preimage* of B under f. We symbolize this set also by

$$\{x : f(x) \in B\}. \tag{3.744}$$

Note 3.31. For any function $f : X \to Y$, we have $\emptyset \subseteq Y$ with (2.43) and $Y \subseteq Y$ with (2.10), so that the preimages of $\emptyset$ and Y under f are defined.

Exercise 3.90 (Inverse image of the empty set and of the codomain). Verify for any function $f : X \to Y$ that

a) the inverse image of the empty set is also empty, that is,

$$f^{-1}[\emptyset] = \emptyset. \tag{3.745}$$

(Hint: Apply Definition 2.6 and Method 1.12.)

b) the inverse image of the codomain is the domain, that is,

$$f^{-1}[Y] = X. \tag{3.746}$$

(Hint: Apply the Equality Criterion for sets.)

The following proposition shows that the 'composition' of image and inverse image does not produce a strict identity like the composition of function and inverse function.

Proposition 3.223. *For any function $f : X \to Y$ and any subset $B \subseteq Y$ it is true that the image of the inverse image of B (under f) is included in B, that is,*

$$\forall X, Y, f, B \left([f \in Y^X \ \land \ B \subseteq Y] \Rightarrow f[f^{-1}[B]] \subseteq B \right). \tag{3.747}$$

Proof. We let X, Y, f and B be arbitrary sets, assume f to be an element of Y^X, and assume B to be a subset of Y. To prove that this implies $f[f^{-1}[B]] \subseteq B$, we verify (using the definition of a subset)

$$\forall y \left(y \in f[f^{-1}[B]] \Rightarrow y \in B \right). \tag{3.748}$$

For this purpose, we let y be arbitrary and assume $y \in f[f^{-1}[B]]$. By definition of an image, the assumption implies $y \in \operatorname{ran}(f \restriction f^{-1}[B])$, and the definition of a range shows then that there exists an element, say $\bar{x}$, such that $(\bar{x}, y) \in f \restriction f^{-1}[B]$ holds. This implies with the definition of a restriction $(\bar{x}, y) \in f$ and $\bar{x} \in f^{-1}[B]$, where the former finding can be written in function notation as $y = f(\bar{x})$ and where the latter implies $f(\bar{x}) \in B$ by definition of an inverse image. Combining these new findings via substitution yields $y \in B$, which completes the proof of the implication in (3.748). As y was arbitrary, we may therefore conclude that (3.748) is true, so that $f[f^{-1}[B]] \subseteq B$ holds indeed, and this in turn proves the implication in (3.747). Finally, since X, Y, f and B are arbitrary, we now conclude that (3.747) holds, as proposed. □

Exercise 3.91. Show that any subset of a set Y is identical with the image of the inverse image of B under any function onto Y, that is,

$$\forall X, Y, f, B \left([f : X \twoheadrightarrow Y \ \land \ B \subseteq Y] \Rightarrow f[f^{-1}[B]] = B \right). \tag{3.749}$$

(Hint: Apply the Axiom of Extension, using (3.747) and (3.630).)

Proposition 3.224. *For any function $f : X \to Y$, any subset A of X and any subset B of Y it is true that, if the image of A (under f) is included in B, then A is included in the inverse image of B (under f), that is,*

$$\forall X, Y, f, A, B \left([f \in Y^X \land A \subseteq X \land B \subseteq Y] \Rightarrow \big(f[A] \subseteq B \Rightarrow A \subseteq f^{-1}[B] \big) \right). \tag{3.750}$$

Proof. We let X, Y be arbitrary sets, f an arbitrary element of Y^X, A an arbitrary subset of X, and B an arbitrary subset of Y. Next, we assume $f[A] \subseteq B$, or equivalently

$$\forall y \left(y \in f[A] \Rightarrow y \in B \right), \tag{3.751}$$

(using the definition of a subset), and verify $A \subseteq f^{-1}[B]$, i.e.

$$\forall x\,(x \in A \Rightarrow x \in f^{-1}[B]), \tag{3.752}$$

For this purpose, we let x be arbitrary, assume $x \in A$, and show that this implies $x \in f^{-1}[B]$. We first observe that the restriction $f \restriction A$ is a function with domain A (see Proposition 3.164). Then, by definition of a function, there exists a unique function value $y = f \restriction A(x)$, so that $(x, y) \in f \restriction A$. This implies on the one hand $(x, y) \in f$ with the definition of a restriction, and thus $y = f(x)$. On the other hand, it follows by definition of a range that $y \in \mathrm{ran}(f \restriction A)$ holds, and therefore $y \in f[A]$ by definition of a direct image. This in turn implies $y \in B$ with (3.751) and consequently $f(x) \in B$ with the previously established equation $y = f(x)$. Now, $f(x) \in B$ implies the desired $x \in f^{-1}[B]$ with the definition of an inverse image. Then, since x is arbitrary, we conclude that (3.752) is true, which proves $A \subseteq f^{-1}[B]$. As X, Y, f, A and B were also arbitrary, we finally conclude that the proposed universal sentence (3.750) is true. □

Exercise 3.92. Verify the following implication for any function $f : X \to Y$, any subset A of X and any subset B of Y. If A is included in the inverse image of B (under f), then the image of A (under f) is included in B, that is,

$$\forall X, Y, f, A, B\,([f \in Y^X \wedge A \subseteq X \wedge B \subseteq Y] \Rightarrow \big(A \subseteq f^{-1}[B] \Rightarrow f[A] \subseteq B\big)). \tag{3.753}$$

Proposition 3.225. *For any constant function $g_c : X \to Y$ and any subset $B \subseteq Y$, it is true that the inverse image of B under g_c is defined. Moreover, $g_c^{-1}[B] = X$ or $g_c^{-1}[B] = \emptyset$ holds.*

Proof. We let X, Y, c, g_c, B be arbitrary and assume that c is an element of Y, that g_c is the constant function from X to $\{c\}$, and that B is a subset of Y. Since $c \in Y$ implies $\{c\} \subseteq Y$ with (2.184), we see that Y is a codomain of g_c, so that we have $g_c : X \to Y$. As we also assumed $B \subseteq Y$ to be true, we then see that $g_c^{-1}[B]$ is defined. We now consider the two cases $c \in B$ and $c \notin B$.

In case $c \in B$ is true, we may establish $g_c^{-1}[B] = X$ by means of the Equality Criterion for sets. To do this, we prove

$$\forall x\,(x \in g_c^{-1}[B] \Leftrightarrow x \in X). \tag{3.754}$$

Letting x be arbitrary, we see on the one hand that assuming $x \in g_c^{-1}[B]$ to be true implies $x \in X$ by definition of an inverse image. Assuming conversely $x \in X$ to hold, we obtain with (3.534) the function value

$g_c(x) = c$ [$\in B$]. Then, $g_c(x) \in B$ implies $x \in g_c^{-1}[B]$ with the definition of an inverse image. Thus, the equivalence in (3.754) is true, and as x was arbitrary, we may therefore conclude that (3.754) holds, so that the sets $g_c^{-1}[B]$ and X are indeed equal.

In the other case that $c \notin B$ is true, we establish $g_c^{-1}[B] = \emptyset$ by verifying

$$\forall x\,(x \notin g_c^{-1}[B]). \tag{3.755}$$

For this purpose, we take an arbitrary x and prove $x \notin g_c^{-1}[B]$ by contradiction, assuming $\neg x \notin g_c^{-1}[B]$ to hold, so that $x \in g_c^{-1}[B]$ follows to be true with the Double Negation Law. We then obtain with the definition of an inverse image $x \in X$ and $g_c(x) \in B$. Here, $x \in X$ yields $g_c(x) = c$ with (3.534). With this equation, $g_c(x) \in B$ gives $c \in B$ via substitution, which is in contradiction to the current case assumption $c \notin B$. Thus, the proof of $x \notin g_c^{-1}[B]$ is complete, and since x is arbitrary, we may infer from this the truth of the universal sentence (3.755), which in turn implies $g_c^{-1}[B] = \emptyset$ with the definition of the empty set.

As X, Y, c, g_c and B were arbitrary, we may finally conclude that the proposition is true. □

Proposition 3.226. *For any sets X, Y and any surjection $f : X \twoheadrightarrow Y$ the inverse image (under f) of the singleton formed by any element c of Y is nonempty, i.e.*

$$\forall X, Y, f\,(f : X \twoheadrightarrow Y \;\Rightarrow \forall c\,(c \in Y \Rightarrow f^{-1}[\{c\}] \neq \emptyset)). \tag{3.756}$$

Proof. We let X and Y be arbitrary sets and f an arbitrary surjection from X to Y, so that $\mathrm{ran}(f) = Y$ holds by definition of a surjection. Next, we let c be an arbitrary element of Y, so that the preceding equation gives $c \in \mathrm{ran}(f)$. By definition of a range, there then exists a constant, say $\bar{x}$, such that $(\bar{x}, c) \in f$ holds, which then implies $\bar{x} \in \mathrm{dom}(f)$ with the definition of a domain, so that $\bar{x} \in X$. We may write $(\bar{x}, c) \in f$ also as $f(\bar{x}) = c$, which gives $f(\bar{x}) \in \{c\}$ with (2.169), and consequently $\bar{x} \in f^{-1}[\{c\}]$ with the definition of an inverse image. This shows that the inverse image $f^{-1}[\{c\}]$ is nonempty. As c is arbitrary, we therefore conclude that the universal sentence with respect to c in (3.756) holds. Because X, Y and f were also arbitrary, we then further conclude that the proposed universal sentence (3.756) holds. □

Exercise 3.93. Show for any function $f : X \to Y$ that the inverse image of the singleton formed by an element of Y which is not in the range of f is empty, that is,

$$\forall X, Y, f, y\,([f \in Y^X \,\wedge\, y \in Y] \Rightarrow [y \notin \mathrm{ran}(f) \Rightarrow f^{-1}[\{y\}] = \emptyset]). \tag{3.757}$$

(Hint: Prove the second implication by contraposition, using (2.42), (2.169) and the Double Negation Law.)

Exercise 3.94. Show for any sets X, Y and any surjection $f : X \twoheadrightarrow Y$ that the inverse image (under f) of any nonempty subset of Y is nonempty, i.e.

$$\forall X, Y, f\,(f : X \twoheadrightarrow Y \;\Rightarrow \forall C\,([C \subseteq Y \,\wedge\, C \neq \emptyset] \Rightarrow f^{-1}[C] \neq \emptyset)). \quad (3.758)$$

(Hint: Proceed similarly as in the proof of Proposition 3.226, using (2.42).)

Proposition 3.227. *For any function $f : X \to Y$ the inverse image of the complement (with respect to the codomain) of a subset B of Y equals the complement (with respect to the domain) of the inverse image of B, that is,*

$$\forall B\,(B \subseteq Y \Rightarrow f^{-1}[B^c] = (f^{-1}[B])^c). \quad (3.759)$$

Proof. We let X, Y, f and B be arbitrary sets, assume $f : X \to Y$, and assume furthermore that B is a subset of Y. Letting x also be arbitrary, we now obtain the true equivalences

$$\begin{aligned} x \in f^{-1}[B^c] &\Leftrightarrow f(x) \in B^c \\ &\Leftrightarrow f(x) \in Y \setminus B \\ &\Leftrightarrow f(x) \in Y \,\wedge\, \neg f(x) \in B \\ &\Leftrightarrow x \in f^{-1}[Y] \,\wedge\, \neg x \in f^{-1}[B] \\ &\Leftrightarrow x \in f^{-1}[Y] \setminus f^{-1}[B] \\ &\Leftrightarrow x \in X \setminus f^{-1}[B] \\ &\Leftrightarrow x \in (f^{-1}[B])^c \end{aligned}$$

using (3.742), the definition of a complement (applied with respect to Y), the definition of a set difference, again (3.742), again the definition of a set difference, then (3.746), and finally the definition of a complement (applied now with respect to X). Because x is arbitrary, we may therefore conclude that the equivalence

$$x \in f^{-1}[B^c] \Leftrightarrow x \in (f^{-1}[B])^c$$

holds for all x, so that $f^{-1}[B^c] = (f^{-1}[B])^c$ follows to be true with the Equality Criterion for sets. Since X, Y, f and B were arbitrary, we may then further conclude that the proposition holds, as claimed. □

Exercise 3.95. Verify for any function $f : X \to Y$ and any subsets A, B of Y that

a) the inverse image of the intersection of A and B is identical with the intersection of the inverse image of A and the inverse image of B, i.e.

$$\forall X, Y, f, A, B\,([f \in Y^X \;\wedge\; A \subseteq Y \;\wedge\; B \subseteq Y] \\ \Rightarrow\; f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]). \tag{3.760}$$

b) the inverse image of the union of A and B is identical with the union of the inverse image of A and the inverse image of B, i.e.

$$\forall X, Y, f, A, B\,([f \in Y^X \;\wedge\; A \subseteq Y \;\wedge\; B \subseteq Y] \\ \Rightarrow\; f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]). \tag{3.761}$$

(Hint: Proceed similarly as in the proof of Proposition 3.227.)

Exercise 3.96. Show for any functions $f : X \to Y$, $g : Y \to Z$ and any subset $B \subseteq Z$ that the inverse image of B under the composition of g and f is identical with the inverse image (under f) of the inverse image (under g) of B, that is,

$$\forall X, Y, Z, f, g, B\,([f \in Y^X \;\wedge\; g \in Z^Y \;\wedge\; B \subseteq Z] \\ \Rightarrow\; (g \circ f)^{-1}[B] = f^{-1}[g^{-1}[B]]). \tag{3.762}$$

(Hint: Apply Notation 3.6.)

Theorem 3.228. *For any function $f : X \to Y$ there is a unique function $f^{\leftarrow}$ from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ such that any $B \subseteq Y$ is mapped to the preimage of B under f.*

Proof. Letting X, Y and f be arbitrary sets such that $f : X \to Y$, we may apply the Axiom of Specification in connection with the Equality Criterion for sets to establish the true uniquely existential sentence

$$\exists! f^{\leftarrow}\, \forall Z\,(Z \in f^{\leftarrow} \Leftrightarrow [Z \in \mathcal{P}(Y) \times \mathcal{P}(X) \wedge \exists B\,(B \subseteq Y \wedge Z = (B, f^{-1}[B]))]).$$

We thus specified the set $f^{\leftarrow}$ such that it satisfies

$$\forall Z\,(Z \in f^{\leftarrow} \Leftrightarrow [Z \in \mathcal{P}(Y) \times \mathcal{P}(X) \;\wedge\; \exists B\,(B \subseteq Y \;\wedge\; Z = (B, f^{-1}[B]))]). \tag{3.763}$$

Clearly, $Z \in f^{\leftarrow}$ implies in particular $Z \in \mathcal{P}(Y) \times \mathcal{P}(X)$ for any Z, so that the inclusion $f^{\leftarrow} \subseteq \mathcal{P}(Y) \times \mathcal{P}(X)$ holds by definition of a subset. We may therefore apply the Function Criterion to prove that the binary relation $f^{\leftarrow}$ is a function from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. To do this, we verify

$$\forall B\,(B \in \mathcal{P}(Y) \Rightarrow \exists! A\,(A \in \mathcal{P}(X) \;\wedge\; (B, A) \in f^{\leftarrow})). \tag{3.764}$$

Letting $\bar{B}$ be arbitrary in $\mathcal{P}(Y)$, so that $\bar{B} \subseteq Y$ holds by definition of a power set, it follows that the inverse image $f^{-1}[\bar{B}]$ is a uniquely specified set according to Theorem 3.222a). Then, the ordered pair $\bar{Z} = (\bar{B}, f^{-1}[\bar{B}])$ is also uniquely determined. Together with $\bar{B} \subseteq Y$, this equation shows that there exists a set B such that the conjunction $B \subseteq Y \wedge Z = (B, f^{-1}[B])$ holds. Let us now observe that the inverse image $f^{-1}[\bar{B}]$ is a subset of X according to Note 3.30, and thus an element of the power set $\mathcal{P}(X)$. Thus, the conjunction $\bar{B} \in \mathcal{P}(Y) \wedge f^{-1}[\bar{B}] \in \mathcal{P}(X)$ is true, which implies $(\bar{B}, f^{-1}[\bar{B}]) \in \mathcal{P}(Y) \times \mathcal{P}(X)$ with the definition of the Cartesian product of two sets. Together with the previously established existential sentence with respect to B, this implies with (3.763) that $(\bar{B}, f^{-1}[\bar{B}]) \in f^{\leftarrow}$ holds. Together with the previously obtained $f^{-1}[\bar{B}] \in \mathcal{P}(X)$, this finding proves the existential part of the uniquely existential sentence in (3.764). Regarding the uniqueness part, we take arbitrary sets A and A', assume $A, A' \in \mathcal{P}(X)$ as well as $(\bar{B}, A), (\bar{B}, A') \in f^{\leftarrow}$ to be true, and show that these assumptions imply $A = A'$. Here, the latter assumption implies with (3.763) in particular that there are sets, say $\bar{\bar{B}}$ and $\bar{\bar{B}}'$, satisfying in particular $(\bar{B}, A) = (\bar{\bar{B}}, f^{-1}[\bar{\bar{B}}])$ and $(\bar{B}, A') = (\bar{\bar{B}}', f^{-1}[\bar{\bar{B}}'])$. These equations give now with the Equality Criterion for ordered pairs $\bar{B} = \bar{\bar{B}}$ and $A = f^{-1}[\bar{\bar{B}}]$, as well as $\bar{B} = \bar{\bar{B}}'$ and $A' = f^{-1}[\bar{\bar{B}}']$. We therefore obtain with the uniqueness of an inverse image

$$A = f^{-1}[\bar{\bar{B}}] = f^{-1}[\bar{B}] = f^{-1}[\bar{\bar{B}}'] = A',$$

which equations yield the desired $A = A'$. This completes the proof of the uniqueness part, and thus the proof of the uniquely existential sentence in (3.764). Since $\bar{B}$ is arbitrary, we may now further conclude that the universal sentence (3.764) holds, which in turn implies $f^{\leftarrow} : \mathcal{P}(Y) \to \mathcal{P}(X)$ (with the Function Criterion). This follows then to be true for any X, Y and f. □

Definition 3.56 (Inverse-image function, preimage function). For any function $f : X \to Y$ we call the function

$$f^{\leftarrow} : \mathcal{P}(Y) \to \mathcal{P}(X), \quad B \mapsto f^{\leftarrow}(B) = f^{-1}[B], \tag{3.765}$$

which maps every subset of Y into the inverse image of B under f, the *inverse-image function* or the *preimage function* of f.

The preimage function of a constant function has a particularly simple form.

Proposition 3.229. *The preimage function of any constant function g_c : $X \to Y$ consists of the domain of f and the empty set, that is,*

$$\forall X, Y, c, g_c \, (g_c : X \to Y \ \Rightarrow \operatorname{ran}(g_c^{\leftarrow}) = \{X, \emptyset\}). \tag{3.766}$$

Proof. We let X, Y, c and g_c be arbitrary and assume that g_c is the constant function $g_c : X \to Y$ with value c. To prove the equation in (3.766), we apply the Equality Criterion for sets and verify the equivalent universal sentence

$$\forall A\,(A \in \operatorname{ran}(g_c^{\leftarrow}) \Leftrightarrow A \in \{\emptyset, X\}). \tag{3.767}$$

For this purpose, we let $\bar{A}$ be an arbitrary set and assume first $A \in \operatorname{ran}(g_c^{\leftarrow})$. By definition of a range, there then exists a set, say $\bar{B}$, such that $(\bar{B}, \bar{A}) \in g_c^{\leftarrow}$ holds. We therefore obtain with the definition of a preimage function

$$\bar{A} = g_c^{\leftarrow}(\bar{B}) = g_c^{-1}[\bar{B}],$$

where $g_c^{-1}[\bar{B}] = X$ or $g_c^{-1}[\bar{B}] = \emptyset$ holds due to Proposition 3.225. Then substitution yields the true disjunction $\bar{A} = X \vee \bar{A} = \emptyset$, so that $\bar{A} \in \{X, \emptyset\}$ follows to be true with the definition of a pair, proving the first part ('$\Rightarrow$') of the equivalence in (3.767). Regarding the second part ('$\Leftarrow$'), we now assume $\bar{A} \in \{X, \emptyset\}$ to be true, so that $\bar{A} = X \vee \bar{A} = \emptyset$ holds (by definition of a pair). On the one hand, if the first part $\bar{A} = X$ of this disjunction is true, then we obtain

$$g_c^{\leftarrow}(Y) = g_c^{-1}[Y] = X\ [= \bar{A}]$$

with the definition of a preimage function (using the fact that Y is evidently element of the domain $\mathcal{P}(Y)$ of $g_c^{\leftarrow}$) and with Exercise 3.90b), which we may also write as $(Y, \bar{A}) \in g_c^{\leftarrow}$. Consequently, $\bar{A} \in \operatorname{ran}(g_c^{\leftarrow})$ follows to be true by definition of a range. On the other hand, if $\bar{A} = \emptyset$ is true, then we obtain

$$g_c^{\leftarrow}(\emptyset) = g_c^{-1}[\emptyset] = \emptyset\ [= \bar{A}]$$

by applying again with the definition of a preimage function (using now the fact that $\emptyset$ is clearly element of the domain $\mathcal{P}(Y)$ of $g_c^{\leftarrow}$) and by applying Exercise 3.90a). We may now write this as $(\emptyset, \bar{A}) \in g_c^{\leftarrow}$, and therefore $\bar{A} \in \operatorname{ran}(g_c^{\leftarrow})$ follows again to be true with the definition of a range. This completes the proof of the equivalence in (3.767), and since $\bar{A}$ is arbitrary, we may infer from this the truth of the universal sentence (3.767), and thus the truth of the desired equation $\operatorname{ran}(g_c^{\leftarrow}) = \{X, \emptyset\}$. This in turn proves the implication in (3.767), and as X, Y, c and g_c were arbitrary, we may then finally conclude that the proposition is true. □

3.4.6. Families

The concept of a functions may be applied to list constants according to a specified index (set).

Notation 3.7 (Family, term, index set, index). For any function $f : I \to Y$ we also write

$$(f_i)_{i\in I} \quad \text{or} \quad (f_i \mid i \in I) \tag{3.768}$$

and speak of a *family* in Y, and we call

$$f_i = f(i) \tag{3.769}$$

for every $i \in I$ a *term* of the family. Then, for any family $(f_i)_{i\in I}$, we call I the *index set* and each $i \in I$ an *index* of the family.

Theorem 3.230 (Criterion for lower & upper bounds for a family). *It is true for any family $f = (f_i)_{i\in I}$ in any set in Y and for any partial ordering $\leq_Y$ of Y that*

a) an element $a \in Y$ is a lower bound for f iff

$$\forall i\,(i \in I \Rightarrow a \leq_Y f_i). \tag{3.770}$$

b) an element $u \in Y$ is an upper bound for f iff

$$\forall i\,(i \in I \Rightarrow f_i \leq_Y u). \tag{3.771}$$

Proof. We let Y be an arbitrary set, $\leq_Y$ an arbitrary reflexive partial ordering of Y, and $f = (f_i)_{i\in I}$ an arbitrary family in Y. Concerning a), we let a be an arbitrary element of Y. To prove the first part ('$\Rightarrow$') of the equivalence, we assume that a is a lower bound for f, which means by definition that a is a lower bound for the range of f. We now let $i \in I$ be arbitrary, so that i is an element of the domain I of f. By definition, there exists then a particular constant $\bar{y}$ such that $(i, \bar{y}) \in f$. We can write the latter also as $\bar{y} = f(i) = f_i$ by using the notations for functions and families. Furthermore, $(i, \bar{y}) \in f$ implies $\bar{y} \in \mathrm{ran}(f)$ with the definition of a range. Since a is a lower bound for that range, it is by definition true that $a \leq_Y \bar{y}$ $[= f_i]$. We thus have $a \leq_Y f_i$, and as i was arbitrary, we may therefore conclude that the universal sentence (3.770) is true, so that the proof of the first part of the equivalence is complete.

To establish the second part ('$\Leftarrow$'), we assume now (3.770) to be true, and we show that a is a lower bound $\mathrm{ran}(f)$. Letting $y \in \mathrm{ran}(f)$ be arbitrary, there evidently exists a particular index $j \in I$ with $(j, y) \in f$, that is, with $y = f_j$. Consequently, $j \in \mathrm{dom}(f)$ $[= I]$ is clearly true, so that

$a \leq_Y f_j$ holds according to (3.770); thus, $a \leq y$. As y was arbitrary, we may therefore conclude that a is a lower bound for $\mathrm{ran}(f)$, completing the proof of the equivalence.

Since a, Y, $\leq_Y$ and $f = (f_i)_{i \in I}$ were initially arbitrary as well, we may further conclude that a) holds. Sentence b) can be proved similarly. □

Exercise 3.97. Prove Part b) of Theorem 3.230.

We now consider the following, often occurring special cases of a family.

Proposition 3.231. *Verify for any sets I, X, Y and for any family $f = (f_i)_{i \in I}$ in Y^X that the family $f^{(x)} = (f_i(x))_{i \in I}$ exists uniquely in Y for any $x \in X$.*

Proof. We let I, X, Y, f and x be arbitrary, assume $f = (f_i)_{i \in I}$ to be a family in Y^X, and assume moreover $x \in X$ to be true. We now apply Function definition by replacement to prove that there exists a unique function $f^{(x)}$ with domain I such that

$$\forall i \, (i \in I \Rightarrow f^{(x)}(i) = f_i(x)) \tag{3.772}$$

holds. For this purpose, we verify

$$\forall i \, (i \in I \Rightarrow \exists! y \, (y = f_i(x))), \tag{3.773}$$

letting i be arbitrary and assuming $i \in I$ to be true. Therefore, there exists the unique term f_i (in Y^X) of the family $f = (f_i)_{i \in I}$, according to the Function Criterion, and this term is a function from X to Y. Then, the initial assumption $x \in X$ implies the existence of the unique function value $f_i(x)$ (in Y), again by virtue of the Function Criterion. Thus, $f_i(x)$ is a constant in Y, so that we obtain the true existential sentence $\exists! y \, (y = f_i(x))$ with (1.109). As i was arbitrary, we may therefore conclude that the universal sentence (3.773) is true, which in turn implies the unique existence of a function $f^{(x)}$ with domain I satisfying (3.772). Thus, $f^{(x)}$ is a family with index set I, which we may write as $f^{(x)} = (f_i(x))_{i \in I}$. We now show that Y is a codomain of this family, i.e. that $\mathrm{ran}(f^{(x)}) \subseteq Y$ holds. To do this, we verify

$$\forall y \, (y \in \mathrm{ran}(f^{(x)}) \Rightarrow y \in Y), \tag{3.774}$$

letting y be arbitrary and assuming $y \in \mathrm{ran}(f^{(x)})$ to hold. By definition of a range, there exists then a constant, say $\bar{k}$, with $(\bar{k}, y) \in f^{(x)}$, which we may write in function/family notation as $y = f^{(x)}(\bar{k}) = f_{\bar{k}}(x)$. We may write now the resulting equation $y = f_{\bar{k}}(x)$ also as $(x, y) \in f_{\bar{k}}$, which shows that y is an element of the range of the function $f_{\bar{k}} : X \to Y$. Here, $\mathrm{ran}(f_{\bar{k}}) \subseteq Y$ holds by definition of a codomain, so that $y \in \mathrm{ran}(f_{\bar{k}})$ gives $y \in Y$ with the definition of a subset. This finding completes the proof of the implication in (3.774), and since y is arbitrary, we may therefore conclude that (3.774) holds, which universal sentence yields $\mathrm{ran}(f^{(x)}) \subseteq Y$ (by definition of a subset). Thus, Y is a codomain of $f^{(x)}$, so that the latter is a family in Y.

Because I, X, Y, f and x were initially arbitrary, we may infer from these findings the truth of the proposed sentence. □

Note 3.32. We call a family $f = (f_i)_{i\in I}$ in a set Y^X also a *family of functions* (from X to Y).

Definition 3.57 (Sequence, family & sequence of sets). We say that a family $(f_i)_{i\in I}$

(1) is a *sequence* if

- $I = n$ for some $n \in \mathbb{N}$, or
- $I = \mathbb{N}$, in which case we write the family also as

$$(f_n)_{n\in\mathbb{N}}, \tag{3.775}$$

or

- $I = \mathbb{N}_+$, in which case we write

$$(f_n)_{n\in\mathbb{N}_+}. \tag{3.776}$$

(2) is a *family of sets* iff it is a family in some set system.

(3) is a *sequence of sets* iff it is a sequence in some set system.

Note 3.33. In Chapter 4, we will introduce sequences with index sets different from those considered in Definition 3.57(1). Therefore, the current definition of a sequence is not bidirectional, as indicated by the one-directional 'if'.

Proposition 3.232. *For any set system $\mathcal{K}$ there is a family $(A_i)_{i\in I}$ with range $\mathcal{K}$, i.e.*

$$\forall \mathcal{K}\, \exists I, A\, (A = (A_i)_{i\in I} \wedge \mathrm{ran}(A) = \mathcal{K}). \tag{3.777}$$

Proof. We let $\mathcal{K}$ be an arbitrary set system, define $I = \mathcal{K}$, and let $A = \mathrm{id}_{\mathcal{K}}$ be the identity function on $\mathcal{K}$. Thus, we may write A as $A : \mathcal{K} \to \mathcal{K}$, or equivalently as $A : I \to \mathcal{K}$, and therefore as $(A_i)_{i\in I}$ (by definition of a family). The identity function $A : \mathcal{K} \to \mathcal{K}$ is a bijection (see Corollary 3.203), thus in particular a surjection (by Property 2 of a bijection), which implies $\mathrm{ran}(A) = \mathcal{K}$ with the definition of a surjection. Therefore, the existential sentence in (3.777) holds for an arbitrary $\mathcal{K}$, so that we may conclude that the universal sentence (3.777) is true. □

Definition 3.58 (Union & intersection of a family of sets). For any family of sets $A = (A_i \mid i \in I)$

a) we call the set

$$\bigcup_{i \in I} A_i = \bigcup \operatorname{ran}(A) \tag{3.778}$$

the *union of* A, and

b) with $I \neq \emptyset$ we call the set

$$\bigcap_{i \in I} A_i = \bigcap \operatorname{ran}(A) \tag{3.779}$$

the *intersection of* A.

Note 3.34. Since $[\operatorname{dom}(A) =]\ I \neq \emptyset$ implies $\operatorname{ran}(A) \neq \emptyset$ due to Exercise 3.19 in connection with the Law of Contraposition, the intersection (3.779) is then indeed specified.

We will use a slightly different notation for such unions and intersections in the case that the family is a sequence of sets with domain $\mathbb{N}$ or $\mathbb{N}_+$.

Notation 3.8. We will write for the *union of a sequence of sets* $A = (A_n)_{n \in \mathbb{N}}$ and $A = (A_n)_{n \in \mathbb{N}_+}$, respectively,

$$\bigcup_{n=0}^{\infty} A_n, \tag{3.780}$$

$$\bigcup_{n=1}^{\infty} A_n. \tag{3.781}$$

Similarly, we write for the *intersection of a sequence of sets* $A = (A_n)_{n \in \mathbb{N}}$ and $A = (A_n)_{n \in \mathbb{N}_+}$, respectively,

$$\bigcap_{n=0}^{\infty} A_n, \tag{3.782}$$

$$\bigcap_{n=1}^{\infty} A_n. \tag{3.783}$$

Based on the family notation, we may now replace the characteristic properties (2.197) and (2.89) of the union and intersection of an arbitrary set system by the following equivalent, easier-to-handle properties.

Theorem 3.233 (Characterization of the union & the intersection of a family of sets). *For any family of sets* $A = (A_i)_{i \in I}$, *a) it is true that an element y is in the union of the family A iff y is in A_i for some index i in I, i.e.*

$$\forall y\,(y \in \bigcup_{i\in I} A_i \Leftrightarrow \exists i\,(i \in I \,\wedge\, y \in A_i)). \tag{3.784}$$

b) with $I \neq \emptyset$ it is true that an y is in the intersection of the family A iff y is in A_i for all indexes i in I, i.e.

$$\forall y\,(y \in \bigcap_{i\in I} A_i \Leftrightarrow \forall i\,(i \in I \,\Rightarrow\, y \in A_i)). \tag{3.785}$$

Proof. We let A and I be arbitrary sets and assume that A is a family of sets with index set I, so that $A = (A_i)_{i\in I}$ is a function with domain I.

Concerning a), we take an arbitrary y and prove the first implication ('⇒') directly, assuming $y \in \bigcup_{i\in I} A_i$ to be true. Thus, $y \in \bigcup \mathrm{ran}(A)$ holds by definition of the union of a family of sets, which in turn implies

$$\exists Y\,(Y \in \mathrm{ran}(A) \,\wedge\, y \in Y)$$

with the definition of the union of a set system. Thus, there exists a set, say $\bar{Y}$, satisfying the conjunction of $\bar{Y} \in \mathrm{ran}(A)$ and $y \in \bar{Y}$. Here, the first part of the conjunction implies by definition of a range that there is an element, say $\bar{i}$, such that $(\bar{i}, \bar{Y}) \in A$ holds; since A was assumed to be a function, we may write this also as $\bar{Y} = A(\bar{i}) = A_{\bar{i}}$. Then, the second part $y \in \bar{Y}$ of the preceding conjunction yields via substitution $y \in A_{\bar{i}}$. Moreover, since I was assumed to be the domain of A, it follows from $(\bar{i}, \bar{Y}) \in A$ that $\bar{i} \in I$ holds. We thus showed that there is some i such that $i \in I$ and $y \in A_i$ are both true, which existential sentence proves the first part of the equivalence in (3.784).

To prove the second implication ('⇐') directly, we now assume that there exists an element, say $\bar{i}$, such that $\bar{i} \in I$ and $y \in A_{\bar{i}}$ are true. Here $\bar{i} \in I$ [$=$ $\mathrm{dom}(A)$] implies by definition of a domain that there exists a set, say $\bar{Y}$, with $(\bar{i}, \bar{Y}) \in A$, which we may also write as $\bar{Y} = A(\bar{i}) = A_{\bar{i}}$; with these equations, $y \in A_{\bar{i}}$ gives $y \in \bar{Y}$. Furthermore, it follows from $(\bar{i}, \bar{Y}) \in A$ with the definition of a range that $\bar{Y} \in \mathrm{ran}(A)$ holds. We thus established the existence of a set Y which satisfies both $Y \in \mathrm{ran}(A)$ and $y \in Y$. This existential sentence then further implies $y \in \bigcup \mathrm{ran}(A)$ by definition of the union of a set system, which yields $y \in \bigcup_{i\in I} A_i$ with the definition of the union of a family of sets. Thus, the proof of the equivalence in (3.784) is complete.

Since A and I were initially arbitrary, we may therefore conclude that a) holds for any set A and any set I. □

Exercise 3.98. Prove Part b) of Theorem 3.233.

Exercise 3.99. Prove that the union of any family of empty sets is empty, that is,

$$\forall I, A\,(A : I \to \{\emptyset\} \Rightarrow \bigcup_{i\in I} A_i = \emptyset). \tag{3.786}$$

Corollary 3.234. *The union of any set system $\mathcal{K}$ can be written as the union of a family of sets in $\mathcal{K}$, that is,*

$$\forall \mathcal{K}\, \exists I, A\,(A : I \to \mathcal{K} \,\wedge\, \bigcup \mathcal{K} = \bigcup_{i\in I} A_i). \tag{3.787}$$

Proof. Letting $\mathcal{K}$ be an arbitrary set system, there exist according to Proposition 3.232 sets, say $\bar{I}$ and $\bar{A}$, such that $\bar{A}$ is a family of sets $(A_i)_{i\in I}$ with $\mathrm{ran}(\bar{A}) = \mathcal{K}$. The latter equation then yields

$$\bigcup \mathcal{K} = \bigcup \mathrm{ran}(\bar{A}) = \bigcup_{i\in I} A_i$$

by applying substitution and the definition of the union of a family of sets, so that the existential sentence in (3.787) is evidently true. Since $\mathcal{K}$ was arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Exercise 3.100. Show that the intersection of any set system $\mathcal{K}$ can be written as the intersection of a family of sets in $\mathcal{K}$, that is,

$$\forall \mathcal{K}\, \exists I, A\,(A : I \to \mathcal{K} \,\wedge\, \bigcap \mathcal{K} = \bigcap_{i\in I} A_i). \tag{3.788}$$

Definition 3.59 (Family of pairwise disjoint sets, pairwise disjoint terms, partition). We say for any index set I, any set X and any system $\mathcal{K}$ of subsets of X that a family of sets $A : I \to \mathcal{K}$ is

(1) a *family of pairwise disjoint sets* in $\mathcal{K}$ (alternatively, that a family of sets $A : I \to \mathcal{K}$ has *pairwise disjoint terms*) iff the terms A_i and A_j are disjoint for any distinct indexes i and j, i.e. iff

$$\forall i, j\,([i, j \in I \,\wedge\, i \neq j] \Rightarrow A_i \cap A_j = \emptyset). \tag{3.789}$$

(2) a *partition* of X iff the terms of the family are pairwise disjoint and the union of the family is identical with X, that is, iff

$$\forall i, j\,([i, j \in I \,\wedge\, i \neq j] \Rightarrow A_i \cap A_j = \emptyset) \,\wedge\, \bigcup_{i\in I} A_i = X. \tag{3.790}$$

Proposition 3.235. *The following equation holds for any sequences of sets* $A = (A_n)_{n\in\mathbb{N}_+}$ *and* $B = (B_n)_{n\in\mathbb{N}_+}$.

$$\bigcup_{n=1}^{\infty}(A_n \cup B_n) = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right). \tag{3.791}$$

Proof. We let A and B be arbitrary sets such that $A = (A_n)_{n\in\mathbb{N}_+}$ and $B = (B_n)_{n\in\mathbb{N}_+}$ are families of sets. To prove the proposed equation, we apply the Equality Criterion for sets and verify

$$\forall y\,(y \in \bigcup_{n=1}^{\infty}(A_n \cup B_n) \Leftrightarrow y \in \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)), \tag{3.792}$$

letting y be arbitrary. Let us first observe the truth of the equivalence

$$y \in \bigcup_{n=1}^{\infty}(A_n \cup B_n) \;\Leftrightarrow\; \exists n\,(n \in \mathbb{N}_+ \;\wedge\; y \in A_n \cup B_n) \tag{3.793}$$

in view of (3.784) in connection with Notation 3.8, and then truth of the equivalences

$$\begin{aligned} & y \in \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right) \\ & \Leftrightarrow\; y \in \bigcup_{n=1}^{\infty} A_n \;\vee\; y \in \bigcup_{n=1}^{\infty} B_n & (3.794)\\ & \Leftrightarrow\; \exists n\,(n \in \mathbb{N}_+ \;\wedge\; y \in A_n) \;\vee\; \exists n\,(n \in \mathbb{N}_+ \;\wedge\; y \in B_n) & (3.795)\\ & \Leftrightarrow\; \exists n\,([n \in \mathbb{N}_+ \;\wedge\; y \in A_n] \;\vee\; [n \in \mathbb{N}_+ \;\wedge\; y \in B_n]) & (3.796) \end{aligned}$$

in light of the definition of the union of two sets, the Characterization of the union of a family of sets (applied to the given sequences) and the Distributive Law for quantification (1.75). Now, to prove the first part ('$\Rightarrow$') of the equivalence in (3.792), we assume

$$y \in \bigcup_{n=1}^{\infty}(A_n \cup B_n) \tag{3.797}$$

to be true, which assumption implies with (3.793) that there exists a constant, say $\bar{n}$, such that the conjunction of $\bar{n} \in \mathbb{N}_+$ and $y \in A_{\bar{n}} \cup B_{\bar{n}}$ holds. Since the equivalences

$$\begin{aligned} \bar{n} \in \mathbb{N}_+ \;\wedge\; y \in A_{\bar{n}} \cup B_{\bar{n}} \;&\Leftrightarrow\; \bar{n} \in \mathbb{N}_+ \;\wedge\; [y \in A_{\bar{n}} \;\vee\; y \in A_{\bar{n}}] & (3.798)\\ &\Leftrightarrow\; [\bar{n} \in \mathbb{N}_+ \;\wedge\; y \in A_{\bar{n}}] \;\vee\; [\bar{n} \in \mathbb{N}_+ \;\wedge\; y \in B_{\bar{n}}] & (3.799) \end{aligned}$$

are true because of the definition of the union of two sets and the Distributive Law for sentences (1.44), we see that the latter disjunction follows to be true, so that the existential sentence in (3.796) holds. Because of the true equivalences (3.794) – (3.796),

$$y \in \left(\bigcup_{n=1}^{\infty} A_n \right) \cup \left(\bigcup_{n=1}^{\infty} B_n \right) \tag{3.800}$$

follows then also to be true, completing the proof of the first part of the equivalence in (3.792).

To prove the second part ('$\Leftarrow$'), we now assume that (3.800) is true, so that the true equivalences (3.794) – (3.796) imply the existence of a particular constant $\bar{n}$ which satisfies the disjunction in (3.799). Because of the equivalences (3.798) – (3.799), $\bar{n}$ therefore satisfies also the conjunction on the left-hand side of the equivalence (3.798), which shows that the existential sentence (3.793) holds, which in turn implies (3.797) with the true equivalence (3.793). Thus, the second part of the equivalence is true as well, and as y is arbitrary, we may therefore conclude that the universal sentence (3.792) is true. Then, the truth of the equation (3.791) follows with the Equality Criterion for sets, and since A and B were initially also arbitrary, we may finally conclude that the proposition holds, as claimed. □

Proposition 3.236. *For any set X and any family of sets $(A_i)_{i \in I}$ with $I \neq \emptyset$ there exists the unique family of sets $(X \cap A_i)_{i \in I}$.*

Proof. We let X, I and A be arbitrary sets and assume that $A = (A_i)_{i \in I}$ is a family of sets with nonempty index set I. We now apply Function definition by replacement to establish the stated families of sets. For this purpose, we first prove

$$\forall i \, (i \in I \Rightarrow \exists! Y \, (Y = X \cap A_i)), \tag{3.801}$$

letting i be arbitrary in I. Since A is a function from I to $\mathrm{ran}(A)$ (using Exercise 3.70a)), the function value $A_i = A(i)$ exists then uniquely according to the Function Criterion. Then, the intersection $X \cap A_i$ is also uniquely specified, so that the uniquely existential sentence in (3.801) follows to be true with (1.109). Since i is arbitrary, we may therefore conclude that the universal sentence (3.801) holds. Therefore, there exists a unique function F with domain I satisfying $F(i) = X \cap A_i$. Thus, F is a family of sets with index set I and terms $F_i = F(i)$, so that we may write $F = (X \cap A_i)_{i \in I}$. As X, I and A were initially arbitrary sets, we may therefore conclude that the proposition is true. □

Exercise 3.101. Verify the following sentences.

a) For any set X and any family of sets $(A_i)_{i\in I}$ there exists the unique family of sets $(X \cup A_i)_{i\in I}$.

b) For any set X and any family of sets $(A_i)_{i\in I}$ there exists the unique family of sets $(A_i \setminus X)_{i\in I}$.

c) For any set X and any family of sets $(A_i)_{i\in I}$ there exist the unique families of sets $(X \times A_i)_{i\in I}$ and $(A_i \times X)_{i\in I}$.

d) For any set Y and any family of sets $(A_i)_{i\in I}$ in $\mathcal{P}(Y)$ there exists the unique family of sets $(A_i^c)_{i\in I}$, where complementation is taken with respect to Y.

e) For any function $f : X \to Y$ and any family of sets $(B_i)_{i\in I}$ in $\mathcal{P}(Y)$ there exists the unique family of sets $(f^{-1}[B_i])_{i\in I}$.

Exercise 3.102. Show for any set X and any family $(A_i)_{i\in I}$ of pairwise disjoint sets with $I \neq \emptyset$ that $(X \cap A_i)_{i\in I}$ is also a family of pairwise disjoint sets.

(Hint: Use Proposition 3.236, (2.62), (2.72) and (2.58).)

Lemma 3.237. *For any sets $I \neq \emptyset$ and X, for any set system $\mathcal{K}$, and for any family $A = (A_i)_{i\in I}$ of sets such that for any index i there exists a set C in $\mathcal{K}$ whose intersection with X constitutes the i-th term of the family, in the sense that*

$$\forall i\,(i \in I \Rightarrow \exists C\,(C \in \mathcal{K} \wedge X \cap C = A_i)), \tag{3.802}$$

it is true that there exists a family $B = (B_i)_{i\in I}$ of sets in $\mathcal{K}$ such that

$$\forall i\,(i \in I \Rightarrow X \cap B_i = A_i). \tag{3.803}$$

Proof. We let $I \neq \emptyset$, X, $\mathcal{K}$ and A be arbitrary sets, assuming A to be a family $A = (A_i)_{i\in I}$ of sets and assuming (3.802) to hold. To begin with the construction, we apply Function definition by replacement to establish unique function K with domain I, i.e. a unique family $K = (K_n)_{i\in I}$, such that any term K_i contains all the sets $C \in \mathcal{K}$ for which $X \cap C = A_i$ holds. For this purpose, we prove

$$\forall i\,(i \in I \Rightarrow \exists! Y\,(\forall C\,(C \in Y \Leftrightarrow [C \in \mathcal{K} \wedge X \cap C = A_i]))). \tag{3.804}$$

We take an arbitrary index i and observe that the uniquely existential sentence can be established by means of the Axiom of Specification and the Equality Criterion for sets (in the usual way). Since i is arbitrary, we may infer from this the truth of the universal sentence (3.804), so that there exists indeed a unique function/family K with domain I such that

$$\forall i\,(i \in I \Rightarrow \forall C\,(C \in K_i \Leftrightarrow [C \in \mathcal{K} \wedge X \cap C = A_i])). \tag{3.805}$$

We may now prove by contradiction that the empty set is not element of the range of that sequence. To do this, we assume the negation $\neg\emptyset \notin \mathrm{ran}(K)$ to be true, so that the Double Negation Law gives the true sentence $\emptyset \in \mathrm{ran}(K)$. By definition of a range, there exists then a constant, say $\bar{k}$, with $(\bar{k}, \emptyset) \in K$. On the one hand, this gives $\bar{k} \in I$ [$= \mathrm{dom}(K)$] by definition of a domain. On the other hand, because we established K already as a function/family, we may write $\emptyset = K_{\bar{k}}$. Now, the former finding implies with (3.802) that there is a particular set $\bar{C} \in \mathcal{K}$ with $X \cap \bar{C} = A_{\bar{k}}$. We therefore obtain $\bar{C} \in K_{\bar{k}}$ with (3.805), so that $K_{\bar{k}}$ is clearly a nonempty set. We thus found $\emptyset \neq K_{\bar{k}}$ and $\emptyset = K_{\bar{k}}$ to be both true, which conjunction constitutes a contradiction, completing the proof of $\emptyset \notin \mathrm{ran}(K)$. This finding in turn implies with the Axiom of Choice that there exists a function, say $\bar{f}$, such that $\bar{f} : \mathrm{ran}(K) \to \bigcup \mathrm{ran}(K)$ and the universal sentence

$$\forall Y\,(Y \in \mathrm{ran}(K) \Rightarrow \bar{f}(Y) \in Y) \tag{3.806}$$

hold. Then, the composition of $\bar{f}$ and K yields the function/family $\bar{f} \circ K : I \to \bigcup \mathrm{ran}(K)$ according to (3.604), which we denote in family notation by $\bar{B} = (\bar{B}_i)_{i \in I} = \bar{f} \circ K$. Let us verify here that $\bigcup \mathrm{ran}(K)$ is included in the set system $\mathcal{K}$, i.e. $\bigcup \mathrm{ran}(K) \subseteq \mathcal{K}$ (which will imply that $\bar{f} \circ K$ is a family in $\mathcal{K}$). Letting C be arbitrary and assuming $C \in \bigcup \mathrm{ran}(K)$ to be true, there is then (by definition of the union of a set system) a particular set $\bar{Y} \in \mathrm{ran}(K)$ with $C \in \bar{Y}$. Therefore, there is (by definition of a range) a particular constant $\bar{k}$ with $(\bar{k}, \bar{Y}) \in K$, which we may write also as $\bar{Y} = K_{\bar{k}}$; thus, substitution yields $C \in K_{\bar{k}}$. Moreover, $(\bar{k}, \bar{Y}) \in K$ shows (in light of the definition of a domain) that $\bar{k} \in I$ [$= \mathrm{dom}(K)$] holds. This and $C \in K_{\bar{k}}$ implies then with (3.805) especially the truth of $C \in \mathcal{K}$. We thus demonstrated that $C \in \bigcup \mathrm{ran}(K)$ implies $C \in \mathcal{K}$, and since C was arbitrary, we may therefore infer from this implication that the inclusion $\bigcup \mathrm{ran}(K) \subseteq \mathcal{K}$ holds (by definition of a subset). Consequently, the function/family $\bar{B} : I \to \bigcup \mathrm{ran}(K)$ has also the codomain $\mathcal{K}$ according to (3.519), i.e. $\bar{B} : I \to \mathcal{K}$ is a family $(\bar{B}_i)_{i \in I}$ of sets in $\mathcal{K}$.

It thus remains for us to prove that the family $\bar{B} = (\bar{B}_i)_{i \in I}$ satisfies

$$\forall i\,(i \in I \Rightarrow X \cap \bar{B}_i = A_i). \tag{3.807}$$

We let i be arbitrary, assuming $i \in I$ to hold, so that the associated term of the family $\bar{B}$ is given by $\bar{B}_i = \bar{f}(K_i)$. Clearly, the term K_i of the family

K is in the range of that sequence, i.e. $K_i \in \operatorname{ran}(K)$, which in turn implies $\bar{f}(K_i) \in K_i$ with (3.806). In view of the preceding equation, we therefore have $\bar{B}_i \in K_i$, and this yields in particular $X \cap \bar{B}_i = A_i$ due to (3.805). Here, i was arbitrary, so that the universal sentence follows to be true.

Since I, X, $\mathcal{K}$ and A were initially arbitrary sets, we may therefore conclude that the stated lemma holds. □

Corollary 3.238. *For any sets $I \neq \emptyset$ and X, for any set system $\mathcal{K}$, and for any family $A = (A_i)_{i\in I}$ of sets such that for any index i there exists a set C in $\mathcal{K}$ whose intersection with X constitutes the i-th term of the family, in the sense that*

$$\forall i\,(i \in I \Rightarrow \exists C\,(C \in \mathcal{K} \wedge X \cap C = A_i)),$$

it is true that there exists a family of sets $(X \cap B_i)_{i\in I}$ such that

$$(A_i)_{i\in I} = (X \cap B_i)_{i\in I}. \tag{3.808}$$

Theorem 3.239 (Associative Laws for families of sets). *The following equations hold for any set X and any family of sets $(A_i)_{i\in I}$ with $I \neq \emptyset$.*

a) The intersection of the family of sets $(X \cap A_i)_{i\in I}$ is identical with the intersection of X and the family $(A_i)_{i\in I}$, i.e.

$$\bigcap_{i\in I}(X \cap A_i) = X \cap \bigcap_{i\in I} A_i. \tag{3.809}$$

b) The union of the family of sets $(X \cup A_i)_{i\in I}$ is identical with the union of X and the family $(A_i)_{i\in I}$, i.e.

$$\bigcup_{i\in I}(X \cup A_i) = X \cup \bigcup_{i\in I} A_i. \tag{3.810}$$

Proof. Concerning b), we let X, A and I be arbitrary sets, assume $I \neq \emptyset$, so that the existential sentence $\exists i\,(i \in I)$ is true according to 2.42, and assume moreover that $A = (A_i)_{i\in I}$ is a family of sets with index set I. Then, the family of sets $(X \cup A_i)_{i\in I}$ is uniquely specified according to Exercise 3.101a). Next, we apply the Equality Criterion for sets to prove the equation in (3.810) and verify accordingly

$$\forall y\,(y \in \bigcup_{i\in I}(X \cup A_i) \Leftrightarrow y \in X \cup \bigcup_{i\in I} A_i). \tag{3.811}$$

Letting y be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming

$$y \in \bigcup_{i \in I} (X \cup A_i) \tag{3.812}$$

to be true, which then implies with the definition of the union of a family of sets

$$\exists i \, (i \in I \wedge y \in X \cup A_i). \tag{3.813}$$

Let us now observe the truth of the equivalences

$$y \in X \cup \bigcup_{i \in I} A_i \Leftrightarrow y \in X \vee \exists i \, (i \in I \wedge y \in A_i) \tag{3.814}$$

$$\Leftrightarrow [\exists i \, (i \in I) \wedge y \in X] \vee \exists i \, (i \in I \wedge y \in A_i) \tag{3.815}$$

$$\Leftrightarrow \exists i \, (i \in I \wedge y \in X) \vee \exists i \, (i \in I \wedge y \in A_i) \tag{3.816}$$

$$\Leftrightarrow \exists i \, ([i \in I \wedge y \in X] \vee [i \in I \wedge y \in A_i]) \tag{3.817}$$

using the definition of the union of two sets in connection with the definition of the union of a family of sets, then the fact that the previously established existential sentence $\exists i \, (i \in I)$ implies the equivalence of the sentence $y \in X$ and the conjunction $\exists i (i \in I) \wedge y \in X$ in view of (1.13), using subsequently (1.58), and using finally the Distributive Law for quantification (1.75). Therefore, to show that (3.813) implies the desired consequent $y \in X \cup \bigcup_{i \in I} A_i$, we may establish equivalently the right-hand side of (3.817), that is,

$$\exists i \, ([i \in I \wedge y \in X] \vee [i \in I \wedge y \in A_i]). \tag{3.818}$$

For this purpose, we notice in light of (3.813) that there is some index, say $\bar{i} \in I$, with $y \in X \cup A_{\bar{i}}$. Therefore, $y \in X \vee y \in A_{\bar{i}}$ holds by definition of the union of two sets. Thus, the conjunction

$$\bar{i} \in I \wedge [y \in X \vee y \in A_{\bar{i}}] \tag{3.819}$$

is true, which now further implies

$$[\bar{i} \in I \wedge y \in X] \vee [\bar{i} \in I \wedge y \in A_{\bar{i}}] \tag{3.820}$$

with the Distributive Law for sentences (1.44). The truth of this disjunction shows that the existential sentence (3.818) is indeed true. This finding implies in view of the true equivalences (3.814) – (3.817) that $y \in X \cup \bigcup_{i \in I} A_i$ holds, proving the first part of the equivalence in (3.811).

To prove the second part ('$\Leftarrow$'), we now assume $y \in X \cup \bigcup_{i \in I} A_i$ to be true, which then implies the truth of the existential sentence (3.818) with the true equivalences (3.814) – (3.817). We now show that (3.813) follows to be true. Because of (3.818), there is some index, say $\bar{i}$, such that (3.820) is true. This disjunction implies then (3.819) with the Distributive Law for sentences (1.44), and then also $\bar{i} \in I \wedge y \in X \cup A_{\bar{i}}$ (applying again the definition of the union of two sets). Thus, the existential sentence (3.813) holds, which in turn implies (3.812) by definition of the union of a family of sets, so that the proof of the second part of the equivalence in (3.811) is also complete.

As y was arbitrary, the universal sentence (3.811) follows then to be true, and this gives the proposed equation (3.810) with the Equality Criterion for sets. Since X, A and I were initially arbitrary sets, we may therefore conclude that the proposed universal sentence b) is true. $\square$

Exercise 3.103. Establish Part a) of Theorem 3.239.

(Hint: Proceed in analogy to the proof of Part b), replacing the two arguments to establish (3.815) and (3.816) by the single argument (1.58).)

Theorem 3.240 (Distributive Laws for families of sets). *The following equations hold for any set X and any family of sets $A = (A_i)_{i \in I}$ with $I \neq \emptyset$.*

a) The pairwise union is distributive with respect to the intersection of a family of sets in the sense that

$$X \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (X \cup A_i). \tag{3.821}$$

b) The pairwise intersection is distributive with respect to the union of a family of sets in the sense that

$$X \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (X \cap A_i). \tag{3.822}$$

c) The set difference is distributive with respect to the union of a family of sets in the sense that

$$\left(\bigcup_{i \in I} A_i \right) \setminus X = \bigcup_{i \in I} (A_i \setminus X). \tag{3.823}$$

d) The pairwise Cartesian product is distributive with respect to the union *of a family of sets in the sense that*

$$X \times \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(X \times A_i), \tag{3.824}$$

$$\left(\bigcup_{i\in I} A_i\right) \times X = \bigcup_{i\in I}(A_i \times X). \tag{3.825}$$

Proof. We let X, I and A be arbitrary sets, assume $I \neq \emptyset$, and assume further that $A = (A_i)_{i\in I}$ is a family of sets with index set I.

Concerning a), we prove the equation in (3.821) by means of the Equality Criterion for sets, i.e. by verifying

$$\forall y\,(y \in X \cup \bigcap_{i\in I} A_i \Leftrightarrow y \in \bigcap_{i\in I}[X \cup A_i]). \tag{3.826}$$

We let y be arbitrary, so that we obtain, on the one hand, the true equivalences

$$y \in X \cup \bigcap_{i\in I} A_i \Leftrightarrow [y \in X \lor \forall i\,(i \in I \Rightarrow y \in A_i)] \tag{3.827}$$

$$\Leftrightarrow \forall i\,(y \in X \lor [i \in I \Rightarrow y \in A_i]) \tag{3.828}$$

by applying the definition of the union of two sets together with the definition of the intersection of a family of sets and then the Neutrality Law for quantification (1.63). On the other hand, we obtain the true equivalence (applying again the definition of the intersection of a family of sets)

$$y \in \bigcap_{i\in I}[X \cup A_i] \;\Leftrightarrow\; \forall i\,(i \in I \Rightarrow y \in X \cup A_i). \tag{3.829}$$

Now, to prove the first part ('$\Rightarrow$') of the equivalence in (3.826), we assume that $y \in X \cup \bigcap_{i\in I} A_i$ is true, which implies the universal sentence

$$\forall i\,(y \in X \lor [i \in I \Rightarrow y \in A_i]) \tag{3.830}$$

with the equivalences (3.827) – (3.828), and we show that the universal sentence

$$\forall i\,(i \in I \Rightarrow y \in X \cup A_i) \tag{3.831}$$

follows to be true. Letting $\bar{i}$ be arbitrary, we now obtain the true equivalences

$$y \in X \lor [\bar{i} \in I \Rightarrow y \in A_{\bar{i}}] \Leftrightarrow y \in X \lor [\bar{i} \notin I \lor y \in A_{\bar{i}}] \tag{3.832}$$

$$\Leftrightarrow \bar{i} \notin I \lor [y \in X \lor y \in A_{\bar{i}}] \tag{3.833}$$

$$\Leftrightarrow \bar{i} \in I \Rightarrow [y \in X \cup A_{\bar{i}}] \tag{3.834}$$

using the Conditional Law (1.31), the Associative Law for the disjunction (1.40), and finally again the Conditional Law (1.31) together with the definition of the union of two sets. Since the disjunction $y \in X \vee [\bar{i} \in I \Rightarrow y \in A_{\bar{i}}]$ is true because of (3.830), the preceding equivalences yield the true implication $\bar{i} \in I \Rightarrow [y \in X \cup A_{\bar{i}}]$. As $\bar{i}$ was arbitrary, we may therefore conclude that the universal sentence (3.831) holds indeed, which implies then in turn $y \in \bigcap_{i \in I}[X \cup A_i]$ with the true equivalence (3.829). Thus, the proof of the first part of the equivalence in (3.826) is complete.

To prove the second part ('$\Leftarrow$'), we now assume $y \in \bigcap_{i \in I}[X \cup A_i]$ to be true, which implies then the truth of the universal sentence (3.831) because of (3.829). Let us now verify that the universal sentence (3.830) follows to be true. Letting $\bar{i}$ be arbitrary, the true universal sentence (3.831) gives the true implication $\bar{i} \in I \Rightarrow [y \in X \cup A_{\bar{i}}]$, which then implies the truth of the disjunction $y \in X \vee [\bar{i} \in I \Rightarrow y \in A_{\bar{i}}]$, because the equivalence (3.832) – (3.834) hold again for the arbitrarily selected $\bar{i}$. We may now infer from this finding the truth of the universal sentence (3.830), which then further implies $y \in X \cup \bigcap_{i \in I} A_i$ with the true equivalences (3.827) – (3.828). This completes the proof of the second part of the equivalence in (3.826), and as y was arbitrary, we may therefore conclude that the universal sentence (3.826) is true.

In view of the Equality Criterion for sets, this shows that the sets $X \cup \bigcap_{i \in I} A_i$ and $\bigcap_{i \in I}(X \cup A_i)$ are identical, so that the equation (3.821) holds. Since X, I and A were initially also arbitrary, we may then further conclude that the proposed sentence a) is true.

Part b) and c) can be proved similarly.

Concerning d), we we prove the equation (3.824) by means of the Equality Criterion for sets, i.e. by verifying

$$\forall y\,(y \in X \times \bigcup_{i \in I} A_i \Leftrightarrow y \in \bigcup_{i \in I}[X \times A_i]). \tag{3.835}$$

We let y be arbitrary, and assume first $y \in X \times \bigcup_{i \in I} A_i$ to be true. By definition of the Cartesian product of two sets, there are then particular constants $\bar{x} \in X$ and $\bar{a} \in \bigcup_{i \in I} A_i$ satisfying $(\bar{x}, \bar{a}) = y$. According to the Characterization of the union of a family of sets, there is then a particular index $\bar{k} \in I$ such that $\bar{a} \in A_{\bar{k}}$ holds. Consequently, the conjunction of $\bar{x} \in X$ and $\bar{a} \in A_{\bar{k}}$ is true, which in turn implies $(\bar{x}, \bar{a}) \in X \times A_{\bar{k}}$ (by definition of the Cartesian product of two sets). We therefore obtain $y \in X \times A_{\bar{k}}$ via substitution, where $X \times A_{\bar{k}}$ is (in view of $\bar{k} \in I$) a term of the family of sets $(X \times A_i)_{i \in I}$ established in Exercise 3.101c). We thus showed that there exists an index $i \in I$ such that $y \in X \times A_i$ holds, which existential sentence implies the desired consequent $y \in \bigcup_{i\in I}(X \times A_i)$ of the first part ('$\Rightarrow$') of the equivalence in (3.835) by means of the Characterization of the union of a family of sets.

To establish the second part ('$\Leftarrow$'), we assume now $y \in \bigcup_{i\in I}(X \times A_i)$ to be true, so that there evidently exists a particular index $\bar{k} \in I$ with $y \in X \times A_{\bar{k}}$. We therefore have $y = (\bar{x}, \bar{a})$ for some particular $\bar{x} \in X$ and $\bar{a} \in A_{\bar{k}}$. The latter shows that $\bar{a} \in A_i$ holds for some $i \in I$, so that $\bar{a} \in \bigcup_{i\in I} A_i$ follows to be true. The simultaneous truth of $\bar{x} \in X$ and $\bar{a} \in \bigcup_{i\in I} A_i$ gives then $(\bar{x}, \bar{a}) \in X \times \bigcup_{i\in I} A_i$ and thus $y \in X \times \bigcup_{i\in I} A_i$, as desired. Having completed the proof of the equivalence in (3.835), we may now infer from this the truth of the universal sentence (3.835), because y was arbitrary. Then, the equation (3.824) follows to be true with the Equality Criterion for sets.

The proof of the other equation (3.825) is based on the same arguments as the proof of (3.824).

Since X, I and A were initially arbitrary sets, we may therefore conclude that the stated theorem is true. □

Exercise 3.104. Prove parts b) – d) of Theorem 3.240.

(Hint: Proceed in analogy to the proof of Part a).)

Theorem 3.241 (De Morgan's Laws for the intersection & the union of a family of sets). *The following equations hold for any set Y and any family of sets $A = (A_i)_{i\in I}$ in $\mathcal{P}(Y)$ with $I \neq \emptyset$.*

a) **De Morgan's Law for the intersection of a family of sets:**

$$\left(\bigcap_{i\in I} A_i^c\right)^c = \bigcup_{i\in I} A_i. \tag{3.836}$$

b) **De Morgan's Law for the union of a family of sets:**

$$\left(\bigcup_{i\in I} A_i^c\right)^c = \bigcap_{i\in I} A_i. \tag{3.837}$$

Proof. We let Y, I and A be arbitrary sets, assume $I \neq \emptyset$, and assume moreover that $A = (A_i)_{i\in I}$ is a family of sets in $\mathcal{P}(Y)$. Then, the family of sets $(A_i^c)_{i\in I}$ is uniquely specified according to Exercise 3.101b).

Concerning a), we may establish the equation in (3.837) via the Equality Criterion for sets and verify for this purpose the universal sentence

$$\forall y \, (y \in \left(\bigcup_{i\in I} A_i^c\right)^c \Leftrightarrow y \in \bigcap_{i\in I} A_i). \tag{3.838}$$

Letting y be arbitrary, we may on the one hand observe the truth of the equivalences

$$y \in \left(\bigcup_{i\in I} A_i^c\right)^c \Leftrightarrow y \notin \bigcup_{i\in I} A_i^c \tag{3.839}$$

$$\Leftrightarrow \neg\exists i\,(i \in I \,\wedge\, y \in A_i^c) \tag{3.840}$$

$$\Leftrightarrow \forall i\,(i \in I \Rightarrow y \notin A_i^c) \tag{3.841}$$

in light of (2.132), the definition of the union of a family of sets and (1.81). On the other hand, the definition of the intersection of a family of sets yields the true equivalence

$$y \in \bigcap_{i\in I} A_i \;\Leftrightarrow\; \forall i\,(i \in I \Rightarrow y \in A_i). \tag{3.842}$$

We now establish the first part ('⇒') of the equivalence in (3.838), assuming $y \in \left(\bigcup_{i\in I} A_i^c\right)^c$ to be true, so that

$$\forall i\,(i \in I \Rightarrow y \notin A_i^c) \tag{3.843}$$

follows to be true with the equivalences (3.839) – (3.841). We now show that this implies

$$\forall i\,(i \in I \Rightarrow y \in A_i). \tag{3.844}$$

For this purpose, we let $\bar{i} \in I$ be arbitrary, which gives $y \notin A_{\bar{i}}^c$ with (3.843), therefore $y \in (A_{\bar{i}}^c)^c$ with (2.132), and consequently $y \in A_{\bar{i}}$ with (2.136), proving the implication in (3.844). As $\bar{i}$ is arbitrary, we may now conclude that (3.844) holds, which in turn implies $y \in \bigcap_{i\in I} A_i$ according to (3.842). This finding proves the first implication '⇒'.

To establish the second part ('⇐'), we now assume that $y \in \bigcap_{i\in I} A_i$ holds, so that (3.844) follows to be true with (3.842). We may now prove that the truth of this universal sentence implies the truth of (3.843). To do this, we take an arbitrary $\bar{i} \in I$, so that (3.844) yields $y \in A_{\bar{i}}$, which further implies $y \in (A_{\bar{i}}^c)^c$ with (2.136) and then also $y \notin A_{\bar{i}}^c$ with (2.132). This negation proves the implication in (3.843), and since $\bar{i}$ is arbitrary, we may therefore conclude that (3.843) holds. This universal sentence in turn implies $y \in \left(\bigcup_{i\in I} A_i^c\right)^c$ with the previously established equivalences (3.839) – (3.841), completing the proof of the second implication '⇐'. Thus, the equivalence in (3.838) holds, and because y is arbitrary, we may then further conclude that (3.838) is also true.

The truth of that universal sentence means according to the Equality Criterion for sets that $\left(\bigcup_{i\in I} A_i^c\right)^c$ and $\bigcap_{i\in I} A_i$ are identical, so that the proof of the equation (3.837) is complete. As Y, I and A were arbitrary sets, the sentence a) follows to be universally true. □

Exercise 3.105. Establish Theorem 3.241b).
(Hint: Proceed similarly as in the proof of Theorem 3.241a).)

We now return to the issue of inverse images and establish a few more laws in the specific context of families of sets.

Proposition 3.242. *For any function $f : X \to Y$ the inverse image of the union of a family $B = (B_i)_{i \in I}$ of subsets of Y equals the union of the inverse images of the terms of B, that is,*

$$\forall X, Y, f, I, B\,([f \in Y^X \,\wedge\, B \in \mathcal{P}(Y)^I] \Rightarrow f^{-1}[\bigcup_{i \in I} B_i] = \bigcup_{i \in I} f^{-1}[B_i]). \tag{3.845}$$

Proof. Letting X, Y, f, I and B be arbitrary sets, we assume $f \in Y^X$ and $B \in \mathcal{P}(Y)^I$ to be true, so that f is a function from X to Y and B a family $(B_i)_{i \in I}$ in $\mathcal{P}(Y)$ (i.e., of subsets of Y). Then, the family of set $(f^{-1}[B_i])_{i \in I}$ is uniquely specified according to Exercise 3.101c). Let us now apply the Equality Criterion for sets and prove the equation in (3.845) by verifying the equivalent

$$\forall x\,(x \in f^{-1}[\bigcup_{i \in I} B_i] \Leftrightarrow x \in \bigcup_{i \in I} f^{-1}[B_i]). \tag{3.846}$$

Letting x be arbitrary, we first assume

$$x \in f^{-1}[\bigcup_{i \in I} B_i], \tag{3.847}$$

which implies with the definition of an inverse image

$$f(x) \in \bigcup_{i \in I} B_i, \tag{3.848}$$

and then with the definition of the union of a family of sets

$$\exists i\,(i \in I \,\wedge\, f(x) \in B_i). \tag{3.849}$$

Thus, there exists an index, say $\bar{i}$, such that

$$\bar{i} \in I \wedge f(x) \in B_{\bar{i}} \tag{3.850}$$

holds. This conjunction further implies with the definition of an inverse image

$$\bar{i} \in I \wedge x \in f^{-1}[B_{\bar{i}}], \tag{3.851}$$

and this conjunction shows that the existential sentence

$$\exists i\,(i \in I \;\wedge\; x \in f^{-1}[B_i]) \tag{3.852}$$

holds, so that the desired consequent

$$x \in \bigcup_{i \in I} f^{-1}[B_i] \tag{3.853}$$

follows to be true by definition of the union of a family of sets. Conversely, we now assume (3.853) to be true, so that (3.852) holds by definition of the union of a family of sets. Thus, there is an index, say $\bar{i}$, satisfying (3.851), which gives (3.850) (with the definition of an inverse image), so that the existential sentence (3.849) holds as well. This implies then first (3.848) with the definition of the union of a family of sets, and subsequently the desired (3.847) with the definition of an inverse image. We thus proved the equivalence in (3.846), and since x is arbitrary, we may therefore conclude that the universal sentence (3.846) is true. This in turn implies the equation in (3.845) with the Equality Criterion for sets, and as X, Y, f, I and B were also arbitrary, we may finally conclude that the proposed universal sentence (3.845) holds. □

Exercise 3.106. Show for any function $f : X \to Y$ that the inverse image of the intersection of a family $B = (B_i)_{i \in I}$ of subsets of Y with nonempty index set is identical with the intersection of the inverse images of the terms of B, that is,

$$\forall X, Y, f, I, B\,([f \in Y^X \wedge I \neq \emptyset \wedge B \in \mathcal{P}(Y)^I] \Rightarrow f^{-1}[\bigcap_{i \in I} B_i] = \bigcap_{i \in I} f^{-1}[B_i]). \tag{3.854}$$

Corollary 3.243 (Inverse image of the union of a family of disjoint sets). *For any function $f : X \to Y$ the inverse image of the union of a family $B = (B_i)_{i \in I}$ of disjoint subsets of Y equals the union of the family of sets $(f^{-1}[B_i])_{i \in I}$, and the terms of this family are all disjoint, that is,*

$$\forall X, Y, f, I, B\,(\Big[f \in Y^X \wedge B \in \mathcal{P}(Y)^I \wedge \forall i, j\,([i, j \in I \wedge i \neq j] \Rightarrow B_i \cap B_j = \emptyset)\Big] \Rightarrow$$
$$\Big[f^{-1}[\bigcup_{i \in I} B_i] = \bigcup_{i \in I} f^{-1}[B_i] \wedge \forall i, j\,([i, j \in I \wedge i \neq j] \Rightarrow f^{-1}[B_i] \cap f^{-1}[B_j] = \emptyset)\Big]). \tag{3.855}$$

Proof. Letting X, Y, f, I and B be arbitrary such that f is a function from X to Y and such that $B = (B_i)_{i \in I}$ is a family of sets in $\mathcal{P}(Y)$ with disjoint terms, satisfying

$$\forall i, j\,([i, j \in I \;\wedge\; i \neq j] \Rightarrow B_i \cap B_j = \emptyset), \tag{3.856}$$

we obtain the equation in (3.855) with (3.845). To show that

$$\forall i,j\,([i,j\in I\;\wedge\; i\neq j]\Rightarrow f^{-1}[B_i]\cap f^{-1}[B_j]=\emptyset) \tag{3.857}$$

also holds, we let i and j be arbitrary and assume $i,j\in I\wedge i\neq j$ to be true, which implies $B_i\cap B_j=\emptyset$ with the assumed (3.856). We then obtain the equations

$$f^{-1}[B_i]\cap f^{-1}[B_j]=f^{-1}[B_i\cap B_j]=f^{-1}[\emptyset]=\emptyset$$

by applying Exercise 3.95a), then substitution based on the previously established equation $B_i\cap B_j=\emptyset$, and finally Exercise 3.90a). These equations give the desired consequent $f^{-1}[B_i]\cap f^{-1}[B_j]=\emptyset$, and since i and j were arbitrary, we may therefore conclude that the universal sentence (3.857) is indeed true. As X, Y, f, I and B were also arbitrary, we may then further conclude that the proposed universal sentence (3.855) also holds. □

We now generalize the concept of a Cartesian product from two sets to a family of sets.

Proposition 3.244. *For any set I and any family $s=(X_i)_{i\in I}$ of sets, it is true that there exists a unique set (system) $\bigtimes_{i\in I}X_i$ such that a set f is element of $\bigtimes_{i\in I}X_i$ iff f is a family (with index set I) in the union of the family $(X_i)_{i\in I}$ and if any term f_i is in X_i, i.e.*

$$\exists!\bigtimes_{i\in I}X_i\;\forall f\,(f\in\bigtimes_{i\in I}X_i\Leftrightarrow[f\in[\bigcup_{i\in I}X_i]^I\;\wedge\;\forall i\,(i\in I\Rightarrow f_i\in X_i)]).$$

Furthermore, the set $\bigtimes_{i\in I}X_i$ satisfies

$$\forall f\,(f\in\bigtimes_{i\in I}X_i\Leftrightarrow[f\text{ is a family with index set }I\;\wedge\;\forall i\,(i\in I\Rightarrow f_i\in X_i)]). \tag{3.858}$$

Proof. Letting I be an arbitrary (index) set and letting s an arbitrary set such that $s=(X_i)_{i\in I}$ is a family of sets, we may evidently apply the Axiom of Specification and the Equality Criterion for sets to establish the stated uniquely existential sentence. Thus, the set $\bigtimes_{i\in I}X_i$ satisfies

$$\forall f\,(f\in\bigtimes_{i\in I}X_i\Leftrightarrow[f\in[\bigcup_{i\in I}X_i]^I\;\wedge\;\forall i\,(i\in I\Rightarrow f_i\in X_i)]). \tag{3.859}$$

To establish (3.858), we take an arbitrary set f and assume first $f\in\bigtimes_{i\in I}X_i$ to be true. This assumption implies with (3.859) in particular

$f \in [\bigcup_{i\in I} X_i]^I$, so that f is a function from I to $\bigcup_{i\in I} X_i$. Thus, f is a family with index set I. As the preceding assumption implies in particular also the truth of the universal sentence $\forall i\,(i \in I \Rightarrow f_i \in X_i)$, the conjunction in (3.858) holds, so that the first part ('$\Rightarrow$') of the equivalence is true.

Regarding the second part ('$\Leftarrow$'), we now assume that f is a family with index set I and that

$$\forall i\,(i \in I \Rightarrow f_i \in X_i) \tag{3.860}$$

holds. Thus, the second part of the conjunction in (3.859) already holds, and we may prove that the first part $f \in [\bigcup_{i\in I} X_i]^I$ also holds. Since we assumed f to be a family with index set I, we have that I is the domain of f. Next, we demonstrate that $\bigcup_{i\in I} X_i$ is a codomain of f, i.e. that the range of f is included in $\bigcup_{i\in I} X_i$. To show this, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall y\,(y \in \mathrm{ran}(f) \Rightarrow y \in \bigcup_{i\in I} X_i). \tag{3.861}$$

We let $\bar{y}$ be arbitrary and assume $\bar{y} \in \mathrm{ran}(f)$, so that there exists by definition of a range a constant, say $\bar{k}$, with $(\bar{k}, \bar{y}) \in f$. Since f is a function/family, we may write this also as $\bar{y} = f_{\bar{k}}$. Furthermore, we obtain with the definition of a domain $\bar{k} \in I$ $[= \mathrm{dom}(f)]$, so that (3.860) gives $[\bar{y} =]\ f_{\bar{k}} \in X_i$. Thus, the existential sentence $\exists i\,(i \in I \wedge \bar{y} \in X_i)$ is true, which in turn implies $\bar{y} \in \bigcup_{i\in I} X_i$ with the Characterization of the union of a family of sets. This finding proves the implication in (3.861), and as $\bar{y}$ was arbitrary, we may therefore conclude that (3.861) holds, which then yields $\mathrm{ran}(f) \subseteq \bigcup_{i\in I} X_i$ (by definition of a subset). Thus, $\bigcup_{i\in I} X_i$ is a codomain of f by definition, so that f is a function from I to $\bigcup_{i\in I} X_i$, and therefore an element of $[\bigcup_{i\in I} X_i]^I$. This completes the proof of the conjunction in (3.859), which in turn implies $f \in \bigtimes_{i\in I} X_i$, completing the proof of the equivalence in (3.858).

Because f is arbitrary, we may now further conclude that $\bigtimes_{i\in I} X_i$ satisfies indeed the universal sentence (3.858). Since I and X were initially arbitrary sets, we may finally infer from this the truth of the proposition. □

Definition 3.60 (Cartesian product of a family of sets). For any set I and any family $(X_i)_{i\in I}$ of sets, we call the unique set system

$$\bigtimes_{i\in I} X_i \tag{3.862}$$

consisting precisely of all families f with index set I and with i-th term f_i in X_i for all $i \in I$ the *Cartesian product* of $(X_i)_{i\in I}$.

Exercise 3.107. Show for any set I and any family $(X_i)_{i\in I}$ of sets that the Cartesian product of that family is identical with the singleton formed by the empty set if I is empty, that is,

$$I = \emptyset \;\Rightarrow\; \bigtimes_{i\in I} X_i = \{\emptyset\}. \tag{3.863}$$

(Hint: Apply the Equality Criterion for sets in connection with (3.858), (3.120) and (2.169).)

Notation 3.9. In the special cases of a sequence of sets $(X_n)_{n\in\mathbb{N}}$ and of a sequence of sets $(X_n)_{n\in\mathbb{N}_+}$, we also write, respectively,

$$\bigtimes_{n=0}^{\infty} X_n \quad \text{or} \quad X_0 \times X_1 \times \ldots \tag{3.864}$$

$$\bigtimes_{n=1}^{\infty} X_n \quad \text{or} \quad X_1 \times X_2 \times \ldots \tag{3.865}$$

Furthermore, if a sequence $(X_n)_{n\in\mathbb{N}_+}$ is constant in the sense that $X_n = X$ holds for any $n \in \mathbb{N}_+$, we write:

$$X^\omega \tag{3.866}$$

Theorem 3.245 (Emptiness Criterion for Cartesian products of families of sets). *It is true for any nonempty set I and any family $X = (X_i)_{i\in I}$ of sets that the Cartesian product of X is empty iff at least one of the terms of the family is nonempty, that is,*

$$\exists i\,(i \in I \,\wedge\, X_i = \emptyset) \;\Leftrightarrow\; \bigtimes_{i\in I} X_i = \emptyset. \tag{3.867}$$

Proof. We let I and X be arbitrary sets and assume that X is a family of sets with index set $I \neq \emptyset$, which family we may therefore write as $(X_i)_{i\in I}$. We prove the first part ('$\Rightarrow$') of the stated equivalence by contraposition, by assuming $\bigtimes_{i\in I} X_i \neq \emptyset$ and demonstrating the truth of the negated existential sentence in (3.867). Because of the Negation Law for existential conjunctions, we may prove equivalently the universal sentence

$$\forall i\,(i \in I \Rightarrow X_i = \emptyset), \tag{3.868}$$

letting $\bar{i}$ be arbitrary and assuming $\bar{i} \in I$ to be true. The preceding assumption $\bigtimes_{i\in I} X_i \neq \emptyset$ implies with (2.42) that there exists an element in the Cartesian product $\bigtimes_{i\in I} X_i$, say $\bar{f}$. It then follows with the definition of the Cartesian product of a family of sets that $\bar{f}$ is a family with index set

I such that $\bar{f}_i \in X_i$ holds for any $i \in I$. With this, $\bar{i} \in I$ implies $\bar{f}_{\bar{i}} \in X_{\bar{i}}$, which clearly shows that $X_{\bar{i}}$ is nonempty, so that the implication in (3.868) is true. Because $\bar{i}$ is arbitrary, we may therefore conclude that the universal sentence (3.868) holds. Consequently, the negation of the existential sentence in (3.867) is true as well, which finding completes the proof of the implication '$\Rightarrow$' by contradiction.

Next, we prove the second part ('$\Leftarrow$') of the equivalence by contraposition, assuming the negation of $\exists i\,(i \in I \wedge X_i = \emptyset)$ to hold and showing that $\bigtimes_{i\in I} X_i \neq \emptyset$ follows to be true. The assumption implies with the Negation Law for existential conjunctions that the universal sentence (3.868) holds, which allows to prove $\emptyset \notin \mathrm{ran}(X)$. For this purpose, we apply (2.5) and verify the equivalent universal sentence

$$\forall Y\,(Y \in \mathrm{ran}(X) \Rightarrow Y \neq \emptyset), \tag{3.869}$$

letting Y be arbitrary and assuming $Y \in \mathrm{ran}(X)$ to be true. By definition of a range, there exists then a constant, say $\bar{k}$, such that $(\bar{k}, Y) \in X$ holds. Since X is a function/family, we may write $Y = X_{\bar{k}}$, and the definition of a domain shows that $\bar{k} \in I$ $[= \mathrm{dom}(X)]$ is true. The latter implies now $[Y =]\ X_{\bar{k}} \neq \emptyset$, which proves the implication in (3.869). Because Y was arbitrary, we may now infer from the truth of this implication the truth of the universal sentence (3.869), and therefore the truth of $\emptyset \notin \mathrm{ran}(X)$. This negation in turn implies with the Axiom of Choice that there exists a particular function $\bar{f} : \mathrm{ran}(X) \to \bigcup \mathrm{ran}(X)$ satisfying

$$\forall Y\,(Y \in \mathrm{ran}(X) \Rightarrow \bar{f}(Y) \in Y). \tag{3.870}$$

Since the family $(X_i)_{i\in I}$ is evidently a function $X : I \to \mathrm{ran}(X)$, we obtain the composition $\bar{f} \circ X : I \to \bigcup \mathrm{ran}(X)$ with Proposition 3.178, which is thus a family with index set I. Furthermore, we may show that the composition satisfies

$$\forall i\,(i \in I \Rightarrow (\bar{f} \circ X)_i \in X_i). \tag{3.871}$$

Letting i be arbitrary and assuming that $i \in I$ is true, the corresponding term X_i is clearly in the codomain/range of X, i.e. we have $X_i \in \mathrm{ran}(X)$. This further implies $\bar{f}(X_i) \in X_i$ with (3.870), which we may write also in the form $(\bar{f} \circ X)_i \in X_i$ by using the notation for compositions. Here, i is arbitrary, so that the universal sentence (3.871) follows indeed to be true. Then, since $\bar{f} \circ X$ is a family with index set I satisfying (3.871), we obtain now $\bar{f} \circ X \in \bigtimes_{i\in I} X_i$ with the definition of the Cartesian product of a family of sets. This finding demonstrates the existence of an element in $\bigtimes_{i\in I} X_i$ and therefore the truth of $\bigtimes_{i\in I} X_i \neq \emptyset$ in light of (2.42), completing the proof of the second part of the equivalence in (3.867) via contraposition.

Since I and X were initially arbitrary sets, we may furthermore conclude that the stated theorem holds. $\square$

The next proposition generalizes Proposition 3.8 (which we stated in the context of the Cartesian product of two sets) to the Cartesian product of a family of sets.

Proposition 3.246. *The following implication holds for any index set I and for any families $A = (A_i)_{i\in I}$ and $B = (B_i)_{i\in I}$ of sets.*

$$\forall i\,(i \in I \Rightarrow A_i \subseteq B_i) \;\Rightarrow\; \bigtimes_{i\in I} A_i \subseteq \bigtimes_{i\in I} B_i \tag{3.872}$$

Proof. We let I, A and B be arbitrary sets and assume $A = (A_i)_{i\in I}$ and $B = (B_i)_{i\in I}$ to be families of sets with index set I. To prove the implication directly, we assume

$$\forall i\,(i \in I \Rightarrow A_i \subseteq B_i) \tag{3.873}$$

and show that this implies the inclusion $\bigtimes_{i\in I} A_i \subseteq \bigtimes_{i\in I} B_i$, which means

$$\forall f\,(f \in \bigtimes_{i\in I} A_i \Rightarrow f \in \bigtimes_{i\in I} B_i) \tag{3.874}$$

by definition of a subset. For this purpose, we let f be arbitrary and assume $f \in \bigtimes_{i\in I} A_i$ to hold, so that

$$\forall i\,(i \in I \Rightarrow f_i \in A_i) \tag{3.875}$$

follows in particular to be true by definition of the Cartesian product of a family of sets. To prove that this implies $f \in \bigtimes_{i\in I} B_i$, we observe first that f is a family with index set I, so that it suffices to establish

$$\forall i\,(i \in I \Rightarrow f_i \in B_i). \tag{3.876}$$

We let i be arbitrary and assume $i \in I$, which gives on the one hand $A_i \subseteq B_i$ with (3.873) and therefore

$$\forall y\,(y \in A_i \Rightarrow y \in B_i) \tag{3.877}$$

with the definition of a subset; on the other hand, $i \in I$ implies $f_i \in A_i$ with (3.875), which in turn yields $f_i \in B_i$ with (3.877), proving the implication in (3.876). Since i is arbitrary, we may therefore conclude that the universal sentence (3.876) is true. Together with the assumption that f is a family with index set I, this implies $f \in \bigtimes_{i\in I} B_i$ according to Proposition 3.244. As f was also arbitrary, we may further conclude that the universal sentence (3.874) also holds, so that the inclusion $\bigtimes_{i\in I} A_i \subseteq \bigtimes_{i\in I} B_i$ follows to be true (by definition of a subset). Because I, A, and B were initially arbitrary sets, the proposition holds, as claimed. $\square$

Lemma 3.247. *The following implication holds for any index set I and for any families $A = (A_i)_{i \in I}$ and $B = (B_i)_{i \in I}$ of nonempty sets.*

$$\bigtimes_{i \in I} A_i \subseteq \bigtimes_{i \in I} B_i \;\Rightarrow\; \forall i \,(i \in I \Rightarrow A_i \subseteq B_i) \tag{3.878}$$

Proof. Letting I, A and B be arbitrary sets, we assume $A = (A_i)_{i \in I}$ and $B = (B_i)_{i \in I}$ to be families such that

$$\forall i \,(i \in I \Rightarrow A_i \neq \emptyset), \tag{3.879}$$

$$\forall i \,(i \in I \Rightarrow B_i \neq \emptyset). \tag{3.880}$$

Moreover, we assume the inclusion $\bigtimes_{i \in I} A_i \subseteq \bigtimes_{i \in I} B_i$ to holds and let then i be arbitrary. We prove now the implication $i \in I \Rightarrow A_i \subseteq B_i$ by contradiction, assuming that $i \in I$ and the negation $\neg A_i \subseteq B_i$ are both true. In view of the definition of a subset and the Negation Law for universal implication, there exists then a particular element $\bar{y}$ in A_i such that $\bar{y} \notin B_i$ holds. Let us consider now the restriction $A \restriction (I \setminus \{i\})$, where $I \setminus \{i\} \subseteq I$ holds according to (2.125), so that this restriction is a function/family with domain/index set $I \setminus \{i\}$ due to (3.566). In view of the preceding inclusion, it is true for any j that $j \in I \setminus \{i\}$ implies $j \in I$ (by definition of a subset) and therefore $A_j \neq \emptyset$ with (3.879). This fact evidently implies with the Emptiness Criterion for Cartesian products of families of sets that $\bigtimes_{j \in I \setminus \{i\}} A_j \neq \emptyset$ holds. Consequently, that Cartesian product has an element, say $\bar{f}$, which constitutes thus a family with index set $I \setminus \{i\}$ and terms satisfying

$$\forall j \,(j \in I \setminus \{i\} \Rightarrow \bar{f}_j \in A_j). \tag{3.881}$$

Forming then the singleton $\{(i, \bar{y})\}$, we have a function with domain $\{i\}$ because of (3.536). Observing now the truth of $\{i\} \cap (I \setminus \{i\}) = \emptyset$ in light of (2.111), we see that the domains of the functions $\{(i, \bar{y})\}$ and $\bar{f}$ are disjoint, so that these functions are compatible (see Exercise 3.73). Consequently, the union

$$u = \{(i, \bar{y})\} \cup \bar{f} \tag{3.882}$$

constitutes a function with domain $\{i\} \cup (I \setminus \{i\})$ (see Proposition 3.176). Here, we note that the assumed $i \in I$ implies $\{i\} \subseteq I$ with (2.184), which inclusion in turn implies

$$\begin{aligned} I &= (I \setminus \{i\}) \cup \{i\} \\ &= \{i\} \cup (I \setminus \{i\}) \end{aligned}$$

with (2.263) and the Commutative Law for the union of two sets. Thus, $\bar{f}$ is a function/family with domain I, which we now prove to be an element of the Cartesian product $\bigtimes_{i\in I} A_i$. For this purpose, we establish the truth of

$$\forall j\,(j \in I \Rightarrow u_j \in A_j), \tag{3.883}$$

letting j be arbitrary and assuming $j \in I$ $[= \{i\} \cup (I \setminus \{i\})]$ to hold. The definition of the union of two sets implies then the truth of the disjunction $j \in \{i\} \vee j \in I \setminus \{i\}$, which we use to prove $u_j \in A_j$ by cases. The first case $j \in \{i\}$ yields $j = i$ with (2.169), so that the previously established $\bar{y} \in A_i$ gives $\bar{y} \in A_j$ by means of substitution. Furthermore, we see in light of (3.882) and the definition of the union of a set system that $(i, \bar{y}) \in u$ holds, which we can write as $\bar{y} = u_i = u_j$, applying the notation for families and again substitution based on $j = i$. Therefore, $\bar{y} \in A_j$ implies the desired consequent $u_j \in A_j$ in the first case. The second case $j \in I \setminus \{i\}$ implies with (3.881) $\bar{f}_j \in A_j$, where evidently $(j, \bar{f}_j) \in \bar{f}$ holds, so that $(j, \bar{f}_j) \in u$ follows to be true with the definition of the union of a set system and (3.882). Writing this finding in function/family notation as $\bar{f}_j = u_j$, we now see that $\bar{f}_j \in A_j$ implies $u_j \in A_j$, as desired. Having thus completed the proof by cases, we can infer from the truth of $u_j \in A_j$ the truth of the universal sentence (3.883), so that the family u (with index set I) is indeed an element of the Cartesian product $\bigtimes_{i\in I} A_i$. Because of the assumed inclusion $\bigtimes_{i\in I} A_i \subseteq \bigtimes_{i\in I} B_i$, it follows then that u is also an element of the Cartesian product $\bigtimes_{i\in I} B_i$, which means that the terms of u satisfy

$$\forall j\,(j \in I \Rightarrow u_j \in B_j).$$

Consequently, the previously made assumption $i \in I$ implies $u_i \in B_i$; noting that (3.882) gives $(i, \bar{y}) \in u$, which we can write as $\bar{y} = u_i$, we thus obtain via substitution $\bar{y} \in B_i$. Recalling that $\bar{y} \notin B_i$ also holds, as arrived at contradiction, so that the proof of the implication $i \in I \Rightarrow A_i \subseteq B_i$ is now complete. Here, i was arbitrary, so that the universal sentence in (3.878) follows then to be true. This in turn proves the implication (3.878), and as I, A and B were initially arbitrary sets, we may therefore conclude that the stated lemma holds. □

We generalize now the Equality Criterion for Cartesian products of two sets to families of sets.

Theorem 3.248 (Equality Criterion for Cartesian products of families of sets). *The following equivalence holds for any (index) set I and for any families $A = (A_i)_{i\in I}$ and $B = (B_i)_{i\in I}$ of nonempty sets.*

$$\forall i\,(i \in I \Rightarrow A_i = B_i) \;\Leftrightarrow\; \bigtimes_{i\in I} A_i = \bigtimes_{i\in I} B_i. \tag{3.884}$$

Proof. We take arbitrary sets I, A and B such that $A = (A_i)_{i\in I}$ and $B = (B_i)_{i\in I}$ are families of sets where

$$\forall i\,(i \in I \Rightarrow A_i \neq \emptyset), \tag{3.885}$$

$$\forall i\,(i \in I \Rightarrow B_i \neq \emptyset). \tag{3.886}$$

To prove the first part ('$\Rightarrow$') of the equivalence, we assume the universal sentence

$$\forall i\,(i \in I \Rightarrow A_i = B_i) \tag{3.887}$$

to be true. This assumption allows us to establish the conjunction

$$\forall i\,(i \in I \Rightarrow A_i \subseteq B_i) \wedge \forall i\,(i \in I \Rightarrow B_i \subseteq A_i). \tag{3.888}$$

Indeed, letting i be arbitrary and assuming $i \in I$ to be true, we obtain $A_i = B_i$ with (3.887), which equation implies $A_i \subseteq B_i$ and $B_i \subseteq A_i$ with (2.22); since i is arbitrary, we may therefore conclude that both of the universal sentences in (3.888) are true. These universal sentences imply now

$$\bigtimes_{i\in I} A_i \subseteq \bigtimes_{i\in I} B_i \wedge \bigtimes_{i\in I} B_i \subseteq \bigtimes_{i\in I} A_i \tag{3.889}$$

with (3.872), so that the desired equation

$$\bigtimes_{i\in I} A_i = \bigtimes_{i\in I} B_i \tag{3.890}$$

follows to be true with (2.22).

To prove the second part ('$\Leftarrow$') of the equivalence in (3.884), we assume (3.890) to be true, which assumption implies then the truth of the conjunction (3.889) with (2.22). These two inclusions further imply the universal sentences in (3.888) because of (3.878), which we use now to establish (3.887). Letting i be arbitrary and assuming $i \in I$ to be true, the universal sentences in (3.888) give the inclusions $A_i \subseteq B_i$ and $B_i \subseteq A_i$, which in turn imply $A_i = B_i$ with (2.22), as desired. Here, i is arbitrary, so that (3.887) follows to be true indeed.

We thus completed the proof of the equivalence (3.884), and as the sets I, A and B were initially arbitrary, we may therefore conclude that the theorem holds. $\square$

Proposition 3.249. *For any set I, any family $(X_i)_{i\in I}$ of sets and any $j \in I$, there exists a unique function π_j with domain $\bigtimes_{i\in I} X_i$ such that*

$$\forall f\,(f \in \bigtimes_{i\in I} X_i \Rightarrow \pi_j(f) = f_j). \tag{3.891}$$

holds. Moreover, X_j is a codomain of this function.

Proof. We let I, $(X_i)_{i\in I}$ and j be an arbitrary such that $(X_i)_{i\in I}$ is a family of sets and such that $j \in I$ holds. We now apply Function definition by replacement and verify for this purpose

$$\forall f\,(f \in \bigtimes_{i\in I} X_i \Rightarrow \exists! y\,(y = f_j)), \tag{3.892}$$

letting f be an arbitrary set and assuming $f \in \bigtimes_{i\in I} X_i$ to be true. Thus, f is a family of sets with index set I, so that the initial assumption $j \in I$ gives the uniquely specified function value f_j. Consequently, the uniquely existential sentence in (3.892) is true according to (1.109). Since f is arbitrary, we may therefore conclude that the universal sentence (3.892) holds, which implies the unique existence of a function π_j with domain $\bigtimes_{i\in I} X_i$ satisfying (3.891). To prove that X_j is a codomain of π_j, that is, to prove the inclusion $\mathrm{ran}(\pi_j) \subseteq X_j$, we verify

$$\forall y\,(y \in \mathrm{ran}(\pi_j) \Rightarrow y \in X_j). \tag{3.893}$$

To do this, we take an arbitrary y and assume $y \in \mathrm{ran}(\pi_j)$, so that there exists a constant, say $\bar{f}$, with $(\bar{f}, y) \in \pi_j$, using the definition of a range. On the one hand, we may write this as $y = \pi_j(\bar{f})$, because π_j is a function. On the other hand, $(\bar{f}, y) \in \pi_j$ gives with the definition of a domain $\bar{f} \in \bigtimes_{i\in I} X_i$ and then $\pi_j(\bar{f}) = \bar{f}_j$ due to (3.891), so that we obtain $y = \bar{f}_j$ via substitution. Let us now observe that $\bar{f} \in \bigtimes_{i\in I} X_i$ implies by definition of the Cartesian product of a family of sets that $\bar{f}_i \in X_i$ holds for any $i \in I$. Thus, the assumed $j \in I$ yields $[y =]\ \bar{f}_j \in X_j$, which finding proves the implication in (3.893). As y is arbitrary, we may infer from this the truth of the universal sentence (3.893), which gives $\mathrm{ran}(\pi_j) \subseteq X_j$ with the definition of a subset. Thus, X_j is indeed a codomain of π_j.

Since I, $(X_i)_{i\in I}$ and j were initially arbitrary, we may therefore conclude that the proposition is true. □

Definition 3.61 (Projection function on the Cartesian product of a family of sets). For any set I, any family $(X_i)_{i\in I}$ of sets and any $j \in I$ we call

$$\pi_j : \bigtimes_{i\in I} X_i \to X_j, \quad f \mapsto \pi_j(f) = f_j \tag{3.894}$$

the *projection function* from $\bigtimes_{i\in I} X_i$ to X_j or the j-th *projection map* on $\bigtimes_{i\in I} X_i$.

Theorem 3.250 (Surjectivity of projection functions with nonempty domain). *It is true for any nonempty index set I, any family $(X_i)_{i\in I}$ of* *nonempty sets and any index $j \in I$ that the j-th projection map on $\bigtimes_{i \in I} X_i$ is a surjection, i.e.*

$$\forall i \,(i \in I \Rightarrow X_i \neq \emptyset) \;\Rightarrow\; \forall j \,(j \in I \Rightarrow \pi_j : \bigtimes_{i \in I} X_i \twoheadrightarrow X_j). \tag{3.895}$$

Proof. We take arbitrary I, $(X_i)_{i \in I}$ and j, assuming $(X_i)_{i \in I}$ to be a family of sets with index set $I \neq \emptyset$ and assuming then also the universal sentence

$$\forall i \,(i \in I \Rightarrow X_i \neq \emptyset) \tag{3.896}$$

to be true. Next, we let also j be arbitrary and assume that $j \in I$ is true, so that the projection function $\pi_j : \bigtimes_{i \in I} X_i \to X_j$ is defined. To prove that it constitutes a surjection with range X_j, we apply the Surjection Criterion and verify accordingly

$$\forall y \,(y \in X_j \Rightarrow \exists f \,(\pi_j(f) = y)), \tag{3.897}$$

letting $y \in X_j$ be arbitrary. We may now apply Function definition by replacement to establish a unique family F with domain I such that

$$\forall i \,(i \in I \Rightarrow ([i = j \Rightarrow F(i) = \{y\}] \,\wedge\, [i \neq j \Rightarrow F(i) = X_i])). \tag{3.898}$$

To do this, we prove accordingly the universal sentence

$$\forall i \,(i \in I \Rightarrow \exists! Y \,([i = j \Rightarrow Y = \{y\}] \,\wedge\, [i \neq j \Rightarrow Y = X_i])), \tag{3.899}$$

letting i be arbitrary and assuming $i \in I$ to be true. Using the fact that the Law of the Excluded Middle gives us the true disjunction $i = j \,\vee\, i \neg j$, we may prove the existential part of the desired uniquely existential sentence by cases. In the first case $i = j$, we observe that replacing the variable Y by the constant $\{y\}$ (which is the uniquely specified singleton formed by the constant y) yields the true implication $i = j \Rightarrow \{y\} = \{y\}$ (in which both the antecedent and the consequent are true) as well as the true implication $i \neq j \Rightarrow Y = X_i$ (which has a false antecedent). Then, the conjunction of these two implications is also true, so that there exists indeed a Y for which the conjunction in (3.899) holds. In the second case $i \neq j$, we replace now Y by the set X_i, so that the first implication $i = j \Rightarrow X_i = \{y\}$ has now a false antecedent and the second implication $i \neq j \Rightarrow X_i = X_i$ both a true antecedent and a true consequent. Thus, both implications are true for the choice X_j, which proves the existential part also for the second case.

To establish the uniqueness part, we prove

$$\begin{aligned} \forall Y, Y' \,([([i = j \Rightarrow Y = \{y\}] \,\wedge\, [i \neq j \Rightarrow Y = X_i]) \\ \wedge\, ([i = j \Rightarrow Y' = \{y\}] \,\wedge\, [i \neq j \Rightarrow Y' = X_i])] \Rightarrow Y = Y'), \end{aligned} \tag{3.900}$$

letting Y and Y' be arbitrary and assuming the two conjunctions

$$[i = j \Rightarrow Y = \{y\}] \wedge [i \neq j \Rightarrow Y = X_i]$$
$$[i = j \Rightarrow Y' = \{y\}] \wedge [i \neq j \Rightarrow Y' = X_i]$$

to be true. Considering the same cases as before, we have that the first case $i = j$ implies $Y = \{y\}$ as well as $Y' = \{y\}$, and that the second case $i \neq j$ implies $Y = X_i$ as well as $Y' = X_i$. Consequently, substitution yields $Y = Y'$ for both cases, and since Y and Y' are arbitrary, we may therefore conclude that the uniqueness part (3.900) holds. Thus, the proof of the uniquely existential sentence is complete, and as i was arbitrary, we may further conclude that (3.899) is true.

Consequently, there exists a unique function F with domain I satisfying (3.898), and we may write F then as the family $(F_i)_{i \in I}$. This family in turn defines the Cartesian product $\bigtimes_{i \in I} F_i$, which we now prove to be nonempty.

For this purpose, we show that

$$\forall i\, (i \in I \Rightarrow F_i \neq \emptyset) \tag{3.901}$$

holds. Letting i be arbitrary and assuming $i \in I$ to hold, we prove the desired consequent $F_i \neq \emptyset$ by the same cases as before. The first case $i = j$ implies $[F_i =]\ F(i) = \{y\}$ with (3.898), for which singleton we have $y \in \{y\}$ according to (2.153) and therefore evidently $[F_i =]\ \{y\} \neq \emptyset$. The second case $i \neq j$ implies $[F_i =]\ F(i) = X_i$ again with (3.898), for which set the assumed $i \in I$ implies $[F_i =]\ X_i \neq \emptyset$ with the initial assumption (3.896). We thus found $F_i \neq \emptyset$ in any case, where i was arbitrary, so that (3.901) follows to be indeed true. This universal sentence implies now with the Negation Law for existential conjunctions $\neg\exists i\, (i \in I \wedge F_i = \emptyset)$, so that the negation $\bigtimes_{i \in I} F_i \neq \emptyset$ follows to be true with the Emptiness Criterion for Cartesian products of families of sets in connection with the Law of Contraposition. This negation in turn demonstrates that there exists an element in $\bigtimes_{i \in I} F_i$, say $\bar{f}$, for which the assumed $j \in I$ implies $\bar{f}_j \in F_j$ by definition of the Cartesian product of a family of sets. As $j \in I$ and $j = j$ imply $[F_j =]\ F(j) = \{y\}$ with (3.898), substitution gives us $\bar{f}_j \in \{y\}$ and therefore $\bar{f}_j = y$ with (2.169).

Our next task is to prove that $\bar{f}$ is an element also of the Cartesian product $\bigtimes_{i \in I} X_i$. We already know that $\bar{f}$ is a family with index set I, so it only remains for us to demonstrate the truth of

$$\forall i\, (i \in I \Rightarrow \bar{f}_i \in X_i). \tag{3.902}$$

Letting i be arbitrary and assuming that $i \in I$ holds, we consider once again the two cases $i = j$ and $i \neq j$. If $i = j$ is true, we obtain $\bar{f}_i = \bar{f}_j = y$, so that the previous assumption $y \in X_j$ gives $\bar{f}_i \in X_j$ and therefore $\bar{f}_i \in X_i$ via substitution based on the current case assumption. If $i \neq j$ is true, we obtain from $i \in I$ and $\bar{f} \in \bigtimes_{i \in I} F_i$ (by definition of the Cartesian product of a family of sets) $\bar{f}_i \in F_i$, where $[F_i =]\ F(i) = X_i$ follows to be true with (3.898), so that substitution yields $\bar{f}_i \in X_i$. Thus, the desired consequent of the implication in (3.902) holds in any case. Because i is arbitrary, we may infer from this the truth of the universal sentence (3.902), which finding completes the verification of $\bar{f}$ as an element also of the Cartesian product $\bigtimes_{i \in I} X_i$. This means that $\bar{f}$ is an element of the domain of the projection function π_j, so that the associated value is given by $\pi_j(\bar{f}) = \bar{f}_j$. Combining this with the previously established equation $\bar{f}_j = y$, we arrive at $\pi_j(\bar{f}) = y$, which in turn proves the existential sentence in (3.897). Here, y was arbitrary, so that the universal sentence (3.897) follows now to be true, and this finally implies that π_j is a surjection from $\bigtimes_{i \in I} X_i$ to X_j. Because I, $(X_i)_{i \in I}$ and j were initially arbitrary, we may therefore conclude that the stated theorem is indeed true. $\square$

3.4.7. Partially ordered sets of functions

In this section, we consider sets of functions from a set X to a set Y, and we will see that any reflexive partial ordering $\leq$ of the codomain Y may be used to define a reflexive partial ordering $\preceq$ of the set of functions Y^X.

Proposition 3.251. *For any nonempty set X, any set Y and any reflexive partial ordering $\leq$ of Y, it is true that there exists a unique set $\preceq$ such that a set Z is in $\preceq$ iff Z is in the Cartesian product $Y^X \times Y^X$ and moreover if Z is the ordered pair (f,g) formed by some functions f and g in Y^X for which the value of f at any x is less than or equal to the value of g at x, i.e. it is true that*

$$\begin{aligned}&\exists!\preceq \forall Z\,(Z\in\preceq\Leftrightarrow[Z\in Y^X\times Y^X\\&\quad\wedge\ \exists f,g\,(f,g\in Y^X\ \wedge\ \forall x\,(x\in X\Rightarrow f(x)\leq g(x)))\ \wedge\ (f,g)=Z)]).\end{aligned}$$

This set $\preceq$ satisfies

$$\forall f,g\,(f,g\in Y^X\Rightarrow[f\preceq g\Leftrightarrow\forall x\,(x\in X\Rightarrow f(x)\leq g(x))]) \tag{3.903}$$

and is a reflexive partial ordering of the set of functions Y^X.

Proof. We let X, Y and $\leq$ be arbitrary sets, assume $X\neq\emptyset$, and assume moreover $\leq$ to be a reflexive partial ordering of Y. We may then evidently apply the Axiom of Specification and the equality Criterion for sets to form the stated uniquely existential sentence. Thus, the set $\preceq$ satisfies the universal sentence

$$\begin{aligned}&\forall Z\,(Z\in\preceq\Leftrightarrow[Z\in Y^X\times Y^X\\&\quad\wedge\ \exists f,g\,(f,g\in Y^X\ \wedge\ \forall x\,(x\in X\Rightarrow f(x)\leq g(x))\ \wedge\ (f,g)=Z)]).\end{aligned} \tag{3.904}$$

Clearly, $Z\in\preceq$ implies in particular $Z\in Y^X\times Y^X$ for any Z, so that we have $\preceq\subseteq Y^X\times Y^X$ by definition of a subset, and this inclusion shows that $\preceq$ is a binary relation on Y^X.

Let us now verify that the binary relation $\preceq$ is also characterized by(3.903), letting f and g be arbitrary sets and assuming $f,g\in Y^X$ to be true. To prove the first part ('$\Rightarrow$') of the equivalence directly, we assume that $f\preceq g$ holds, which we may write also as $(f,g)\in\preceq$. This in turn implies with (3.904) in particular that there are elements of Y^X, say $\bar{f}$ and $\bar{g}$ such that the universal sentence

$$\forall x\,(x\in X\Rightarrow\bar{f}(x)\leq\bar{g}(x)) \tag{3.905}$$

and $(\bar{f},\bar{g})=(f,g)$ are true. The latter equation yields with the Equality Criterion for ordered pairs $\bar{f}=f$ and $\bar{g}=g$. Since the assumptions $f,g\in$

Y^X and the previously established $\bar{f}, \bar{g} \in Y^X$ show that $\bar{f}$ and f as well as $\bar{g}$ and g are functions with domain X, we may apply the Equality Criterion for functions to infer from the preceding equations the truth of the universal sentences

$$\forall x\, (x \in X \Rightarrow \bar{f}(x) = f(x)), \tag{3.906}$$
$$\forall x\, (x \in X \Rightarrow \bar{g}(x) = g(x)). \tag{3.907}$$

After this preparatory work, we are now in the position to establish the desired consequent

$$\forall x\, (x \in X \Rightarrow f(x) \leq g(x)) \tag{3.908}$$

of the first implication '⇒' in (3.903). Letting $\bar{x}$ be arbitrary and assuming $\bar{x} \in X$ to be true, we obtain on the one hand with (3.905) the inequality $\bar{f}(\bar{x}) \leq \bar{g}(\bar{x})$, and on the other hand with (3.906) – (3.907) the equations $\bar{f}(\bar{x}) = f(\bar{x})$ and $\bar{g}(\bar{x}) = g(\bar{x})$. Applying now substitutions based on these equations to the preceding inequality, we obtain $f(\bar{x}) \leq g(\bar{x})$, proving the implication in (3.908). As $\bar{x}$ was arbitrary, we may therefore conclude that the universal sentence (3.908) is true, so that the proof of the first part of the equivalence in (3.903) is complete.

Regarding the second part ('⇐'), we now assume (3.908) to be true. On the one hand, as we assumed $f, g \in Y^X$ to be true, we obtain for the ordered pair $Z = (f, g)$

$$Z \in Y^X \times Y^X \tag{3.909}$$

with the definition of the Cartesian product of two sets. On the other hand, we see that there exist constants f and g which satisfy the multiple conjunction

$$f, g \in Y^X \,\wedge\, \forall x\, (x \in X \Rightarrow f(x) \leq g(x)) \,\wedge\, (f, g) = Z.$$

Then, the conjunction of (3.909) and the preceding existential sentence implies with (3.904) $[Z =]\ (f, g) \in \preceq$, which we may write also as $f \preceq g$ (since $\preceq$ is a binary relation). Thus, the second part of the equivalence in (3.903) also holds, so that the proof of the equivalence is complete. As f and g were arbitrary, we may now further conclude that the proposed universal sentence (3.903) is indeed true.

Next, we prove that the binary relation $\preceq$ on Y^X is reflexive, by verifying

$$\forall f\, (f \in Y^X \Rightarrow f \preceq f). \tag{3.910}$$

We take an arbitrary set f, assume $f \in Y^X$ to be true (so that f is a function from X to Y), and we establish the truth of

$$\forall x\, (x \in X \Rightarrow f(x) \leq f(x)), \tag{3.911}$$

which will imply the desired consequent $f \preceq f$ with (3.903). Indeed, letting $x \in X$ be arbitrary, so that the corresponding function value $f(x)$ is an element of Y according to the Function Criterion, we obtain $f(x) \leq f(x)$ with the fact that partial ordering $\leq$ of Y is reflexive. Because x is arbitrary, we may therefore conclude that (3.911) holds, so that $f \preceq f$ follows to be true with (3.903). As f is also arbitrary, we may now infer from this finding the truth of (3.910), which means that $\preceq$ is reflexive.

To establish the antisymmetry of $\preceq$, we verify

$$\forall f, g\,(f, g \in Y^X \Rightarrow [(f \preceq g \wedge g \preceq f) \Rightarrow f = g]), \tag{3.912}$$

letting f and g be arbitrary and assuming first $f, g \in Y^X$ and then the conjunction $f \preceq g \wedge g \preceq f$ to be true. The latter inequalities give with (3.903)

$$\forall x\,(x \in X \Rightarrow f(x) \leq g(x)), \tag{3.913}$$
$$\forall x\,(x \in X \Rightarrow g(x) \leq f(x)), \tag{3.914}$$

which universal sentences allow us to establish

$$\forall x\,(x \in X \Rightarrow f(x) = g(x)). \tag{3.915}$$

To do this, we let x be arbitrary and assume $x \in X$, which yields with (3.913) and (3.914) $f(x) \leq g(x)$ and $g(x) \leq f(x)$. These two inequalities in turn imply $f(x) = g(x)$ with the antisymmetry of the reflexive partial ordering $\leq$ of Y. As x is arbitrary, we may therefore conclude that (3.915) holds, and this universal sentence implies the desired $f = g$ with the Equality Criterion for functions. Because f and g are arbitrary, we may now further conclude that the universal sentence (3.912) is true, which shows that $\preceq$ is antisymmetric.

It only remains for us to show that $\preceq$ is transitive, i.e. that $\preceq$ has the definite property

$$\forall f, g, h\,(f, g, h \in Y^X \Rightarrow [(f \preceq g \wedge g \preceq h) \Rightarrow f \preceq h]). \tag{3.916}$$

We take arbitrary f, g and h, assume $f, g, h \in Y^X$, and assume moreover $f \preceq g$ and $g \preceq h$, which inequalities imply

$$\forall x\,(x \in X \Rightarrow f(x) \leq g(x)), \tag{3.917}$$
$$\forall x\,(x \in X \Rightarrow g(x) \leq h(x)) \tag{3.918}$$

with (3.903). To establish the desired consequent $f \preceq h$, we prove

$$\forall x\,(x \in X \Rightarrow f(x) \leq h(x)), \tag{3.919}$$

letting x be arbitrary in X. Consequently, we obtain $f(x) \leq g(x)$ and $g(x) \leq h(x)$ with (3.917) and (3.918), respectively, and these two inequalities imply $f(x) \leq h(x)$ because of the transitivity of the partial ordering $\leq$. This finding proves already the implication in (3.919), and since x is arbitrary, we may therefore conclude that (3.919) holds. The truth of this universal sentence implies now the truth of $f \preceq h$ with (3.903), from which we may then infer the truth of the universal sentence (3.916), because f, g and h were arbitrary. Therefore, the reflexive and antisymmetric binary relation $\preceq$ on Y^X is also transitive, and constitutes thus a reflexive partial ordering of Y^X. Since X, Y and $\leq$ were initially arbitrary sets, we may finally conclude that the proposition is true. □

Theorem 3.252 (Characterization of lower & upper bounds for a family of functions). *For any sets I, X and Y, any reflexive partial ordering $\leq$ of Y, any family $f = (f_i)_{i\in I}$ of functions from X to Y and any function g from X to Y, it is true that*

a) g is a lower bound for $(f_i)_{i\in I}$ (with respect to the reflexive partial ordering $\preceq$ of Y^X) iff $g(x)$ is a lower bound for $f^{(x)} = (f_i(x))_{i\in I}$ (with respect to the reflexive partial ordering $\leq$ of Y) for any $x \in X$.

b) g is an upper bound for $(f_i)_{i\in I}$ iff $g(x)$ is an upper bound for $f^{(x)} = (f_i(x))_{i\in I}$ for any $x \in X$.

Proof. We let I, X, Y, $\leq$, f and g be arbitrary such that $\leq$ is a reflexive partial ordering of Y, such that $f = (f_i)_{i\in I}$ is a family in Y^X with index set I, and such that $g \in Y^X$ hold. Regarding the first part ('$\Rightarrow$') of the equivalence in b), we assume that g is an upper bound for $\mathrm{ran}(f)$, so that

$$\forall h\,(h \in \mathrm{ran}(f) \Rightarrow h \preceq g) \tag{3.920}$$

holds. We then take arbitrary $\bar{x}$, assume $\bar{x} \in X$ to be true, and show that this implies

$$\forall y\,(y \in \mathrm{ran}(f^{(\bar{x})}) \Rightarrow y \leq g(\bar{x})). \tag{3.921}$$

For this purpose, we let y be arbitrary and assume $y \in \mathrm{ran}(f^{(\bar{x})})$ to be true, so that there exists by definition of a range a constant, say $\bar{k}$, with $(\bar{k}, y) \in f^{(\bar{x})}$. This yields on the one hand in function/family notation $y = f^{(\bar{x})}(\bar{k}) = f_{\bar{k}}(\bar{x})$, and on the other hand $\bar{k} \in I$ $[= \mathrm{dom}(f^{(\bar{x})})]$ by definition of a domain. Let us observe here that $\bar{k} \in I$ implies the unique existence of the term $f_{\bar{k}} = f(\bar{k})$, which equation we may write in function notation as $(\bar{k}, f_{\bar{k}}) \in f$. By definition of a range, we therefore obtain $f_{\bar{k}} \in \mathrm{ran}(f)$, which implies $f_{\bar{k}} \preceq g$ with (3.920), and therefore

$$\forall x\,(x \in X \Rightarrow f_{\bar{k}}(x) \leq g(x)) \tag{3.922}$$

with (3.903). Then, the assumed $\bar{x} \in X$ yields $[y =]\ f_{\bar{k}}(\bar{x}) \leq g(\bar{x})$, and this finding proves the implication in (3.921). As y is arbitrary, we may therefore conclude that the universal sentence (3.921) holds, so that $g(\bar{x})$ is an upper bound for (the range of) the sequence $f^{(\bar{x})}$. Since $\bar{x}$ was also arbitrary, we may now further conclude that the first part of the stated equivalence holds.

Regarding the second part ('$\Leftarrow$'), we now assume that $g(x)$ is an upper bound for the sequence $f^{(x)}$ for all $x \in X$, and we show that g follows to be an upper bound for $\mathrm{ran}(f)$, i.e. that (3.920) follows to be true. For this purpose, we let h be arbitrary and assume $h \in \mathrm{ran}(f)$ to be true, so that there is (by definition of a range) a particular constant $\bar{k}$ with $(\bar{k}, h) \in f$. We therefore obtain $h = f_{\bar{k}}$ and $\bar{k} \in I$ $[= \mathrm{dom}(f)]$, where the former finding implies

$$\forall x\, (x \in X \Rightarrow h(x) = f_{\bar{k}}(x)) \tag{3.923}$$

with the Equality Criterion for functions. To establish the desired consequent $h \preceq g$, we verify now

$$\forall x\, (x \in X \Rightarrow h(x) \leq g(x)), \tag{3.924}$$

letting $\bar{x} \in X$ be arbitrary, so that $g(\bar{x})$ is (by assumption) an upper bound for the range of the sequence $f^{(\bar{x})}$; thus, the constant $\bar{x}$ satisfies the universal sentence (3.921). Furthermore, $\bar{x} \in X$ implies with (3.923) that $h(\bar{x}) = f_{\bar{k}}(\bar{x}) = f^{(\bar{x})}(\bar{k})$ holds. We may write the resulting equation $h(\bar{x}) = f^{(\bar{x})}(\bar{k})$ also as $(\bar{k}, h(\bar{x})) \in f^{(\bar{x})}$, so that $h(\bar{x}) \in \mathrm{ran}(f^{(\bar{x})})$ follows to be true by definition of a range, and this implies $h(\bar{x}) \leq g(\bar{x})$ with (3.921). Thus, the proof of the implication in (3.924) is complete, and since $\bar{x}$ was arbitrary, we may therefore conclude that (3.924) is true. This universal sentence in turn implies $h \preceq g$ with (3.903), proving the implication in (3.920). Because h was arbitrary, we may infer from this the truth of the universal sentence (3.920), which means that g is indeed an upper bound for $\mathrm{ran}(f)$.

Thus, the proof of the equivalence b) is complete, and since I, X, Y, $\leq$, f and g were arbitrary, we may further conclude that Part b) of the theorem is true. □

Exercise 3.108. Prove Part a) of Theorem 3.252.

(Hint: Proceed in analogy to the proof of Part b).)

Exercise 3.109. Show for any nonempty set X, any set Y and any irreflexive partial ordering $<$ of Y that there exists a unique set $\prec$ such that a set Z is in $\prec$ iff Z is in the Cartesian product $Y^X \times Y^X$ and moreover if Z is the ordered pair (f, g) formed by some functions f and g in Y^X for which the value of f at any x is less than the value of g at x, i.e. such that

$$\begin{aligned}&\exists! \prec \forall Z\,(Z \in \prec \Leftrightarrow [Z \in Y^X \times Y^X \\ &\quad \wedge \exists f, g\,(f, g \in Y^X \wedge \forall x\,(x \in X \Rightarrow f(x) < g(x))) \wedge (f, g) = Z)]).\end{aligned}$$

Prove that this set $\prec$ satisfies

$$\forall f, g\,(f, g \in Y^X \Rightarrow [f \prec g \Leftrightarrow \forall x\,(x \in X \Rightarrow f(x) < g(x))]) \tag{3.925}$$

and that $\prec$ is an irreflexive partial ordering of the set of functions Y^X. (Hint: Apply a proof by contradiction to establish the irreflexivity of $\prec$.)

Exercise 3.110. Verify for any sets X and Y and any reflexive partial ordering $\leq$ of Y (inducing the irreflexive partial ordering $<$ of Y) that a function f from X to Y is less than or equal to a function g from X to Y if f is less than g, i.e. that

$$\forall f, g\,(f, g \in Y^X \Rightarrow [f \prec g \Rightarrow f \preceq g]) \tag{3.926}$$

holds, where $\preceq$ and $\prec$ are defined by $\leq$ and $<$, respectively.

Theorem 3.253 (Generation of complete lattices of functions). *For any set X and any complete lattice $(Y, \leq)$, it is true that the partially ordered set $(Y^X, \preceq)$ is itself a complete lattice.*

Proof. We let X, Y and $\leq$ be arbitrary sets and assume $(Y, \leq)$ to be a complete lattice. Since $\leq$ is then especially a reflexive partial ordering of Y, we may indeed define the reflexive partial ordering $\preceq$ of Y^X according to Proposition 3.251. Let us observe here that the set Y is nonempty because we assumed $(Y, \leq)$ to be a lattice. Consequently, the set of function Y^X is also nonempty in view of Proposition 3.155. We now verify that the partially ordered set $(Y^X, \preceq)$ has the definite property

$$\forall A\,(A \subseteq Y^X \Rightarrow \exists S, I\,(S, I \in Y^X \wedge S = \overset{\preceq}{\sup} A \wedge I = \overset{\preceq}{\inf} A)), \tag{3.927}$$

letting A be arbitrary and assuming $A \subseteq Y^X$ to be true. Thus, every element of A is a function from X to Y. We first demonstrate that there exists, for any element x of the domain X of the functions in A, a unique set F_x consisting of the values (within the codomain Y) at x of all the functions in A, i.e.

$$\forall x\,(x \in X \Rightarrow \exists! F_x\, \forall y\,(y \in F_x \Leftrightarrow [y \in Y \wedge \exists f\,(f \in A \wedge y = f(x))])). \tag{3.928}$$

Letting x be arbitrary and assuming $x \in X$ to be true, we may evidently establish the uniquely existential sentence by applying the Axiom of Specification and the Equality Criterion for sets. Thus, the set F_x satisfies

$$\forall y\,(y \in F_x \Leftrightarrow [y \in Y \wedge \exists f\,(f \in A \wedge y = f(x))]), \tag{3.929}$$

which shows in particular that $y \in F_x$ implies $y \in Y$ for any y, so that $F_x \subseteq Y$ holds by definition of a subset. As x was arbitrary, we may therefore conclude that the universal sentence (3.928) is true. Consequently, the universal sentence (3.929) and the inclusion $F_x \subseteq Y$ hold then also for any $x \in X$. After this preparation, we show via Function definition by replacement that there are (unique) functions S and I with domain X satisfying

$$\forall x\,(x \in X \Rightarrow S(x) = \overset{\leq}{\sup} F_x), \tag{3.930}$$

$$\forall x\,(x \in X \Rightarrow I(x) = \overset{\leq}{\inf} F_x), \tag{3.931}$$

which task we accomplish by verifying

$$\forall x\,(x \in X \Rightarrow \exists! y\,(y = \overset{\leq}{\sup} F_x)), \tag{3.932}$$

$$\forall x\,(x \in X \Rightarrow \exists! y\,(y = \overset{\leq}{\inf} F_x)). \tag{3.933}$$

We take an arbitrary $x \in X$, so that the inclusion $F_x \subseteq Y$ holds, as mentioned before. Then, since $(Y, \leq)$ is a complete lattice, there are elements of Y, say $\bar{S}_x$ and $\bar{I}_x$, such that $\bar{S}_x = \sup^{\leq} F_x$ and $\bar{I}_x = \inf^{\leq} F_x$. Consequently, the uniquely existential sentences in (3.932) and (3.933) are both true according to (1.109). As x was arbitrary, both of the universal sentences (3.932) and (3.933) follow then to be also true. Consequently, there are unique functions S and I with domain X such that (3.930) and (3.931) hold.

Let us check now that Y is a codomain of S and I, i.e. that the inclusions $\mathrm{ran}(S) \subseteq Y$ and $\mathrm{ran}(I) \subseteq Y$ hold. Letting y and y' be arbitrary and assuming first $y \in \mathrm{ran}(S)$ as well as $y' \in \mathrm{ran}(I)$ to be true, there exist by definition of a range constants, say $\bar{x}$ and $\bar{x}$', with $(\bar{x}, y) \in S$ and $(\bar{x}', y') \in I$. Because we established S and I already as functions (with domain $\mathrm{dom}(S) = \mathrm{dom}(S) = X$), we obtain on the one hand $\bar{x} \in X$ [$= \mathrm{dom}(S)$] as well as $\bar{x}' \in X$ [$= \mathrm{dom}(I)$] with the definition of a domain, and on the other hand we may write $y = S(\bar{x})$ as well as $y' = I(\bar{x}')$. Here, $\bar{x} \in X$ and $\bar{x}' \in X$ imply [$y =$] $S(\bar{x}) = \sup^{\leq} F_{\bar{x}}$ as well as [$y' =$] $I(\bar{x}') = \inf^{\leq} F_{\bar{x}}$ with (3.930) and (3.931), respectively. As mentioned earlier, the preceding supremum and infimum are both in Y because of the completeness of the lattice $(Y, \leq)$. We thus showed that $y \in \mathrm{ran}(S)$ implies $y \in Y$ and that $y' \in \mathrm{ran}(I)$ implies $y' \in Y$. Since y and y' were arbitrary, we may infer from the truth of these implications the truth of the two inclusions $\mathrm{ran}(S) \subseteq Y$ and $\mathrm{ran}(I) \subseteq Y$, using the definition of a subset. Thus, S and I are both functions from X to Y.

Our next task is to demonstrate that the two functions $S : X \to Y$ and $I : X \to Y$ satisfy also the equations in (3.927), that is, $S = \sup^{\preceq} A$ and $I = \inf^{\preceq} A$. We first show that S is an upper bound and I a lower bound for A with respect to $\preceq$, i.e. that

$$\forall h\,(h \in A \Rightarrow h \preceq S), \tag{3.934}$$

$$\forall h\,(h \in A \Rightarrow I \preceq h) \tag{3.935}$$

hold. We let h be arbitrary, assume $h \in A$ to be true, so that the initial assumption $A \subseteq Y^X$ yields $h \in Y^X$ by definition of a subset, and therefore $h : X \Rightarrow Y$. We now show that this implies $h \preceq S$ as well as $I \preceq h$, i.e.

$$\forall x\,(x \in X \Rightarrow h(x) \leq S(x)), \tag{3.936}$$

$$\forall x\,(x \in X \Rightarrow I(x) \leq h(x)), \tag{3.937}$$

using (3.903). We let x be arbitrary and assume $x \in X$ to be true, which yields $S(x) = \sup^{\leq} F_x$ with (3.930) and $I(x) = \inf^{\leq} F_x$ with (3.931). Since the supremum $S(x)$ is an upper bound and the infimum $I(x)$ a lower bound for F_x with respect to $\leq$, we have the true universal sentences

$$\forall y\,(y \in F_x \Rightarrow y \leq S(x)), \tag{3.938}$$

$$\forall y\,(y \in F_x \Rightarrow I(x) \leq y). \tag{3.939}$$

Recalling $h : X \to Y$ and the assumption $x \in X$, we thus have that the function value $y = h(x)$ is in Y. In view of the assumption $h \in A$, this shows that the existential sentence $\exists f\,(f \in A \wedge y = f(x))$ is true, which then implies together with $[h(x) =]\ y \in Y$ that $[h(x) =]\ y \in F_x$ holds, according to (3.929). Thus, (3.938) and (3.939) give the desired inequalities $y \leq S(x)$ as well as $I(x) \leq y$. As x was arbitrary, we may therefore conclude that the universal sentences (3.936) and (3.937) are true, which in turn imply $h \preceq S$ as well as $I \preceq h$, as desired. Because h was arbitrary as well, we may now further conclude that the universal sentences (3.934) and (3.935) also hold, so that S is indeed an upper and I a lower bound for A with respect to $\preceq$.

To prove that S is the least upper and I the greatest lower bound for A, we apply the Characterization of the supremum & infimum and let S', I' be arbitrary such that S' is an upper and I' a lower bound for A with respect to $\preceq$, so that the universal sentences

$$\forall h\,(h \in A \Rightarrow h \preceq S'), \tag{3.940}$$

$$\forall h\,(h \in A \Rightarrow I' \preceq h), \tag{3.941}$$

hold. We need to show that $S \preceq S'$ as well as $I' \preceq I$ follow to be true, which inequalities we may write equivalently as

$$\forall x\,(x \in X \Rightarrow S(x) \leq S'(x)), \tag{3.942}$$

$$\forall x\,(x \in X \Rightarrow I'(x) \leq I(x)). \tag{3.943}$$

Letting now x be arbitrary and assuming $x \in X$ to be true, we obtain with (3.930) – (3.931) $S(x) = \sup^{\leq} F_x$ and $I(x) = \inf^{\leq} F_x$. We now verify that $S'(x)$ is an upper and $I'(x)$ a lower bound for F_x with respect to $\leq$, i.e.

$$\forall y\,(y \in F_x \Rightarrow y \leq S'(x)), \tag{3.944}$$

$$\forall y\,(y \in F_x \Rightarrow I'(x) \leq y). \tag{3.945}$$

We let y be arbitrary and assume $y \in F_x$ to be true, which implies with (3.929) $y \in Y$ and moreover that there exists an element in A, say $\bar{f}$, with $y = \bar{f}(x)$. Here, $\bar{f} \in A$ implies with (3.940) and (3.941) the inequalities $\bar{f} \preceq S'$ and $I' \preceq \bar{f}$, so that the universal sentences

$$\forall x\,(x \in X \Rightarrow \bar{f}(x) \leq S'(x))$$

$$\forall x\,(x \in X \Rightarrow I'(x) \leq \bar{f}(x))$$

are true. Consequently, the assumed $x \in X$ implies $[y =]\ \bar{f}(x) \leq S'(x)$ as well as $I'(x) \leq \bar{f}(x)\ [= y]$. We thus obtained the desired consequents in (3.944) and (3.945), and since y was arbitrary, we may therefore conclude that the universal sentences (3.944) – (3.945) are true. Thus, $S'(x)$ is indeed an upper and $I'(x)$ a lower bound for F_x with respect to $\leq$. We already established $S(x)$ as the supremum and $I(x)$ as the infimum of F_x (with respect to $\leq$), so that the Characterization of the supremum & infimum yields the inequalities $S(x) \leq S'(x)$ as well as $I'(x) \leq I(x)$, as desired. As x was arbitrary, we may infer from these findings the truth of the universal sentences (3.942) – (3.943). These in turn imply $S \preceq S'$ as well as $I' \preceq I$, and because S', I' are arbitrary here, we see that the upper bound S for A is the least one and that the lower bound I for A is the greatest one (with respect to $\preceq$).

Recalling that S and I are elements of Y^X, we thus established the existential sentence in (3.927). Since A was arbitrary, we may therefore conclude that the universal sentence (3.927) holds, so that the partially ordered set $(Y^X, \preceq)$ is a complete lattice by definition. Because X, Y and $\leq$ were initially arbitrary sets, we may now finally conclude that the proposition holds. □

Theorem 3.254 (Generation of lattices of functions). *For any set X and any lattice $(Y, \leq)$, it is true that the partially ordered set $(Y^X, \preceq)$ is itself a lattice.*

Exercise 3.111. Prove Theorem 3.254.

(Hint: Carry out a similar proof as for Theorem 3.254; here, it is not necessary to apply Function definition by replacement, because one may utilize directly the definition of a pair.)

Definition 3.62 (Supremum & infimum of a function). For any sets X and Y, any partial ordering $\leq$ of Y and any function $f : X \to Y$, we say that

(1) an element S in Y is the supremum f iff S is the supremum of the range of f.

(2) an element I in Y is the infimum f iff I is the infimum of the range of f.

Notation 3.10. For any partially ordered set $(Y, \leq_Y)$ and any sequence $f = (a_n)_{n \in \mathbb{N}}$ in Y such that

(1) the supremum of the range of $(a_n)_{n\in\mathbb{N}}$ exists, we also write

$$\sup_{n\in\mathbb{N}} a_n = \overset{\leq_Y}{\sup_{n\in\mathbb{N}}} a_n = \overset{\leq_Y}{\sup} \operatorname{ran}(f) = \sup \operatorname{ran}(f). \tag{3.946}$$

(2) the infimum of the range of $(a_n)_{n\in\mathbb{N}}$ exists, we also write

$$\inf_{n\in\mathbb{N}} a_n = \overset{\leq_Y}{\inf_{n\in\mathbb{N}}} a_n = \overset{\leq_Y}{\inf} \operatorname{ran}(f) = \inf \operatorname{ran}(f). \tag{3.947}$$

Theorem 3.255 (Characterization of the supremum & of the infimum of a sequence of functions). *For any sets X and Y, any reflexive partial ordering $\leq$ of Y, any sequence $f = (f_n)_{n\in\mathbb{N}_+}$ of functions from X to Y and any function g from X to Y, it is true that*

a) g is the supremum of the sequence $f = (f_n)_{n\in\mathbb{N}_+}$ with respect to $\preceq$ iff $g(x)$ is the supremum of the sequence $f^{(x)} = (f_n(x))_{n\in\mathbb{N}_+}$ with respect to $\leq$ for any $x \in X$, i.e.

$$g = \overset{\preceq}{\sup} \operatorname{ran}(f) \;\Leftrightarrow\; \forall x\, (x \in X \Rightarrow g(x) = \overset{\leq}{\sup} \operatorname{ran}(f^{(x)})). \tag{3.948}$$

b) g is the infimum of $f = (f_n)_{n\in\mathbb{N}_+}$ with respect to $\preceq$ iff $g(x)$ is the infimum of $f^{(x)} = (f_n(x))_{n\in\mathbb{N}_+}$ with respect to $\leq$ for any $x \in X$, i.e.

$$g = \overset{\preceq}{\inf} \operatorname{ran}(f) \;\Leftrightarrow\; \forall x\, (x \in X \Rightarrow g(x) = \overset{\leq}{\inf} \operatorname{ran}(f^{(x)})). \tag{3.949}$$

Proof. We let X, Y, $\leq$, f and g be arbitrary sets, we assume $(Y, \leq)$ to be a partially ordered set, $f = (f_n)_{n \in \mathbb{N}_+}$ to be a sequence in Y^X, and g to be an element of Y^X.

To prove the first part ('$\Rightarrow$') of the equivalence in a), we assume that $g = \sup^{\preceq} \mathrm{ran}(f)$ is true, so that g as a supremum is an upper bound for (the range of) f, which thus satisfies

$$\forall g' \, (g' \text{ is an upper bound for } \mathrm{ran}(f) \Rightarrow g \preceq g'). \tag{3.950}$$

We now prove the desired universal sentence

$$\forall x \, (x \in X \Rightarrow g(x) = \overset{\leq}{\sup} \, \mathrm{ran}(f^{(x)})), \tag{3.951}$$

letting $\bar{x}$ be arbitrary and assuming $\bar{x} \in X$ to be true. Then, since we found g to be an upper bound for $\mathrm{ran}(f)$, it follows with the Characterization of upper bounds for the range of a sequence of functions in particular that $g(\bar{x})$ is an upper bound for the sequence $f^{(\bar{x})}$. To complete the proof that $g(\bar{x})$ is the supremum of $f^{(\bar{x})}$, we take an arbitrary upper bound S' for $f^{(\bar{x})}$ and show that $g(\bar{x}) \leq S'$ follows to be true. For this purpose, we construct a function $\bar{g}' : X \to Y$ such that $\bar{g}'(\bar{x}) = S'$ and such that $\bar{g}'(x)$ is some upper bound for the sequence $f^{(x)} = (f_n(x))_{n \in \mathbb{N}_+}$ for all $x \neq \bar{x}$ (which evidently implies that $\bar{g}'$ is an upper bound for f). To do this, we apply first Function definition by replacement to establish the unique existence of a function F with domain X satisfying

$$\begin{aligned} \forall x \, (x \in X \Rightarrow [(x = \bar{x} \Rightarrow F(x) = \{S'\}) \\ \wedge \, (x \neq \bar{x} \Rightarrow F(x) \text{ consists of all upper bounds for } \mathrm{ran}(f^{(x)}))]), \end{aligned} \tag{3.952}$$

and we prove for this purpose

$$\begin{aligned} \forall x \, (x \in X \Rightarrow \exists! U \, ([(x = \bar{x} \Rightarrow U = \{S'\}) \\ \wedge \, (x \neq \bar{x} \Rightarrow U \text{ consists of all upper bounds for } \mathrm{ran}(f^{(x)}))])). \end{aligned} \tag{3.953}$$

We let x be arbitrary, assume $x \in X$ to be true, and begin with the existential part, which we prove by considering the two cases based on the disjunction $x = \bar{x} \vee x \neq \bar{x}$ (which is true according to the Law of the Excluded Middle). In the first case of $x = \bar{x}$, replacing the variable U by the constant $\{S'\}$ yields the true implication

$$x = \bar{x} \Rightarrow \{S'\} = \{S'\}$$

(which has a true antecedent and consequent) and the true implication

$$x \neq \bar{x} \Rightarrow \{S'\} \text{ consists of all upper bounds for } \mathrm{ran}(f^{(x)})$$

(being based on a false antecedent); thus, the existential part holds for the first case. In the second case $x \neq \bar{x}$, we see that $f^{(x)}$ is a sequence in Y due to Proposition 3.231, so that $\mathrm{ran}(f^{(x)}) \subseteq Y$ holds by definition of a codomain; since $(Y, \leq)$ is partially ordered, we then observe in light of Exercise 3.41a) that there exists a (unique) set $\bar{U}$ consisting of all upper bounds for $\mathrm{ran}(f^{(x)})$ (with respect to $\leq$). Thus, the implication

$$x = \bar{x} \Rightarrow \bar{U} = \{S'\}$$

(which has a false antecedent) and the implication

$$x \neq \bar{x} \Rightarrow \bar{U} \text{ consists of all upper bounds for } \mathrm{ran}(f^{(x)})$$

are true, so that the existential part holds also in the second case. Regarding the uniqueness part, we take an arbitrary constant $\bar{U}$ satisfying the implication

$$\begin{aligned}&(x = \bar{x} \Rightarrow \bar{U} = \{S'\})\\ &\quad \wedge\ (x \neq \bar{x} \Rightarrow \bar{U} \text{ consists of all upper bounds for } \mathrm{ran}(f^{(x)}))\end{aligned}$$

and an arbitrary constant $\bar{U}'$ satisfying the implication

$$\begin{aligned}&(x = \bar{x} \Rightarrow \bar{U}' = \{S'\})\\ &\quad \wedge\ (x \neq \bar{x} \Rightarrow \bar{U}' \text{ consists of all upper bounds for } \mathrm{ran}(f^{(x)})),\end{aligned}$$

and we show that $\bar{U} = \bar{U}'$ follows to be true. Then, the first case $x = \bar{x}$ implies $\bar{U} = \{S'\}$ as well as $\bar{U}' = \{S'\}$, which equations yield via substitution $\bar{U} = \bar{U}'$, as desired. The other case $x \neq \bar{x}$ implies that $\bar{U}$ and $\bar{U}'$ both consist of all upper bounds for $\mathrm{ran}(f^{(x)})$, which also gives the desired $\bar{U} = \bar{U}'$ because the set of all upper bounds for $\mathrm{ran}(f^{(x)})$ is unique (again according to Exercise 3.41a)). We thus established also the uniqueness part for both cases, and as x was arbitrary, we may therefore conclude that the universal sentence (3.953) is true, which then implies that there is a unique function F with domain X such that (3.952) holds. Since the range of F is a subset of itself (according to Proposition 2.4), we have that $\mathrm{ran}(F)$ is a codomain of F, so that

$$F : X \to \mathrm{ran}(F) \tag{3.954}$$

holds. Let us now prove that $\emptyset \notin \mathrm{ran}(F)$ holds, by verifying the equivalent universal sentence (according to (2.5))

$$\forall U\, (U \in \mathrm{ran}(F) \Rightarrow U \neq \emptyset). \tag{3.955}$$

We let U be arbitrary and assume $U \in \mathrm{ran}(F)$, so that there is (by definition of a range) a constant, say $\bar{x}$, with $(\bar{x}, U) \in F$. Because F is a function with domain X, we obtain $U = F(\bar{x})$ and $\bar{x} \in X$. In view of (3.952), it then follows from these findings that U consists of all upper bounds for $f^{(\bar{x})}$. Since g is an upper bound for f, it follows with the Characterization of upper bounds for a sequence of functions in particular that $g(\bar{x})$ is an upper bound for the sequence $f^{(\bar{x})}$, so that $g(\bar{x}) \in U$ is true. Thus, there exists an element in U, so that the desired negation $U \neq \emptyset$ holds according to (2.42). As U was arbitrary, we may now conclude that (3.955) is indeed true. Because this universal sentence implies $\emptyset \notin \mathrm{ran}(F)$, we may now apply the Axiom of Choice to infer from this negation that there exists a function from $\mathrm{ran}(F)$ to the union of that range, say $\bar{G} : \mathrm{ran}(F) \to \bigcup \mathrm{ran}(F)$, such that

$$\forall U \,(U \in \mathrm{ran}(F) \Rightarrow \bar{G}(U) \in U) \tag{3.956}$$

holds. We may now show that Y is also a codomain of $\bar{G}$. For this purpose, we establish $\bigcup \mathrm{ran}(F) \subseteq Y$ by verifying the equivalent

$$\forall u \,(u \in \bigcup \mathrm{ran}(F) \Rightarrow u \in Y). \tag{3.957}$$

We let u be arbitrary and assume $u \in \bigcup \mathrm{ran}(F)$. By definition of the union of a set system, there exists then an element of $\mathrm{ran}(F)$, say $\bar{U}$, such that $u \in \bar{U}$ is true. Here, $\bar{U} \in \mathrm{ran}(F)$ evidently implies the existence of a particular element $\bar{x}$ of the domain X of F such that $(\bar{x}, \bar{U}) \in F$ holds. We may write for the latter also $[u \in]\ \bar{U} = F(\bar{x})$, which shows in light of (3.952) that u is an upper bound for $\mathrm{ran}(f^{(\bar{x})})$ (with respect to the partial ordering $\leq$ of Y). Thus, $u \in Y$ holds by definition of an upper bound, as desired, and as u was arbitrary, we may therefore conclude that the universal sentence (3.957) is true. Consequently, we obtain the inclusion $\bigcup \mathrm{ran}(F) \subseteq Y$ with the definition of a subset, and because the inclusion $\mathrm{ran}(\bar{G}) \subseteq \bigcup \mathrm{ran}(F)$ is also true by definition of a codomain, the transitivity of $\subseteq$ yields $\mathrm{ran}(\bar{G}) \subseteq Y$. Thus, Y is indeed a codomain of $\bar{G}$, so that

$$\bar{G} : \mathrm{ran}(F) \to Y. \tag{3.958}$$

In view of (3.954) and (3.958), the composition $\bar{g}' = \bar{G} \circ F$ follows then to be a function from X to Y according to Proposition 3.178, i.e. $\bar{g}' : X \to Y$. Let us quickly check that $\bar{g}'$ has all of the desired properties. Letting $x \in X$ be arbitrary and considering the case of $x = \bar{x}$, this function yields

$$[\bar{g}'(\bar{x}) =]\ \bar{G}(F(\bar{x})) = \bar{G}(\{S'\}) \in \{S'\}$$

according to (3.952) and (3.956), and therefore indeed $\bar{g}'(\bar{x}) = S'$ with (2.169). In the other case of $x \neq \bar{x}$, we have

$$[\bar{g}'(x) =]\ \bar{G}(F(x)) \in F(x)$$

where $F(x)$ consists of all upper bounds for $\mathrm{ran}(f^{(x)})$, as specified in (3.952); thus, $\bar{g}'(x)$ is indeed an upper bound for the sequence $f^{(x)} = (f_n(x))_{n\in\mathbb{N}_+}$ for any $x \neq \bar{x}$. Having accomplished the task of establishing the upper bound $\bar{g}'$ for $\mathrm{ran}(f)$, we may now apply (3.950) to infer the truth of $g \preceq \bar{g}'$, which yields in particular $g(\bar{x}) \leq \bar{g}'(\bar{x})$ $[= S']$. This finding completes the proof of $g(\bar{x}) \leq S'$, and as S' was an arbitrary upper bound for $f^{(\bar{x})}$, we may conclude that $g(\bar{x})$ is the supremum of $f^{(\bar{x})}$. Because $\bar{x}$ was also arbitrary, we may furthermore conclude that the universal sentence (3.951) holds, completing the proof of the first part of the equivalence in a).

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume that $g(x)$ is the supremum of $f^{(x)}$ for all $x \in X$. Thus, $g(x)$ is an upper bound for $f^{(x)}$ for all $x \in X$, so that g is evidently an upper bound for f. To demonstrate that g is the least upper bound, we let g' be an arbitrary upper bound for f and show that $g \preceq g'$ follows to be true. To establish the latter, we use (3.903) and let x be arbitrary in X. Now, since g' is an upper bound for f, we have that $g'(x)$ is an upper bound for $f^{(x)}$; as $g(x)$ is the least upper bound for that sequence, we see that $g(x) \leq g'(x)$ holds. Because x is arbitrary, the preceding inequality is then true for all $x \in X$, so that $g \preceq g'$ holds indeed. This completes the proof that g is the supremum of the sequence f, which means that the second part of the proposed equivalence is true as well.

Since $X, Y, \leq, f$ and g were initially arbitrary, we may therefore conclude that Part a) of the stated theorem holds. □

Notation 3.11. In case the supremum/infimum of the range of a sequence $f = (f_n)_{n\in\mathbb{N}_+}$ of functions from X to Y exists, we write respectively

$$\sup_{n\in\mathbb{N}_+} f_n \,\mathrm{pointwise} = \overset{\preceq}{\sup}\,\mathrm{ran}(f) \tag{3.959}$$

and

$$\inf_{n\in\mathbb{N}_+} f_n \,\mathrm{pointwise} = \overset{\preceq}{\inf}\,\mathrm{ran}(f), \tag{3.960}$$

and we speak of the *pointwise supremum/pointwise infimum* of (the range of) f.

3.4.8. Monotone functions, order-embeddings, and order-isomorphisms

In the following we will use irreflexive partial orderings alongside their induced reflexive counterparts.

Definition 3.63 (Increasing / order-preserving / isotone function, strictly increasing function, decreasing / order-reversing / antitone function, strictly decreasing function, monotone function, strictly monotone function). For any partially ordered sets $(X, <_X)$ and $(Y, <_Y)$ and for any function $f : X \to Y$, we say that

(1) f is *increasing* or *order-preserving* or *isotone* (with respect to $\leq_Y$) iff

$$\forall x, y\,([x, y \in X \,\wedge\, x <_X y] \Rightarrow f(x) \leq_Y f(y)). \tag{3.961}$$

(2) f is *strictly increasing* (with respect to $<_Y$) iff

$$\forall x, y\,([x, y \in X \,\wedge\, x <_X y] \Rightarrow f(x) <_Y f(y)). \tag{3.962}$$

(3) f is *decreasing/order-reversing/antitone* (with respect to $\leq_Y$) iff

$$\forall x, y\,([x, y \in X \,\wedge\, x <_X y] \Rightarrow f(x) \geq_Y f(y)). \tag{3.963}$$

(4) f is *strictly decreasing* (with respect to $<_Y$) iff

$$\forall x, y\,([x, y \in X \,\wedge\, x <_X y] \Rightarrow f(x) >_Y f(y)). \tag{3.964}$$

(5) f is *monotone* iff f is increasing or decreasing.

(6) f is *strictly monotone* iff f is strictly increasing or strictly decreasing.

Note 3.35. We will use the terms *isotone* and *antitone* only in cases of set systems which are partially ordered by inclusion according to Theorem 3.64.

Proposition 3.256. *For any partially ordered sets $(X, <_X)$, $(Y, <_X)$ and for any element $c \in Y$, it is true that the constant function $g_c : X \to Y$ is increasing.*

Proof. Letting X, $<_X$, Y, $<_Y$ and c be arbitrary such that $(X, <_X)$ and $(Y, <_X)$ are partially ordered sets and such that $c \in Y$ holds, we now show that the constant function g_c on X satisfies (3.961). For this purpose, we take arbitrary x, y and assume the conjunction of $x, y \in X$ and $x <_X y$ to be true. We then obtain with Corollary 3.154 the equations $g_c(x) = c$ and $g_c(y) = c$, so that substitution yields $g_c(x) = g_c(y)$. Consequently, the disjunction $g_c(x) <_Y g_c(y) \vee g_c(x) = g_c(y)$ is also true, which in turn gives $g_c(x) \leq_Y g_c(y)$ by definition of an induced reflexive partial ordering. Since x and y are arbitrary, we may infer from this finding that the constant function g_c is indeed increasing, by definition. As X, $<_X$, Y, $<_Y$ and c were initially arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Exercise 3.112. Prove for any partially ordered sets $(X, <_X)$, $(Y, <_X)$ and for any element $c \in Y$ that the constant function $g_c : X \to Y$ is decreasing.

Exercise 3.113. Verify for any partially ordered set $(Y, <_X)$ that the empty function $\emptyset : \emptyset \to Y$ is increasing, strictly increasing, decreasing, strictly decreasing, monotone, and strictly monotone.

(Hint: Apply (2.40) with Proposition 3.151 and Corollary 3.66.)

There exist convenient alternative characterizations of increasing and decreasing functions.

Proposition 3.257. *For any partially ordered sets* $(X, \leq_X)$ *and* $(Y, \leq_Y)$ *and for any function* $f : X \to Y$*, it is true that* f *is increasing iff*

$$\forall x, y\,([x, y \in X \;\wedge\; x \leq_X y] \Rightarrow f(x) \leq_Y f(y)). \tag{3.965}$$

Proof. We let X, Y, $<_X$, $\leq_X$, $<_Y$, $\leq_Y$ and f be arbitrary sets, assume that $<_X$ is an irreflexive partial ordering and that $\leq_X$ is a reflexive partial ordering of X inducing each other, assume moreover that $<_Y$ is an irreflexive partial ordering and that $\leq_Y$ is a reflexive partial ordering of Y inducing each other, and we also assume that f is a function from X to Y.

To prove the first part ('$\Rightarrow$') of the proposed equivalence, we assume that f is increasing, i.e. that (3.961) holds. To show that this implies (3.965), we let x and y be arbitrary and assume $x, y \in X$ as well as $x \leq_X y$ to be true. The latter assumption implies with Theorem 3.81 (or Theorem 3.86) the truth of the disjunction $x <_X y \vee x = y$. On the one hand, if $x <_X y$ holds, then the conjunction of this and the assumption $x, y \in X$ implies the desired $f(x) \leq_Y f(y)$ with the assumption (3.961). On the other hand, if $x = y$ is true, then the assumed $x, y \in X$ $[= \mathrm{dom}(f)]$ implies together with $x = y$ the truth of $f(x) = f(y)$ because of (3.526). Consequently, the disjunction $f(x) <_Y f(y) \vee f(x) = f(y)$ is also true (irrespective of the truth value of $f(x) <_Y f(y)$), which gives the desired $f(x) \leq_Y f(y)$ again with Theorem 3.81 (or Theorem 3.86). We thus showed that $x, y \in X$ and $x \leq_X y$ imply $f(x) \leq_Y f(y)$, and as x and y are arbitrary, we may therefore conclude that (3.965) holds, proving the first part of the proposed equivalence.

To prove the second part ('$\Leftarrow$'), we now assume (3.965) to be true and demonstrate that (3.961) follows to be true. For this purpose, we let x and y be arbitrary, assume $x, y \in X$ and $x <_X y$ to hold, and apply the arguments used in the proof of '$\Rightarrow$'. To begin with, the assumed inequality $x <_X y$ implies the disjunction $x <_X y \vee x = y$ (no matter if $x = y$ is true or false), and therefore $x \leq_X y$. Then, the conjunction of $x, y \in X$ and $x \leq_X y$ implies the desired $f(x) \leq_Y f(y)$ with (3.965). As x and y are arbitrary, it follows that (3.961) holds, so that f is increasing by definition. Thus, the proof of the equivalence is complete.

Since X, Y, $<_X$, $\leq_X$, $<_Y$, $\leq_Y$ and f were initially arbitrary sets, we may finally conclude that the proposition is true. $\square$

Exercise 3.114. Show for any partially ordered sets $(X, <_X)$ and $(Y, <_Y)$ that a function $f : X \to Y$ is decreasing iff

$$\forall x, y\,([x, y \in X \wedge x \leq_X y] \Rightarrow f(x) \geq_Y f(y)). \tag{3.966}$$

The expectation that a strictly increasing/decreasing/monotone function should be the special case of an increasing/decreasing/monotone function is confirmed straightforwardly.

Exercise 3.115. Verify the following implications for any partially ordered sets $(X, <_X)$, $(Y, <_Y)$ and for any function $f : X \to Y$.

a) If f is strictly increasing, then f is increasing.

b) If f is strictly decreasing, then f is decreasing.

c) If f is strictly monotone, then f is monotone.

Exercise 3.116. Prove for any partially ordered sets $(X, <_X)$ and $(Y, <_Y)$, for any function $f : X \to Y$ and for any set $A \subseteq X$ that

a) the restriction of f to A is increasing if f is increasing, i.e.

$$\begin{aligned} &\forall x, y\,([x, y \in X \wedge x <_X y] \Rightarrow f(x) \leq_Y f(y)) \\ &\quad \Rightarrow \forall x, y\,([x, y \in A \wedge x <_X y] \Rightarrow f \restriction A(x) \leq_Y f \restriction A(y)). \end{aligned} \tag{3.967}$$

b) the restriction of f to A is strictly increasing if f is strictly increasing, i.e.

$$\begin{aligned} &\forall x, y\,([x, y \in X \wedge x <_X y] \Rightarrow f(x) <_Y f(y)) \\ &\quad \Rightarrow \forall x, y\,([x, y \in A \wedge x <_X y] \Rightarrow f \restriction A(x) <_Y f \restriction A(y)). \end{aligned} \tag{3.968}$$

(Hint: Use (3.567).)

Definition 3.64 (Order-embedding, order-isomorphism, order-isomorphic partially ordered sets). For any posets $(X, \leq_X)$, $(Y, \leq_Y)$ and for any function $f : X \to Y$, we say that

(1) f is an *order-embedding* from $(X, \leq_X)$ to $(Y, \leq_Y)$, symbolically

$$f : (X, \leq_X) \hookrightarrow (Y, \leq_Y), \tag{3.969}$$

if, and only if,

$$\forall x, y \, (x, y \in X \Rightarrow [x \leq_X y \Leftrightarrow f(x) \leq_Y f(y)]). \tag{3.970}$$

(2) f is an *order-isomorphism* from $(X, \leq_X)$ to $(Y, \leq_Y)$, symbolically

$$f : (X, \leq_X) \rightleftarrows (Y, \leq_Y), \tag{3.971}$$

iff f is a surjection and an order-embedding. Then, we say that $(X, \leq_X)$ and $(Y, \leq_Y)$ are *order-isomorphic* iff there exists an order-isomorphism from $(X, \leq_X)$ to $(Y, \leq_Y)$.

We now verify that an order-embedding is indeed an injection, as suggested by the symbol '$\hookrightarrow$' in (3.969).

Proposition 3.258. *For any order-embedding $f : (X, \leq_X) \hookrightarrow (Y, \leq_Y)$ it is true that f is an increasing injection from X to Y.*

Proof. Letting X, Y, $\leq_X$, $<_X$, $\leq_Y$, $<_Y$ and f be arbitrary sets, we assume that $\leq_X$ is a reflexive and $<_X$ an irreflexive partial ordering of X inducing each other, assume furthermore that $\leq_Y$ is a reflexive and $<_Y$ an irreflexive partial ordering of Y inducing each other, and assume in addition that f is an order-embedding from $(X, \leq_X)$ to $(Y, \leq_Y)$. To show that f is an injection from X to Y, we verify the defining property (3.615). To do this, we let x and x' be arbitrary and assume both $x, x' \in X$ and $f(x) = f(x')$ to be true. This equation implies the disjunctions

$$f(x) <_Y f(x') \lor f(x) = f(x')$$
$$f(x') <_Y f(x) \lor f(x') = f(x)$$

(irrespective of the truth values of $f(x) <_Y f(x')$ and $f(x') <_Y f(x)$, respectively) and therefore the inequalities $f(x) \leq_Y f(x')$ and $f(x') \leq_Y f(x)$ with Theorem 3.81 (or Theorem 3.86). Since f was assumed to be an order-embedding from $(X, <_X)$ to $(Y, <_Y)$, these two inequalities imply, respectively, $x \leq_X x'$ and $x' \leq_X x$ according to (3.970). Then, the conjunction of these two inequalities further implies $x = x'$ with the fact that the reflexive partial ordering $\leq_X$ is antisymmetric. Because x and x' are arbitrary, we may therefore conclude that f satisfies (3.615), so that f is an injection.

To show that f is in addition increasing, we verify that f satisfies (3.965), letting $x, y \in X$ be arbitrary with $x \leq_X y$. Since f is an order-embedding from $(X, <_X)$ to $(Y, <_Y)$, it follows from $x, y \in X$ that the equivalence $x \leq_X y \Leftrightarrow f(x) \leq_Y f(y)$ is true, so that the assumed $x \leq_X y$ implies the desired $f(x) \leq_Y f(y)$. Because x and y are arbitrary, we may then infer from this that f satisfies (3.965), so that f is increasing.

As the sets X, Y, $\leq_X$, $<_X$, $\leq_Y$, $<_Y$ and f were arbitrary, it follows that the proposed sentence holds, as claimed. □

Combining this finding with the fact that an order-isomorphism is a surjection by definition, we see that an order-isomorphism is a bijection, so that the use of the symbol '$\rightleftarrows$' in (3.971) is justified.

Corollary 3.259. *For any order-isomorphism $f : (X, \leq_X) \rightleftarrows (Y, \leq_Y)$ it is true that f is an increasing bijection from X to Y.*

In cases of totally or linearly ordered sets it is conversely also possible to establish an increasing in-/bijection as an order-embedding/-isomorphism.

Proposition 3.260. *For any totally ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ and for any function $f : X \to Y$, it is true that f is an order-embedding from $(X, \leq_X)$ to $(Y, \leq_Y)$ if, and only if, f is an increasing injection.*

Proof. We let X, Y, $\leq_X$, $<_X$, $\leq_Y$, $<_Y$ and f be arbitrary sets, assume that $\leq_X$ is a total and $<_X$ a linear ordering of X inducing each other, assume also that $\leq_Y$ is a total and $<_Y$ a linear ordering of Y inducing each other, and assume furthermore that f is a function from X to Y.

To prove the first part ('$\Rightarrow$') of the proposed equivalence, we assume that f is an order-embedding from $(X, \leq_X)$ to $(Y, \leq_Y)$. Then, since the totally ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ are in particular partially ordered, we may apply Proposition 3.258 to infer that f is an increasing injection from X to Y.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume f to be an increasing injection and show that this implies that f is also an order-embedding from $(X, \leq_X)$ to $(Y, \leq_Y)$. For this purpose, we verify that f satisfies

$$\forall x, y\, (x, y \in X \Rightarrow [x \leq_X y \Leftrightarrow f(x) \leq_Y f(y)]) \tag{3.972}$$

letting x and y be arbitrary in X.

Regarding the first part ('$\Rightarrow$') of the equivalence in (3.972), we assume $x \leq_X y$, so that the disjunction $x <_X y \vee x = y$ holds with Theorem 3.81 (or Theorem 3.86). If $x <_X y$ is true, then we obtain the desired $f(x) \leq_Y f(y)$ with the assumption that f is increasing. If $x = y$ is true, then (3.526) yields $f(x) = f(y)$, so that the disjunction $f(x) <_Y f(y) \vee f(x) = f(y)$ also holds, and consequently $f(x) \leq_Y f(y)$ (using again one of the previously mentioned theorems). We thus showed that $x \leq_X y$ implies $f(x) \leq_Y f(y)$, which completes the proof of the first part of the equivalence in (3.970).

We now prove the second part ('$\Leftarrow$') of the equivalence by contradiction, assuming $f(x) \leq_Y f(y)$ and $\neg x \leq_X y$ to be both true. The latter implies $y <_X x$ with the Negation Formula for $\leq$, so that the disjunction $y <_X x \vee y = x$ also holds (regardless of the truth value of $y = x$). Evidently, this disjunction gives $y \leq_X x$, which then further implies $f(y) \leq_Y f(x)$ with the previously established first part of the equivalence in (3.972). Now, the conjunction of this and the assumed $f(x) \leq_Y f(y)$ in turn implies $f(y) = f(x)$ with the antisymmetry of the total ordering $\leq_Y$, and this equation yields $y = x$ with the definition of an injection. Thus, $y = x$ and $y <_X x$ are both true, which contradicts the fact that the linear ordering $<_X$ is comparable.

This completes the proof of the equivalence in (3.972), and since x and y are arbitrary, the universal sentence (3.972) follows then to be true. Thus, f is an order-embedding (by definition) from $(X, \leq_X)$ to $(Y, \leq_Y)$, which finding completes the proof of the proposed equivalence. Since X, Y, $\leq_X$, $<_X$, $\leq_Y$, $<_Y$ and f were arbitrary sets, we may finally conclude that the proposition holds. □

If the injectivity condition is replaced by bijectivity in the preceding proposition, the order-embedding turns into an order-isomorphism, by definition.

Corollary 3.261. *For any totally ordered sets $(X, \leq_X)$, $(Y, \leq_Y)$ and for any function $f : X \to Y$, it is true that f is an order-isomorphism from $(X, \leq_X)$ to $(Y, \leq_Y)$ if, and only if, f is an increasing bijection.*

Proposition 3.262. *The following implications are true for any complete lattices $(X, \leq_X)$ and $(Y, \leq_Y)$ and any function $f : X \to Y$.*

a) If f is an increasing function from X to Y, then

$$\forall A\,(A \subseteq X \Rightarrow \sup f[A] \leq_Y f(\sup A)). \tag{3.973}$$

b) If f is an order-isomorphism from $(X, \leq_X)$ to $(Y, \leq_Y)$, then

$$\forall A\,(A \subseteq X \Rightarrow \sup f[A] = f(\sup A)). \tag{3.974}$$

Proof. We let X, Y, $\leq_X$, $\leq_Y$, f be arbitrary sets such that $(X, \leq_X)$ and $(Y, \leq_Y)$ are complete lattices and such that f is a function from X to Y.

Concerning a), we assume that f is increasing, and then also that A is an arbitrary subset of X. Since $(X, \leq_X)$ is a complete lattice, $\sup A$ exists in X, and this supremum is by definition an upper bound for A, so that

$$\forall x\,(x \in A \Rightarrow x \leq_X \sup A) \tag{3.975}$$

holds. We now show that $f(\sup A)$ is an upper bound for $f[A]$, i.e. that

$$\forall y\,(y \in f[A] \Rightarrow y \leq_Y f(\sup A)) \tag{3.976}$$

is true. For this purpose, we let y be arbitrary and assume $y \in f[A]$ to be true, which means by definition of a direct image that $y \in \operatorname{ran}(f \upharpoonright A)$ holds. By definition of a range, there then exists an element, say $\bar{x}$, such that $(\bar{x}, y) \in f \upharpoonright A$ is true. Since $\operatorname{dom}(f \upharpoonright A) = A$ holds according to Proposition 3.164, it follows with the definition of a domain that $\bar{x} \in A$ holds, which further implies $\bar{x} \leq_X \sup A$ with (3.975). Since f is increasing, this inequality yields $f(\bar{x}) \leq_X f(\sup A)$ with (3.965). Now, the previously established $(\bar{x}, y) \in f \upharpoonright A$ implies also $(\bar{x}, y) \in f$ by definition of a restriction, which we may write as $y = f(\bar{x})$, so that substitution in the preceding inequality yields the desired $y \leq_Y f(\sup A)$. Because y is arbitrary, we may therefore conclude that (3.976) holds, so that $f(\sup A)$ is indeed an upper bound for the image $f[A]$. Furthermore, as f is a function with codomain Y, we have (again due to Proposition 3.164) that the restriction $f \upharpoonright A$ is also a function with codomain Y, so that $\operatorname{ran}(f \upharpoonright A) \subseteq Y$. Thus, the image $f[A] = \operatorname{ran}(f \upharpoonright A)$ is a subset of Y, and its supremum $\sup f[A]$ exists then in Y since $(Y, \leq_Y)$ is by assumption a complete lattice. Recalling that $f(\sup A)$ is an upper bound for $f[A]$, we now obtain with the definition of a supremum the inequality $\sup f[A] \leq_Y f(\sup A)$, as desired. Since A is arbitrary, it then follows that (3.973) is true, so that the proof of a) is complete.

Concerning b), we assume that f is an order-isomorphism from $(X, \leq_X)$ to $(Y, \leq_Y)$, so that f is then an increasing bijection in view of Corollary 3.261. Next, we let $A \subseteq X$ be arbitrary and prove the equation in (3.973) by showing that $f(\sup A)$ is the supremum of $f[A]$. Since f is increasing, it follows as in the proof of a) that (3.976) holds, i.e. that $f(\sup A)$ is an upper bound for $f[A]$. To prove that this is the least upper bound, we apply the definition of a supremum, letting S' be an arbitrary upper bound for $f[A]$ and showing that this implies $f(\sup A) \leq_Y S'$. By definition of an upper bound, we thus have

$$\forall y\,(y \in f[A] \Rightarrow y \leq_Y S'). \tag{3.977}$$

Recalling that f is a bijection from X to Y, it follows with 3.683d) that the inverse function f^{-1} is a bijection from Y to X. Based on this observation, we now verify that $f^{-1}(S')$ is an upper bound for A, i.e. that

$$\forall x\,(x \in A \Rightarrow x \leq_X f^{-1}(S')) \tag{3.978}$$

holds. For this purpose, we let x be arbitrary, assume $x \in A$ to be true, and recall that $A = \operatorname{dom}(f \upharpoonright A)$ holds, so that there exists by definition of a domain an element, say $\bar{y}$, such that $(x,\bar{y}) \in f \upharpoonright A$. By definition of a range, this implies $\bar{y} \in \operatorname{ran}(f \upharpoonright A)$, so that $\bar{y} \in f[A]$ holds (by definition of a direct image), and therefore $\bar{y} \leq_Y S'$ due to (3.977). Now, since $(x,\bar{y}) \in f \upharpoonright A$ also implies $(x,\bar{y}) \in f$ (by definition of a restriction) and therefore $\bar{y} = f(x)$, the preceding inequality becomes after substitution $f(x) \leq_Y S'$. As the equations

$$S' = \mathrm{id}_Y(S') = f(f^{-1}(S')) \tag{3.979}$$

hold by definition of an identity function and with (3.680), the preceding inequality yields $f(x) \leq_Y f(f^{-1}(S'))$, which then implies the desired $x \leq_X f^{-1}(S')$ with (3.970), applying the fact that the assumed order-isomorphism f is an order-embedding by definition. As x is arbitrary, we may therefore conclude that (3.978) is true, which means that $f^{-1}(S')$ is indeed an upper bound for A. Since $\sup A$ is the least upper bound for A, we obtain the inequality $\sup A \leq_X f^{-1}(S')$ with the definition of a supremum. Then, this inequality implies $f(\sup A) \leq_X f(f^{-1}(S'))$ with (3.970), and therefore $f(\sup A) \leq_X S'$ with the equations (3.979). Because S' is an arbitrary upper bound for $f[A]$, it follows now from the preceding inequality that $f(\sup A)$ is the supremum of $f[A]$, so that the equation in (3.974) holds. Since A is arbitrary, we may then further conclude that b) is true.

In the proofs of a) and b) the sets X, Y, $\leq_X$, $\leq_Y$ and f were arbitrary, so that we may finally conclude that the proposed universal sentences are true. □

Exercise 3.117. Prove the following implications for any complete lattices $(X, \leq_X)$, $(Y, \leq_Y)$ and for any function $f : X \to Y$.

a) If f is an increasing function from X to Y, then

$$\forall A\,(A \subseteq X \Rightarrow f(\inf A) \leq_Y \inf f[A]). \tag{3.980}$$

b) If f is an order-isomorphism from $(X, \leq_X)$ to $(Y, \leq_Y)$, then

$$\forall A\,(A \subseteq X \Rightarrow f(\inf A) = \inf f[A]). \tag{3.981}$$

(Hint: Proceed in analogy to the proof of Proposition 3.262.)

Part II.

Counting Data

Chapter 4.

Counting Sets

4.1. Counting Domains $(C, s, 0_C)$

We begin this chapter with a formal and general description of *counting*. For this purpose, we define a certain structure in form of an ordered triple consisting of (1) a set C whose elements may be regarded as the possible outcomes of a counting activity, (2) a function s each of whose values represents the unique possible outcome of a single counting step after having counted up to a certain element of C, and (3) an element 0_C in C which indicates that no counting (step) has been carried out yet.

Definition 4.1 (Counting domain, counting set, successor function, beginning, Peano's Axioms). For any set C, any function $s : C \to C$ and any element $0_C \in C$, we say that the ordered triple $(C, s, 0_C)$ is a *counting domain* iff

1. the range of the function s does not contain 0_C, i.e.

$$\forall n \, (n \in C \Rightarrow s(n) \neq 0_C), \tag{4.1}$$

2. the function s is an injection, i.e.

$$\forall m, n \, ([m, n \in C \, \wedge \, s(m) = s(n)] \Rightarrow m = n), \tag{4.2}$$

   and

3. any subset M of C, containing 0_C and the value of the successor function at any element of M, is identical with C, i.e.

$$\forall M \, ([M \subseteq C \, \wedge \, 0_C \in M \, \wedge \, \forall n \, (n \in M \Rightarrow s(n) \in M)] \Rightarrow M = C). \tag{4.3}$$

We then call C the *counting set*, s the *successor function*, and 0_C the *beginning* of C. Furthermore, we call the definite properties 1. – 3. *Peano's axioms*.

Theorem 4.1 (Characterization of the elements of a counting set). *The following sentence holds for any counting domain* $(C, s, 0_C)$.

$$\forall n\,(n \in C \Rightarrow [n = 0_C \ \vee\ \exists m\,(m \in C \ \wedge\ m \neq n \ \wedge\ s(m) = n)]). \tag{4.4}$$

Proof. We let C, s and 0_C be arbitrary, assume that $(C, s, 0_C)$ is a counting domain, and apply the Axiom of Specification in connection with the Equality Criterion for sets to obtain the true sentence

$$\exists! M\, \forall n\,(n \in M \Leftrightarrow [n \in C \wedge (n = 0_C \vee \exists m\,(m \in C \wedge m \neq n \wedge s(m) = n))]). \tag{4.5}$$

Thus, M satisfies

$$\forall n\,(n \in M \Leftrightarrow [n \in C \ \wedge\ (n = 0_C \ \vee\ \exists m\,(m \in C \ \wedge\ m \neq n \ \wedge\ s(m) = n))]). \tag{4.6}$$

Let us observe here that $n \in M$ implies in particular $n \in C$ for any n, which means by definition of a subset

$$M \subseteq C. \tag{4.7}$$

Furthermore, the truth of the equation $n = 0_C$ implies $n \in C$ since 0_C holds by definition of a counting domain. Then, the disjunction in (4.6) is true no matter if the existential sentence involved is true or false. Consequently, the conjunction of the true $0_C \in C$ and the preceding disjunction also holds, which implies with (4.6)

$$0_C \in M. \tag{4.8}$$

Next, we verify

$$\forall n\,(n \in M \Rightarrow s(n) \in M), \tag{4.9}$$

which – together with (4.7) and (4.8) – will imply $M = C$ with Property 3 of a counting domain. To prove this universal sentence, we let n be arbitrary, and we prove the implication directly, assuming $n \in M$ to be true. As observed earlier, the latter implies $n \in C$, and $s(n) \in C$ is then also true because C is codomain of s. Let us now use the disjunction $n = 0_C \ \vee\ n \neq 0_C$, which is true according to the Law of the Excluded Middle, to prove the sentence

$$\exists m\,(m \in C \ \wedge\ m \neq s(n) \ \wedge\ s(m) = s(n)) \tag{4.10}$$

by cases. Since $n, 0_C \in C$ is true, the first case $n = 0_C$ implies $s(n) = s(0_C)$ with (3.526). Moreover, $s(n) \neq 0_C$ is also true according to Property 1 of a counting domain, so that the multiple conjunction

$$0_C \in C \ \wedge\ 0_C \neq s(n) \ \wedge\ s(0_C) = s(n)$$

evidently holds, which shows that the existential sentence (4.10) is true in the first case. Since the assumed $n \in M$ implies with (4.6) in particular

$$n = 0_C \lor \exists m\,(m \in C \land m \neq n \land s(m) = n),$$

the second case $n \neq 0_C$ shows that the first part of this disjunction is false, so that the second part part of the disjunction (i.e., the existential sentence) is true. Thus, there exists a constant, say $\bar{m}$, such that the multiple conjunction

$$\bar{m} \in C \land \bar{m} \neq n \land s(\bar{m}) = n \tag{4.11}$$

is true. This yields in particular $\bar{m} \neq n$, which then further implies $s(\bar{m}) \neq s(n)$ with (4.2) in connection with the Injection Criterion. As (4.11) implies also in particular $s(\bar{m}) = n$, it follows with the preceding inequality via substitution that $n \neq s(n)$ holds. Thus, the multiple conjunction

$$n \in C \land n \neq s(n) \land s(n) = s(n)$$

is evidently true, so that the existential sentence (4.10) holds also in the second case, completing the proof by cases. Then, the disjunction

$$s(n) = 0_C \lor \exists m\,(m \in C \land m \neq s(n) \land s(m) = s(n))$$

is also true (even though $s(n) = 0_C$ is false). Recalling the previously found $s(n) \in C$, we now see that the conjunction

$$s(n) \in C \land (s(n) = 0_C \lor \exists m\,(m \in C \land m \neq s(n) \land s(m) = s(n)))$$

holds, which in turn implies the desired consequent $s(n) \in M$ with (4.6). As n was arbitrary, we may therefore conclude that (4.9) is true. The conjunction of (4.7), (4.8) and (4.9) then implies $M = C$ with (4.3). We are now in a position to verify the proposed universal sentence. Letting $n \in C$ be arbitrary, we may apply substitution based on the preceding equation to obtain $n \in M$, which implies with (4.6) in particular

$$n = 0_C \lor \exists m\,(m \in C \land m \neq n \land s(m) = n).$$

As this is the desired consequent of the implication in (4.4), and since n was arbitrary, we may therefore conclude that (4.4) is true. Because C, s and 0_C were initially arbitrary, we may finally conclude that the theorem holds. □

Proposition 4.2. *The successor function in any counting domain $(C, s, 0_C)$ is a bijection from C to $C \setminus \{0_C\}$, i.e.*

$$s : C \rightleftarrows C \setminus \{0_C\}. \tag{4.12}$$

Proof. We let C, s and 0_C be arbitrary, assume that $(C, s, 0_C)$ is a counting domain, and prove first that the range of s is identical with $C \setminus \{0_C\}$. For this purpose, we may prove the equivalent (applying the Equality Criterion for sets)

$$\forall n\,(n \in \operatorname{ran}(s) \Leftrightarrow n \in C \setminus \{0_C\}). \tag{4.13}$$

We let n be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $n \in \operatorname{ran}(s)$ is true. This assumption implies by definition of a range that there exists an element, say $\bar{m}$, such that $(\bar{m}, n) \in s$ holds. Then, $\bar{m} \in \operatorname{dom}(s)$ holds by definition of a domain, so that we obtain $\bar{m} \in C$ by definition of s. Furthermore, $(\bar{m}, n) \in s$ means $s(\bar{m}) = n$ due to Notation 3.4, so that $s(\bar{m}) \neq 0_C$ holds because of Property 1 of a counting domain. Therefore, the preceding equation gives $n \neq 0_C$, which implies $n \notin \{0_C\}$ with (2.169). Now, since s is a function with codomain C, we have that $\operatorname{ran}(s) \subseteq C$ holds (by definition of a codomain). With this, the assumption $n \in \operatorname{ran}(s)$ implies $n \in C$ by definition of a subset. Thus, $n \in C$ and $n \notin \{0_C\}$ are both true, so that we obtain the desired $n \in C \setminus \{0_C\}$ by applying the definition of a set difference.

To prove the second part ('$\Leftarrow$') of the equivalence in (4.13), we now assume $n \in C \setminus \{0_C\}$ to be true, which evidently means that $n \in C$ and $n \notin \{0_C\}$ are both true. Since the latter implies $n \neq 0_C$ with (2.169) and the Law of Contraposition, it follows with (4.4) from $n \in C$ that there exists an element of C, say $\bar{m}$, such that $\bar{m} \neq n$ and $s(\bar{m}) = n$ hold. The latter equation means $(\bar{m}, n) \in s$, so that the desired $n \in \operatorname{ran}(s)$ holds (by definition of a range). Thus, the proof of the equivalence is complete, and as n is arbitrary, we therefore conclude that (4.13) is true, which in turn proves the equation $\operatorname{ran}(s) = C \setminus \{0_C\}$. Consequently, s is a surjection from C to $C \setminus \{0_C\}$; in view of Note 3.23, we then see that the injection $s : C \hookrightarrow C$ is also an injection from C to $C \setminus \{0_C\}$. These two findings imply that s is a bijection from C to $C \setminus \{0_C\}$. Since C, s and 0_C were arbitrary, we may now further conclude that the proposition holds. $\square$

Proposition 4.3. *For any counting domain $(C, s, 0_C)$ it is true that the successor of any element of C is different from that element, that is,*

$$\forall n\,(n \in C \Rightarrow s(n) \neq n). \tag{4.14}$$

Proof. We let C, s and 0_C be arbitrary and assume that $(C, s, 0_C)$ is a counting domain. Then, we let n also be arbitrary and prove the implication by cases. Regarding the first case, we prove the implication

$$(n \in C \land n = 0_C) \Rightarrow s(n) \neq n \tag{4.15}$$

by contradiction, assuming $n \in C$, $n = 0_C$ and $\neg s(n) \neq n$ to be true, where the latter implies $s(n) = n$ with the Double negation Law. Together with the evident fact $0_C \in C$, the assumed $n \in C$ and $n = 0_C$ imply $s(n) = s(0_C)$ because s is a function (see Corollary 3.150). Since $s(0_C) \neq 0_C$ also holds according to Property 1 of a counting domain, we obtained a contradiction, so that the proof of the implication (4.15) is complete. Regarding the second case, we now prove the corresponding implication

$$(n \in C \wedge n \neq 0_C) \Rightarrow s(n) \neq n, \tag{4.16}$$

by contradiction, assuming that $n \in C$, $n \neq 0_C$ and $\neg s(n) \neq n$ hold, so that $s(n) = n$ follows again to be true. In view of the Characterization of the elements of a counting set, the assumed $n \in C$ implies the truth of the disjunction

$$n = 0_C \vee \exists m \, (m \in C \wedge m \neq n \wedge s(m) = n), \tag{4.17}$$

whose first part $n = 0_C$ is false by virtue of the assumption $n \neq 0_C$, so that the second part (i.e., the existential sentence) is true. Thus, there exists an element, say $\bar{m}$, such that the multiple conjunction

$$\bar{m} \in C \wedge \bar{m} \neq n \wedge s(\bar{m}) = n \tag{4.18}$$

is true. The third part $s(\bar{m}) = n$ implies with the previously established $s(n) = n$ the equation $s(\bar{m}) = s(n)$ via substitution. This equation in turn implies $\bar{m} = n$ with the injectivity of s (see (4.2)), which equation is in contradiction to the true second part $\bar{m} \neq n$ of the multiple conjunction (4.18). This completes the proof of the implication (4.16) and thus the proof of the implication in (4.14) by cases. Since n, C, s and 0_C were arbitrary, the proposition follows to be true. $\square$

The following section shows that the set $\mathbb{N}$ of natural numbers is suitable for the purpose of formal counting, which is a fact we apply routinely in daily life.

4.1.1. The counting domain $(\mathbb{N}, s^+, 0)$

Lemma 4.4. *The following sentences are true regarding* $\mathbb{N}$*.*

a) There exists a unique set $\in_{\mathbb{N}}$ *consisting of all the elements* Y *in* $\mathbb{N} \times \mathbb{N}$ *such* Y *is an ordered pair formed by some natural numbers* m *and* n *where* m *is element of* n*, and this set is a binary relation on* $\mathbb{N} \times \mathbb{N}$ *satisfying*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \in_{\mathbb{N}} n \Leftrightarrow m \in n]). \tag{4.19}$$

b) Furthermore, the binary relation $\in_{\mathbb{N}}$ *is transitive, that is,*

$$\forall a, b, n\,(a, b, n \in \mathbb{N} \Rightarrow [(a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n) \Rightarrow a \in_{\mathbb{N}} n]). \tag{4.20}$$

c) Moreover, the binary relation $\in_{\mathbb{N}}$ *is irreflexive, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow \neg n \in_{\mathbb{N}} n). \tag{4.21}$$

Proof. Concerning a), we may apply the Axiom of Specification in connection with the Equality Criterion for sets (in analogy to the proof of Theorem 2.15) to obtain the true uniquely existential sentence

$$\exists! \in_{\mathbb{N}} \forall Y\,(Y \in \in_{\mathbb{N}} \Leftrightarrow [Y \in \mathbb{N} \times \mathbb{N} \,\wedge\, \exists m, n\,(m \in n \,\wedge\, (m, n) = Y)]).$$

Since $Y \in \in_{\mathbb{N}}$ implies in particular $Y \in \mathbb{N} \times \mathbb{N}$ for any Y, it follows by definition of a subset that $\in_{\mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N}$ holds. Thus, $\in_{\mathbb{N}}$ is a binary relation on $\mathbb{N} \times \mathbb{N}$, which satisfies

$$\forall Y\,(Y \in \in_{\mathbb{N}} \Leftrightarrow [Y \in \mathbb{N} \times \mathbb{N} \,\wedge\, \exists m, n\,(m \in n \,\wedge\, (m, n) = Y)]). \tag{4.22}$$

To show that $\in_{\mathbb{N}}$ satisfies also (4.19), we let m and n be arbitrary in $\mathbb{N}$. To prove the first part ('$\Rightarrow$') of the equivalence in (4.19), we assume $m \in_{\mathbb{N}} n$, which we may also write as $(m, n) \in \in_{\mathbb{N}}$. In view of (4.22), this implies then in particular that there exist constants, say $\bar{m}$ and $\bar{n}$, such that $\bar{m} \in \bar{n}$ and $(\bar{m}, \bar{n}) = (m, n)$ hold. Since this equation yields $\bar{m} = m$ and $\bar{n} = n$ with the Equality Criterion for ordered pairs, we may apply substitutions to write $\bar{m} \in \bar{n}$ as $m \in n$, so that the proof of the first part of the equivalence in (4.19) is complete. To prove the second part ('$\Leftarrow$'), we now assume $m \in n$ to be true. Forming now the ordered pair $Y = (m, n)$, we clearly see that the existential sentence in (4.22) holds. In addition, the initial assumptions $m \in \mathbb{N}$ and $n \in \mathbb{N}$ imply with the definition of the Cartesian product of two sets $[(m, n) =]\ Y \in \mathbb{N} \times \mathbb{N}$, which in turn implies – together with the previously established existential sentence – that $[Y =]\ (m, n) \in \in_{\mathbb{N}}$ holds. Since we may write this as $m \in_{\mathbb{N}} n$, the second part of the equivalence in (4.19) also holds. Thus, the proof of the implication in (4.19) is also complete, and since m and n were arbitrary, we may therefore conclude that the universal sentence (4.19) is true.

Concerning b), to prove that the binary relation $\in_{\mathbb{N}}$ is transitive and satisfies therefore Property 2 of an irreflexive partial ordering, we verify (4.20), which we may write as

$$\forall a, b\,(a, b \in \mathbb{N} \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow [(a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n) \Rightarrow a \in_{\mathbb{N}} n])), \tag{4.23}$$

by using Exercise 1.16. First we let a and b be arbitrary in $\mathbb{N}$ and prove then

$$\forall n\,(n \in \mathbb{N} \Rightarrow [(a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n) \Rightarrow a \in_{\mathbb{N}} n]) \tag{4.24}$$

by mathematical induction. Considering the base case ($n = 0 = \emptyset$), we note that $b \in n$ is false by definition of $\emptyset$, so that the equivalent $b \in_{\mathbb{N}} n$ (using (4.19)) is also false. Then, the conjunction $a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n$ is false as well (irrespective of the truth value of its first part). Therefore, the implication $[a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n] \Rightarrow a \,\wedge\, n$ has a false antecedent and is thus true (for the base case $n = 0$). Concerning the induction step, we let n be arbitrary in $\mathbb{N}$ and verify

$$[(a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n) \Rightarrow a \in_{\mathbb{N}} n] \Rightarrow [(a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n^+) \Rightarrow a \in_{\mathbb{N}} n^+]. \tag{4.25}$$

For this purpose, we make the induction assumption

$$(a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n) \Rightarrow a \in_{\mathbb{N}} n \tag{4.26}$$

and show that this implies

$$(a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n^+) \Rightarrow a \in_{\mathbb{N}} n^+. \tag{4.27}$$

To prove this implication directly, we assume $a \in_{\mathbb{N}} b$ and $b \in_{\mathbb{N}} n^+$, so that the equivalent $a \in b$ and $b \in n^+$ (applying again (4.19)) hold. The latter means by the definition of a successor $b \in n \cup \{n\}$, so that the conjunction $a \in b \wedge b \in n \cup \{n\}$ is true. We then obtain with the definition of the union of a pair

$$a \in b \,\wedge\, (b \in n \,\vee\, b \in \{n\}),$$

and then with the Distributive Law for sentences (1.44) also

$$(a \in b \,\wedge\, b \in n) \,\vee\, (a \in b \,\wedge\, b \in \{n\}). \tag{4.28}$$

Based on this true disjunction, we now prove the sentence $a \in n$ by cases. If the first part $a \in b \,\wedge\, b \in n$ holds (which we may write equivalently as $a \in_{\mathbb{N}} b \,\wedge\, b \in_{\mathbb{N}} n$ because of (4.19)), then $a \in_{\mathbb{N}} n$ follows to be true with the induction assumption (4.26), so that the equivalent $a \in n$ holds as well. If the second part $a \in b \wedge b \in \{n\}$ of the disjunction (4.28) is true, then we obtain in particular the true $b \in \{n\}$, which further implies $b = n$ with (2.169). The preceding conjunction also implies in particular $a \in b$, which then gives $a \in n$ with the true equation $b = n$, completing the proof by cases of the sentence $a \in n$. Let us now observe that $n \subseteq n^+$ holds according to (2.306); with this, we see in light of the definition of a subset that $a \in n$ implies $a \in n^+$. Then, the equivalent $a \in_{\mathbb{N}} n^+$ is also true, which finding completes the proof of the implication (4.27) and thus also the proof of the implication (4.25). Since n was arbitrary, we may therefore conclude that the induction step holds as well, so that the proof of (4.24) by mathematical induction is complete. As a and b were also arbitrary, we may finally conclude that (4.20) holds, so that $\in_{\mathbb{N}}$ is transitive.

Concerning c), we carry out a proof by mathematical induction in combination with a proof by contradiction. Regarding the base case ($n = 0 = \emptyset$), we immediately see that $\emptyset \notin \emptyset$ is true by definition of the empty set. To prove the induction step, we let n be arbitrary in $\mathbb{N}$ and verify the implication

$$n \notin n \Rightarrow n^+ \notin n^+ \tag{4.29}$$

by contradiction. To do this, we make the induction assumption

$$n \notin n \tag{4.30}$$

and further assume

$$\neg n^+ \notin n^+. \tag{4.31}$$

The latter means $\neg(\neg n^+ \in n^+)$, which implies $n^+ \in n^+$ with the Double Negation Law. Furthermore, $n^+ = n \cup \{n\}$ holds by definition of a successor, so that (4.31) gives $n^+ \in n \cup \{n\}$ via substitution. By definition of the union of a pair, this further implies $n^+ \in n \vee n^+ = n$, so that the conjunction of the induction assumption (4.30) and the preceding disjunction is true, that is,

$$n \notin n \wedge (n^+ \in n \vee n^+ = n).$$

This conjunction in turn implies

$$(n \notin n \wedge n^+ \in n) \vee (n \notin n \wedge n^+ = n), \tag{4.32}$$

with the Distributive Law for sentences (1.44). Next, we use this true disjunction to prove the sentence $n \in n$ by cases. In case $n \notin n \wedge n^+ \in n$ is true, we have that the second part $n^+ \in n$ of that conjunction is true in particular, which we may also write as $n^+ \in_{\mathbb{N}} n$ in view of (4.19). Let us now recall that $n \in n^+$ also holds because of (2.305), which we may equivalently write as $n \in_{\mathbb{N}} n^+$. Thus, the conjunction of $n \in_{\mathbb{N}} n^+$ and $n^+ \in_{\mathbb{N}} n$ is true, which then implies $n \in_{\mathbb{N}} n$ with the transitivity (4.20) of $\in_{\mathbb{N}}$, and consequently also the equivalent $n \in n$, as desired. If the second part $n \notin n \wedge n^+ = n$ of the disjunction (4.32) is true, then $n^+ = n$ holds in particular. We may therefore apply substitution to the previously established fact $n \in n^+$ to obtain $n \in n$ again, completing the proof by cases. As this result clearly contradicts the induction assumption (4.30), the proof of the implication (4.29) is complete. Then, as n is arbitrary, we further conclude that the induction step holds, which completes the proof of (4.21) via mathematical induction. □

Recalling the definition of an irreflexive partial ordering, we may immediately restate the parts b) and c) of the preceding lemma as the following sentence.

Corollary 4.5. *The binary relation of belonging $\in_{\mathbb{N}}$ is an irreflexive partial ordering of $\mathbb{N}$.*

Theorem 4.6 (Counting domain of natural numbers). *The following sentences are true.*

a) The function

$$s^+ : \mathbb{N} \to \mathbb{N}, \quad n \mapsto s^+(n) = n^+ \quad (= n \cup \{n\}) \tag{4.33}$$

exists uniquely.

b) Furthermore, the ordered triple $(\mathbb{N}, s^+, 0)$ constitutes a counting domain.

Proof. Concerning a), let us first recall that $\mathbb{N}$ is an inductive set according to Theorem 2.104d), so that $\emptyset \in \mathbb{N}$ holds and $n^+ = n \cup \{n\}$ is a uniquely specified set in $\mathbb{N}$ for any $n \in \mathbb{N}$. Consequently, in view of Theorem 3.160, there exists the unique function s^+ with domain $\mathbb{N}$ such that $s^+(n) = n^+$ holds for any $n \in \mathbb{N}$. To prove that $\mathbb{N}$ is a codomain of s^+, we verify $\mathrm{ran}(s^+) \subseteq \mathbb{N}$. For this purpose, we apply the definition of a subset and prove the equivalent

$$\forall m\, (m \in \mathrm{ran}(s^+) \Rightarrow m \in \mathbb{N}). \tag{4.34}$$

We let m be an arbitrary element of $\mathrm{ran}(s^+)$, so that, by definition of a range, there exists an element, say $\bar{n}$, such that $(\bar{n}, m) \in s^+$ is true. Since s^+ is a function, we may write this also as $m = s^+(\bar{n})$. By definition of s^+, we obtain $(m =)\ s^+(\bar{n}) = \bar{n}^+$, so that $m = \bar{n}^+$. Then, $(m =)\ \bar{n}^+ = \bar{n} \cup \{\bar{n}\}$ is, by definition of $\mathbb{N}$, an element of $\mathbb{N}$, which gives the desired $m \in \mathbb{N}$. Since

m was arbitrary, it follows that (4.34) is true, which means that the range of s^+ is a subset of $\mathbb{N}$. By definition of a codomain, we therefore have that s^+ is a function with codomain $\mathbb{N}$.

Concerning b), we first verify that $(\mathbb{N}, s^+, 0)$ satisfies (4.1), i.e.

$$\forall n\,(n \in \mathbb{N} \Rightarrow n^+ \neq 0). \tag{4.35}$$

For this purpose, we let n be arbitrary in $\mathbb{N}$. Since $n \in \{n\}$ holds according to (2.153), it evidently follows with (2.210) that $n \in n \cup \{n\}$ $[= n^+]$ holds, which shows that there exists an element in n^+. Thus, n^+ is clearly not the empty set $\emptyset$ $[= 0]$, so that $n^+ \neq 0$ is true. This proves the implication in (4.35), and since n is arbitrary, we may therefore conclude that (4.35) holds, which means that Property 1 of a counting domain is satisfied by $\mathbb{N}$, s^+ and 0.

Regarding Property 2, we prove

$$\forall m, n\,([m, n \in \mathbb{N} \wedge m^+ = n^+] \Rightarrow m = n). \tag{4.36}$$

For this purpose, we carry out a proof of a universal sentence and a proof by contradiction. Letting m and n be arbitrary, we assume the multiple conjunction

$$m, n \in \mathbb{N} \wedge m^+ = n^+ \wedge m \neq n \tag{4.37}$$

and show that this implies a contradiction. This conjunction implies in particular $m^+ = n^+$. With the fact that that $m \in m^+$ holds in view of (2.305), this yields $m \in n^+$. The latter means $m \in n \cup \{n\}$ (by definition of a successor) and therefore $m \in n \vee m \in \{n\}$ (by definition of the union of a pair), and equivalently $m \in n \vee m = n$ with (2.169). Since (4.37) implies in particular $m \neq n$, so that the second part $m = n$ of the preceding true disjunction is false, we may infer from this that the first part $m \in n$ of that disjunction is true. Similarly, the previously obtained equation $m^+ = n^+$ implies with the previously established fact $n \in n^+$ that $n \in m^+$ $(= m \cup \{m^+\})$ holds (using also the definition of a successor), and therefore $n \in m \vee n \in \{m\}$ (applying the definition of the union of a pair). Here, we may substitute $n \in \{m\}$ by the equivalent $n = m$ (using (2.169)) to obtain the disjunction $n \in m \vee n = m$. Recalling that $m = n$ is false, we see that the first part $n \in m$ of this true disjunction holds. We thus showed that $n \in m$ and $m \in n$ are both true, for which we may write equivalently $n \in_{\mathbb{N}} m$ and $m \in_{\mathbb{N}} n$ due to (4.19). Consequently, the transitivity (4.20) of $\in_{\mathbb{N}}$ yields $n \in_{\mathbb{N}} n$, in contradiction to the fact that $\neg n \in_{\mathbb{N}} n$ holds because of the irreflexivity (4.21) of $\in_{\mathbb{N}}$. Thus, the proof of the implication in (4.36) is complete, and since m and n are arbitrary, we may therefore conclude that

(4.36) holds, so that Property 2 of a counting domain is also satisfied.

Property 3 reads

$$\forall M\,([M \subseteq \mathbb{N} \,\wedge\, 0 \in M \,\wedge\, \forall n\,(n \in M \Rightarrow n^+ \in M)] \Rightarrow M = \mathbb{N}), \qquad (4.38)$$

which we recognize to be the Principle of Mathematical Induction (applying the definition of an inductive set). Thus, the proof of b) is complete. □

We may apply Theorem 4.1 immediately to the counting domain $(\mathbb{N}, s^+, 0)$ in order to characterize the natural numbers accordingly.

Corollary 4.7. *A natural number is* 0 *or the successor of another natural number, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow [n = 0 \,\vee\, \exists m\,(m \in \mathbb{N} \,\wedge\, m \neq n \,\wedge\, m^+ = n)]). \qquad (4.39)$$

The following two results immediately follow by applying the Propositions 4.2 and 4.3 to the counting domain of natural numbers.

Corollary 4.8. *The successor function of the counting domain* $(\mathbb{N}, s^+, 0)$ *is a bijection from* $\mathbb{N}$ *to* $\mathbb{N}_+$*, that is,*

$$s^+ : \mathbb{N} \rightleftarrows \mathbb{N}_+. \qquad (4.40)$$

Proof. Since $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ holds by definition of the set of positive natural numbers, (4.40) follows directly from (4.12). □

Corollary 4.9. *Any natural number is different from its successor, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow n^+ \neq n). \qquad (4.41)$$

Exercise 4.1. Verify that the successor of any natural number is element of the set of positive natural numbers, that is,

$$\forall n\,(n \in \mathbb{N} \Rightarrow n^+ \in \mathbb{N}_+). \qquad (4.42)$$

(Hint: Use (4.35), (2.169), (4.33), and (2.307).)

Note 4.1. Since (4.35) is equivalent to

$$\neg\exists n\,(n \in \mathbb{N} \,\wedge\, n^+ = 0) \qquad (4.43)$$

due to the Negation Law for existential conjunctions, this means that 0 has no predecessor.

Note 4.2. As every natural number is a subset of $\mathbb{N}$ according to Proposition 2.110), we see that every natural number is an element of the power set of $\mathbb{N}$, so that $\mathcal{P}(\mathbb{N})$ may be taken as the codomain of the successor function s^+ on $\mathbb{N}$, i.e. we may write instead of (4.33)

$$s^+ : \mathbb{N} \to \mathcal{P}(\mathbb{N}), \quad n \mapsto s^+(n) = n^+ \quad (= n \cup \{n\}). \qquad (4.44)$$

4.1.2. Mathematical induction and recursion in counting domains

Note 4.3. In view of the similarity of Property 3 of a counting domain $(C, s, 0_C)$ and the Principle of Mathematical Induction for $(\mathbb{N}, s^+, 0)$, the former may be viewed as a generalization of the latter. Bearing in mind that the former is a defining property whereas the latter is a proven fact, we will nevertheless speak also of the Principle of Mathematical Induction in the context of an arbitrary counting domain. Similarly, we will call a set M which satisfies

$$0_C \in M \wedge \forall n \, (n \in M \Rightarrow s(n) \in M) \tag{4.45}$$

an *inductive set* with respect to $(C, s, 0_C)$, indicating the analogy to the definite property (2.289) of an inductive set. It will then also be convenient to call 0_C the *initial element* of M or of C and $s(n)$ the *successor* of n (in C).

Since Proposition 2.107 and the related Method 2.2 (i.e., proof by mathematical induction) essentially involves the elements of the specific counting domain $(\mathbb{N}, s^+, 0)$, their generalization is straightforward.

Exercise 4.2. Establish for any counting domain $(C, s, 0_C)$ and for an arbitrary formula $\varphi(n)$ the truth of the following sentences.

a) There exists a unique set

$$X = \{n : n \in C \wedge \varphi(n)\} \tag{4.46}$$

which contains precisely all the elements n in C for which $\varphi(n)$ is true, and this set X is a subset of C.

b) If this set X is inductive, then $\varphi(n)$ is true for all $n \in C$, that is,

$$[0_C \in X \wedge \forall n \, (n \in X \Rightarrow s(n) \in X)] \Rightarrow \forall n \, (n \in \mathbb{N} \Rightarrow \varphi(n)). \tag{4.47}$$

c) If

$$\varphi(0_C) \tag{4.48}$$

and

$$\forall n \, (n \in C \Rightarrow [\varphi(n) \Rightarrow \varphi(s(n))]) \tag{4.49}$$

are true, then the set X specified in a) satisfies (4.45), i.e. X is an inductive set.

(Hint: Proceed as in the proof of Proposition 2.107.)

Method 4.1 (Proof by mathematical induction for counting domains). To prove for an arbitrary counting domain $(C, s, 0_C)$ that a given formula $\varphi(n)$ holds for all elements of C, we show that the so-called *base case* (4.48) and *induction step* (4.49) are true, where $\varphi(n)$ is called the *induction assumption*.

Exercise 4.3. Prove Theorem 4.1 alternatively by applying a proof by mathematical induction.

Next, we introduce a method for defining a new function whose values are determined successively in the following *recursive* sense. Except for the initial value, which is specified to be a given element of the underlying set, we determine each value of the new function by evaluating a given function at two arguments, where the first argument is obtained by counting the steps taken to determine the preceding value, which value itself constitutes the second argument. The following theorem establishes the uniqueness of a function defined by this method (which we then state formally afterwards).

Theorem 4.10 (Recursion Theorem). *The following sentences are true for any counting domain $(C, s, 0_C)$, any set A, any function $f : C \times A \to A$, and any $a \in A$.*

a) Letting $\varphi(X)$ be the definite property

$$(0_C, a) \in X \wedge \forall n, b\,((n, b) \in X \Rightarrow (s(n), f(n, b)) \in X), \tag{4.50}$$

there exists a unique set

$$\mathcal{X} = \{X : X \subseteq C \times A \wedge \varphi(X)\} \tag{4.51}$$

consisting of all the subsets of $C \times A$ which satisfy $\varphi(X)$. Then, the intersection

$$u = \bigcap \mathcal{X} \tag{4.52}$$

satisfies $\varphi(u)$.

b) Furthermore, u satisfies the universal sentence

$$\forall b\,(b \in A \Rightarrow [(0_C, b) \in u \Rightarrow b = a]). \tag{4.53}$$

c) Moreover, u satisfies the universal sentence

$$\begin{aligned} &\forall n, b'\,([\exists! b\,(b \in A \wedge (n, b) \in u) \wedge (s(n), b') \in u] \\ &\quad \Rightarrow \forall b\,([b \in A \wedge (n, b) \in u] \Rightarrow b' = f(n, b)). \end{aligned} \tag{4.54}$$

d) Finally, there exists a unique function $u : C \to A$ such that

$$u(0_C) = a \wedge \forall n\,(n \in C \Rightarrow u(s(n)) = f(n, u(n))). \tag{4.55}$$

Proof. We let C, s, 0_C, A, f and a be arbitrary. Here, we assume that $(C, s, 0_C)$ is a counting domain, that f is a function from $\mathbb{N} \times A$ to A, and moreover that a is an element of A.

Concerning a), we notice that the subsets of $C \times A$ are elements of the power set $\mathcal{P}(C \times A)$, so that we may apply the Axiom of Specification and the Equality Criterion for sets to obtain the true sentence

$$\exists! \mathcal{X}\, \forall X\, (X \in \mathcal{X} \Leftrightarrow [X \in \mathcal{P}(C \times A) \wedge \varphi(X)]).$$

Thus, the set $\mathcal{X}$ satisfies

$$\forall X\, (X \in \mathcal{X} \Leftrightarrow [X \in \mathcal{P}(C \times A) \wedge \varphi(X)]), \tag{4.56}$$

and the intersection $u = \bigcap \mathcal{X}$ is uniquely specified, where

$$\forall y\, (y \in u \Leftrightarrow \forall X\, (X \in \mathcal{X} \Rightarrow y \in X)) \tag{4.57}$$

holds by definition of the intersection of a set system. We now prove that $\varphi(u)$ is true, i.e. we prove the conjunction

$$(0_C, a) \in u \wedge \forall n, b\, ((n, b) \in u \Rightarrow (s(n), f(n, b)) \in u). \tag{4.58}$$

Regarding the first part of this conjunction, we show that

$$\forall X\, (X \in \mathcal{X} \Rightarrow (0_C, a) \in X) \tag{4.59}$$

is true, which will imply the desired $(0_C, a) \in u$ with (4.57). Letting X be arbitrary, we assume $X \in \mathcal{X}$, which implies with (4.56) the conjunction of $X \in \mathcal{P}(C \times A)$ and $\varphi(X)$, and therefore in particular $\varphi(X)$. Thus, X satisfies the conjunction (4.50), so that $(0_C, a) \in X$ holds in particular. Since X is arbitrary, we therefore conclude that the universal sentence (4.59) is true, which implies $(0_C, a) \in u$ in view of (4.57). To prove the second part of the conjunction (4.58), we let n and b be arbitrary and assume $(n, b) \in u$, which implies

$$\forall X\, (X \in \mathcal{X} \Rightarrow (n, b) \in X) \tag{4.60}$$

with (4.57). We now verify the truth of

$$\forall X\, (X \in \mathcal{X} \Rightarrow (s(n), f(n, b)) \in X). \tag{4.61}$$

To do this, we let X be arbitrary in $\mathcal{X}$, which implies $(n, b) \in X$ with (4.60). Furthermore, $X \in \mathcal{X}$ implies in particular $\varphi(X)$ because of (4.56) and therefore in particular

$$\forall n, b\,((n, b) \in X \Rightarrow (s(n), f(n, b)) \in X).$$

Due to $(n, b) \in X$ it follows from this universal sentence that $(s(n), f(n, b)) \in X$. As X is arbitrary, we therefore conclude that (4.61) is indeed true. The latter further implies $(s(n), f(n, b)) \in u$ with (4.57). As n and b were arbitrary, we then conclude that the second part of the conjunction, and thus the entire conjunction (4.58) holds, which means that $\varphi(u)$ is true.

Concerning b), we let b be arbitrary, assume $b \in A$, and prove then the implication

$$(0_C, b) \in u \Rightarrow b = a$$

by contradiction. To do this, we assume $(0_C, b) \in u$ and $b \neq a$, and we first verify $\varphi(u \setminus \{(0_C, b)\})$, i.e. that $u \setminus \{(0_C, b)\}$ satisfies the conjunction (4.50). Concerning the first part of the conjunction, let us observe that the equivalence

$$(0_C, a) = (0_C, b) \;\Leftrightarrow\; (0_C = 0_C \,\wedge\, a = b)$$

is true due to (3.3); here, $a = b$ is false by assumption, so that the conjunction is also false. Then, the left-hand side $(0_C, a) = (0_C, b)$ of the equivalence is also false. This implies with (2.169) that $(0_C, a) \in \{(0_C, b)\}$ false as well, which means that $(0_C, a) \notin \{(0_C, b)\}$ is true. Now, recalling from a) that $\varphi(u)$ is true, so that $(0_C, a) \in u$ holds in particular, we obtain the conjunction

$$(0_C, a) \in u \,\wedge\, (0_C, a) \notin \{(0_C, b)\},$$

which is equivalent to $(0_C, a) \in u \setminus \{(0_C, b)\}$, by definition of the set difference. Thus, $u \setminus \{(0_C, b)\}$ satisfies the first part of the conjunction (4.50). Concerning the second part, we let n and $\bar{b}$ be arbitrary and assume $(n, \bar{b}) \in u \setminus \{(0_C, b)\}$. Since $u \setminus \{(0_C, b)\} \subseteq u$ holds due to (2.125), we have that $(n, \bar{b}) \in u \setminus \{(0_C, b)\}$ implies $(n, \bar{b}) \in u$ with the definition of a subset. As $\varphi(u)$ is true, so that u satisfies in particular the second part of the conjunction (4.50), it follows from $(n, \bar{b}) \in u$ that $(s(n), f(n, \bar{b})) \in u$. Let us now notice the truth of the equivalence

$$(s(n), f(n, \bar{b})) = (0_C, b) \;\Leftrightarrow\; (s(n) = 0_C \,\wedge\, f(n, \bar{b}) = b)$$

in light of (3.3). Here, $s(n) = 0_C$ is false because of Property 1 of a counting domain, so that the conjunction is false as well. Thus, the left-hand side $(s(n), f(n, \bar{b})) = (0_C, b)$ of the equivalence is also false. Consequently,

$(s(n), f(n, \bar{b})) \notin \{(0_C, b)\}$ (applying again (2.169). We therefore obtain with the previously established $(s(n), f(n, \bar{b})) \in u$ the conjunction

$$(s(n), f(n, \bar{b})) \in u \wedge (s(n), f(n, \bar{b})) \notin \{(0_C, b)\},$$

which means $(s(n), f(n, \bar{b})) \in u \setminus \{(0_C, b)\}$ (by definition of the set difference). As n and $\bar{b}$ were arbitrary, we therefore conclude that $u \setminus \{(0_C, b)\}$ satisfies the second part of the conjunction (4.50), which completes the proof of $\varphi(u \setminus \{(0_C, b)\})$. Next, we verify that

$$u \setminus \{(0_C, b)\} \subseteq u \subseteq X \subseteq C \times A.$$

holds for any $X \in \mathcal{X}$. We already established the first of these inclusions before. Letting X be arbitrary in $\mathcal{X}$, the second inclusion $u \subseteq X$ holds with (4.52) and Proposition 2.29. The third inclusion holds with the definition (4.51) of $\mathcal{X}$. Since $\subseteq$ is a partial ordering (see Theorem 3.64) and therefore transitive, we obtain $u \setminus \{(0_C, b)\} \subseteq C \times A$ (which then holds for any $X \in \mathcal{X}$). The conjunction of this and $\varphi(u \setminus \{(0_C, b)\})$ now implies with (4.51) that $u \setminus \{(0_C, b)\}$ is an element of $\mathcal{X}$. As noted before, $u \subseteq X$ holds for any $X \in \mathcal{X}$, so that $u \subseteq u \setminus \{(0_C, b)\}$ holds in particular. Consequently, $(0_C, b) \in u$ (which we assumed initially) implies $(0_C, b) \in u \setminus \{(0_C, b)\}$, which means

$$(0_C, b) \in u \wedge (0_C, b) \notin \{(0_C, b)\}.$$

Here, $(0_C, b) \notin \{(0_C, b)\}$ contradicts the evident fact that $(0_C, b) \in \{(0_C, b)\}$ is true, so that the preceding implication constitutes a contradiction, which completes the proof of the implication $(0_C, b) \in u \Rightarrow b = a$. As b was arbitrary, we therefore conclude that b) holds, as claimed.

Concerning c), we let n and b' be arbitrary, assume that there exists a unique $b \in A$ with $(n, b) \in u$ and that $(s(n), b') \in u$ holds, and let then b be arbitrary. We now prove the implication by contradiction, assuming that $b \in A$, $(n, b) \in u$ and $b' \neq f(n, b)$ are true. Due to the assumed existential sentence, the arbitrarily selected b is in fact unique. Next, we verify that $\varphi(u \setminus \{(s(n), b')\})$ holds, i.e. that $u \setminus \{(s(n), b')\}$ satisfies (4.50). Regarding the first part of the conjunction, we notice the truth of the equivalence

$$(0_C, a) = (s(n), b') \Leftrightarrow (0_C = s(n) \wedge a = b') \tag{4.62}$$

in view of (3.3), where $0_C = s(n)$ is false by Property 1 of a counting domain. Therefore, the conjunction in (4.62) and thus also the left-hand side $(0_C, a) = (s(n), b')$ of the equivalence is false, which implies with (2.169) that $(0_C, a) \in \{(s(n), b')\}$ is false as well; thus, $(0_C, a) \notin \{(s(n), b')\}$ is true. Because u satisfies $\varphi(u)$ (as shown in a)) and therefore in particular $(0_C, a) \in u$, we obtain the conjunction

$$(0_C, a) \in u \wedge (0_C, a) \notin \{(s(n), b')\},$$

which means $(0_C, a) \in u \setminus \{(s(n), b')\}$ (by definition of the set difference). To show that $u \setminus \{(s(n), b')\}$ satisfies also the second part of the conjunction (4.50), we let $\bar{n}$ and $\bar{b}$ be arbitrary, assume $(\bar{n}, \bar{b}) \in u \setminus \{(s(n), b')\}$, and show that this implies

$$(s(\bar{n}), f(\bar{n}, \bar{b})) \in u \setminus \{(s(n), b')\},$$

or equivalently

$$(s(\bar{n}), f(\bar{n}, \bar{b})) \in u \wedge (s(\bar{n}), f(\bar{n}, \bar{y})) \notin \{(s(n), b')\}. \tag{4.63}$$

Due to $u \setminus \{(s(n), b')\} \subseteq u$ (again with (2.125)), we have that $(\bar{n}, \bar{b}) \in u \setminus \{(s(n), b')\}$ implies $(\bar{n}, \bar{b}) \in u$ (by definition of a subset). Since $\varphi(u)$ is true, u satisfies in particular the second part of the conjunction (4.50); with this, $(\bar{n}, \bar{b}) \in u$ implies $(s(\bar{n}), f(\bar{n}, \bar{b})) \in u$, so that the first part of the conjunction (4.63) holds. To prove the second part, which is equivalent to

$$(s(\bar{n}), f(\bar{n}, \bar{b})) \neq (s(n), b') \tag{4.64}$$

because of (2.169), let us observe the truth of the equivalence (using again (3.3))

$$(s(\bar{n}), f(\bar{n}, \bar{b})) = (s(n), b') \Leftrightarrow (s(\bar{n}) = s(n) \wedge f(\bar{n}, \bar{b}) = b'). \tag{4.65}$$

We now consider the two exhaustive cases $\bar{n} \neq n$ and $\bar{n} = n$. Regarding the first case, we notice that the implication $s(\bar{n}) = s(n) \Rightarrow \bar{n} = n$ is true by Property 2 of a counting domain. By definition of an implication, the case assumption that $\bar{n} = n$ is false then implies that $s(\bar{n}) = s(n)$ is also false. Consequently, the conjunction in (4.65) is also false, and therefore the left-hand side of the equivalence is false as well, which means that (4.64) is true. Regarding the second case $\bar{n} = n$, we see that the previously established $(\bar{n}, \bar{b}) \in u$ is then equivalent to $(n, \bar{b}) \in u$. Since b was initially assumed to be the unique element satisfying $(n, b) \in u$, we obtain $\bar{b} = b$. Therefore, $f(\bar{n}, \bar{b}) = b'$ is equivalent to $f(n, b) = b'$, which false by our initial assumption. Consequently, the conjunction in (4.65) and thus the left-hand side of the equivalence is false also in the second case, so that (4.64) is true in any case. This completes the proof of the conjunction (4.63), and as $\bar{n}$ and $\bar{b}$ are arbitrary, it follows that $u \setminus \{(s(n), b')\}$ satisfies (4.50), i.e. $\varphi(u \setminus \{(s(n), b')\})$. We already demonstrated the truth of the inclusions

$$u \setminus \{(s(n), b')\} \subseteq u \subseteq X \subseteq C \times A.$$

for any $X \in \mathcal{X}$ (see also the proof of b)), so that $u \setminus \{(s(n), b')\}$ is a subset of $C \times A$ (using again the transitivity of $\subseteq$). The conjunction of this and the previously established true sentence $\varphi(u \setminus \{(s(n), b')\})$ now implies $u \setminus \{(s(n), b')\} \in \mathcal{X}$ with (4.51). We already noted before that $u \subseteq X$ holds for any $X \in \mathcal{X}$, so that in particular $u \subseteq u \setminus \{(s(n), b')\}$. Then, the initial assumption $(s(n), b') \in u$ implies $(s(n), b') \in u \setminus \{(s(n), b')\}$, which means

$$(s(n), b') \in u \wedge (s(n), b') \notin \{(s(n), b')\}.$$

Here, $(s(n), b') \notin \{(s(n), b')\}$ contradicts the fact that $(s(n), b')$ is an element of $\{(s(n), b')\}$, so that the preceding conjunction is a contradiction. Thus, the proof of the implication in (4.54) is complete; as n and b were arbitrary, we therefore conclude that c) is true.

Concerning d), we begin with the existential part and show that the binary relation u, specified by (4.52) based on (4.51), is a function from C to A satisfying (4.55). We first prove by means of the Function Criterion that $u : C \to A$, i.e. we prove

$$\forall n \, (n \in C \Rightarrow \exists! b \, (b \in A \wedge (n, b) \in u)). \tag{4.66}$$

For this purpose, we apply a proof by mathematical induction. Concerning the base case ($n = 0_C$), we observe that the uniquely existential sentence in (4.66) with $n = 0_C$ is, in view of the Criterion for unique existence, equivalent to

$$\begin{aligned} &\exists b \, (b \in A \wedge (0_C, b) \in u) \\ &\quad \wedge \forall b, b' \, ([b \in A \wedge (0_C, b) \in u \wedge b' \in A \wedge (0_C, b') \in u] \Rightarrow b = b'). \end{aligned} \tag{4.67}$$

We already demonstrated in a) that $(0_C, a) \in u$ is true (where $a \in A$), so that existential part in (4.67) holds. Regarding the uniqueness part, we let b and b' be arbitrary and assume $b \in A$, $(0_C, b) \in u$, $b' \in A$, and $(0_C, b')$. These assumptions evidently imply $b = a$ and $b' = a$ with (4.53), which equations give the desired $b = b'$ after substitution. As b and b' are arbitrary, we therefore conclude that the second part of the conjunction (4.67) also holds, which completes the proof of the uniquely existential sentence in (4.66) for $n = 0_C$. Regarding the induction step, we let n be arbitrary in C, make the induction assumption

$$\exists! b \, (b \in A \wedge (n, b) \in u), \tag{4.68}$$

and prove

$$\exists! b \, (b \in A \wedge (s(n), b) \in u),$$

which is equivalent to

$$\begin{aligned}&\exists b\,(b \in A \,\wedge\, (s(n), b) \in u) &(4.69)\\&\quad\wedge\, \forall b', b''\,([b' \in A \,\wedge\, (s(n), b') \in u \,\wedge\, b'' \in A \,\wedge\, (s(n), b'') \in u] \Rightarrow b' = b'').\end{aligned}$$

We let b be the unique element in A satisfying $(n, b) \in u$ according to the induction assumption. This implies $(s(n), f(n, b)) \in u$ since u satisfies (4.50), where $f(n, b) \in A$ holds by definition of f. Thus, the existential part in (4.69) is true. Regarding the uniqueness part, we let b' and b'' be arbitrary and assume $b' \in A$, $(s(n), b') \in u$, $b'' \in A$, and $(s(n), b'') \in u$. Then, the conjunction of the induction assumption (4.68) and $(s(n), b') \in u$ implies the universal sentence

$$\forall b\,([b \in A \,\wedge\, (n, b) \in u] \Rightarrow b' = f(n, b)),$$

so that the conjunction of $b \in A$ and $(n, b) \in u$ further implies $b' = f(n, b)$. Similarly, the conjunction of (4.68) and $(s(n), b'') \in u$ implies

$$\forall b\,([b \in A \,\wedge\, (n, b) \in u] \Rightarrow b'' = f(n, b)),$$

so that the conjunction of $b \in A$ and $(n, b) \in u$ now gives $b'' = f(n, b)$ $(= b')$. Thus, $b' = b''$ holds, and since b' and b'' are arbitrary, it follows that the second part of the conjunction (4.69) is true. As n was arbitrary, we therefore conclude that the induction step holds, so that the proof of (4.66) is complete. We thus proved that u is a function with domain C and codomain A. We may then apply Notation 3.4 to write the previously established fact $(0_C, a) \in u$ as $u(0_C) = a$, which shows that u satisfies the first part of the conjunction (4.55). With respect to the second part, we let n be arbitrary in C. Because of the Function Criterion, $u : C \to A$ then implies that there exists the unique b in A with $(n, b) \in u$, that is, with $u(n) = b$. As u satisfies (4.50), $(n, b) \in u$ implies $(s(n), f(n, b)) \in u$, that is, $(s(n), f(n, u(n))) \in u$, that is, $u(s(n)) = f(n, u(n))$. Since n is arbitrary, we conclude that u satisfies also the second part of the conjunction (4.55).

It remains for us to prove the uniqueness part of the sentence d). For this purpose, we verify

$$\begin{aligned}&\forall u, u'\,([u : C \to A \,\wedge\, u(0_C) = a \,\wedge\, \forall n\,(n \in C \Rightarrow u(s(n)) = f(n, u(n)))\\&\qquad\wedge\, u' : C \to A \,\wedge\, u'(0_C) = a \,\wedge\, \forall n\,(n \in C \Rightarrow u'(s(n)) = f(n, u'(n)))]\\&\quad\Rightarrow u = u').\end{aligned}$$

Letting u and u' be arbitrary functions from C to A such that the conjunctions

$$u(0_C) = a \wedge \forall n\,(n \in C \Rightarrow u(s(n)) = f(n, u(n))) \tag{4.70}$$

$$u'(0_C) = a \wedge \forall n\,(n \in C \Rightarrow u'(s(n)) = f(n, u'(n))) \tag{4.71}$$

hold, we now prove $u = u'$ by applying Method 3.2. To do this, we demonstrate that $u(n) = u'(n)$ holds for all $n \in C$. For this purpose, we apply a proof by mathematical induction. Concerning the base case ($n = 0_C$), we have $u(0_C) = a = u'(0_C)$ due to (4.70) – (4.71) and therefore $u(0_C) = u'(0_C)$. Concerning the induction step, we let n be arbitrary in C, make the induction assumption $u(n) = u'(n)$, and show that this implies $u(s(n)) = u'(s(n))$. Let us observe that $n \in C$ implies with (4.70) on the one hand $u(s(n)) = f(n, u(n))$, which is equivalent to

$$u(s(n)) = f(n, u'(n)) \tag{4.72}$$

because of the induction assumption $u(n) = u'(n)$. On the other hand, $n \in C$ implies $u'(s(n)) = f(n, u'(n))$ with (4.71), so that (4.72) is equivalent to the desired $u(s(n)) = u'(s(n))$. As n is arbitrary, we therefore conclude that the induction step holds, which completes the proof by mathematical induction of $u = u'$. Since u and u' were also arbitrary, we further conclude that the uniqueness part of the sentence d) is true as well. This completes the proof of d).

As C, s, 0_C, A, f and a were initially arbitrary, we may finally conclude that the theorem holds, as claimed. □

Method 4.2 (Definition by recursion). To define for an arbitrary counting domain $(C, s, 0_C)$, an arbitrary set A, an arbitrary function $f : C \times A \to A$ and an arbitrary $a \in A$ the function $u : C \to A$ which satisfies (4.55), we will write

$$(1) \quad u(0_C) = a, \tag{4.73}$$

$$(2) \quad u(s(n)) = f(n, u(n)) \quad \text{for any } n \in C. \tag{4.74}$$

In case of the counting domain $(\mathbb{N}, s^+, 0)$, we will define by recursion as follows.

Method 4.3 (Definition by recursion (for $\mathbb{N}$)). To define for an arbitrary set A, an arbitrary function $f : \mathbb{N} \times A \to A$ and an arbitrary $a \in A$ the function $u : \mathbb{N} \to A$ which satisfies (4.55) with respect to the particular counting domain $(\mathbb{N}, s^+, 0)$, we will write

$$(1) \quad u_0 = a, \tag{4.75}$$

$$(2) \quad u_{n^+} = f(n, u_n) \quad \text{for any } n \in \mathbb{N}. \tag{4.76}$$

4.2. Initial Segments of Counting Sets

We now intend to apply the preceding method in order to recursively define a function u whose values are certain subsets of a counting set C. We prepare this definition by specifying a function f (as required by the Recursion Theorem and the Definition by recursion), which determines how these subsets are to be formed.

Proposition 4.11. *For any counting domain* $(C, s, 0_C)$ *there exists a unique set*

$$f = \{((n, B), B \cup \{s(n)\}) : n \in C \wedge B \subseteq C\}$$

consisting of all the ordered triples (n, B, V) *with* $n \in C$ *and* $B \subseteq C$ *such that* V *is the union of* B *and the singleton* $\{s(n)\}$, *and this set* f *is a function from* $C \times \mathcal{P}(C)$ *to* $\mathcal{P}(C)$, *i.e.*

$$f : C \times \mathcal{P}(C) \to \mathcal{P}(C), \quad (n, B) \mapsto B \cup \{s(n)\}. \tag{4.77}$$

Proof. We let $(C, s, 0_C)$ be an arbitrary counting domain. We may then apply the Axiom of Specification in connection with the Equality Criterion for sets to obtain the true uniquely existential sentence

$$\exists! f \, \forall z \, (z \in f \Leftrightarrow [z \in C \times \mathcal{P}(C) \times \mathcal{P}(C)$$
$$\wedge \, \exists n, B, V \, (V = B \cup \{s(n)\} \wedge (n, B, V) = z)]).$$

Thus, the set f satisfies

$$\forall z \, (z \in f \Leftrightarrow [z \in C \times \mathcal{P}(C) \times \mathcal{P}(C)$$
$$\wedge \, \exists n, B, V \, (V = B \cup \{s(n)\} \wedge ((n, B), V) = z)]). \tag{4.78}$$

We first observe that f is a subset of $(C \times \mathcal{P}(C)) \times \mathcal{P}(C)$, because $z \in f$ evidently implies in particular $z \in (C \times \mathcal{P}(C)) \times \mathcal{P}(C)$ for any z. We may then view f as a binary relation included in $X \times \mathcal{P}(C)$, where $X = C \times \mathcal{P}(C)$. Therefore, we may apply the Function Criterion to prove that $f : X \to \mathcal{P}(C)$. For this purpose, we demonstrate the truth of

$$\forall x \, (x \in C \times \mathcal{P}(C) \Rightarrow \exists! y \, (y \in \mathcal{P}(C) \wedge (x, y) \in f)). \tag{4.79}$$

We let x be arbitrary in $C \times \mathcal{P}(C)$, so that (by definition of a Cartesian product) there exists an element of C, say $\bar{n}$, and an element of $\mathcal{P}(C)$, say $\bar{B}$, such that $(\bar{n}, \bar{B}) = x$. We now show that this implies the uniquely existential sentence in (4.79), which is equivalent to

$$\exists y \, (y \in \mathcal{P}(C) \wedge (x, y) \in f)) \tag{4.80}$$
$$\wedge \, \forall y, y' \, (y \in \mathcal{P}(C) \wedge (x, y) \in f \wedge y' \in \mathcal{P}(C) \wedge (x, y') \in f)$$

because of the Criterion for unique existence. Regarding the existential part, we recall thet s is a function from C to C (by definition of a counting domain), so that the function value $s(\bar{n})$ exists in C. It then follows that the singleton $\{s(\bar{n})\}$ is a subset of $\mathcal{P}(C)$ (see Exercise 2.22); as $\bar{B} \in \mathcal{P}(C)$ implies $\bar{B} \subseteq C$ by definition of a power set, we have that the conjunction of $\bar{B} \subseteq C$ and $\{s(\bar{n})\} \subseteq C$ implies $\bar{B} \cup \{s(\bar{n})\} \subseteq C$ due to (2.252); consequently,

$$\bar{B} \cup \{s(\bar{n})\} \in \mathcal{P}(C). \tag{4.81}$$

We then obtain

$$(\bar{n}, \bar{B}, \bar{B} \cup \{s(\bar{n})\}) \in C \times \mathcal{P}(C) \times \mathcal{P}(C),$$

and there exist n, B, V such that $V = B \cup \{s(n)\}$ and $(n, B, V) = (\bar{n}, \bar{B}, \bar{B} \cup \{s(\bar{n})\})$ are both true (namely $\bar{n}$, $\bar{B}$ and $\bar{B} \cup \{s(\bar{n})\}$). Therefore,

$$[(\bar{n}, \bar{B}, \bar{B} \cup \{s(\bar{n})\}) =] \quad (x, \bar{B} \cup \{s(\bar{n})\}) \in f \tag{4.82}$$

holds with (4.78). The conjunction of (4.81) and (4.82) evidently proves the existential part of (4.80). Regarding the uniqueness part, we let y and y' be arbitrary and assume the multiple conjunction

$$y \in \mathcal{P}(C) \,\wedge\, (x, y) \in f \,\wedge\, y' \in \mathcal{P}(C) \,\wedge\, (x, y') \in f.$$

This conjunction implies in particular $(x, y) \in f$ and $(x, y') \in f$. The former further implies with (4.78) in particular that there exist n, B, V, say $\bar{n}'$, $\bar{B}'$ and $\bar{V}'$, such that

$$\bar{V}' = \bar{B}' \cup \{s(\bar{n}')\} \,\wedge\, ((\bar{n}', \bar{B}'), \bar{V}') = (x, y) \quad [= ((\bar{n}, \bar{B}), y)]$$

holds. This gives $\bar{n}' = \bar{n}$, $\bar{B}' = \bar{B}$ and $\bar{V}' = y$, and therefore $y = \bar{B} \cup \{s(\bar{n})\}$. Similarly, $(x, y') \in f$ implies with (4.78) in particular that there exist n, B, V, say $\bar{n}''$, $\bar{B}''$ and $\bar{V}''$, such that

$$\bar{V}'' = \bar{B}'' \cup \{s(\bar{n}'')\} \,\wedge\, (\bar{n}'', \bar{B}'', \bar{V}'') = (x, y') \quad [= ((\bar{n}, \bar{B}), y')]$$

is true. This gives $\bar{n}'' = \bar{n}$, $\bar{B}'' = \bar{B}$ and $\bar{V}'' = y'$, and therefore $y' = \bar{B} \cup \{s(\bar{n})\}$ $(= y)$. Thus, $y = y'$ holds, and since y and y' are arbitrary, it follows that the second part of the conjunction (4.80) is true as well. This completes the proof of the uniquely existential sentence in (4.79). As x was arbitrary, we then conclude that (4.79) holds, which implies $f : C \times \mathcal{P}(C) \to \mathcal{P}(C)$ with the Function Criterion. Since $(C, s, 0_C)$ was an arbitrary counting domain, we finally conclude that the proposition holds, as claimed. □

Using the denotations $A = \mathcal{P}(C)$ and $a = \{0_C\}$ and using the function f defined in the preceding proposition, we may evidently apply the method of definition by recursion to obtain the following new function u_C.

Definition 4.2 (Initial segment of a counting set). For any counting domain $(C, s, 0_C)$ we define the function $u_C : C \to \mathcal{P}(C)$ recursively by

$$(1) \quad u_C(0_C) = \{0_C\}, \tag{4.83}$$

$$(2) \quad u_C(s(n)) = f(n, u_C(n)) = u_C(n) \cup \{s(n)\} \quad \text{for any } n \in C, \tag{4.84}$$

where f is the function in (4.77). For any $n \in C$ (including $n = 0_C$), we then write for $u_C(n)$ also

$$\{0_C, \ldots, n\}, \tag{4.85}$$

and we call this set the *initial segment of* C (up to n).

Note 4.4. In view of the notation (4.85), the recursion equation (4.84) means that the initial segment of C up to the successor of an $n \in C$ equals the union of the initial segment of C up to n and the singleton formed by that successor, that is,

$$\forall n \, (n \in C \Rightarrow \{0_C, \ldots, s(n)\} = \{0_C, \ldots, n\} \cup \{s(n)\}). \tag{4.86}$$

Notation 4.1. In the particular instance of the counting domain $(\mathbb{N}, s^+, 0)$, we obtain for the *initial segment of* $\mathbb{N}$ (up to n)

$$(1) \quad u_{\mathbb{N}}(0) = \{0\}, \tag{4.87}$$

$$(2) \quad u_{\mathbb{N}}(n^+) = u_{\mathbb{N}}(n) \cup \{n^+\} \quad \text{for any } n \in \mathbb{N}, \tag{4.88}$$

and we write for any $n \in \mathbb{N}$ (including $n = 0$) instead of $u_{\mathbb{N}}(n)$ also

$$\{0, \ldots, n\}.$$

Note 4.5. Equation (4.88) means that the initial segment of $\mathbb{N}$ up to the successor of a natural number n equals the union of the initial segment of $\mathbb{N}$ up to n and the singleton formed by that successor, that is,

$$\forall n \, (n \in \mathbb{N} \Rightarrow \{0, \ldots, n^+\} = \{0, \ldots, n\} \cup \{n^+\}). \tag{4.89}$$

Proposition 4.12. *In case of the counting domain* $(\mathbb{N}, s^+, 0)$*, the function* $u_{\mathbb{N}}$ *is identical with the successor function* s^+*, i.e.*

$$u_{\mathbb{N}} = s^+. \tag{4.90}$$

Proof. According to Definition 4.2, the function $u_{\mathbb{N}} : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ satisfies (4.87) – (4.88). Due to Note 4.2, the successor function in $(\mathbb{N}, s^+, 0)$ may be viewed as the function $s^+ : \mathbb{N} \to \mathcal{P}(\mathbb{N})$. As $u_{\mathbb{N}}$ and s^+ share the same domain we may apply the Equality Criterion for functions to prove $u_{\mathbb{N}} = s^+$. For this purpose, we verify

$$\forall n \, (n \in \mathbb{N} \Rightarrow u_{\mathbb{N}}(n) = s^+(n)) \tag{4.91}$$

by mathematical induction. Concerning the base case ($n = 0$), we obtain the equations

$$s^+(0) = 0^+ = 0 \cup \{0\} = \emptyset \cup \{0\} = \{0\} = u_{\mathbb{N}}(0)$$

using (4.44), (2.290), (2.49), (2.216), and finally (4.87). Concerning the induction step, we let n be arbitrary in $\mathbb{N}$, make the induction assumption $u_{\mathbb{N}}(n) = s^+(n)$, and show that this implies $u_{\mathbb{N}}(n^+) = s^+(n^+)$. We now obtain the equations

$$s^+(n^+) = (n^+)^+ = n^+ \cup \{n^+\} = s^+(n) \cup \{n^+\} = u_{\mathbb{N}}(n) \cup \{n^+\} = u_{\mathbb{N}}(n^+)$$

with (4.44), (2.290), again (4.44), then the induction assumption, and finally (4.84). As n is arbitrary, we therefore conclude that the induction step is true, which completes the proof by mathematical induction of (4.91), and thus the proof of $u_{\mathbb{N}} = s^+$. □

Notation 4.2. We therefore have three ways at our disposal to denote the initial segment of $\mathbb{N}$ up to a natural number n, viz.

$$u_{\mathbb{N}}(n) = n^+ = \{0, \ldots, n\}, \tag{4.92}$$

The dots within the notation $\{0_C, \ldots, n\}$ for an initial segment of C up to n suggest that such a set contains, besides 0_C and n, all of the elements that 'succeed' 0_C and 'precede' n. The task for the remainder of this section is to bring out more clearly what is meant by these notions.

Proposition 4.13. *For any counting domain $(C, s, 0_C)$ it is true that the initial segment of C up to an $n \in C$ contains n, i.e.*

$$\forall n \, (n \in C \Rightarrow n \in \{0_C, \ldots, n\}). \tag{4.93}$$

Proof. We let C, s and 0_C be arbitrary, assume $(C, s, 0_C)$ to be a counting domain, and prove the universal sentence by mathematical induction. Concerning the base case ($n = 0_C$), we notice that

$$0_C \in u_C(0_C) \quad [= \{0_C, \ldots, n\}]$$

holds with (4.83). Concerning the induction step, we let n be arbitrary in C and prove the implication

$$n \in \{0_C, \ldots, n\} \Rightarrow s(n) \in \{0_C, \ldots, s(n)\}) \tag{4.94}$$

directly. We make the induction assumption

$$n \in u_C(n) \quad [= \{0_C, \ldots, n\}]$$

and prove

$$s(n) \in u_C(s(n)) \quad [= \{0_C, \ldots, s(n)\}].$$

In view of (4.84), we have

$$u_C(s(n)) = u_C(n) \cup \{s(n)\},$$

where $s(n) \in u_C(n) \cup \{s(n)\}$ holds by definition of the union of a pair. Then, the preceding equation gives the desired $s(n) \in u_C(s(n))$ after substitution. As n was arbitrary, we may therefore conclude that the induction step is true as well. Since C, s and 0_C were initially arbitrary, it follows that the proposition holds, as claimed. □

Note 4.6. Proposition 4.13 shows in particular that every value of u_C is a nonempty set.

Exercise 4.4. Verify for any counting domain $(C, s, 0_C)$ that the initial segment of C up to an $n \in C$ contains the beginning 0_C, i.e.

$$\forall n\,(n \in C \Rightarrow 0_C \in \{0_C, \ldots, n\}). \tag{4.95}$$

(Hint: Proceed in analogy to the proof of Proposition 4.13.)

Corollary 4.14. *The initial segment of* $\mathbb{N}$ *up to a natural number* n *contains* n *and* 0*, i.e.*

$$\forall n\,(n \in \mathbb{N} \Rightarrow n \in \{0, \ldots, n\}), \tag{4.96}$$

$$\forall n\,(n \in \mathbb{N} \Rightarrow 0 \in \{0, \ldots, n\}). \tag{4.97}$$

Note 4.7. Since $u_{\mathbb{N}}(n) = s^+(n) = n^+$ holds for any $n \in \mathbb{N}$, the sentences (4.96) and (4.97) are equivalent to the previously found sentences (2.305 and (2.319).

Proposition 4.15. *The following sentences are true for any counting domain* $(C, s, 0_C)$*.*

a) If the successor of an $m \in C$ *is element of the initial segment of* C *up to an* $n \in C$*, then* m *is also element of this initial segment, i.e.*

$$\forall m, n\,(m, n \in C \Rightarrow [s(m) \in \{0_C, \ldots, n\} \Rightarrow m \in \{0_C, \ldots, n\}]). \tag{4.98}$$

b) The successor of an $n \in C$ *is not an element of the initial segment of* C *up to* n*, i.e.*

$$\forall n\,(n \in C \Rightarrow s(n) \notin \{0_C, \ldots, n\}). \tag{4.99}$$

c) If an $m \in C$ is element of the initial segment of C up to an $n \in C$ where m and n are different, then the successor of m is also element of this initial segment, i.e.

$$\begin{aligned}&\forall m, n\,(m, n \in C \Rightarrow \\ &\quad [(m \neq n \,\wedge\, m \in \{0_C, \ldots, n\}) \Rightarrow s(m) \in \{0_C, \ldots, n\}]).\end{aligned} \tag{4.100}$$

d) The function u_C is an injection, i.e.

$$\forall m, n\,(m, n \in C \Rightarrow [\{0_C, \ldots, m\} = \{0_C, \ldots, n\} \Rightarrow m = n]). \tag{4.101}$$

e) If the initial segment of C up to an $m \in C$ is a proper subset of the initial segment of C up to an $n \in C$, then the initial segment of C up to the successor of m is included in the initial segment of C up to n, i.e.

$$\begin{aligned}&\forall m, n\,([m, n \in C \Rightarrow \\ &\quad [\{0_C, \ldots, m\} \subset \{0_C, \ldots, n\} \Rightarrow \{0_C, \ldots, s(m)\} \subseteq \{0_C, \ldots, n\}]).\end{aligned} \tag{4.102}$$

f) The initial segment of C up to any $n \in C$ is a proper subset of the initial segment of C up to the successor of n, i.e.

$$\forall n\,(n \in C \Rightarrow \{0_C, \ldots, n\} \subset \{0_C, \ldots, s(n)\}). \tag{4.103}$$

Proof. We let $(C, s, 0_C)$ be an arbitrary counting domain and $u_C : C \to \mathcal{P}(C)$ the function satisfying (4.83) and (4.84).

Concerning a), let us rewrite (4.98) equivalently as

$$\forall m\,(m \in C \Rightarrow \forall n\,(n \in C \Rightarrow [s(m) \in u_C(n) \Rightarrow m \in u_C(n)])) \tag{4.104}$$

using Exercise 1.16. Now, we let m be arbitrary, assume $m \in C$, and carry out a proof by mathematical induction with respect to n. Concerning the base case ($n = 0_C$), we prove the implication

$$s(m) \in u_C(0_C) \Rightarrow m \in u_C(0_C). \tag{4.105}$$

The antecedent $s(m) \in u_C(0_C)$ is equivalent to $s(m) \in \{0_C\}$ due to (4.83); the latter is also equivalent to $s(m) = 0_C$ because of (2.169). By Property 1 of a counting domain, $s(m) = 0_C$ is false, so that the equivalent antecedent $s(m) \in u_C(0_C)$ is also false. Therefore, the implication (4.105) is true. Concerning the induction step, we let n be arbitrary in C, make the induction assumption

$$s(m) \in u_C(n) \Rightarrow m \in u_C(n), \tag{4.106}$$

and show that this implies

$$s(m) \in u_C(s(n)) \Rightarrow m \in u_C(s(n)). \tag{4.107}$$

To prove this implication directly, we assume $s(m) \in u_C(s(n))$, which implies $s(m) \in u_C(n) \cup \{s(n)\}$ with (4.84), and therefore

$$s(m) \in u_C(n) \,\vee\, s(m) \in \{s(n)\} \tag{4.108}$$

by definition of the union of two sets. We now use this true disjunction to prove the sentence $m \in u_C(n)$ by cases. If $s(m) \in u_C(n)$ is true, then it follows with the induction assumption (4.106) that $m \in u_C(n)$. If $s(m) \in \{s(n)\}$ holds, then it follows with (2.169) that $s(m) = s(n)$, which implies $m = n$ with Property 2 of a counting domain; in view of Proposition 4.13, $n \in u_C(n)$ is true and then after substitution also $m \in u_C(n)$. Thus, the proof of $m \in u_C(n)$ by cases is complete. This finding evidently implies the disjunction $m \in u_C(n) \vee m \in \{s(n)\}$ (no matter if $m \in \{s(n)\}$ is true or false), and therefore $m \in u_C(n) \cup \{s(n)\}$ (by definition of the union of two sets). Due to (4.84), this means $m \in u_C(s(n))$, which proves the implication (4.107). As n is arbitrary, we therefore conclude that the induction step is true, which completes the proof by mathematical induction. Since m was also arbitrary, we may further conclude that (4.104) holds, so that the proof of (4.98) is complete.

Concerning b), we carry out a proof by mathematical induction. Regarding the base case ($n = 0_C$), we prove $s(0_C) \notin u_C(0_C)$. By Property 1 of a counting domain, we have $s(0_C) \neq 0_C$ and therefore $s(0_C) \notin \{0_C\}$ with (2.169). Using (4.83), we obtain the desired $s(0_C) \notin u_C(0_C)$. Regarding the induction step

$$\forall n\, (n \in C \Rightarrow [s(n) \notin u_C(n) \Rightarrow s(s(n)) \notin u_C(s(n))]), \tag{4.109}$$

we let $n \in C$ be arbitrary and prove the inner implication by contradiction, assuming

$$s(n) \notin u_C(n) \,\wedge\, s(s(n)) \in u_C(s(n)). \tag{4.110}$$

Due to (4.84), this conjunction is equivalent to

$$s(n) \notin u_C(n) \,\wedge\, s(s(n)) \in u_C(n) \cup \{s(n)\}$$

and then also to (applying the definition of the union of a pair)

$$s(n) \notin u_C(n) \,\wedge\, [s(s(n)) \in u_C(n) \,\vee\, s(s(n)) \in \{s(n)\}].$$

An application of the Distributive Law for sentences (1.44) then gives the equivalently true

$$[s(n) \notin u_C(n) \wedge s(s(n)) \in u_C(n)] \vee [s(n) \notin u_C(n) \wedge s(s(n)) \in \{s(n)\}]. \tag{4.111}$$

Next, we use this true disjunction to prove the sentence $s(n) \in u_C(n)$ by cases. In case

$$s(n) \notin u_C(n) \wedge s(s(n)) \in u_C(n)$$

is true, then it follows in particular that $s(s(n)) \in u_C(n)$ holds, which in turn implies the desired $s(n) \in u_C(n)$ with (4.98). In the other case that the second part

$$s(n) \notin u_C(n) \wedge s(s(n)) \in \{s(n)\}$$

of the disjunction (4.111) is true, it follows in particular that $s(s(n)) \in \{s(n)\}$ holds, and this yields $s(s(n)) = s(n)$ with (2.169). This equation implies by Property 2 of a counting domain $s(n) = n$, so that the fact $n \in u_C(n)$ from Proposition 4.13 gives via substitution $s(n) \in u_C(n)$ once again, completing the proof by cases. Let us now observe that this sentence's negation $s(n) \notin u_C(n)$ is also true in view of the assumed conjunction (4.110). We thus obtained a contradiction, which completes the proof of the inner implication in (4.109), and then also the direct proof of the outer implication (based on the antecedent $n \in C$). As n was arbitrary, we may therefore conclude that the induction step (4.109) holds besides the base case, so that the proof of (4.99) via methematical induction is complete.

Concerning c), we first rewrite (4.100) equivalently as

$$\forall m\,(m \in C \Rightarrow \forall n\,(n \in C \Rightarrow [(m \neq n \wedge m \in u_C(n)) \Rightarrow s(m) \in u_C(n)])). \tag{4.112}$$

by applying Exercise 1.16, let then m be arbitrary in C, and apply a proof by mathematical induction with respect to n. Regarding the base case ($n = 0_C$), we prove the implication

$$[m \neq 0_C \wedge m \in u_C(0_C)] \Rightarrow s(m) \in u_C(0_C). \tag{4.113}$$

Here, we obtain the equivalences

$$m \in u_C(0_C) \Leftrightarrow m \in \{0_C\} \Leftrightarrow m = 0_C$$

with (4.83) and (2.169); according to the Substitution Rule for conjunctions (1.22), the conjunction in (4.113) follows then to be equivalent to the contradiction $m \neq 0_C \wedge m = 0_C$, so that the antecedent of the implication (4.113) is also false. Thus, this implication is true (no matter if the consequent is true or false), which proves the base case.

Regarding the induction step, we let n be arbitrary in C, make the induction assumption

$$[m \neq n \wedge m \in u_C(n)] \Rightarrow s(m) \in u_C(n), \tag{4.114}$$

and prove

$$[m \neq s(n) \wedge m \in u_C(s(n))] \Rightarrow s(m) \in u_C(s(n)). \tag{4.115}$$

We now obtain the true equivalences

$$\begin{aligned}
& m \neq s(n) \wedge m \in u_C(s(n)) \\
& \quad \Leftrightarrow m \neq s(n) \wedge m \in u_C(n) \cup \{s(n)\} \\
& \quad \Leftrightarrow m \neq s(n) \wedge [m \in u_C(n) \vee m \in \{s(n)\}] \\
& \quad \Leftrightarrow m \neq s(n) \wedge [m \in u_C(n) \vee m = s(n)] \\
& \quad \Leftrightarrow [m \neq s(n) \wedge m \in u_C(n)] \vee [m \neq s(n) \wedge m = s(n)] \\
& \quad \Leftrightarrow [m \neq s(n) \wedge m \in u_C(n)]
\end{aligned}$$

using the definition of an initial segment, the definition of the union of a pair, (2.169), the Distributive Law for sentences (1.44), and finally the Contradiction Law in connection with the evident fact that the conjunction $m \neq s(n) \wedge m = s(n)$ is a contradiction. In view of the Substitution Rule for implications (1.26), we may therefore prove the implication

$$[m \neq s(n) \wedge m \in u_C(n)] \Rightarrow s(m) \in u_C(s(n)) \tag{4.116}$$

equivalently to (4.115). We do this by considering the two exhaustive cases $m = n$ and $m \neq n$. In the first case, we assume the conjunction of $m \neq s(n)$, $m \in u_C(n)$ and $m = n$. Since $s(n) \in u_C(s(n))$ is true because of Proposition 4.13 (using the fact that $s(n) \in C$ holds by definition of the function s), it then follows with $m = n$ that $s(m) \in u_C(s(n))$. In the second case, we assume the conjunction of $m \neq s(n)$, $m \in u_C(n)$ and $m \neq n$. The second and third part imply with the induction assumption (4.114) that $s(m) \in u_C(n)$ holds, which then further implies the disjunction $s(m) \in u_C(n) \vee s(m) \in \{s(n)\}$ (no matter if $s(m) \in \{s(n)\}$ is true or not), and therefore $s(m) \in u_C(n) \cup \{s(n)\}$ (applying the definition of the union of a pair), so that $s(m) \in u_C(s(n))$ follows to be true by definition of an initial segment. Thus, $s(m) \in u_C(s(n))$ holds in any case, which proves the implication (4.116), and thus (4.116). As n is arbitrary, we therefore conclude that the induction step holds, which completes the proof by mathematical induction. Since m was also arbitrary, we then further conclude that (4.112) is true, which completes the proof of (4.100).

Concerning d), we let m and n be arbitrary in C and prove then the implication $u_C(m) = u_C(n) \Rightarrow m = n$ by contradiction. To do this, we assume $u_C(m) = u_C(n)$ and $m \neq n$. Recalling that $n \in u_C(n)$ holds with Proposition 4.13, we obtain $n \in u_C(m)$ with the assumed equation. On the one hand, the conjunction of the assumption $n \neq m$ and $n \in u_C(m)$ implies $s(n) \in u_C(m)$ with the previously established (4.100); on the other hand, the fact that $s(n) \notin u_C(n)$ is true in view of (4.99) implies – together with the assumed equation $u_C(m) = u_C(n)$ – that $s(n) \notin u_C(m)$ holds, which contradicts the previously obtained $s(n) \in u_C(m)$. This proves the stated implication, and since m as well as n was arbitrary, we my therefore conclude that (4.101) is true.

Concerning e), we let m and n be arbitrary in C and prove the implication

$$u_C(m) \subset u_C(n) \Rightarrow u_C(s(m)) \subseteq u_C(n) \tag{4.117}$$

directly, assuming $u_C(m) \subset u_C(n)$, which means by definition of a proper subset (in connection with the definition of a subset)

$$\forall k\,(k \in u_C(m) \Rightarrow k \in u_C(n)) \,\wedge\, u_C(m) \neq u_C(n). \tag{4.118}$$

To show that this implies $u_C(s(m)) \subseteq u_C(n)$, which is equivalent to

$$\forall k\,(k \in u_C(s(m)) \Rightarrow k \in u_C(n)) \tag{4.119}$$

by definition of a subset, we let k be arbitrary and prove the implication in (4.119) directly, which is equivalent to the original implication (4.117) according to the Substitution rule for implications (1.27). To do this, we assume $k \in u_C(s(m))$, which implies $k \in u_C(m) \cup \{s(m)\}$ by definition of an initial segment, and therefore

$$k \in u_C(m) \vee k \in \{s(m)\} \tag{4.120}$$

by definition of the union of a pair. We now use this true disjunction to prove the desired consequent $k \in u_C(n)$ by cases. In case $k \in u_C(m)$ is true, then this implies the desired $k \in u_C(n)$ with the first part of the assumed conjunction (4.118). If the second part $k \in \{s(m)\}$ of the disjunction (4.120) is true, then this implies $k = s(m)$ with (2.169); since u_C is a function, the second part $u_C(m) \neq u_C(n)$ of the assumed conjunction (4.118) implies $m \neq n$ with Proposition 3.149. Furthermore, the fact that $m \in u_C(m)$ holds due to Proposition 4.13 implies $m \in u_C(n)$ with the first part of the assumed conjunction (4.118). Now, the conjunction of these findings $m \neq n$ and $m \in u_C(n)$ implies with (4.100) that $s(m) \in u_C(n)$ holds, so that substitution based on the previously found equation $k = s(m)$ yields $k \in u_C(n)$, as desired. Thus, the proof by cases of $k \in u_C(n)$ is complete, and this finding proves the implication in (4.119). As k is arbitrary, we therefore conclude that (4.119) is true, which means that $u_C(s(m)) \subseteq u_C(n)$ holds. This in turn proves the implication (4.117), and since m and n were arbitrary, we then conclude that (4.102) holds, as claimed.

Concerning f), we let n be arbitrary in C. In view of (2.201) in connection with the definition of a pair we see on the one hand that $u_C(n) \subseteq u_C(n) \cup \{s(n)\}$ holds, and therefore $u_C(n) \subseteq u_C(s(n))$ using the definition of an initial segment. On the other hand, the fact that $n \neq s(n)$ holds with Proposition 4.3 implies $u_C(n) \neq u_C(s(n))$ with the Injection Criterion and the previously established fact (4.101) that u_C is an injection. Consequently, $u_C(n) \subset u_C(s(n))$ holds by definition of a proper subset. As n was arbitrary, we therefore conclude that (4.103) is true.

Since C, s and 0_C were arbitrary in the proofs of the sentences a) – f), we may infer from these the truth of the proposition. □

We now derive from the preceding proposition the particular results for the counting domain $(\mathbb{N}, s^+, 0)$, where we use the two notations n^+ and $\{0, \ldots, n\}$ for the initial segment $u_{\mathbb{N}}(n)$ according to (4.92).

Corollary 4.16. *The following sentences are true for the counting domain* $(\mathbb{N}, s^+, 0)$.

a) *If the successor of a natural number m is element of the initial segment of $\mathbb{N}$ up to a natural number n, then m is also element of this initial segment, i.e.*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m^+ \in \{0, \ldots, n\} \Rightarrow m \in \{0, \ldots, n\}]). \quad (4.121)$$

In other words, if the successor of a natural number m is element of the successor of a natural number n, then m is also element of that successor, i.e.

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m^+ \in n^+ \Rightarrow m \in n^+]). \quad (4.122)$$

b) *The successor of a natural number n is not an element of the initial segment of $\mathbb{N}$ up to n, i.e.*

$$\forall n\,(n \in \mathbb{N} \Rightarrow n^+ \notin \{0, \ldots, n\}). \quad (4.123)$$

In other words, the successor of a natural number n is not an element of itself, i.e.

$$\forall n\,(n \in \mathbb{N} \Rightarrow n^+ \notin n^+). \quad (4.124)$$

c) If a natural number m is element of the initial segment of $\mathbb{N}$ up to a natural number n where m and n are different, then the successor of m is also element of this initial segment, i.e.

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [(m \neq n \wedge m \in \{0, \ldots, n\}) \Rightarrow m^+ \in \{0, \ldots, n\}]). \tag{4.125}$$

In other words, if a natural number m is element of the successor of a natural number n where m and n are different, then the successor of m is also element of the successor of n, i.e.

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [(m \neq n \wedge m \in n^+) \Rightarrow m^+ \in n^+]). \tag{4.126}$$

d) The function $u_{\mathbb{N}}$ is an injection, i.e.

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [\{0, \ldots, m\} = \{0, \ldots, n\} \Rightarrow m = n]). \tag{4.127}$$

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m^+ = n^+ \Rightarrow m = n]). \tag{4.128}$$

e) If the initial segment of $\mathbb{N}$ up to a natural number m is a proper subset of the initial segment of $\mathbb{N}$ up to a natural number n, then the initial segment of $\mathbb{N}$ up to the successor of m is included in the initial segment of $\mathbb{N}$ up to n, i.e.

$$\begin{aligned}&\forall m, n\,([m, n \in \mathbb{N} \Rightarrow \\ &\quad [\{0, \ldots, m\} \subset \{0, \ldots, n\} \Rightarrow \{0, \ldots, m^+\} \subseteq \{0, \ldots, n\}]).\end{aligned} \tag{4.129}$$

In other words, if the successor of a natural number m is a proper subset of the successor of a natural number n, then the successor of m^+ is included in the successor of n, i.e.

$$\forall m, n\,([m, n \in \mathbb{N} \Rightarrow [m^+ \subset n^+ \Rightarrow m^{++} \subseteq n^+]). \tag{4.130}$$

f) The initial segment of $\mathbb{N}$ up to any natural number n is a proper subset of the initial segment of $\mathbb{N}$ up to the successor of n, i.e.

$$\forall n\,(n \in \mathbb{N} \Rightarrow \{0, \ldots, n\} \subset \{0, \ldots, n^+\}). \tag{4.131}$$

In other words, the successor of any natural number n is a proper subset of the successor of n^+, i.e.

$$\forall n\,(n \in \mathbb{N} \Rightarrow n^+ \subset n^{++}). \tag{4.132}$$

Note 4.8. We already stated (4.128) in (4.36) and proved this sentence as part of Theorem 4.6 independently from the results of Proposition 4.15.

We may now use the recursion equation (4.86) and the preceding proposition to find further relationships for initial segments.

Proposition 4.17. *The following sentences are true for any counting domain* $(C, s, 0_C)$*.*

a) The initial segment of C *up to an* $n \in C$ *and the singleton formed by the successor of* n *are disjoint, that is,*

$$\forall n\,(n \in C \Rightarrow \{0_C, \ldots, n\} \cap \{s(n)\} = \emptyset). \tag{4.133}$$

b) The initial segment of C *up to an* $n \in C$ *equals the difference of the initial segment of* C *up to the successor of* n *and the singleton formed by that successor, that is,*

$$\forall n\,(n \in C \Rightarrow \{0_C, \ldots, n\} = \{0_C, \ldots, s(n)\} \setminus \{s(n)\}). \tag{4.134}$$

c) The singleton formed by the successor of an $n \in C$ *equals the difference of the initial segment of* C *up to the successor of* n *and the initial segment up to* n*, i.e.*

$$\forall n\,(n \in C \Rightarrow \{s(n)\} = \{0_C, \ldots, s(n)\} \setminus \{0_C, \ldots, n\}). \tag{4.135}$$

Proof. We let $(C, s, 0_C)$ be an arbitrary counting domain. Concerning a), we let n be arbitrary and prove the implication by contradiction, assuming $n \in C$ and $\{0_C, \ldots, n\} \cap \{s(n)\} \neq \emptyset$ to be true. The latter implies with Proposition 2.11 that there exists an element in $\{0_C, \ldots, n\} \cap \{s(n)\}$, say $\bar{m}$. Then, $\bar{m} \in \{0_C, \ldots, n\}$ and $\bar{m} \in \{s(n)\}$ are both true by definition of the intersection of two sets. Here, $\bar{m} \in \{s(n)\}$ implies $\bar{m} = s(n)$ with (2.169); applying now substitution based on this equation, the previously established $\bar{m} \in \{0_C, \ldots, n\}$ gives $s(n) \in \{0_C, \ldots, n\}$, which evidently contradicts the fact $s(n) \notin \{0_C, \ldots, n\}$ resulting from (4.99). Thus, the proof of the implication (4.133) via contradiction is complete, and since n was arbitrary, we may therefore conclude that b) is true.

Concerning b), we let $n \in C$ be arbitrary and observe in light of (2.262) that the conjunction of the equations in (4.86) and (4.133) implies the equation in (4.134). As n is arbitrary, we may therefore further conclude that c) also holds.

Finally, concerning c), we again let $n \in C$ be arbitrary. On the one hand, we may apply the Commutative Law for the union of two sets in connection with the equation in (4.86) to obtain the true equation

$$\{0_C, \ldots, s(n)\} = \{s(n)\} \cup \{0_C, \ldots, n\}. \tag{4.136}$$

On the other hand, we obtain from the equation in (4.133) with the Commutative Law for the intersection of two sets the equivalent equation

$$\{s(n)\} \cap \{0_C, \ldots, n\} = \emptyset. \tag{4.137}$$

Then, we may again employ (2.262) to infer from the truth of (4.136) and of (4.137) the truth of the equation in (4.135). Since n is arbitrary, d) follows then to be true as well.

As C, s and 0_C were initially arbitrary, we may finally conclude that the proposition holds, as claimed. □

Corollary 4.18. *The following sentences are true for the counting domain* $(\mathbb{N}, s^+, 0)$.

a) The initial segment of N up to a natural number n and the singleton formed by the successor of n are disjoint, that is,

$$\forall n\, (n \in \mathbb{N} \Rightarrow \{0, \ldots, n\} \cap \{n^+\} = \emptyset). \tag{4.138}$$

b) The initial segment of N up to a natural number n equals the difference of the initial segment up to the successor of n and the singleton formed by that successor, that is,

$$\forall n\, (n \in \mathbb{N} \Rightarrow \{0, \ldots, n\} = \{0, \ldots, n^+\} \setminus \{n^+\}). \tag{4.139}$$

c) The singleton formed by the successor of a natural number n equals the difference of the initial segment of N up to the successor of n and the initial segment of N up to n, that is,

$$\forall n\, (n \in \mathbb{N} \Rightarrow \{n^+\} = \{0, \ldots, n^+\} \setminus \{0, \ldots, n\}). \tag{4.140}$$

4.3. Linearly Ordered Sets $(C, <_C)$ and the Linearly Ordered Set $(\mathbb{N}, <_{\mathbb{N}})$

Recalling that the binary relation $\subseteq$ may be used to define a reflexive partial ordering of an arbitrary set system, it is now used in connection with the initial segments of a counting set to define a reflexive partial ordering of that counting set. This ordering turns out to be total, so that the induced irreflexive partial ordering is linear.

Theorem 4.19 (Total & linear ordering of a counting set). *The following sentences are true for any counting domain $(C, s, 0_C)$.*

a) There exists a unique set $\leq_C$ consisting of all ordered pairs (m, n) satisfying $u_C(m) \subseteq u_C(n)$. This set $\leq_C$ is a binary relation satisfying

$$\forall m, n\,(m, n \in C \Rightarrow [m \leq_C n \Leftrightarrow \{0_C, \ldots, m\} \subseteq \{0_C, \ldots, n\}]) \tag{4.141}$$

and is furthermore a total ordering of C.

b) Moreover, the linear ordering $<_C$ of C induced by $\leq_C$ satisfies

$$\forall n\,(n \in C \Rightarrow n <_C s(n)). \tag{4.142}$$

Proof. We let C, s and 0_C be arbitrary and assume that $(C, s, 0_C)$ is a counting domain. Concerning a), we recall from the definition of an initial segment that the domain of u_C equals C, so that every ordered pair (m, n) satisfying $u_C(m) \subseteq u_C(n)$ is an element of the Cartesian product $C \times C$. We may therefore apply the Axiom of Specification together with the Equality Criterion for sets to obtain the true sentence

$$\begin{aligned}&\exists!\, \leq_C\ \forall Y\,(Y \in \leq_C \\ &\quad \Leftrightarrow [Y \in C \times C\ \wedge\ \exists m, n\,(u_C(m) \subseteq u_C(n)\ \wedge\ (m, n) = Y)]).\end{aligned}$$

Thus, the set $\leq_C$ satisfies

$$\forall Y\,(Y \in \leq_C \Leftrightarrow [Y \in C \times C\ \wedge\ \exists m, n\,(u_C(m) \subseteq u_C(n)\ \wedge\ (m, n) = Y)]). \tag{4.143}$$

Let us now check that $\leq_C$ is a subset of $C \times C$, i.e. that

$$\forall Y\,(Y \in \leq_C \Rightarrow Y \in C \times C)$$

holds by definition of a subset. Letting Y be arbitrary, we see that $Y \in \leq_C$ implies the conjunction of $Y \in C \times C$ and the existential sentence in (4.143), so that $Y \in \leq_C$ implies in particular $Y \in C \times C$. Then, as Y is arbitrary, we conclude that $\leq_C \subseteq C \times C$ holds indeed. Thus, $\leq_C$ is a binary relation on C, which we now demonstrate to satisfy (4.141). We let m and n be arbitrary in C and prove the first part ('$\Rightarrow$') of the equivalence directly by assuming $m \leq_C n$. This implies with (4.143) in particular that there exist constants, say $\bar{m}$ and $\bar{n}$, such that $u_C(\bar{m}) \subseteq u_C(\bar{n})$ and $(\bar{m}, \bar{n}) = (m, n)$ hold. In view of the Equality Criterion for ordered pairs, the preceding equation implies $\bar{m} = m$ and $\bar{n} = n$. With these equations, it follows after substitution in $u_C(\bar{m}) \subseteq u_C(\bar{n})$ that $u_C(m) \subseteq u_C(n)$ holds, which is the desired consequent of the first part of the equivalence. To prove the second part ('$\Leftarrow$'), we now assume $u_C(m) \subseteq u_C(n)$. Denoting the ordered pair (m, n) by Y, we then see that the existential sentence

$$\exists m, n\,(u_C(m) \subseteq u_C(n) \,\wedge\, (m, n) = Y)$$

is true, and furthermore that $[(m, n) =]\ Y \in C \times C$ holds due to $m, n \in C$ and the definition of a Cartesian product. It then follows with (4.143) that $Y \in\ \leq_C$, which means $m \leq_C n$. Since m and n were arbitrary, we therefore conclude that $\leq_C$ satisfies indeed (4.141).

Next, we prove that the binary relation $\leq_C$ on C is a total ordering. We begin with the verification that $\leq_C$ is reflexive, i.e.

$$\forall a\,(a \in C \Rightarrow a \leq_C a). \tag{4.144}$$

For this purpose, we let a be arbitrary in C and observe from (4.141) that $a \leq_C a$ is equivalent to $u_C(a) \subseteq u_C(a)$. We already know from Theorem 3.64 that $\subseteq$ is a reflexive partial ordering of any set system, in particular of $\mathcal{P}(C)$, so that $u_C(a) \subseteq u_C(a)$ is true. Thus, $a \leq_C a$ also holds, and since a was arbitrary, we therefore conclude that (4.144) is true, which means that $\leq_C$ is reflexive.

To prove that $\leq_C$ is antisymmetric, that is,

$$\forall a, b\,(a, b \in C \Rightarrow [(a \leq_C b \,\wedge\, b \leq_C a) \Rightarrow a = b]), \tag{4.145}$$

we let a and b be arbitrary in C and assume $a \leq_C b$ as well as $b \leq_C a$. Because of (4.141) the assumptions mean that $u_C(a) \subseteq u_C(b)$ and $u_C(b) \subseteq u_C(a)$. We already established $\subseteq$ as a reflexive partial ordering, so that $\subseteq$ is antisymmetric, which yields $u_C(a) = u_C(b)$ (this equation follows also directly with the Axiom of Extension). This equation implies $a = b$ with (4.101); as a and b are arbitrary, we therefore conclude that (4.145) holds, so that $\leq_C$ is also antisymmetric.

We now show that $\leq_C$ is transitive, i.e. that $\leq_C$ that satisfies

$$\forall a, b, c\,(a, b, c \in C \Rightarrow [(a \leq_C b \,\wedge\, b \leq_C c) \Rightarrow a \leq_C c]). \tag{4.146}$$

Letting a, b, c be arbitrary in C and assuming both $a \leq_C b$ and $b \leq_C c$ to be true, it follows with (4.141) that the conjunction of $u_C(a) \subseteq u_C(b)$ and $u_C(b) \subseteq u_C(c)$ holds, which implies $u_C(a) \subseteq u_C(c)$ with the transitivity of the reflexive partial ordering $\subseteq$ (see also Proposition 2.5). The latter then implies the desired $a \leq_C c$ (applying once again (4.141)). Since a, b, c were arbitrary, we conclude that (4.146) is true, which means that $\leq_C$ is indeed transitive.

Thus, as a reflexive, antisymmetric and transitive binary relation on C, $\leq_C$ is a reflexive partial ordering of C. We now verify that $\leq_C$ is also total, i.e.

$$\forall a, b\,(a, b \in C \Rightarrow [a \leq_C b \,\vee\, b \leq_C a]),$$

by proving the equivalent (using Exercise 1.16)

$$\forall a\,(a \in C \Rightarrow \forall b\,(b \in C \Rightarrow [a \leq_C b \,\vee\, b \leq_C a])). \tag{4.147}$$

We let a be arbitrary in C and carry out a proof by mathematical induction with respect to b. Concerning the base case ($b = 0_C$), we prove that $a \leq_C 0_C$ or $0_C \leq_C a$ is true. Since $0_C \in u_C(a)$ holds with Exercise 4.4 and therefore $\{0_C\} \subseteq u_C(a)$ with with (2.184), we obtain with the equation $u_C(0_C) = \{0_C\}$ in (4.83) after substitution $u_C(0_C) \subseteq u_C(a)$. This implies $0_C \leq_C a$ with (4.141)), so that the disjunction of $0_C \leq_C a$ and $a \leq_C 0_C$ holds as well (no matter if $a \leq_C 0_C$ is true or not), which proves the base case. Regarding the induction step, we let b be an arbitrary element of C, make the induction assumption $a \leq_C b \vee b \leq_C a$, and use this true disjunction to prove

$$a \leq_C s(b) \,\vee\, s(b) \leq_C a \tag{4.148}$$

by cases.

On the one hand, if $a \leq_C b$ holds, then also $u_C(a) \subseteq u_C(b)$ because of (4.141)). Since $u_C(b) \subseteq u_C(b) \cup \{s(b)\}$ also holds in view of (2.201), where $u_C(b) \cup \{s(b)\} = u_C(s(b))$ according to (4.84), we obtain from $u_C(a) \subseteq u_C(b) \subseteq u_C(s(b))$ with the transitivity of $\subseteq$ that $u_C(a) \subseteq u_C(s(b))$. This implies $a \leq_C s(b)$ with (4.141)) and then also the disjunction (4.148) (no matter if $s(b) \leq_C a$ is true or false).

On the other hand, if the second part $b \leq_C a$ of the induction assumption is true, then also $u_C(b) \subseteq u_C(a)$ due to (4.141), so that the disjunction $u_C(b) \subset u_C(a) \vee u_C(b) = u_C(a)$ holds (see Proposition 2.8). Let us now use this true disjunction to prove (4.148) by cases. In case $u_C(b) \subset u_C(a)$ is true, then it follows with (4.102) that $u_C(s(b)) \subseteq u_C(a)$. Consequently, $s(b) \leq_C a$ is true, and therefore also the disjunction (4.148) (no matter if $a \leq_C s(b)$ is true or false). In the other case that $u_C(b) = u_C(a)$ is true, the disjunction $u_C(a) \subset u_C(b) \vee u_C(a) = u_C(b)$ holds also (no matter if

$u_C(a) \subset u_C(b)$ is true or false), so that we obtain $u_C(a) \subseteq u_C(b)$ (using again Proposition 2.8), which we already showed in the case $a \leq_C b$ to imply (4.148).

Thus, the two nested proofs by cases are complete (4.148), and since b was arbitrary, we therefore conclude that the induction step is true. This completes the proof by mathematical induction, and as a is also arbitrary, we further conclude that (4.147) holds, so that $\leq_C$ is a total binary relation. Thus, $\leq_C$ is a total ordering of C.

Concerning b), since $(C, \leq_C)$ is a totally ordered set, it follows with Theorem 3.86 that there exists the induced binary relation $<_C$, which is a linear ordering of C because of Theorem 3.89. We now show that $<_C$ satisfies

$$\forall n \, (n \in C \Rightarrow (n, s(n)) \in {<_C}). \tag{4.149}$$

For this purpose, we let n be arbitrary in C and observe that $u_C(n) \subseteq u_C(n) \cup \{s(n)\}$ and therefore $u_C(n) \subseteq u_C(s(n))$ holds (using Proposition 2.67 and (4.84)). It then follows with (4.141) that $n \leq_C s(n)$. We furthermore have $s(n) \neq n$ with Proposition 4.3, so that the conjunction $n \leq_C s(n) \wedge n \neq s(n)$ holds. We now define the ordered pair $u = (n, s(n))$, which is an element of $C \times C$ by definition of the successor function s. Thus, there exist a, b with $a \leq_C b$, $a \neq b$, and $(a, b) = u$. The conjunction of this existential sentence and the previously established $u \in C \times C$ then implies $u \in {<_C}$ with (3.254), which means that $(n, s(n)) \in {<_C}$ holds. As n is arbitrary, we therefore conclude that (4.149) is true, completing the proof of b).

Since C, s and 0_C were arbitrary, we may finally infer from these findings the truth of the stated theorem. □

Theorem 4.20 (Characterization of the linear ordering of a counting set). *For any counting domain $(C, s, 0_C)$ the linear ordering $<_C$ of C induced by $\leq_C$ satisfies*

$$\forall m, n \, (m, n \in C \Rightarrow [m <_C n \Leftrightarrow \{0_C, \ldots, m\} \subset \{0_C, \ldots, n\}]). \tag{4.150}$$

Proof. We let C, s and 0_C be arbitrary, assume that $(C, s, 0_C)$ is a counting domain, let $m, n \in C$ be arbitrary, and observe the truth of the equivalences

$$\begin{aligned} m <_C n \;&\Leftrightarrow\; m \leq_C n \,\wedge\, m \neq n \\ &\Leftrightarrow\; \{0_C, \ldots, m\} \subseteq \{0_C, \ldots, n\} \,\wedge\, m \neq n \\ &\Leftrightarrow\; \{0_C, \ldots, m\} \subseteq \{0_C, \ldots, n\} \,\wedge\, u_C(m) \neq u_C(n) \\ &\Leftrightarrow\; \{0_C, \ldots, m\} \subseteq \{0_C, \ldots, n\} \,\wedge\, \{0_C, \ldots, m\} \neq \{0_C, \ldots, n\} \\ &\Leftrightarrow\; \{0_C, \ldots, m\} \subset \{0_C, \ldots, n\} \end{aligned}$$

in light of (3.252), (4.141), Corollary 3.186 in connection with Proposition 4.15d), the definition of an initial segment, and the definition of a proper subset. Since m and n were arbitrary, we may therefore conclude that the universal sentence (4.150) is true, and as C, s and 0_C were also arbitrary, we may then infer from this the truth of the stated theorem. $\square$

Definition 4.3 (Standard total & linear ordering of a counting set). For any counting domain $(C, s, 0_C)$ we call $\leq_C$ the *standard total ordering of* C and $<_C$ the *standard linear ordering of* C.

We now state the findings of Theorem 4.19 and of Theorem 4.20 specifically for the counting domain of natural numbers.

Corollary 4.21. *The following sentences are true for the counting domain* $(\mathbb{N}, s^+, 0)$.

a) The total ordering $\leq_\mathbb{N}$ *of* $\mathbb{N}$ *(defined according to Theorem 4.19) satisfies*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \leq_\mathbb{N} n \Leftrightarrow \{0, \ldots, m\} \subseteq \{0, \ldots, n\}]), \tag{4.151}$$

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \leq_\mathbb{N} n \Leftrightarrow m^+ \subseteq n^+]), \tag{4.152}$$

and moreover

$$\forall n\,(n \in \mathbb{N} \Rightarrow n <_\mathbb{N} n^+). \tag{4.153}$$

b) Then, the linear ordering $<_\mathbb{N}$ *of* $\mathbb{N}$ *induced by* $\leq_\mathbb{N}$ *satisfies*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m <_\mathbb{N} n \Leftrightarrow \{0, \ldots, m\} \subset \{0, \ldots, n\}]), \tag{4.154}$$

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m <_\mathbb{N} n \Leftrightarrow m^+ \subset n^+]). \tag{4.155}$$

Definition 4.4 (Standard total & linear ordering of $\mathbb{N}$). We call $\leq_\mathbb{N}$ the *standard total ordering of* $\mathbb{N}$ and $<_\mathbb{N}$ the *standard linear ordering of* $\mathbb{N}$.

Notation 4.3. Whenever the context is unambiguous, we will also write $\leq$ instead of $\leq_\mathbb{N}$ and $<$ instead of $<_\mathbb{N}$.

Exercise 4.5. Verify for any counting domain $(C, s, 0_C)$ that, if an $m \in C$ is less than an $n \in C$, then the successor of m is less than or equal to n, i.e.

$$\forall m, n\,(m, n \in C \Rightarrow [m <_C n \Rightarrow s(m) \leq_C n]) \tag{4.156}$$

(Hint: Use Theorem 4.20, Proposition 4.15e), and Theorem 4.19a).)

Corollary 4.22. *The following sentence is true for the counting domain* $(\mathbb{N}, s^+, 0)$.

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m <_\mathbb{N} n \Rightarrow m^+ \leq_\mathbb{N} n]) \tag{4.157}$$

Exercise 4.6. Prove for any counting domain $(C, s, 0_C)$ that, if the successor of an $m \in C$ is less than or equal to the successor of an $n \in C$, then m is less than or equal to n, i.e.

$$\forall m, n\, (m, n \in C \Rightarrow [s(m) \leq_C s(n) \Rightarrow m \leq_C n]). \tag{4.158}$$

(Hint: Apply Method 1.11 in connection with the Negation Formulas for $\leq$ and $<$, (4.156), and Property 3 of a reflexive partial ordering.

Corollary 4.23. *If the successor of a natural number m is less than or equal to the successor of a natural number n, then m is less than or equal to n, that is,*

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow [m^+ \leq_{\mathbb{N}} n^+ \Rightarrow m \leq_{\mathbb{N}} n]). \tag{4.159}$$

Proposition 4.24. *For any counting domain $(C, s, 0_C)$ it is true that, if the successor of an $m \in C$ is less than the successor of an $n \in C$, then m is less than n, i.e.*

$$\forall m, n\, (m, n \in C \Rightarrow [s(m) <_C s(n) \Rightarrow m <_C n]). \tag{4.160}$$

Proof. We let C, s and 0_C be arbitrary and assume that $(C, s, 0_C)$ is a counting domain. Next, we let m and n be arbitrary in C and assume $s(m) <_C s(n)$ to be true. The disjunction $s(m) <_C s(n) \vee s(m) = s(n)$ then also holds, which gives with (3.253) $s(m) \leq_C s(n)$, which in turn implies $m \leq_C n$ with (4.158). Moreover, since the linear ordering $<_C$ is comparable, the truth of $s(m) <_C s(n)$ implies $s(m) \neq s(n)$, and then also $m \neq n$ with the fact that s is a function (see Proposition 3.149). We thus showed that $m \leq_C n$ and $m \neq n$ are both true, so that the desired consequent $m <_C n$ follows to be true with (3.252). Since m, n, C, s and 0_C were arbitrary, we may therefore conclude that the stated universal sentence holds, as claimed. □

Corollary 4.25. *If the successor of a natural number m is less than the successor of a natural number n, then m is less than n, that is,*

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow [m^+ <_{\mathbb{N}} n^+ \Rightarrow m <_{\mathbb{N}} n]). \tag{4.161}$$

Proposition 4.26. *For any counting domain $(C, s, 0_C)$ and any elements m and n of the counting set, it is true that m is less than or equal to n iff the successor of m is less than or equal to the successor of n, i.e.*

$$\forall m, n\, (m, n \in C \Rightarrow [m \leq_C n \Leftrightarrow s(m) \leq_C s(n)]). \tag{4.162}$$

Proof. Letting C, s and 0_C be arbitrary such that $(C, s, 0_C)$ is a counting domain and letting $m, n \in C$ be arbitrary, we have on the one hand that the assumption $m \leq_C n$ and the fact $n <_{\mathbb{N}} s(n)$ (see Theorem 4.19b)) imply with the Transitivity Formula for $\leq$ and $<$ the inequality $m <_{\mathbb{N}} s(n)$, which in turn implies $s(m) \leq_C s(n)$ with (4.156). On the other hand, the assumption $s(m) \leq_{\mathbb{N}} s(n)$ implies $m \leq_C n$ with (4.158). Thus, the proposed equivalence is true, which in turn proves the stated implication. Since m and n were arbitrary, the universal sentence (4.162) follows then to be also true, and as C, s and 0_C were initially also arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Corollary 4.27. *For any natural numbers m and n it is true that m is less than or equal to n if, and only if, the successor of m is less than or equal to the successor of n, that is,*

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow [m \leq_{\mathbb{N}} n \Leftrightarrow m^+ \leq_{\mathbb{N}} n^+]). \tag{4.163}$$

We are now in a position to state some facts about natural numbers, which we consider to be basic in the remainder of this exposition (so that we will usually apply them without explicitly stating a reference to the following exercise).

Exercise 4.7. Verify the following sentences.

a) 0 is less than 1, i.e.

$$0 <_{\mathbb{N}} 1, \tag{4.164}$$

and 0 is unequal to 1, i.e.

$$0 \neq 1. \tag{4.165}$$

(Hint: Apply (2.291), Corollary 4.21, and Property 2* of a linear ordering.)

b) 1 is less than 2, i.e.

$$1 <_{\mathbb{N}} 2, \tag{4.166}$$

and 1 is unequal to 2, i.e.

$$1 \neq 2. \tag{4.167}$$

c) 0 is less than 2, i.e.

$$0 <_{\mathbb{N}} 2, \tag{4.168}$$

and 0 is unequal to 2, i.e.

$$0 \neq 2. \tag{4.169}$$

(Hint: Use Property 3 and Property 2* of a linear ordering)

The standard linear ordering of a counting set induces a lattice structure of the underlying set C.

Proposition 4.28. *For any counting domain* $(C, s, 0_C)$ *the linearly ordered set* $(C, \leq_C)$ *is a lattice.*

Proof. We let C, s and 0_C be arbitrary such that $(C, s, 0_C)$ is a counting domain and show that $(C, \leq_C)$ satisfies (3.303). To do this, we let $m, n \in C$ be arbitrary and recall that $\leq_C$ is the total ordering of C which corresponds to the linear ordering $<_C$ (see Theorem 4.19). Consequently, the supremum S and the infimum I of the pair $\{m, n\}$ exist because of Proposition 3.113 and Exercise 3.52, respectively. As m and n are arbitrary, we may therefore conclude that $(C, <_C)$ satisfies (3.303), which ordered pair thus constitutes a lattice. Since C, s and 0_C were also arbitrary, the proposition follows then to be true. □

Corollary 4.29. *The linearly ordered set* $(\mathbb{N}, \leq_{\mathbb{N}})$ *is a lattice.*

We now see that not all subsets of a counting set have a supremum (so that counting sets do not give rise to complete lattices).

Proposition 4.30. *The counting set* C *of any counting domain* $(C, s, 0_C)$ *is not bounded from above, i.e. it is true that*

$$\neg \exists u \, (u \in C \, \land \, \forall n \, (n \in C \Rightarrow n \leq_C u)). \tag{4.170}$$

Proof. Letting C, s and 0_C be arbitrary such that $(C, s, 0_C)$ is a counting domain, we prove the sentence (4.170) by contradiction, assuming its negation to be true, so that the existential sentence in (4.170) follows to be true with the Double Negation Law. Thus, there exists an element of C, say $\bar{u}$, with

$$\forall n \, (n \in C \Rightarrow n \leq_C \bar{u}).$$

Let us now observe that $\bar{u} \in C$ implies the tautology $\bar{u} <_C s(\bar{u})$ with (4.142). Now, since $\bar{u} \in C$ gives $s(\bar{u}) \in C$ with the definition of a successor function, we obtain with the preceding universal sentence the inequality $s(\bar{u}) \leq_C \bar{u}$, which in turn implies $\neg \bar{u} <_C s(\bar{u})$ with the Negation Formula for $<$, completing the proof by contradiction. Since C, s and 0_C were initially arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Corollary 4.31. *The set of natural numbers does not have an upper bound.*

Proposition 4.32. *It is true for any counting domain* $(C, s, 0_C)$ *that any element of* C *which is greater than or equal to an* $m \in C$ *and less than the successor of* m *is identical to* m*, i.e.*

$$\forall m, n \, (m, n \in C \Rightarrow [(m \leq_C n \, \land \, n <_C s(m)) \Rightarrow n = m]). \tag{4.171}$$

Proof. We let $(C, s, 0_C)$ be an arbitrary counting domain, let m and n be arbitrary in C, and assume then both $m \leq_C n$ and $n <_C s(m)$ to be true. Here, $n <_C s(m)$ implies $\{0_C, \ldots, n\} \subset \{0_C, \ldots, s(m)\}$ with (4.150) and then $\{0_C, \ldots, s(n)\} \subseteq \{0_C, \ldots, s(m)\}$ with (4.102). This finding yields $s(n) \leq_C s(m)$ with (4.141) and consequently $n \leq_C m$ because of (4.162). The conjunction of this and the assumption $m \leq_C n$ then implies the desired equation $n = m$ with the antisymmetry of the partial ordering $\leq_C$. Since m and n are arbitrary, we may therefore conclude that the proposed universal sentence (4.171) is true. □

Exercise 4.8. Show for any counting domain $(C, s, 0_C)$ that any element of C which is greater than an $m \in C$ and less than or equal to the successor of m is identical to that successor, i.e.

$$\forall m, n\,(m, n \in C \Rightarrow [m <_C n \wedge n \leq_C s(m)] \Rightarrow n = s(m)). \tag{4.172}$$

(Hint: Use similar arguments as in the proof of Proposition 4.32.)

Note 4.9. The findings of the preceding proposition and exercise show for any counting domain $(C, s, 0_C)$ that the successor of an $m \in C$ is the 'next greater' element after m.

Corollary 4.33. *The successor m^+ of a natural number m is the next greater natural number after m, in the sense that*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \leq_{\mathbb{N}} n \wedge n <_{\mathbb{N}} m^+] \Rightarrow n = m), \tag{4.173}$$

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m <_{\mathbb{N}} n \wedge n \leq_{\mathbb{N}} m^+] \Rightarrow n = m^+). \tag{4.174}$$

The following theorem provides, retrospectively, the justification for calling each value $u_C(n)$ of the recursively defined function u_C an initial segment of C up to n, as expressed by the notation $\{0_C, \ldots, n\}$.

Theorem 4.34 (Characterization of initial segments). *For any counting domain $(C, s, 0_C)$ it is true that the initial segment of C up to an $n \in C$ contains precisely all elements of C which are less than or equal to n, i.e.*

$$\forall m, n\,(m, n \in C \Rightarrow [m \in \{0_C, \ldots, n\} \Leftrightarrow m \leq_C n]). \tag{4.175}$$

Moreover, it is true for any $n \in C$ that 0_C is the least element and n the greatest element of the initial segment of C up to n, that is,

$$\min \{0_C, \ldots, n\} = 0_C, \tag{4.176}$$

$$\max \{0_C, \ldots, n\} = n. \tag{4.177}$$

Proof. Letting C, s and 0_C be arbitrary such that $(C, s, 0_C)$ is a counting domain and letting $m, n \in C$ also be arbitrary, we now prove the first part ('$\Rightarrow$') of the equivalence by contradiction, assuming both $m \in \{0_C, \ldots, n\}$ and $\neg m \leq_C n$ to be true. The latter implies $m >_C n$ with the Negation Formula for $\leq$, so that $m = n$ and $m <_C n$ are both false according to the Characterization of comparability (since $<_C$ is a linear ordering). Thus, $m \neq n$ and the assumed $m \in \{0_C, \ldots, n\}$ are both true, so that $s(m) \in \{0_C, \ldots, n\}$ follows to be true with (4.100). Now, since the previously established $m >_C n$ yields $n <_C m$, the disjunction $n <_C m \vee n = m$ is then also true, so that $n \leq_C m$ holds in view of Theorem 3.86b). Consequently, we obtain $\{0_C, \ldots, n\} \subseteq \{0_C, \ldots, m\}$ with (4.141). With this inclusion, the previously established $s(m) \in \{0_C, \ldots, n\}$ implies $s(m) \in \{0_C, \ldots, m\}$ (using the definition of a subset), which contradicts (4.99).

To prove the second part ('$\Leftarrow$') of the proposed equivalence, we assume $m \leq_C n$, which yields $\{0_C, \ldots, m\} \subseteq \{0_C, \ldots, n\}$ with (4.141). Since $m \in \{0_C, \ldots, m\}$ holds according to Proposition 4.13, it follows with the preceding inclusion (with the definition of a subset) that $m \in \{0_C, \ldots, n\}$. This completes the proof of the equivalence, which in turn proves the stated implication. As m and n are arbitrary, we may therefore conclude that the universal sentence (4.175) is true.

Next, we let $n \in C$ be arbitrary and show first that 0_C is a lower bound for $\{0_C, \ldots, n\}$. To do this, we verify sentences

$$\forall m\, (m \in \{0_C, \ldots, n\} \Rightarrow 0_C \leq_C m), \tag{4.178}$$

letting m be arbitrary and assuming $m \in \{0_C, \ldots, n\}$ $[= u_C(n)]$ to be true. Observing now that $u_C(n)$ is by Definition 4.2 of the function $u_C : C \to \mathcal{P}(C)$ and by the Function Criterion a (unique) value of the codomain $\mathcal{P}(C)$ and therefore a subset of C (by definition of a power set), we see that the preceding assumption $m \in u_C(n)$ implies $m \in C$ with the definition of a subset. Consequently, $u_C(m) = \{0_C, \ldots, m\}$ is also a uniquely specified value of the function u_C. It then follows with (4.95) that $0_C \in \{0_C, \ldots, m\}$ holds, which in turn implies the first desired consequent $0_C \leq_C m$ with the Characterization of initial segments. As m is arbitrary, we may therefore conclude that the universal sentence (4.178) holds, so that 0_C is a lower bound for $\{0_C, \ldots, n\}$. Since this lower bound 0_C is an element of that initial segment according to (4.95), we have that 0_C is the minimum element of $\{0_C, \ldots, n\}$, proving (4.176).

Similarly, we may show that n is an upper bound for $\{0_C, \ldots, n\}$, satisfying thusly

$$\forall m\, (m \in \{0_C, \ldots, n\} \Rightarrow m \leq_C n). \tag{4.179}$$

Letting $m \in \{0_C, \ldots, n\}$ be arbitrary, it follows with (4.175) that $m \leq_C n$ holds, which finding proves already the implication in (4.179). As m was arbitrary, we may therefore conclude that (4.179) is true, which means that n is an upper bound for the initial segment $\{0_C, \ldots, n\}$. Moreover, since $n \in \{0_C, \ldots, n\}$ holds with (4.93), the upper bound n is the maximum element of that initial segment, which proves (4.177). Because n, C, s, and 0_C were arbitrary, we may infer from these findings the truth of the stated theorem. □

Corollary 4.35. *The initial segment of* $\mathbb{N}$ *up to a natural number* n *(in other words, the successor of a natural number* n*) consists of all natural numbers that are less than or equal to* n*, that is,*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \in \{0, \ldots, n\} \Leftrightarrow m \leq_{\mathbb{N}} n]), \tag{4.180}$$

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \in n^+ \Leftrightarrow m \leq_{\mathbb{N}} n]). \tag{4.181}$$

Moreover, it is true for any natural number n *that all elements of the initial segment of* $\mathbb{N}$ *up to* n *(in other words, that all elements of the successor of* n*) are between* 0 *and* n*, where* 0 *is the least and* n *the greatest element of that initial segment (successor), that is,*

$$\min\{0, \ldots, n\} = \min n^+ = 0, \tag{4.182}$$

$$\max\{0, \ldots, n\} = \max n^+ = n. \tag{4.183}$$

Proposition 4.36. *The following sentences are true for any counting domain* $(C, s, 0_C)$*.*

a) The beginning 0_C *is a lower bound for* C*, i.e.*

$$\forall n\,(n \in C \Rightarrow 0_C \leq_C n), \tag{4.184}$$

and 0_C *is also the least element of* C*.*

b) For any $n \in C$ *it is false that* n *is less than the beginning* 0_C*, that is,*

$$\forall n\,(n \in C \Rightarrow \neg n <_C 0_C). \tag{4.185}$$

c) There is no element of C *which is less than the initial element, i.e.*

$$\neg\exists n\,(n \in C \wedge n <_C 0_C). \tag{4.186}$$

Proof. Letting C, s and 0_C be arbitrary such that $(C, s, 0_C)$ is a counting domain and letting n be an arbitrary element of C, the inequality $0_C \leq_C n$ follows with (4.95) and the Characterization of initial segments. As n is arbitrary, it then follows that (4.184) holds, so that 0_C is a lower bound for C by definition. Furthermore, since 0_C is an element of C by definition of a counting domain, the lower bound 0_C for C is the least element of C by definition. Thus, a) is true.

Letting now $n \in C$ be arbitrary, we obtain with a) the inequality $0_C \leq_C n$, which implies $\neg n <_C 0_C$ with the Negation Formula for $<$ (recalling that $<_C$ is a linear ordering). Since n is arbitrary, we may therefore conclude that the universal sentence b) is true.

Finally, we observe that b) implies c) with the Negation Law for existential conjunctions. Since C, s and 0_C were initially arbitrary, we may then conclude that the proposition is true. □

Corollary 4.37. *The following sentences are true for* $(\mathbb{N}, s^+, 0)$*.*

a) 0 *is a lower bound for* $\mathbb{N}$*, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow 0 \leq_{\mathbb{N}} n), \tag{4.187}$$

and 0 *is also the least element of* $\mathbb{N}$*.*

b) For any natural number n *it is false that* n *is less than* 0*, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow \neg n <_{\mathbb{N}} 0). \tag{4.188}$$

c) There is no natural number which is less than 0*, that is,*

$$\neg\exists n\,(n \in \mathbb{N} \wedge n <_{\mathbb{N}} 0). \tag{4.189}$$

The particular specification of the set of natural numbers admits the following characterization of its linear ordering, alternatively to the characterization in terms of initial segments.

Lemma 4.38 (Characterization of $<_{\mathbb{N}}$ by $\in_{\mathbb{N}}$). *A natural number* m *is less than a natural number* n *iff* m *is element of* n*, that is,*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m <_{\mathbb{N}} n \Leftrightarrow m \in n]). \tag{4.190}$$

Proof. Let us first rewrite (4.190) equivalently as

$$\forall m\,(m \in \mathbb{N} \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow [m <_{\mathbb{N}} n \Leftrightarrow m \in n])) \tag{4.191}$$

by applying (1.90). We now let m be arbitrary in $\mathbb{N}$ and carry out a proof by mathematical induction with respect to n. Concerning the base case $(n = 0)$, we prove $m <_{\mathbb{N}} 0 \Leftrightarrow m \in 0$, which is by definition of the zero 0 equivalent to

$$m <_{\mathbb{N}} 0 \Leftrightarrow m \in \emptyset.$$

Here, the first part $m <_{\mathbb{N}} 0 \Rightarrow m \in \emptyset$ of the equivalence is true since $m <_{\mathbb{N}} 0$ is false due to (4.188); the second part $m \in \emptyset \Rightarrow m <_{\mathbb{N}} 0$ of the equivalence is also true because $m \in \emptyset$ is false by definition of the empty set. Concerning the induction step, we let n be arbitrary in $\mathbb{N}$, make the induction assumption $m <_{\mathbb{N}} n \Leftrightarrow m \in n$, and show that this implies

$$m <_{\mathbb{N}} n^+ \Leftrightarrow m \in n^+. \tag{4.192}$$

To prove the first part ('$\Rightarrow$') of this equivalence, we assume $m <_{\mathbb{N}} n^+$, which implies $m^+ \leq_{\mathbb{N}} n^+$ with (4.157). The latter further implies $m \leq_{\mathbb{N}} n$ with (4.159), which inequality then gives the desired $m \in n^+$ with (4.181).

To prove the second part ('$\Leftarrow$') of the equivalence (4.192), we now assume $m \in n^+$ to be true, which implies $m \leq_{\mathbb{N}} n$ again with (4.181). Let us now recall that $n <_{\mathbb{N}} n^+$ holds due to (4.153). Then, the conjunction of $m \leq_{\mathbb{N}} n$ and $n <_{\mathbb{N}} n^+$ implies $m <_{\mathbb{N}} n^+$ with the Transitivity Formula for $\leq$ and $<$, which completes the proof of the equivalence (note that we did not have to use the induction assumption). As n is arbitrary, we therefore conclude that the induction step is true, so that the proof by mathematical induction of the universal sentence with respect to n in (4.191) is complete. Since m was also arbitrary, it then follows that the lemma (4.190) holds. □

Theorem 4.39. *The binary relation of belonging* $\in_{\mathbb{N}}$ *is a linear ordering of* $\mathbb{N}$.

Proof. We already established $\in_{\mathbb{N}}$ as an irreflexive partial ordering of $\mathbb{N}$ (see Corollary 4.5). It therefore remains to show that $\in_{\mathbb{N}}$ is connex, i.e. that $\in_{\mathbb{N}}$ satisfies

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \in n \,\vee\, n \in m \,\vee\, m = n]). \tag{4.193}$$

For this purpose, we let $m, n \in \mathbb{N}$ be arbitrary and observe that the linear ordering $<_{\mathbb{N}}$ is connex, so that

$$m <_{\mathbb{N}} n \,\vee\, n <_{\mathbb{N}} m \,\vee\, m = n \tag{4.194}$$

holds. Now, in light of (4.190), we see that $m <_{\mathbb{N}} n$ is equivalent to $m \in n$ and that $n <_{\mathbb{N}} m$ is equivalent to $n \in m$. Therefore, the true multiple disjunction (4.194) implies the truth of the equivalent multiple disjunction in (4.193). As m and n are arbitrary, we may therefore conclude that (4.193) is true, so that the irreflexive partial ordering $\in_{\mathbb{N}}$ is indeed connex, and thus by definition a linear ordering of $\mathbb{N}$. □

We already explored the presence of extremal elements of a counting set and of an initial segment. Our next task is to extend these investigations to more general subsets of a counting set.

Proposition 4.40. *The following sentences are true for any counting domain* $(C, s, 0_C)$.

a) Every nonempty subset of the initial segment of C *up to an* $n \in C$ *has a greatest element, that is,*

$$\forall n\,(n \in C \Rightarrow \forall A\,([A \subseteq \{0_C, \ldots, n\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \max A))). \tag{4.195}$$

b) Any nonempty and bounded-from-above subset of C *has a greatest element, that is,*

$$\begin{aligned}&\forall A\,([A \subseteq C \,\wedge\, A \neq \emptyset \,\wedge\, \exists u\,(u \in C \,\wedge\, \forall k\,(k \in A \Rightarrow k \leq_C u)] \\ &\quad \Rightarrow \exists m\,(m = \max A))).\end{aligned} \tag{4.196}$$

Proof. We let C, s and 0_C be arbitrary and assume $(C, s, 0_C)$ to be a counting domain. Concerning a), we carry out a proof by mathematical induction. Regarding the base case ($n = 0_C$), we prove

$$\forall A\,([A \subseteq \{0_C, \ldots, 0_C\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \max A)). \tag{4.197}$$

Letting A be arbitrary and assuming both $A \subseteq \{0_C, \ldots, 0_C\}$ and $A \neq \emptyset$ to be true, we obtain the equations

$$\{0_C, \ldots, 0_C\} = u_C(0_C) = \{0_C\}$$

by using the definition of an initial segment, so that the assumed $A \subseteq \{0_C, \ldots, 0_C\}$ gives $A \subseteq \{0_C\}$ via substitution. The disjunction of the latter inclusion and the assumed $A \neq \emptyset$ further implies $A = \{0_C\}$ with (2.186). Since $\max\{0_C\} = 0_C$ holds according to Corollary 3.96, it follows with $A = \{0_C\}$ that $\max A = 0_C$, which evidently proves the existential sentence in (4.197), and thus the implication in (4.197). Because A is arbitrary, we may therefore conclude that the universal sentence (4.197) holds, completing the proof of the base case, holds.

Regarding the induction step, we let $n \in C$ be arbitrary, make the induction assumption

$$\forall A\,([A \subseteq \{0_C, \ldots, n\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \max A)), \tag{4.198}$$

and prove

$$\forall A\,([A \subseteq \{0_C, \ldots, s(n)\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \max A)). \tag{4.199}$$

To do this, we let A be arbitrary and prove the implication by cases. In the first case, we assume $A \subseteq \{0_C, \ldots, s(n)\}$ and $A \neq \emptyset$ as well as $s(n) \in A$, and we prove the consequent (i.e. the existential sentence) in (4.199) by demonstrating that $s(n) = \max A$ holds. For this purpose, we first verify that $s(n)$ is an upper bound for A, that is,

$$\forall m\,(m \in A \Rightarrow m \leq_C s(n)). \tag{4.200}$$

Letting m be arbitrary in A, it follows with the assumed inclusion $A \subseteq \{0_C, \ldots, s(n)\}$ and the definition of a subset that $m \in \{0_C, \ldots, s(n)\}$. Since $s(n) = \max\{0_C, \ldots, s(n)\}$ holds with (4.177) and the definition of an initial segment, we have in particular that $m \leq_C s(n)$, because the maximum element $s(n)$ is an upper bound for $\{0_C, \ldots, s(n)\}$. As m is arbitrary, we may therefore conclude that (4.200) is true, which then gives together with the case assumption $s(n) \in A$ with the definition of a greatest element the desired $s(n) = \max A$.

In the second case, we assume the conjunction of $A \subseteq \{0_C, \ldots, s(n)\}$, $A \neq \emptyset$ and $s(n) \notin A$. Here, the conjunction of the first and the last part implies $A \subseteq \{0_C, \ldots, s(n)\} \setminus \{s(n)\}$ with (2.188), and therefore $A \subseteq \{0_C, \ldots, n\}$ with (4.134). The conjunction of this and the assumption $A \neq \emptyset$ then implies $\exists m\,(m = \max A)$ with the induction assumption (4.198), which completes the proof of the implication in (4.199) by cases. Since A is arbitrary, we may therefore conclude that (4.199) is true; as n was also arbitrary, we may then further conclude that the induction step holds, which completes the proof of (4.195) via mathematical induction.

Concerning b), we let A be arbitrary and prove the implication directly, assuming that A is a nonempty subset of C and that there exists an upper bound for A in C, say $\bar{u}$. Let us first prove that $A \subseteq \{0_C, \ldots, \bar{u}\}$ holds, i.e. (using the definition of a subset)

$$\forall k\,(k \in A \Rightarrow k \in \{0_C, \ldots, \bar{u}\}), \tag{4.201}$$

which will – together with the assumption $A \neq \emptyset$ – imply the desired consequent. We let k be arbitrary and prove the implication in (4.201) directly, assuming that $k \in A$ holds, which implies on the one hand $k \in C$ with the initial assumption $A \subseteq C$ (by definition of a subset). On the other hand, $k \in A$ implies $k \leq_C \bar{u}$ with the assumption that $\bar{u}$ is an upper bound for A. It then follows with the Characterization of initial segments that $k \in \{0_C, \ldots, \bar{u}\}$, which proves the implication in (4.201). Since k is arbitrary, we therefore conclude that the universal sentence (4.201) holds, which means $A \subseteq \{0_C, \ldots, \bar{u}\}$. Then, the conjunction of this and the initial assumption $A \neq \emptyset$ implies with (4.195) the existential sentence $\exists m\,(m = \max A)$, which in turn proves the implication in (4.196). Since A was arbitrary, we may then conclude that (4.196) holds.

As C, s and 0_C were initially arbitrary in the proofs of a) and b), it finally follows that the proposition is true. □

Corollary 4.41. *The following sentences are true for the counting domain* $(\mathbb{N}, s^+, 0)$.

a) Every nonempty subset of the initial segment of $\mathbb{N}$ *up to a natural number* n *has a greatest element, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow \forall A\,([A \subseteq \{0, \ldots, n\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \max A))). \tag{4.202}$$

b) Any nonempty and bounded-from-above subset of $\mathbb{N}$ *has a greatest element, that is,*

$$\begin{aligned}&\forall A\,([A \subseteq \mathbb{N} \,\wedge\, A \neq \emptyset \,\wedge\, \exists u\,(u \in \mathbb{N} \,\wedge\, \forall k\,(k \in A \Rightarrow k \leq_{\mathbb{N}} u)]\\ &\quad \Rightarrow \exists m\,(m = \max A))).\end{aligned} \tag{4.203}$$

Since every counting set has a least element (viz. its beginning), the conditions for the existence of least elements for subsets of a counting set are less restrictive than those for the existence of greatest elements.

Proposition 4.42. *The following sentences are true for any counting domain* $(C, s, 0_C)$.

a) Every nonempty subset of the initial segment of C *up to an* $n \in C$ *has a least element, that is,*

$$\forall n\,(n \in C \Rightarrow \forall A\,([A \subseteq \{0_C, \ldots, n\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \min A))). \tag{4.204}$$

b) Any nonempty subset of C *has a least element, that is,*

$$\forall A\,([A \subseteq C \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \min A))). \tag{4.205}$$

Proof. We let C, s and 0_C be arbitrary such that $(C, s, 0_C)$ is a counting domain. Concerning a), we carry out a proof by mathematical induction. Regarding the base case ($n = 0_C$), we let A be an arbitrary nonempty subset of $\{0_C, \ldots, 0_C\} = u_C(0_C) = \{0_C\}$. Then, the conjunction of $A \subseteq \{0_C\}$ and $A \neq \emptyset$ implies $A = \{0_C\}$ with (2.186). As $\min\{0_C\} = 0_C$ is true due to Corollary 3.96, we have that $\min A = 0_C$, which proves the existential sentence and thus the inner implication in (4.204) for $n = 0_C$. As A is arbitrary, we therefore conclude that the base case holds. Regarding the induction step, we let n be arbitrary in C, make the induction assumption

$$\forall A\,([A \subseteq \{0_C, \ldots, n\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \min A)), \tag{4.206}$$

and prove

$$\forall A\,([A \subseteq \{0_C, \ldots, s(n)\} \wedge A \neq \emptyset] \Rightarrow \exists m\,(m = \min A)). \tag{4.207}$$

For this purpose, we let A be arbitrary and prove the implication by cases, considering the two exhaustive cases $A \cap \{0_C, \ldots, n\} = \emptyset$ and $A \cap \{0_C, \ldots, n\} \neq \emptyset$. In the first case, we assume

$$A \subseteq \{0_C, \ldots, s(n)\} \wedge A \neq \emptyset \wedge A \cap \{0_C, \ldots, n\} = \emptyset$$

and show first that this implies $A \subseteq \{s(n)\}$, i.e. (using the definition of a subset)

$$\forall y\,(y \in A \Rightarrow y \in \{s(n)\}). \tag{4.208}$$

To do this, we let $\bar{y}$ be arbitrary and assume $\bar{y} \in A$, which implies with the assumption $A \subseteq \{0_C, \ldots, s(n)\}$ (and the definition of a subset) that $\bar{y} \in \{0_C, \ldots, s(n)\}$, and therefore with (4.86) $\bar{y} \in \{0_C, \ldots, n\} \cup \{s(n)\}$, thus (by definition of the union of two sets)

$$\bar{y} \in \{0_C, \ldots, n\} \vee \bar{y} \in \{s(n)\}. \tag{4.209}$$

Let us now observe that the case assumption $A \cap \{0_C, \ldots, n\} = \emptyset$ means (by definition of the empty set)

$$\forall y\,(y \notin A \cap \{0_C, \ldots, n\}),$$

so that we obtain in particular $\neg \bar{y} \in A \cap \{0_C, \ldots, n\}$ and therefore (applying the definition of the intersection of two sets and De Morgan's Law for sets (2.259))

$$\bar{y} \notin A \vee \bar{y} \notin \{0_C, \ldots, n\}.$$

Now, as $\bar{y} \in A$ is true by assumption, $\bar{y} \notin A$ is false and consequently the second part $\bar{y} \notin \{0_C, \ldots, n\}$ of the preceding disjunction is true; thus, the first part of the disjunction (4.209) is false, so that the second part $\bar{y} \in \{s(n)\}$ of that true disjunction is true. Since $\bar{y}$ is arbitrary, we may therefore conclude that the universal sentence (4.208) holds, which means that $A \subseteq \{s(n)\}$ is indeed true. Now, the conjunction of this inclusion and the assumed $A \neq \emptyset$ implies $A = \{s(n)\}$ with (2.186). It then follows with Corollary 3.96 that $s(n)$ is the least element of A, which proves the existential sentence in (4.207).

We now prove the implication in (4.207) for the second case, assuming accordingly

$$A \subseteq \{0_C, \ldots, s(n)\} \wedge A \neq \emptyset \wedge A \cap \{0_C, \ldots, n\} \neq \emptyset.$$

Applying (2.74), we have

$$(\emptyset \neq)\ A \cap \{0_C, \ldots, n\} \subseteq \{0_C, \ldots, n\},$$

which implies with the induction assumption (4.206) that the least element $\bar{m}$ of $A \cap \{0_C, \ldots, n\}$ exists. By definition of a least element, we thus have that $\bar{m}$ is element of that intersection, so that $\bar{m} \in A$ and $\bar{m} \in \{0_C, \ldots, n\}$ are both true according to the definition of the intersection of two sets. Let us now verify that $\bar{m}$ is also a lower bound for A, that is,

$$\forall y\, (y \in A \Rightarrow \bar{m} \leq_C y). \tag{4.210}$$

Letting $\bar{y}$ be arbitrary and assuming $\bar{y} \in A$ to be true, we may apply the same arguments as in the first case to infer from this in view of the assumption $A \subseteq \{0_C, \ldots, s(n)\}$ the truth of the disjunction (4.209) for the current constant $\bar{y}$. We may now use this true disjunction to prove the sentence $\bar{m} \leq_C \bar{y}$. Firstly, if $\bar{y} \in \{0_C, \ldots, n\}$ is true, then this implies together with $\bar{y} \in A$ (with the definition of the intersection of two sets) that $\bar{y} \in A \cap \{0_C, \ldots, n\}$ holds, which gives $\bar{m} \leq_C \bar{y}$ since $\bar{m}$ is the least element of and thus a lower bound for $A \cap \{0_C, \ldots, n\}$. Secondly, if $\bar{y} \in \{s(n)\}$ is true, which implies $\bar{y} = s(n)$ with (2.169), we may establish the inequalities

$$\bar{m} \leq_C n <_C \bar{y} \quad [= s(n)]. \tag{4.211}$$

Here, the first inequality results from the fact that $\bar{m} \in A \cap \{0_C, \ldots, n\}$ implies in particular $\bar{m} \in \{0_C, \ldots, n\}$ (by definition of the intersection of two sets) and then $\bar{m} \leq_C n$ with the Characterization of initial segments. The second inequality is true since $n <_C s(n)$ holds according to Theorem 4.19b). The inequalities (4.211) further imply $\bar{m} <_C \bar{y}$ with the Transitivity Formula for $\leq$ and $<$, and the disjunction $\bar{m} <_C \bar{y} \vee \bar{m} = \bar{y}$ is then also true, so that we obtain $\bar{m} \leq_C \bar{y}$ with (3.253). Thus, the proof by cases of the preceding inequality is complete, which in turn proves the implication in (4.210). Since $\bar{y}$ was arbitrary, we may therefore conclude that the universal sentence (4.210) holds, which means that $\bar{m}$ is indeed a lower bound for A. We already showed that $\bar{m}$ is an element of A, so that the lower bound $\bar{m}$ is the least element of A. Thus, the existential sentence in (4.207) is true also in the second case.

As A was arbitrary, we may therefore conclude that (4.207) is true, and since n was also arbitrary, we may now further conclude that the induction step holds. Thus, the proof by mathematical induction of (4.204) is complete.

Concerning b), we let A be arbitrary such that $A \subseteq C$ and $A \neq \emptyset$ both hold. The latter implies with (2.42) that there is an element in A, say $\bar{n}$.

Applying the definition of a subset to the assumption $A \subseteq C$, we therefore obtain $\bar{n} \in C$ and then also $\bar{n} \in \{0_C, \ldots, \bar{n}\}$ with Proposition 4.13. The conjunction of this and the previously established $\bar{n} \in A$ further implies $\bar{n} \in A \cap \{0_C, \ldots, \bar{n}\}$ by definition of the intersection of two sets. Thus, the set $A \cap \{0_C, \ldots, \bar{n}\}$ is evidently nonempty and furthermore the inclusion $A \cap \{0_C, \ldots, \bar{n}\} \subseteq \{0_C, \ldots, \bar{n}\}$ holds due to (2.74). It then follows with a) from these findings that the least element $\bar{m}$ of $A \cap \{0_C, \ldots, \bar{n}\}$ exists. Let us observe here that the least element $\bar{m}$ is by definition an element of the preceding intersection and is therefore both in A and in $\{0_C, \ldots, \bar{n}\}$. To show that $\bar{m}$ is also the least element of A, it therefore suffices (in view of $\bar{m} \in A$) to show that $\bar{m}$ is a lower bound for A, i.e.

$$\forall y \, (y \in A \Rightarrow \bar{m} \leq_C y). \tag{4.212}$$

For this purpose, we let y be arbitrary and assume $y \in A$ to be true. This implies with the assumption $A \subseteq C$ (and the definition of a subset) that $y \in C$ holds. Since $\leq_C$ is total, the disjunction $y \leq_C \bar{n} \vee \bar{n} \leq_C y$ is true, which we now use to prove the sentence $\bar{m} \leq_C y$ by cases. On the one hand, if $y \leq_C \bar{n}$ is true, then $y \in \{0, \ldots, \bar{n}\}$ holds due to the Characterization of initial segments. The conjunction of this and $y \in A$ then implies $y \in A \cap \{0, \ldots, \bar{n}\}$ (by definition of the intersection of two sets), and therefore $\bar{m} \leq_C y$ (since $\bar{m}$ is the least element of $A \cap \{0, \ldots, \bar{n}\}$). On the other hand, if $\bar{n} \leq_C y$ is true, then we may combine this inequality with the inequality $\bar{m} \leq_C \bar{n}$, which we obtain from the previous observation $\bar{m} \in \{0_C, \ldots, \bar{n}\}$ with the Characterization of initial segments, to form $\bar{m} \leq_C \bar{n} \leq_C y$. This conjunction implies now the desired $\bar{m} \leq_C y$ with the transitivity of $\leq_C$, so that the proof of this inequality by cases is complete. Since y is arbitrary, we may therefore conclude that (4.212) is true, which shows that $\bar{m}$ is indeed a lower bound for A. Since $\bar{m}$ is in A, the lower bound $\bar{m}$ is the least element of A. Thus, the existential sentence in (4.205) is evidently true, and since A was arbitrary, we may infer from this the truth of the universal sentence (4.205).

As C, s and 0_C were arbitrary in the proofs of a) and b), we may finally conclude that the proposition holds, as claimed. □

Corollary 4.43. *For any counting domain $(C, s, 0_C)$ it is true that $(C, \leq_C)$ is a well ordered set.*

Proof. Letting $(C, s, 0_C)$ be an arbitrary counting domain, we have that $(C, \leq_C)$ is a totally ordered set. Together with Proposition 4.42b), this means that $(C, \leq_C)$ is by definition a well ordered set. This is then true for any counting domain. □

The preceding proposition and corollary are immediately applicable, respectively, to the counting domain and to the lattice based on the set of natural numbers.

Corollary 4.44. *The following sentences are true for the counting domain* $(\mathbb{N}, s^+, 0)$.

a) Every nonempty subset of the initial segment of $\mathbb{N}$ *up to a natural number* n *has a least element, that is,*

$$\forall n \, (n \in \mathbb{N} \Rightarrow \forall A \, ([A \subseteq \{0, \ldots, n\} \, \wedge \, A \neq \emptyset] \Rightarrow \exists m \, (m = \min A))). \tag{4.213}$$

b) Any nonempty subset of $\mathbb{N}$ *has a least element, that is,*

$$\forall A \, ([A \subseteq \mathbb{N} \, \wedge \, A \neq \emptyset] \Rightarrow \exists m \, (m = \min A))). \tag{4.214}$$

Corollary 4.45. *The lattice* $(\mathbb{N}, \leq_{\mathbb{N}})$ *is well-ordered.*

4.4. The Counting Domain $(\mathbb{N}_+, s^+ \upharpoonright \mathbb{N}_+, 1)$ and the Linearly Ordered Set $(\mathbb{N}_+, <_{\mathbb{N}_+})$

Given any counting domain, we may construct a new one by removing any initial segment from the given counting set, by letting the new beginning be the successor of the greatest element from the removed initial segment, and by restricting the given successor function to the new counting set.

Theorem 4.46 (Generation of counting domains from restricted counting sets). *For any counting domain $(C, s, 0_C)$ and any $m \in C$ the ordered triple*

$$(C \setminus \{0_C, \ldots, m\}, s \upharpoonright C \setminus \{0_C, \ldots, m\}, s(m)) \tag{4.215}$$

is also a counting domain.

Proof. Letting $(C, s, 0_C)$ be an arbitrary counting domain and m an arbitrary element of C we now verify that $(\bar{C}, s \upharpoonright \bar{C}, s(m))$ with $\bar{C} = C \setminus \{0_C, \ldots, m\}$ satisfies the Properties 1 – 3 of a counting domain.

Regarding Property 1, we demonstrate the truth of

$$\forall n \, (n \in \bar{C} \Rightarrow s \upharpoonright \bar{C}(n) \neq s(m)). \tag{4.216}$$

For this purpose, we let n be arbitrary in $\bar{C} = C \setminus \{0_C, \ldots, m\}$, so that $n \in C$ and $\neg n \in \{0_C, \ldots, m\}$ holds by definition of a set difference. Since $n \in \{0_C, \ldots, m\}$ is equivalent to $n \leq_C m$ in view of the Characterization of initial segments, it follows with the Substitution Rule for negations from the preceding negation that $\neg n \leq_C m$ is true, which inequality in turn implies $m <_C n$ with the Negation Formula for $\leq$. We therefore obtain $s(m) \leq_C n$ by applying (4.156), and $n <_C s(n)$ also holds according to (4.142). Now, the conjunction of $s(m) \leq_C n$ and $n <_C s(n)$ implies $s(m) <_C s(n)$ with the Transitivity Formula for $\leq$ and $<$, and therefore $s(m) \neq s(n)$ because of the Characterization of comparability in connection with the linear ordering $<_C$. Furthermore, since $\bar{C} = C \setminus \{0_C, \ldots, m\}$ implies $\bar{C} \subseteq C$ with (2.125), it follows from $n \in \bar{C}$ in connection with the definition of the successor function that the equation $s \upharpoonright \bar{C}(n) = s(n)$ holds because of Corollary 3.165. With this, the previously established inequality $s(m) \neq s(n)$ yields $s(m) \neq s \upharpoonright \bar{C}(n)$ via substitution, proving the implication in (4.216). As n was arbitrary, we may therefore conclude that (4.216) is true, which means that $(\bar{C}, s \upharpoonright \bar{C}, s(m))$ satisfies Property 1 of a counting domain.

Regarding Property 2, we prove accordingly

$$\forall m, n \, ([m, n \in \bar{C} \, \wedge \, s \upharpoonright \bar{C}(m) = s \upharpoonright \bar{C}(n)] \Rightarrow m = n). \tag{4.217}$$

To do this, we let m and n be arbitrary in $\bar{C}$ such that $s \restriction \bar{C}(m) = s \restriction \bar{C}(n)$ holds. Since the equations $s \restriction \bar{C}(m) = s(m)$ and $s \restriction \bar{C}(n) = s(n)$ are true (again by Corollary 3.165), we obtain after substitution $s(m) = s(n)$. As the successor function s of the counting domain $(C, s, 0_C)$ satisfies (4.2), it follows from the preceding equation that $m = n$. Because m and n are arbitrary, we may then conclude that $(\bar{C}, s \restriction \bar{C}, s(m))$ satisfies indeed (4.217), and thus Property 2.

Regarding Property 3, we verify

$$\forall M\, ([M \subseteq \bar{C} \,\wedge\, s(m) \in M \,\wedge\, \forall n\, (n \in M \Rightarrow s \restriction \bar{C}(n) \in M)] \Rightarrow M = \bar{C}). \tag{4.218}$$

We let M be an arbitrary subset of $\bar{C}$ such that $s(m) \in M$ and the universal sentence

$$\forall n\, (n \in M \Rightarrow s \restriction \bar{C}(n) \in M) \tag{4.219}$$

hold. The former assumption means $M \subseteq C \setminus \{0_C, \ldots, m\}$ (by definition of $\bar{C}$), which implies with (2.251)

$$M \cup \{0_C, \ldots, m\} \;\subseteq\; (C \setminus \{0_C, \ldots, m\}) \cup \{0_C, \ldots, m\}. \tag{4.220}$$

As the initial segment $\{0_C, \ldots, m\}$ is a subset of C, we may apply (2.263) to obtain the equation

$$(C \setminus \{0_C, \ldots, m\}) \cup \{0_C, \ldots, m\} = C,$$

so that (4.220) becomes

$$M \cup \{0_C, \ldots, m\} \subseteq C. \tag{4.221}$$

Then,

$$0_C \in M \cup \{0_C, \ldots, m\} \tag{4.222}$$

holds by definition of the union of a pair. We now verify

$$\forall n\, (n \in M \cup \{0_C, \ldots, m\} \Rightarrow s(n) \in M \cup \{0_C, \ldots, m\}). \tag{4.223}$$

For this purpose, we let n be arbitrary in $M \cup \{0_C, \ldots, m\}$, so that the disjunction

$$n \in M \,\vee\, n \in \{0_C, \ldots, m\}$$

follows to be true (by definition of the union of two sets), which we now use to prove the sentence $s(n) \in M \cup \{0_C, \ldots, m\}$ by cases.

On the one hand, if $n \in M$ is true, then $s \restriction \bar{C}(n) \in M$ also holds due to (4.219), and therefore $s(n) \in M$ with $s \restriction \bar{C}(n) = s(n)$ (applying Corollary 3.165). Then, $s(n) \in M$ implies that the disjunction

$$s(n) \in M \,\vee\, s(n) \in \{0_C, \ldots, m\} \tag{4.224}$$

is true (no matter if $s(n) \in \{0_C, \ldots, m\}$ is true or false), and this disjunction gives (by definition of the union of two sets) the desired consequent $s(n) \in M \cup \{0_C, \ldots, m\}$ of the implication in (4.223).

On the other hand, if $n \in \{0_C, \ldots, m\}$ is true, then it follows with the Characterization of initial segments that $n \leq_C m$ holds, which yields $n <_C m \vee n = m$ since the induced linear ordering $<_C$ satisfies (3.253). We now carry out a proof of the sentence (4.224) by cases based on the preceding true disjunction. In case $n <_C m$ holds, we obtain the inequality $s(n) \leq_C m$ with (4.156), and therefore $s(n) \in \{0_C, \ldots, m\}$ with the Characterization of initial segments, so that the disjunction (4.224) holds (irrespective of the truth value of $s(n) \in M$). In the other case that $n = m$ is true, we obtain $s(n) = s(m)$ because s – being a function – satisfies Corollary 3.150. With the latter equation, the initial assumption $s(m) \in M$ gives $s(n) \in M$ via substitution, so that the disjunction (4.224) is again true. Thus, the proof of this disjunction by cases is complete, so that $s(n) \in M \cup \{0_C, \ldots, m\}$ follows to be true. This completes the proof by cases of this sentence, and this completes also the proof of the implication in (4.223). Since n is arbitrary, we may therefore conclude that the universal sentence (4.223) is true. Now, the conjunction of (4.221), (4.222) and (4.223) implies

$$M \cup \{0_C, \ldots, m\} = C \tag{4.225}$$

with (4.3), using the fact that $(C, s, 0_C)$ satisfies Property 3 of a counting domain. Then, as the assumption $M \subseteq C \setminus \{0_C, \ldots, m\}$ implies $M \cap \{0_C, \ldots, m\} = \emptyset$ with (2.117), we may apply (2.262) to obtain from (4.225) the equation $M = C \setminus \{0_C, \ldots, m\}$, which gives $M = \bar{C}$. Because M was arbitrary, we may therefore conclude that the universal sentence (4.218) is true, so that $(\bar{C}, s \upharpoonright \bar{C}, s(m))$ satisfies also Property 3 of a counting domain. Since m, C, s and 0_C were initially arbitrary, we may finally conclude that the stated theorem is indeed true. □

The preceding theorem immediately yields the counting domain based on the set of positive natural numbers.

Corollary 4.47. *The ordered triple*

$$(\mathbb{N}_+, s^+ \upharpoonright \mathbb{N}_+, 1) \tag{4.226}$$

constitutes a counting domain.

Proof. Since $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ holds by definition of the set of positive natural numbers, where $\{0\}$ is by definition the initial segment of $\mathbb{N}$ up to 0, the corollary holds in view of $s^+(0) = 0^+ = 1$ and Theorem 4.46. □

Note 4.10. In the proof of Theorem 4.46, we used the fact that $s \upharpoonright \bar{C}(n) = s(n)$ holds for an arbitrary $n \in \bar{C}$, where s is the successor function with domain C. In the particular instance of $C = \mathbb{N}$, we have the successor function s^+, whose restriction to $\bar{C} = \mathbb{N}_+$ then yields, according to the preceding equation, the values

$$s^+ \upharpoonright \mathbb{N}_+(n) = s^+(n) = n^+. \tag{4.227}$$

Thus, we may use the notation n^+ both for the successor of a natural number n and for the successor of a positive natural number n.

Exercise 4.9. Verify that the ordered triple

$$(\mathbb{N}_+ \setminus \{1\}, s^+ \upharpoonright \mathbb{N}_+ \setminus \{1\}, 2) \tag{4.228}$$

is also a counting domain.

In view of the preceding notation for the successor of an element of the counting set $\mathbb{N}_+$, the following three results immediately follow by applying Theorem 4.1, Proposition 4.2 and 4.3 to the counting domain $(\mathbb{N}_+, s^+ \upharpoonright \mathbb{N}_+, 1)$.

Corollary 4.48. *A positive natural number is* 1 *or the successor of another positive natural number, that is,*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow [n = 1 \lor \exists m\,(m \in \mathbb{N}_+ \land m \neq n \land m^+ = n)]). \tag{4.229}$$

Corollary 4.49. *The successor function of the particular counting domain* $(\mathbb{N}_+, s^+ \upharpoonright \mathbb{N}_+, 1)$ *is a bijection from* $\mathbb{N}_+$ *to* $\mathbb{N}_+ \setminus \{1\}$*, that is,*

$$s^+ \upharpoonright \mathbb{N}_+ : \mathbb{N}_+ \rightleftarrows \mathbb{N}_+ \setminus \{1\}. \tag{4.230}$$

Corollary 4.50. *Any positive natural number is different from its successor, that is,*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow n^+ \neq n). \tag{4.231}$$

Because we established the proof method of mathematical induction for general counting domains, we now see that this method may not only be applied with respect to the counting set $\mathbb{N}$, but also to with respect to $\mathbb{N}_+$.

Method 4.4 (Proof by mathematical induction (for $\mathbb{N}_+$)). To prove that a given formula $\varphi(n)$ holds for all positive natural numbers, we may show that the base case

$$\varphi(1) \tag{4.232}$$

and the induction step

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow [\varphi(n) \Rightarrow \varphi(n^+)]) \tag{4.233}$$

are true.

Exercise 4.10. Establish a proof method based on mathematical induction for $\mathbb{N}_+ \setminus \{1\}$.

Next, we specialize Method 4.2 to $\mathbb{N}_+$.

Method 4.5 (Definition by recursion (for $\mathbb{N}_+$)). To define for an arbitrary set A, an arbitrary function $f : \mathbb{N}_+ \times A \to A$ and an arbitrary $a \in A$ the function $u : \mathbb{N}_+ \to A$ which satisfies (4.55) with respect to the particular counting domain $(\mathbb{N}_+, s^+ \restriction \mathbb{N}_+, 1)$, we will write

$$(1) \quad u_1 = a, \tag{4.234}$$

$$(2) \quad u_{n^+} = f(n, u_n) \quad \text{for any } n \in \mathbb{N}_+. \tag{4.235}$$

We now characterize initial segments of the counting set $\mathbb{N}_+$ according to the previous findings for a generic counting set.

Notation 4.4. In the particular instance of the counting domain $(\mathbb{N}_+, s^+ \restriction \mathbb{N}_+, 1)$, we obtain for the *initial segment of* $\mathbb{N}_+$ (up to n)

$$(1) \quad u_{\mathbb{N}_+}(1) = \{1\}, \tag{4.236}$$

$$(2) \quad u_{\mathbb{N}_+}(n^+) = u_{\mathbb{N}_+}(n) \cup \{n^+\} \quad \text{for any } n \in \mathbb{N}_+, \tag{4.237}$$

and we write for any $n \in \mathbb{N}_+$ (including $n = 1$)

$$\{1, \ldots, n\} = u_{\mathbb{N}_+}(n). \tag{4.238}$$

Furthermore, we write in case of $n = 0$ ($\notin \mathbb{N}_+$) also

$$\{1, \ldots, n\} = \{1, \ldots, 0\} = \emptyset. \tag{4.239}$$

Exercise 4.11. Show that the initial segment of $\mathbb{N}_+$ up to any natural number n is included both in the set of natural numbers and in the set of positive natural numbers, that is,

$$\forall n \, (n \in \mathbb{N} \Rightarrow [\{1, \ldots, n\} \subseteq \mathbb{N}_+ \, \wedge \, \{1, \ldots, n\} \subseteq \mathbb{N}]). \tag{4.240}$$

(Hint: Apply Method 1.9 based on (2.310).)

Proposition 4.51. *The initial segment of* $\mathbb{N}_+$ *up to the successor of a natural number* n *is identical with the union of the initial segment of* $\mathbb{N}_+$ *up to* n *and the singleton formed by that successor, that is,*

$$\forall n \, (n \in \mathbb{N} \Rightarrow \{1, \ldots, n^+\} = \{1, \ldots, n\} \cup \{n^+\}). \tag{4.241}$$

Proof. We let $n \in \mathbb{N}$ be arbitrary, so that the disjunction $n = 0 \lor n \in \mathbb{N}_+$ follows to be true with (2.310), which we now use to prove the stated equation by cases.

In case of $n = 0$, we obtain the equations

$$\begin{aligned}
\{1, \ldots, n^+\} &= \{1, \ldots, 0^+\} \\
&= \{1, \ldots, 1\} \\
&= \{1\} \\
&= \emptyset \cup \{1\} \\
&= \{1, \ldots, 0\} \cup \{0^+\} \\
&= \{1, \ldots, n\} \cup \{n^+\}
\end{aligned}$$

by applying substitution, (2.291), (4.236) together with (4.238), (2.216), (4.239) together with (2.291), and finally again substitution based on the case assumption. In the other case of $n \in \mathbb{N}_+$, we obtain the desired equation immediately with (4.237) and (4.238). Thus, the proof by cases is complete, and since n was initially arbitrary, we may therefore conclude that the universal sentence (4.241) holds, as claimed. □

Exercise 4.12. Show that the initial segment of $\mathbb{N}_+$ up to 2 is identical with the pair formed by 1 and 2, that is,

$$\{1, \ldots, 2\} = \{1, 2\}. \tag{4.242}$$

(Hint: Use (2.292), (4.241), Notation 4.4 and (2.226).)

Exercise 4.13. Verify the following sentence.

$$\forall n\, (n \in \mathbb{N} \Rightarrow \{1, \ldots, n\} \cap \{n^+\} = \emptyset). \tag{4.243}$$

(Hint: Apply Method 1.9 in connection with (2.310), and use (4.239), (2.62), as well as (4.133).)

We now establish two useful relationships between initial segments of $\mathbb{N}$ and of $\mathbb{N}_+$.

Proposition 4.52. *The following sentences are true.*

a) For any $n \in \mathbb{N}_+$ it is true that the initial segment of $\mathbb{N}_+$ up to n is identical with the initial segment of $\mathbb{N}$ up to n without 0, i.e.

$$\{1, \ldots, n\} = \{0, \ldots, n\} \setminus \{0\}. \tag{4.244}$$

b) *For any* $n \in \mathbb{N}$ *it is true that the initial segment of* $\mathbb{N}$ *up to* n *is identical with the initial segment of* $\mathbb{N}_+$ *up to* n *adjoined by* 0*, i.e.*

$$\{0, \ldots, n\} = \{1, \ldots, n\} \cup \{0\}. \tag{4.245}$$

Proof. Concerning a), we carry out a proof by mathematical induction according to Method 4.4. Regarding the base case ($n = 1$), we obtain the equations

$$\begin{aligned}\{1, \ldots, 1\} &= u_{\mathbb{N}_+}(1) = \{1\} = \{0, 1\} \setminus \{0\} = (\{0\} \cup \{1\}) \setminus \{0\} \\ &= (u_{\mathbb{N}}(0) \cup \{0^+\}) \setminus \{0\} = u_{\mathbb{N}}(1) \setminus \{0\} \\ &= \{0, \ldots, 1\} \setminus \{0\}\end{aligned}$$

using (4.238), (4.236), (2.178) in connection with (4.165), then (2.226), (4.87) together with (2.291), subsequently (4.88), and finally (4.92). Thus, the base case holds.

Regarding the induction step, we let n be arbitrary in $\mathbb{N}_+$ (so that $n \in \mathbb{N}$ follows to be true with (2.308) and the definition of a subset), make the induction assumption

$$\{1, \ldots, n\} = \{0, \ldots, n\} \setminus \{0\},$$

and show that this implies $\{1, \ldots, n^+\} = \{0, \ldots, n^+\} \setminus \{0\}$. Let us now observe the truth of the equations

$$\begin{aligned}\{1, \ldots, n^+\} &= \{1, \ldots, n\} \cup \{n^+\} \\ &= (\{0, \ldots, n\} \setminus \{0\}) \cup \{n^+\} \\ &= (\{0, \ldots, n\} \cup \{n^+\}) \setminus \{0\} \\ &= \{0, \ldots, n^+\} \setminus \{0\}\end{aligned}$$

using (4.241), the induction assumption, (2.223) based on the fact that the previously established $n \in \mathbb{N}$ implies $0 \neq n^+$ according to (4.35) and therefore $\{0\} \cap \{n^+\} = \emptyset$ with (2.174), and finally (4.89). As n was arbitrary, we may therefore conclude that the induction step is also true, which completes the proof by mathematical induction of the universal sentence (4.244).

Concerning b), We let $n \in \mathbb{N}$ be arbitrary, so that (2.310) gives the disjunction $n = 0 \lor n \in \mathbb{N}_+$, which we now use to prove the sentence (4.245) by cases. The first case $n = 0$ gives

$$\begin{aligned}\{0, \ldots, n\} &= u_{\mathbb{N}}(n) = u_{\mathbb{N}}(0) = \{0\} = \emptyset \cup \{0\} \\ &= \{1, \ldots, n\} \cup \{0\}\end{aligned}$$

by applying (4.92), substitution based on the case assumption $n = 0$, (4.87), (2.216), and (4.239). The other case $n \in \mathbb{N}_+$ implies $n \in \mathbb{N}$ (as noted already in the proof of a)) and then $0 \in \{0, \ldots, n\}$ with (4.97), so that $\{0\} \subseteq \{0, \ldots, n\}$ follows to be true with (2.184). We then obtain

$$\begin{aligned}\{0, \ldots, n\} &= (\{0, \ldots, n\} \setminus \{0\}) \cup \{0\} \\ &= \{1, \ldots, n\} \cup \{0\}\end{aligned}$$

by using (2.263) in connection with the preceding inclusion and (4.244). Thus, (4.245) is true in any case, and since n was arbitrary, it follows that b) also holds. □

Corollary 4.53. *For any $n \in \mathbb{N}$ the initial segment of $\mathbb{N}_+$ up to n is included in the initial segment of $\mathbb{N}$ up to n, i.e.*

$$\{1, \ldots, n\} \subseteq \{0, \ldots, n\}. \tag{4.246}$$

Proof. Letting n be an arbitrary natural number, the inclusion of $\{1, \ldots, n\}$ in $\{0, \ldots, n\}$ follows immediately with (2.201) from (4.245), which is then true for any n. □

The following corollary is derived from Proposition 4.13 and Exercise 4.4.

Corollary 4.54. *The initial segment of $\mathbb{N}_+$ up to a positive natural number n contains n and 1, i.e.*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow n \in \{1, \ldots, n\}), \tag{4.247}$$

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow 1 \in \{1, \ldots, n\}). \tag{4.248}$$

Next, we restate the properties listed in Proposition 4.15 and in Proposition 4.17 for the counting domain $(\mathbb{N}_+, s^+ \upharpoonright \mathbb{N}_+, 1)$.

Corollary 4.55. *The following sentences are true.*

a) If the successor of a positive natural number m is element of the initial segment of $\mathbb{N}_+$ up to a positive natural number n, then m is also element of this initial segment, i.e.

$$\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow [m^+ \in \{1, \ldots, n\} \Rightarrow m \in \{1, \ldots, n\}]). \tag{4.249}$$

b) The successor of a positive natural number n is not an element of the initial segment of $\mathbb{N}_+$ up to n, i.e.

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow n^+ \notin \{1, \ldots, n\}). \tag{4.250}$$

c) *If a positive natural number m is element of the initial segment of $\mathbb{N}_+$ up to a positive natural number n where m and n are different, then the successor of m is also element of this initial segment, i.e.*

$$\begin{aligned}&\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow \\ &\quad [(m \neq n \wedge m \in \{1, \ldots, n\}) \Rightarrow m^+ \in \{1, \ldots, n\}]).\end{aligned} \tag{4.251}$$

d) *The function $u_{\mathbb{N}_+}$ is an injection, i.e.*

$$\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow [\{1, \ldots, m\} = \{1, \ldots, n\} \Rightarrow m = n]). \tag{4.252}$$

e) *If the initial segment of $\mathbb{N}_+$ up to a positive natural number m is a proper subset of the initial segment of $\mathbb{N}_+$ up to a positive natural number n, then the initial segment of $\mathbb{N}_+$ up to the successor of m is included in the initial segment of $\mathbb{N}_+$ up to n, i.e.*

$$\begin{aligned}&\forall m, n\,([m, n \in \mathbb{N}_+ \Rightarrow \\ &\quad [\{1, \ldots, m\} \subset \{1, \ldots, n\} \Rightarrow \{1, \ldots, m^+\} \subseteq \{1, \ldots, n\}]).\end{aligned} \tag{4.253}$$

f) *The initial segment of $\mathbb{N}_+$ up to any positive natural number n is a proper subset of the initial segment of $\mathbb{N}_+$ up to the successor of n, i.e.*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \{1, \ldots, n\} \subset \{1, \ldots, n^+\}). \tag{4.254}$$

g) *The initial segment of $\mathbb{N}_+$ up to a positive natural number n and the singleton formed by the successor of n are disjoint, that is,*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \{1, \ldots, n\} \cap \{n^+\} = \emptyset). \tag{4.255}$$

h) *The initial segment of $\mathbb{N}_+$ up to a positive natural number n equals the difference of the initial segment of $\mathbb{N}_+$ up to the successor of n and the singleton formed by that successor, that is,*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \{1, \ldots, n\} = \{1, \ldots, n^+\} \setminus \{n^+\}). \tag{4.256}$$

i) *The singleton formed by the successor of a positive natural number n equals the difference of the initial segment of $\mathbb{N}_+$ up to the successor of n and the initial segment up to n, i.e.*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \{n^+\} = \{1, \ldots, n^+\} \setminus \{1, \ldots, n\}). \tag{4.257}$$

In the following, we restate the findings of Theorem 4.19 and Theorem 4.20, for the counting domain based on the positive natural numbers, which yield the subsequent definitions for the orderings $\leq_{\mathbb{N}_+}$ and $<_{\mathbb{N}_+}$.

Corollary 4.56. *The following sentences are true.*

a) The total ordering $\leq_{\mathbb{N}_+}$ *of* $\mathbb{N}_+$ *(defined according to Theorem 4.19) satisfies*

$$\forall m,n\,(m,n\in\mathbb{N}_+\Rightarrow[m\leq_{\mathbb{N}_+}n\Leftrightarrow\{1,\ldots,m\}\subseteq\{1,\ldots,n\}]),\tag{4.258}$$

and

$$\forall n\,(n\in\mathbb{N}_+\Rightarrow n<_{\mathbb{N}_+}n^+).\tag{4.259}$$

b) Then, the linear ordering $<_{\mathbb{N}_+}$ *of* $\mathbb{N}_+$ *induced by* $\leq_{\mathbb{N}_+}$ *satisfies*

$$\forall m,n\,(m,n\in\mathbb{N}_+\Rightarrow[m<_{\mathbb{N}_+}n\Leftrightarrow\{1,\ldots,m\}\subset\{1,\ldots,n\}]),\tag{4.260}$$

Definition 4.5 (Standard total & linear ordering of $\mathbb{N}_+$**).** We call $\leq_{\mathbb{N}_+}$ the *standard total ordering of* $\mathbb{N}_+$ and $<_{\mathbb{N}_+}$ the *standard linear ordering of* $\mathbb{N}_+$.

The relationships established in Proposition 4.52 allow us to bring out clearly the useful fact that it is irrelevant whether we use the total ordering $\leq_{\mathbb{N}_+}$ or $\leq_{\mathbb{N}}$ to compare two positive natural numbers.

Proposition 4.57. *The following sentence is true.*

$$\forall m,n\,(m,n\in\mathbb{N}_+\Rightarrow[m\leq_{\mathbb{N}_+}n\Leftrightarrow m\leq_{\mathbb{N}}n]).\tag{4.261}$$

Proof. We let m and n be arbitrary in $\mathbb{N}_+$ and observe first the truth of the equivalence

$$m\leq_{\mathbb{N}_+}n\;\Leftrightarrow\;\{1,\ldots,m\}\subseteq\{1,\ldots,n\}\tag{4.262}$$

in light of (4.258). Similarly, we obtain

$$m\leq_{\mathbb{N}}n\;\Leftrightarrow\;\{0,\ldots,m\}\subseteq\{0,\ldots,n\}\tag{4.263}$$

by applying (4.152). We now prove the first part ('$\Rightarrow$') of the proposed equivalence, assuming $m\leq_{\mathbb{N}_+}n$, which implies $\{1,\ldots,m\}\subseteq\{1,\ldots,n\}$ with (4.262), and then furthermore

$$\{1,\ldots,m\}\cup\{0\}\subseteq\{1,\ldots,n\}\cup\{0\}$$

with (2.251). This yields $\{0,\ldots,m\}\subseteq\{0,\ldots,n\}$ by applying substitution based on (4.245), and therefore $m\leq_{\mathbb{N}}n$ with (4.263), which proves the first part of the equivalence in (4.261).

To prove the second part ('$\Leftarrow$'), we now assume $m \leq_{\mathbb{N}} n$ to be true, which implies $\{0, \ldots, m\} \subseteq \{0, \ldots, n\}$ with (4.263). It then follows with (2.190) that

$$\{0, \ldots, m\} \setminus \{0\} \subseteq \{0, \ldots, n\} \setminus \{0\}$$

holds, so that we obtain $\{1, \ldots, m\} \subseteq \{1, \ldots, n\}$ with (4.244), which further implies $m \leq_{\mathbb{N}_+} n$ with (4.262), completing the proof of the equivalence. As m and n were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Exercise 4.14. Establish the following sentence.

$$\forall m, n\, (m, n \in \mathbb{N}_+ \Rightarrow [m <_{\mathbb{N}_+} n \Leftrightarrow m <_{\mathbb{N}} n]). \tag{4.264}$$

(Hint: Use (4.261) in connection with Theorem 3.86 and Theorem 3.49.)

Proposition 4.58. *The standard total ordering of* $\mathbb{N}_+$ *is included in the standard total ordering* $\mathbb{N}$*, that is,*

$$\leq_{\mathbb{N}_+} \;\subseteq\; \leq_{\mathbb{N}} . \tag{4.265}$$

Proof. We use the definition of a subset and prove equivalently

$$\forall z\, (z \in \leq_{\mathbb{N}_+} \Rightarrow z \in \leq_{\mathbb{N}}). \tag{4.266}$$

For this purpose, we let $z \in \leq_{\mathbb{N}_+}$ be arbitrary, so that there exist by definition of a binary relation constants, say $\bar{m}$ and $\bar{n}$, such that $z = (\bar{m}, \bar{n})$ holds. Thus, we have $(\bar{m}, \bar{n}) \in \leq_{\mathbb{N}_+}$, which we may write as $\bar{m} \leq_{\mathbb{N}_+} \bar{n}$, using Notation 3.2. As a partial ordering of $\mathbb{N}_+$, we have that $\leq_{\mathbb{N}_+}$ is a binary relation on $\mathbb{N}_+$ and therefore a subset of the Cartesian product $\mathbb{N}_+ \times \mathbb{N}_+$. It then follows from $(\bar{m}, \bar{n}) \in \leq_{\mathbb{N}_+}$ with the definition of a subset that $(\bar{m}, \bar{n}) \in \mathbb{N}_+ \times \mathbb{N}_+$ holds, which implies $\bar{m}, \bar{n} \in \mathbb{N}_+$ by definition of the Cartesian product of two sets. With this finding, the previously established $\bar{m} \leq_{\mathbb{N}_+} \bar{n}$ yields $\bar{m} \leq_{\mathbb{N}} \bar{n}$ with (4.261), which we may write as $[z =]\ (\bar{m}, \bar{n}) \in \leq_{\mathbb{N}}$. Thus, $z \in \leq_{\mathbb{N}}$ holds, which proves the implication in (4.266). Since z was arbitrary, we may therefore conclude that the universal sentence (4.266) is true, so that the equivalent inclusion (4.265) holds, as claimed. □

Exercise 4.15. Show that the standard linear ordering of $\mathbb{N}_+$ is included in the standard linear ordering $\mathbb{N}$, that is,

$$<_{\mathbb{N}_+} \;\subseteq\; <_{\mathbb{N}} . \tag{4.267}$$

(Hint: Proceed in analogy to the proof of Proposition 4.58, using now (4.264).)

Proposition 4.59. *The successor function s^+ with respect to the counting domain $(\mathbb{N}, s^+, 0)$ is an order-isomorphism from $(\mathbb{N}, \leq_{\mathbb{N}})$ to $(\mathbb{N}_+, \leq_{\mathbb{N}_+})$, i.e.*

$$s^+ : (\mathbb{N}, \leq_{\mathbb{N}}) \rightleftarrows (\mathbb{N}_+, \leq_{\mathbb{N}_+}). \tag{4.268}$$

Proof. Let us first recall that s^+ is a bijection from $\mathbb{N}$ to $\mathbb{N}_+$, according to Corollary 4.8. To show that s^+ is also an order-embedding from $(\mathbb{N}, \leq_{\mathbb{N}})$ to $(\mathbb{N}_+, \leq_{\mathbb{N}_+})$, we verify

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \leq_{\mathbb{N}} n \Leftrightarrow s^+(m) \leq_{\mathbb{N}_+} s^+(n)]). \tag{4.269}$$

We let m and n be arbitrary in $\mathbb{N}$ and notice that $s^+(m)$ and $s^+(n)$ are elements of the range $\mathbb{N}_+$ of s^+. We then obtain the equivalences

$$\begin{aligned} m \leq_{\mathbb{N}} n &\Leftrightarrow m^+ \leq_{\mathbb{N}} n^+ \\ &\Leftrightarrow m^+ \leq_{\mathbb{N}_+} n^+ \\ &\Leftrightarrow s^+(m) \leq_{\mathbb{N}_+} s^+(n) \end{aligned}$$

using (4.163), (4.261), and the notation in (4.33) for the successor of a natural number. As m and n were arbitrary, it follows that (4.269) is true, which proves that s^+ is an order-embedding from $(\mathbb{N}, \leq_{\mathbb{N}})$ to $(\mathbb{N}_+, \leq_{\mathbb{N}_+})$. Together with the finding that s^+ is a bijection, this gives (4.268), by definition of an order-isomorphism. □

We now use the fact that the total orderings $\leq_{\mathbb{N}_+}$ and $\leq_{\mathbb{N}}$ as well as the linear orderings $<_{\mathbb{N}_+}$ and $<_{\mathbb{N}}$ coincide for ordered pairs of positive natural number (according to Proposition 4.57 and Exercise 4.14, respectively) to restate for the counting domain $(\mathbb{N}_+, s^+ \upharpoonright \mathbb{N}_+, 1)$ the generic findings of

- Exercises 4.5, 4.6,
- Propositions 4.24, 4.26, 4.28, 4.30,
- Exercise 4.8,
- Theorem 4.34 (Characterization of initial segments),
- Propositions 4.36, 4.40, 4.42,
- and Corollary 4.43.

Corollary 4.60. *The following sentences are true.*

a) If a positive natural number m is less than a positive natural number n, then the successor of m is less than or equal to n, that is,

$$\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow [m <_{\mathbb{N}} n \Rightarrow m^+ \leq_{\mathbb{N}} n]). \tag{4.270}$$

b) If the successor of a positive natural number m is less than or equal to the successor of a positive natural number n, then m is less than or equal to n, i.e.

$$\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow [m^+ \leq_{\mathbb{N}} n^+ \Rightarrow m \leq_{\mathbb{N}} n]). \tag{4.271}$$

c) If the successor of a positive natural number m is less than the successor of a positive natural number n, then m is less than n, i.e.

$$\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow [m^+ <_{\mathbb{N}} n^+ \Rightarrow m <_{\mathbb{N}} n]). \tag{4.272}$$

d) A positive natural number m is less than or equal to a positive natural number n iff the successor of m is less than or equal to the successor of n, i.e.

$$\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow [m \leq_{\mathbb{N}} n \Leftrightarrow m^+ \leq_{\mathbb{N}} n^+]). \tag{4.273}$$

e) The linearly ordered set $(\mathbb{N}_+, \leq_{\mathbb{N}_+})$ is a lattice.

f) The set of positive natural numbers does not have an upper bound with respect to $\leq_{\mathbb{N}}$.

g) The successor m^+ of a positive natural number m is the next greater positive natural number after m, i.e.

$$\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow [m <_{\mathbb{N}} n \wedge n \leq_{\mathbb{N}} m^+] \Rightarrow n = m^+). \tag{4.274}$$

h) The initial segment of $\mathbb{N}_+$ up to a positive natural number n consists of all positive natural numbers that are less than or equal to n, i.e.

$$\forall m, n\,(m, n \in \mathbb{N}_+ \Rightarrow [m \in \{1, \ldots, n\} \Leftrightarrow m \leq_{\mathbb{N}} n]). \tag{4.275}$$

Moreover, it is true for any positive natural number n that all elements of the initial segment of $\mathbb{N}_+$ up to n are between 1 and n, where 1 is the least and n the greatest element of that initial segment, i.e.

$$\min\{1, \ldots, n\} = 1, \tag{4.276}$$

$$\max\{1, \ldots, n\} = n. \tag{4.277}$$

i) 1 is a lower bound for $\mathbb{N}_+$, i.e.

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow 1 \leq_{\mathbb{N}} n), \tag{4.278}$$

and 1 is also the least element of $\mathbb{N}_+$.

j) *For any positive natural number* n *it is false that* n *is less than* 1, *i.e.*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \neg n <_{\mathbb{N}} 1). \tag{4.279}$$

k) *There is no positive natural number which is less than* 1, *i.e.*

$$\neg\exists n\,(n \in \mathbb{N}_+ \,\wedge\, n <_{\mathbb{N}} 1). \tag{4.280}$$

l) *Every nonempty subset of the initial segment of* $\mathbb{N}_+$ *up to a positive natural number* n *has a greatest element, i.e.*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \forall A\,([A \subseteq \{1,\ldots,n\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \max A))). \tag{4.281}$$

Moreover, any nonempty and bounded-from-above subset of $\mathbb{N}_+$ *has a greatest element, i.e.*

$$\begin{aligned}&\forall A\,([A \subseteq \mathbb{N}_+ \,\wedge\, A \neq \emptyset \,\wedge\, \exists u\,(u \in \mathbb{N}_+ \,\wedge\, \forall k\,(k \in A \Rightarrow k \leq_{\mathbb{N}} u)]\\ &\quad\Rightarrow \exists m\,(m = \max A))).\end{aligned} \tag{4.282}$$

m) *Every nonempty subset of the initial segment of* $\mathbb{N}_+$ *up to a positive natural number* n *has a least element, i.e.*

$$\forall n\,(n \in \mathbb{N} \Rightarrow \forall A\,([A \subseteq \{0,\ldots,n\} \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \min A))). \tag{4.283}$$

Furthermore, any nonempty subset of $\mathbb{N}_+$ *has a least element, i.e.*

$$\forall A\,([A \subseteq \mathbb{N}_+ \,\wedge\, A \neq \emptyset] \Rightarrow \exists m\,(m = \min A))). \tag{4.284}$$

n) *The lattice* $(\mathbb{N}_+, \leq_{\mathbb{N}_+})$ *is well-ordered.*

Proposition 4.61. *The restriction of the successor function* s^+ *to a natural number* n *is a bijection from* n *to the initial segment of* $\mathbb{N}_+$ *up to* n, *i.e.*

$$\forall n\,(n \in \mathbb{N} \Rightarrow s^+ \upharpoonright n : n \rightleftarrows \{1,\ldots,n\}). \tag{4.285}$$

Proof. We let n be arbitrary in $\mathbb{N}$ and recall from (4.40) that s^+ is the bijection $s^+ : \mathbb{N} \rightleftarrows \mathbb{N}_+$, so that s^+ is in particular an injection from n to $\mathbb{N}_+$. According to (2.320), we have $n \subseteq \mathbb{N}$, so that $s^+ \upharpoonright n$ is an injection from n to $\mathbb{N}_+$ due to Proposition 3.187.

Next, we verify that the range of $s^+ \upharpoonright n$ is identical with $\{1,\ldots,n\}$, by considering the two cases $n = 0$ and $n \neq 0$. In the first case, we have $n = 0 = \emptyset$ and then $\{1,\ldots,n\} = \emptyset$ with (4.239); furthermore, the domain of the restriction $s^+ \upharpoonright n = s^+ \upharpoonright \emptyset$ is then the empty set due to Proposition

3.164, so that this restriction is itself identical with $\emptyset$ due Proposition 3.151. As the empty function is a bijection from $\emptyset$ to $\emptyset$ (see Corollary 3.201), we then see in light of the previous findings that $s^+ \upharpoonright n$ is a bijection from n to $\{1, \ldots, n\}$ in case of $n = 0$.

In the other case of $n \neq 0$, we have $n \neq \emptyset$ by definition of the zero 0, so that the domain n of $s^+ \upharpoonright n$ is now nonempty. It then follows with Exercise 3.19 that the range of (the binary relation) $s^+ \upharpoonright n$ is also nonempty. We also notice that $n \neq 0$ implies $n \in \mathbb{N}_+$ with (2.310). We therefore obtain with Proposition 4.13 $n \in \{1, \ldots, n\}$, which shows that the stated initial segment is also nonempty in case of $n \neq 0$. Based on these observations, we now apply the Equality Criterion for sets and prove $\mathrm{ran}(s^+ \upharpoonright n) = \{1, \ldots, n\}$ by verifying the equivalent

$$\forall m \, (m \in \mathrm{ran}(s^+ \upharpoonright n) \Leftrightarrow m \in \{1, \ldots, n\}). \tag{4.286}$$

To do this, we let m be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $m \in \mathrm{ran}(s^+ \upharpoonright n)$ to be true (recalling our previous observation that the range indeed has an element). We already showed that $s^+ \upharpoonright n$ is a function with codomain $\mathbb{N}_+$, so that $\mathrm{ran}(s^+ \upharpoonright n) \subseteq \mathbb{N}_+$ holds. With this, the assumption $m \in \mathrm{ran}(s^+ \upharpoonright n)$ implies $m \in \mathbb{N}_+$ with the definition of a subset. Moreover, it follows by definition of a range that there exists a constant, say $\bar{k}$, such that $(\bar{k}, m) \in s^+ \upharpoonright n$ holds, which in turn implies $(\bar{k}, m) \in s^+$ and $\bar{k} \in n$ by definition of a restriction. On the one hand, the former may be written as $m = s^+(\bar{k}) = \bar{k}^+$, using the notations for a function and for a successor. On the other hand, $\bar{k} \in n$ implies $\bar{k} \in \mathbb{N}$ with the previously mentioned inclusion $n \subseteq \mathbb{N}$, so that we obtain $\bar{k} <_{\mathbb{N}} n$ with (4.190), which inequality implies $\bar{k}^+ \leq_{\mathbb{N}} n$ with (4.157). Let us now observe that $\bar{k} \in \mathbb{N}$ gives $\bar{k}^+ \neq 0$ with Property 1 of a counting domain and therefore $\bar{k}^+ \in \mathbb{N}_+$ according to (2.310). Since $n \in \mathbb{N}_+$ also holds, $\bar{k}^+ \leq_{\mathbb{N}} n$ implies $\bar{k}^+ \in \{1, \ldots, n\}$ with the Characterization of initial segments (4.275), and thus the desired $m \in \{1, \ldots, n\}$ via substitution based on the previously established equation $m = \bar{k}^+$. Thus, the first part of the equivalence in (4.286) holds.

To prove the second part ('$\Leftarrow$'), we now assume $m \in \{1, \ldots, n\}$ to be true, which implies $m \in \mathbb{N}_+$ (by definition of an initial segment of $\mathbb{N}_+$ for $n \in \mathbb{N}_+$) and then also $m \leq_{\mathbb{N}} n$ with (4.275). Here, $m \in \mathbb{N}_+$ implies $m \in \mathbb{N}$ and $m \neq 0$ (by definition of $\mathbb{N}_+$). It then follows from the latter two findings with (4.39) that there exists an element of $\mathbb{N}$, say $\bar{k}$, such that $\bar{k} \neq m$ and $\bar{k}^+ = m$ hold. We may write the latter equation as $s^+(\bar{k}) = m$ (using the notation for successors), and then that equation as

$$(\bar{k}, m) \in s^+. \tag{4.287}$$

Furthermore, $\bar{k} \in \mathbb{N}$ implies $\bar{k} <_{\mathbb{N}} \bar{k}^{+}$ with (4.153), so that substitution based on the previously established equation $\bar{k}^{+} = m$ yields $\bar{k} <_{\mathbb{N}} m$. The conjunction of this and the previously obtained $m \leq_{\mathbb{N}} n$ then implies $\bar{k} <_{\mathbb{N}} n$ with the Transitivity Formula for $<$ and $\leq$, and therefore

$$\bar{k} \in n \tag{4.288}$$

with (4.190). Now, the conjunction of (4.287) and (4.288) implies with the definition of a restriction$(\bar{k}, m) \in s^{+} \upharpoonright n$, and the definition of a range then gives the desired $m \in \mathrm{ran}(s^{+} \upharpoonright n)$, completing the proof of the equivalence. As m is arbitrary, we may therefore conclude that the universal sentence (4.286) is true, so that the equation $\mathrm{ran}(s^{+} \upharpoonright n) = \{1, \ldots, n\}$ also holds, according to the Equality Criterion for sets. Consequently, the function

$$s^{+} \upharpoonright n : n \to \{1, \ldots, n\} \tag{4.289}$$

is a surjection by definition. Since $s^{+} \upharpoonright n : n \to \mathbb{N}_{+}$ is an injection, it follows with Note 3.23 that (4.289) is also an injection. Thus, (4.289) is a bijection by definition, which proves the implication in (4.285) also in the case of $n \neq 0$. As n was arbitrary, it then follows that the proposed universal sentence is true. □

Proposition 4.62. *For any $m, n \in \mathbb{N}$ there exists a unique set $\{m^{+}, \ldots, n\}$ consisting of all natural numbers between m^{+} and n in the sense of*

$$\forall k \, (k \in \{m^{+}, \ldots, n\} \Leftrightarrow [k \in \mathbb{N} \wedge m^{+} \leq_{\mathbb{N}} k \leq_{\mathbb{N}} n]). \tag{4.290}$$

and this set is identical with the difference of the initial segment of $\mathbb{N}_{+}$ up to n and the initial segment of $\mathbb{N}_{+}$ up to m, i.e.

$$\{m^{+}, \ldots, n\} = \{1, \ldots, n\} \setminus \{1, \ldots, m\}. \tag{4.291}$$

Proof. Letting m and n be arbitrary natural numbers, we may apply the Axiom of Specification in connection with the Equality Criterion for sets to obtain the true uniquely existential sentence

$$\exists! \{m^{+}, \ldots, n\} \, \forall k \, (k \in \{m^{+}, \ldots, n\} \Leftrightarrow [k \in \mathbb{N} \wedge m^{+} \leq_{\mathbb{N}} k \wedge k \leq_{\mathbb{N}} n]). \tag{4.292}$$

Thus, the set $\{m^{+}, \ldots, n\}$ satisfies (4.290).

Next, to establish the equation (4.291), we apply the Equality Criterion for sets and accordingly verify the universal sentence

$$\forall k \, (k \in \{m^{+}, \ldots, n\} \Leftrightarrow k \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}), \tag{4.293}$$

letting k be arbitrary. To prove the first part ('$\Rightarrow$') of the equivalence directly, we assume $k \in \{m^+, \ldots, n\}$, so that the inequalities

$$[m <_{\mathbb{N}}] \quad m^+ \leq_{\mathbb{N}} k \leq_{\mathbb{N}} n$$

follow to be true with (4.290) and (4.153). Here, $k \leq_{\mathbb{N}} n$ implies $k \in \{0, \ldots, n\}$ with (4.180) according to the Characterization of initial segments, and the inequalities $m <_{\mathbb{N}} m^+ \leq_{\mathbb{N}} k$ imply $m <_{\mathbb{N}} k$ with the Transitivity Formula for $<$ and $\leq$. The latter further implies $\neg k \leq_{\mathbb{N}} m$ with the Negation Formula for $\leq$ and consequently $\neg k \in \{0, \ldots, m\}$ (using again the Characterization of initial segments). Since $\{1, \ldots, m\}$ is included in $\{0, \ldots, m\}$ according to Corollary 4.53, it then follows from the preceding finding $k \notin \{0, \ldots, m\}$ that $k \notin \{1, \ldots, m\}$ holds, applying (2.9). Now, we also have that $m^+ \neq 0$ is true because of Property 1 of a counting domain; since $m^+ <_{\mathbb{N}} 0$ is also false in view of (4.188), it follows with the comparability of the linear ordering $<_{\mathbb{N}}$ that $0 <_{\mathbb{N}} m^+ \; [\leq_{\mathbb{N}} k]$ holds. These two inequalities give $0 <_{\mathbb{N}} k$ with the Transitivity Formula for $<$ and $\leq$, so that we obtain $k \neq 0$ (utilizing again the comparability of $<_{\mathbb{N}}$), which inequality then implies $k \notin \{0\}$ with (2.169). As $k \in \{0, \ldots, n\}$ and $k \notin \{0\}$ are simultaneously true, it follows with (4.244) in connection with the definition of a set difference that $k \in \{1, \ldots, n\}$ holds. Together with the previously established $k \notin \{1, \ldots, m\}$, this yields $k \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$ by definition of a set difference, proving the first part of the equivalence in (4.293).

We now prove the sentence '$\Leftarrow$' by cases, based on the fact that the disjunction $n = 0 \vee n \neq 0$ is true according to the Law of the Excluded Middle. In the first case of $n = 0$, we obtain $\{1, \ldots, n\} = \emptyset$ with (4.239) and therefore

$$\{1, \ldots, n\} \setminus \{1, \ldots, m\} = \emptyset \setminus \{1, \ldots, m\} = \emptyset \tag{4.294}$$

with (2.105). Thus, the antecedent $k \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$ $(= \emptyset)$ is false by definition of the empty set, so that the implication ('$\Leftarrow$') itself is true.

The other case $n \neq 0$ implies $n \notin \{0\}$ with (2.169) and the Law of Contraposition, which inequality yields with the initial assumption $n \in \mathbb{N}$ and the definition of a set difference $n \in \mathbb{N} \setminus \{0\}$. This gives $n \in \mathbb{N}_+$ with the definition of the set of positive natural numbers, so that $\{1, \ldots, n\}$ is nonempty by definition of an initial segment. Now, since $n \leq_{\mathbb{N}} m \vee \neg n \leq_{\mathbb{N}} m$ is true by the Law of the Excluded Middle, we may use this true disjunction to prove the implication '$\Leftarrow$' by cases.

If $n \leq_{\mathbb{N}} m$ holds, then we may prove by contradiction that the antecedent $k \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$ of the implication ('$\Leftarrow$') to be proven is again false. Assuming $k \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$ to be true, it follows (by definition of a set difference) that $k \in \{1, \ldots, n\}$ and $k \notin \{1, \ldots, m\}$ both hold.

The former implies $k \leq_{\mathbb{N}} n$ and the latter $\neg k \leq_{\mathbb{N}} m$ with (4.275). Here, the latter negation further implies $m <_{\mathbb{N}} k$ with the Negation Formula for $\leq$. The conjunction of this inequality and the previously established $k \leq_{\mathbb{N}} n$ then yields $m <_{\mathbb{N}} n$ with the Transitivity Formula for $<$ and $\leq$, and therefore $\neg n \leq_{\mathbb{N}} m$ (again with the Negation Formula for $\leq$). This negation evidently contradicts the subcase assumption $n \leq_{\mathbb{N}} m$, completing the proof that the antecedent of the implication ('$\Leftarrow$') in (4.293) is false. Consequently, the implication itself is true for the first subcase.

If the second subcase $\neg n \leq_{\mathbb{N}} m$ holds, then also $m <_{\mathbb{N}} n$ because of the Negation Formula for $\leq$, and we may prove the implication ('$\Leftarrow$') directly. Assuming the antecedent $k \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$ to be true, we obtain – as for the first subcase – the true conjunction of $k \in \{1, \ldots, n\}$ and $k \notin \{1, \ldots, m\}$, again with the consequences $k \leq_{\mathbb{N}} n$ and $m <_{\mathbb{N}} k$. Since the latter implies $m^+ \leq_{\mathbb{N}} k$ with (4.157), we have in combination with the former inequality $k \leq_{\mathbb{N}} n$ that $m^+ \leq_{\mathbb{N}} k \leq_{\mathbb{N}} n$ holds, which evidently yields $k \in \{m^+, \ldots, n\}$ with (4.290). This completes the direct proof of the implication ('$\Leftarrow$') for the second subcase and thus the two proofs nested proofs by cases. Since k was arbitrary, the universal sentence (4.293) follows therefore to be true. Consequently, the sets $\{m^+, \ldots, n\}$ and $\{1, \ldots, n\} \setminus \{1, \ldots, m\}$ turn out to be identical (in view of the Equality Criterion for sets), so that the proof of the equation (4.291) is complete. Since m and n were arbitrary, we may now finally conclude that the proposition holds. $\square$

Note 4.11. A comparison of (4.292) with (3.351) reveals for any $m, n \in \mathbb{N}$ that the set $\{m^+, \ldots, n\}$ is identical with the closed interval $[m^+, n]$ in $\mathbb{N}$ with respect to $\mathbb{N}$, that is,

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow \{m^+, \ldots, n\} = [m^+, n]). \tag{4.295}$$

We call $\{m^+, \ldots, n\}$ also the *intermediate segment of* $\mathbb{N}_+$ *from* m^+ *to* n.

Corollary 4.63. *The following sentences are true.*

$$\forall m\, (m \in \mathbb{N} \Rightarrow \{m^+, \ldots, 0\} = \emptyset), \tag{4.296}$$

$$\forall m, n\, ([m, n \in \mathbb{N} \wedge n \leq_{\mathbb{N}} m] \Rightarrow \{m^+, \ldots, n\} = \emptyset). \tag{4.297}$$

Proof. On the one hand, letting $m \in \mathbb{N}$ be arbitrary, the truth of (4.296) follows from (4.294) and (4.291). On the other hand, letting $m, n \in \mathbb{N}$ be arbitrary such that $n \leq_{\mathbb{N}} m$ holds, the truth of

$$\forall k\, (k \notin \{m^+, \ldots, n\})$$

follows by letting k be arbitrary and recalling from the proof of Proposition 4.291 that $n \leq_{\mathbb{N}} m$ implies that $k \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$ is false, so that $k \notin \{m^+, \ldots, n\}$ follows to be true with (4.291). Since k is arbitrary, we may therefore conclude in view of the definition of the empty set that $\{m^+, \ldots, n\} = \emptyset$ holds, which proves (4.297). As m and n were also arbitrary, it then follows that the universal sentences a) and b) are true. $\square$

Corollary 4.64. *The intermediate segment (of* $\mathbb{N}_+$*) from the successor of a natural number* m *to a natural number* n *is included in the initial segment of* $\mathbb{N}_+$ *up to* n*, that is,*

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow \{m^+, \ldots, n\} \subseteq \{1, \ldots, n\}). \tag{4.298}$$

Proof. Letting $m, n \in \mathbb{N}$ be arbitrary, we obtain the inclusion in (4.291) immediately from (4.291) by applying (2.125). $\square$

Proposition 4.65. *The following implication holds for any* $m, n \in \mathbb{N}$.

$$m \leq_{\mathbb{N}} n \;\Rightarrow\; \{1, \ldots, m\} \cup \{m^+, \ldots, n\} = \{1, \ldots, n\}. \tag{4.299}$$

Proof. We let m and n be arbitrary natural numbers, assume $m \leq_{\mathbb{N}} n$ to be true, and we prove the equation in (4.299) by means of the Equality Criterion for sets, verifying accordingly

$$\forall k\, (k \in \{1, \ldots, m\} \cup \{m^+, \ldots, n\} \Leftrightarrow k \in \{1, \ldots, n\}). \tag{4.300}$$

For this purpose, we take an arbitrary k and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $k \in \{1, \ldots, m\} \cup \{m^+, \ldots, n\}$ to be true. By definition of the union of two sets, the disjunction

$$k \in \{1, \ldots, m\} \vee k \in \{m^+, \ldots, n\} \tag{4.301}$$

is then also true, which we now use to prove the desired consequent $k \in \{1, \ldots, n\}$ by cases. The first case $k \in \{1, \ldots, m\}$ implies the truth of $k \in \mathbb{N}$, $1 \leq_{\mathbb{N}} k$ and $k \leq_{\mathbb{N}} m$ with (4.290), where the latter yields together with the initial assumption $m \leq_{\mathbb{N}} n$ in view of the transitivity of the standard total ordering $\leq_{\mathbb{N}}$ the inequality $k \leq_{\mathbb{N}} n$. Thus, $k \in \mathbb{N}$, $1 \leq_{\mathbb{N}} k$ and $k \leq_{\mathbb{N}} n$ are all true, so that $k \in \{1, \ldots, n\}$ follows to be true according to (4.290). The second case $k \in \{m^+, \ldots, n\}$ gives with the already established inclusion (4.298) and the definition of a subset $k \in \{1, \ldots, n\}$, which thus holds in any case, so that the proof of the first part of the equivalence is complete.

To prove the second part ('$\Leftarrow$') in (4.300) , we now assume $k \in \{1, \ldots, n\}$ to be true, so that we evidently have $k \in \mathbb{N}$, $1 \leq_{\mathbb{N}} k$ and $k \leq_{\mathbb{N}} n$. Observing that the Law of the Excluded Middle gives the true disjunction $k \leq_{\mathbb{N}} m \vee \neg k \leq_{\mathbb{N}} m$, we now prove the disjunction (4.301) by cases. The first case $k \leq_{\mathbb{N}} m$ implies together with $k \in \mathbb{N}$ and $1 \leq_{\mathbb{N}} k$ that $k \in \{1, \ldots, m\}$ holds, so that the disjunction to be proven is also true. The second case $\neg k \leq_{\mathbb{N}} m$ implies with the Negation Formula for $\leq$ first $m <_{\mathbb{N}} k$ and then evidently $m^+ \leq_{\mathbb{N}} k$ with (4.270). This inequality in turn implies together with $k \in \mathbb{N}$ and $k \leq_{\mathbb{N}} n$ that $k \in \{m^+, \ldots, n\}$ holds. Then, the disjunction (4.301) is again true, so that the proof by cases is complete. Consequently, the desired consequent of the second part ('$\Leftarrow$') of the equivalence in (4.300) follows to be true by definition of the union of two sets.

As k was arbitrary, we may therefore conclude that the universal sentence (4.300) holds, so that the equation in (4.299) follows to be true. Thus, the proof of the implication (4.299) is complete as well. As m and n were arbitrary, we may conclude that the proposed universal sentence is true. $\square$

Exercise 4.16. Show that the initial segment of $\mathbb{N}_+$ up to a natural number m and the intermediate segment (of $\mathbb{N}_+$) from the successor of m to a natural number n are disjoint, that is,

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow \{1, \ldots, m\} \cap \{m^+, \ldots, n\} = \emptyset). \tag{4.302}$$

(Hint: Use Definition 2.6 in connection with Method 1.12.)

4.5. Monotone Sequences and Subsequences

Theorem 4.66 (Monotony Criterion for increasing sequences). *For any partially ordered set $(Y, \leq_Y)$ it is true that a sequence $f = (f_n)_{n\in\mathbb{N}}$ in Y is increasing iff any term of the sequence is less than or equal to the succeeding term, i.e. iff*

$$\forall n\,(n \in \mathbb{N} \Rightarrow f_n \leq_Y f_{n^+}). \tag{4.303}$$

Proof. We let Y, $\leq_Y$ and f be arbitrary, assume that $\leq_Y$ is a reflexive partial ordering of Y, and assume that $f = (f_n)_{n\in\mathbb{N}}$ is a sequence in Y.

To prove the first part ('$\Rightarrow$') of the proposed equivalence, we assume that $(f_n)_{n\in\mathbb{N}}$ is increasing, so that

$$\forall m, n\,([m, n \in \mathbb{N} \wedge m <_{\mathbb{N}} n] \Rightarrow f_m \leq_Y f_n) \tag{4.304}$$

holds with (3.961). To show that this implies (4.303), we let n be arbitrary in $\mathbb{N}$ and recall that $n \in \mathbb{N}$ implies $n <_{\mathbb{N}} n^+$ with (4.153). Then, the conjunction of $n, n^+ \in \mathbb{N}$ and $n <_{\mathbb{N}} n^+$ implies the desired $f_n \leq_Y f_{n^+}$ with (4.304). As n is arbitrary, we therefore conclude that (4.303) is true, which proves the first part of the proposed equivalence.

To prove the second part ('$\Leftarrow$'), we now assume (4.303) to be true and verify (4.304), which we may equivalently write as

$$\forall m\,(m \in \mathbb{N} \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow [m <_{\mathbb{N}} n \Rightarrow f_m \leq_Y f_n])) \tag{4.305}$$

by using (1.90). To prove this sentence, we let m be arbitrary in $\mathbb{N}$ and carry out a proof by mathematical induction with respect to n. Regarding the base case ($n = 0$), we need to prove the implication

$$m <_{\mathbb{N}} 0 \;\Rightarrow\; f_m \leq_Y f_0.$$

Let us observe here that the antecedent $m <_{\mathbb{N}} 0$ is false because of (4.188), so that the implication itself is true. Regarding the induction step, we let n be arbitrary in $\mathbb{N}$, make the induction assumption

$$m <_{\mathbb{N}} n \;\Rightarrow\; f_m \leq_Y f_n, \tag{4.306}$$

and show that this implies

$$m <_{\mathbb{N}} n^+ \;\Rightarrow\; f_m \leq_Y f_{n^+}. \tag{4.307}$$

We prove the latter implication directly, assuming $m <_{\mathbb{N}} n^+$, which implies $m^+ \leq_{\mathbb{N}} n^+$ with (4.157). The latter further implies $m \leq_{\mathbb{N}} n$ with (4.159), so that $m <_{\mathbb{N}} n \vee m = n$ holds according to the Characterization of induced irreflexive partial orderings. We now use this true disjunction to prove the sentence $f_m \leq_Y f_{n^+}$ by cases. On the one hand, if $m <_{\mathbb{N}} n$ is true, then the induction assumption (4.306) gives $f_m \leq_Y f_n$; as $n \in \mathbb{N}$ implies $f_n \leq_Y f_{n^+}$ with the assumption (4.303), the desired inequality $f_m \leq_Y f_{n^+}$ follows from the previous two inequalities with the transitivity of the reflexive partial ordering $\leq_Y$. On the other hand, if $m = n$ is true, then we obtain $f_m = f_n$ (using the fact that f is a function in connection with Corollary 3.150), so that the previously established $f_n \leq_Y f_{n^+}$ gives $f_m \leq_Y f_{n^+}$ via substitution. Thus, the proof by cases is complete, and the truth of the preceding inequality implies then the truth of the implication (4.307). As n was arbitrary, we may therefore conclude that the induction step holds, so that the proof of the universal sentence with respect to n in (4.305) via mathematical induction is now also complete. Since m was also arbitrary, we may further conclude that the universal sentence (4.305) is true, so that the equivalent sentence (4.304) also holds. Thus, the proof of the second part of the proposed equivalence is complete, which equivalence is therefore true. Because Y, $\leq_Y$ and f were initially arbitrary sets, we may finally conclude that the stated theorem is true. □

Theorem 4.67 (Monotony Criterion for strictly increasing sequences). *For any partially ordered set $(Y, <_Y)$ it is true that a sequence $(f_n)_{n \in \mathbb{N}}$ in Y is strictly increasing iff any term of the sequence is less than the succeeding term, i.e. iff*

$$\forall n\,(n \in \mathbb{N} \Rightarrow f_n <_Y f_{n^+}). \tag{4.308}$$

Exercise 4.17. Prove the Monotony Criterion for strictly increasing sequences.

(Hint: Proceed as in the proof of the Monotony Criterion for increasing sequences.)

Corollary 4.68. *The identity function* $\mathrm{id}_{\mathbb{N}}$ *is a strictly increasing sequence.*

Proof. We may view the identity function $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ as the sequence $\mathrm{id}_{\mathbb{N}} = (n)_{n \in \mathbb{N}}$ since $\mathrm{id}_{\mathbb{N}}(n) = n$ holds for any $n \in \mathbb{N}$. We also have that $(\mathbb{N}, <_{\mathbb{N}})$ is a partially ordered and moreover that $n <_{\mathbb{N}} n^+$ holds for any $n \in \mathbb{N}$ because of (4.153), so that Theorem 4.67 shows that $\mathrm{id}_{\mathbb{N}} = (n)_{n \in \mathbb{N}}$ is indeed strictly increasing. □

Theorem 4.69 (Monotony Criterion for decreasing sequences). *For any partially ordered set $(Y, \leq_Y)$ it is true that a sequence $f = (f_n)_{n \in \mathbb{N}}$ in Y is decreasing iff any term of the sequence is greater than or equal to the succeeding term, i.e. iff*

$$\forall n\,(n \in \mathbb{N} \Rightarrow f_n \geq_Y f_{n^+}). \tag{4.309}$$

Exercise 4.18. Establish the Monotony Criterion for decreasing sequences.

We now consider in particular monotone sequences of functions.

Theorem 4.70 (Monotony Criteria for sequences of functions). *The following equivalences are true for any sets X and Y, any irreflexive partial ordering $\leq$ of Y and any sequence $f = (f_n)_{n\in\mathbb{N}}$ of functions from X to Y.*

a) The sequence $(f_n)_{n\in\mathbb{N}}$ is increasing with respect to the reflexive partial ordering $\preceq$ of Y^X (as defined in Proposition 3.251) iff the sequence $(f_n(x))_{n\in\mathbb{N}}$ is increasing with respect to $\leq$ for all x in X.

b) The sequence $(f_n)_{n\in\mathbb{N}}$ is decreasing with respect to $\preceq$ iff the sequence $(f_n(x))_{n\in\mathbb{N}}$ is decreasing with respect to $\leq$ for all x in X.

Proof. We let X, Y, $\leq$ and f be arbitrary sets such that $(Y, \leq)$ is partially ordered and such that $f = (f_n)_{n\in\mathbb{N}}$ is a sequence in Y^X. To prove the first part ('$\Rightarrow$') of the proposed equivalence, we assume that $(f_n)_{n\in\mathbb{N}}$ is increasing with respect to the reflexive partial ordering $\preceq$ of Y^X, so that the universal sentence

$$\forall n \, (n \in \mathbb{N} \Rightarrow f_n \preceq f_{n^+}) \tag{4.310}$$

holds according to the Monotony Criterion for increasing sequences. We then let $\bar{x} \in X$ be arbitrary, and we show that $(f_n(\bar{x}))_{n\in\mathbb{N}}$ is increasing with respect to $\leq$, i.e.

$$\forall n \, (n \in \mathbb{N} \Rightarrow f_n(\bar{x}) \leq f_{n^+}(\bar{x})). \tag{4.311}$$

For this purpose, we let $\bar{n} \in \mathbb{N}$ be arbitrary, which implies $f_{\bar{n}} \preceq f_{\bar{n}^+}$ with (4.310), so that the equivalent universal sentence

$$\forall x \, (x \in X \Rightarrow f_{\bar{n}}(x) \leq f_{\bar{n}^+}(x)) \tag{4.312}$$

follows to be true with (3.903). With this, the assumed $\bar{x} \in X$ yields the inequality $f_{\bar{n}}(\bar{x}) \leq f_{\bar{n}^+}(\bar{x})$, proving the the implication in (4.311). As $\bar{n}$ is arbitrary, we may therefore conclude that (4.311) holds, which shows in light of the Monotony Criterion for increasing sequences that the sequence $(f_n(\bar{x}))_{n\in\mathbb{N}}$ is increasing. Since $\bar{x}$ was also arbitrary, we may now infer from this finding the truth of the first part of the proposed equivalence.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume that $(f_n(x))_{n\in\mathbb{N}}$ is increasing for all $x \in X$, and we show that the universal sentence (4.310) follows to be true. To do this, we take an arbitrary $\bar{n} \in \mathbb{N}$ and establish the desired consequent $f_{\bar{n}} \preceq f_{\bar{n}^+}$ by proving the equivalent universal sentence (4.312). Letting $\bar{x} \in X$ be arbitrary, we now see that the particular sequence $(f_n(\bar{x}))_{n\in\mathbb{N}}$ is (by assumption) increasing, so that (4.311) holds. Thus, $\bar{n} \in \mathbb{N}$ implies $f_{\bar{n}}(\bar{x}) \leq f_{\bar{n}^+}(\bar{x})$, and since $\bar{x}$ is arbitrary, we may therefore conclude that the universal sentence (4.312) is indeed true. Consequently, we obtain the inequality $f_{\bar{n}} \preceq f_{\bar{n}^+}$ with (3.903), and as $\bar{n}$ is arbitrary, we may now further conclude that the universal sentence (4.310) also holds. It then follows from this with the Monotony Criterion for increasing sequences that the sequence $(f_n)_{n\in\mathbb{N}}$ is increasing.

We thus completed the proof of the proposed equivalence, and as X, Y, $\leq$ and f were initially arbitrary sets, we may infer from this the truth of Part a) of the stated theorem. □

Exercise 4.19. Establish Part b) of Theorem 4.70.

According to Theorem 3.64, $\subseteq$ defines a reflexive partial ordering of any set system, which fact immediately gives the following particular instances of the monotony criteria for increasing & decreasing sequences.

Corollary 4.71 (Monotony Criteria for sequences of sets). *For any set system $\mathcal{K}$ and any sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{K}$, it is true that*

a) $(A_n)_{n\in\mathbb{N}}$ is isotone (with respect to $\subseteq_{\mathcal{K}}$) iff any term of the sequence is included in the succeeding term, i.e. iff

$$\forall n\, (n \in \mathbb{N} \Rightarrow A_n \subseteq A_{n^+}). \tag{4.313}$$

b) $(A_n)_{n\in\mathbb{N}}$ is antitone (with respect to $\subseteq_{\mathcal{K}}$) iff

$$\forall n\, (n \in \mathbb{N} \Rightarrow A_{n^+} \subseteq A_n). \tag{4.314}$$

Exercise 4.20. Verify that we may replace $\mathbb{N}$ by $\mathbb{N}_+$ in the monotony criteria for increasing and decreasing sequences on $\mathbb{N}$.

Throughout the current exposition we will establish various concepts for the 'limit' of a given sequence, with an increasing degree of generality. At this point, we introduce a first basic notion of 'limit' which we will apply in the specific context of monotone sequences.

Definition 4.6 (Limit of an increasing & decreasing sequence, increasingly & decreasingly convergent sequence). For any partially ordered set $(Y, \leq_Y)$, any sequence $s = (a_n)_{n\in\mathbb{N}}$ in Y and any element $L \in Y$, if $(a_n)_{n\in\mathbb{N}}$ is

(1) increasing and the supremum of the range of $(a_n)_{n\in\mathbb{N}}$ (i.e., $\sup \operatorname{ran}(s)$) exists, then we say that L is the *limit* of $(a_n)_{n\in\mathbb{N}}$, symbolically

$$L = \overset{\leq_Y}{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} a_n, \tag{4.315}$$

if, and only if,

$$L = \overset{\leq_Y}{\sup} \operatorname{ran}(s), \tag{4.316}$$

in which case we say that $(a_n)_{n\in\mathbb{N}}$ *converges increasingly to* L. We also say that an increasing sequence $f = (a_n)_{n\in\mathbb{N}}$ in Y is increasingly convergent iff there exists an $L \in Y$ satisfying (4.316).

(2) decreasing and the infimum of the range of $(a_n)_{n\in\mathbb{N}}$ (i.e., $\inf \operatorname{ran}(s)$) exists, then we say that L is the *limit* of $(a_n)_{n\in\mathbb{N}}$, symbolically

$$L = \overset{\leq_Y}{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} a_n, \tag{4.317}$$

if, and only if,

$$L = \overset{\leq_Y}{\inf} \operatorname{ran}(s), \tag{4.318}$$

in which case we say that $(a_n)_{n\in\mathbb{N}}$ *converges decreasingly to* L. We also say that a decreasing sequence $f = (a_n)_{n\in\mathbb{N}}$ in Y is decreasingly convergent iff there exists an $L \in Y$ which satisfies (4.318).

Note 4.12. In case of a sequence of sets $s = (a_n)_{n\in\mathbb{N}}$, we will usually say that s *converges isotonely/antitonely* rather than s converges increasingly/decreasingly.

Note 4.13. The previous sentences may evidently all be established by replacing $\mathbb{N}$ with $\mathbb{N}_+$.

Corollary 4.72. *The following sentence is true for any set* Ω.

a) Any isotone sequence $s = (A_n)_{n\in\mathbb{N}_+}$ *in the power set of* Ω *converges isotonely to its union, i.e.*

$$\overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty} A_n, \tag{4.319}$$

and

b) any antitone sequence $s = (A_n)_{n\in\mathbb{N}_+}$ *in the power set of* Ω *converges antitonely to its intersection, i.e.*

$$\overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} A_n, \tag{4.320}$$

where the limits are taken with respect to the reflexive partial ordering of inclusion ($\subseteq$) of $\mathcal{P}(\Omega)$.

Proof. We let Ω and s be arbitrary sets and assume that $s = (A_n)_{n\in\mathbb{N}_+}$ is a sequence of sets in the power set of Ω, that is, we assume that $s : \mathbb{N}_+ \to \mathcal{P}(\Omega)$ holds. Thus, $\mathrm{ran}(s) \subseteq \mathcal{P}(\Omega)$ holds by definition of a codomain, so that we obtain with Proposition 3.100

$$\bigcup \mathrm{ran}(s) = \sup \mathrm{ran}(s), \tag{4.321}$$

$$\bigcap \mathrm{ran}(s) = \inf \mathrm{ran}(s), \tag{4.322}$$

where the supremum and the infimum are taken with respect to the reflexive partial ordering of inclusion of $\mathcal{P}(\Omega)$. Let us also observe that the definition of the union and of the intersection of a sequence of sets gives

$$\bigcup_{n=1}^{\infty} A_n = \bigcup \mathrm{ran}(s), \tag{4.323}$$

$$\bigcap_{n=1}^{\infty} A_n = \bigcap \mathrm{ran}(s), \tag{4.324}$$

so that substitution yields

$$\bigcup_{n=1}^{\infty} A_n = \sup \mathrm{ran}(s), \tag{4.325}$$

$$\bigcap_{n=1}^{\infty} A_n = \sup \mathrm{ran}(s), \tag{4.326}$$

Now, if the sequence s is isotone, we obtain with (4.325) in view of Definition 4.6(1) the equation $\bigcup_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n$. Similarly, if s is antitone, we obtain because of (4.326) the equation $\bigcap_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n$ by applying Definition 4.6(2). Thus, the proposed equations (4.319) and (4.320) are true, and since Ω and s were initially arbitrary sets, we may therefore conclude that the proposed sentence holds, as claimed. □

Proposition 4.73. *For any set Ω, for any subset $\mathcal{K} \subseteq \mathcal{P}(\Omega)$, for any sequence of sets $s = (A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{K}$ which is isotone with respect to the the reflexive partial ordering of inclusion $\subseteq_{\mathcal{K}}$, and for any $L \in \mathcal{K}$, it is true that L is the limit of $(A_n)_{n\in\mathbb{N}_+}$ with respect to $\subseteq_{\mathcal{K}}$ on $\mathcal{K}$ iff L is the limit of $(A_n)_{n\in\mathbb{N}_+}$ with respect to the reflexive partial ordering of inclusion $\subseteq_{\mathcal{P}(\Omega)}$, i.e.*

$$L = \overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n \;\Leftrightarrow\; L = \overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} A_n. \tag{4.327}$$

Proof. We take arbitrary sets Ω, $\mathcal{K}$, s and L, we assume that $\mathcal{K}$ is included in the power set of Ω, we assume $s = (A_n)_{n\in\mathbb{N}_+}$ to be a sequence of sets in $\mathcal{K}$ such that it is isotone with respect to $\subseteq_{\mathcal{K}}$, and we assume that L is a set in $\mathcal{K}$. Observing here that $\mathcal{K}$ is a codomain of the sequence s, so that the inclusions

$$\mathrm{ran}(s) \subseteq \mathcal{K} \subseteq \mathcal{P}(\Omega) \tag{4.328}$$

hold with the definition of a codomain and the first assumption made, we obtain $\mathrm{ran}(s) \subseteq \mathcal{P}(\Omega)$ with (2.13), which shows that $\mathcal{P}(\Omega)$ is also a codomain of the sequence; thus, $s = (A_n)_{n\in\mathbb{N}_+}$ is a sequence both in $\mathcal{K}$ and in $\mathcal{P}(\Omega)$. We now see in light of the Monotony Criteria for sequences of sets and Note 4.13 that the isotony of $(A_n)_{n\in\mathbb{N}_+}$ with respect to $\subseteq_{\mathcal{K}}$ implies the truth of the universal sentence $\forall n\,(n \in \mathbb{N}_+ \Rightarrow A_n \subseteq A_{n^+})$, which in turn implies the isotony of $(A_n)_{n\in\mathbb{N}_+}$ (viewed now as a sequence in $\mathcal{P}(\Omega)$) with respect to $\subseteq_{\mathcal{P}(\Omega)}$. Therefore, we obtain for the limit with respect to $\subseteq_{\mathcal{P}(\Omega)}$

$$\overset{\subseteq_{\mathcal{P}(\Omega)}}{\sup}\,\mathrm{ran}(s) = \overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty} A_n = \bigcup \mathrm{ran}(s). \tag{4.329}$$

by using the definition of the limit of an increasing sequence, Corollary 4.72a), and the definition of the union of a sequence of sets.

We now establish the first part ('$\Rightarrow$') of the equivalence (4.327), assuming

$$L = \overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n \quad \left[\overset{\subseteq_{\mathcal{K}}}{\sup}\,\mathrm{ran}(s)\right] \tag{4.330}$$

to be true (using also the definition of the limit of an increasing sequence). Thus, the element $L \in \mathcal{K}$ is the supremum of $\mathrm{ran}(s)$ with respect to $\subseteq_{\mathcal{K}}$, which finding implies with the inclusions (4.328) because of Exercise 3.44b) the equation $L = \bigcup \mathrm{ran}(s)$. Applying now substitution based on the equations (4.329) yields now the desired consequent

$$L = \overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} A_n. \tag{4.331}$$

To prove the second part ('$\Leftarrow$'), we assume conversely (4.331) to be true, which gives $L = \bigcup \mathrm{ran}(s)$ with (4.329). Since we assumed initially L to be in $\mathcal{K}$, we obtain via substitution $\bigcup \mathrm{ran}(s) \in \mathcal{K}$, and therefore Exercise 3.44b) gives

$$L = \overset{\subseteq_{\mathcal{K}}}{\sup}\,\mathrm{ran}(s).$$

Since $s = (A_n)_{n\in\mathbb{N}_+}$ is isotone in $\mathcal{K}$, we may therefore apply the definition of the limit of an increasing sequence to obtain for the limit with respect to $\subseteq_{\mathcal{K}}$

$$\overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n = \overset{\subseteq_{\mathcal{K}}}{\sup}\,\mathrm{ran}(s) \quad [= L].$$

Thus, the second part of the equivalence also holds. Because Ω, $\mathcal{K}$, s and L were initially arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Exercise 4.21. Show for any set Ω, for any subset $\mathcal{K} \subseteq \mathcal{P}(\Omega)$, for any sequence of sets $s = (A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{K}$ which is antitone with respect to the reflexive partial ordering of inclusion $\subseteq_{\mathcal{K}}$, and for any $L \in \mathcal{K}$ that L is the limit of $(A_n)_{n\in\mathbb{N}_+}$ with respect to $\subseteq_{\mathcal{K}}$ on $\mathcal{K}$ iff L is the limit of $(A_n)_{n\in\mathbb{N}_+}$ with respect to the reflexive partial ordering of inclusion $\subseteq_{\mathcal{P}(\Omega)}$, i.e.

$$L = \overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n \Leftrightarrow L = \overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} A_n. \tag{4.332}$$

Corollary 4.74. *The following implication holds for any set Ω, any subset $\mathcal{K} \subseteq \mathcal{P}(\Omega)$ and any sequence of sets $s = (A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{K}$ which is isotone with respect to $\subseteq_{\mathcal{K}}$.*

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{K} \Rightarrow \overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty} A_n. \tag{4.333}$$

Proof. Letting Ω, $\mathcal{K}$ and s be arbitrary sets, assuming $\mathcal{K} \subseteq \mathcal{P}(\Omega)$ to hold and $s = (A_n)_{n\in\mathbb{N}_+}$ to be a sequence of sets in $\mathcal{K}$ such that s is isotone with respect to $\subseteq_{\mathcal{K}}$, and assuming moreover

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{K} \tag{4.334}$$

to be true, we recall the truth of the equations (4.329), so that

$$\bigcup_{n=1}^{\infty} A_n = \overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} A_n$$

holds. Due to the assumption (4.334), we then obtain with Proposition 4.73

$$\bigcup_{n=1}^{\infty} A_n = \overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n,$$

which proves the implication (4.333). Since Ω, $\mathcal{K}$ and s were initially arbitrary sets, we may therefore conclude that the proposed universal sentence is true. □

Exercise 4.22. Verify that the following implication holds for any set Ω, any subset $\mathcal{K} \subseteq \mathcal{P}(\Omega)$ and any sequence of sets $s = (A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{K}$ which is isotone with respect to $\subseteq_{\mathcal{K}}$.

$$\bigcap_{n=1}^{\infty} \in \mathcal{K} \Rightarrow \lim_{n\to\infty}^{\subseteq_{\mathcal{K}}} A_n = \bigcap_{n=1}^{\infty} A_n. \tag{4.335}$$

One of the great benefits of complete lattices lies in the fact that a monotone sequence (in the set forming the complete lattice) always has a limit.

Proposition 4.75. *For any complete lattice $(Y, \leq_Y)$, it is true that any increasing sequence $(a_n)_{n\in\mathbb{N}_+}$ in Y converges increasingly.*

Proof. We let Y, $\leq_Y$ and s be arbitrary sets such that $(Y, \leq_Y)$ is a complete lattice and such that s is an increasing sequence $(a_n)_{n\in\mathbb{N}_+}$ in Y. Since the range of s is a subset of its codomain Y, the supremum $\bar{L} = \sup \mathrm{ran}(s)$ (with respect to $\leq_Y$) exists (in Y) by definition of a complete lattice. Then, the existence of an element L in Y satisfying $L = \sup \mathrm{ran}(s)$ implies that the increasing sequence $(a_n)_{n\in\mathbb{N}_+}$ in Y is increasingly convergent, by definition. As Y, $\leq_Y$ and s were arbitrary, we may therefore conclude that Part a) of the proposition is true. □

Exercise 4.23. Show for any complete lattice $(Y, \leq_Y)$ that any decreasing sequence $(a_n)_{n\in\mathbb{N}_+}$ in Y converges decreasingly.

Proposition 4.76. *For any partially ordered set $(Y, <_Y)$ and any element $c \in Y$, it is true that the constant sequence $g_c = (a_n)_{n\in\mathbb{N}_+} = (c)_{n\in\mathbb{N}_+}$ converges increasingly to c.*

Proof. We take arbitrary Y, $<_Y$ and c, assume $c \in Y$ to be true, and assume moreover that the ordered pair $(Y, <_Y)$ is partially ordered. Now, since $\mathbb{N}_+ \neq \emptyset$ is clearly true, it follows with Proposition 3.193 that the constant function $g_c : \mathbb{N}_+ \to \{c\}$ is a surjection, so that we have $\mathrm{ran}(g_c) = \{c\}$; thus, g_c constitutes a sequence $(a_n)_{n\in\mathbb{N}_+}$ in Y. Furthermore, Corollary 3.106 shows that $c = \sup\{c\}$ is true, so that we obtain $c = \sup \mathrm{ran}(g_c)$ via substitution. As the constant sequence g_c is increasing according to Proposition 3.256, recalling that the ordered pair $(\mathbb{N}_+, <_{\mathbb{N}_+})$ based on the standard linear ordering of $\mathbb{N}_+$ is a partially ordered set, we have that g_c converges increasingly to the limit c. Because Y, $<_Y$ and c are arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Exercise 4.24. Show for any partially ordered set $(Y, <_Y)$ and any element $c \in Y$ that the constant sequence $g_c = (a_n)_{n\in\mathbb{N}_+} = (c)_{n\in\mathbb{N}_+}$ converges decreasingly to c.

(Hint: Proceed in analogy to the proof of Proposition 4.76, using Exercise 3.112.)

Proposition 4.77 (Characterization of the pointwise limit of an increasing sequence of functions). *For any sets X and Y, any reflexive partial ordering $\leq$ of Y, any increasing sequence $f = (f_n)_{n\in\mathbb{N}_+}$ of functions from X to Y and any function g from X to Y, it is true that g is the limit of the sequence $(f_n)_{n\in\mathbb{N}_+}$ iff $g(x)$ is the limit of the sequence $f^{(x)} = (f_n(x))_{n\in\mathbb{N}_+}$ for every x in X, i.e.*

$$g = \lim_{n\to\infty} f_n \;\Leftrightarrow\; \forall x\,(x \in X \Rightarrow g(x) = \lim_{n\to\infty} f_n(x)). \tag{4.336}$$

Proof. We let X, Y, $\leq$ and f be arbitrary sets, and we assume $(Y, \leq)$ to be partially ordered, $f = (f_n)_{n\in\mathbb{N}_+}$ to be an increasing sequence of functions from X to Y and g to be a function from X to Y. Consequently, we see in light of Proposition 4.70a) that the sequence $f^{(x)} = (f_n(x))_{n\in\mathbb{N}_+}$ is also increasing for any $x \in X$. Regarding the first part ('$\Rightarrow$') of the stated equivalence, we assume

$$g = \lim_{n\to\infty} f_n \quad [= \sup \operatorname{ran}(f) \text{ pointwise}]$$

and let $x \in X$ be arbitrary, so that we obtain the desired result

$$g(x) = \sup \operatorname{ran}(f^{(x)}) = \lim_{n\to\infty} f_n(x)$$

with Theorem 3.255a) and the definition of the limit applied to the increasing sequence $(f_n(x))_{n\in\mathbb{N}_+}$. To establish the second part ('$\Leftarrow$'), we now assume

$$\forall x\,(x \in X \Rightarrow g(x) = \lim_{n\to\infty} f_n(x) = \sup \operatorname{ran}(f),$$

which implies

$$g = \sup \operatorname{ran}(f) \text{ pointwise} = \lim_{n\to\infty} f_n$$

again with Theorem 3.255a). This completes the proof of the equivalence; since X, Y, $\leq$, f and g were initially arbitrary sets, we may therefore conclude that the proposition is true. □

Exercise 4.25 (Characterization of the pointwise limit of a decreasing sequence of functions). Show that the equivalence (4.336) holds also for any decreasing sequence $f = (f_n)_{n\in\mathbb{N}_+}$ of functions from X to Y (for any sets X and Y, any reflexive partial ordering $\leq$ of Y and any function g from X to Y).

Notation 4.5. We call the limit f of a monotone sequence $(f_n)_{n\in\mathbb{N}_+}$ of functions also the *pointwise limit* of that sequence. Then, we also say that $(f_n)_{n\in\mathbb{N}_+}$ *converges pointwise* to f, symbolically

$$f = \lim_{n\to\infty} f_n \text{ pointwise,} \tag{4.337}$$

Note 4.14. In case of a complete lattice of functions $(Y^X, \preceq)$ both the pointwise supremum and the pointwise infimum of the range of any sequence $(f_n)_{n\in\mathbb{N}_+}$ of functions in Y^X exists. Then, in case the sequence is monotone, the pointwise limit also exists.

Definition 4.7 (Subsequence). For any set Y and any sequence $f = (a_n)_{n\in\mathbb{N}}$ in Y, we say that a sequence $f' = (a'_n)_{n\in\mathbb{N}}$ is a *subsequence* of $(a_n)_{n\in\mathbb{N}}$ iff f' is the composition of f and some strictly increasing sequence $g = (m_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, i.e. iff

$$\exists g\,(g \in \mathbb{N}^{\mathbb{N}} \wedge g \text{ is strictly increasing } \wedge f' = f \circ g). \tag{4.338}$$

Exercise 4.26. Show for any set Y, any sequence $f = (a_n)_{n\in\mathbb{N}}$ in Y and any strictly increasing sequence $g = (m_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$

a) that the composition of f and g exists as the sequence $f' = (a'_n)_{n\in\mathbb{N}} = (a_{m_n})_{n\in\mathbb{N}}$ in Y, i.e.

$$(f \circ g) : \mathbb{N} \to Y, \quad n \mapsto a'_n = a_{m_n}, \tag{4.339}$$

b) and that any term a'_n of the sequence f' occurs in the original sequence f at an index m_n greater than or equal to n, i.e.

$$\forall n\,(n \in \mathbb{N} \Rightarrow n \leq_{\mathbb{N}} m_n). \tag{4.340}$$

(Hint: Concerning a), apply Proposition 3.178; concerning b), carry out a proof by mathematical induction and apply the Transitivity Formula for $\leq$ and $<$ as well as (4.157) within the induction step.)

Corollary 4.78. *For any set Y and any sequence $f = (a_n)_{n\in\mathbb{N}}$ in Y, it is true that any subsequence of f is also a sequence in Y.*

Corollary 4.79. *Any sequence $f = (a_n)_{n\in\mathbb{N}}$ is a subsequence of itself.*

Proof. We let Y and f be arbitrary sets such that $f = (a_n)_{n\in\mathbb{N}}$ is a sequence in Y. Since the identity function $\mathrm{id}_{\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ according to Corollary 4.68, and because $f = f \circ \mathrm{id}_{\mathbb{N}}$ holds according to (3.610), we see that f is a subsequence of f, by definition. □

Proposition 4.80. *For any partially ordered set $(Y, \leq_Y)$, any sequence $f = (a_n)_{n\in\mathbb{N}}$ in Y, any subsequence $f' = (a'_n)_{n\in\mathbb{N}}$ of f and any element $a \in Y$, it is true that a is a lower bound for f' if a is a lower bound for f.*

Proof. We let Y, $\leq$, f, f' and a be arbitrary such that $(Y, \leq_Y)$ is partially ordered, such that $f = (a_n)_{n\in\mathbb{N}}$ is a sequence in Y and $f' = (a'_n)_{n\in\mathbb{N}}$ a subsequence of f, and such that a is an element of Y. Then, there exists by definition of a subsequence a particular strictly increasing sequence $\bar{g} = (\bar{m}_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ with $f' = f \circ \bar{g}$. Now, to prove the proposed implication directly, we assume that a is a lower bound for the sequence f, which means by definition that

$$\forall y\,(y \in \mathrm{ran}(f) \Rightarrow a \leq_Y y) \tag{4.341}$$

holds. To prove that f' is a also bounded from below by a, we show that a is a lower bound for the range of f', i.e. that

$$\forall y\,(y \in \mathrm{ran}(f') \Rightarrow a \leq_Y y) \tag{4.342}$$

is true. For this purpose, we let y be arbitrary and assume $y \in \mathrm{ran}(f')$ to be true. By definition of a range, there then exists a constant, say $\bar{n}$, such that $(\bar{n}, y) \in f'$ holds. Since f' is a function/sequence, we may write this as

$$f'(\bar{n}) = a'_{\bar{n}} = y. \tag{4.343}$$

Due to Exercise 4.26b), we then obtain

$$a'_{\bar{n}} = a_{\bar{m}_{\bar{n}}} \quad [= f(\bar{m}_{\bar{n}})], \tag{4.344}$$

so that $(\bar{m}_{\bar{n}}, a_{\bar{m}_{\bar{n}}}) \in f$ holds. Consequently, $a_{\bar{m}_{\bar{n}}} \in \mathrm{ran}(f)$ is true (using the definition of a range), which finding further implies $a \leq_Y a_{\bar{m}_{\bar{n}}}$ with (4.341), and therefore $a \leq_Y a'_{\bar{n}}$ with (4.344). This in turn implies the desired inequality $a \leq_Y y$ with (4.343), and since y is arbitrary, we may infer from this the truth of (4.342). This proves that $f' = (a'_n)_{n\in\mathbb{N}}$ is a (sub)sequence bounded from below by a. As Y, $\leq_Y$, f, f' and a were also arbitrary, we may now finally conclude that the proposition holds. □

Exercise 4.27. Show for any partially ordered set $(Y, \leq_Y)$, any sequence $f = (a_n)_{n\in\mathbb{N}}$ in Y, any subsequence $f' = (a'_n)_{n\in\mathbb{N}}$ of f and any $u \in Y$ that u is an upper bound for f' if u is an upper bound for f.

(Hint: Proceed in analogy to the proof of Proposition 4.82.)

Corollary 4.81. *For any partially ordered set $(Y, \leq_Y)$ it is true that any subsequence of any bounded sequence in Y is also bounded.*

Proof. Letting $(Y, \leq_Y)$ be an arbitrary partially ordered set, $f = (a_n)_{n\in\mathbb{N}}$ an arbitrary bounded sequence in Y and $f' = (a'_n)_{n\in\mathbb{N}}$ an arbitrary subsequence of f, we have by definition of a bounded function that f is both bounded from below and bounded from above. Accordingly, to prove that f' is a bounded sequence, we show that f' is both bounded from below and bounded from above. Since f is bounded from below, there exists a lower bound for f, say $\bar{a}$, and it follows then with Proposition 4.80 that the subsequence f' of f is also bounded from below (by $\bar{a}$). Furthermore, as f is bounded from above, there exists an upper bound for f, say $\bar{u}$, and it follows then with Exercise 4.27 that the subsequence f' is itself bounded from above (by $\bar{u}$). Since f' is both bounded from below and bounded from above, it therefore follows that f' is bounded. Since $(Y, \leq_Y)$, f and f' were arbitrary, we conclude that the corollary is true. □

Proposition 4.82. *For any partially ordered set $(Y, \leq_Y)$ it is true that any subsequence of any increasing sequence in Y is also increasing.*

Proof. We let Y and $\leq_Y$ be arbitrary such that $(Y, \leq_Y)$ is a partially ordered set, let f be arbitrary such that $f = (a_n)_{n\in\mathbb{N}}$ is an increasing sequence in Y, so that

$$\forall M, N\,([M, N \in \mathbb{N} \wedge M <_{\mathbb{N}} N] \Rightarrow a_M \leq_Y a_N) \tag{4.345}$$

holds by definition of an increasing function/sequence, and we let f' be arbitrary such that $f' = (a'_n)_{n\in\mathbb{N}}$ is a subsequence of f. To show that f' is increasing, we verify (using again the definition of an increasing function)

$$\forall M, N\,([M, N \in \mathbb{N} \wedge M <_{\mathbb{N}} N] \Rightarrow a'_M \leq_Y a'_N) \tag{4.346}$$

For this purpose, we let M and N be arbitrary in $\mathbb{N}$ such that $M <_{\mathbb{N}} N$ holds. Let us now observe in light of the definition of a subsequence that, since f' is a subsequence of f, there exists a strictly increasing sequence of indexes in $\mathbb{N}$, say $(\bar{m}_n)_{n\in\mathbb{N}}$, such that $a'_M = a_{\bar{m}_M}$ as well as $a'_N = a_{\bar{m}_N}$ holds. Since the index sequence is strictly increasing, the initial assumption $M <_{\mathbb{N}} N$ implies $\bar{m}_M <_{\mathbb{N}} \bar{m}_N$. Together with the fact that $\bar{m}_N, \bar{m}_M \in \mathbb{N}$, this implies $a_{\bar{m}_M} \leq_Y a_{\bar{m}_N}$ with (4.345), which inequality we may also write as $a'_M \leq_Y a'_N$ (applying substitution). This proves the implication in (4.346), and since M and N were arbitrary, we therefore conclude that (4.346) is true, which means that the (sub)sequence f' is increasing. As Y, $\leq_Y$, f and f' were also arbitrary, it then follows that the proposition holds, as claimed. □

Exercise 4.28. Prove for any partially ordered set $(Y, \leq_Y)$ that any subsequence of any increasing sequence in Y is decreasing.

(Hint: Proceed in analogy to the proof of Proposition 4.82.)

Corollary 4.83. *For any partially ordered set* $(Y, \leq_Y)$ *it is true that any subsequence of any monotone sequence in* Y *is monotone.*

Exercise 4.29. Verify that all of the sentences in the current Section 4.5 remain valid when $\mathbb{N}$ is replaced by $\mathbb{N}_+$.

4.6. Induction and Recursion Theorems Involving Initial Segments

We begin this section with a useful variant of the Recursion Theorem.

Theorem 4.84 (Recursion Theorem for initial segments). *The following sentences are true for any counting domain $(C, s, 0_C)$, any set A, any $n \in C$, any function $f : \{0_C, \ldots, n\} \times A \to A$, and any $a \in A$.*

a) For an arbitrary constant $\bar{a}$ satisfying $\bar{a} \notin A$, it is true that there exists a unique function

$$\bar{f} : C \times A \cup \{\bar{a}\} \to A \cup \{\bar{a}\}, \quad (n, b) \mapsto \begin{cases} f(n, b) & \text{if } (n, b) \in \mathrm{dom}(f), \\ \bar{a} & \text{if } (n, b) \notin \mathrm{dom}(f) \end{cases} \tag{4.347}$$

b) Then, there is a unique function $\bar{u} : C \to A \cup \{\bar{a}\}$ such that

$$(1) \;\; \bar{u}(0_C) \quad = a, \tag{4.348}$$

$$(2) \;\; \bar{u}(s(m)) = \bar{f}(m, \bar{u}(m)) \quad \text{for any } m \in C. \tag{4.349}$$

c) Moreover, for any argument $m \in C$ which is less than or equal to n, it is true that the corresponding value of $\bar{u}$ is in A, that is,

$$\forall m \, (m \in C \Rightarrow [m \leq_C s(n) \Rightarrow \bar{u}(m) \in A]). \tag{4.350}$$

d) Furthermore, the restriction $u = \bar{u} \restriction \{0_C, \ldots, s(n)\}$ is a function from $\{0_C, \ldots, s(n)\}$ to A with

$$(1) \;\; u(0_C) \;\; = a, \tag{4.351}$$

$$(2) \;\; u(s(i)) = f(i, u(i)) \quad \text{for any } i \in \{0_C, \ldots, n\}. \tag{4.352}$$

Proof. We let C, s, 0_C, A, n, f and a be arbitrary and assume $(C, s, 0_C)$ to be a counting domain, n to be an element of C, f to be a function from $\{0_C, \ldots, n\} \times A$ to A, and a to be an element of A.

Concerning a), we first verify the universal sentence

$$\forall x \, (x \in C \times A \cup \{\bar{a}\} \Rightarrow \exists! y \, ([x \in \{0_C, \ldots, n\} \times A \Rightarrow y = f(x)] \\ \wedge \; [x \notin \{0_C, \ldots, n\} \times A \Rightarrow y = \bar{a}])). \tag{4.353}$$

For this purpose, we let x be arbitrary, assume $x \in C \times A \cup \{\bar{a}\}$ to be true, and prove the existential part first. To do this, we consider the two cases $x \in \{0_C, \ldots, n\} \times A$ and $x \notin \{0_C, \ldots, n\} \times A$.

In the first case $x \in \{0_C, \ldots, n\} \times A$, it follows from the initial assumption $f : \{0_C, \ldots, n\} \times A \to A$ with the Function Criterion that there exists a unique element $\bar{y}$ in A such that $\bar{y} = f(x)$ holds. Thus, the implication

$$x \in \{0_C, \ldots, n\} \times A \ \Rightarrow \ \bar{y} = f(x) \tag{4.354}$$

is true; furthermore, as $x \notin \{0_C, \ldots, n\} \times A$ is clearly false in the present first case, the implication

$$x \notin \{0_C, \ldots, n\} \times A \ \Rightarrow \ \bar{y} = \bar{a} \tag{4.355}$$

(having a false antecedent) is also true, so that the existential part holds. To prove the uniqueness part, we let y and y' be arbitrary such that

$$\begin{aligned}&([x \in \{0_C, \ldots, n\} \times A \Rightarrow y = f(x)] \wedge [x \notin \{0_C, \ldots, n\} \times A \Rightarrow y = \bar{a}]) \\ &\wedge \ ([x \in \{0_C, \ldots, n\} \times A \Rightarrow y' = f(x)] \wedge [x \notin \{0_C, \ldots, n\} \times A \Rightarrow y' = \bar{a}])\end{aligned} \tag{4.356}$$

holds. The current case assumption implies in particular $y = f(x)$ and $y' = f(x)$, which equations then give $y = y'$, proving the uniqueness part. Thus, the uniquely existential sentence in (4.353) is true in the first case.

In the second case $x \notin \{0_C, \ldots, n\} \times A$, we observe in light of (1.109) that there exists the unique element $\bar{y}$ such that $\bar{y} = \bar{a}$ holds. Thus, the implication (4.355) is then true, and the implication (4.354) also holds since the antecedent $x \in \{0_C, \ldots, n\} \times A$ is false in current case. This proves the existential part, and to prove the uniqueness part, we let y and y' be arbitrary such that the multiple conjunction (4.356) is satisfied. The current case assumption then yields in particular $y = \bar{a}$ and $y' = \bar{a}$, so that we obtain $y = y'$, which proves the uniqueness part. Therefore, the uniquely existential sentence in (4.353) is true in any case. As x was arbitrary, it then follows with Theorem 3.160 that there exists a unique function $\bar{f}$ with domain $C \times A \cup \{\bar{a}\}$ satisfying

$$\begin{aligned}\forall x\, (x \in C \times A \cup \{\bar{a}\} \ \Rightarrow ([&x \in \{0_C, \ldots, n\} \times A \Rightarrow \bar{f}(x) = f(x)] \\ &\wedge \ [x \notin \{0_C, \ldots, n\} \times A \Rightarrow \bar{f}(x) = \bar{a}])),\end{aligned} \tag{4.357}$$

which mapping we may also write as

$$x = (n, b) \mapsto \bar{f}(x) = \bar{f}(n, b) = \begin{cases} f(n, b) & \text{if } (n, b) \in \mathrm{dom}(f), \\ \bar{a} & \text{if } (n, b) \notin \mathrm{dom}(f) \end{cases} \tag{4.358}$$

It remains for us to verify that $A \cup \{\bar{a}\}$ is a codomain of $\bar{f}$, i.e. that $\mathrm{ran}(\bar{f}) \subseteq A \cup \{\bar{a}\}$ holds. For this purpose, we prove the equivalent (using the definition of a subset)

$$\forall y\, (y \in \mathrm{ran}(\bar{f}) \Rightarrow y \in A \cup \{\bar{a}\}). \tag{4.359}$$

Letting y be arbitrary and assuming $y \in \mathrm{ran}(\bar{f})$ to be true, we then see that there exists (by definition of a range) a constant, say $\bar{x}$, such that $(\bar{x}, y) \in \bar{f}$, which we may also write as $y = \bar{f}(\bar{x})$. We now use the fact that the disjunction $\bar{x} \in \mathrm{dom}(f) \vee \bar{x} \notin \mathrm{dom}(f)$ is true according to the Law of the Excluded Middle to prove the sentence

$$\bar{f}(\bar{x}) \in A \cup \{\bar{a}\} \tag{4.360}$$

by cases. On the one hand, if $\bar{x} \in \mathrm{dom}(f)$ is true, then (4.358) yields the equation $\bar{f}(\bar{x}) = f(\bar{x})$. Thus, $\bar{f}(\bar{x})$ is (according to the Function Criterion) an element of the codomain A of f, and this finding $\bar{f}(\bar{x}) \in A$ further implies (4.360) by definition of the union of two sets.

On the other hand, if $\bar{x} \notin \mathrm{dom}(f)$ is true, then (4.358) gives $\bar{f}(\bar{x}) = \bar{a}$, which implies and $\bar{f}(\bar{x}) \in \{\bar{a}\}$ with (2.169), and then (4.360), using again the definition of the union of two sets.

Thus, the proof by cases is complete, and (4.360) in turn proves the implication in (4.359). Since y is arbitrary, we may therefore conclude that the universal sentence (4.359) is true, so that the inclusion $\mathrm{ran}(\bar{f}) \subseteq A \cup \{\bar{a}\}$ holds indeed. Thus, the set $A \cup \{\bar{a}\}$ is a codomain of the function $\bar{f}$, so that the proof of a) is complete.

Concerning b), we observe that the element $a \in A$ is also in $\bar{A} = A \cup \{\bar{a}\}$, so that we may apply the Recursion Theorem to define $\bar{u}$ in (4.348) – (4.349) by using the function $\bar{f} : C \times \bar{A} \to \bar{A}$.

Concerning c), we apply a proof by mathematical induction. Regarding the base case ($m = 0_C$), we prove

$$0_C \leq_C s(n) \;\Rightarrow\; \bar{u}(0_C) \in A \tag{4.361}$$

On the one hand, we see that the initially assumed $n \in C$ implies

$$0_C \leq_C n <_C s(n)$$

with (4.184) and (4.142), and therefore $0_C <_C s(n)$ with the Transitivity Formula for $\leq$ and $<$. Then, the disjunction $0_C <_C s(n) \vee 0_C = s(n)$ is also true, which in turn implies $0_C \leq_C s(n)$ with the Characterization of induced irreflexive partial orderings. On the other hand, we have $\bar{u}(0_C) = a$ $(\in A)$ with (4.348) and the initial assumption $a \in A$, so that $\bar{u}(0_C) \in A$ holds. Thus, the antecedent and the consequent of the implication (4.361) are true, so that the implication itself is true in the base case.

Regarding the induction step, we let m be arbitrary in C, make the induction assumption

$$m \leq_C s(n) \;\Rightarrow\; \bar{u}(m) \in A, \tag{4.362}$$

and show that this implies

$$s(m) \leq_C s(n) \;\Rightarrow\; \bar{u}(s(m)) \in A. \tag{4.363}$$

We now prove the latter implication directly, assuming $s(m) \leq_C s(n)$ to be true. Since $m \in C$ implies $m <_C s(m)$ with (4.142), we obtain $m <_C s(n)$ with the Transitivity Formula for $<$ and $\leq$. Then, the disjunction $m <_C s(n) \lor m = s(n)$ also holds, which implies $m \leq_C s(n)$ with the Characterization of induced irreflexive partial orderings. Now, the preceding inequality gives $\bar{u}(m) \in A$ with the induction assumption (4.362). Furthermore, the assumed $s(m) \leq_C s(n)$ implies $m \leq_C n$ with (4.158) and then $m \in \{0_C, \ldots, n\}$ with the Characterization of initial segments. Thus, the two preceding findings $m \in \{0_C, \ldots, n\}$ and $\bar{u}(m) \in A$ yield

$$(m, \bar{u}(m)) \in \{0_C, \ldots, n\} \times A \quad [= \mathrm{dom}(f)]$$

with the definition of the Cartesian product of two sets (and the initial assumption for f), that is, $(m, \bar{u}(m)) \in \mathrm{dom}(f)$. Therefore, we obtain

$$\begin{aligned} \bar{u}(s(m)) &= \bar{f}(m, \bar{u}(m)) \\ &= f(m, \bar{u}(m)) \quad [\in A] \end{aligned}$$

with (4.349), (4.347), and the initial assumption that A is a codomain of f. These equations give $\bar{u}(s(m)) \in A$, which proves (4.363). As m is arbitrary, we may therefore conclude that the induction step is also true, so that the proof of (4.350) via mathematical induction is complete.

Concerning d), we first notice that the restriction $u = \bar{u} \restriction \{0_C, \ldots, s(n)\}$ is a function from $\{0_C, \ldots, s(n)\}$ to $A \cup \{\bar{a}\}$ in view of Proposition 3.164. Furthermore, we obtain

$$\begin{aligned} u(0_C) &= (\bar{u} \restriction \{0_C, \ldots, s(n)\})(0_C) \\ &= \bar{u}(0_C) \\ &= a \end{aligned}$$

with the preceding notation for u, the definition of a restriction, and (4.348), which proves (4.351). To verify (4.352), we let $i \in \{0_C, \ldots, n\}$ be arbitrary, so that we obtain $i \leq_C n$ with the Characterization of initial segments, which in turn implies $s(i) \leq_C s(n)$ with (4.162), and this inequality yields $s(i) \in \{0_C, \ldots, s(n)\}$ [$= \mathrm{dom}(u)$] again with the Characterization of initial segments; thus, $s(i) \in \mathrm{dom}(u)$ holds. We then obtain the equations

$$\begin{aligned} u(s(i)) &= \bar{u} \restriction \{0_C, \ldots, s(n)\})(s(i)) \\ &= \bar{u}(s(i)) \\ &= \bar{f}(i, \bar{u}(i)) \end{aligned} \tag{4.364}$$

using the notation for u, the definition of a restriction, and (4.349). Let us now observe that the previously established inequalities $i \leq_C n$ and $n <_C s(n)$ imply with the Transitivity Formula for $\leq$ and $<$ the truth of $i <_C s(n)$, which inequality then yields $\bar{u}(i) \in A$ with (4.350) as well as $i \in \{0_C, \ldots, s(n)\}$ with the Characterization of initial segments. Consequently, we obtain the equations

$$\begin{aligned} u(s(i)) &= \bar{f}(i, \bar{u}(i)) \\ &= f(i, \bar{u}(i)) \\ &= f(i, \bar{u} \upharpoonright \{0_C, \ldots, s(n)\}(i)) \\ &= f(i, u(i)) \end{aligned}$$

with (4.364), (4.347) in connection with the fact that $i \in \{0_C, \ldots, n\}$ and $\bar{u}(i) \in A$ evidently give $(i, \bar{u}(i)) \in \mathrm{dom}(f)$, then the definition of a restriction in connection with the the previously found $i \in \{0_C, \ldots, s(n)\}$, and finally the notation for u. We thus showed that $u(s(i)) = f(i, u(i))$ is true, and since i is arbitrary, we may therefore conclude that (4.352) also holds. These findings will also show that A is a codomain of u, i.e. that the range of u is included in A. To establish $\mathrm{ran}(u) \subseteq A$, we may apply the definition of a subset and prove equivalently

$$\forall y \, (y \in \mathrm{ran}(u) \Rightarrow y \in A). \tag{4.365}$$

Letting y be arbitrary and assuming $y \in \mathrm{ran}(u)$ to be true, there exists (by definition of a range) a constant, say $\bar{m}$, such that $(\bar{m}, y) \in u$ holds. Thus, $\bar{m}$ is an element of the domain $\{0_C, \ldots, s(n)\}$ of u, so that $\bar{m} \leq_C s(n)$ follows to be true with the Characterization of initial segments, and this inequality yields $\bar{u}(\bar{m}) \in A$ with (4.350). Furthermore, we may write $(\bar{m}, y) \in u$ on the one hand as $y = u(\bar{m})$, and on the other hand as $(\bar{m}, y) \in \bar{u} \upharpoonright \{0_C, \ldots, s(n)\}$ $[= u]$. The latter implies $(\bar{m}, y) \in \bar{u}$ with the definition of a restriction, which we may then also write as $y = \bar{u}(\bar{m})$. Combining the previous two equations for y via substitution, we obtain $u(\bar{m}) = \bar{u}(\bar{m})$, so that the previously established $\bar{u}(\bar{m}) \in A$ gives $[y =]\, u(\bar{m}) \in A$, and thus the desired $y \in A$. Since y is arbitrary, we may therefore conclude that the universal sentence (4.365), from which we infer with the definition of a subset that the range of u is indeed included in A. Thus, A is a codomain of u, which finding completes the proof of d).

Since C, s, 0_C, A, n, f and a were initially arbitrary in the proofs of a) – d), we may finally conclude that the stated theorem is true. □

Note 4.15. For any counting domain $(C, s, 0_C)$, any set A, any $n \in C$, any function $f : \{0_C, \ldots, n\} \times A \to A$ and any $a \in A$, it is true that there exists a unique function $u : \{0, \ldots, n^+\} \to A$ satisfying (4.351) and (4.352).

Before we restate this note for the particular counting domains $(\mathbb{N}, s^+, 0)$ and $(\mathbb{N}_+, s^+ \restriction \mathbb{N}_+, 1)$, we introduce the following alternative definitions of a sequence.

Definition 4.8 (Sequence on an initial segment of $\mathbb{N}$ & $\mathbb{N}_+$). We say that a family $(a_i \,|\, i \in I)$ is a sequence if

- $I = \{0, \ldots, n\}$ or
- $I = \{1, \ldots, n\}$

holds for some $n \in \mathbb{N}$.

Note 4.16. Since every initial segment $\{0, \ldots, m\}$ is identical with the natural number m^+ according to (4.92), it makes sense to speak of a sequence also in case of

- $I = n$,

where the natural number $n = 0 = \emptyset$ represents the empty index set, and where $n \neq 0$ represents (according to the Characterization of the elements of a counting set) the successor $n = m^+$ of some natural number m defining the initial segment $\{0, \ldots, m\}$.

We give here a first simple example of a sequence on an initial segment.

Proposition 4.85. *It is true for any set Y and any constants $y_1, y_2 \in Y$ that there exists a unique sequence $s = (a_i \,|\, i \in \{1, 2\})$ in Y with terms $a_1 = y_1$ and $a_2 = y_2$.*

Proof. We let Y, y_1 and y_2 be arbitrary and assume $y_1, y_2 \in Y$ to be true. According to Corollary 3.156, the singletons $\{(1, y_1)\}$ and $\{(2, y_2)\}$ are functions with domains $\{1\}$ and $\{2\}$, respectively. Since $1 \neq 2$ is true according to (4.167), we obtain now $\{1\} \cap \{2\} = \emptyset$ with (2.174), so that $\{(1, y_1)\}$ and $\{(2, y_2)\}$ are compatible functions because of Exercise 3.73. Consequently, the pair formed by $\{(1, y_1)\}$ and $\{(2, y_2)\}$ constitutes a compatible set of functions due to Proposition 3.174. We then obtain for its union

$$s = \bigcup\{\{(1, y_1)\}, \{(2, y_2)\}\} = \{(1, y_1)\} \cup \{(2, y_2)\} = \{(1, y_1), (2, y_2)\} \tag{4.366}$$

using (2.209) and (2.226), which constitutes a function with domain

$$\mathrm{dom}(s) = \bigcup\{\{1\}, \{2\}\} = \{1\} \cup \{2\} = \{1, 2\} = \{1, \ldots, 2\}$$

according to Theorem 3.175, (2.209), (2.226) and (4.242). Thus, we may write s as a sequence $(a_i \,|\, i \in \{1, 2\})$ with terms $a_i = s(i)$ for all $i \in \{1, 2\}$.

Observing in light of (2.151) and (4.366) that $(1, y_1) \in s$ and $(2, y_2) \in s$ are true, we obtain for the terms of s $y_1 = s_1$ $[= a_1]$ and $y_2 = s_2$ $[= a_2]$. It now remains for us to prove that s is a sequence in Y, i.e. that $\mathrm{ran}(s) \subseteq Y$ holds. For this purpose, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall y \, (y \in \mathrm{ran}(s) \Rightarrow y \in Y), \tag{4.367}$$

letting y be arbitrary and assuming $y \in \mathrm{ran}(s)$ to be true. Consequently, there is by definition of a range a constant $\bar{i}$ with $(\bar{i}, y) \in s$. This finding gives $y = s_{\bar{i}}$ (using sequence notation) as well as $\bar{i} \in \{1, 2\}$ $[= \mathrm{dom}(s)]$ (by definition of a domain). The latter yields the true disjunction $\bar{i} = 1 \vee \bar{i} = 2$ with the definition of a pair, which we use to prove $y \in Y$ by cases. The first case $\bar{i} = 1$ yields $y = s_1 = y_1$ $[\in Y]$, and the second case $\bar{i} = 2$ implies $y = s_2 = y_2$ $[\in Y]$. Thus, $y \in Y$ holds in any case. Since y was arbitrary, we may therefore conclude that the universal sentence (4.367) is true, so that the inclusion $\mathrm{ran}(s) \subseteq Y$ follows to be true (by definition of a subset). Thus, s is indeed a sequence in Y (by definition of a codomain), having the terms $a_1 = y_1$ and $a_2 = y_2$. Because Y, y_1 and y_2 were initially arbitrary, we may infer from this the truth of the proposition. □

Proposition 4.86. *For any set Y there exists a unique set $Y^{<\mathbb{N}}$ containing precisely every sequence in Y whose domain is some natural number.*

Proof. Letting Y be an arbitrary set, we may apply the Axiom of Specification in connection with the Equality Criterion for sets to obtain the true uniquely existential sentence

$$\exists! Y^{<\mathbb{N}} \forall f \, (f \in Y^{<\mathbb{N}} \Leftrightarrow f \in \mathcal{P}(\mathbb{N} \times Y) \wedge \exists n \, (n \in \mathbb{N} \wedge f \in Y^n)).$$

Thus, the set $Y^{<\mathbb{N}}$ satisfies

$$\forall f \, (f \in Y^{<\mathbb{N}} \Leftrightarrow [f \in \mathcal{P}(\mathbb{N} \times Y) \wedge \exists n \, (n \in \mathbb{N} \wedge f \in Y^n)]). \tag{4.368}$$

We may now show that

$$\forall f \, (f \in Y^{<\mathbb{N}} \Leftrightarrow \exists n \, (n \in \mathbb{N} \wedge f \in Y^n)) \tag{4.369}$$

also holds. Letting f be arbitrary we notice that assuming $f \in Y^{<\mathbb{N}}$ implies with (4.368) in particular the existential sentence on the right-hand side of the equivalence, so that the first part ('$\Rightarrow$') of the equivalence holds. Conversely, assuming now that existential sentence to be true, there exists a natural number, say $\bar{n}$, with $f \in Y^{\bar{n}}$. Thus, f is the function $f : \bar{n} \to Y$, and f therefore is a subset of the Cartesian product $\bar{n} \times Y$ according to

Proposition 3.146. Since $\bar{n} \subseteq \mathbb{N}$ and $Y \subseteq Y$ are true because of (2.320) and (2.10), respectively, we then obtain $\bar{n} \times Y \subseteq \mathbb{N} \times Y$ with Proposition 3.8. Together with $f \subseteq \bar{n} \times Y$, this inclusion implies $f \subseteq \mathbb{N} \times Y$ with the transitivity of $\subseteq$ (in the sense of Proposition 2.5). Consequently, $f \in \mathcal{P}(\mathbb{N} \times Y)$ holds by definition of a power set, and the conjunction of this and the assumed existential sentence further implies $f \in Y^{<\mathbb{N}}$ with (4.368). This proves the second part ('$\Leftarrow$') of the equivalence in (4.369), and since f was arbitrary, we may therefore conclude that the universal sentence (4.369) is true. Thus, $Y^{<\mathbb{N}}$ containing precisely any sequence $f : n \to Y$ where n is some natural number. As Y was initially arbitrary, we thus see that the proposed sentence is true. □

Notation 4.6. For any natural numbers k and n with $k \leq n$ and

- for any sequence $s = (a_i \,|\, i \in \{0, \ldots, n\})$, we will write
$$(a_i \,|\, i \in \{0, \ldots, k\}) = s \upharpoonright \{0, \ldots, k\}. \tag{4.370}$$
- for any sequence $s = (a_i \,|\, i \in \{1, \ldots, n\})$, we will write
$$(a_i \,|\, i \in \{1, \ldots, k\}) = s \upharpoonright \{1, \ldots, k\}. \tag{4.371}$$

Then preceding notations immediately result in the following inclusions by means of Corollary 3.23 and Proposition 3.34.

Corollary 4.87. *For any natural numbers k and n with $k \leq n$ and*

a) for any sequence $s = (a_i \,|\, i \in \{0, \ldots, n\})$, it is true that
$$(a_i \,|\, i \in \{0, \ldots, k\}) \subseteq (a_i \,|\, i \in \{0, \ldots, n\}), \tag{4.372}$$
$$\mathrm{ran}((a_i \,|\, i \in \{0, \ldots, k\})) \subseteq \mathrm{ran}((a_i \,|\, i \in \{0, \ldots, n\})). \tag{4.373}$$

b) for any sequence $s = (a_i \,|\, i \in \{1, \ldots, n\})$, it is true that
$$(a_i \,|\, i \in \{1, \ldots, k\}) \subseteq (a_i \,|\, i \in \{1, \ldots, n\}), \tag{4.374}$$
$$\mathrm{ran}((a_i \,|\, i \in \{1, \ldots, k\})) \subseteq \mathrm{ran}((a_i \,|\, i \in \{1, \ldots, n\})). \tag{4.375}$$

Notation 4.7. For any $n \in \mathbb{N}$ we will write for the *union of a sequence of sets* $s = (A_i \,|\, i \in \{0, \ldots, n\})$ and $s = (A_i \,|\, i \in \{1, \ldots, n\})$, respectively,
$$\bigcup_{i=0}^{n} A_i, \tag{4.376}$$
$$\bigcup_{i=1}^{n} A_i. \tag{4.377}$$

Similarly, we write for the *intersection of a sequence of sets s*, respectively,

$$\bigcap_{i=0}^{n} A_i, \tag{4.378}$$

$$\bigcap_{i=1}^{n} A_i. \tag{4.379}$$

In the same way we denote the *Cartesian product of a sequence of sets s*, respectively, by

$$\bigtimes_{i=0}^{n} A_i, \tag{4.380}$$

$$\bigtimes_{i=1}^{n} A_i. \tag{4.381}$$

Here, we call every element of $\bigtimes_{i=1}^{n} A_i$ an *n-tuple*, symbolically

$$(a_1, \ldots, a_n), \tag{4.382}$$

or an *n-vector* symbolized by

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \tag{4.383}$$

In case of the *Cartesian product of a constant sequence of sets s* on $\{1, \ldots, n\}$ with value A, we also use the notation

$$\bigtimes_{i=1}^{n} A = A^n \tag{4.384}$$

and speak then of the n-th Cartesian power of A.

The following proposition shows that Cartesian powers constitute relatively simply structured sets of functions.

Proposition 4.88. *It is true for any natural number n that the n-th Cartesian power of a set A is identical with the set of functions from the initial segment of $\mathbb{N}_+$ up to n to A, that is,*

$$\forall n, A\,(n \in \mathbb{N} \Rightarrow A^n = A^{\{1,\ldots,n\}}). \tag{4.385}$$

Proof. We take an arbitrary natural number n and an arbitrary set A, and we prove the equation by means of the Equality Criterion for sets. For this purpose, we prove the universal sentence

$$\forall s\,(s \in A^n \Leftrightarrow s \in A^{\{1,\ldots,n\}}), \tag{4.386}$$

letting s be arbitrary. To prove the first part ('$\Rightarrow$') of the equivalence, we assume $s \in A^n$, which shows in light of (4.384) and the definition of the Cartesian product of a family of sets that s is a family with index set $\{1,\ldots,n\}$ and terms satisfying

$$\forall i\,(i \in \{1,\ldots,n\} \Rightarrow s_i \in A). \tag{4.387}$$

Thus, s is a function with domain $\{1,\ldots,n\}$, and we can also show that A is a codomain of s, i.e. that the inclusion $\mathrm{ran}(s) \subseteq A$ holds. To do this, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall y\,(y \in \mathrm{ran}(s) \Rightarrow y \in A). \tag{4.388}$$

We take an arbitrary set y and assume that $y \in \mathrm{ran}(s)$ holds. The definition of a range gives us then a particular constant $\bar{k}$ satisfying $(\bar{k}, y) \in s$, which we can write in function/family notation as $y = s_{\bar{k}}$. Furthermore, $(\bar{k}, y) \in s$ implies with the definition of a domain $\bar{k} \in \{1,\ldots,n\}$, and this further implies $s_{\bar{k}} \in A$ with (4.387). Applying now substitution based on the preceding equation results in $y \in A$, which is the desired consequent of the implication in (4.388). Since y is arbitrary, we may therefore conclude that the universal sentence (4.388) holds, so that the equivalent inclusion $\mathrm{ran}(s) \subseteq A$ is indeed true. This means that A is a codomain of the function s (with domain $\{1,\ldots,n\}$), so that s turns out to be an element of the set of functions $A^{\{1,\ldots,n\}}$, as desired.

Regarding the second part ('$\Leftarrow$') of the equivalence in (4.386), we assume now $s \in A^{\{1,\ldots,n\}}$ to be true, so that s is a function with domain $\{1,\ldots,n\}$ and codomain A. Thus, we may view s as a family with index set $\{1,\ldots,n\}$. In addition, we see that s satisfies (4.387), since letting i be arbitrary and assuming i to be an element of the domain $\{1,\ldots,n\}$ of s implies with the Function Criterion that the uniquely existing function value s_i is in the codomain A of s; because i is arbitrary, (4.387) follows then indeed to be true. These findings demonstrate now in view of the definition of the Cartesian product of a family of sets and in view of the notation for Cartesian powers that $s \in A^n$ holds.

We thus proved the equivalence in (4.386), in which s was arbitrary, so that the universal sentence (4.386) follows to be true. This in turn completes the proof of the equation $A^n = A^{\{1,\ldots,n\}}$, and as n and A were initially both arbitrary, we conclude that the stated proposition holds. □

We now give a first useful application of Cartesian powers in the context of intervals.

Exercise 4.30. Define for any partially ordered set $(Y, \leq_Y)$, any $n \in \mathbb{N}$ and any n-tuples $a = (a_i \,|\, i \in \{1, \ldots, n\})$ and $b = (b_i \,|\, i \in \{1, \ldots, n\})$ in Y the n-tuples $([a_i, b_i]_Y \,|\, i \in \{1, \ldots, n\})$ and $((a_i, b_i)_Y \,|\, i \in \{1, \ldots, n\})$.

(Hint: Apply Function definition by replacement.)

Theorem 4.89 (Characterization of closed & open intervals in Y^n as Cartesian products of intervals in Y). *For any partially ordered set $(Y, \leq_Y)$ and any $n \in \mathbb{N}$, it is true that*

a) each closed interval in Y^n with respect to the reflexive partial ordering $\preceq$ of Y^n can be written as a Cartesian product of a family of closed intervals in Y, in the sense that

$$\forall a, b\,(a, b \in Y^n \Rightarrow [a, b]_{Y^n} = \bigtimes_{i=1}^{n} [a_i, b_i]_Y). \tag{4.389}$$

b) each open interval in Y^n with respect to the irreflexive partial ordering $\prec$ of Y^n (derived from the induced irreflexive partial ordering $<_Y$) can be written as a Cartesian product of a family of open intervals in Y, in the sense that

$$\forall a, b\,(a, b \in Y^n \Rightarrow (a, b)_{Y^n} = \bigtimes_{i=1}^{n} (a_i, b_i)_Y). \tag{4.390}$$

Proof. We let $(Y, \leq_Y)$ be an arbitrary partially ordered set (where $\leq_Y$ is reflexive) and n an arbitrary natural number. Then, the reflexive partial ordering $\preceq$ of $Y^n = Y^{1,\ldots,n}$ is defined through Proposition 3.251 (applying also Proposition 4.88. Letting now $a, b \in Y^n$ be arbitrary, we observe that these constants can be written as the families $a = (a_i \,|\, i \in \{1, \ldots, n\})$ and $b = (b_i \,|\, i \in \{1, \ldots, n\})$. Exercise 4.30 demonstrates that the n-tuple $([a_i, b_i]_Y \,|\, i \in \{1, \ldots, n\})$ is defined. We apply the Equality Criterion for sets to establish the stated equation and prove accordingly

$$\forall x\,(x \in [a, b]_{Y^n} \Leftrightarrow x \in \bigtimes_{i=1}^{n} [a_i, b_i]_Y). \tag{4.391}$$

We taking an arbitrary x. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $x \in [a, b]_{Y^n}$. The definition of a closed interval with respect to the reflexive partial ordering $\preceq$ of Y^n shows on the one hand that x is an element of Y^n, which may therefore be written as the family $x =$

$(x_i \,|\, i \in \{1, \ldots, n\})$. On the other hand, the inequalities $a \preceq x \preceq b$ hold. Consequently, the universal sentences

$$\forall i \,(i \in \{1, \ldots, n\} \Rightarrow a_i \leq_Y x_i), \tag{4.392}$$

$$\forall i \,(i \in \{1, \ldots, n\} \Rightarrow x_i \leq_Y b_i) \tag{4.393}$$

hold according to (3.903), which allow us to establish the truth of

$$\forall i \,(i \in \{1, \ldots, n\} \Rightarrow x_i \in [a_i, b_i]_Y). \tag{4.394}$$

Letting $i \in \{1, \ldots, n\}$ be arbitrary, we obtain from (4.392) and (4.393) the inequalities $a_i \leq_Y x_i \leq_Y b_i$. Therefore, $x_i \in [a_i, b_i]_Y$ holds by definition of a closed interval in Y, as desired. As i was arbitrary, we may infer from this the truth of the universal sentence (4.394). As x is a family with index set $\{1, \ldots, n\}$, we obtain $x \in \bigtimes_{i=1}^{n} [a_i, b_i]_Y$ due to (3.858), which completes the proof of the first part of the equivalence in (4.391).

To establish the second part ('$\Leftarrow$'), we assume now $x \in \bigtimes_{i=1}^{n} [a_i, b_i]_Y$ to be true. Clearly, this implies the truth of (4.394). For an arbitrary $i \in \{1, \ldots, n\}$, that universal sentence yields $x_i \in [a_i, b_i]_Y$, so that the inequalities $a_i \leq_Y x_i$ and $x_i \leq_Y b_i$ evidently hold. We may therefore conclude that the universal sentences (4.392) and (4.392) are both true. Clearly, these imply the inequalities $a \preceq x$ and $x \preceq b$ and therefore $x \in [a, b]_{Y^n}$, as desired. Thus, the proof of the equivalence is now complete, in which x is arbitrary, so that the universal sentence (4.391) also holds. Consequently, the sets $[a, b]_{Y^n}$ and $\bigtimes_{i=1}^{n} [a_i, b_i]_Y$ are identical. Since a and b were arbitrary, we may infer from the truth of this equation the truth of the universal sentence (4.389). As the sets $(Y, \leq_Y)$ and n were initially also arbitrary, we may now finally conclude that Part a) of the stated theorem is true. Part b) can be proved similarly. $\square$

Exercise 4.31. Establish the Characterization of open intervals in Y^n as Cartesian products of intervals in Y.

(Hint: Induce $<_Y$ from $\leq_Y$ and apply Note 3.14) regarding the irreflexive partial ordering $\prec$ defined as in Exercise 3.109.

Exercise 4.32. Show for any set Y that

a) there exists a unique set $Y^{<\mathbb{N}_+}$ containing precisely every sequence in Y whose domain is some initial segment of $\mathbb{N}_+$, in the sense that

$$\forall f \,(f \in Y^{<\mathbb{N}_+} \Leftrightarrow \exists n \,(n \in \mathbb{N} \,\wedge\, f \in Y^{\{1,\ldots,n\}})) \tag{4.395}$$

(Hint: Proceed in analogy to the proof of Proposition 4.86), using now the power set $\mathcal{P}(\mathbb{N}_+ \times Y)$ in connection with (4.240).)

b) there exists the unique sequence $(Y^n)_{n\in\mathbb{N}}$ of Cartesian powers of Y. (Hint: Use replacement in connection with (1.109).)

c) the set $Y^{<\mathbb{N}_+}$ is identical with the union of the sequence of Cartesian powers of Y, that is,

$$Y^{<\mathbb{N}_+} = \bigcup_{n=0}^{\infty} Y^n. \tag{4.396}$$

(Hint: Apply the Equality Criterion for sets in connection with (4.395), (4.385) and the Characterization of the union of a family of sets.)

d) the set $Y^{<\mathbb{N}_+}$ is identical with the singleton formed by the empty set if Y is itself empty, that is,

$$Y = \emptyset \Rightarrow Y^{<\mathbb{N}_+} = \{\emptyset\}. \tag{4.397}$$

(Hint: Apply the Equality Criterion for sets in connection with Proposition 3.152, (4.239), Proposition 3.151 and (2.169).)

Theorem 4.90 (Recursive evaluation of the union & intersection of a sequence of sets on an initial segment of $\mathbb{N}_+$). *The following equations hold for any $n \in \mathbb{N}$ and any sequence $A = (A_i \,|\, i \in \{1,\ldots,n^+\})$.*

$$\bigcup_{i=1}^{n^+} A_i = \left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n^+}, \tag{4.398}$$

$$\bigcap_{i=1}^{n^+} A_i = \left(\bigcap_{i=1}^{n} A_i\right) \cap A_{n^+}. \tag{4.399}$$

Proof. We let n be an arbitrary natural number and A be an arbitrary set such that A is a sequence $(A_i \,|\, i \in \{1,\ldots,n^+\})$. We now establish the proposed equation (4.398) by applying the Equality Criterion for sets, i.e. by proving the universal sentence

$$\forall y\,(y \in \bigcup_{i=1}^{n^+} A_i \Leftrightarrow y \in \left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n^+}). \tag{4.400}$$

For this purpose, we let y be arbitrary and assume first

$$y \in \bigcup_{i=1}^{n^+} A_i \tag{4.401}$$

to be true, which assumption implies the truth of the existential sentence

$$\exists i\,(i \in \{1,\ldots,n^+\} \,\wedge\, y \in A_i) \tag{4.402}$$

with the Characterization of the union of a family of sets. Thus, there is a particular $\bar{k} \in \{1,\ldots,n^+\}$ with $y \in A_{\bar{k}}$. We therefore obtain $\bar{k} \in \{1,\ldots,n\} \cup \{n^+\}$ with (4.241), which we may write also as the disjunction of $\bar{k} \in \{1,\ldots,n\}$ and $\bar{k} \in \{n^+\}$, according to the definition of the union of two sets. We now use this true disjunction to prove

$$y \in \bigcup_{i=1}^{n} A_i \,\vee\, y \in A_{n^+} \tag{4.403}$$

by cases. In the first case of $\bar{k} \in \{1,\ldots,n\}$, we see in light of the previously established $y \in A_{\bar{k}}$ that the existential sentence

$$\exists i\,(i \in \{1,\ldots,n\} \,\wedge\, y \in A_i) \tag{4.404}$$

is true. Since $\bar{k} \in \{1,\ldots,n\}$ implies that $A_{\bar{k}}$ is a term of the restricted sequence $(A_i \,|\, i \in \{1,\ldots,n\})$, the preceding existential sentence yields with the Characterization of the union of a family of sets

$$y \in \bigcup_{i=1}^{n} A_i, \tag{4.405}$$

so that the desired disjunction (4.403) holds indeed in the first case. The second case $\bar{k} \in \{n^+\}$ gives $\bar{k} = n^+$ with (2.169), and therefore $y \in A_{\bar{k}}$ implies $y \in A_{n^+}$ via substitution. Thus, the disjunction (4.403) holds also for the second case, and this disjunction yields now (by definition of the union of two sets)

$$y \in \left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n^+}, \tag{4.406}$$

so that the first part ('$\Rightarrow$') of the equivalence in (4.400) holds.

To establish the second part ('$\Leftarrow$'), we now assume (4.406) to be true, which evidently gives the disjunction (4.403). Let us now apply this disjunction to prove (4.402) by cases. On the one hand, if $y \in \bigcup_{i=1}^{n} A_i$ holds, then the Characterization of the union of a family of sets implies the existence of a particular $\bar{k} \in \{1,\ldots,n\}$ with $y \in A_{\bar{k}}$. Because the inclusion

$$\{1,\ldots,n\} \subseteq \{1,\ldots,n\} \cup \{n^+\} \quad [= \{1,\ldots,n^+\}]$$

is true according to (2.245), we then obtain $\bar{k} \in \{1,\ldots,n^+\}$ by definition of a subset. In view of $y \in A_{\bar{k}}$, the existential sentence (4.402) is thus clearly true in the first case. On the other hand, if $y \in A_{n^+}$ holds, we see in light of the evident fact $n^+ \in \{n^+\}$ that the desired existential sentence is again true, so that the proof by cases is complete. Consequently, we obtain (4.401) with the Characterization of the union of a family of sets, which finding completes the proof of the equivalence.

As y was arbitrary, we may therefore conclude that the universal sentence (4.400) holds, so that the equation (4.398) follows to be true.

We now establish the proposed equation (4.399), applying again the Equality Criterion for sets, i.e. we prove the universal sentence

$$\forall y \, (y \in \bigcap_{i=1}^{n^+} A_i \Leftrightarrow y \in \left(\bigcap_{i=1}^{n} A_i\right) \cap A_{n^+}). \tag{4.407}$$

To do this, we take an arbitrary y and assume first

$$y \in \bigcap_{i=1}^{n^+} A_i \tag{4.408}$$

to be true, which assumption implies

$$\forall i \, (i \in \{1, \ldots, n^+\} \Rightarrow y \in A_i). \tag{4.409}$$

with the Characterization of the intersection of a family of sets. Let us observe now that the desired consequent is equivalent to

$$\forall i \, (i \in \{1, \ldots, n\} \Rightarrow y \in A_i) \, \wedge \, y \in A_{n^+} \tag{4.410}$$

in view of the the Characterization of the intersection of a family of sets and the definition of the intersection of two sets. To prove the first part of this conjunction, we let i be arbitrary, assume $i \in \{1, \ldots, n\}$ to be true, and show that $y \in A_i$ also holds. Since $\{1, \ldots, n\}$ is included in $\{1, \ldots, n^+\}$, the preceding assumption implies $i \in \{1, \ldots, n^+\}$ and therefore $y \in A_i$ with (4.409), as desired. As i was arbitrary, we may therefore conclude that the first part of the conjunction (4.410) holds. Regarding the second part, we observe the truth of $n^+ \in \{1, \ldots, n^+\}$, so that the desired $y \in A_{n^+}$ follows immediately to be true with (4.409). Thus, the first part of the equivalence to be proven holds.

Concerning the second part ('$\Leftarrow$'), we assume $y \in (\bigcap_{i=1}^{n} A_i) \cap A_{n^+}$ to be true, which gives the equivalent conjunction (4.410). We now show that the universal sentence (4.409) follows to be true, letting i be arbitrary and assuming

$$i \in \{1, \ldots, n^+\} \quad [= \{1, \ldots, n\} \cup \{n^+\}]$$

to hold. Consequently, the disjunction $i \in \{1, \ldots, n\} \vee i \in \{n^+\}$ holds, which we now use to prove (4.409) by cases. If $i \in \{1, \ldots, n\}$ is true, then the first part of the conjunction (4.410) gives $y \in A_i$, as required, and if $i \in \{n^+\}$ is true, so that $i = n^+$ holds, then we obtain again $y \in A_i$ via substitution from the second part of the conjunction. Thus, the proof by cases is complete, and since i is arbitrary, we may therefore conclude that the universal sentence (4.409) is true. This finding means that (4.408) holds, so that the equivalence (4.407) is true.

Because y is arbitrary, we may infer from this the truth of (4.407), which universal sentence implies the truth of the the equation (4.399). Because n and A were initially arbitrary, we may now further conclude that the stated theorem is indeed true. □

Exercise 4.33. Establish the following equations for any $n \in \mathbb{N}$ and any sequence $A = (A_i \,|\, i \in \{0, \ldots, n^+\})$.

$$\bigcup_{i=0}^{n^+} A_i = \left(\bigcup_{i=0}^{n} A_i \right) \cup A_{n^+}, \tag{4.411}$$

$$\bigcap_{i=0}^{n^+} A_i = \left(\bigcap_{i=0}^{n} A_i \right) \cap A_{n^+}. \tag{4.412}$$

Corollary 4.91. *For any set A, any $a \in A$, any $n \in \mathbb{N}$ and any function $f : \{0, \ldots, n\} \times A \to A$, it is true that there exists a unique sequence $u = (u_i \,|\, i \in \{0, \ldots, n^+\})$ in A satisfying*

$$(1) \quad u_0 \;= a, \tag{4.413}$$

$$(2) \quad u_{i^+} = f(i, u_i) \quad \text{for any } i \in \{0, \ldots, n\}. \tag{4.414}$$

Corollary 4.92. *For any set A, any $a \in A$, any $n \in \mathbb{N}_+$ and any function $f : \{1, \ldots, n\} \times A \to A$, it is true that there exists a unique sequence $u = (u_i \,|\, i \in \{1, \ldots, n^+\})$ in A satisfying*

$$(1) \quad u_1 \;= a, \tag{4.415}$$

$$(2) \quad u_{i^+} = f(i, u_i) \quad \text{for any } i \in \{1, \ldots, n\}. \tag{4.416}$$

We now introduce another technique for mathematical induction.

Proposition 4.93. *The following implication holds for an arbitrary formula $\varphi(n)$ and for any counting domain $(C, s, 0_C)$. If*

$$\forall n \, (n \in C \Rightarrow [\forall m \, ([m \in C \wedge m <_C n] \Rightarrow \varphi(m)) \Rightarrow \varphi(n)]) \tag{4.417}$$

is true, then the set $\{n : n \in C \wedge \varphi(n)\}$ of all $n \in C$ for which $\varphi(n)$ holds is inductive.

Proof. We let C, s and 0_C be arbitrary such that $(C, s, 0_C)$ is a counting domain and prove the stated implication directly, assuming (4.417) to be true. To show that the set $X = \{n : n \in C \wedge \varphi(n)\}$ (see (4.46) in Exercise 4.2a)) is inductive with respect to $(C, s, 0_C)$, we verify

$$0_C \in X \wedge \forall n\,(n \in X \Rightarrow s(n) \in X) \tag{4.418}$$

according to Note 4.3. To prove the first part of this conjunction, we observe that the fact $0_C \in C$ implies with the assumption (4.417) that

$$\forall m\,([m \in C \wedge m <_C 0_C] \Rightarrow \varphi(m)) \;\Rightarrow\; \varphi(0_C), \tag{4.419}$$

is true. We now verify that the antecedent (i.e., the universal sentence) is true, which will then imply the truth of $\varphi(0_C)$. Letting m be arbitrary, we have that $m \in C \wedge m <_C 0_C$ is false due to (4.186), so that the implication

$$[m \in C \wedge m <_C 0_C] \Rightarrow \varphi(m)$$

is true. As m was arbitrary, we may therefore conclude that the universal sentence in (4.419) holds, which then implies $\varphi(0_C)$ because the implication (4.419) is true. Consequently, the conjunction of $0_C \in C$ and $\varphi(0_C)$ implies the desired $0_C \in X$ by definition of X.

To verify the second part of the conjunction (4.418), we let $n \in X$ be arbitrary, which means that $n \in C$ and $\varphi(n)$ are both true by definition of X. Since the successor function s is a function from C to C, we see that $n \in C$ implies $s(n) \in C$, which further implies

$$\forall m\,([m \in C \wedge m <_C s(n)] \Rightarrow \varphi(m)) \Rightarrow \varphi(s(n)) \tag{4.420}$$

with the assumed antecedent (4.417). Furthermore, since $n \in C$ implies $n <_C s(n)$ according to (4.142), the conjunction $n \in C \wedge n <_C s(n)$ is true; since $\varphi(n)$ also holds, the implication

$$[n \in C \wedge n <_C s(n)] \;\Rightarrow\; \varphi(n) \tag{4.421}$$

is also true. Now, as the implication

$$[m \in C \wedge m <_C s(n)] \Rightarrow \varphi(m)) \;\Rightarrow\; \varphi(s(n))$$

in (4.420) is true for the particular constant $m = n$, we see that the truth of the antecedent (4.421) then implies the truth of $\varphi(s(n))$. Thus, the conjunction $s(n) \in C \wedge \varphi(s(n))$ is true, which in turn gives the desired $s(n) \in X$ (by definition of X). This completes the proof of the conjunction (4.418), so that the set $X = \{n : n \in C \wedge \varphi(n)\}$ is indeed inductive with respect to $(C, s, 0_C)$. Since C, s and 0_C were initially arbitrary, we may therefore conclude that the proposition is true. □

Corollary 4.94. *The following implication holds for an arbitrary formula $\varphi(n)$ and for any counting domain $(C, s, 0_C)$. If (4.417) is true, then $\varphi(n)$ holds for all $n \in C$.*

Proof. Letting $(C, s, 0_C)$ be an arbitrary counting domain, we assume that (4.417) is true, which then implies with Proposition 4.93 that the set $X = \{n : n \in C \land \varphi(n)\}$ is inductive with respect to $(C, s, 0_C)$. It then follows with Exercise 4.2b) that $\varphi(n)$ holds for all elements of C. □

Method 4.6 (Proof by strong induction for counting domains). To prove that a given formula $\varphi(n)$ holds for all elements of the counting set C in a counting domain $(C, s, 0_C)$, we may show that the *induction step* (4.417) is true.

Corollary 4.95. *The following implications hold for an arbitrary formula $\varphi(n)$.*

a) If

$$\forall n\, (n \in \mathbb{N} \Rightarrow [\forall m\, ([m \in \mathbb{N} \land m <_{\mathbb{N}} n] \Rightarrow \varphi(m)) \Rightarrow \varphi(n)]) \tag{4.422}$$

is true, then $\varphi(n)$ is true for all $n \in \mathbb{N}$.

b) If

$$\forall n\, (n \in \mathbb{N}_+ \Rightarrow [\forall m\, ([m \in \mathbb{N}_+ \land m <_{\mathbb{N}_+} n] \Rightarrow \varphi(m)) \Rightarrow \varphi(n)]) \tag{4.423}$$

is true, then $\varphi(n)$ is true for all $n \in \mathbb{N}_+$.

Method 4.7 (Proof by strong induction (for $\mathbb{N}$)). To prove that a given formula $\varphi(n)$ holds for all natural numbers, we may show that the induction step (4.422) is true.

Note 4.17. The assumption of a base case is not necessary since $\varphi(0)$ is implied by (4.422).

Method 4.8 (Proof by strong induction (for $\mathbb{N}_+$)). To prove that a given formula $\varphi(n)$ holds for all positive natural numbers, we may show that the induction step (4.423) is true.

Theorem 4.96 (Strong Recursion Theorem). *The following sentences are true for any set B and any function $F : B^{<\mathbb{N}} \to B$.*

a) There exists a unique function f with domain $\mathbb{N} \times B^{<\mathbb{N}}$ such that

$$x = (n, g) \mapsto \begin{cases} g \cup \{(n, F(g))\} & \text{if } \mathrm{dom}(g) = n \\ \emptyset & \text{if } \mathrm{dom}(g) \neq n \end{cases} \tag{4.424}$$

b) Then, there exists a unique sequence $u = (u_n)_{n\in\mathbb{N}}$ *in* $B^{<\mathbb{N}}$ *satisfying*

$$(1)\ \ u_0 \ = \emptyset, \tag{4.425}$$
$$(2)\ \ u_{n^+} = f(n, u_n) \quad \text{for any } n \in \mathbb{N}, \tag{4.426}$$

and any term u_n *is a sequence in* B *with domain* n*, i.e.*

$$\forall n\, (n \in \mathbb{N} \Rightarrow u_n \in B^n). \tag{4.427}$$

c) Furthermore, the sequence $u = (u_n)_{n\in\mathbb{N}}$ *is isotone, and its range is a compatible set of functions satisfying*

$$\forall u'\, (u' \in \operatorname{ran}(u) \Rightarrow \exists m\, (m \in \mathbb{N} \wedge u' = u_m \wedge u' \in B^m)). \tag{4.428}$$

d) Moreover, that range's union

$$v = \bigcup \operatorname{ran}(u) \tag{4.429}$$

is a sequence in B *with domain* $\mathbb{N}$*, and every term* u_n *of the sequence* $(u_n)_{n\in\mathbb{N}}$ *is identical with the restriction of* v *to* n*, i.e.*

$$\forall n\, (n \in \mathbb{N} \Rightarrow v \restriction n = u_n). \tag{4.430}$$

e) Finally, the sequence $v = (v_n)_{n\in\mathbb{N}}$ *satisfies*

$$\forall n\, (n \in \mathbb{N} \Rightarrow v_n = F(v \restriction n)), \tag{4.431}$$

and v *is the only sequence in* B *with domain* $\mathbb{N}$ *satisfying (4.431), i.e.*

$$\forall v'\, ([v' \in B^{\mathbb{N}} \wedge \forall n\, (n \in \mathbb{N} \Rightarrow v'_n = F(v \restriction n))] \Rightarrow v = v'). \tag{4.432}$$

Proof. We let B and F be arbitrary sets and assume that F is a function from $B^{<\mathbb{N}}$ to B.

Concerning a), we apply Function definition by replacement and prove for this purpose

$$\forall x\, (x \in \mathbb{N} \times B^{<\mathbb{N}} \Rightarrow \exists! y\, (\varphi(x, y))), \tag{4.433}$$

where $\varphi(x, y)$ is the formula

$$\begin{aligned} \exists n, g\, (x = (n, g) \ \wedge\ & [\operatorname{dom}(g) = n \ \Rightarrow\ y = g \cup \{(n, F(g))\}] \\ \wedge\ & [\operatorname{dom}(g) \neq n \ \Rightarrow\ y = \emptyset]). \end{aligned} \tag{4.434}$$

We let $\bar{x}$ be arbitrary in $\mathbb{N} \times B^{<\mathbb{N}}$, so that there exists (by definition of a Cartesian product) an element of $\mathbb{N}$, say $\bar{n}$, and an element of $B^{<\mathbb{N}}$, say $\bar{g}$, such that

$$\bar{x} = (\bar{n}, \bar{g}). \tag{4.435}$$

We now prove the existential part of $\exists! y\,(\varphi(\bar{x}, y))$. According to the Law of the Excluded Middle, the disjunction $\mathrm{dom}(\bar{g}) = \bar{n} \vee \mathrm{dom}(\bar{g}) \neq \bar{n}$ is true, which we now use to prove the sentence $\exists y\,(\varphi(\bar{x}, y))$ by cases. In the first case that $\mathrm{dom}(\bar{g}) = \bar{n}$ holds, we observe in light of (1.109) that there exists a (unique) set $\bar{y}$ such that the equation $\bar{y} = \bar{g} \cup \{(\bar{n}, F(\bar{g}))\}$ holds. Then, the implication

$$\mathrm{dom}(\bar{g}) = \bar{n} \;\Rightarrow\; \bar{y} = \bar{g} \cup \{(\bar{n}, F(\bar{g}))\} \tag{4.436}$$

is true because the antecedent and the consequent are both true. Furthermore, the current case assumption implies with the Double Negation Law that $\mathrm{dom}(\bar{g}) \neq \bar{n}$ is false, so that the implication

$$\mathrm{dom}(\bar{g}) \neq \bar{n} \;\Rightarrow\; \bar{y} = \emptyset \tag{4.437}$$

is true because of the false antecedent. Thus, the conjunction of (4.435), (4.436) and (4.437) is true, so that the formula (4.434) becomes a true existential sentence for $\bar{x}$ and $\bar{y}$. This means that $\varphi(\bar{x}, \bar{y})$ is true, which shows that the existential sentence $\exists y\,(\varphi(\bar{x}, y))$ holds (in the first case). In the second case that $\mathrm{dom}(\bar{g}) \neq \bar{n}$ holds, there exists in view of (1.109) a (unique) set $\bar{y}$ such that $\bar{y} = \emptyset$ holds, so that the implication (4.437) is evidently true (for the current constant $\bar{y} = \emptyset$). Moreover, since the current case assumption means that $\mathrm{dom}(\bar{g}) = \bar{n}$ is false, the implication (4.436) now has a false antecedent and is thus true (for the current constant $\bar{y}$). Consequently, the conjunction of (4.435), (4.436) and (4.437) is true for $\bar{y} = \emptyset$, so that (4.434) constitutes a true existential sentence for $\bar{x}$ and $\bar{y} = \emptyset$. This means that $\varphi(\bar{x}, \bar{y})$ is true for $\bar{y} = \emptyset$, which shows that the existential sentence $\exists y\,(\varphi(\bar{x}, y))$ holds also in the second case, completing the proof of that sentence by cases and thus the proof of the existential part.

To prove the uniqueness part, we verify

$$\forall y', y''\,([\varphi(\bar{x}, y') \,\wedge\, \varphi(\bar{x}, y'')] \Rightarrow y' = y''), \tag{4.438}$$

letting y' and y'' be arbitrary such that $\varphi(\bar{x}, y')$ and $\varphi(\bar{x}, y'')$ are both true. These two sentences mean in view of (4.434) that there are particular constants $\bar{n}', \bar{g}'$ as well as $\bar{n}'', \bar{g}''$ satisfying, respectively,

$$\begin{aligned} \bar{x} = (\bar{n}', \bar{g}') \;\wedge\; & [\mathrm{dom}(\bar{g}') = \bar{n}' \;\Rightarrow\; y' = \bar{g}' \cup \{(\bar{n}', F(\bar{g}'))\}] \\ \wedge\; & [\mathrm{dom}(\bar{g}') \neq \bar{n}' \;\Rightarrow\; y' = \emptyset] \end{aligned} \tag{4.439}$$

and

$$\bar{x} = (\bar{n}'', \bar{g}'') \;\wedge\; [\mathrm{dom}(\bar{g}'') = \bar{n}'' \;\Rightarrow\; y'' = \bar{g}'' \cup \{(\bar{n}'', F(\bar{g}''))\}] \\ \wedge\; [\mathrm{dom}(\bar{g}'') \neq \bar{n}'' \;\Rightarrow\; y'' = \emptyset]. \tag{4.440}$$

Recalling (4.435), the equations

$$(\bar{n}, \bar{g}) = \bar{x} = (\bar{n}', \bar{g}') = (\bar{n}'', \bar{g}'')$$

are then true, so that the equations

$$\bar{n} = \bar{n}' = \bar{n}'' \tag{4.441}$$

$$\bar{g} = \bar{g}' = \bar{g}'' \tag{4.442}$$

follow to be true with the Equality Criterion for ordered pairs. Let us now consider again the cases $\mathrm{dom}(\bar{g}) = \bar{n}$ and $\mathrm{dom}(\bar{g}) \neq \bar{n}$ to prove the sentence $y' = y''$. If $\mathrm{dom}(\bar{g}) = \bar{n}$ holds, then substitution based on (4.441) – (4.442) evidently yields the true equations

$$\mathrm{dom}(\bar{g}') = \bar{n}',$$
$$\mathrm{dom}(\bar{g}'') = \bar{n}'',$$

which then imply with the true implications in (4.439) – (4.440)

$$y' = \bar{g}' \cup \{(\bar{n}', F(\bar{g}'))\},$$
$$y'' = \bar{g}'' \cup \{(\bar{n}'', F(\bar{g}''))\}.$$

We may now carry out substitutions based on the equations (4.441) – (4.442) and based on the fact that $\bar{g} = \bar{g}'$ as well as $\bar{g} = \bar{g}''$ give $F(\bar{g}) = F(\bar{g}')$ as well as $F(\bar{g}) = F(\bar{g}'')$ (using the fact that F is a function in connection with Corollary 3.150), to obtain

$$y' = \bar{g} \cup \{(\bar{n}, F(\bar{g}))\}$$
$$y'' = \bar{g} \cup \{(\bar{n}, F(\bar{g}))\},$$

which equations yield the desired $y' = y''$ via substitution. If on the other hand $\mathrm{dom}(\bar{g}) \neq \bar{n}$ holds, then substitution based on (4.441) – (4.442) results in the true inequalities

$$\mathrm{dom}(\bar{g}') \neq \bar{n}',$$
$$\mathrm{dom}(\bar{g}'') \neq \bar{n}'',$$

which in turn imply with the true implications in (4.439) – (4.440)

$$y' = \emptyset,$$
$$y'' = \emptyset.$$

We therefore obtain $y' = y''$ again, so that the proof of this equation by cases is complete. Since y' and y'' were arbitrary, we may therefore conclude that (4.438) is true. Thus, the uniqueness part of $\exists! y\,(\varphi(\bar{x}, y))$ also holds, completing the proof of this uniquely existential sentence. Because $\bar{x}$ was also arbitrary, we may now further conclude that (4.433) is true, which universal sentence sentence then implies with Theorem 3.160 that there exists a unique function f with domain $\mathbb{N} \times B^{<\mathbb{N}}$ such that $\varphi(x, f(x))$ is true for all $x \in \mathbb{N} \times B^{<\mathbb{N}}$. Thus, every element $x \in \mathbb{N} \times B^{<\mathbb{N}}$ is associated with the unique value $f(x)$, where $x = (n, g)$ (for some n and g) is then evidently mapped according to (4.424) because of (4.434).

Concerning b), let us observe $\emptyset$ is a function from $\emptyset$ to B according to Proposition 3.151, where have $0 = \emptyset$ by definition of the number zero 0; thus, there is a natural number n such that $\emptyset$ is a function from n to B, so that $\emptyset \in B^{<\mathbb{N}}$ holds in view of Proposition 4.86. Together with the previously established defined function $f : \mathbb{N} \times B^{<\mathbb{N}} \to B^{<\mathbb{N}}$, this allows us to apply Method 4.3 to define the sequence $u : \mathbb{N} \to B^{<\mathbb{N}}$ by the recursion

$$\begin{aligned} &(1)\ \ u_0 &&= \emptyset, \\ &(2)\ \ u_{n^+} &&= f(n, u_n) \quad \text{for any } n \in \mathbb{N}. \end{aligned}$$

We now prove (4.427) by mathematical induction. Regarding the base case ($n = 0$), we recall the previously made observation that $u_0 = \emptyset$ is a function from 0 to B, thus an element of B^0. Regarding the induction step, we let n be arbitrary in $\mathbb{N}$, make the induction assumption $u_n \in B^n$, and show that this implies $u_{n^+} \in B^{n^+}$. Using (4.426) and then the definition of the mapping f in (4.424) in connection with the fact that the induction assumption $u_n \in B^n$ gives $u_n : n \to B$ and therefore $\mathrm{dom}(u_n) = n$, we obtain the equations

$$u_{n^+} = f(n, u_n) = u_n \cup \{(n, F(u_n))\}. \tag{4.443}$$

Since $\in_{\mathbb{N}}$ is irreflexive (according to Corollary 4.5) so that $n \notin n$ holds (because of Lemma 4.4c)), we see that n is not an element of the domain n of u_n. Therefore, we may apply Proposition 3.177 to observe that

$$[u_{n^+} =] \quad u_n \cup \{(n, F(u_n))\} \tag{4.444}$$

is a function with domain $n \cup \{n\}$ $(= n^+)$ (recalling the definition of a successor), so that u_{n^+} is a function with domain n^+. Next, we verify that B is a codomain of u_{n^+}, by showing that $\mathrm{ran}(u_{n^+}) \subseteq B$ holds. To do this, we apply the definition of a subset and prove the equivalent

$$\forall y\,(y \in \mathrm{ran}(u_{n^+}) \Rightarrow y \in B). \tag{4.445}$$

We let y be arbitrary and assume $y \in \operatorname{ran}(u_{n^+})$, so that there exists by definition of a range a constant, say $\bar{n}$, such that $(\bar{n}, y) \in u_{n^+}$ holds. In view of (4.444), this implies with the definition of the union of two sets

$$(\bar{n}, y) \in u_n \lor (\bar{n}, y) \in \{(n, F(u_n))\},$$

and we may now use this true disjunction to prove the sentence $y \in B$ by cases. In the first case $(\bar{n}, y) \in u_n$, we obtain $y \in \operatorname{ran}(u_n)$ $[\subseteq B]$ with the definition of a range (recalling that B is codomain of u_n), and this finding implies the desired $y \in B$ with the definition of a subset. In the other case $(\bar{n}, y) \in \{(n, F(u_n))\}$, we obtain $(\bar{n}, y) = (n, F(u_n))$ with (2.169) and therefore in particular $y = F(u_n)$ with the Equality Criterion for ordered pairs. Since B is a codomain of F, we have $F(u_n) \in B$ according to the Function Criterion, so that substitution based on the preceding equation yields $y \in B$ also for the second case. This completes the proof by cases, and the truth of $y \in B$ proves in turn the implication in (4.445). Since y was arbitrary, we may therefore conclude that (4.445) holds, so that the range of u_{n^+} is indeed included in B. Thus, B is codomain also of u_{n^+}, so that u_{n^+} is a function from n^+ to B, and therefore u_{n^+} is an element of B^{n^+}. As n was arbitrary, we may now further conclude that the induction step holds, which completes the proof of (4.427) via by mathematical induction.

Concerning c), we apply the Monotony Criterion for isotone sequences of sets and prove

$$\forall n\, (n \in \mathbb{N} \Rightarrow u_n \subseteq u_{n^+}). \tag{4.446}$$

To do this, we let n be arbitrary in $\mathbb{N}$, so that (4.443) holds. Then, (2.201) shows that $u_n \subseteq u_n \cup \{(n, F(u_n))\}$ $[= u_{n^+}]$ is true. Thus, $u_n \subseteq u_{n^+}$ holds, which proves the implication in (4.446). As n is arbitrary, we may therefore infer from this the truth of the universal sentence (4.446). Due to Corollary 4.71a), it then follows that the sequence $(u_n)_{n \in \mathbb{N}}$ is indeed isotone, and this finding implies

$$\forall m, n\, ([m, n \in \mathbb{N} \land m <_{\mathbb{N}} n] \Rightarrow u_m \subseteq u_n) \tag{4.447}$$

by definition of an increasing sequence.

We now establish (4.428), letting u' be arbitrary and assuming $u' \in \operatorname{ran}(u)$ to be true. Consequently, there is (by definition of a range) a constant, say $\bar{m}$, such that $(\bar{m}, u') \in u$ holds. On the one hand, we may write the latter also in function/sequence notation as $u' = u_{\bar{m}}$. On the other hand, observing that $\operatorname{dom}(u) = \mathbb{N}$ holds, we obtain (by definition of a domain) $\bar{m} \in \mathbb{N}$, which in turn implies $[u' =]\; u_{\bar{m}} \in B^{\bar{m}}$ with (4.427). We thus showed that the conjunction $\bar{m} \in \mathbb{N} \land u' = u_{\bar{m}} \land u' \in B^{\bar{m}}$ is true, so that the desired existential sentence in (4.428) holds. Since u' was arbitrary, we may therefore conclude that (4.428) is true.

Next, we verify that the previously established facts imply the compatibility of the range of the sequence $u = (u_n)_{n\in\mathbb{N}}$ in $B^{<\mathbb{N}}$. For this purpose, we take arbitrary $u', u'' \in \mathrm{ran}(u)$, so that there are in view of (4.428) natural numbers, say $\bar{m}$ and $\bar{n}$, with $u' = u_{\bar{m}}$ $[\in B^{\bar{m}}]$ and $u'' = u_{\bar{n}}$ $[\in B^{\bar{n}}]$. Thus, $u_{\bar{m}}$ and $u_{\bar{n}}$ are functions (with codomain B), and we now show that $u_{\bar{m}}$ and $u_{\bar{n}}$ are compatible. To do this, we apply Proposition 3.172 and show accordingly that one of the functions is a subset of the other one. Using now the standard total ordering $\leq_{\mathbb{N}}$ of $\mathbb{N}$, we have the true disjunction

$$\bar{m} \leq_{\mathbb{N}} \bar{n} \vee \bar{n} \leq_{\mathbb{N}} \bar{m}, \tag{4.448}$$

which we now use to prove by cases that $u_{\bar{m}}$ and $u_{\bar{n}}$ are compatible. In the first case that $\bar{m} \leq_{\mathbb{N}} \bar{n}$, we obtain $\bar{m} <_{\mathbb{N}} \bar{n} \vee \bar{m} = \bar{n}$ with the Characterization of induced irreflexive partial orderings, and we now prove the sentence $u_{\bar{m}} \subseteq u_{\bar{n}}$ by cases based on that true disjunction. If $\bar{m} <_{\mathbb{N}} \bar{n}$ holds, then that inclusion follows immediately to be true with (4.447). If $\bar{m} = \bar{n}$ holds, then we obtain $u_{\bar{m}} = u_{\bar{n}}$ because u is a function (which thus satisfies Corollary 3.150); then, the disjunction $u_{\bar{m}} \subset u_{\bar{n}} \vee u_{\bar{m}} = u_{\bar{n}}$ is also true (regardless of the truth value of its first part), and this disjunction yields the desired $u_{\bar{m}} \subseteq u_{\bar{n}}$ with (2.26), completing this proof by cases. The preceding inclusion then implies with Proposition 3.172 that the functions $u_{\bar{m}}$ and $u_{\bar{n}}$ are compatible (in case of $\bar{m} \leq_{\mathbb{N}} \bar{n}$). In case the second part $\bar{n} \leq_{\mathbb{N}} \bar{m}$ of the disjunction (4.448) is true, so that we now obtain $\bar{n} <_{\mathbb{N}} \bar{m} \vee \bar{n} = \bar{m}$, we may apply a proof by cases (similarly to the one for the first case) to establish $u_{\bar{n}} \subseteq u_{\bar{m}}$. If $\bar{n} <_{\mathbb{N}} \bar{m}$ holds, then (4.447) yields already the desired inclusion. If $\bar{n} = \bar{m}$ holds, then $u_{\bar{n}} = u_{\bar{m}}$ also holds (exploiting again the fact that u is a function), and the disjunction $u_{\bar{n}} \subset u_{\bar{m}} \vee u_{\bar{n}} = u_{\bar{m}}$ is therefore true as well, which implies the desired inclusion $u_{\bar{n}} \subseteq u_{\bar{m}}$ (using again (2.26)). Thus, the preceding inclusion holds in any case, so that we may now apply once again Proposition 3.172 to infer from this finding that $u_{\bar{n}}$ and $u_{\bar{m}}$ are compatible. Because of Corollary 3.171, it then follows that $u_{\bar{m}}$ and $u_{\bar{n}}$ are compatible, as in the first case. This completes the proof by cases based on the disjunction (4.448), and we apply now substitutions to infer from the compatibility of $u_{\bar{m}}$ and $u_{\bar{n}}$ the compatibility of u' and u''. Since u' and u'' were arbitrary, we may therefore conclude that the range of $u = (u_n)_{n\in\mathbb{N}}$ is a compatible set of functions.

Concerning d), we may now apply Theorem 3.175 to the range of $u = (u_n)_{n\in\mathbb{N}}$ to obtain the function

$$v = \bigcup \mathrm{ran}(u).$$

We now verify that the domain of v equals $\mathbb{N}$. According to Theorem 3.175, we have $\mathrm{dom}(v) = \bigcup \mathcal{D}$, where $\mathcal{D}$ consists of the domains of all the functions in $(u_n)_{n\in\mathbb{N}}$, i.e. where

$$\mathcal{D} = \{D : \exists u' \,(u' \in \mathrm{ran}(u) \,\wedge\, \mathrm{dom}(u') = D)\}. \tag{4.449}$$

Due to Equality Criterion for sets, we may prove [$\mathrm{dom}(v) =$] $\bigcup\mathcal{D} = \mathbb{N}$ by verifying equivalently

$$\forall n \,(n \in \bigcup\mathcal{D} \Leftrightarrow n \in \mathbb{N}). \tag{4.450}$$

To do this, we let n be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $n \in \bigcup\mathcal{D}$. By definition of the union of a set system, there then exists a set, say $\bar{D}$, such that $\bar{D} \in \mathcal{D}$ and $n \in \bar{D}$ hold. In light of the definition of $\mathcal{D}$, we now see that $\bar{D} \in \mathcal{D}$ implies that there exists an element of the range of u, say $\bar{u}'$, with $\mathrm{dom}(\bar{u}') = \bar{D}$. With this equation, the previously established $n \in \bar{D}$ yields $n \in \mathrm{dom}(\bar{u}')$. Furthermore, $\bar{u}' \in \mathrm{ran}(u)$ implies with (4.428) the existence of a particular $\bar{m} \in \mathbb{N}$ [$= \mathrm{dom}(u)$] with $\bar{u}' = u_{\bar{m}}$ and $u_{\bar{m}} \in B^{\bar{m}}$. Thus, $\mathrm{dom}(\bar{u}') = \bar{m}$, and therefore the previously established $n \in \mathrm{dom}(\bar{u}')$ yields $n \in \bar{m}$. Since $\bar{m} \in \mathbb{N}$ also implies $\bar{m} \subseteq \mathbb{N}$ with (2.320), we have that $n \in \bar{m}$ implies the desired $n \in \mathbb{N}$ by definition of a subset. This finding proves the first part of the equivalence in (4.450).

To prove the second part ('$\Leftarrow$'), we now assume $n \in \mathbb{N}$, which implies on the one hand $n \in n^+$ with (2.305) and on the other hand $n^+ \in \mathbb{N}$ with (2.297). The latter yields $u_{n^+} \in B^{n^+}$ with (4.427), so that u_{n^+} is a function with domain n^+ (and codomain B). Since u_{n^+} is a term of the sequence u with index n^+, so that $(n^+, u_{n^+}) \in u$ holds, we obtain $u_{n^+} \in \mathrm{ran}(u)$ by definition of a range. Thus, the conjunction of $u_{n^+} \in \mathrm{ran}(u)$ and $\mathrm{dom}(u_{n^+}) = n^+$ is true, which shows that the existential sentence in (4.449) is satisfied by $\bar{D} = n^+$. This equation gives on the one hand $\bar{D} \in \mathcal{D}$ by definition of $\mathcal{D}$ in (4.449), and on the other hand it allows us to apply substitution to the previously established $n \in n^+$ to obtain $n \in \bar{D}$. Now, since $\bar{D} \in \mathcal{D}$ and $n \in \bar{D}$ are both true, we see that there exists a set D which satisfies $D \in \mathcal{D} \wedge n \in D$, and this existential sentences further implies $n \in \bigcup\mathcal{D}$ by definition of the union of a set system. This completes the proof of the second part of the equivalence in (4.450), and since n was arbitrary, we may therefore conclude that the (4.450) is true. This universal sentence in turn implies the equation [$\mathrm{dom}(v) =$] $\bigcup\mathcal{D} = \mathbb{N}$ with the Equality Criterion for sets. We therefore obtain $\mathrm{dom}(v) = \mathbb{N}$, which shows that v is a sequence with domain $\mathbb{N}$.

Next, we verify that v is a sequence in B, i.e. that $\mathrm{ran}(v) \subseteq B$ holds. For this purpose, we prove the equivalent (applying the definition of a subset)

$$\forall y \,(y \in \mathrm{ran}(v) \Rightarrow y \in B). \tag{4.451}$$

To do this, we let y be arbitrary and assume $y \in \operatorname{ran}(v)$ to be true. By definition of a range, there then exists a constant, say $\bar{n}$, such that $(\bar{n}, y) \in v$ holds. With the equation (4.429), it then follows that $(\bar{n}, y) \in \bigcup \operatorname{ran}(u)$, which implies (by definition of the union of a set system) that there exists a set, say $\bar{u}'$, with $\bar{u}' \in \operatorname{ran}(u)$ and $(\bar{n}, y) \in \bar{u}'$ are true. Here, the latter shows that y is an element of the range of $\bar{u}'$. Due to (4.428), we have that $\bar{u}' \in \operatorname{ran}(u)$ implies the existence of a particular constant $\bar{m}$ such that $\bar{m} \in \mathbb{N}$, $\bar{u}' = u_{\bar{m}}$ and $\bar{u}' \in B^{\bar{m}}$ hold, so that $\bar{u}'$ is a function from $\bar{m}$ to B. Thus, $\operatorname{ran}(\bar{u}') \subseteq B$ holds by definition of a codomain, and therefore the previously found $y \in \operatorname{ran}(\bar{u}')$ gives $y \in B$ (with the definition of a subset). As y is arbitrary, we may now conclude that (4.451) is true, which universal sentence yields (again by definition of a subset) $\operatorname{ran}(v) \subseteq B$, so that v is a function/sequence with codomain B.

We now prove (4.430), letting n be arbitrary in $\mathbb{N}$. To demonstrate the truth of the equation $v \upharpoonright n = u_n$, we use again the Equality Criterion for sets and verify equivalently

$$\forall z\,(z \in v \upharpoonright n \Leftrightarrow z \in u_n). \tag{4.452}$$

For this purpose, we let z be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $z \in v \upharpoonright n$. This implies by definition of a binary relation that there exist constants, say $\bar{k}$ and $\bar{y}$ such that $z = (\bar{k}, \bar{y})$. Thus, $(\bar{k}, \bar{y}) \in v \upharpoonright n$ holds, where n is the domain of the restriction $v \upharpoonright n$ according to Proposition 3.164, so that we obtain $\bar{k} \in n$ (with the definition of a domain). Because the assumed $n \in \mathbb{N}$ implies $u_n \in B^n$ with (4.427), we see that n is also the domain of u_n, so that $\bar{k} \in n$ yields $\bar{k} \in \operatorname{dom}(u_n)$. Then, by definition of a domain, there exists a constant, say $\bar{y}'$, such that

$$(\bar{k}, \bar{y}') \in u_n. \tag{4.453}$$

Moreover, the previously established $(\bar{k}, \bar{y}) \in v \upharpoonright n$ implies $(\bar{k}, \bar{y}) \in v$ with the definition of a restriction, so that we see in light of (4.429) and the definition of the union of a set system that there exists an element of the range of $u = (u_n)_{n \in \mathbb{N}}$, say $\bar{u}'$, such that $(\bar{k}, \bar{y}) \in \bar{u}'$ holds. Here, $\bar{u}' \in \operatorname{ran}(u)$ implies with (4.428) that $\bar{u}' = u_{\bar{m}}$ and $\bar{u}' \in B^{\bar{m}}$ hold for a particular $\bar{m} \in \mathbb{N}$, so that we obtain via substitution

$$(\bar{k}, \bar{y}) \in u_{\bar{m}}. \tag{4.454}$$

We now show that the conjunction of (4.453) and (4.454) implies $\bar{y}' = \bar{y}$. We could establish this equation by performing two nested proofs by cases in analogy to the compatibility proof in c). For a change, we use here

Method 1.13 as a variant of the proof by cases, utilizing the connexity of the standard linear ordering $<_{\mathbb{N}}$, so that the disjunction

$$\bar{m} <_{\mathbb{N}} n \lor n <_{\mathbb{N}} \bar{m} \lor \bar{m} = n$$

is true. In the first case $\bar{m} <_{\mathbb{N}} n$, the inclusion $u_{\bar{m}} \subseteq u_n$ follows with (4.447), so that (4.454) implies $(\bar{k}, \bar{y}) \in u_n$ with the definition of a subset. Then, the conjunction of this and (4.453) implies $\bar{y} = \bar{y}'$ because u_n is a function, as noted earlier. Similarly, we obtain in the second case $n <_{\mathbb{N}} \bar{m}$ the inclusion $u_n \subseteq u_{\bar{m}}$, so that (4.453) implies $(\bar{k}, \bar{y}') \in u_{\bar{m}}$. The conjunction of this and (4.454) then yields $\bar{y}' = \bar{y}$ because the previously established $\bar{u}' \in B^{\bar{m}}$ shows that $u_{\bar{m}}$ $[= \bar{u}']$ is a function. Finally, the third case $\bar{m} = n$ yields $u_{\bar{m}} = u_n$ with the fact that u is a function/sequence (which thus satisfies Corollary 3.150). Then, applying substitution based on that equation to (4.454), we obtain $(\bar{k}, \bar{y}) \in u_n$, so that $\bar{y} = \bar{y}'$ follows to be true as in the first case of $\bar{m} <_{\mathbb{N}} n$. Thus, the proof by cases is complete, so that substitution of the preceding equation into (4.453) gives $[(\bar{k}, \bar{y}) =]\ z \in u_n$, which finding proves the first part of the equivalence in (4.452).

To prove the second part ('$\Leftarrow$'), we now assume $z \in u_n$, so that (by definition of a binary relation) there exist constants, say $\bar{k}$ and $\bar{y}$, such that $(\bar{k}, \bar{y}) \in u_n$. This implies together with $u_n \in \mathrm{ran}(u)$ (which evidently holds since u_n is a value of the function u) the truth of $(\bar{k}, \bar{y}) \in \bigcup \mathrm{ran}(u)$ (using the definition of the union of a set system), so that $(\bar{k}, \bar{y}) \in v$ holds in view of (4.429). By definition of a domain, it also follows from $(\bar{k}, \bar{y}) \in u_n$ that $\bar{k} \in \mathrm{dom}(u_n)$, which yields $\bar{k} \in n$ because n is the domain of u_n according to (4.427). Then, since $(\bar{k}, \bar{y}) \in v$ and $\bar{k} \in n$ are both true, it follows with the definition of a restriction that $(\bar{k}, \bar{y}) \in v \upharpoonright n$, which proves the second part of the equivalence in (4.452). As z is arbitrary, we may therefore conclude that (4.452) is true, which means that $v \upharpoonright n$ and u_n are identical sets. Since n was also arbitrary, we may then further conclude that (4.430) holds.

Concerning e), we first show that the sequence v defined by (4.429) satisfies (4.431). To do this, we let $n \in \mathbb{N}$ be arbitrary and observe first the truth of the equations

$$v \upharpoonright n^+ = u_{n^+} = f(n, u_n) = u_n \cup \{(n, F(u_n))\} \tag{4.455}$$

using firstly (4.430) with the fact that $n \in \mathbb{N}$ implies $n^+ \in \mathbb{N}$ with (2.297), secondly (4.426), and thirdly the definition of the function f in (4.424) in connection with the fact that $\mathrm{dom}(u_n) = n$ holds because $n \in \mathbb{N}$ implies $u_n \in B^n$ according to (4.427). Since $(n, F(u_n)) \in \{(n, F(u_n))\}$ holds in view of (2.153), it then follows with the definition of the union of two sets that $(n, F(u_n))$ is an element of $u_n \cup \{(n, F(u_n))\}$, which yields $(n, F(u_n)) \in$

$v \restriction n^+$ via substitution based on the equations in (4.455). This implies with the definition of a restriction that $(n, F(u_n)) \in v$ holds, which we may also write

$$v_n = F(u_n) \tag{4.456}$$

due to the fact established in d) that v is a sequence/function. Since $n \in \mathbb{N}$ also implies $u_n = v \restriction n$ with (4.430), we obtain $F(u_n) = F(v \restriction n)$ with the initial assumption that F is a function (which thus satisfies Corollary 3.150). Combining now the preceding equation with the equation (4.456), we therefore arrive at the desired equation $v_n = F(v \restriction n)$. As n is arbitrary, we may therefore conclude that the universal sentence (4.431) is true for the sequence $(v_n)_{n\in\mathbb{N}} = v$ defined in (4.429).

To prove that v is the only sequence satisfying (4.429), we verify (according to Method 1.18) the universal sentence (4.432). We let v' be arbitrary such that v' is a sequence in B with domain $\mathbb{N}$ satisfying (4.431). To prove that this implies $v = v'$, we apply the Equality Criterion for functions and demonstrate accordingly the truth of

$$\forall n\,(n \in \mathbb{N} \Rightarrow v_n = v'_n). \tag{4.457}$$

We now prove this universal sentence by strong induction, based on the induction step

$$\forall n\,(n \in \mathbb{N} \Rightarrow [\forall m\,([m \in \mathbb{N} \wedge m <_{\mathbb{N}} n] \Rightarrow v_m = v'_m) \Rightarrow v_n = v'_n]). \tag{4.458}$$

To prove this sentence, we let n be arbitrary in $\mathbb{N}$, make the induction assumption

$$\forall m\,([m \in \mathbb{N} \wedge m <_{\mathbb{N}} n] \Rightarrow v_m = v'_m), \tag{4.459}$$

and show that this implies $v_n = v'_n$. We begin with the proof that the induction assumption implies $v \restriction n = v' \restriction n$, which is equivalent to

$$\forall z\,(z \in v \restriction n \Leftrightarrow z \in v' \restriction n) \tag{4.460}$$

due to the Equality Criterion for sets. Letting z be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $z \in v \restriction n$ is true. Since $v \restriction n$ is a binary relation, there exist constants, say $\bar{k}$ and $\bar{y}$, such that $z = (\bar{k}, \bar{y})$. Thus, $(\bar{k}, \bar{y}) \in v \restriction n$, which implies $(\bar{k}, \bar{y}) \in v$ and $\bar{k} \in n$ with the definition of a restriction in connection with the previously mentioned fact that n is the domain of $v \restriction n$. Then, $\bar{k} \in n$ implies with (4.190) that $\bar{k} <_{\mathbb{N}} n$, which further implies $v_{\bar{k}} = v'_{\bar{k}}$ with the induction assumption. As the previously obtained $(\bar{k}, \bar{y}) \in v$ may be written in function/sequence notation as $v_{\bar{k}} = \bar{y}$, we obtain after substitution $v'_{\bar{k}} = \bar{y}$, which then means $(\bar{k}, \bar{y}) \in v'$, and thus $z \in v'$. Now, the conjunction of this and the previously established $\bar{k} \in n$ yields (by definition of a restriction) the desired $z \in v' \restriction n$, which proves the first part of the equivalence in (4.460).

The second part ('$\Leftarrow$') is proved in analogy to the first part. Assuming now $z \in v' \restriction n$ to be true, we again have $z = (\bar{k}, \bar{y})$ for some particular $\bar{k}$ and $\bar{y}$, so that $(\bar{k}, \bar{y}) \in v' \restriction n$ holds. This implies $(\bar{k}, \bar{y}) \in v'$ and $\bar{k} \in n$, where the latter further implies $\bar{k} <_{\mathbb{N}} n$ and therefore $v_{\bar{k}} = v'_{\bar{k}}$ with the induction assumption. Now, $(\bar{k}, \bar{y}) \in v'$ means $v'_{\bar{k}} = \bar{y}$, so that $v_{\bar{k}} = \bar{y}$ holds, which we may write as $(\bar{k}, \bar{y}) \in v$. This equation yields $z \in v$, which then implies, together with $\bar{k} \in n$, the desired $z \in v \restriction n$, completing the proof of the equivalence. As z is arbitrary, we may therefore conclude that (4.460) is true, so that

$$v \restriction n = v' \restriction n \tag{4.461}$$

holds indeed. Let us then observe the truth of the equations

$$v_n = F(v \restriction n) = F(v' \restriction n) = v'_n,$$

using (4.431), then Corollary 3.526 in connection with (4.461) and the assumption that F is a function, and finally again (4.431). These equations give the desired $v_n = v'_n$, and since n was arbitrary, we may therefore conclude that (4.458) holds. This completes the proof of (4.457) via strong induction, so that $v = v'$ follows to be true. This equation in turn proves the implication in (4.432), and as v' was arbitrary, we may finally conclude that the universal sentence (4.432) holds, completing the proof of e).

Since B and F were initially arbitrary sets in the proofs of a) – e), the stated theorem follows then to be true. □

In view Proposition 1.22 we may summarize the Strong Recursion Theorem as follows.

Corollary 4.97. *For any set B and any function $F : B^{<\mathbb{N}} \to B$ there exists a unique sequence $v = (v_n)_{n \in \mathbb{N}}$ in B satisfying*

$$\forall n\,(n \in \mathbb{N} \Rightarrow v_n = F(v \restriction n)).$$

4.7. Denumerations

Definition 4.9 (Denumeration, length of a denumeration). For any set A and any $n \in \mathbb{N}$ we say that a set c is a *denumeration* of A of *length* n iff c is a bijection from the initial segment of $\mathbb{N}_+$ up to n to A, i.e. iff

$$c : \{1, \ldots, n\} \rightleftarrows A. \tag{4.462}$$

Note 4.18. In the case of $A = \emptyset$, we have that $\emptyset : \{1, \ldots, 0\} \rightleftarrows A$ is a bijection due to (4.239) and Corollary 3.201; thus, the denumeration of the empty set has the anticipated length $n = 0$.

4.7.1. Finite sets

Apparently, if the domain of a denumeration of some set A 'ends' at some natural number n, then A cannot have an 'infinite' number of elements.

Definition 4.10 (Finite & infinite set). We say that a set A is *finite* iff there is a natural number n such that there exists a denumeration of A of length n, i.e. iff

$$\exists n, c\, (n \in \mathbb{N} \wedge c : \{1, \ldots, n\} \rightleftarrows A). \tag{4.463}$$

Furthermore, we say that a set A is *infinite* iff A is not a finite set, i.e. iff the negation of (4.463) is true.

Corollary 4.98. *Every natural number is a finite set, that is,*

$$\forall n\, (n \in \mathbb{N} \Rightarrow n \text{ is finite}). \tag{4.464}$$

Proof. Letting m be an arbitrary natural number, we observe in light of Proposition 4.61 that $s^+ \upharpoonright m$ is a bijection from m to $\{1, \ldots, m\}$, i.e.

$$s^+ \upharpoonright m : m \rightleftarrows \{1, \ldots, m\}. \tag{4.465}$$

It then follows with Theorem 3.212 that the inverse function $(s^+ \upharpoonright m)^{-1}$ is a bijection from $\{1, \ldots, m\}$ to m, i.e.

$$(s^+ \upharpoonright m)^{-1} : \{1, \ldots, m\} \rightleftarrows m. \tag{4.466}$$

Thus, there exists a natural number n and a bijection c from $\{1, \ldots, n\}$ to m, so that m is a finite set by definition. □

Proposition 4.99. *The union of a finite set and a singleton is finite, i.e.*

$$\forall A, y\, (A \text{ is finite } \Rightarrow A \cup \{y\} \text{ is finite}). \tag{4.467}$$

Proof. We let A and y be arbitrary and prove the implication by cases, considering the two exhaustive cases $y \in A$ and $y \notin A$.

In the first case, we assume that A is finite and that $y \in A$ holds. The latter implies $\{y\} \subseteq A$ with (2.184), which then further implies $\{y\} \cup A = A$ with (2.247), and therefore $A \cup \{y\} = A$ using (2.214). Since A is finite, we thus have that $A \cup \{y\}$ is also finite.

In the second case, we assume that A is finite where $y \notin A$. By definition of a finite set, the former assumption implies that there exist a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}\}$ to A, say $\bar{c}$. Here, $\bar{n} \in \mathbb{N}$ implies $\bar{n} = 0 \lor \bar{n} \in \mathbb{N}_+$ with (2.310), which disjunction we now use to prove by cases that the union $A \cup \{y\}$ is finite. If $\bar{n} = 0$ is true, we have the bijection $\bar{c} : \{1, \ldots, 0\} \rightleftarrows A$; thus, the domain of $\bar{c}$ is empty according to the notation for initial segments of $\mathbb{N}_+$, so that $1 \notin \emptyset$ is true by definition of the empty set. Together with the assumed $y \notin A$, we obtain with (3.667) the bijection $\bar{c} \cup \{(1, y)\}$ from $\emptyset \cup \{1\}$ to $A \cup \{y\}$, i.e.

$$\bar{c} \cup \{(1, y)\} : \{1, \ldots, \bar{n}^+\} \rightleftarrows A \cup \{y\}, \tag{4.468}$$

using (2.216) and (2.291) with the definition of an initial segment. Since $\bar{n}^+ = 1$ is a natural number, this bijection shows that $A \cup \{y\}$ is a finite set for $\bar{n} = 0$.

If $\bar{n} \in \mathbb{N}_+$ is true, then we have that $\bar{n}^+ \notin \{1, \ldots, \bar{n}\}$ holds according to Corollary 4.55b). Together with the assumption $y \notin A$, this implies with (3.667) that $\bar{c} \cup \{(\bar{n}^+, y)\}$ is a bijection from $\{1, \ldots, \bar{n}\} \cup \{\bar{n}^+\}$ to $A \cup \{y\}$, so that we obtain by means of (4.241)

$$\bar{c} \cup \{(\bar{n}^+, y)\} : \{1, \ldots, \bar{n}^+\} \rightleftarrows A \cup \{y\}. \tag{4.469}$$

As $\bar{n} \in \mathbb{N}$ implies $\bar{n}^+ \in \mathbb{N}$ with (2.297), (4.469) shows that there are a natural number n and a bijection from $\{1, \ldots, n\}$ to $A \cup \{y\}$, so that $A \cup \{y\}$ is finite by definition. This completes the proof of the implication by cases, and since A and y were arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Corollary 4.100. *Any singleton is finite, that is,*

$$\forall y \, (\{y\} \text{ is finite}). \tag{4.470}$$

Proof. Letting y be arbitrary, we obtain $\{y\} = \emptyset \cup \{y\}$ with (2.216), where $\emptyset$ is a finite set by definition. We then see in light of (4.467) that $\emptyset \cup \{y\}$ and thus the singleton $\{y\}$ is finite. Since y was arbitrary, we may therefore conclude that (4.470) is true. □

Exercise 4.34. Show for any $m \in \mathbb{N}$ that the initial segment $A = \{1, \ldots, m\}$ of $\mathbb{N}_+$ up to m is finite.

(Hint: Apply Corollary 3.203.)

Lemma 4.101. *For any natural number m and any positive natural number n, it is true that there is no injection from the initial segment $\{1, \ldots, m^+\}$ to the initial segment $\{1, \ldots, n\}$ if n is less than m^+, that is,*

$$\forall m, n\,([m \in \mathbb{N} \wedge n \in \mathbb{N}_+] \Rightarrow$$
$$[n < m^+ \Rightarrow \neg \exists f\,(f \in \{1, \ldots, n\}^{\{1,\ldots,m^+\}} \wedge f \text{ is an injection})]). \quad (4.471)$$

Proof. Let us rewrite the proposed sentence equivalently as

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \forall m\,(m \in \mathbb{N} \Rightarrow \quad (4.472)$$
$$[n < m^+ \Rightarrow \forall f\,(f \in \{1, \ldots, n\}^{\{1,\ldots,m^+\}} \Rightarrow f \text{ is not an injection})])).$$

using (1.90) and (1.81). We now prove this sentence by mathematical induction. Regarding the base case ($n = 1$), we prove accordingly

$$\forall m\,(m \in \mathbb{N} \Rightarrow \quad (4.473)$$
$$[1 < m^+ \Rightarrow \forall f\,(f \in \{1, \ldots, 1\}^{\{1,\ldots,m^+\}} \Rightarrow f \text{ is not an injection})]).$$

For this purpose, we let m be arbitrary in $\mathbb{N}$, assume $1 < m^+$, let then f be arbitrary, and prove the inner implication by contradiction, assuming that f is a function from $\{1, \ldots, m^+\}$ to $\{1, \ldots, 1\}$, i.e. from $\{1, \ldots, m^+\}$ to $\{1\}$, and that f is an injection, so that

$$\forall x, x'\,([x, x' \in \{1, \ldots, m^+\} \wedge x \neq x'] \Rightarrow f(x) \neq f(x')) \quad (4.474)$$

holds according to the Injection Criterion. Let us now observe that $m \in \mathbb{N}$ implies $m^+ \in \mathbb{N}_+$ with (4.40), and that $1, m^+ \in \{1, \ldots, m^+\}$ holds then according to Corollary 4.54. Thus, the domain of f is nonempty, so that $\mathrm{ran}(f) \neq \emptyset$ follows to be true with (3.119) and the Law of Contraposition. Since the codomain $\{1\}$ of f is a singleton, $\exists! y(y \in \{1\})$ holds according to (2.180). Now, as 1 and m^+ are elements of the domain $\{1, \ldots, m^+\}$ of the function f, it follows with the Function Criterion that there exist the unique values $y = f(1)$ and $y' = f(m^+)$ in $\{1\}$. Then, $y \in \{1\}$ and $y' \in \{1\}$ imply $y = y'$ with the preceding uniquely existential sentence, so that substitution yields $f(1) = f(m^+)$. Furthermore, the assumption $1 < m^+$ implies $1 \neq m^+$ with the comparability of the linear ordering $<$ of $\mathbb{N}$. It then follows with (4.474) that $f(1) \neq f(m^+)$ is true, which contradicts the previously established equation $f(1) = f(m^+)$. This completes the proof by contradiction, and since f is arbitrary, we may therefore conclude that the universal sentence in (4.473) with respect to f is true. Because m was also arbitrary, we may then further conclude that (4.473) holds, which proves the base case.

Regarding the induction step, we let n be arbitrary in $\mathbb{N}_+$, make the induction assumption

$$\forall m\,(m \in \mathbb{N} \Rightarrow \tag{4.475}$$
$$[n < m^+ \Rightarrow \forall f\,(f \in \{1,\ldots,n\}^{\{1,\ldots,m^+\}} \Rightarrow f \text{ is not an injection})]),$$

and show that this implies

$$\forall m\,(m \in \mathbb{N} \Rightarrow \tag{4.476}$$
$$[n^+ < m^+ \Rightarrow \forall f\,(f \in \{1,\ldots,n^+\}^{\{1,\ldots,m^+\}} \Rightarrow f \text{ is not an injection})]).$$

To prove the latter, we let m be arbitrary in $\mathbb{N}$, assume $n^+ < m^+$, and let f be an arbitrary function from $\{1,\ldots,m^+\}$ to $\{1,\ldots,n^+\}$. Since m^+ and n^+ are elements of $\mathbb{N}_+$, we may apply (4.272) alongside (4.187) to infer from the assumed $n^+ < m^+$ the truth of $[0 \leq]\ n < m$, so that the Transitivity Formula for $\leq$ and $<$ as well as the comparability of $<$ yield first $0 < m$ and then $0 \neq m$, which inequality clearly shows that $m \in \mathbb{N}_+$ is also true. Let us now observe that the disjunction

$$\operatorname{ran}(f) \subseteq \{1,\ldots,n\} \vee \neg\operatorname{ran}(f) \subseteq \{1,\ldots,n\} \tag{4.477}$$

is true according to the Law of the Excluded Middle, which disjunction we now use to prove by cases that f is not an injection.

In case $\operatorname{ran}(f) \subseteq \{1,\ldots,n\}$ is true, then f is a function from $\{1,\ldots,m^+\}$ to $\{1,\ldots,n\}$ by definition of a codomain. The conjunction of the fact $n < n^+$ (see 4.153) and the assumption $n^+ < m^+$ implies $n < m^+$ with the transitivity of the linear ordering $<$, which inequality then further implies together with the preceding finding $f \in \{1,\ldots,n\}^{\{1,\ldots,m^+\}}$ and the induction assumption that $f : \{1,\ldots,m^+\} \to \{1,\ldots,n\}$ is not an injection. It then follows with Note 3.23 that $f : \{1,\ldots,m^+\} \to \{1,\ldots,n^+\}$ is not an injection either (because the definite property of injectivity is independent of the choice for the codomain).

In case the second part $\neg\operatorname{ran}(f) \subseteq \{1,\ldots,n\}$ of the disjunction (4.477) is true, this means by definition of a subset that

$$\neg\forall y\,(y \in \operatorname{ran}(f) \Rightarrow y \in \{1,\ldots,n\})$$

holds. In view of the Negation Law for universal implications, there then exists an element of $\operatorname{ran}(f)$, say $\bar{y}$, such that $\bar{y} \notin \{1,\ldots,n\}$. Furthermore,

$\bar{y} \in \operatorname{ran}(f)$ implies $\bar{y} \in \{1, \ldots, n^+\}$, because we assumed that f is a function with codomain $\{1, \ldots, n^+\}$, so that $\operatorname{ran}(f) \subseteq \{1, \ldots, n^+\}$ holds. Thus, the conjunction

$$\bar{y} \in \{1, \ldots, n^+\} \land \bar{y} \notin \{1, \ldots, n\}$$

holds, which means $\bar{y} \in \{1, \ldots, n^+\} \setminus \{1, \ldots, n\}$ by definition of a set difference, and therefore $\bar{y} \in \{n^+\}$ is true according to (4.257). Then, this finding implies $\bar{y} = n^+$ with (2.169). As n^+ is thus an element of $\operatorname{ran}(f)$, there exists (by definition of a range) a constant, say $\bar{x}^+$, such that $(\bar{x}^+, n^+) \in f$ holds, which we may write as $f(\bar{x}^+) = n^+$; then, we obtain with the definition of a domain $\bar{x}^+ \in \{1, \ldots, m^+\}$. To make further progress, we notice in light of the Law of the Excluded Middle that the disjunction of

$$\exists k \left([k \in \{1, \ldots, m^+\} \land \bar{x}^+ \neq k] \land f(\bar{x}^+) = f(k) \right) \tag{4.478}$$

and the negation of (4.478) is true, which disjunction we apply to prove by cases that f is not an injection.

If (4.478) is true, then there is a particular constant $\bar{k} \in \{1, \ldots, m^+\}$ with $\bar{x}^+ \neq \bar{k}$ and $f(\bar{x}^+) = f(\bar{k})$. Recalling the previously established $\bar{x}^+ \in \{1, \ldots, m^+\}$, we obtain the existential sentence

$$\exists x, x' \left(x, x' \in \{1, \ldots, m^+\} \land x \neq x' \land f(x) = f(x') \right),$$

which in turn implies with Exercise 3.77 that f is not an injection (as desired).

If the negation of (4.478) is true, then the Negation Law for existential conjunction yields the universal sentence

$$\forall k \left([k \in \{1, \ldots, m^+\} \land \bar{x}^+ \neq k] \Rightarrow n^+ \neq f(k) \right). \tag{4.479}$$

it is apparently true that the range of $f \setminus \{(\bar{x}^+, n^+)\}$ now excludes n^+, because no element of $\{1, \ldots, m^+\}$ distinct from $\bar{x}^+$ is associated with n^+. We now utilize this idea to define via replacement a new function g with domain $\{1, \ldots, m\}$ which satisfies

$$x \mapsto g(x) = \begin{cases} f(x) & \text{if } x \in \{1, \ldots, \bar{x}\}, \\ f(x^+) & \text{if } x \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\}. \end{cases} \tag{4.480}$$

We prove for this purpose

$$\begin{aligned} &\forall x \left(x \in \{1, \ldots, m\} \Rightarrow \exists! y \left([x \in \{1, \ldots, \bar{x}\} \Rightarrow y = f(x)] \right.\right. \\ &\quad \left.\left. \land [x \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\} \Rightarrow y = f(x^+)] \right) \right), \end{aligned} \tag{4.481}$$

letting x be arbitrary and assuming $x \in \{1, \ldots, m\}$ to be true, where the initial segment $\{1, \ldots, m\}$ of $\mathbb{N}_+$ is indeed nonempty due to the previously established fact that $m \in \mathbb{N}_+$ holds; thus $x \in \mathbb{N}_+$. Let us now observe that the previously found $\bar{x}^+ \in \{1,\ldots,m^+\}$ implies $\bar{x}^+ \leq m^+$ with (4.275), which inequality further implies $\bar{x} \leq m$ with (4.273), and this gives $\{1,\ldots,\bar{x}\} \subseteq \{1,\ldots,m\}$ with (4.258) and (4.261). It then follows with (2.263) that

$$\{1,\ldots,m\} = (\{1,\ldots,m\} \setminus \{1,\ldots,\bar{x}\}) \cup \{1,\ldots,\bar{x}\},$$

so that the assumption $x \in \{1,\ldots,m\}$ implies

$$x \in \{1,\ldots,\bar{x}\} \ \vee \ x \in \{1,\ldots,m\} \setminus \{1,\ldots,\bar{x}\} \tag{4.482}$$

with the definition of the union of two sets and the Commutative Law for the disjunction. We now use this true disjunction to prove the existential part regarding the uniquely existential sentence in (4.481) by cases.

On the one hand, if $x \in \{1,\ldots,\bar{x}\}$ is true, we observe that $\{1,\ldots,m\} \subset \{1,\ldots,m^+\}$ holds with (4.254), so that $\{1,\ldots,m\} \subseteq \{1,\ldots,m^+\}$ follows to be true with (2.26). Therefore, the assumed $x \in \{1,\ldots,m\}$ implies $x \in \{1,\ldots,m^+\}$ by definition of a subset, which shows that x is an element of the domain of the function f. Consequently, the function value $\bar{y} = f(x)$ exists, so that the implication

$$x \in \{1,\ldots,\bar{x}\} \ \Rightarrow \ \bar{y} = f(x) \tag{4.483}$$

is true (since both the antecedent and the consequent are true). Furthermore, as $x \in \{1,\ldots,\bar{x}\}$ was assumed to be true, the negation $x \notin \{1,\ldots,\bar{x}\}$ is false, and therefore the conjunction of $x \in \{1,\ldots,m\}$ and $x \notin \{1,\ldots,\bar{x}\}$ is also false, which shows in light of the definition of a set difference that the antecedent of the implication

$$x \in \{1,\ldots,m\} \setminus \{1,\ldots,\bar{x}\} \ \Rightarrow \ \bar{y} = f(x^+) \tag{4.484}$$

is false. Thus, the implications (4.483) and (4.484) are both true in the first case.

On the other hand, if the second part $x \in \{1,\ldots,m\} \setminus \{1,\ldots,\bar{x}\}$ of the disjunction (4.482) is true, we have in particular $x \in \{1,\ldots,m\}$, which implies $x \leq m$ with (4.275) and therefore also $x^+ \leq m^+$ with (4.273). This finding further implies

$$x^+ \in \{1,\ldots,m^+\} \tag{4.485}$$

again with (4.275), so that x^+ is an element of the domain of f. Consequently, the function value $\bar{y} = f(x^+)$ exists, and the truth of this equation and of the current case assumption clearly shows that the implication (4.484) is true for $\bar{y} = f(x^+)$. Since the case assumption implies (with the definition of a set difference) in particular that $x \in \{1,\ldots,\bar{x}\}$ is false, we see that the implication (4.483) – being based on a false antecedent – is true for $\bar{y} = f(x^+)$ as well.

Thus, there exists in any case a constant y which satisfies the conjunction of the two implications in (4.481), so that the proof of the existential part by cases is complete. Regarding the uniqueness part, we let y and y' be arbitrary constants satisfying that conjunction of implications, and we prove $y = y'$ by cases, based on the true disjunction (4.482).

On the one hand, if $x \in \{1, \ldots, \bar{x}\}$ holds, then the true implications

$$x \in \{1, \ldots, \bar{x}\} \Rightarrow y = f(x)$$
$$x \in \{1, \ldots, \bar{x}\} \Rightarrow y' = f(x)$$

give $y = f(x)$ and $y' = f(x)$, so that the desired $y = y'$ follows via substitution.

On the other hand, if $x \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\}$ holds, then the other two true implications

$$x \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\} \Rightarrow y = f(x^+)$$
$$x \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\} \Rightarrow y' = f(x^+)$$

yield $y = f(x^+)$ and $y' = f(x^+)$, so that $y = y'$ follows again to be true. This completes the proof of the uniqueness part by cases, and the resulting truth of the uniquely existential sentence in (4.481) proves in turn the implication based on the antecedent $x \in \{1, \ldots, m\}$. Since x was arbitrary, we may therefore conclude that the universal sentence (4.481) holds, which then implies with Theorem 3.160 that there exists a unique function g with domain $\{1, \ldots, m\}$ such that $g(x) = f(x)$ holds for any $x \in \{1, \ldots, \bar{x}\}$ and such that $g(x) = f(x^+)$ holds for any $x \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\}$. We thus obtained the mapping (4.480).

We now demonstrate that $\{1, \ldots, n\}$ is a codomain of g, i.e. that $\mathrm{ran}(g) \subseteq \{1, \ldots, n\}$ holds. To do this, we apply the definition of a subset and verify

$$\forall y\, (y \in \mathrm{ran}(g) \Rightarrow y \in \{1, \ldots, n\}). \tag{4.486}$$

We let y be an arbitrary element of the range of g, so that there exists (by definition of a range) a particular constant $\bar{\bar{x}}$ with $(\bar{\bar{x}}, y) \in g$, which we may also write as $g(\bar{\bar{x}}) = y$ and which shows (in view of the definition of a domain) that $\bar{\bar{x}} \in \{1, \ldots, m\}$ [$= \mathrm{dom}(g)$] holds. We may now apply exactly the same line of arguments used earlier for establishing (4.482) to infer from $\bar{\bar{x}} \in \{1, \ldots, m\}$ the truth of the disjunction

$$\bar{\bar{x}} \in \{1, \ldots, \bar{x}\} \vee \bar{\bar{x}} \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\}. \tag{4.487}$$

Let us now prove the sentence $g(\bar{\bar{x}}) \in \{1, \ldots, n\}$ by cases by using this disjunction.

On the one hand, if $\bar{\bar{x}} \in \{1, \ldots, \bar{x}\}$ holds, then we obtain $\bar{\bar{x}} \leq \bar{x}$ with (4.275), where $\bar{x} < \bar{x}^+$ holds according to (4.153). The previous two inequalities imply $\bar{\bar{x}} < \bar{x}^+$ with the Transitivity Formula for $\leq$ and $<$, so that $\bar{\bar{x}} \neq \bar{x}^+$ is true due to the comparability of $<$. Moreover, in view of the previously established inclusions $\{1, \ldots, \bar{x}\} \subseteq \{1, \ldots, m\} \subseteq \{1, \ldots, m^+\}$, we see that the current case assumption $\bar{\bar{x}} \in \{1, \ldots, \bar{x}\}$ yields $\bar{\bar{x}} \in \{1, \ldots, m^+\}$ with the transitivity of $\subseteq$ and the definition of a subset. Thus, $\bar{\bar{x}}$ is an element of the domain $\{1, \ldots, m^+\}$ of f, and the conjunction of $\bar{\bar{x}} \in \{1, \ldots, m^+\}$ and $\bar{\bar{x}} \neq \bar{x}^+$ is true, which implies $n^+ \neq f(\bar{\bar{x}})$ with (4.479), and therefore $f(\bar{\bar{x}}) \notin \{n^+\}$ with (2.169). As $\{1, \ldots, n^+\}$ was assumed to be the codomain of f, we therefore have the conjunction

$$f(\bar{\bar{x}}) \in \{1, \ldots, n^+\} \wedge f(\bar{\bar{x}}) \notin \{n^+\}.$$

By definition of a set difference, this means $f(\bar{\bar{x}}) \in \{1, \ldots, n^+\} \setminus \{n^+\}$, which then yields $f(\bar{\bar{x}}) \in \{1, \ldots, n\}$ with (4.256). Since $\bar{\bar{x}} \in \{1, \ldots, \bar{x}\}$ implies $g(\bar{\bar{x}}) = f(\bar{\bar{x}})$ by definition of the function g, we obtain the desired $g(\bar{\bar{x}}) \in \{1, \ldots, n\}$ via substitution.

On the other hand, if $\bar{\bar{x}} \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\}$ holds, then we have $\bar{\bar{x}} \notin \{1, \ldots, \bar{x}\}$ (by definition of a set difference), and we therefore obtain $\neg \bar{\bar{x}} \leq \bar{x}$ with (4.275), so that $\bar{x} < \bar{\bar{x}}$ follows to be true with the Negation Formula for $\leq$. The latter inequality implies $\bar{x} \neq \bar{\bar{x}}$ with the comparability of $<$, which finding then further implies $\bar{x}^+ \neq \bar{\bar{x}}^+$ with (4.128) and the Law of Contraposition. We may now infer from the current case assumption the truth of $\bar{\bar{x}}^+ \in \{1, \ldots, m^+\}$ by applying the same arguments that we used in the course of establishing (4.485). Thus, the conjunction of $\bar{\bar{x}}^+ \in \{1, \ldots, m^+\}$ and $\bar{\bar{x}}^+ \neq \bar{x}^+$ is true, and this conjunction implies $n^+ \neq f(x^+)$ with (4.479); consequently, $f(\bar{\bar{x}}^+) \notin \{n^+\}$ follows to be true with (2.169). Recalling that $\{1, \ldots, n^+\}$ is the codomain of f, we therefore have the true conjunction

$$f(\bar{\bar{x}}^+) \in \{1, \ldots, n^+\} \wedge f(\bar{\bar{x}}^+) \notin \{n^+\},$$

which evidently yields $f(\bar{\bar{x}}^+) \in \{1, \ldots, n^+\} \setminus \{n^+\}$ and therefore $f(\bar{\bar{x}}^+) \in \{1, \ldots, n\}$. As the case assumption $\bar{\bar{x}} \in \{1, \ldots, m\} \setminus \{1, \ldots, \bar{x}\}$ implies $g(\bar{\bar{x}}) = f(x^+)$ by definition of g, we obtain again $g(\bar{\bar{x}}) \in \{1, \ldots, n\}$, completing the proof by cases.

Recalling the equation $g(\bar{\bar{x}}) = y$, we thus showed that $y \in \operatorname{ran}(g)$ implies $y \in \{1, \ldots, n\}$. As y was arbitrary, we may therefore conclude that (4.486) holds, so that $\{1, \ldots, n\}$ constitutes indeed a codomain of g. We thus showed that g is a function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$, and we now demonstrate that – as a consequence of this finding – the assumed function

$f:\{1,\ldots,m^+\}\to\{1,\ldots,n^+\}$ is not an injection. To begin with, the initial assumption $n\in\mathbb{N}_+$ implies $1\le n$ with (4.278), which gives – together with the previously established $n<m$ – the inequality $1<m$ by applying the Transitivity Formula for $\le$ and $<$. This inequality yields $1\neq m$ with the comparability of $<$, so that there exists because of 4.229 an element $\bar{k}$ in $\mathbb{N}_+$ (and thus in $\mathbb{N}$) such that $\bar{k}^+=m$. Thus, $\bar{k}\in\mathbb{N}$, $n<\bar{k}^+$ and $g\in\{1,\ldots,n\}^{\{1,\ldots,\bar{k}^+\}}$ are true, so that g is not an injection because of the induction assumption (4.475). Consequently, there exist because of Exercise 3.77 elements of the domain of g, say $\bar{x}',\bar{x}''\in\{1,\ldots,m\}$, such that the conjunction

$$\bar{x}'\neq\bar{x}''\ \wedge\ g(\bar{x}')=g(\bar{x}'') \tag{4.488}$$

holds. To prove that $f:\{1,\ldots,m^+\}\to\{1,\ldots,n^+\}$ is not an injection, we verify

$$\exists x,x'\,(x,x'\in\{1,\ldots,m^+\}\ \wedge\ x\neq x'\ \wedge\ f(x)=f(x')) \tag{4.489}$$

according to (3.620). We notice that $\bar{x}',\bar{x}''\in\{1,\ldots,m\}$ implies the truth of the disjunctions

$$\bar{x}'\in\{1,\ldots,\bar{x}\}\ \vee\ \bar{x}'\in\{1,\ldots,m\}\setminus\{1,\ldots,\bar{x}\} \tag{4.490}$$

$$\bar{x}''\in\{1,\ldots,\bar{x}\}\ \vee\ \bar{x}''\in\{1,\ldots,m\}\setminus\{1,\ldots,\bar{x}\} \tag{4.491}$$

and of $\bar{x}'^+,\bar{x}''^+\in\{1,\ldots,m^+\}$ for the same reasons as the selection of an arbitrary $x\in\{1,\ldots,m\}$ implied, respectively, (4.482) and (4.485). It will also be useful to apply the previously mentioned inclusion $\{1,\ldots,m\}\subseteq\{1,\ldots,m^+\}$ (in connection with the definition of a subset) to infer from the assumed $\bar{x}',\bar{x}''\in\{1,\ldots,m\}$ the truth of $\bar{x}',\bar{x}''\in\{1,\ldots,m^+\}$. Thus, the constants $\bar{x}'$ and $\bar{x}''$ are also elements of the domain of f, as required within the proof of the non-injectivity of f based on (4.489). We now prove that existential sentence by cases, using first the true disjunction (4.490).

If $\bar{x}'\in\{1,\ldots,\bar{x}\}$ holds, then the definition of g yields $g(\bar{x}')=f(\bar{x}')$. We now include another proof by cases (of the desired existential sentence) based on the other true disjunction (4.491). On the one hand, in case $\bar{x}''\in\{1,\ldots,\bar{x}\}$ holds, which evidently yields $g(\bar{x}'')=f(\bar{x}'')$, we obtain with the equation in (4.488) after substitution $f(\bar{x}')=f(\bar{x}'')$. In view of the previously established $\bar{x}',\bar{x}''\in\{1,\ldots,m^+\}$ and in view of the inequality in (4.488), we thus see that the conjunction

$$\bar{x}',\bar{x}''\in\{1,\ldots,m^+\}\ \wedge\ \bar{x}'\neq\bar{x}''\ \wedge\ f(\bar{x}')=f(\bar{x}'') \tag{4.492}$$

is true, so that the existential sentence (4.489) to be proven holds indeed. On the other hand, the case $\bar{x}''\in\{1,\ldots,m\}\setminus\{1,\ldots,\bar{x}\}$ yields $g(\bar{x}'')=$

$f(\bar{x}''^{+})$ by definition of g, and we obtain therefore with the equation in (4.488) via substitution $f(\bar{x}') = f(\bar{x}''^{+})$. Furthermore, $\bar{x}'' \in \{1,\ldots,m\} \setminus \{1,\ldots,\bar{x}\}$ implies $\bar{x}'' \notin \{1,\ldots,\bar{x}\}$ (by definition of a set difference) and therefore $\bar{x}''^{+} \notin \{1,\ldots,\bar{x}\}$ with (4.249) and the Law of Contraposition. Thus, the conjunction of $\bar{x}' \in \{1,\ldots,\bar{x}\}$ and $\bar{x}''^{+} \notin \{1,\ldots,\bar{x}\}$ is true, which yields $\bar{x}' \neq \bar{x}''^{+}$ with Proposition 2.1. Recalling $\bar{x}' \in \{1,\ldots,m^{+}\}$ and $\bar{x}''^{+} \in \{1,\ldots,m^{+}\}$, these findings show that the conjunction

$$\bar{x}', \bar{x}''^{+} \in \{1,\ldots,m^{+}\} \wedge \bar{x}' \neq \bar{x}''^{+} \wedge f(\bar{x}') = f(\bar{x}''^{+}) \tag{4.493}$$

is true, and consequently the existential sentence (4.489) holds again. Thus, the proof by cases based on the disjunction (4.491) is complete.

If $\bar{x}' \in \{1,\ldots,m\}\setminus\{1,\ldots,\bar{x}\}$ holds, then the definition of g yields $g(\bar{x}') = f(\bar{x}'^{+})$. Let us consider once again the two cases of the disjunction (4.491). On the one hand, if $\bar{x}'' \in \{1,\ldots,\bar{x}\}$ holds, which yields $g(\bar{x}'') = f(\bar{x}'')$ by definition of g, then we obtain with the equation in (4.488) $f(\bar{x}'^{+}) = f(\bar{x}'')$. Moreover, $\bar{x}' \in \{1,\ldots,m\} \setminus \{1,\ldots,\bar{x}\}$ gives $\bar{x}' \notin \{1,\ldots,\bar{x}\}$ (by definition of a set difference) and then $\bar{x}'^{+} \notin \{1,\ldots,\bar{x}\}$ with (4.249) and the Law of Contraposition. As $\bar{x}'' \in \{1,\ldots,\bar{x}\}$ and $\bar{x}'^{+} \notin \{1,\ldots,\bar{x}\}$ are thus both true, it follows that $\bar{x}'' \neq \bar{x}'^{+}$ (using again Proposition 2.1). We therefore have the true conjunction

$$\bar{x}'', \bar{x}'^{+} \in \{1,\ldots,m^{+}\} \wedge \bar{x}'' \neq \bar{x}'^{+} \wedge f(\bar{x}'') = f(\bar{x}'^{+}), \tag{4.494}$$

so that the existential sentence (4.489) holds, as desired. On the other hand, if $\bar{x}'' \in \{1,\ldots,m\} \setminus \{1,\ldots,\bar{x}\}$ holds, which gives $g(\bar{x}'') = f(\bar{x}''^{+})$ by definition of g, then we obtain with the equation in (4.488) $f(\bar{x}'^{+}) = f(\bar{x}''^{+})$. Due to the inequality in (4.488), we see that $\bar{x}'$ and $\bar{x}''$ are distinct elements of the same set $\{1,\ldots,m\} \setminus \{1,\ldots,\bar{x}\}$. The preceding inequality $\bar{x}' \neq \bar{x}''$ then implies $s^{+}(\bar{x}') \neq s^{+}(\bar{x}'')$ with the Injection Criterion (recalling that the successor function is an injection), which inequality we may write also as $\bar{x}'^{+} \neq \bar{x}''^{+}$. Therefore, the conjunction

$$\bar{x}'^{+}, \bar{x}''^{+} \in \{1,\ldots,m^{+}\} \wedge \bar{x}'^{+} \neq \bar{x}''^{+} \wedge f(\bar{x}'^{+}) = f(\bar{x}''^{+}) \tag{4.495}$$

is now true, so that the existential sentence (4.489) holds in any case.

This completes the proof that f is not an injection for the case that the negation of (4.478) is true. We thus completed also the proof that f is not an injection if the second part $\neg\mathrm{ran}(f) \subseteq \{1,\ldots,n\}$ of the disjunction (4.477) holds. Since f is arbitrary, we may therefore conclude that the universal sentence with respect to f in (4.476) is true. Because m is also arbitrary, we may now further conclude that the universal sentence (4.476) holds. Then, as n was arbitrary as well, we may finally conclude that the induction step is true (besides the base case), which completes the proof of (4.472) by mathematical induction, and thus also the proof of the proposed sentence (4.471). □

Exercise 4.35. Show that (4.471) also holds for $n = 0$, that is,

$$\forall m, n\,([m \in \mathbb{N} \wedge n = 0] \Rightarrow$$
$$[n < m^+ \Rightarrow \neg\exists f\,(f \in \{1, \ldots, n\}^{\{1,\ldots,m^+\}} \wedge f \text{ is an injection})]). \quad (4.496)$$

(Hint: Apply first Method 1.6 and then Method 1.11, using (3.528).)

Theorem 4.102 (Dirichlet's Drawer Principle/Pigeonhole Principle). *For any natural numbers m and n, it is true that there is no injection from the initial segment $\{1, \ldots, m\}$ to the initial segment $\{1, \ldots, n\}$ if n is less than m, that is,*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [n < m \Rightarrow \neg\exists f\,(f : \{1, \ldots, m\} \hookrightarrow \{1, \ldots, n\})]). \quad (4.497)$$

Proof. Letting m and n be arbitrary, we assume first $m, n \in \mathbb{N}$ and subsequently also $n < m$ to be true. Since $0 \leq n$ also holds according to (4.187), we obtain with the Transitivity Formula for $\leq$ and $<$ the inequality $0 < n$ and therefore $n \neq 0$ with the comparability of the linear ordering $<$ of $\mathbb{N}$. It then follows from this with (4.39) that there exists a particular constant $\bar{k} \in \mathbb{N}$ with $\bar{k} \neq m$ and $\bar{k}^+ = m$. Thus, the initial assumption $n < m$ yields $n < \bar{k}^+$ via substitution. Furthermore, we see in light of (2.310) that the disjunction $n = 0 \vee n \in \mathbb{N}_+$ is true, which we now use to prove the sentence

$$\neg\exists f\,(f : \{1, \ldots, \bar{k}^+\} \hookrightarrow \{1, \ldots, n\}) \quad (4.498)$$

by cases. If $n = 0$ holds, then (4.498) follows from $n < \bar{k}^+$ with (4.496), and if $n \in \mathbb{N}_+$ holds, we obtain (4.498) with (4.471). Thus, the negated existential sentence in (4.497) is also true, because we may simply substitute m for $\bar{k}^+$ in (4.498). Since m and n were initially arbitrary, we may therefore conclude that the universal sentence (4.497) is true. □

Note 4.19. According to (4.260), the inequality $n < m$ is equivalent to $\{1, \ldots, n\} \subset \{1, \ldots, m\}$. Dirichlet's Drawer Principle thus shows that it is not possible that distinct elements of the domain of a function are always associated with distinct values if the range is a proper subset of the domain. This finding may be viewed as a mathematical analogue to the observable fact that a collection of objects/pigeons cannot be put into different drawers/holes if there are 'less drawers/holes than objects/pigeons'.

Before we can use the length n of a denumeration of a given (finite) set A as a valid counting of the number of elements in A, we must show that all other possible denumerations of A have the same length n.

Theorem 4.103 (Denumeration theorem). *For any finite set A the length of any denumeration of A is unique, that is,*

$$\forall A\,(A \text{ is a finite set } \Rightarrow\ \exists! n\,(n \in \mathbb{N} \wedge \exists c\,(c : \{1,\dots,n\} \rightleftarrows A))). \tag{4.499}$$

Proof. We let A be an arbitrary set, assume A to be finite, and observe that the existential part

$$\exists n\,(n \in \mathbb{N} \wedge \exists c\,(c : \{1,\dots,n\} \rightleftarrows A))$$

of the uniquely existential sentence is equivalent to

$$\exists n\,\exists c\,(n \in \mathbb{N} \wedge c : \{1,\dots,n\} \rightleftarrows A)$$

because of the Neutrality Law for quantification (1.57), which is then true by definition of a finite set. To verify the uniqueness part, we prove

$$\begin{aligned}\forall n, n'\,([(n \in \mathbb{N} \wedge \exists c\,(c : \{1,\dots,n\} \rightleftarrows A))\\ \wedge\ (n' \in \mathbb{N} \wedge \exists c\,(c : \{1,\dots,n'\} \rightleftarrows A))] \Rightarrow n = n').\end{aligned} \tag{4.500}$$

For this purpose, we let n and n' be arbitrary and prove the implication directly, assuming that n and n' are natural numbers, assuming that there exists a bijection from $\{1,\dots,n\}$ to A, say

$$\bar{c} : \{1,\dots,n\} \rightleftarrows A,$$

and assuming moreover that there exists a bijection from $\{1,\dots,n'\}$ to A, say

$$\bar{c}' : \{1,\dots,n'\} \rightleftarrows A.$$

Then, the corresponding inverse functions are also bijections because of (3.683), that is,

$$\begin{aligned}\bar{c}^{-1} &: A \rightleftarrows \{1,\dots,n\},\\ \bar{c}'^{-1} &: A \rightleftarrows \{1,\dots,n'\}.\end{aligned}$$

It then follows with (3.672) that the compositions $\bar{c}'^{-1} \circ \bar{c}$ and $\bar{c}^{-1} \circ \bar{c}'$ are the bijections

$$\begin{aligned}\bar{c}'^{-1} \circ \bar{c} &: \{1,\dots,n\} \rightleftarrows \{1,\dots,n'\},\\ \bar{c}^{-1} \circ \bar{c}' &: \{1,\dots,n'\} \rightleftarrows \{1,\dots,n\}.\end{aligned}$$

These compositions are thus in particular injections, which shows that the existential sentence

$$\exists f\,(f : \{1,\ldots,n\} \hookrightarrow \{1,\ldots,n'\}), \tag{4.501}$$
$$\exists f\,(f : \{1,\ldots,n'\} \hookrightarrow \{1,\ldots,n\}). \tag{4.502}$$

are true. Now, since the standard linear ordering $<$ of $\mathbb{N}$ is connex, the disjunction

$$n < n' \lor n' < n \lor n = n' \tag{4.503}$$

is true. We may now prove by contradiction that $\neg n < n'$ is true. For this purpose, we assume the negation of that negation to be true, so that $n < n'$ follows to be true with the Double Negation Law. It then follows with (4.497) that the negation of (4.502) is also true, so that we obtained a contradiction. Thus, $\neg n < n'$ is indeed true.

On the other hand, we may prove by contradiction that $\neg n' < n$ holds. To do this, we assume the double negation to be true, so that $n' < n$ follows now to be true. We then see in light of Dirichlet's Drawer Principle that the negation of (4.501) turns out to be true as well. Thus, we have a contradiction again, so that $\neg n' < n$ is also true (besides $\neg n < n'$).

Consequently, the third part $n = n'$ of the true disjunction (4.503) holds, which finding completes the direct proof of the implication in (4.500). As n and n' are arbitrary, we may therefore conclude that the universal sentence (4.500) is true, completing the proof of the uniquely existential sentence in (4.499). Since A was initially an arbitrary set, we now finally conclude that the theorem holds, as claimed. □

This unique length of the denumerations of a given (finite) set A is defined to be its *number of elements*.

Definition 4.11 (Cardinality/number of elements). For any finite set A we say that a natural number n is the *cardinality* or the *number of elements* of A, symbolically

$$n = |A|, \tag{4.504}$$

iff there exists a denumeration of A of length n, i.e. iff

$$\exists c\,(c : \{1,\ldots,n\} \rightleftarrows A) \tag{4.505}$$

Corollary 4.104. *The initial segment of* $\mathbb{N}_+$ *up to an* $n \in \mathbb{N}$ *has* n *elements, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow |\{1,\ldots,n\}| = n). \tag{4.506}$$

Proof. Letting n be arbitrary in $\mathbb{N}$, we see that the identity function $\mathrm{id}_{\{1,\ldots,n\}}$ is a bijection from $\{1,\ldots,n\}$ to $\{1,\ldots,n\}$ due to Corollary 3.203, so that there exists a denumeration of $\{1,\ldots,n\}$ of length n. Consequently, $n = |\{1,\ldots,n\}|$ holds by definition of a cardinality, which equation proves the implication in (4.506), and since n was initially arbitrary, we may therefore conclude that the proposed universal sentence holds. $\square$

Note 4.20. In case of $m = 0$, we have $\{1,\ldots,m\} = \emptyset$, and we may view $\mathrm{id}_{\{1,\ldots,m\}}$ as the empty bijection $\emptyset : \emptyset \rightleftarrows \emptyset$.

The following result is an immediate consequence of the proof of Proposition 4.99, which involves the existence of the bijections (4.468) and (4.469).

Corollary 4.105. *The union of a finite set A with cardinality $\bar{n}$ and the singleton formed by a constant y which is not in A is a finite set with cardinality $\bar{n}^+$, that is,*

$$\forall A, y, \bar{n}\,([y \notin A \,\wedge\, A \text{ is finite } \wedge\, |A| = \bar{n}] \\ \Rightarrow [A \cup \{y\} \text{ is finite } \wedge\, |A \cup \{y\}| = \bar{n}^+]). \tag{4.507}$$

Proposition 4.106. *The Cartesian product of the initial segment of $\mathbb{N}_+$ up to a natural number m and the singleton formed by a natural number n has cardinality m, that is,*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow |\{1,\ldots,m\} \times \{n\}| = m). \tag{4.508}$$

Proof. We let m and n be arbitrary natural numbers and show that there exists a denumeration c from $\{1,\ldots,m\}$ to $\{1,\ldots,m\} \times \{n\}$. First, we apply Function definition by replacement and verify for this purpose

$$\forall k\,(k \in \{1,\ldots,m\} \Rightarrow \exists! y\,(y = (k, n))). \tag{4.509}$$

Letting k be arbitrary, we assume $k \in \{1,\ldots,m\}$ to be true. By definition of an ordered pair, (k, n) is a uniquely specified set, so that the uniquely existential sentence in (4.509) follows to be true with (1.109). Since k is arbitrary, we may therefore conclude that the universal sentence (4.509) holds, which then further implies with Theorem 3.160 that there exists the unique function $\bar{c}$ with domain $\{1,\ldots,m\}$ satisfying

$$\forall k\,(k \in \{1,\ldots,m\} \Rightarrow \bar{c}(k) = (k, n)). \tag{4.510}$$

Next, we show that $\bar{c}$ is a bijection from $\{1,\ldots,m\}$ to $\{1,\ldots,m\} \times \{n\}$. To prove that $\bar{c}$ is a surjection from $\{1,\ldots,m\}$ to $\{1,\ldots,m\} \times \{n\}$, we demonstrate that $\{1,\ldots,m\} \times \{n\}$ is the range of $\bar{c}$ by verifying

$$\forall y\,(y \in \{1,\ldots,m\} \times \{n\} \Rightarrow \exists k\,(k \in \{1,\ldots,m\} \,\wedge\, \bar{c}(k) = y)). \tag{4.511}$$

according to the Surjection Criterion. We let y be arbitrary and assume $y \in \{1, \ldots, m\} \times \{n\}$ to be true. By definition of the Cartesian product of two sets, there then exists an element in $\{1, \ldots, m\}$, say $\bar{k}$, and there exists the unique element n in $\{n\}$ (according to Exercise 2.20), such that $(\bar{k}, n) = y$. Here, $\bar{k} \in \{1, \ldots, m\}$ implies $\bar{c}(\bar{k}) = (\bar{k}, n) \, [= y]$ with (4.510), which equations yield $\bar{c}(\bar{k}) = y$. We thus showed that the existential sentence in (4.511) holds, which proves the implication in (4.511). As y is arbitrary, we may therefore conclude that the universal sentence (4.511) holds. We thus showed that $\bar{c}$ is a surjection from $\{1, \ldots, m\}$ to $\{1, \ldots, m\} \times \{n\}$.

It remains for us to verify that $\bar{c}$ is also an injection from $\{1, \ldots, m\}$ to $\{1, \ldots, m\} \times \{n\}$. For this purpose, we prove

$$\forall k, k' \, ([k, k' \in \{1, \ldots, m\} \wedge \bar{c}(k) = \bar{c}(k')] \Rightarrow k = k'). \tag{4.512}$$

Letting k and k' be arbitrary in $\{1, \ldots, m\}$ such that $\bar{c}(k) = \bar{c}(k')$ holds, we infer from the latter equation with (4.510) that $(k, n) = (k', n)$ is true. This equation implies in particular $k = k'$ with the Equality Criterion for ordered pairs, and since k, k' are arbitrary, it follows then that the universal sentence (4.512) holds. Thus, the surjection $\bar{c} : \{1, \ldots, m\} \twoheadrightarrow \{1, \ldots, m\} \times \{n\}$ is also an injection, and therefore a bijection by definition. Because m is a natural number, the existence of this denumeration then evidently gives

$$m = |\{1, \ldots, m\} \times \{n\}|,$$

in view of the definition of a cardinality, proving the implication in (4.508). As m and n were arbitrary, we may consequently conclude that the proposed universal sentence is true. □

The notion of *number of elements* of a set suggests the following terminology.

Definition 4.12 (Equinumerous/equivalent finite sets). We say that two finite sets A and B are *equinumerous* or *equivalent* iff they have the same number of elements, i.e. iff

$$|A| = |B|. \tag{4.513}$$

The idea of counting a finite set via a bijective association/comparison with an initial segment of $\mathbb{N}_+$ is now generalized to compare any two given sets with respect to their number of elements.

Proposition 4.107. *Two finite sets A and B are equinumerous iff there exists a one-to-one correspondence from A to B, that is,*

$$\forall A, B \, (|A| = |B| \;\Leftrightarrow\; \exists f \, (f : A \rightleftarrows B)). \tag{4.514}$$

Proof. We let A and B arbitrary sets and assume A and B to be finite, so that there exist natural numbers, say $\bar{m}$ and $\bar{n}$, and denumerations of A and B of length $\bar{m}$ and $\bar{n}$, respectively, say

$$\begin{aligned} \bar{c} &: \{1,\dots,\bar{m}\} \rightleftarrows A, \\ \bar{d} &: \{1,\dots,\bar{n}\} \rightleftarrows B. \end{aligned} \tag{4.515}$$

These denumerations give the cardinalities $\bar{m} = |A|$ and $\bar{n} = |B|$. To prove the first implication ($\Rightarrow$), we assume $|A| = |B|$, so that substitution yields first $\bar{m} = \bar{n}$ and then

$$\bar{c} : \{1,\dots,\bar{n}\} \rightleftarrows A.$$

Because of (3.683), we obtain the inverse bijection

$$\bar{c}^{-1} : A \rightleftarrows \{1,\dots,\bar{n}\},$$

and (3.672) gives then the bijective composition

$$\bar{d} \circ \bar{c}^{-1} : A \rightleftarrows B,$$

so that the existential sentence in (4.514) is true, as desired.

To prove the second implication ($\Leftarrow$), we assume the preceding existential sentence to be true and take in particular $\bar{f} : A \rightleftarrows B$. Then, we may form the composition of $\bar{f}$ and (4.515), which is the bijection $\bar{f} \circ \bar{c} : \{1,\dots,\bar{m}\} \rightleftarrows B$ according to (3.672). This bijection is a denumeration of B of length $\bar{m}$, so that $\bar{m} = |B|$ holds. Since $\bar{m} = |A|$ is also true, we may apply substitution to obtain the desired equation $|A| = |B|$.

Thus, the proof of the equivalence is complete, and since A and B were arbitrary sets, we may therefore conclude that the proposition holds. □

Definition 4.13 (Permutation). For any finite set X we call any one-to-one correspondence

$$\pi : X \rightleftarrows X \tag{4.516}$$

a *permutation of* X.

Note 4.21. Permutations are invertible transformations, so that the set of all permutations of a (finite) set X is a more specific set of invertible transformations. In contrast to a permutation, the domain of an invertible transformation may be an infinite set.

Definition 4.14 (Set of permutations). We call the set of invertible transformations on any finite set X also the *set of permutations* of X, symbolically

$$\Pi(X). \tag{4.517}$$

Note 4.22. As identity functions are bijective (see Corollary 3.203), the identity function on any finite set constitutes a permutation of that set.

Corollary 4.108. *Any natural number n and the initial segment of $\mathbb{N}_+$ up to n are equinumerous sets, that is,*

$$|n| = |\{1, \ldots, n\}|, \tag{4.518}$$

$$|\{1, \ldots, n\}| = |n|. \tag{4.519}$$

Furthermore, the cardinality of a natural number n is identical with n, i.e.

$$\forall n \, (n \in \mathbb{N} \Rightarrow |n| = n). \tag{4.520}$$

Proof. Letting n be an arbitrary natural number, the existence of the bijections (4.465) and (4.466) prove (4.518) and (4.519), respectively. Since $|\{1, \ldots, n\}| = n$ holds according to (4.506), the equation (4.518) yields $|n| = n$ via substitution. Since n is arbitrary, the stated universal sentences follow then to be true. □

Proposition 4.109. *The Cartesian product of a finite set A and a singleton is itself finite, and this Cartesian product and the set A are equinumerous, that is,*

$$\forall A, a \, (A \text{ is finite } \Rightarrow [A \times \{a\} \text{ is finite } \wedge \, |A \times \{a\}| = |A|]). \tag{4.521}$$

Proof. Letting A and a be arbitrary, we first establish a bijection c from A to $A \times \{a\}$. To begin with, we introduce the notation $B = A \times \{a\}$ and then apply the Axiom of Specification in connection with the Axiom of Extension to obtain the true uniquely existential sentence

$$\exists! \bar{c} \forall z \, (z \in \bar{c} \Leftrightarrow [z \in A \times (A \times \{a\}) \, \wedge \, \exists x \, (z = (x, (x, a)))]).$$

Thus, $z \in \bar{c}$ implies in particular $z \in A \times (A \times \{a\})$ for any z, so that $\bar{c}$ may be viewed as a binary relation included in $A \times B$ (applying the definitions of a binary relation and of a subset). This binary relation $\bar{c}$ thus satisfies

$$\forall z \, (z \in \bar{c} \Leftrightarrow [z \in A \times (A \times \{a\}) \, \wedge \, \exists x \, (z = (x, (x, a)))]). \tag{4.522}$$

We now demonstrate that $\bar{c}$ is a function from A to $A \times \{a\}$. To do this, we apply the Function Criterion and verify accordingly

$$\forall x \, (x \in A \Rightarrow \exists! y \, (y \in A \times \{a\} \, \wedge \, (x, y) \in \bar{c})). \tag{4.523}$$

We let $\bar{x} \in A$ be arbitrary and prove first the existential part of the uniquely existential sentence with respect to y. Since $a \in \{a\}$ is evidently true and since the ordered pair $\bar{y} = (\bar{x}, a)$ is uniquely specified, we see that there exist x and b such that $x \in A$, $b \in \{a\}$ and $(x, b) = \bar{y}$ are true. It then follows with the definition of the Cartesian product of two sets that $\bar{y} \in A \times \{a\}$ holds. Furthermore, the ordered pair

$$\bar{z} = (\bar{x}, \bar{y}) = (\bar{x}, (\bar{x}, a))$$

is uniquely specified, which equations show that there exists an x with $\bar{z} = (x, (x, a))$. We also see that there exist x and y with $x \in A$, $y \in A \times \{a\}$ and $(x, y) = \bar{z}$, which implies (again by definition of the Cartesian product of two sets) that $\bar{z} \in A \times (A \times \{a\})$ holds. It then follows from these findings with (4.522) that $[(\bar{x}, \bar{y}) =]\ \bar{z} \in \bar{c}$ is true, which gives $(\bar{x}, \bar{y}) \in \bar{c}$. Together with the previously established $\bar{y} \in A \times \{a\}$, this proves the existential part in (4.523). To prove the uniqueness part, we verify

$$\forall y, y' \, ([y \in A \times \{a\} \wedge (\bar{x}, y) \in \bar{c} \wedge y' \in A \times \{a\} \wedge (\bar{x}, y') \in \bar{c}] \Rightarrow y = y'). \tag{4.524}$$

We let $\bar{y}$ and $\bar{y}'$ be arbitrary in $A \times \{a\}$ such that $(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{y}')$ are elements of $\bar{c}$. Here, $(\bar{x}, \bar{y}) \in \bar{c}$ implies with (4.522) in particular that there exists an element, say $\bar{\bar{x}}$, with $(\bar{x}, \bar{y}) = (\bar{\bar{x}}, (\bar{\bar{x}}, a))$; this equation gives $\bar{x} = \bar{\bar{x}}$ and $\bar{y} = (\bar{\bar{x}}, a)$ with (3.3), so that substitution yields $\bar{y} = (\bar{x}, a)$. Similarly, $(\bar{x}, \bar{y}') \in \bar{c}$ implies in particular that there is an element, say $\bar{\bar{x}}'$, with $(\bar{x}, \bar{y}') = (\bar{\bar{x}}', (\bar{\bar{x}}', a))$; consequently, $\bar{x} = \bar{\bar{x}}'$ and $\bar{y}' = (\bar{\bar{x}}', a)$, so that

$$\bar{y}' = (\bar{x}, a) \quad [= \bar{y}].$$

These equations yield $\bar{y} = \bar{y}'$, and since $\bar{y}$ and $\bar{y}'$ were arbitrary, we may therefore conclude that (4.524) holds, completing the proof of the uniquely existential sentence in (4.523). As x was arbitrary, we may then further conclude that (4.523) is true, which proves due to the Function Criterion that $\bar{c}$ is a function from A to $A \times \{a\}$.

Next, we show that $\bar{c}$ is a surjection, i.e. that $\mathrm{ran}(\bar{c}) = A \times \{a\}$ holds. To do this, we verify $A \times \{a\} \subseteq \mathrm{ran}(\bar{c})$ first, i.e. (using the definition of a subset)

$$\forall y \, (y \in A \times \{a\} \Rightarrow y \in \mathrm{ran}(\bar{c})). \tag{4.525}$$

Letting $\bar{y} \in A \times \{a\}$ be arbitrary, it follows by definition of the Cartesian product of two sets that there exist elements, say $\bar{x}$ and $\bar{a}$, with $\bar{x} \in A$, $\bar{a} \in \{a\}$ and $(\bar{x}, \bar{a}) = \bar{y}$. Here, $\bar{a} \in \{a\}$ implies $\bar{a} = a$ with (2.169), so that we obtain $\bar{y} = (\bar{x}, a)$. Moreover, $\bar{x} \in A$ implies with (4.523) the existence of the unique $y \in A \times \{a\}$ satisfying $(\bar{x}, y) \in \bar{c}$. This further implies with (4.522) that there is an element, say $\bar{\bar{x}}$, such that $(\bar{x}, y) = (\bar{\bar{x}}, (\bar{\bar{x}}, a))$. This gives $\bar{x} = \bar{\bar{x}}$ and $y = (\bar{\bar{x}}, a)$ in view of (3.3), with the consequence that

$$y = (\bar{x}, a) \quad [= \bar{y}].$$

Thus, $y = \bar{y}$ holds, so that the previously obtained $(\bar{x}, y) \in \bar{c}$ can be written as $(\bar{x}, \bar{y}) \in \bar{c}$. It now follows from this by definition of a range that $\bar{y} \in \operatorname{ran}(\bar{c})$ is true, as desired. Since $\bar{y}$ is arbitrary, we therefore conclude that (4.525) holds, so that $A \times \{a\} \subseteq \operatorname{ran}(\bar{c})$ follows to be true by definition of a subset. As $A \times \{a\}$ is a codomain of $\bar{c}$, we also have that $\operatorname{ran}(\bar{c}) \subseteq A \times \{a\}$. The conjunction of the previous two inclusions then gives $\operatorname{ran}(\bar{c}) = A \times \{a\}$ with the Axiom of Extension, so that the function $\bar{c} : A \to A \times \{a\}$ is a surjection by definition.

We now show that $\bar{c} : A \to A \times \{a\}$ is also an injection and verify accordingly

$$\forall x, x' \, ([x, x' \in A \wedge \bar{c}(x) = \bar{c}(x')] \Rightarrow x = x'). \tag{4.526}$$

We let $x, x' \in A$ be arbitrary and assume $[y =]\ \bar{c}(x) = \bar{c}(x')$. Here, we may write $y = \bar{c}(x)$ and $y = \bar{c}(x')$ also as $(x, y) \in \bar{c}$ and $(x', y) \in \bar{c}$. Here, $(x, y) \in \bar{c}$ implies with (4.522) in particular that there exists an element, say $\bar{x}$, with $(x, y) = (\bar{x}, (\bar{x}, a))$, which equation implies $x = \bar{x}$ and $y = (\bar{x}, a)$ with (3.3), and therefore $y = (x, a)$. Similarly, $(x', y) \in \bar{c}$ implies in particular that there exists an element, say $\bar{x}'$, with $(x', y) = (\bar{x}', (\bar{x}', a))$, which equation yields $x' = \bar{x}'$ and

$$[(x, a) =] \quad y = (\bar{x}', a) = (x', a).$$

We therefore obtain $(x, a) = (x', a)$, which then implies in particular the desired $x = x'$, applying once again (3.3). This proves the implication in (4.526), and since x and x' are arbitrary, we therefore conclude that (4.526) is true. Thus, the surjection $\bar{c} : A \twoheadrightarrow A \times \{a\}$ is also an injection, so that $\bar{c}$ is a bijection from A to $A \times \{a\}$, that is,

$$\bar{c} : A \rightleftarrows A \times \{a\}.$$

Now, since A is by assumption a finite set, there exist a natural number, say $\bar{n}$, and a denumeration from $\{1, \ldots, \bar{n}\}$ to A, say

$$\bar{d} : \{1, \ldots, \bar{n}\} \rightleftarrows A.$$

This shows that there exist a set n and a set d satisfying $n \in \mathbb{N}$, $d : \{1, \ldots, n\} \rightleftarrows A$ and $\bar{n} = n$, which existential sentence implies $\bar{n} = |A|$ with the definition of a cardinality, since A is finite by assumption. Now, the composition of $\bar{c}$ and $\bar{d}$ is the bijection

$$\bar{c} \circ \bar{d} : \{1, \ldots, \bar{n}\} \rightleftarrows A \times \{a\}$$

according to (3.672). This shows that there exist a set n and a set e such that $n \in \mathbb{N}$ and $e : \{1, \ldots, n\} \rightleftarrows A \times \{a\}$ hold. This existential sentence then implies that $A \times \{a\}$ is a finite set, by definition. Clearly, such an n satisfies also $\bar{n} = n$, so that

$$[|A| =] \quad \bar{n} = |A \times \{a\}|$$

follows by definition of a cardinality. These equations yield $|A \times \{a\}| = |A|$, so that the proof of the implication in (4.521) is complete. Since A and a were arbitrary, we finally conclude that the proposition is true. □

Proposition 4.110. *The cardinality of a finite set A is 0 iff the set A is empty, that is,*

$$\forall A\,(A \text{ is finite } \Rightarrow [|A| = 0 \Leftrightarrow A = \emptyset]). \tag{4.527}$$

Proof. We let A be an arbitrary finite set, so that there then exists a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}\}$ to A, say $\bar{c}$; thus, we have $\bar{n} = |A|$ by definition of a cardinality.

We now prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $|A| = 0$ holds, so that substitution yields $\bar{n} = 0$, and therefore $\bar{c}$ is a bijection from $\{1, \ldots, 0\}$ to A, i.e. from $\emptyset$ to A in view of (4.239). As $\emptyset$ is the only function from $\emptyset$ to A according to Proposition 3.151, we have $\bar{c} = \emptyset$. Since the bijection $\bar{c}$ is a surjection, the range of $\bar{c}$ is identical with the codomain A, i.e. $\mathrm{ran}(\bar{c}) = A$; furthermore, $\bar{c} = \emptyset$ implies $\mathrm{ran}(\bar{c}) = \emptyset$ with (3.121), so that substitution gives $A = \emptyset$, as desired.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume $A = \emptyset$. This assumption implies with Corollary 3.201 that $\emptyset$ is a bijection from $\emptyset$ to A, which bijection we may also write as $\emptyset : \{1, \ldots, 0\} \rightleftarrows A$, using again (4.239). The existence of this bijection then implies $0 = |A|$ by definition of a cardinality, which finding completes the proof of the equivalence in (4.527).

Since A was arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Proposition 4.111. *The following sentences are true.*

a) The singleton formed by any y is a finite set with cardinality 1, i.e.

$$\forall y\,(\{y\} \text{ is finite } \wedge |\{y\}| = 1). \tag{4.528}$$

b) The cardinality of a finite set is 1 iff A is the singleton formed by some y, that is,

$$\forall A\,(A \text{ is finite } \Rightarrow [|A| = 1 \Leftrightarrow \exists y\,(A = \{y\})]). \tag{4.529}$$

Proof. Concerning a), we let y be arbitrary, so that $\{(1,y)\}$ is a bijection from $\{1\}$ to $\{y\}$ (see Corollary 3.202). Thus, by definition of an initial segment of $\mathbb{N}_+$, we have $\{1\} = \{1,\ldots,1\}$, so that the existential sentence

$$\exists n, c\,(n \in \mathbb{N} \,\wedge\, c : \{1,\ldots,n\} \rightleftarrows \{y\})$$

is true. Consequently, $\{y\}$ is finite with $1 = |\{y\}|$ by definition of a cardinality. As y is arbitrary, it follows that a) is true.

Concerning b), we let A be arbitrary and assume that A is finite, so that there exists a natural number, say $\bar{n}$, and a bijection from $\{1,\ldots,\bar{n}\}$ to A, say $\bar{c}$. Here, $\bar{n} = |A|$ holds by definition of a cardinality. To prove the first part ('$\Rightarrow$') of the equivalence, we further assume $|A| = 1$, so that $\bar{n} = 1$ follows to be true. Thus, $\bar{c}$ is a bijection from $\{1,\ldots,1$ to A, i.e. from $\{1\}$ to A, and therefore in particular a surjection from $\{1\}$ to A. It then follows with (3.648) that $A = \{\bar{c}(1)\}$ holds, so that there exists an y with $A = \{y\}$, which existential sentence proves the first part of the equivalence.

To prove the second part ('$\Leftarrow$') of the equivalence in (4.529), we now assume that there exists an element, say $\bar{y}$, such that $A = \{\bar{y}\}$. It then follows with (4.528) that the singleton $A = \{\bar{y}\}$ is finite with cardinality $|A| = 1$. This completes the proof of the equivalence, and since A was arbitrary, we may therefore conclude that b) also holds. □

Lemma 4.112. *For any positive natural number n and any finite set A with cardinality n^+, it is true that the removing an element a from the set A yields a finite set with cardinality n, that is,*

$$\begin{aligned}&\forall n, A, a\,([n \in \mathbb{N}_+ \,\wedge\, A \text{ is finite} \,\wedge\, |A| = n^+ \,\wedge\, a \in A] \\ &\quad \Rightarrow [A \setminus \{a\} \text{ is finite } \wedge\, |A \setminus \{a\}| = n]).\end{aligned} \tag{4.530}$$

Proof. We let n, A and a be arbitrary, assume n to be a positive natural number, A to be a finite set with cardinality $|A| = n^+$, and a to be an element of A. By definition of a cardinality, there then exists a bijection from $\{1,\ldots,n^+\}$ to A, say $\bar{c}$. Let us now observe that the disjunction $\bar{c}(n^+) = a \vee \bar{c}(n^+) \neq a$ is true according to the Law of the Excluded Middle, so that we may prove the desired consequent in (4.530) by cases.

In the first case that $\bar{c}(n^+) = a$ is true, which we may also write as $(n^+, a) \in \bar{c}$, we have with Proposition 3.205 that $\bar{c} \setminus \{(n^+, a)\}$ is a bijection from $\{1,\ldots,n^+\} \setminus \{n^+\}$ to $A \setminus \{a\}$. Consequently, $\bar{c} \setminus \{(n^+, a)\}$ is a bijection from $\{1,\ldots,n\}$ to $A \setminus \{a\}$ in view of (4.256). Since $n \in \mathbb{N}_+$ evidently implies that n is a natural number, this bijection shows that $A \setminus \{a\}$ is a finite set with cardinality $n = |A \setminus \{a\}|$.

In the second case that $\bar{c}(n^+) \neq a$ is true, we see that $\bar{a} = \bar{c}(n^+)$ is an element of the codomain A of $\bar{c}$, where $\bar{a} \neq a$ holds. As the bijection $\bar{c}$

is in particular a surjection, we have that A is the range of $\bar{c}$. Then, the initial assumption $a \in A$ implies by definition of a range that there exists an element, say $\bar{m}$, with $(\bar{m}, a) \in \bar{c}$. We may write this also as $\bar{c}(\bar{m}) = a$, where $\bar{m} \in \{1, \ldots, n^+\}$ [$= \mathrm{dom}(\bar{c})$] holds by definition of a domain. Since $\bar{a} \neq a$ implies $\bar{c}(n^+) \neq \bar{c}(\bar{m})$ via substitution, we obtain with Proposition 3.149 the inequality $n^+ \neq \bar{m}$. We now define a function f with domain $\{1, \ldots, n^+\}$ by replacement, where the formula $\varphi(x, y)$ is given by

$$\begin{aligned} &[x = n^+ \Rightarrow y = a] \\ \wedge\, &[x = \bar{m} \Rightarrow y = \bar{a}] \\ \wedge\, &[x \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\} \Rightarrow y = \bar{c}(x)]. \end{aligned} \tag{4.531}$$

To do this, we prove

$$\forall x\, (x \in \{1, \ldots, n^+\} \Rightarrow \exists! y\, (\varphi(x, y))), \tag{4.532}$$

letting x be arbitrary and assuming $x \in \{1, \ldots, n^+\}$ to be true. Let us now establish the truth of the disjunction

$$x = n^+ \vee x = \bar{m} \vee x \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}. \tag{4.533}$$

To begin with, since $n^+ \in \{1, \ldots, n^+\}$ is true because of (4.247) and as $\bar{m} \in \{1, \ldots, n^+\}$ also holds (as shown before), we obtain $\{n^+\} \subseteq \{1, \ldots, n^+\}$ and $\{\bar{m}\} \subseteq \{1, \ldots, n^+\}$ with (2.184), which inclusions further implies $\{n^+\} \cup \{\bar{m}\} \subseteq \{1, \ldots, n^+\}$ with (2.252), where $\{n^+\} \cup \{\bar{m}\} = \{n^+, \bar{m}\}$ holds with (2.226), so that substitution yields $\{n^+, \bar{m}\} \subseteq \{1, \ldots, n^+\}$. It then follows with (2.263) that

$$\begin{aligned} \{1, \ldots, n^+\} &= [\{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}] \cup \{n^+, \bar{m}\} \\ &= [\{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}] \cup \{n^+\} \cup \{\bar{m}\} \\ &= \{n^+\} \cup \{\bar{m}\} \cup \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}, \end{aligned}$$

where we applied the commutativity and the associativity of the union of two sets to obtain the last equation. Therefore, the assumed $x \in \{1, \ldots, n^+\}$ implies with the definition of the union of two sets the multiple disjunction

$$x \in \{n^+\} \vee x \in \{\bar{m}\} \vee x \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\},$$

which in turn gives (4.533) in view of (2.169). We now use this true disjunction to prove the desired uniquely existential sentence in (4.532) by cases.

If the first part $x = n^+$ of the disjunction (4.533) is true, then the antecedent of the first implication

$$x = n^+ \Rightarrow y = a \tag{4.534}$$

is true, so that the implication itself is true iff the consequent $y = a$ is true. Indeed, there exists a unique y such that $y = a$ holds (see Proposition 1.21), which is the constant $y = a$. Now, the previously established inequality $n^+ \neq \bar{m}$ implies $x \neq \bar{m}$, so that $x = \bar{m}$ is false. Then, the implication

$$x = \bar{m} \Rightarrow y = \bar{a} \tag{4.535}$$

is true for $y = a$, because the antecedent is false. Furthermore, $x = n^+$ implies $x \in \{n^+, \bar{m}\}$ by definition of a pair; consequently, $x \notin \{n^+, \bar{m}\}$ is false, which implies that the conjunction $x \in \{1, \ldots, n^+\} \wedge x \notin \{n^+, \bar{m}\}$ is also false, which means by definition of a set difference that $x \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}$ is false. With this, we see that the third implication

$$x \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\} \Rightarrow y = \bar{c}(x) \tag{4.536}$$

is true for $y = a$, as the antecedent is false. We thus showed that there exists a unique y such that $\varphi(x, y)$ holds, if $x = n^+$ is true.

Similarly, if the second part $x = \bar{m}$ of the disjunction (4.533) is true, then the antecedent of the second implication (4.535) is true, so that the implication itself is true iff the consequent $y = \bar{a}$ is true. Because of Proposition 1.21 there indeed exists a unique y satisfying $y = \bar{a}$, which is the constant $y = \bar{a}$. Now, $n^+ \neq \bar{m}$ implies $x \neq n^+$, so that $x = n^+$ is false. Then, the first implication (4.534) with $y = \bar{a}$ has a false antecedent and is therefore true. Moreover, $x = \bar{m}$ evidently implies $x \in \{n^+, \bar{m}\}$ as before, so that $x \in \{1, \ldots, n^+\} \setminus \{\bar{n}, \bar{m}\}$ turns out to be false and the third implication (4.536) to be true. Thus, there also exists a unique y such that $\varphi(x, y)$ is true if $x = \bar{m}$ holds.

Finally, if the third part $x \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}$ of the disjunction (4.533) is true, then the antecedent of the third implication (4.536) is true, so that the implication itself is true iff $y = \bar{c}(x)$ holds. This equation is evidently satisfied by the unique constant $y = \bar{c}(x)$. The assumption $x \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}$ also implies that $x \notin \{n^+, \bar{m}\}$, which means by definition of a pair in connection with De Morgan's Law for sentences (1.51) that $x \neq n^+$ and $x \neq \bar{m}$ are both true. Then, as the equations $x = n^+$ and $x = \bar{m}$ are false, the implications (4.534) and (4.535) have false antecedents and are therefore true for $y = \bar{c}(x)$. Thus, there exists also in this case a unique y such that $\varphi(x, y)$ holds.

This proves the implication in (4.532), and since x was arbitrary, we may therefore conclude that the universal (4.532) is true. It then follows with Theorem 3.160 that there exists a unique function f with domain $\{1,\ldots,n^+\}$ such that $\varphi(x,f(x))$ holds for any $x\in\{1,\ldots,n^+\}$, and we may write this mapping as

$$x\mapsto f(x)=\begin{cases} a & \text{if } x=n^+\\ \bar{a} & \text{if } x=\bar{m}\\ \bar{c}(x) & \text{if } x\in\{1,\ldots,n^+\}\setminus\{n^+,\bar{m}\}\end{cases}$$

We now verify that f is an injection, i.e. that f satisfies (3.615) with $X=\{1,\ldots,n^+\}$. Let us first rewrite that sentence equivalently as

$$\forall x,x'\,(x,x'\in\{1,\ldots,n^+\}\Rightarrow[f(x)=f(x')\Rightarrow x=x']) \tag{4.537}$$

by applying (1.49). We let x and x' be arbitrary in $\{1,\ldots,n^+\}$ and prove the implication

$$f(x)=f(x')\Rightarrow x=x'. \tag{4.538}$$

Let us recall that $x\in\{1,\ldots,n^+\}$ implies the truth of the disjunction (4.533).

If $x=n^+$ holds, then the definition of f gives $f(x)=a$. Observing now that $x'\in\{1,\ldots,n^+\}$ also holds, we evidently obtain the true disjunction (4.533), which we established for an arbitrary element of $\{1,\ldots,n^+\}$. Then, in case of $x'=n^+$ $(=x)$, we obtain $f(x')=a$ $(=f(x))$ by definition of f, so that $f(x)=f(x')$ and $x=x'$ are both true; consequently, the implication (4.538) takes the value 'true'. In case of $x'=\bar{m}$, we obtain $f(x')=\bar{a}$ $(\neq a)$, so that $f(x')\neq f(x)$ is true, so that the implication (4.538) has a false antecedent and is thus true. Finally, in case of $x'\in\{1,\ldots,n^+\}\setminus\{n^+,\bar{m}\}$, we obtain $f(x')=\bar{c}(x')$ (by definition of f) as well as $x'\notin\{n^+,\bar{m}\}$ and therefore $x'\neq\bar{m}$ (by definition of a pair). Recalling that $\bar{c}(\bar{m})=a$ $(=f(x))$ holds and noting that $x'\neq\bar{m}$ implies $\bar{c}(x')\neq\bar{c}(\bar{m})$ with the Injection Criterion and the injectivity of $\bar{c}$, we see that $f(x')\neq f(x)$ is again true; thus, the implication (4.538) holds also in this case.

If $x=\bar{m}$ holds, then $f(x)=\bar{a}$. Now, in case of $x'=n^+$, we have $f(x')=a$ $(\neq\bar{a})$ and therefore $f(x')\neq f(x)$, so that the implication (4.538) is true. In case of $x'=\bar{m}$ $(=x)$, we have $f(x')=\bar{a}$ $(=f(x))$; consequently, $f(x')=f(x)$ and $x'=x$ are both true, so that (4.538) holds again. Finally, in case of $x'\in\{1,\ldots,n^+\}\setminus\{n^+,\bar{m}\}$, we have $f(x')=\bar{c}(x')$ as well as $x'\notin\{n^+,\bar{m}\}$, and therefore $x'\neq n^+$ (by definition of a pair). Recalling that $\bar{c}(n^+)=\bar{a}$ $(=f(x))$ holds and observing that $x'\neq n^+$ implies $\bar{c}(x')\neq\bar{c}(\bar{n})$ with the Injection Criterion, we have that $f(x')\neq f(x)$ is true once again, with the consequence that (4.538) is true also in the present case.

If $x\in\{1,\ldots,n^+\}\setminus\{n^+,\bar{m}\}$ holds, then $f(x)=\bar{c}(x)$ and also $x\notin\{n^+,\bar{m}\}$, so that $x\neq n^+$ and $x\neq\bar{m}$ are both true. In case of $x'=n^+$, which gives $f(x') = a$ $(= \bar{c}(\bar{m}))$, we observe that $x \neq \bar{m}$ implies $\bar{c}(x) \neq \bar{c}(\bar{m})$ with the Injection Criterion, so that $f(x) \neq f(x')$ and consequently (4.538) holds. In case of $x' = \bar{m}$, which gives $f(x') = \bar{a}$ $(= \bar{c}(n^+))$, we notice that $x \neq n^+$ now implies $\bar{c}(x) \neq \bar{c}(n^+)$ with the injectivity of $\bar{c}$; thus, $f(x) \neq f(x')$ is true, and therefore also (4.538). Finally, in case of $x' \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}$, we obtain $f(x') = \bar{c}(x')$. Now, the disjunction $\bar{c}(x) = \bar{c}(x') \vee \bar{c}(x) \neq \bar{c}(x')$ is true because of the Law of the Excluded Middle. On the one hand, if $\bar{c}(x) = \bar{c}(x')$ is true, then substitution yields the true equation $f(x) = f(x')$, and it follows with the injectivity of $\bar{c}$ that $x = x'$ holds. Thus, the implication (4.538) takes the value 'true'. On the other hand, if $\bar{c}(x) \neq \bar{c}(x')$ is true, then this gives $f(x) \neq f(x')$ by substitution, and the implication (4.538) turns out to be true, being based on a false antecedent.

We thus completed the proof of (4.538) by cases; as x and x' were arbitrary, we may therefore conclude that the universal sentence (4.537) holds, so that f is indeed an injection.

Next, we verify that the range of f is identical with A. For this purpose, we apply the Equality Criterion for sets and prove accordingly

$$\forall y\,(y \in \operatorname{ran}(f) \Leftrightarrow y \in A). \tag{4.539}$$

We let y be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $y \in \operatorname{ran}(f)$ to be true. Then, by definition of a range, there exists an element, say $\bar{x}$, such that $(\bar{x}, y) \in f$. Consequently, $\bar{x}$ is by definition of a domain an element of $\operatorname{dom}(f) = \{1, \ldots, n^+\}$. This evidently implies the disjunction (4.533), in which we replace the arbitrary x by $\bar{x}$.

Now, if $\bar{x} = n^+$ holds, then the definition of f gives $f(\bar{x}) = a$. Since $(\bar{x}, y) \in f$ may be written in function notation as $f(\bar{x}) = y$, we obtain after substitution $y = a$. The initial assumption $a \in A$ thus gives the desired $y \in A$.

If $\bar{x} = \bar{m}$ holds, then we obtain $f(\bar{x}) = \bar{a}$ by definition of f, so that $y = \bar{a}$. Then, the previously established $\bar{a} \in A$ also gives $y \in A$.

Finally, if $\bar{x} \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}$ holds, then $f(\bar{x}) = \bar{c}(\bar{x})$ is true by definition of f. Since $\bar{c}$ is a bijection from $\{1, \ldots, n^+\}$ to A, we evidently have $\bar{c}(\bar{x}) \in A$, and therefore $f(\bar{x}) \in A$ due to the preceding equation. Recalling $f(\bar{x}) = y$, we thus see that $y \in A$ holds also in this case, completing the proof of the first part of the equivalence in (4.539).

To prove the second part ('$\Leftarrow$'), we now assume that $y \in A$ is true. Recalling that A is the range of the bijection/surjection $\bar{c}$, we see that there exists an element, say $\bar{x}$, such that $(\bar{x}, y) \in \bar{c}$ holds. Consequently, $\bar{x}$ is an element of the domain $\{1, \ldots, n^+\}$ of $\bar{c}$. Let us also recall that $\bar{c}(n^+) = \bar{a}$ and $\bar{c}(\bar{m}) = a$ hold, and that $(\bar{x}, y) \in \bar{c}$ means $\bar{c}(\bar{x}) = y$.

Now, if $\bar{x} = n^+$ holds, then substitution yields $\bar{c}(n^+) = y$ and then $\bar{a} = y$. By definition of f, we have $f(\bar{m}) = \bar{a}$, which implies $f(\bar{m}) = y$. This means $(\bar{m}, y) \in f$, so that y is an element of the range of f.

If $\bar{x} = \bar{m}$ holds, then we obtain $\bar{c}(\bar{m}) = y$ and then $a = y$. Here, $f(n^+) = a$ holds by definition of f, which implies $f(n^+) = y$. Clearly, y is then an element of $\mathrm{ran}(f)$.

Finally, if $\bar{x} \in \{1, \ldots, n^+\} \setminus \{n^+, \bar{m}\}$ holds, then the definition of f gives $f(\bar{x}) = \bar{c}(\bar{x})$ $(= y)$, so that $f(\bar{x}) = y$ is true. We therefore have $y \in \mathrm{ran}(f)$ also in this case.

Thus, the proof of the second part of the equivalence in (4.539) is complete. As y was arbitrary, we may then further conclude that the universal sentence (4.539) is true, which implies $\mathrm{ran}(f) = A$ with the Equality Criterion for sets. Then, by definition of a surjection, we have $f : \{1, \ldots, n^+\} \twoheadrightarrow A$. Since f is also an injection with domain $\{1, \ldots, n^+\}$, for which we may use any codomain (see Note 3.23), we see that f is a bijection from $\{1, \ldots, n^+\}$ to A. By definition of f, it is true that $f(n^+) = a$. We may therefore proceed in analogy to the already established case of $\bar{c}(n^+) = a$. Thus, observing that $f(n^+) = a$ may be written also as $(n^+, a) \in f$, it follows with Proposition 3.205 that $f \setminus \{(n^+, a)\}$ is a bijection from $\{1, \ldots, n^+\} \setminus \{n^+\}$ to $A \setminus \{a\}$, and therefore a bijection from $\{1, \ldots, n\}$ to $A \setminus \{a\}$ in view of (4.256). This bijection clearly shows that the set $A \setminus \{a\}$ is finite, having again the cardinality $|A \setminus \{a\}| = n$, completing the proof by cases of the consequent in (4.530). As n, A and a were initially arbitrary, we may now finally conclude that the proposed universal sentence (4.530) is true. □

Lemma 4.113. *The union of a finite set A and a finite set B with cardinality n is itself finite, that is,*

$$\forall n, A, B\,([n \in \mathbb{N}_+ \,\wedge\, A \text{ is finite} \,\wedge\, B \text{ is finite} \,\wedge\, |B| = n] \Rightarrow A \cup B \text{ is finite}). \tag{4.540}$$

Proof. Let us first rewrite (4.540) equivalently as (using (1.90))

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \forall A, B\,([A \text{ is finite} \wedge B \text{ is finite} \wedge |B| = n] \Rightarrow A \cup B \text{ is finite})). \tag{4.541}$$

We now carry out a proof by mathematical induction. To prove the base case ($n = 1$), we let A and B be arbitrary such that A and B are finite and such that $|B| = 1$ holds. The latter implies with (4.529) that there exists an element, say $\bar{y}$, such that $B = \{\bar{y}\}$ is true. Then, the finiteness of A implies the finiteness of $A \cup \{\bar{y}\}$ with (4.467), so that $A \cup B$ follows to be finite via substitution based on the preceding equation. This finding proves the base case.

Regarding the induction step, we let n be arbitrary in $\mathbb{N}_+$, make the induction assumption

$$\forall A, B\,([A \text{ is finite } \wedge\ B \text{ is finite } \wedge\ |B| = n] \Rightarrow A \cup B \text{ is finite}), \tag{4.542}$$

and demonstrate that this implies

$$\forall A, B\,([A \text{ is finite } \wedge\ B \text{ is finite } \wedge\ |B| = n^+] \Rightarrow A \cup B \text{ is finite}). \tag{4.543}$$

For this purpose, we let A and B be arbitrary such that A and B are finite with $|B| = n^+$. Since $n \in \mathbb{N}_+$ evidently implies $n \in \mathbb{N}$ and therefore $n^+ \neq 0$ with (4.35), we may apply substitution based on the equation $|B| = n^+$ to infer from the preceding inequality the truth of $|B| \neq 0$. This inequality yields $B \neq \emptyset$ with (4.527), so that there exists in view of (2.42) an element in B, say $\bar{y}$. Thus, the conjunction

$$n \in \mathbb{N}_+\ \wedge\ B \text{ is finite } \wedge\ |B| = n^+\ \wedge\ \bar{y} \in B$$

is true, which implies with (4.530) that $B \setminus \{\bar{y}\}$ is a finite set with cardinality $|B \setminus \{\bar{y}\}| = n$. This finding, together with the assumption that A is finite, implies with the induction assumption (4.542) that $A \cup (B \setminus \{\bar{y}\})$ is a finite set. Then, the union

$$[A \cup (B \setminus \{\bar{y}\})] \cup \{\bar{y}\}$$

is also finite due to (4.467), and the preceding union is identical with

$$A \cup [(B \setminus \{\bar{y}\}) \cup \{\bar{y}\}] \tag{4.544}$$

because of the Associative Law for the union of two sets, so that the latter set is also finite. Then, the fact $\bar{y} \in B$ implies $\{\bar{y}\} \subseteq B$ with (2.184), which in turn implies

$$(B \setminus \{\bar{y}\}) \cup \{\bar{y}\} = B$$

with (2.263). Consequently, the finite set in (4.544) is identical with $A \cup B$, which proves the finiteness of the latter. Since A and B are arbitrary, we may therefore conclude that the universal sentence (4.543) is true. As n was also arbitrary, we may now further conclude that the induction step holds (besides the base case), so that the proof of (4.541) by mathematical induction is complete. Consequently, the proposed equivalent sentence (4.540) is also true. □

Theorem 4.114 (Finiteness of the union of two finite sets). *The union of two finite sets is finite, that is,*

$$\forall A, B\,([A \text{ is finite } \wedge\ B \text{ is finite}] \Rightarrow A \cup B \text{ is finite}). \tag{4.545}$$

Proof. We let A and B be sets and assume that A and B are finite. By definition of a finite set, there then exist a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}\}$ to B, say $\bar{c}$. Furthermore, the definition of a cardinality gives $\bar{n} = |B|$. Noting that $\bar{n} \in \mathbb{N}$ implies the true disjunction $\bar{n} = 0 \lor \bar{n} \in \mathbb{N}_+$ with (2.310), we may apply now a proof by cases.

If $\bar{n} = 0$ holds, then we obtain $|B| = 0$ via substitution and therefore $B = \emptyset$ with (4.527), which gives $A \cup B = A \cup \emptyset = A$ with (2.216). Since A is by assumption finite, we thus see that $A \cup B$ is finite.

If $\bar{n} \in \mathbb{N}_+$ holds, then the conjunction

$$\bar{n} \in \mathbb{N}_+ \land A \text{ is finite} \land B \text{ is finite} \land |B| = \bar{n}$$

is true, which then implies with (4.540) that $A \cup B$ is finite.

Thus, $A \cup B$ is finite in any case, and since A and B were arbitrary, we may therefore conclude that (4.545) is true. □

Proposition 4.115. *For any natural number n, any finite set A with cardinality n and any proper subset S of A, it is true that S is a finite set with cardinality less than n, that is,*

$$\forall n, A, S\,([n \in \mathbb{N} \land A \text{ is finite} \land |A| = n \land S \subset A] \Rightarrow [S \text{ is finite} \land |S| < n]). \tag{4.546}$$

Proof. Let us first rewrite (4.546) equivalently as

$$\forall n\,(n \in \mathbb{N} \Rightarrow \forall A, S\,([A \text{ is finite} \land |A| = n \land S \subset A] \Rightarrow [S \text{ is finite} \land |S| < n])) \tag{4.547}$$

by means of (1.90). We now prove the universal sentence with respect to n by mathematical induction. Regarding the base case ($n = 0$), we prove

$$\forall A, S\,([A \text{ is finite} \land |A| = 0 \land S \subset A] \Rightarrow [S \text{ is finite} \land |S| < 0]), \tag{4.548}$$

letting A and S be arbitrary. Let us now prove by contradiction that the antecedent of the implication in (4.548) is false, i.e. that the negation of the antecedent is true. For this purpose, we assume

$$\neg\neg[A \text{ is finite} \land |A| = 0 \land S \subset A]$$

to be true, so that we obtain with the Double Negation Law in particular the true sentences $|A| = 0$ and $S \subset A$. The preceding equation yields $A = \emptyset$ by definition of a cardinality, so that $S \subset A$ is false due to (2.47), which finding is in contradiction to $S \subset A$. Thus, the implication in (4.548) has a false antecedent and is therefore true. Since A and S were arbitrary, we may infer from this the truth of the base case.

Regarding induction step, we let n be arbitrary in $\mathbb{N}$, make the induction assumption

$$\forall A, S\,([A \text{ is finite} \wedge |A| = n \wedge S \subset A] \Rightarrow [S \text{ is finite} \wedge |S| < n]), \quad (4.549)$$

and show that this implies

$$\forall A, S\,([A \text{ is finite} \wedge |A| = n^+ \wedge S \subset A] \Rightarrow [S \text{ is finite} \wedge |S| < n^+]). \quad (4.550)$$

To prove the latter sentence, we let A and S be arbitrary and prove the implication directly, assuming that A is a finite set with cardinality $|A| = n^+$ and that S is a proper subset of A. Thus, there exists (by definition of a cardinality) a particular bijection

$$\bar{c} : \{1, \ldots, n^+\} \rightleftarrows A. \quad (4.551)$$

Let us observe that $n \in \mathbb{N}$ implies $n^+ \in \mathbb{N}_+$ with (4.42) and therefore $n^+ \in \{1, \ldots, n^+\}$ with (4.247). Since $\{1, \ldots, n^+\}$ is the domain of $\bar{c}$, there is an element in the codomain/range A, say $\bar{y}$, such that $(n^+, \bar{y}) \in \bar{c}$ holds. Then, we may apply Proposition 3.205 to obtain from (4.551) the bijection

$$\bar{c} \setminus \{(n^+, \bar{y})\} : \{1, \ldots, n^+\} \setminus \{n^+\} \rightleftarrows A \setminus \{\bar{y}\}. \quad (4.552)$$

We now prove by cases that S is a finite set with cardinality $|S| = n^+$, based on the disjunction $n = 0 \vee n \in \mathbb{N}_+$ implied by $n \in \mathbb{N}$ with (2.310).

If $n = 0$ holds, then $n^+ = 1$ is true according to (2.291), then we have the bijection $\bar{c} : \{1, \ldots, 1\} \rightleftarrows A$, which we may write as $\bar{c} : \{1\} \rightleftarrows A$ by applying the definition of an initial segment. This shows with Exercise 3.81 that A is the singleton $\{\bar{c}(1)\}$. Then, $A = \{\bar{c}(1)\}$ implies together with the assumption $S \subset A$ that $S \subset \{\bar{c}(1)\}$. This further implies $S = \emptyset$ with (2.187), so that S is a finite set by definition, and we obtain $|S| = 0$ with (4.527). Since $n \in \mathbb{N}$ implies the inequalities $0 \leq n < n^+$ with (4.187) and (4.153), so that $0 < n^+$ follows to be true with the Transitivity Formula for $\leq$ and $<$, we obtain the desired $|S| < n^+$ after substitution.

If $n \in \mathbb{N}_+$ holds, then this gives the equation $\{1, \ldots, n\} = \{1, \ldots, n^+\} \setminus \{n^+\}$ with (4.256), so that we may write (4.552) as

$$\bar{c} \setminus \{(n^+, \bar{y})\} : \{1, \ldots, n\} \rightleftarrows A \setminus \{\bar{y}\}.$$

This shows that $A \setminus \{\bar{y}\}$ is a finite set with cardinality $|A \setminus \{\bar{y}\}| = n$, so that the assumed $|A| = n^+$ yields via substitution

$$|A| = n^+ = |A \setminus \{\bar{y}\}|^+.$$

We now consider the two subcases bases on the disjunction $\bar{y} \notin S \vee \bar{y} \in S$, which evidently holds according to the Law of the Excluded Middle.

Regarding the first subcase $\bar{y} \notin S$, we note that $S \subset A$ implies in particular $S \subseteq A$ by definition of a proper subset. Then, the conjunction of this and the case assumption $\bar{y} \notin S$ implies $S \subseteq A \setminus \{\bar{y}\}$ with Proposition 2.63, so that the disjunction of $S \subset A \setminus \{\bar{y}\}$ and $S = A \setminus \{\bar{y}\}$ is true due to (2.26). We may use this true disjunction to prove by cases that S is a finite set with cardinality $|S| < n$. On the one hand, if $S \subset A \setminus \{\bar{y}\}$ holds, this implies – together with the previously established fact that $A \setminus \{\bar{y}\}$ is finite with $|A \setminus \{\bar{y}\}| = n$ – due to the induction assumption (4.549) that S is a finite set with cardinality $|S| < n$. On the other hand, if $S = A \setminus \{\bar{y}\}$ holds, then the previously mentioned fact that $A \setminus \{\bar{y}\}$ is finite with $|A \setminus \{\bar{y}\}| = n$ implies via substitution that S is finite with $|S| = n$, completing the current proof by cases. Recalling that $n \in \mathbb{N}$ implies $n < n^+$ due to (4.153), we then obtain the desired inequality $|S| < n^+$ with the transitivity of the linear ordering $<$ of $\mathbb{N}$.

The second subcase $\bar{y} \in S$ implies together with the assumption $S \subset A$

$$S \setminus \{\bar{y}\} \subset A \setminus \{\bar{y}\},$$

according to Proposition 2.64. Together with the fact that $A \setminus \{\bar{y}\}$ is finite with $|A \setminus \{\bar{y}\}| = n$, this implies with the induction assumption (4.549) that $S \setminus \{\bar{y}\}$ is a finite set with $|S \setminus \{\bar{y}\}| < n$. Consequently, there exists by definition of a finite set a natural number, say $\bar{m}$, and a bijection from $\{1, \ldots, \bar{m}\}$ to $S \setminus \{\bar{y}\}$, say $\bar{d}$, so that $|S \setminus \{\bar{y}\}| = \bar{m}$. holds by definition of a cardinality. It then follows from $|S \setminus \{\bar{y}\}| < n$ that $\bar{m} < n$ holds, which further implies $\bar{m}^+ \leq n$ $(< n^+)$ with (4.157) and therefore $m^+ < n^+$ with the Transitivity Formula for $\leq$ and $<$. Since $S \setminus \{\bar{y}\}$ is a finite set with $|S \setminus \{\bar{y}\}| = \bar{m}$ and as $\bar{y} \notin S \setminus \{\bar{y}\}$ also holds according to (2.179), it furthermore follows with (4.507) that $(S \setminus \{\bar{y}\}) \cup \{\bar{y}\}$ is a finite set with cardinality $\bar{m}^+$. Because the assumed $\bar{y} \in S$ gives $\{\bar{y}\} \subseteq S$ with (2.184), we see in light of (2.263) that $(S \setminus \{\bar{y}\}) \cup \{\bar{y}\}$ is identical with the set S, so that S is finite with $|S| = \bar{m}^+$, which successor we already showed to be less than n^+. Thus, S is finite and $|S| < n^+$ is also true, completing the two nested proofs by cases.

Since A and S were arbitrary, we may therefore conclude that the universal sentence (4.550) is true. As n was also arbitrary, we may then further conclude that the induction step holds, so that the proof by mathematical induction of (4.547) is complete; thus, the sentence (4.546) is also true. □

Corollary 4.116. *Any proper subset S of a finite set A is itself a finite set with cardinality less than the cardinality of A, that is,*

$$\forall A, S\,([A \text{ is finite} \wedge S \subset A] \Rightarrow [S \text{ is finite} \wedge |S| < |A|]). \tag{4.553}$$

Proof. We let A and S be arbitrary sets, assume that A is finite, and assume moreover that S is a proper subset of A. Since A is finite, there exist a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}$ to A, say $\bar{c}$. The existence of this bijection shows in light of the definition of a cardinality that $\bar{n} = |A|$ holds. Thus, the multiple conjunction

$$\bar{n} \in \mathbb{N} \,\wedge\, A \text{ is finite} \,\wedge\, |A| = \bar{n} \,\wedge\, S \subset A$$

is true, which then implies with (4.546) that S is finite and that $|S| < \bar{n}$ holds. Applying now substitution to this inequality based on the previously established equation $\bar{n} = |A|$ yiels $|S| < |A|$. These findings prove the implication in (4.553), and since A and S were arbitrary, we may therefore conclude that the corollary is true. □

Proposition 4.117. *For any positive natural number n and any finite subset A of $\mathbb{N}$ with cardinality n it is true that A is bounded from above, that is,*

$$\begin{aligned}&\forall n, A\,([n \in \mathbb{N}_+ \,\wedge\, A \text{ is finite} \,\wedge\, A \subseteq \mathbb{N} \,\wedge\, |A| = n] \\ &\quad \Rightarrow A \text{ is bounded from above}).\end{aligned} \tag{4.554}$$

Proof. We first apply (1.90) to rewrite (4.554) equivalently as

$$\begin{aligned}&\forall n\,(n \in \mathbb{N}_+ \Rightarrow \forall A\,([A \text{ is finite} \,\wedge\, A \subseteq \mathbb{N} \,\wedge\, |A| = n] \\ &\quad \Rightarrow A \text{ is bounded from above})),\end{aligned} \tag{4.555}$$

and we then prove this universal sentence via mathematical induction. Regarding the case $(n = 1)$, we let A be an arbitrary set and assume A to be a finite subset of $\mathbb{N}$ with cardinality $|A| = 1$. These assumptions imply with (4.529) that there exists a constant, say $\bar{y}$, such that $A = \{\bar{y}\}$ holds. Consequently, $\bar{y}$ is an upper bound for the singleton $A = \{\bar{y}\}$ according to Proposition 3.91 – applied to the totally ordered set $(\mathbb{N}, \leq)$. Thus, an upper bound for A exists, so that the set A is bounded from above. Since A is arbitrary, we may therefore conclude that the base case holds.

Regarding the induction step, we let $n \in \mathbb{N}_+$ be arbitrary, make the induction assumption

$$\forall A\,([A \text{ is finite} \wedge A \subseteq \mathbb{N} \wedge |A| = n] \Rightarrow A \text{ is bounded from above}), \tag{4.556}$$

and show that this implies

$$\forall A\,([A \text{ is finite} \,\wedge\, A \subseteq \mathbb{N} \,\wedge\, |A| = n^+] \Rightarrow A \text{ is bounded from above}). \tag{4.557}$$

To prove this universal sentence, we take an arbitrary set A and assume A to be a finite subset of $\mathbb{N}$ with cardinality $|A| = n^+$. Evidently, $n^+ \neq 0$ holds with (4.35), so that $|A| \neq 0$ is true, which then further implies $A \neq \emptyset$ with (4.527). Since A is nonempty, there clearly exists an element in A, say $\bar{y}$. Thus, the conjunction

$$n \in \mathbb{N}_+ \wedge A \text{ is finite } \wedge |A| = n^+ \wedge \bar{y} \in A$$

holds, which implies with (4.530) that $A \setminus \{\bar{y}\}$ is a finite set with $|A \setminus \{\bar{y}\}| = n$. Let us also observe that $A \setminus \{\bar{y}\} \subseteq A$ holds due to (2.125), which then implies together with the assumption $A \subseteq \mathbb{N}$ and the transitivity of $\subseteq$ that $A \setminus \{\bar{y}\} \subseteq \mathbb{N}$. These findings then imply with the induction assumption (4.556) that $A \setminus \{\bar{y}\}$ is bounded from above, so that there exists an element of $\mathbb{N}$, say $\bar{u}$, such that

$$\forall y \, (y \in A \setminus \{\bar{y}\} \Rightarrow y \leq \bar{u}) \tag{4.558}$$

holds. Let us notice that the fact $\bar{y} \in A$ implies $\{\bar{y}\} \subseteq A$ with (2.184), so that we obtain

$$(A \setminus \{\bar{y}\}) \cup \{\bar{y}\} = A \tag{4.559}$$

with (2.263). We now prove by cases that A is bounded from above, based on the true disjunction $\bar{y} \leq \bar{u} \vee \bar{y} \leq \bar{u} \leq \bar{y}$ (applying the totality of the standard total ordering $\leq$ of $\mathbb{N}$).

In the first case, we assume $\bar{y} \leq \bar{u}$ and demonstrate the truth of

$$\forall y \, (y \in A \Rightarrow y \leq \bar{u}). \tag{4.560}$$

Letting y be arbitrary in A, it follows with (4.559) and the definition of the union of a pair that the disjunction $y \in A \setminus \{\bar{y}\} \vee y \in \{\bar{y}\}$ holds, which we now use to prove $y \leq \bar{u}$ by cases. On the one hand, if $y \in A \setminus \{\bar{y}\}$ is true, then it follows with (4.558) that $y \leq \bar{u}$ holds, as desired. On the other hand, if $y \in \{\bar{y}\}$ is true, then (2.169) gives $y = \bar{y}$, so that the desired $y \leq \bar{u}$ follows from the current case assumption $\bar{y} \leq \bar{u}$. We thus completed the proof of $y \leq \bar{u}$ by cases, and since y was arbitrary, we may therefore conclude that the universal sentence (4.560) holds. Thus, $\bar{u}$ is an upper bound for A in the first case.

In the second case, we assume $\bar{u} \leq \bar{y}$ and verify now

$$\forall y \, (y \in A \Rightarrow y \leq \bar{y}), \tag{4.561}$$

assuming that y is an arbitrary element of A. As in the first case, this means that $y \in A \setminus \{\bar{y}\}$ or $y \in \{\bar{y}\}$ is true. On the one hand, if $y \in A \setminus \{\bar{y}\}$ holds, then (4.558) gives $y \leq \bar{u}$. The conjunction of this and the case assumption

$\bar{u} \leq \bar{y}$ then implies $y \leq \bar{y}$ with the transitivity of the standard total ordering $\leq$ of $\mathbb{N}$, which was to be shown. If, on the other hand, $y \in \{\bar{y}\}$ holds, which yields $y = \bar{y}$ as before, then the disjunction $y < \bar{y} \vee y = \bar{y}$ is also true (irrespective of the truth value of $y < \bar{y}$), which evidently means that the desired inequality $y \leq \bar{y}$ holds again. As y is arbitrary, we may therefore conclude that (4.561) is true, which shows that $\bar{y}$ is now an upper bound for A in the second case.

As there exists an upper bound for A in any case, it follows that A is bounded from above, so that the proof of the implication in (4.557) is complete. Since A is arbitrary, we may then further conclude that (4.557) holds. As n was also arbitrary, we may infer from this the truth of the induction step, which completes the proof of (4.555) via mathematical induction and thus the proof of the proposition. □

Corollary 4.118. *Any finite subset of* $\mathbb{N}$ *is bounded from above, that is,*

$$\forall A\,([A \text{ is finite } \wedge\ A \subseteq \mathbb{N}] \Rightarrow A \text{ is bounded from above}). \tag{4.562}$$

Proof. Letting A be an arbitrary set such that A is finite and a subset of $\mathbb{N}$, we have that there exists a particular bijection from $\{1, \ldots, \bar{n}\}$ to A, where $\bar{n}$ is a particular natural number and moreover the cardinality of A. Let us now observe the truth of $\bar{n} = 0 \vee \bar{n} \in \mathbb{N}_+$ in light of (2.310) and let us now use this disjunction to prove by cases that A is bounded from above.

In case of $\bar{n} = 0$, we obtain $|A| = 0$, which implies $A = \emptyset$ with (4.527), so that any natural number (say, 1) is an upper bound for $A = \emptyset$ according to Proposition 3.90 – applied to the totally ordered set $(\mathbb{N}, \leq)$. Thus, an upper bound for A exists, so that the set A is bounded from above if its cardinality is $\bar{n} = 0$.

In the other case of $\bar{n} \in \mathbb{N}_+$, we may apply (4.554) to obtain the desired result that A is bounded from above.

Since A was initially arbitrary, we may therefore conclude that the corollary holds. □

Corollary 4.119. *For any finite, nonempty subset A of* $\mathbb{N}$ *it is true that the maximum of A exists.*

Proof. We let A be an set and assume that A is a finite and nonempty subset of $\mathbb{N}$. It then follows with the preceding Corollary 4.118 that A is bounded from above. Since A is a nonempty and bounded-from-above subset of $\mathbb{N}$, it follows with Corollary 4.41b) that the greatest element of A exists. Since A is arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Lemma 4.120. *For any positive natural number n, any finite set A and any finite set B with cardinality n, it is true that the Cartesian product of A and B is finite, that is,*

$$\forall n, A, B\,([n \in \mathbb{N}_+ \,\wedge\, A \text{ is finite} \,\wedge\, B \text{ is finite} \,\wedge\, |B| = n] \Rightarrow A \times B \text{ is finite}). \tag{4.563}$$

Proof. We first apply (1.90) to write (4.563) equivalently as

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \forall A, B\,([A \text{ is finite} \wedge B \text{ is finite} \wedge |B| = n] \Rightarrow A \times B \text{ is finite})) \tag{4.564}$$

and apply then a proof by mathematical induction.

Regarding the base case ($n = 1$), we let A and B be arbitrary finite sets with $|B| = 1$, and show that this implies the finiteness of $A \times B$. Here, the finiteness of B implies together with $|B| = 1$ that there exists an element, say $\bar{y}$, with $B = \{\bar{y}\}$ (see Proposition 4.111b)), so that substitution yields $A \times B = A \times \{\bar{y}\}$. In view of Proposition 4.109, the finiteness of A implies then that the preceding Cartesian product is itself finite, which proves the base case.

Regarding the induction step, we let n be arbitrary in $\mathbb{N}_+$, make the induction assumption

$$\forall A, B\,([A \text{ is finite} \,\wedge\, B \text{ is finite} \,\wedge\, |B| = n] \Rightarrow A \times B \text{ is finite}), \tag{4.565}$$

and show that this implies

$$\forall A, B\,([A \text{ is finite} \,\wedge\, B \text{ is finite} \,\wedge\, |B| = n^+] \Rightarrow A \times B \text{ is finite}). \tag{4.566}$$

To prove this universal sentence, we let A and B be arbitrary finite sets such that $|B| = n^+$ holds. The initial assumption $n \in \mathbb{N}_+$ clearly implies $n \in \mathbb{N}$ and then $n^+ \neq 0$ with Property 1 of a counting domain – applied to $(\mathbb{N}, s^+, 0)$. Then, the assumed $|B| = n^+$ gives with the preceding inequality after substitution $|B| \neq 0$, so that $B \neq \emptyset$ holds due to (4.527) in connection with the Law of Contraposition. Consequently, there exists an element in B, say $\bar{y}$, according to Proposition 2.11. The previous findings show that the multiple conjunction

$$n \in \mathbb{N}_+ \,\wedge\, B \text{ is finite} \,\wedge\, |B| = n^+ \,\wedge\, \bar{y} \in B$$

holds, which implies with (4.530) that $B \setminus \{\bar{y}\}$ is finite and that $|B \setminus \{\bar{y}\}| = n$. Together with the assumption that A is finite, this equation implies with the induction assumption (4.565) that the Cartesian product $A \times (B \setminus \{\bar{y}\})$ is finite. Let us now observe that $\bar{y} \in B$ implies $\{\bar{y}\} \subseteq B$ with (2.184), so that (2.263) gives the equation $(B \setminus \{\bar{y}\}) \cup \{\bar{y}\} = B$. With this and (3.62), we then obtain

$$A \times B = A \times [(B \setminus \{\bar{y}\}) \cup \{\bar{y}\}] = [A \times (B \setminus \{\bar{y}\})] \cup [A \times \{\bar{y}\}]. \quad (4.567)$$

It follows with Proposition 4.109 that $A \times \{\bar{y}\}$ is a finite set. Together with the previous finding that $A \times (B \setminus \{\bar{y}\})$ is finite, this implies with Theorem 4.114 that the union (4.567), i.e. $A \times B$, is itself finite, which proves the implication in (4.566). Since A and B are arbitrary, we therefore conclude that the universal sentence (4.566) is true, and as n is also arbitrary, we then further conclude that the induction step holds. Therefore, (4.564) is true, so that the proof of the lemma is complete. □

Theorem 4.121 (Finiteness of the Cartesian product of two finite sets). *The Cartesian product of any two finite sets is itself finite, that is,*

$$\forall A, B\,([A \text{ is finite } \wedge \ B \text{ is finite}] \Rightarrow A \times B \text{ is finite}). \quad (4.568)$$

Exercise 4.36. Prove Theorem 4.121.

(Hint: Use (2.310), (3.27) and (4.563).)

In the following, we establish the equivalence of two-fold Cartesian products $(A \times B) \times C$ and $A \times (B \times C)$ as well as of 'reversed' Cartesian products $A \times B$ and $B \times A$.

Proposition 4.122. *The following sentences are true for any finite sets A, B and C.*

a) There exists a unique set $\bar{f}$ such that an element z is in $\bar{f}$ iff z is in the Cartesian product of $(A \times B) \times C$ and $A \times (B \times C)$ and moreover if z is for some constants a, b and c the ordered pair formed by the ordered triples $((a, b), c)$ and $(a, (b, c))$, that is,

$$\exists! \bar{f}\, \forall z\,(z \in \bar{f} \Leftrightarrow [z \in [(A \times B) \times C] \times [A \times (B \times C)] \\ \wedge \exists a, b, c\,(z = (((a, b), c), (a, (b, c))))]), \quad (4.569)$$

and this set is a function from $(A \times B) \times C$ to $A \times (B \times C)$, that is,

$$\bar{f} : (A \times B) \times C \to A \times (B \times C). \quad (4.570)$$

b) The function $\bar{f}$ is a surjection from $A \times B$ to $B \times A$, that is,

$$\bar{f} : (A \times B) \times C \twoheadrightarrow A \times (B \times C). \quad (4.571)$$

c) *The surjection $\bar{f}$ is an injection from $A \times B$ to $B \times A$, that is,*

$$\bar{f} : (A \times B) \times C \hookrightarrow A \times (B \times C). \tag{4.572}$$

d) *The Cartesian products $(A\times B)\times C$ and $A\times(B\times C)$ are equinumerous, that is,*

$$|(A \times B) \times C| = |A \times (B \times C)|. \tag{4.573}$$

Proof. Letting A, B and C be arbitrary sets and assuming these sets to be finite, we establish in a) – d) a bijection $\bar{f}$ from $(A\times B)\times C$ to $A\times(B\times C)$. We will then also utilize that fact that the finiteness of A, B and C implies first the finiteness of the Cartesian products $A \times B$ and $B \times C$, and then also the finiteness of the Cartesian products $(A \times B) \times C$ and $A \times (B \times C)$ in view of Theorem 4.121.

Concerning a), we apply the Axiom of Specification and the Equality Criterion for sets (in analogy to the proof of Theorem 2.15) to obtain the true uniquely existential sentence (4.569). Thus, the unique set $\bar{f}$ satisfies

$$\begin{aligned} \forall z\,(z \in \bar{f} \Leftrightarrow [z \in [(A \times B) \times C] \times [A \times (B \times C)] \\ \wedge\ \exists a, b, c\,(z = (((a, b), c), (a, (b, c))))]). \end{aligned} \tag{4.574}$$

Thus, $z \in \bar{f}$ implies in particular $z \in [(A \times B) \times C] \times [A \times (B \times C)]$ for any z, so that $\bar{f}$ may be viewed as a binary relation included in $[(A\times B)\times C]\times[A\times(B\times C)]$ (applying the definitions of a binary relation and of a subset). We now prove that the binary relation $\bar{f}$ is a function from $(A \times B) \times C$ to $A \times (B \times C)$ by applying the Function Criterion, i.e. by verifying

$$\forall x\,(x \in (A \times B) \times C \Rightarrow \exists! y\,(y \in A \times (B \times C) \wedge (x, y) \in \bar{f})). \tag{4.575}$$

We let $\bar{x}$ be arbitrary and assume $\bar{x} \in (A \times B) \times C$ to be true, so that there exist by definition of the Cartesian product of two sets constants, say $\bar{u}$ and $\bar{c}$, such that $\bar{u} \in A \times B$, $\bar{c} \in C$ and $(\bar{u}, \bar{c}) = \bar{x}$ hold. Then, $\bar{u} \in A\times B$ implies (again by definition of the Cartesian product of two sets) that there are elements, say $\bar{a}$ and $\bar{b}$, with $\bar{a} \in A$, $\bar{b} \in B$ and $(\bar{a}, \bar{b}) = \bar{u}$; thus, $\bar{x} = ((\bar{a}, \bar{b}), \bar{c})$. We now establish the existential part of the uniquely existential sentence in (4.575). To begin with, $\bar{b} \in B$ and $\bar{c} \in C$ evidently imply that that the ordered pair $\bar{v} = (\bar{b}, \bar{c})$ is in the Cartesian product of B and C, that is, $\bar{v} \in B \times C$. Furthermore, the truth of $\bar{a} \in A$ and $\bar{v} \in B \times C$ clearly shows that the ordered pair $\bar{y} = (\bar{a}, \bar{v})$ is in the Cartesian product of A and $B \times C$, which means that $\bar{y} \in A \times (B \times C)$ holds. Then, since $\bar{x} \in (A \times B) \times C$ and $\bar{y} \in A \times (B \times C)$ are both true, it follows that the ordered pair

$$\bar{z} = (\bar{x}, \bar{y}) = (((\bar{a}, \bar{b}), \bar{c}), (\bar{a}, (\bar{b}, \bar{c})) \tag{4.576}$$

is in the Cartesian product of $(A \times B) \times C$ and $A \times (B \times C)$, that is,

$$\bar{z} \in [(A \times B) \times C] \times [A \times (B \times C)]. \tag{4.577}$$

The equations (4.576) show that there exist constants a, b and c such that $\bar{z} = (((a,b),c),(a,(b,c)))$ holds, which existential sentence implies now – together with (4.577) – in view of (4.574) that $[(\bar{x},\bar{y}) =]\ \bar{z} \in \bar{f}$ is true, so that $(\bar{x},\bar{y}) \in \bar{f}$ holds. Together with the previously obtained $\bar{y} \in A \times (B \times C)$, this finding proves the existential sentence

$$\exists y\,(y \in A \times (B \times C)\ \wedge\ (\bar{x},y) \in \bar{f})$$

and thus the existential part of the uniquely existential sentence in (4.575).

Regarding the uniqueness part, we prove

$$\forall y,y'\,([y \in A \times (B \times C)\ \wedge\ (\bar{x},y) \in \bar{f}\ \wedge\ y' \in A \times (B \times C)\ \wedge\ (\bar{x},y') \in \bar{f}] \\ \Rightarrow y = y'). \tag{4.578}$$

We take two arbitrary $\bar{y}$ and $\bar{y}'$ from the set $A \times (B \times C)$ such that $(\bar{x},\bar{y})$ and $(\bar{x},\bar{y}')$ are in $\bar{f}$. On the one hand, $(\bar{x},\bar{y}) \in \bar{f}$ implies with (4.574) in particular that there are constants, say $\bar{a}$, $\bar{b}$ and $\bar{c}$, with $(\bar{x},\bar{y}) = (((\bar{a},\bar{b}),\bar{c}),(\bar{a},(\bar{b},\bar{c})))$, which equation then yields $\bar{x} = ((\bar{a},\bar{b}),\bar{c})$ and $\bar{y} = (\bar{a},(\bar{b},\bar{c}))$ with the Equality Criterion for ordered pairs. On the other hand, $(\bar{x},\bar{y}') \in \bar{f}$ implies with (4.574) in particular that there exist constants, say $\bar{a}'$, $\bar{b}'$ and $\bar{c}'$, such that $(\bar{x},\bar{y}') = (((\bar{a}',\bar{b}'),\bar{c}'),(\bar{a}',(\bar{b}',\bar{c}')))$, with the consequence that

$$[((\bar{a},\bar{b}),\bar{c}) =]\quad \bar{x} = ((\bar{a}',\bar{b}'),\bar{c}') \tag{4.579}$$

and

$$\bar{y}' = (\bar{a}',(\bar{b}',\bar{c}')) \tag{4.580}$$

hold because of the Equality Criterion for ordered pairs. It follows from (4.579) that $((\bar{a},\bar{b}),\bar{c}) = ((\bar{a}',\bar{b}'),\bar{c}')$ is true, so that further applications of the Equality Criterion for ordered pairs give first $(\bar{a},\bar{b}) = (\bar{a}',\bar{b}')$ and $\bar{c} = \bar{c}'$, and subsequently $\bar{a} = \bar{a}'$ as well as $\bar{b} = \bar{b}'$. Therefore, (4.580) becomes after performing substitutions

$$\bar{y}' = (\bar{a},(\bar{b},\bar{c})) = \bar{y},$$

so that the desired $\bar{y} = \bar{y}'$ follows to be true. As $\bar{y}$ and $\bar{y}'$ are arbitrary, the universal sentence (4.578) follows to be true, completing the proof of the uniquely existential sentence in (4.575). Since $\bar{x}$ is also arbitrary, we may therefore conclude that (4.575) holds, so that $\bar{f}$ is a function from $(A \times B) \times C$ to $A \times (B \times C)$ because of the Function Criterion.

Concerning b), we now apply the Surjection Criterion to prove that $\bar{f}$: $(A \times B) \times C \to A \times (B \times C)$ is a surjection. For this purpose, we verify

$$\forall y \, (y \in A \times (B \times C) \Rightarrow \exists x \, (\bar{f}(x) = y)). \tag{4.581}$$

To do this, we let $\bar{y}$ be arbitrary and assume $\bar{y} \in A \times (B \times C)$ to hold, so that there exist (by definition of the Cartesian product of two sets) constants, say $\bar{a}$ and $\bar{v}$, with $\bar{a} \in A$, $\bar{v} \in B \times C$ and $(\bar{a}, \bar{v}) = \bar{y}$. Then, $\bar{v} \in B \times C$ implies (again by definition of the Cartesian product of two sets) that there are elements, say $\bar{b}$ and $\bar{c}$, satisfying $\bar{b} \in B$, $\bar{c} \in C$ and $(\bar{b}, \bar{c}) = \bar{v}$. We therefore obtain the identity $\bar{y} = (\bar{a}, (\bar{b}, \bar{c}))$. The ordered pair $\bar{u} = (\bar{a}, \bar{b})$ is then also specified, so that the conjunction of $\bar{a} \in A$ and $\bar{b} \in B$ evidently gives $\bar{u} \in A \times B$. With this, the ordered pair

$$\bar{x} = (\bar{u}, \bar{c}) = ((\bar{a}, \bar{b}), \bar{c})$$

exists also, and therefore the conjunction of $\bar{u} \in A \times B$ and $\bar{c} \in C$ yields $\bar{x} \in (A \times B) \times C$. This in turn implies with (4.575) the unique existence of the element $y \in A \times (B \times C)$ with $(\bar{x}, y) \in \bar{\bar{c}}$. It then follows with (4.574) in particular that there exist elements, say $\bar{\bar{a}}$, $\bar{\bar{b}}$ and $\bar{\bar{c}}$, such that $(\bar{x}, y) = (((\bar{\bar{a}}, \bar{\bar{b}}), \bar{\bar{c}}), (\bar{\bar{a}}, (\bar{\bar{b}}, \bar{\bar{c}})))$. Consequently, we obtain with the Equality Criterion for ordered pairs

$$[((\bar{a}, \bar{b}), \bar{c}) =] \quad \bar{x} = ((\bar{\bar{a}}, \bar{\bar{b}}), \bar{\bar{c}}) \tag{4.582}$$

and

$$y = (\bar{\bar{a}}, (\bar{\bar{b}}, \bar{\bar{c}})). \tag{4.583}$$

The equations in (4.582) yield $((\bar{a}, \bar{b}), \bar{c}) = ((\bar{\bar{a}}, \bar{\bar{b}}), \bar{\bar{c}})$ and therefore evidently first $(\bar{a}, \bar{b}) = (\bar{\bar{a}}, \bar{\bar{b}})$ as well as $\bar{c} = \bar{\bar{c}}$, and then $\bar{a} = \bar{\bar{a}}$ as well as $\bar{b} = \bar{\bar{b}}$. With these two equations, (4.583) can be written equivalently as

$$y = (\bar{a}, (\bar{b}, \bar{c})) \quad [= \bar{y}].$$

Thus, $y = \bar{y}$ holds, and therefore the previously found $(\bar{x}, y) \in \bar{c}$ yields $(\bar{x}, \bar{y}) \in \bar{f}$, which we may write also as $\bar{f}(\bar{x}) = \bar{y}$. This equation shows that the existential sentence in (4.581) is true, and since $\bar{y}$ was arbitrary, the universal sentence (4.581) follows to be true. This shows in light of the Surjection Criterion that the function $\bar{f} : (A \times B) \times C \to A \times (B \times C)$ is indeed a surjection.

Concerning c), the next task is to prove that $\bar{f} : (A \times B) \times C \to A \times (B \times C)$ is an injection. To do this, we demonstrate the truth of

$$\forall x, x' \, ([x, x' \in (A \times B) \times C \, \wedge \, \bar{f}(x) = \bar{f}(x')] \Rightarrow x = x'). \tag{4.584}$$

We let x and x' be arbitrary elements of $(A\times B)\times C$ such that $[y=]\ \bar{f}(x) = \bar{f}(x')$, which equations we may then also write as $(x,y)\in\bar{f}$ and $(x',y)\in\bar{f}$. On the one hand, $(x,y)\in\bar{f}$ implies with (4.574) in particular that there are elements, say $\bar{a}$, $\bar{b}$ and $\bar{c}$, such that $(x,y) = (((\bar{a},\bar{b}),\bar{c}),(\bar{a},(\bar{b},\bar{c})))$ is true, which equation then gives $x = ((\bar{a},\bar{b}),\bar{c})$ and $y = (\bar{a},(\bar{b},\bar{c}))$ with the Equality Criterion for ordered pairs. On the other hand, $(x',y)\in\bar{f}$ implies again with (4.574) that there exist elements, say $\bar{a}'$, $\bar{b}'$ and $\bar{c}'$, with $(x',y) = (((\bar{a}',\bar{b}'),\bar{c}'),(\bar{a}',(\bar{b}',\bar{c}')))$, and this equation yields

$$x' = ((\bar{a}',\bar{b}'),\bar{c}') \tag{4.585}$$

and

$$[(\bar{a},(\bar{b},\bar{c}))=]\quad y = (\bar{a}',(\bar{b}',\bar{c}')),$$

again (3.3) with the Equality Criterion for ordered pairs. The latter gives $(\bar{a},(\bar{b},\bar{c})) = (\bar{a}',(\bar{b}',\bar{c}'))$ and therefore first $\bar{a}=\bar{a}'$ as well as $(\bar{b},\bar{c}) = (\bar{b}',\bar{c}')$, and then $\bar{b}=\bar{b}'$ as well as $\bar{c}=\bar{c}'$. Carrying out substitutions in (4.585) based on the preceding equations, we obtain

$$x' = ((\bar{a},\bar{b}),\bar{c})\quad [=x]$$

and therefore the desired equality $x = x'$. As x and x' were arbitrary, we may therefore conclude that (4.584) holds, so that the surjection $\bar{f} : (A\times B)\times C \twoheadrightarrow A\times(B\times C)$ is also an injection from $(A\times B)\times C$ to $A\times(B\times C)$.

Concerning d), we observe that $\bar{f}$ is an injection as well as a surjection, and thus a bijection from $(A\times B)\times C$ to $A\times(B\times C)$. We thus established the existence of a bijection from the finite set $(A\times B)\times C$ to the finite set $A\times(B\times C)$, so that $|(A\times B)\times C| = |A\times(B\times C)|$ follows to be true with Proposition 4.107.

Then, since A, B and C were arbitrary, we may finally conclude that the proposition is true. □

Exercise 4.37. Show that the following sentences are true for any finite sets A and B.

a) There exists a unique set $\bar{g}$ such that an element z is in $\bar{g}$ iff z is in the Cartesian product of $A\times B$ and $B\times A$ and moreover if z is for some constants a and b the ordered pair formed by the ordered pairs (a,b) and (b,a), that is,

$$\exists!\bar{g}\,\forall z\,(z\in\bar{g}\Leftrightarrow[z\in(A\times B)\times(B\times A)\wedge\exists a,b\,(z=((a,b),(b,a)))]), \tag{4.586}$$

and this set is a function from $A\times B$ to $B\times A$, i.e.

$$\bar{g} : A\times B\to B\times A. \tag{4.587}$$

b) The function $\bar{g}$ is a surjection from $A \times B$ to $B \times A$, i.e.

$$\bar{g} : A \times B \twoheadrightarrow B \times A. \tag{4.588}$$

c) The surjection $\bar{g}$ is an injection from $A \times B$ to $B \times A$, i.e.

$$\bar{g} : A \times B \hookrightarrow B \times A. \tag{4.589}$$

d) The Cartesian product of A and B and the Cartesian product of B and A are equinumerous, that is,

$$|A \times B| = |B \times A|. \tag{4.590}$$

(Hint: Proceed in analogy to the proof of Proposition 4.122.)

Lemma 4.123. *For any set A it is true that,*

a) if A is finite, then there exist a natural number n and a surjection from the initial segment of $\mathbb{N}_+$ up to n to A, i.e.

$$A \text{ is finite } \Rightarrow \exists n, f\,(n \in \mathbb{N} \wedge f : \{1, \ldots, n\} \twoheadrightarrow A). \tag{4.591}$$

b) if there exist a natural number n and a surjection from the initial segment of $\mathbb{N}_+$ up to n to A, then there exist a natural number n and an injection from A to the initial segment of $\mathbb{N}_+$ up to n, i.e.

$$\begin{aligned} &\exists n, f\,(n \in \mathbb{N} \wedge f : \{1, \ldots, n\} \twoheadrightarrow A) \\ &\quad \Rightarrow \exists n, f\,(n \in \mathbb{N} \wedge f : A \hookrightarrow \{1, \ldots, n\}). \end{aligned} \tag{4.592}$$

Proof. We let A be an arbitrary set. Concerning a), we assume that A is finite. By definition of a finite set, there then exist a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}\}$ to A, say $\bar{c}$. Thus, $\bar{c}$ is in particular a surjection from $\{1, \ldots, \bar{n}\}$ to A (by definition of a bijection), so that the existential sentence in (4.591) is true, completing the proof of the implication a).

Regarding b), we assume that there exist a natural number, say $\bar{n}$, and a surjection from $\{1, \ldots, \bar{n}\}$ to A, say $\bar{f}$. Thus, $\mathrm{ran}(\bar{f}) = A$ holds by definition of a surjection. We now prove the desired consequent by cases, based on the fact that the disjunction of $\bar{n} = 0$ and $\bar{n} \in \mathbb{N}_+$ holds according to (2.310).

The first case $\bar{n} = 0$ implies $\mathrm{dom}(\bar{f}) = \{1, \ldots, \bar{n}\} = \emptyset$ according to the notation for the empty initial segment of $\mathbb{N}_+$, and therefore $A = \mathrm{ran}(\bar{f}) = \emptyset$

with (3.119), so that $\bar{f}$ is an injection from $\emptyset$ to $\emptyset$ because of Exercise 3.76. We may write this injection via substitution also as $\bar{f} : A \hookrightarrow \{1, \ldots, \bar{n}\}$. Since $\bar{n}$ is a natural number we have that the consequent of the implication (4.601) holds (in case of $\bar{n} = 0$).

The second case $\bar{n} \in \mathbb{N}_+$ shows that the domain $\{1, \ldots, \bar{n}\}$ of $\bar{f}$ is now nonempty (by definition of an initial segment of $\mathbb{N}_+$), which implies with (3.119) that the range A of $\bar{f}$ is also nonempty. We now define the function

$$g : A \to \{1, \ldots, \bar{n}\}, \qquad a \mapsto g(a) = \min \bar{f}^{-1}[\{a\}]. \tag{4.593}$$

replacement. To do this, we verify

$$\forall a\, (a \in A \Rightarrow \exists! m\, (m = \min \bar{f}^{-1}[\{a\}])), \tag{4.594}$$

letting $a \in A$ be arbitrary. We first notice that, since $\bar{f}$ is a surjection, the inverse image $\bar{f}^{-1}[\{a\}]$ is nonempty according to Proposition 3.226; furthermore, as $\bar{f}^{-1}[\{a\}]$ is (by definition of an inverse image) a subset of the domain $\{1, \ldots, \bar{n}\}$ of $\bar{f}$, we see in light of (4.284) that the least element of the inverse image exists. Consequently, the uniquely existential sentence in (4.593) follows to be true with Proposition 4.594, and since a was arbitrary, we may therefore conclude that (4.594) holds. This universal sentence implies then that there exists a unique function g with domain A such that $g(a) = \min \bar{f}^{-1}[\{a\}]$ holds for any $a \in A$. As every inverse image $\bar{f}^{-1}[\{a\}]$ is a subset of $\{1, \ldots, \bar{n}\}$, it follows by definition of a least element for any $a \in A$ that the unique function value $g(a)$ is an element of $\{1, \ldots, \bar{n}\}$. Thus, g is according to the Function Criterion a function from A to $\{1, \ldots, \bar{n}\}$, which we may write as (4.594).

We now apply the Injection Criterion and verify that g is an injection, i.e. that g satisfies

$$\forall a, a'\, ([a, a' \in A \wedge a \neq a'] \Rightarrow g(a) \neq g(a')). \tag{4.595}$$

To do this, we let a and a' be arbitrary distinct elements of A and show that this implies $g(a) \neq g(a')$. Here, $a \neq a'$ implies $\{a\} \cap \{a'\} = \emptyset$ with (2.174). We therefore obtain the equations

$$\begin{aligned} \emptyset &= \bar{f}^{-1}[\emptyset] \\ &= \bar{f}^{-1}[\{a\} \cap \{a'\}] \\ &= \bar{f}^{-1}[\{a\}] \cap \bar{f}^{-1}[\{a'\}], \end{aligned}$$

by applying Exercise 3.90a), then substitution, and finally Exercise 3.106. Consequently,

$$\forall y\, (\neg y \in \bar{f}^{-1}[\{a\}] \cap \bar{f}^{-1}[\{a'\}]) \tag{4.596}$$

holds by definition of the empty set. We now demonstrate that this finding implies the truth of the universal sentence

$$\forall y\,(\neg y \in \bar{f}^{-1}[\{a\}] \,\vee\, \neg y \in \bar{f}^{-1}[\{a'\}]). \tag{4.597}$$

Taking an arbitrary y, the negation in (4.596) implies the disjunction in (4.597) with the definition of the intersection of two sets and De Morgan's Law for sentences (1.51); since y is arbitrary, we may therefore infer from this indeed the truth of (4.597). Now, by definition of a least element, we have the true sentence

$$\min \bar{f}^{-1}[\{a\}] \in \bar{f}^{-1}[\{a\}], \tag{4.598}$$

which in turn implies with (4.597) the truth of the negation

$$\neg \min \bar{f}^{-1}[\{a'\}] \in \bar{f}^{-1}[\{a\}]. \tag{4.599}$$

The conjunction of (4.598) and (4.599) gives the inequality

$$\min \bar{f}^{-1}[\{a\}] \neq \min \bar{f}^{-1}[\{a'\}]$$

with (2.4), which yields the desired $g(a) \neq g(a')$ with the definition of the function g in (4.593). As a and a' were arbitrary, we may therefore conclude that (4.595) is true, so that the function $g : A \to \{1, \ldots, \bar{n}\}$ is an injection. Since $\bar{n}$ is a natural number, the existential sentence on the right-hand side of the implication (4.592) holds therefore also in the second case, which completes the proof of b) by cases. □

Theorem 4.124 (Finiteness Criteria). *For any set A it is true that*

a) *A is finite iff there exists a natural number n and a surjection from the initial segment of $\mathbb{N}_+$ up to n to A, i.e.*

$$A \text{ is finite} \;\Leftrightarrow\; \exists n, f\,(n \in \mathbb{N} \,\wedge\, f : \{1, \ldots, n\} \twoheadrightarrow A). \tag{4.600}$$

b) *A is finite iff there exists a natural number n and an injection from A to the initial segment of $\mathbb{N}_+$ up to n, i.e.*

$$A \text{ is finite} \;\Leftrightarrow\; \exists n, f\,(n \in \mathbb{N} \,\wedge\, f : A \hookrightarrow \{1, \ldots, n\}). \tag{4.601}$$

Proof. We let A be an arbitrary set and observe that the first part ('$\Rightarrow$') of the equivalence a) holds in light of (4.591). Furthermore, because the conjunction of (4.591) and (4.592) is true, it then follows with the Law of the Hypothetical Syllogism that the first part ('$\Rightarrow$') of the equivalence (4.601) holds as well.

We now prove the second part ('$\Leftarrow$') of the equivalence (4.601), assuming that there exist a natural number, say $\bar{n}$, and an injection from A to $\{1,\ldots,\bar{n}\}$, say $\bar{f}: A \hookrightarrow \{1,\ldots,\bar{n}\}$. Here, the initial segment $\{1,\ldots,\bar{n}\}$ is a finite set due to Exercise 4.34. Moreover, it follows with Corollary 3.204 that $\bar{f}: A \to \mathrm{ran}(\bar{f})$ is a bijection, where $\mathrm{ran}(\bar{f}) \subseteq \{1,\ldots,\bar{n}\}$ holds by definition of a range. Consequently, the disjunction

$$\mathrm{ran}(\bar{f}) \subset \{1,\ldots,\bar{n}\} \vee \mathrm{ran}(\bar{f}) = \{1,\ldots,\bar{n}\} \tag{4.602}$$

holds with (2.26), which we now use prove by cases that A is a finite set.

The first case $\mathrm{ran}(\bar{f}) \subset \{1,\ldots,\bar{n}\}$ implies together with the finiteness of $\{1,\ldots,\bar{n}\}$ that $\mathrm{ran}(\bar{f})$ is a finite set, according to (4.553). This finding in turn implies (by definition of a finite set) that there exist a natural number, say $\bar{N}$, and a bijection from $\{1,\ldots,\bar{N}\}$ to $\mathrm{ran}(\bar{f})$, say

$$\bar{d}: \{1,\ldots,\bar{N}\} \rightleftarrows \mathrm{ran}(\bar{f}).$$

Then, we also have $\bar{N} = |\mathrm{ran}(\bar{f})|$ (by definition of a cardinality). The fact that $\bar{f}: A \to \mathrm{ran}(\bar{f})$ is a bijection implies with Theorem 3.212 that its inverse is the bijection

$$\bar{f}^{-1}: \mathrm{ran}(\bar{f}) \rightleftarrows A.$$

It then follows with Theorem 3.207 that the composition $\bar{f}^{-1} \circ \bar{d}$ is the bijection

$$(\bar{f}^{-1} \circ \bar{d}): \{1,\ldots,\bar{N}\} \rightleftarrows A.$$

Thus, there exist a natural number n and a bijection from $\{1,\ldots,n\}$ to A, so that A is a finite set (in the first case).

The second case $\mathrm{ran}(\bar{f}) = \{1,\ldots,\bar{n}\}$ allows us to apply substitution to the previously established bijection $\bar{f}$ from A to $\mathrm{ran}(\bar{f})$, which we may thus write as $\bar{f}: A \rightleftarrows \{1,\ldots,\bar{n}\}$. Then, we obtain with Theorem 3.212 the bijection

$$\bar{f}^{-1}: \{1,\ldots,\bar{n}\} \rightleftarrows A.$$

Thus, there exist also in this case a natural number n and a bijection from $\{1,\ldots,n\}$ to A, so that A is finite in any case.

This completes the proof of the second part ('$\Leftarrow$') of the equivalence (4.601), so that b) is true. It now remains for us to establish the second part ('$\Leftarrow$') of the equivalence (4.600). For this purpose, we observe that the conjunction of (4.592) and the second part ('$\Leftarrow$') of the equivalence (4.601) is true. We may then apply the Law of the Hypothetical Syllogism again to infer from this true conjunction indeed the truth of the desired second part ('$\Leftarrow$') of the equivalence (4.600), so that a) is also true. Since A was initially arbitrary, we may therefore conclude that the theorem holds, as claimed. $\square$

We demonstrate now the application of the Finiteness Criterion based on the construction of a surjection.

Corollary 4.125. *It is true for any set Y and any natural number n that the range of any sequence in $Y^{\{1,\ldots,n\}}$ is finite, that is,*

$$\forall Y, s\,([n \in \mathbb{N} \,\wedge\, s \in Y^{\{1,\ldots,n\}}] \Rightarrow \mathrm{ran}(s) \text{ is finite}). \tag{4.603}$$

Proof. We let Y be an arbitrary set, n an arbitrary natural number and s an arbitrary element of $Y^{\{1,\ldots,n\}}$. By definition of this set of functions, we then have $s : \{1,\ldots,n\} \to Y$, and consequently $s : \{1,\ldots,n\} \twoheadrightarrow \mathrm{ran}(s)$ by definition of a surjection. This shows that the existential sentence

$$\exists n, f\,(n \in \mathbb{N} \,\wedge\, f : \{1,\ldots,n\} \twoheadrightarrow Y)$$

is true, which finding implies with the Finiteness Criterion (4.600) that $\mathrm{ran}(s)$ is a finite set. Since Y, n and s were arbitrary, we may therefore conclude that the universal sentence (4.603) holds. $\square$

Recalling that every set of functions $Y^{\{1,\ldots,n\}}$ is identical with the Cartesian power Y^n, as shown by (4.385), we can immediately formulate the preceding corollary also in the following form.

Corollary 4.126. *It is true for any set Y and any natural number n that the range of any element of the n-th Cartesian power of Y is finite, that is,*

$$\forall Y, n, s\,([n \in \mathbb{N} \,\wedge\, s \in Y^n] \Rightarrow \mathrm{ran}(s) \text{ is finite}). \tag{4.604}$$

Corollary 4.127. *It is true for any set Y that the range of any sequence in $Y^{<\mathbb{N}_+}$ is finite, that is,*

$$\forall Y, s\,(s \in Y^{<\mathbb{N}_+} \Rightarrow \mathrm{ran}(s) \text{ is finite}). \tag{4.605}$$

Proof. We let Y be an arbitrary set and s an arbitrary element of $Y^{<\mathbb{N}_+}$. By definition of this set of function (see Exercise 4.32), there is then a particular $\bar{n} \in \mathbb{N}$ with $s \in Y^{\{1,\ldots,\bar{n}\}}$. Thus, $\mathrm{ran}(s)$ follows to be a finite set with (4.603). Since Y and s are arbitrary, we may therefore conclude that (4.605) holds. $\square$

Proposition 4.128. *The domain of any finite binary relation is finite.*

Proof. We let R be an arbitrary set, and we assume R to be both a finite set and a binary relation. The former assumption implies then with the definition of a finite set that there exist sets, say $\bar{n}$ and $\bar{c}$, such that $\bar{n} \in \mathbb{N}$ and

$\bar{c}: \{1, \ldots, \bar{n}\} \rightleftarrows R$. We may now apply Function definition by replacement to establish a unique function f with domain $\{1, \ldots, \bar{n}\}$ satisfying

$$\forall i\,(i \in \{1, \ldots, \bar{n}\} \Rightarrow \exists b\,(\,(i, (f(i), b)) \in \bar{c}\,)). \tag{4.606}$$

For this purpose, we prove the universal sentence

$$\forall i\,(i \in \{1, \ldots, \bar{n}\} \Rightarrow \exists! y\,(\exists b\,(\,(i, (y, b)) \in \bar{c}\,))), \tag{4.607}$$

letting i be arbitrary and assuming $i \in \{1, \ldots, \bar{n}\}$ $[= \mathrm{dom}(\bar{c})]$ to be true. By definition of a domain, there is then a constant, say $\bar{z}$, with $(i, \bar{z}) \in \bar{c}$. According to the definition of a range, we therefore have $\bar{z} \in \mathrm{ran}(\bar{c})$, which range is identical with the codomain R of $\bar{c}$ since the bijection $\bar{c}$ is in particular a surjection. Thus, substitution yields $\bar{z} \in R$, which implies with the definition of a binary relation that there are constants, say $\bar{a}$ and $\bar{b}$, such that $\bar{z} = (\bar{a}, \bar{b})$. With this equation, the previously established $(i, \bar{z}) \in \bar{c}$ gives us $(i, (\bar{a}, \bar{b})) \in \bar{c}$ via substitution. This finding demonstrates the truth of the existential sentence

$$\exists b\,(\,(i, (\bar{a}, b)) \in \bar{c}\,),$$

and this shows in turn that the existential part

$$\exists y\,(\exists b\,(\,(i, (y, b)) \in \bar{c}\,))$$

of the uniquely existential sentence in (4.607). To establish the uniqueness part, we prove

$$\forall y, y'\,([\exists b\,(\,(i, (y, b)) \in \bar{c}\,) \wedge \exists b'\,(\,(i, (y', b')) \in \bar{c}\,)] \Rightarrow y = y'), \tag{4.608}$$

letting y and y' be arbitrary and assuming the existence of particular constants $\bar{b}$ and $\bar{b}'$ satisfying $(i, (y, \bar{b})) \in \bar{c}$ as well as $(i, (y', \bar{b}'))$. Since $\bar{c}$ is a function, the conjunction of the latter two findings implies $(y, \bar{b}) = (y', \bar{b}')$, and this equation further implies $y = y'$ with the Equality Criterion for ordered pairs. As y and y' were arbitrary, we may now infer from the truth of this equation the truth of the (4.608) and consequently the truth of the uniquely existential sentence in (4.607). Here, i was arbitrary, so that the universal sentence (4.607) follows to be also true. This in turn implies the existence of a unique function f having the domain $\{1, \ldots, \bar{n}\}$ and the definite property (4.606).

Next, we prove that the domain of R is given by the range of f, by demonstrating the truth of the universal sentence

$$\forall y\,(y \in \mathrm{ran}(f) \Leftrightarrow y \in \mathrm{dom}(R)). \tag{4.609}$$

We take an arbitrary y and assume first $y \in \mathrm{ran}(f)$ to be true, so that the definition of a range gives us a particular constant $\bar{k}$ such that $(\bar{k}, y) \in f$ holds. Here, the definition of a domain shows that $\bar{k} \in \{1, \ldots, \bar{n}\}$ [$=$ $\mathrm{dom}(f)$] is true, which in turn implies with (4.606) the existence of a particular constant $\bar{b}$ such that $(\bar{k}, (f(\bar{k}), \bar{b})) \in \bar{c}$. Using again the definition of a range, we obtain then $(f(\bar{k}), \bar{b}) \in \mathrm{ran}(\bar{c})$ [$= R$], so that $f(\bar{k}) \in \mathrm{dom}(R)$ follows to be true by definition of a domain. Observing now that $(\bar{k}, y) \in f$ can be written in function notation as $y = f(\bar{k})$, we arrive at the desired consequent $y \in \mathrm{dom}(R)$ by means of substitution. We thus proved the implication '$\Rightarrow$' in (4.609).

To establish the converse implication '$\Leftarrow$', we now assume $y \in \mathrm{dom}(R)$ to be true, with the evident consequence that $(y, \bar{b}) \in R$ [$= \mathrm{ran}(\bar{c})$] holds for a particular constant $\bar{b}$. We therefore have also $(\bar{k}, (y, \bar{b})) \in \bar{c}$ for a particular constant $\bar{k}$, and this constant is thus in the domain $\{1, \ldots, \bar{n}\}$ of $\bar{c}$. According to (4.606), $\bar{k} \in \{1, \ldots, \bar{n}\}$ implies $(\bar{k}, (f(\bar{k}), \bar{b}')) \in \bar{c}$ for some particular constant $\bar{b}'$, which yields in conjunction with the preceding finding $(\bar{k}, (y, \bar{b})) \in \bar{c}$ the truth of $(f(\bar{k}), \bar{b}') = (y, \bar{b})$, because $\bar{c}$ is a function. Consequently, we obtain with the Equality Criterion for ordered pairs especially the equation $f(\bar{k}) = y$, which we may write in the form $(\bar{k}, y) \in f$. This shows that y is an element of the range of f, so that the implication '$\Leftarrow$' holds as well.

Having thus completed the proof of the equivalence in (4.609), we may now infer from this the truth of the universal sentence (4.609), since y was arbitrary, and subsequently the truth of the proposed equation $\mathrm{ran}(f) = \mathrm{dom}(R)$. This finding demonstrates that f is a surjection from $\{1, \ldots, \bar{n}\}$ to $\mathrm{dom}(R)$ and therefore that the existential sentence

$$\exists n, f\, (n \in \mathbb{N} \wedge f : \{1, \ldots, n\} \twoheadrightarrow \mathrm{dom}(R))$$

holds. It then follows with the Finiteness Criterion (4.600) that $\mathrm{dom}(R)$ is a finite set, and as R was arbitrary here, we may therefore conclude that the proposed universal sentence is true. □

Exercise 4.38. Prove that the range of any finite binary relation is finite.

(Hint: Proceed in analogy to the proof of Proposition 4.128.)

Exercise 4.39. Prove that the range of any function is finite if its domain is finite.

(Hint: Apply (4.600) in connection with Definition 3.48 and (3.649).)

Exercise 4.40. Apply Theorem 4.124b) to show that any subset of any finite set is itself finite, i.e.

$$\forall A, B\, (A \subseteq B \Rightarrow [B \text{ is finite } \Rightarrow A \text{ is finite}]). \tag{4.610}$$

(Hint: Use (3.624).)

We will also apply the following alternative formulations of the Finiteness Criteria.

Theorem 4.129 (Finiteness Criteria (variant)). *For any set A it is true that*

a) A is finite iff there exists a natural number n and a surjection from n to A, i.e.

$$A \text{ is finite} \;\Leftrightarrow\; \exists n, f\,(n \in \mathbb{N} \,\wedge\, f : n \twoheadrightarrow A). \tag{4.611}$$

b) A is finite iff there exists a natural number n and an injection from A to to n, i.e.

$$A \text{ is finite} \;\Leftrightarrow\; \exists n, f\,(n \in \mathbb{N} \,\wedge\, f : A \hookrightarrow n). \tag{4.612}$$

Proof. We let A be an arbitrary set. Concerning a), to prove the first part ('$\Rightarrow$') of the equivalence, we assume that A is finite, which implies with (4.600) that there exists a natural number, say $\bar{n}$, and a surjection from $\{1, \ldots, \bar{n}\}$ to A, say

$$\bar{f} : \{1, \ldots, \bar{n}\} \twoheadrightarrow A.$$

Then, we have the bijection $s^+ \restriction \bar{n}$ from $\bar{n}$ to $\{1, \ldots, \bar{n}\}$ according to (4.465), which is in particular a surjection, i.e.

$$s^+ \restriction \bar{n} : \bar{n} \twoheadrightarrow \{1, \ldots, \bar{n}\}.$$

It now follows with Theorem 3.199 that the composition of the surjections $\bar{f}$ and $s^+ \restriction \bar{n}$ is itself a surjection, that is,

$$\bar{f} \circ (s^+ \restriction \bar{n}) : \bar{n} \twoheadrightarrow A.$$

This proves the existence of a natural number n and of a surjection f from n to A, and thus the first part of the equivalence (4.611).

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume the existential sentence in (4.611) to be true, so that there is a natural number, say $\bar{m}$, and a surjection from $\bar{m}$ to A, say

$$\bar{g} : \bar{m} \twoheadrightarrow A.$$

We then have the bijection $(s^+ \restriction \bar{m})^{-1}$ from $\{1, \ldots, \bar{m}\}$ to $\bar{m}$ due to (4.466), which is thus a surjection, i.e.

$$(s^+ \restriction \bar{m})^{-1} : \{1, \ldots, \bar{m}\} \twoheadrightarrow \bar{m}.$$

Therefore, the composition of the surjections $\bar{g}$ and $(s^+ \restriction \bar{n})^{-1}$ is itself a surjection, that is,

$$\bar{g} \circ (s^+ \restriction \bar{n})^{-1} : \{1, \ldots, \bar{m}\} \twoheadrightarrow A.$$

This evidently proves the existential sentence in (4.600), which then implies that A is finite. Since A was arbitrary, we may therefore conclude that a) holds. □

Exercise 4.41. Establish Part b) of Theorem 4.129.

(Hint: Proceed in analogy to the proof of Part a) and use now Theorem 3.190.)

Exercise 4.42. Show for any set Y that the range of any sequence in $Y^{<\mathbb{N}}$ is finite, that is,

$$\forall Y, s\,(s \in Y^{<\mathbb{N}} \Rightarrow \mathrm{ran}(s) \text{ is finite}). \tag{4.613}$$

(Hint: Proceed similarly as in the proof of Corollary 4.125, using now Proposition 4.86) and (4.611).)

Note 4.23. The preceding Corollary shows, for any Y, that any sequence $s : n \to Y$ in $Y^{<\mathbb{N}}$ has a finite number of terms. We therefore call $Y^{<\mathbb{N}}$ also the *set of finite sequences* in Y.

4.7.2. Infinite sets

Exercise 4.43. Show that an infinite set is nonempty, that is,

$$\forall A\,(A \text{ is infinite } \Rightarrow A \neq \emptyset). \tag{4.614}$$

(Hint: Prove the implication by contradiction.)

Exercise 4.44. Show that any set that includes an infinite set is itself infinite, i.e.

$$\forall A, B\,(A \subseteq B \Rightarrow [A \text{ is infinite } \Rightarrow B \text{ is infinite}]). \tag{4.615}$$

(Hint: Use (4.610).)

Proposition 4.130. *The difference of an infinite set A and a finite subset of A is infinite, that is,*

$$\forall A, B\,([A \text{ is infinite } \wedge\ B \text{ is finite } \wedge\ B \subseteq A] \Rightarrow A \setminus B \text{ is infinite}). \tag{4.616}$$

Proof. We let A and B be arbitrary and prove the implication by contradiction, assuming that A is infinite, that B is a finite subset of A, and that $A \setminus B$ is finite. The finiteness of $A \setminus B$ and of B then implies with (4.545) that the union $(A \setminus B) \cup B$ is finite. As the assumption $B \subseteq A$ implies $(A \setminus B) \cup B = A$ with (2.263), it follows from the finiteness of $(A \setminus B) \cup B$ that A is finite, which contradicts the assumption that A is infinite. This proves the implication in (4.616), and since A and B were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

We now demonstrate that an infinite subset A of $\mathbb{N}$ can be 'sorted' in such a way that the first element of the sorted set is the least element of A, the second element the second lowest element of A, etc. This sorting will accordingly be expressed as an increasing function, where every natural number n is associated with the n-th lowest element of A. We first define the ranking of an arbitrary (finite or infinite) set.

Definition 4.15 (Ranking of a set, rank of an element). For any linearly ordered set $(A, <_A)$ we say that

(1) a set R is a *strict ranking* of A iff R is a strictly increasing bijective sequence in A.

(2) a set R is a *ranking* of A iff R is an increasing surjective sequence in A.

We then call any index i of R the *rank* of $R(i)$.

Lemma 4.131 (Ranking of an infinite subset of $\mathbb{N}$). *The following sentences are true for any infinite subset A of $\mathbb{N}$.*

a) There exists a unique function

$$F : A^{<\mathbb{N}} \to A, \qquad s \mapsto \min[A \setminus \operatorname{ran}(s)]. \tag{4.617}$$

b) Then, there exists a unique sequence

$$f : \mathbb{N} \to A, \qquad n \mapsto f_n = \min[A \setminus \operatorname{ran}(f \restriction n)]. \tag{4.618}$$

c) The sequence f is an injection, that is,

$$f : \mathbb{N} \hookrightarrow A. \tag{4.619}$$

d) The sequence f is strictly increasing.

e) Every term of the sequence f is greater than or equal to its index, i.e.

$$\forall n \, (n \in \mathbb{N} \Rightarrow n \leq f_n). \tag{4.620}$$

f) The sequence f is a surjection, that is,

$$f : \mathbb{N} \twoheadrightarrow A. \tag{4.621}$$

g) The sequence f is a strict ranking of A.

Proof. We let A be an arbitrary set, we assume that A is infinite, and we assume moreover that $A \subseteq \mathbb{N}$ holds.

Concerning a), we define the function F by replacement, letting $\varphi(s, m)$ be the formula

$$m = \min[A \setminus \operatorname{ran}(s)] \tag{4.622}$$

and verifying

$$\forall s \, (s \in A^{<\mathbb{N}} \Rightarrow \exists! m \, (\varphi(s, m))). \tag{4.623}$$

To prove this universal sentence, we let s be arbitrary and assume s to be an element of $A^{<\mathbb{N}}$ (so that s is a finite sequence in A). Then, $s \in A^{<\mathbb{N}}$ implies with (4.613) that $\operatorname{ran}(s)$ is a finite set. Furthermore, since A is codomain of s, the inclusion $\operatorname{ran}(s) \subseteq A$ holds. Now, as $\operatorname{ran}(s)$ a finite subset of the infinite set A, it follows with (4.616) that the set difference $A \setminus \operatorname{ran}(s)$ is an infinite set, and this further implies $A \setminus \operatorname{ran}(s) \neq \emptyset$ with (4.614). Observing that the inclusions

$$A \setminus \operatorname{ran}(s) \subseteq A \subseteq \mathbb{N} \tag{4.624}$$

hold with (2.125) and the initial assumption, we then see in view of the transitivity (2.13) of $\subseteq$ that the nonempty set difference $A \setminus \operatorname{ran}(s)$ is a subset of $\mathbb{N}$. Since $(\mathbb{N}, \leq)$ is a well-ordered set according to Corollary 4.45, it follows that the minimum of $A \setminus \operatorname{ran}(s)$ exists (see also Corollary 4.44). Thus, the existential part of the uniquely existential sentence in (4.623) holds. Now, the range of s is a uniquely specified set, and the set difference of A and this range is also uniquely specified. Moreover, the minimum of this set difference is a unique element of that set difference. Consequently, there exists a unique m such that m is identical with that minimum, which proves the uniqueness part. Let us also notice that this minimum is then by definition an element of $A \setminus \operatorname{ran}(s)$, which further implies

$$\min[A \setminus \operatorname{ran}(s)] \in A \tag{4.625}$$

with the first inclusion in (4.624), applying the definition of a subset. Since s is arbitrary, we may therefore conclude that (4.623) is true and furthermore that (4.625) holds for all $s \in A^{<\mathbb{N}}$. It then follows with Theorem 3.160 that there exists a unique function F with domain $A^{<\mathbb{N}}$ satisfying

$$\forall s \, (s \in A^{<\mathbb{N}} \Rightarrow F(s) = \min[A \setminus \operatorname{ran}(s)]).$$

Since all of the values of F are in A, as noted previously, it also follows with the Function Criterion that A is codomain of F, so that we obtained the desired mapping (4.617).

Concerning b), we observe in light of the Strong Recursion Theorem and Corollary 4.97 that the function (4.617) implies the unique existence of a sequence $f = (f_n)_{n\in\mathbb{N}}$ in A such that

$$f_n = F(f \upharpoonright n) = \min[A \setminus \operatorname{ran}(f \upharpoonright n)] \tag{4.626}$$

holds for all $n \in \mathbb{N}$, which sequence we may evidently write as (4.618). Let us observe here that A is a codomain of f, so that $\operatorname{ran}(f) \subseteq A$ holds. Together with initial assumption $A \subseteq \mathbb{N}$, this implies $\operatorname{ran}(f) \subseteq \mathbb{N}$ with the transitivity (2.13) of $\subseteq$, which inclusion shows that $\mathbb{N}$ is also a codomain of f.

Concerning c), we prove that $f : \mathbb{N} \to A$ is an injection by applying the Injection Criterion, i.e. by verifying

$$\forall m, n\,([m, n \in \mathbb{N} \wedge m \neq n] \Rightarrow f_m \neq f_n). \tag{4.627}$$

To do this, we let m and n be arbitrary in $\mathbb{N}$ such that $m \neq n$ holds. Since the standard linear ordering $<$ of $\mathbb{N}$ is connex, the disjunction of $m < n$, $n < m$ and $m = n$ is true. Because $m = n$ is by assumption false, we have that $m < n$ or $n < m$ is true, so that we may apply a proof by cases to establish the desired consequent $f_m \neq f_n$.

In the first case that $m < n$ is true, we see that $m \in n$ holds in view of (4.190). Since the restriction $f \upharpoonright n$ has the domain n according to Proposition 3.164, it follows from $m \in n$ that $m \in \operatorname{dom}(f \upharpoonright n)$ holds. Let us observe now that the assumption $m \in \mathbb{N}$ and the fact that $\mathbb{N}$ is the domain of f imply with the Function Criterion that there exists a unique element k in A with $(m, k) \in f$, which we may also write in function/sequence notation as $f_m = k$. Then, the conjunction of the previously established $m \in n$ and $(m, k) \in f$ implies with the definition of a restriction that $(m, k) \in f \upharpoonright n$ holds, which we may write as $(f \upharpoonright n)_m = k$. Combining the two equations for k, we obtain $f_m = (f \upharpoonright n)_m$. Furthermore, $(m, k) \in f \upharpoonright n$ implies (with the definition of a range) that $k \in \operatorname{ran}(f \upharpoonright n)$ holds, and therefore

$$f_m \in \operatorname{ran}(f \upharpoonright n) \tag{4.628}$$

after substitution based on the previously established equation $f_m = k$. Next, let us observe in view of (4.626) that f_n by definition of a minimum) is an element of $A \setminus \operatorname{ran}(f \upharpoonright n)$, so that (by definition of a set difference)

$$f_n \notin \operatorname{ran}(f \upharpoonright n) \tag{4.629}$$

is true in particular. Then, the conjunction of (4.628) and (4.629) implies the desired inequality $f_m \neq f_n$ with (2.4).

In the other case that $n < m$ is true, we may proceed by applying the same chain of arguments as in the first case. Thus, $n < m$ implies $n \in m$, and therefore $n \in \mathrm{dom}(f \upharpoonright m)$. Let us also note that the assumption $n \in \mathbb{N}$ $(= \mathrm{dom}(f))$ gives $n \in \mathrm{dom}(f)$. Thus, there exists a unique k with $(n, k) \in f$, so that $f_n = k$. Then, the conjunction of $n \in m$ and $(n, k) \in f$ implies $(n, k) \in f \upharpoonright m$, so that also $(f \upharpoonright m)_n = k$. The previous two equations for k may be combined to $f_n = (f \upharpoonright m)_n$. In addition, $(n, k) \in f \upharpoonright m$ implies the truth of $k \in \mathrm{ran}(f \upharpoonright m)$, and therefore also of

$$f_n \in \mathrm{ran}(f \upharpoonright m). \tag{4.630}$$

Now, as f_m is defined as the minimum of the set $A \setminus \mathrm{ran}(f \upharpoonright m)$, it is an element of that set difference, so that

$$f_m \notin \mathrm{ran}(f \upharpoonright m) \tag{4.631}$$

holds in particular. Then, the conjunction of (4.630) and (4.631) implies the inequality $f_n \neq f_m$, so that $f_m \neq f_n$ is true in any case. As m and n are arbitrary, we therefore conclude that the universal sentence (4.627) holds, which shows that f is an injection.

Concerning d), we now verify that the sequence $(f_n)_{n \in \mathbb{N}}$ is strictly increasing, by proving

$$\forall n\, (n \in \mathbb{N} \Rightarrow f_n < f_{n^+}) \tag{4.632}$$

according to the Monotony Criterion for strictly increasing sequences. To do this, we let n be arbitrary in $\mathbb{N}$ and recall that $n \in \mathbb{N}$ implies $n \subseteq n^+$ with (2.306). This inclusion further implies

$$f \upharpoonright n \subseteq f \upharpoonright n^+$$

with (3.89). This in turn implies

$$\mathrm{ran}(f \upharpoonright n) \subseteq \mathrm{ran}(f \upharpoonright n^+)$$

with (3.113), and then with (2.122) also

$$A \setminus \mathrm{ran}(f \upharpoonright n) \subseteq A \setminus \mathrm{ran}(f \upharpoonright n^+). \tag{4.633}$$

By definition of a minimum, the terms $f_n = \min[A \setminus \mathrm{ran}(f \upharpoonright n)]$ and $f_{n^+} = \min[A \setminus \mathrm{ran}(f \upharpoonright n^+)]$ defined in (4.626) are lower bounds contained in the corresponding set differences. In view of Exercise 3.46a), these lower bounds are infima, which thus exist. We may therefore apply Exercise 3.331 to infer from (4.633) the inequality

$$\inf[A \setminus \mathrm{ran}(f \restriction n)] \leq \inf[A \setminus \mathrm{ran}(f \restriction n^+)],$$

where the infima are minima (as noted previously). Consequently, we have the inequality

$$\min[A \setminus \mathrm{ran}(f \restriction n)] \leq \min[A \setminus \mathrm{ran}(f \restriction n^+)],$$

which gives $f_n \leq f_{n^+}$ after substitution based on (4.626). This inequality means that $f_n < f_{n^+}$ or $f_n = f_{n^+}$ is true. Now, the fact that $n, n^+ \in \mathbb{N}$ and $n \neq n^+$ (see Corollary 4.9) are both true implies $f_n \neq f_{n^+}$ with (4.627), so that $f_n = f_{n^+}$ is false. Therefore, the first part $f_n < f_{n^+}$ of the preceding disjunction is true. As n is arbitrary, we therefore conclude that (4.632) holds, so that the sequence $(f_n)_{n \in \mathbb{N}}$ is strictly increasing.

Concerning e), we apply a proof by mathematical induction. Regarding the base case ($n = 0$), we recall from b) that $\mathbb{N}$ is a codomain of f, so that f_0 is evidently a natural number, which fact implies $0 \leq f_0$ with (4.187), as desired. Regarding the induction step, we let n be arbitrary in $\mathbb{N}$, make the induction assumption $n \leq f_n$, and show that this implies $n^+ \leq f_{n^+}$. As f is strictly increasing, $f_n < f_{n^+}$ is true. Then, the conjunction of this and the induction assumption implies $n < f_{n^+}$ with the Transitivity Formula for $<$ and $\leq$, which inequality in turn implies the desired $n^+ \leq f_{n^+}$ with (4.157). Since n was arbitrary, we may therefore conclude that the induction also holds, which completes the proof of e) via mathematical induction.

Concerning f), we recall that f is a function with codomain A and prove the surjectivity condition $\mathrm{ran}(f) = A$ by contradiction, assuming that $\mathrm{ran}(f) \neq A$ holds. By definition of a codomain, we have $\mathrm{ran}(f) \subseteq A$, so that $\mathrm{ran}(f) \subset A$ or $\mathrm{ran}(f) = A$ is true according to (2.26). Since $\mathrm{ran}(f) = A$ is false by assumption, the first part $\mathrm{ran}(f) \subset A$ of the preceding disjunction is true. This proper inclusion implies $A \setminus \mathrm{ran}(f) \neq \emptyset$ with (2.127). Let us now observe that the inclusions

$$A \setminus \mathrm{ran}(f) \subseteq A \subseteq \mathbb{N} \tag{4.634}$$

hold with (2.125) and the initial assumption, so that the transitivity of $\subseteq$ yields $(\emptyset \neq)\ A \setminus \mathrm{ran}(f) \subseteq \mathbb{N}$. It then follows with the well-ordering of $\mathbb{N}$ that the minimum

$$m = \min A \setminus \mathrm{ran}(f) \tag{4.635}$$

exists. We now prove by contradiction the sentence

$$m \in \mathrm{ran}(f \restriction m^+), \tag{4.636}$$

assuming its negation

$$m \notin \operatorname{ran}(f \restriction m^+) \tag{4.637}$$

to be true. By definition of a minimum, we have due to (4.635) that m is an element of $A \setminus \operatorname{ran}(f)$, which implies in particular $m \in A$ (by definition of a set difference). The conjunction of this and (4.637) implies (again with the definition of a set difference)

$$m \in A \setminus \operatorname{ran}(f \restriction m^+). \tag{4.638}$$

Because of (4.626), we also have

$$f_{m^+} = \min\,[A \setminus \operatorname{ran}(f \restriction m^+)], \tag{4.639}$$

so that f_{m^+} is by definition of a minimum a lower bound for the set $A \setminus \operatorname{ran}(f \restriction m^+)$. Then, by definition of a lower bound, (4.638) implies $f_{m^+} \leq m$. Together with the fact that $f_m < f_{m^+}$ holds according d), these inequalities imply $f_m < m$ with the Transitivity Formula for $<$ and $\leq$, and therefore $\neg m \leq f_m$ with the Negation Formula for $\leq$. Since $m \leq f_m$ is also true because of e), we obtained a contradiction according to (1.11), which finding completes the proof of (4.636). It then follows with the definition of a range that there exists an element, say $\bar{k}$, such that $(\bar{k}, m) \in f \restriction m^+$ holds. By definition of a restriction, this implies in particular $(\bar{k}, m) \in f$, so that $m \in \operatorname{ran}(f)$ holds (again by definition of a range). As noted earlier, it is true that $m \in A \setminus \operatorname{ran}(f)$, which implies (by definition of a set difference) in particular that $m \notin \operatorname{ran}(f)$ holds. We thus showed that $m \in \operatorname{ran}(f)$ and $m \notin \operatorname{ran}(f)$ are both true, so that we evidently obtained a contradiction, which completes the proof of $\operatorname{ran}(f) = A$ by contradiction. Thus, the function $f : \mathbb{N} \to A$ is a surjection.

Concerning g), we notice that f is a bijection from $\mathbb{N}$ to A since f is both an injection and a surjection, according to c) and f). Because of d), f is therefore a strictly increasing bijective sequence in A. Furthermore, Since $(\mathbb{N}, <)$ is a linearly ordered set, it follows with Theorem 3.76 that the subset A of $\mathbb{N}$ is also linearly ordered (with respect to a unique subset $<_A$ of $<$). Thus, f is a strict ranking of A, by definition.

Since A was arbitrary in the proofs of a) – g), it follows that the lemma is true. □

Proposition 4.132. *The following equations hold for any infinite subset A of $\mathbb{N}$ and for the function f in (4.618).*

$$f_0 = \min A, \tag{4.640}$$

$$f_1 = \min[A \setminus \{\min A\}], \tag{4.641}$$

$$f_2 = \min[A \setminus \{\min A, \min[A \setminus \{\min A\}]\}]. \tag{4.642}$$

Proof. Letting A be an arbitrary set such that A is infinite and a subset of $\mathbb{N}$, we obtain the equations

$$\begin{aligned} f_0 &= \min[A \setminus \mathrm{ran}(f \restriction 0)] \\ &= \min[A \setminus \mathrm{ran}(f \restriction \emptyset)] \\ &= \min[A \setminus \mathrm{ran}(\emptyset)] \\ &= \min[A \setminus \emptyset] \\ &= \min A \end{aligned}$$

with (4.618), (2.49), (3.85), (3.121) and (2.102). Next, we observe the truth of the equations

$$\begin{aligned} f_1 &= \min[A \setminus \mathrm{ran}(f \restriction 1)] \\ &= \min[A \setminus \mathrm{ran}(f \restriction \{0\})] \\ &= \min[A \setminus \{f_0\}] \\ &= \min[A \setminus \{\min A\}] \end{aligned}$$

in light of (4.618), (2.154), (3.642) based on the facts $f \in A^{\mathbb{N}}$ and $0 \in \mathbb{N}$, and (4.640). Finally, we obtain

$$\begin{aligned} f_2 &= \min[A \setminus \mathrm{ran}(f \restriction 2)] \\ &= \min[A \setminus \mathrm{ran}(f \restriction \{0, 1\})] \\ &= \min[A \setminus \{f_0, f_1\}] \\ &= \min[A \setminus \{\min A, \min[A \setminus \{\min A\}]\}] \end{aligned}$$

by applying (4.618), (2.156), (3.576) based on the facts $f \in A^{\mathbb{N}}$ and $0, 1 \in \mathbb{N}$, and (4.641). Since A was arbitrary, we may therefore conclude that the proposition holds. □

Proposition 4.133. *For any natural number n it is true that there is no injection from the set of positive natural numbers to the initial segment of $\mathbb{N}_+$ up to n, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow \neg \exists f\,(f : \mathbb{N}_+ \hookrightarrow \{1, \ldots, n\})). \tag{4.643}$$

Proof. We let n be arbitrary and prove the implication by contradiction, assuming that $n \in \mathbb{N}$ holds and assuming that there exists an injection from $\mathbb{N}_+$ to $\{1, \ldots, n\}$, say $\bar{f}$. Let us now observe that $n \in \mathbb{N}$ implies $n^+ \in \mathbb{N}$ with (2.302) and then also $n^+ \in \mathbb{N}_+$ with (2.308), so that $\{1, \ldots, n^+\}$ is a subset of $\mathbb{N}_+$ by definition of an initial segment of $\mathbb{N}_+$. In light of Proposition 3.187, we then see that the restriction of $\bar{f} : \mathbb{N}_+ \hookrightarrow \{1, \ldots, n\}$ to the subset

$\{1,\ldots,n^+\}$ of $\mathbb{N}_+$ is an injection with domain $\{1,\ldots,n^+\}$ and codomain $\{1,\ldots,n\}$. Thus, the existential sentence

$$\exists f\,(f:\{1,\ldots,n^+\}\hookrightarrow\{1,\ldots,n\}) \tag{4.644}$$

holds. Besides $n, n^+ \in \mathbb{N}$, the inequality $n < n^+$ is also true according to (4.153), and therefore Dirichlet's Drawer Principle implies that the negation of (4.644) is true as well, so that we evidently obtained a contradiction. This finding completes the proof of the implication in (4.643) via contradiction, and as n was arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Exercise 4.45. Show for any natural number n that there is no injection from the set of natural numbers to the initial segment of $\mathbb{N}_+$ up to n, i.e.

$$\forall n\,(n\in\mathbb{N}\Rightarrow\neg\exists f\,(f:\mathbb{N}\hookrightarrow\{1,\ldots,n\})). \tag{4.645}$$

(Hint: Proceed in analogy to the proof of Proposition 4.133, observing that $\{1,\ldots,n^+\}$ is also a subset of $\mathbb{N}$.)

Proposition 4.134. *The set* $\mathbb{N}_+$ *is infinite.*

Proof. We prove the proposed sentence by contradiction, assuming $\mathbb{N}_+$ to be not infinite, so that $\mathbb{N}_+$ follows to be finite with the Double Negation Law. Thus, there exist a natural number, say $\bar{n}$, and a bijection from $\{1,\ldots,\bar{n}\}$ to $\mathbb{N}_+$, say $\bar{c}$. It then follows with Theorem 3.212 that the inverse function $\bar{c}^{-1}$ is a bijection from $\mathbb{N}_+$ to $\{1,\ldots,\bar{n}\}$, so that $\bar{c}^{-1}$ is especially an injection from $\mathbb{N}_+$ to $\{1,\ldots,\bar{n}\}$. Thus, the existential sentence

$$\exists f\,(f:\mathbb{N}_+\hookrightarrow\{1,\ldots,\bar{n}\}) \tag{4.646}$$

is true. Since $\bar{n}\in\mathbb{N}$ implies the negation of (4.646) with (4.643), we evidently have a contradiction, so that the proof of the proposed sentence is complete. □

Exercise 4.46. Prove that the set $\mathbb{N}$ is infinite.

(Hint: Proceed in analogy to the proof of Proposition 4.134, using Exercise 4.45.)

Theorem 4.135 (Characterization of infinite sets). *The following sentences are true.*

a) A set A is infinite if there exists a bijection from $\mathbb{N}_+$ to A, that is,

$$\forall A\,(\exists f\,(f:\mathbb{N}_+\rightleftarrows A)\Rightarrow A\text{ is infinite}). \tag{4.647}$$

b) A set A is infinite if there exists a bijection from $\mathbb{N}$ to A, that is,

$$\forall A\,(\exists f\,(f : \mathbb{N} \rightleftarrows A) \Rightarrow A \text{ is infinite}). \tag{4.648}$$

Proof. Concerning a), we let A be arbitrary and prove the implication by contradiction, assuming that there exists a bijection from $\mathbb{N}_+$ to A, say

$$\bar{f} : \mathbb{N}_+ \rightleftarrows A,$$

and assuming moreover that A is a finite set, so that there exist a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}\}$ to A, say

$$\bar{c} : \{1, \ldots, \bar{n}\} \rightleftarrows A.$$

Then, the inverse function $\bar{c}^{-1}$ is a bijection from A to $\{1, \ldots, \bar{n}\}$ due to Theorem 3.212, i.e.

$$\bar{c}^{-1} : A \rightleftarrows \{1, \ldots, \bar{n}\}.$$

Consequently, the composition $\bar{c}^{-1} \circ \bar{f}$ is a bijection from $\mathbb{N}_+$ to $\{1, \ldots, \bar{n}\}$ in view of Theorem 3.207, so that $\bar{c}^{-1} \circ \bar{f}$ is in particular an injection. Thus, the existential sentence

$$\exists f\,(f : \mathbb{N}_+ \hookrightarrow \{1, \ldots, \bar{n}\})$$

is true. Since $\bar{n} \in \mathbb{N}$ implies with (4.643) the negation of the preceding existential sentence, we have a contradiction, which proves the implications in (4.648). Since A was arbitrary, we may therefore conclude that a) holds. □

Exercise 4.47. Establish Part b) of Theorem 4.135.

(Hint: Proceed in analogy to the proof of a), applying now (4.645).)

We now extend the concept of counting a finite set (which ends by producing a natural number n) to the unending counting of an infinite set.

Definition 4.16 (Infinite denumeration). For any set A we say that a set d is an *infinite denumeration* of A iff d is a bijection from the set of natural numbers to A, i.e. iff

$$d : \mathbb{N} \rightleftarrows A. \tag{4.649}$$

Note 4.24. If, for a given set A, we have a function c satisfying (4.649), then A and thus the range of d is infinite because of the Characterization of infinite sets; it is therefore indeed justified to speak of an 'infinite' denumeration.

4.7.3. Countable sets

Definition 4.17 (Countable set, countably infinite set). We say that a set A is *countable* iff

(1) there exists a denumeration of A of length n for some $n \in \mathbb{N}$, i.e.

$$\exists n, c\,(n \in \mathbb{N} \wedge c : \{1, \ldots, n\} \rightleftarrows A), \tag{4.650}$$

(i.e. A is a finite set) or

(2) there exists an infinite denumeration of A, i.e.

$$\exists d\,(d : \mathbb{N} \rightleftarrows A), \tag{4.651}$$

in which case we call A a *countably infinite set*.

Furthermore, we say that a set A is *uncountable* iff A is not countable, i.e. iff the negation of the disjunction of (1) and (2) holds.

Corollary 4.136. *A set A is uncountable iff A is infinite and there is no bijection from $\mathbb{N}$ to A.*

Proof. We let A be an arbitrary set. In view of De Morgan's Law (1.52) we see that A is uncountable iff the conjunction of the negations of (1) and (2) holds. Here, the negation of (1) means that A is infinite. □

Exercise 4.48. Show for any countable set A that either (4.650) or (4.651) holds, i.e. that (4.650) and (4.651) are not simultaneously true.

(Hint: Apply Method 1.12 to prove that the negation of the conjunction of (4.650) and (4.651) is true, and apply similar arguments as in the proof of (4.647).)

The set of natural numbers can be counted (without end).

Corollary 4.137. *The set $\mathbb{N}$ is countably infinite.*

Proof. The identity function on $\mathbb{N}$ is a bijection from $\mathbb{N}$ to $\mathbb{N}$ according to Corollary 3.203. Thus, the set $\mathbb{N}$ satisfies (4.651). □

Countability either of a finite or an infinite set A can now be verified by associating its elements with distinct elements of $\mathbb{N}$, either by means of an injection from the considered set A to the counting set $\mathbb{N}$ or via a surjection from the counting set $\mathbb{N}$ to A. The resulting criteria (which we state as the following theorem) may be viewed as generalizations of the previously established Finiteness Criteria.

Theorem 4.138 (Countability Criteria). *The following sentences are true.*

a) A set A is countable iff there exists an injection from A to the set of natural numbers, that is,

$$\forall A\,(A \text{ is countable } \Leftrightarrow \exists f\,(f : A \hookrightarrow \mathbb{N})). \tag{4.652}$$

b) A set A is countable iff A is empty or there exists a surjection from the set of natural numbers to A, that is,

$$\forall A\,(A \text{ is countable } \Leftrightarrow [A = \emptyset \vee \exists f\,(f : \mathbb{N} \twoheadrightarrow A)]). \tag{4.653}$$

Proof. Concerning a), we let A be an arbitrary set. To prove the first part ('$\Rightarrow$') of the stated equivalence, we assume that A is countable, which means that there (either) exists a denumeration $c : \{1, \ldots, n\} \rightleftarrows A$ for some $n \in \mathbb{N}$ or an infinite denumeration $c : \mathbb{N} \rightleftarrows A$ of A. We now use this disjunction to prove the desired consequent by cases.

In the first case, there is a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}\}$ to A, say

$$\bar{c} : \{1, \ldots, \bar{n}\} \rightleftarrows A.$$

Then, we obtain the bijective inverse function

$$\bar{c}^{-1} : A \rightleftarrows \{1, \ldots, \bar{n}\}$$

with Theorem 3.212, which is especially an injection, that is,

$$\bar{c}^{-1} : A \hookrightarrow \{1, \ldots, \bar{n}\}.$$

Observing that $\bar{n} \in \mathbb{N}$ implies $\{1, \ldots, \bar{n}\} \subseteq \mathbb{N}$ with (4.240), we may now apply Exercise 3.78 to obtain the injection

$$\bar{c}^{-1} : A \hookrightarrow \mathbb{N},$$

so that the existential sentence in (4.652) holds for the first case.

In the second case, there is a bijection from $\mathbb{N}$ to A, say

$$\bar{d} : \mathbb{N} \rightleftarrows A,$$

which gives the bijection

$$\bar{d}^{-1} : A \rightleftarrows \mathbb{N}$$

(again with Theorem 3.212); this bijection is in particular an injection, i.e.

$$\bar{d}^{-1} : A \hookrightarrow \mathbb{N},$$

which shows that the existential sentence in (4.652) holds in any case. Thus, the the proof of the first part of the equivalence in (4.652) is complete.

To prove the second part ('$\Leftarrow$') of the proposed equivalence, we assume that there exists an injection from A to $\mathbb{N}$, say

$$\bar{f} : A \hookrightarrow \mathbb{N}.$$

We then obtain in view of (3.664) the bijection

$$\bar{f} : A \rightleftarrows \mathrm{ran}(\bar{f}),$$

so that the inverse function is (again in view of Theorem 3.212) the bijection

$$\bar{f}^{-1} : \mathrm{ran}(\bar{f}) \rightleftarrows A$$

Here, the Law of the Excluded Middle shows that the range $\mathrm{ran}(\bar{f})$ is a finite set or an infinite set. We may therefore carry out a proof by cases based on this disjunction.

On the one hand, if $\mathrm{ran}(\bar{f}$ is finite, then there is a particular natural number $\bar{m}$ and a particular bijection

$$\bar{\bar{c}} : \{1, \ldots, \bar{m}\} \rightleftarrows \mathrm{ran}(\bar{f}).$$

Consequently, we see in light of Theorem 3.207 that the composition of $\bar{f}^{-1}$ and $\bar{\bar{c}}$ is the bijection

$$\bar{f}^{-1} \circ \bar{\bar{c}} : \{1, \ldots, \bar{m}\} \rightleftarrows A.$$

This finding proves the existence of a denumeration of A of length n for some $n \in \mathbb{N}$, so that A is finite and therefore countable.

On the other hand, if $\mathrm{ran}(\bar{f}$ is infinite, then $\mathrm{ran}(\bar{f}$ is an infinite subset of $\mathbb{N}$, because this range is included in the codomain $\mathbb{N}$ of $\bar{f}$. It follows from this with Lemma 4.131 that there exists the (bijective) strict ranking of A

$$\bar{\bar{d}} : \mathbb{N} \rightleftarrows A,$$

so that we obtain the bijection

$$\bar{f}^{-1} \circ \bar{\bar{d}} : \mathbb{N} \rightleftarrows A$$

(using again Theorem 3.207). Thus, there exists an infinite denumeration of A, so that A is countable also in the second case. This completes the proof of the second part of the equivalence in (4.652).

Since A was initially arbitrary, we may therefore conclude that a) is true.

Concerning b), we let A be an arbitrary set, and we assume first that A is countable. Consequently, there exists in view of the Countability Criterion (4.652) an injection from A to $\mathbb{N}$, say

$$\bar{f} : A \hookrightarrow \mathbb{N}. \tag{4.654}$$

Observing now that the disjunction $A = \emptyset \lor A \neq \emptyset$ is true according to the Law of the Excluded Middle, we may now prove the disjunction in (4.653) by cases. In case of $A = \emptyset$, that disjunction is clearly true. The other case $A \neq \emptyset$ implies with (2.42) that there exists an element in A, say $\bar{a}$. Thus, the conjunction of (4.654) and $\bar{a} \in A$ is true, which implies with (3.692) that the union

$$\bar{f}^{-1} \cup [(\mathbb{N} \setminus \mathrm{ran}(\bar{f})) \times \{\bar{a}\}]$$

is a surjection from $\mathbb{N}$ to A. This proves $\exists f\,(\mathbb{N} \twoheadrightarrow A)$, and the disjunction in (4.653) is then also true. Thus, the proof of the first part ('$\Rightarrow$') of the equivalence in (4.653) is complete.

To prove the second part ('$\Leftarrow$'), we now assume the disjunction in (4.653) to be true, and we prove by cases that A is countable. On the one hand, if $A = \emptyset$ holds, then A is by definition a finite set and therefore countable. On the other hand, if there exists a surjection from $\mathbb{N}$ to A, say $\bar{f} : \mathbb{N} \twoheadrightarrow A$, then it follows with (3.700) that there exists an injection from A to $\mathbb{N}$, so that A follows to be countable with the Countability Criterion (4.652). Thus, the desired consequent holds in any case, so that the proof of the equivalence in (4.653) is complete. Since A was arbitrary, we may therefore conclude that b) is true. □

Corollary 4.139. *Any subset of any countable set is countable, that is,*

$$\forall A\,(A \text{ is countable} \Rightarrow \forall B\,(B \subseteq A \Rightarrow B \text{ is countable})). \tag{4.655}$$

Proof. Letting A be an arbitrary set and assuming A to be countable, it follows with the Countability Criterion (4.652) that there exists an injection from A to $\mathbb{N}$, say

$$\bar{f} : A \hookrightarrow \mathbb{N}.$$

Letting now B be an arbitrary set such that $B \subseteq A$ holds, it follows with Proposition 3.187 that the restriction $\bar{f} \upharpoonright B$ is an injection from B to $\mathbb{N}$, that is,

$$\bar{f} \upharpoonright B : B \hookrightarrow \mathbb{N}.$$

Thus, there exists an injection from B to $\mathbb{N}$, so that B follows to be countable with the Countability Criterion (4.652). Since A and B were arbitrary, we may therefore conclude that the corollary is true. □

Corollary 4.140. *It is true that the difference of any countable set B and any set A constitutes a countable set, that is,*

$$\forall A, B\,(B \text{ is countable} \Rightarrow\; B \setminus A \text{ is countable}). \tag{4.656}$$

Proof. We take arbitrary sets A and B such that B is countable and observe the truth of the inclusion $B \setminus A \subseteq B$ in light of (2.125), so that $B \setminus A$ follows to be countable with (4.655). Here, A and B are arbitrary, so that (4.656) holds indeed. □

Proposition 4.141. *It is true that the range of any function is countable if its domain is countable.*

Proof. Letting X and g be arbitrary sets such that g is a function with countable domain X, we prove the countability of $\operatorname{ran}(g)$ by cases, based on the true disjunction $X = \emptyset \vee X \neq \emptyset$. In the first case $X = \emptyset$, the domain of g is empty, so that the range of g is also empty because of (3.119). Then, the disjunction

$$\operatorname{ran}(g) = \emptyset \,\vee\, \exists f\,(f : \mathbb{N} \twoheadrightarrow \operatorname{ran}(g)) \tag{4.657}$$

is true as well, and this disjunction implies with the Countability Criterion (4.653) that $\operatorname{ran}(g)$ is countable.

In the second case $X \neq \emptyset$, it follows from the countability of X with the preceding Countability Criterion that there exists a set, say $\bar{f}$, such that $\bar{f} : \mathbb{N} \twoheadrightarrow X$. Noting that the given function g is a surjection from X to its range, i.e. $g : X \twoheadrightarrow \operatorname{ran}(g)$, we obtain the surjective composition $g \circ \bar{f} : \mathbb{N} \twoheadrightarrow \operatorname{ran}(g)$ with (3.649). We thus proved the existence of a surjection f from $\mathbb{N}$ to $\operatorname{ran}(g)$, so that the disjunction (4.657) is true again. Therefore, $\operatorname{ran}(g)$ is a countable set also in the second case.

Having completed the proof by cases, we may now infer from this the truth of the proposed universal sentence, because the sets X and g were initially arbitrary. □

Definition 4.18 (Equinumerous/equivalent infinite sets). We say that two infinite sets A and B are *equinumerous* or *equivalent* iff there exists a one-to-one correspondence between them, that is, iff

$$\exists f\,(f : A \rightleftarrows B). \tag{4.658}$$

Notation 4.8. In view of Proposition 4.107 and the previous definition, we may say that any two finite or infinite sets A and B are equinumerous iff there exists a bijection $f : A \rightleftarrows B$. We then also write

$$A \sim B. \tag{4.659}$$

Note 4.25. A set A is then countably infinite iff the set of natural numbers and the set A are equinumerous, i.e. iff

$$\mathbb{N} \sim A. \tag{4.660}$$

The fact that the successor function s^+ of the counting domain $(\mathbb{N}, s^+, 0)$ is a bijection from $\mathbb{N}$ to $\mathbb{N}_+$ according to (4.40) immediately gives the following result.

Corollary 4.142. *The set of natural numbers and the set of positive natural numbers are equinumerous, that is,*

$$\mathbb{N} \sim \mathbb{N}_+. \tag{4.661}$$

Note 4.26. $\mathbb{N}_+$ is countably infinite in view of (4.661) and Note 4.25.

Exercise 4.49. Show that a set A is countable iff

(1) A is equivalent to the initial segment of $\mathbb{N}_+$ up to some $n \in \mathbb{N}$, i.e.

$$\exists n\,(n \in \mathbb{N} \wedge A \sim \{1, \ldots, n\}), \tag{4.662}$$

or

(2) A is equivalent to the set of natural numbers, i.e.

$$A \sim \mathbb{N}. \tag{4.663}$$

(Hint: Apply Method 1.9 to establish both parts of the equivalence, using Theorem 3.212.)

We now justify the notion of *equivalent sets* by establishing a corresponding equivalence relation.

Exercise 4.50. Verify following sentences.

a) Any two identical sets are equinumerous, that is,

$$\forall A\,(A \sim A). \tag{4.664}$$

b) If two sets A and B are equinumerous, then B and A are also equinumerous, i.e.

$$\forall A, B\,(A \sim B \Rightarrow B \sim A). \tag{4.665}$$

c) If two sets A and B as well as B and a set C are equinumerous, then A and C are also equinumerous, that is,

$$\forall A, B, C\,([A \sim B \wedge B \sim C] \Rightarrow A \sim C). \tag{4.666}$$

(Hint: Apply Corollary 3.203, Theorem 3.212, and Theorem 3.207.)

We may apply (4.665) directly to (4.661) to obtain another example of equinumerous sets.

Corollary 4.143. *The set of positive natural numbers and the set of natural numbers are equinumerous, that is,*

$$\mathbb{N}_+ \sim \mathbb{N}. \tag{4.667}$$

Theorem 4.144 (Equivalence relation of equinumerosity). *The following sentences are true for any set system $\mathcal{K}$.*

a) There exists a unique set $R_\sim$ consisting of all the ordered pairs (A, B) such that A and B are equinumerous sets in $\mathcal{K}$, in the sense that

$$\exists! R_\sim \,\forall Z \,(Z \in R_\sim \Leftrightarrow [Z \in \mathcal{K} \times \mathcal{K} \,\wedge\, \exists A, B \,(A \sim B \,\wedge\, (A, B) = Z)]). \tag{4.668}$$

b) The set $R_\sim$ is a binary relation on $\mathcal{K}$ satisfying

$$\forall A, B \,(A, B \in \mathcal{K} \Rightarrow [A \, R_\sim \, B \Leftrightarrow A \sim B]). \tag{4.669}$$

c) Furthermore, the binary relation $R_\sim$ is an equivalence relation on $\mathcal{K}$.

Proof. Letting $\mathcal{K}$ be an arbitrary set system, we may apply the Axiom of Specification and the Equality Criterion for sets to obtain the true uniquely existential sentence (4.668). Thus, the set $R_\sim$ satisfies

$$\forall Z \,(Z \in R_\sim \Leftrightarrow [Z \in \mathcal{K} \times \mathcal{K} \,\wedge\, \exists A, B \,(A \sim B \,\wedge\, (A, B) = Z)]), \tag{4.670}$$

which shows that $Z \in R_\sim$ implies in particular $Z \in \mathcal{K} \times \mathcal{K}$ for any Z, so that the inclusion $R_\sim \subseteq \mathcal{K} \times \mathcal{K}$ follows to be true by definition of a subset. Thus, the set $R_\sim$ is a binary relation on $\mathcal{K}$, and every ordered pair in $R_\sim$ is (by definition of the Cartesian product of two sets) formed by two sets in $\mathcal{K}$.

Next, we establish (4.669), letting $\bar{A}$ and $\bar{B}$ be arbitrary and assuming $\bar{A}, \bar{B} \in \mathcal{K}$ to be true. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume furthermore that $\bar{A} \, R_\sim \, \bar{B}$ holds, and we write this as $(\bar{A}, \bar{B}) \in R_\sim$. This implies with (4.670) in particular that there exist sets, say $\bar{\bar{A}}$ and $\bar{\bar{B}}$, such that the conjunction of $\bar{\bar{A}} \sim \bar{\bar{B}}$ and $(\bar{\bar{A}}, \bar{\bar{B}}) = (\bar{A}, \bar{B})$ holds. The latter equation gives $\bar{\bar{A}} = \bar{A}$ and $\bar{\bar{B}} = \bar{B}$ with the Equality Criterion for ordered pairs, so that $\bar{\bar{A}} \sim \bar{\bar{B}}$ yields $\bar{A} \sim \bar{B}$ via substitution. Thus, the first part of the equivalence in (4.669) holds. Regarding the second part ('$\Leftarrow$'), we now assume $\bar{A} \sim \bar{B}$ to be true, which shows in light of the basic fact $(\bar{A}, \bar{B}) = (\bar{A}, \bar{B})$ that the existential sentence

$$\exists A, B\,(A \sim B \,\wedge\, (A, B) = (\bar{A}, \bar{B})) \tag{4.671}$$

is true. Moreover, since $\bar{A}, \bar{B} \in \mathcal{K}$ means $\bar{A} \in \mathcal{K} \,\wedge\, \bar{B} \in \mathcal{K}$, we obtain with the definition of the Cartesian product of two sets

$$(\bar{A}, \bar{B}) \in \mathcal{K} \times \mathcal{K},$$

which implies together with (4.671) the truth of $(\bar{A}, \bar{B}) \in R_{\sim}$ with (4.670). As we may write this finding as $\bar{A}\, R_{\sim}\, \bar{B}$, the proof of the second part of the equivalence in (4.669) is complete. Since $\bar{A}$ and $\bar{B}$ are arbitrary sets, we may therefore conclude that the universal sentence (4.669) is true.

We now verify that the binary relation $R_{\sim}$ is reflexive, that is,

$$\forall A\,(A \in \mathcal{K} \Rightarrow A\, R_{\sim}\, A). \tag{4.672}$$

Letting $\bar{A}$ be arbitrary and assuming $\bar{A} \in \mathcal{K}$ to be true, it follows with the Idempotent Law for the conjunction that $\bar{A} \in \mathcal{K} \wedge \bar{A} \in \mathcal{K}$ also holds, which yields $\bar{A}, \bar{A} \in \mathcal{K}$. Now, since $\bar{A} \sim \bar{A}$ holds according to (4.664), we obtain the desired $\bar{A}\, R_{\sim}\, \bar{A}$ with (4.669). Because $\bar{A}$ is arbitrary, we may therefore conclude that (4.672) holds, which universal sentence shows that the binary relation $R_{\sim}$ on $\mathcal{K}$ is reflexive.

Next, we show that $R_{\sim}$ is symmetric and verify for this purpose

$$\forall A, B\,(A, B \in \mathcal{K} \Rightarrow [A\, R_{\sim}\, B \Rightarrow B\, R_{\sim}\, A]). \tag{4.673}$$

We take two arbitrary elements $\bar{A}, \bar{B} \in \mathcal{K}$ and assume $\bar{A}\, R_{\sim}\, \bar{B}$. This assumption yields $\bar{A} \sim \bar{B}$ in view of (4.669), and this further implies $\bar{B} \sim \bar{A}$ with (4.665), which in turn gives $\bar{B}\, R_{\sim}\, \bar{A}$ by applying (4.669) again. Since $\bar{A}$ and $\bar{B}$ were arbitrary, we may therefore conclude that (4.673) is true, which means that $R_{\sim}$ is a symmetric binary relation on $\mathcal{K}$.

Finally, we establish the transitivity of $R_{\sim}$ by verifying

$$\forall A, B, C\,(A, B, C \in \mathcal{K} \Rightarrow [(A\, R_{\sim}\, B \,\wedge\, B\, R_{\sim}\, C) \Rightarrow A\, R_{\sim}\, C]). \tag{4.674}$$

Letting $\bar{A}, \bar{B}, \bar{C} \in \mathcal{K}$ be arbitrary and assuming $\bar{A}\, R_{\sim}\, \bar{B}$ as well as $\bar{B}\, R_{\sim}\, \bar{C}$ to be true, we then obtain $\bar{A} \sim \bar{B}$ and $\bar{B} \sim \bar{C}$ with (4.669). These two findings in turn imply $\bar{A} \sim \bar{C}$ with (4.666), which then evidently further implies $\bar{A}\, R_{\sim}\, \bar{C}$, as desired. Since $\bar{A}, \bar{B}, \bar{C}$ are arbitrary, we may infer from this the truth of (4.674).

This completes the proof that the binary relation $R_{\sim}$ on $\mathcal{K}$ is reflexive, symmetric and transitive, so that $R_{\sim}$ constitutes an equivalence class. $\square$

Proposition 4.145. *If two sets A and B as well as two sets C and D are equinumerous, then the Cartesian products $A \times C$ and $B \times D$ are also equinumerous, that is,*

$$\forall A, B, C, D\,([A \sim B \wedge C \sim D] \Rightarrow A \times C \sim B \times D). \tag{4.675}$$

Proof. We let A, B, C and D be arbitrary sets and assume $A \sim B$ as well as $C \sim D$, so that there exist a bijection from A to B and a bijection from C to D, say $f_1 : A \rightleftarrows B$ and $f_2 : C \rightleftarrows D$, according to Notation 4.8. Let us now apply the Axiom of Specification in connection with the Equality Criterion for sets to establish the unique existence of a set F which satisfies

$$\begin{aligned}&\forall z\,(z \in F \Leftrightarrow [z \in (A \times C) \times (B \times D)\\ &\quad \wedge\ \exists a, c\,(a \in A \wedge c \in C \wedge z = ((a,c),(f_1(a), f_2(c))))]).\end{aligned} \tag{4.676}$$

Since $z \in F$ implies evidently in particular $z \in (A \times C) \times (B \times D)$ for any z, it follows with the definition of a subset that $F \subseteq (A \times C) \times (B \times D)$ holds, so that we may view F as a binary relation $F \subseteq X \times Y$ with $X = A \times C$ and $Y = B \times D$. Next, we apply the Function Criterion to prove that F is a function with domain $X = A \times C$ and codomain $Y = B \times D$. For this purpose, we verify

$$\forall x\,(x \in A \times C \Rightarrow \exists! y\,(y \in B \times D \wedge (x, y) \in F)). \tag{4.677}$$

Letting x be arbitrary and assuming $\bar{x} \in A \times C$ to be true, it follows by definition of the Cartesian product of two sets that there is an element in A, say $\bar{a}$, and an element in C, say $\bar{c}$, with $(\bar{a}, \bar{c}) = \bar{x}$. Therefore, the function values $f_1(\bar{a})$ and $f_2(\bar{c})$ are uniquely specified elements of B and D, respectively, according to the Function Criterion. Consequently, we may define first the ordered pair $\bar{y} = (f_1(\bar{a}), f_2(\bar{c}))$ and then the ordered pair

$$\bar{z} = (\bar{x}, \bar{y}) = ((\bar{a}, \bar{c}), (f_1(\bar{a}), f_2(\bar{c}))),$$

and we see in light of $\bar{a} \in A$ and $\bar{c} \in C$ as well as the preceding equations that $\bar{z}$ satisfies the existential sentence in (4.676). Furthermore, $f_1(\bar{a}) \in B$ and $f_2(\bar{c}) \in D$ imply $[\bar{y} =]\ (f_1(\bar{a}), f_2(\bar{c})) \in B \times D$ by definition of the Cartesian product of two sets. As $[\bar{x} =]\ (\bar{a}, \bar{c}) \in A \times C$ also holds, we obtain then (again by definition of the Cartesian product of two sets)

$$[\bar{z} = (\bar{x}, \bar{y}) =] \quad ((\bar{a}, \bar{c}), (f_1(\bar{a}), f_2(\bar{c}))) \in (A \times C) \times (B \times D).$$

The conjunction of $\bar{z} \in (A \times C) \times (B \times D)$ and the previously established existential sentence (which $\bar{z}$ satisfies) implies now $\bar{z} \in F$ with (4.676).

Thus, $\bar{y} \in B \times D$ and $(\bar{x}, \bar{y}) \in F$ are both true, so that the existential part of the uniquely existential sentence in (4.677) holds.

To establish the uniqueness part, we take arbitrary y and y' such that $y, y' \in B \times D$ and $(\bar{x}, y), (\bar{x}, y') \in F$ are true. It then follows with (4.676) that there constants, say $\bar{a}, \bar{c}$ as well as $\bar{a}', \bar{c}'$ satisfying $\bar{a}, \bar{a}' \in A$ and $\bar{c}, \bar{c}' \in C$ as well as the equations

$$(\bar{x}, y) = ((\bar{a}, \bar{c}), (f_1(\bar{a}), f_2(\bar{c}))),$$
$$(\bar{x}, y') = ((\bar{a}', \bar{c}'), (f_1(\bar{a}'), f_2(\bar{c}'))).$$

These equations imply with the Equality Criterion for ordered pairs $\bar{x} = (\bar{a}, \bar{c})$ and $y = (f_1(\bar{a}), f_2(\bar{c}))$ as well as $\bar{x} = (\bar{a}', \bar{c}')$ and $y' = (f_1(\bar{a}'), f_2(\bar{c}'))$. Combining now the two equations for $\bar{x}$ via substitution yields first $(\bar{a}, \bar{c}) = (\bar{a}', \bar{c}')$, then with the Equality Criterion for ordered pairs $\bar{a} = \bar{a}'$ and $\bar{c} = \bar{c}'$. Next, we apply substitutions based on these two equations to obtain $f_1(\bar{a}') = f_1(\bar{a})$ and $f_2(\bar{c}') = f_2(\bar{c})$, so that the Equality Criterion for ordered pairs gives now

$$[y' =] \quad (f_1(\bar{a}'), f_2(\bar{c}')) = (f_1(\bar{a}), f_2(\bar{c})) \quad [= y].$$

These equations yield $y = y'$, and since y and y' are arbitrary, we may therefore conclude that the uniqueness part of the uniquely existential sentence in (4.677) holds as well. Then, since $\bar{x}$ was also arbitrary, we may further conclude that the universal sentence (4.677) is true. Consequently, F is a function from $A \times C$ to $B \times D$ according to the Function Criterion, so that F is the mapping

$$F : A \times C \to B \times D, \qquad (a, c) \mapsto F((a, c)) = (f_1(a), f_2(c)). \tag{4.678}$$

We now verify that F is a bijection by demonstrating that F is both an injection and a surjection. Concerning injectivity, we take arbitrary x and x', assume $x, x' \in A \times C$ and $F(x) = F(x')$ to be true, and we show that this implies $x = x'$. The former assumption implies (by definition of the Cartesian product of two sets) on the one hand that there exist constants, say $\bar{a}$ and $\bar{c}$, with $\bar{a} \in A$, $\bar{c} \in C$ and $(\bar{a}, \bar{c}) = x$, and on the other hand that there are constants, say $\bar{a}'$ and $\bar{c}'$, with $\bar{a}' \in A$, $\bar{c}' \in C$ and $(\bar{a}', \bar{c}') = x'$. Thus, we may write the latter assumption $F(x) = F(x')$ equivalently as $F((\bar{a}, \bar{c})) = F((\bar{a}', \bar{c}'))$ by applying substitutions, and then also as $(f_1(\bar{a}), f_2(\bar{c})) = (f_1(\bar{a}'), f_2(\bar{c}'))$ according to (4.678). The preceding equation now implies $f_1(\bar{a}) = f_1(\bar{a}')$ and $f_2(\bar{c}) = f_2(\bar{c}')$ with the Equality Criterion for ordered pairs. As the bijections f_1 and f_2 are in particular injections, these equations imply $\bar{a} = \bar{a}'$ as well as $\bar{c} = \bar{c}'$, which further imply (again with the Equality Criterion for ordered pairs)

$$[x =] \quad (\bar{a}, \bar{c}) = (\bar{a}', \bar{c}') \quad [= x'],$$

so that $x = x'$ is indeed true. Since x and x' were arbitrary, we may infer from this finding that F is an injection.

Concerning the surjectivity of $F : A \times C \to B \times D$, we apply the Surjection Criterion and take for this purpose an arbitrary y, assume $y \in B \times D$ to be true, and establish the truth of $F(x) = y$ for some x. Let us first observe that the assumed $y \in B \times D$ implies (by definition of the Cartesian product of two sets) the existence of two particular elements $\bar{b} \in B$ and $\bar{d} \in D$ satisfying $(\bar{b}, \bar{d}) = y$. As the bijections f_1 and f_2 are in particular surjections $f_1 : A \twoheadrightarrow B$ and $f_2 : C \twoheadrightarrow D$, it follows with the Surjection Criterion from $\bar{b} \in B$ and from $\bar{d} \in D$ that there are particular constants $\bar{a}$ and $\bar{c}$ which satisfy $f_1(\bar{a}) = \bar{b}$ and $f_2(\bar{b}) = \bar{d}$. Consequently, the previously established $y = (\bar{b}, \bar{d})$ yields

$$y = (f_1(\bar{a}), f_2(\bar{b})) = F((\bar{a}, \bar{c}))$$

by applying substitutions and by using the equation in (4.678). Thus, $F(x) = y$ holds for some x, and since y was arbitrary, we may therefore conclude that (the injection) $F : A \times C \to B \times D$ is a surjection, according to the Surjection Criterion. This completes the proof that the function F in (4.678) is a bijection. In view of Notation 4.8, $A \times C$ and $B \times D$ are then equivalent sets, as claimed. Since A, B, C and D were initially arbitrary sets, we may now finally conclude that the proposition is true. □

Since $\mathbb{N}_+ \sim \mathbb{N}$ and $\mathbb{N} \sim \mathbb{N}_+$ are true according to (4.667) and (4.661), an application of the Idempotent Law for the conjunction yields in view of the preceding proposition the following set equivalences.

Corollary 4.146. *The Cartesian products* $\mathbb{N}_+ \times \mathbb{N}_+$ *and* $\mathbb{N} \times \mathbb{N}$ *are equinumerous, that is,*

$$\mathbb{N}_+ \times \mathbb{N}_+ \sim \mathbb{N} \times \mathbb{N}, \tag{4.679}$$

$$\mathbb{N} \times \mathbb{N} \sim \mathbb{N}_+ \times \mathbb{N}_+. \tag{4.680}$$

The following proposition is about a relationship between the Cartesian product of a family of sets and the Cartesian product of two sets.

Proposition 4.147. *It is true for any natural number n and for any sequence of sets $A = (A_i \,|\, i \in \{1, \ldots, n+1\})$ that the Cartesian product of the sequence A and the Cartesian product of the restricted sequence $A \restriction \{1, \ldots, n\} = (A_i \,|\, i \in \{1, \ldots, n\})$ and the term A_{n+1} are equinumerous, that is,*

$$\forall n, A\left([n \in \mathbb{N} \wedge A = (A_i \,|\, i \in \{1, \ldots, n+1\})] \Rightarrow \bigtimes_{i=1}^{n+1} A_i \sim \left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}\right). \tag{4.681}$$

Proof. We take arbitrary sets n and A such that n is a natural number and such that A is a sequence of sets with index set $\{1,\ldots,n+1\}$. To prove the stated equivalence, we establish a bijection from the Cartesian product $\bigtimes_{i=1}^{n+1} A_i$ to the Cartesian product $\left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}$. To do this, we first verify

$$\forall x\,(x \in \bigtimes_{i=1}^{n+1} A_i \Rightarrow \exists! y\,(y = (x \upharpoonright \{1,\ldots,n\}, x_{n+1}))), \tag{4.682}$$

letting x be arbitrary and assuming $x \in \bigtimes_{i=1}^{n+1} A_i$ to be true. By definition of the Cartesian product of a family of sets, x is then a family with index set $\{1,\ldots,n+1\}$, so that the term x_{n+1} is clearly specified. Moreover, the assumption $n \in \mathbb{N}$ gives rise to the initial segment $\{1,\ldots,n\}$ of $\mathbb{N}_+$. We may therefore define the ordered pair $(x \upharpoonright \{1,\ldots,n\}, x_{n+1})$, which thus constitutes a constant. Consequently, the uniquely existential sentence in (4.682) follows to be true with (1.109). Since x is arbitrary, we may then conclude that the universal sentence (4.682) is true. According to Function definition by replacement, there is now a unique function f with domain $\bigtimes_{i=1}^{n+1} A_i$ such that

$$\forall x\,(x \in \bigtimes_{i=1}^{n+1} A_i \Rightarrow f(x) = (x \upharpoonright \{1,\ldots,n\}, x_{n+1})). \tag{4.683}$$

We can show here that $\left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}$ constitutes the range of the function f, i.e. that the range of the function f is included in the preceding Cartesian product. For this purpose, we demonstrate the truth of

$$\forall y\,(y \in \operatorname{ran}(f) \Leftrightarrow y \in \left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}). \tag{4.684}$$

We take an arbitrary constant y and assume first $y \in \operatorname{ran}(f)$ to be true, so that the definition of a range gives us a particular constant $\bar{x}$ such that $(\bar{x}, y) \in f$ holds. Recalling that $\bigtimes_{i=1}^{n+1} A_i$ is the domain of f, we thus have that $\bar{x}$ is an element of that Cartesian product. This means that $\bar{x}$ is a family with index set $\{1,\ldots,n+1\}$ whose terms satisfy

$$\forall i\,(i \in \{1,\ldots,n+1\} \Leftrightarrow \bar{x}_i \in A_i). \tag{4.685}$$

As indicated by Notation 4.6, the restriction $\bar{x} \upharpoonright \{1,\ldots,n\}$ is therefore a family/sequence with index set $\{1,\ldots,n\}$ and terms satisfying

$$\forall i\,(i \in \{1,\ldots,n\} \Leftrightarrow [\bar{x} \upharpoonright \{1,\ldots,n\}]_i \in A_i),$$

so that

$$\bar{x} \restriction \{1, \ldots, n\} \in \bigtimes_{i=1}^{n} A_i \tag{4.686}$$

holds. Since (4.685) shows clearly that $\bar{x}_{n+1} \in A_{n+1}$ is also true, we obtain with (4.686) and the definition of the Cartesian product of two sets

$$(\bar{x} \restriction \{1, \ldots, n\}, \bar{x}_{n+1}) \in \left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}. \tag{4.687}$$

Let us observe now that the previously established $(\bar{x}, y) \in f$ can be written in function notation as $y = f(\bar{x})$, and that the previous finding $\bar{x} \in \bigtimes_{i=1}^{n+1} A_i$ implies the truth of $f(\bar{x}) = (\bar{x} \restriction \{1, \ldots, n\}, \bar{x}_{n+1})$ with (4.683). Due to (4.687), substitutions give us then the desired consequent

$$y \in \left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1} \tag{4.688}$$

of the first part ('$\Rightarrow$') of the equivalence in (4.684).

To establish the second part ('$\Leftarrow$'), we assume the antecedent (4.688) to be true. By definition of the Cartesian product of two sets, there are then particular elements $\bar{a} \in \bigtimes_{i=1}^{n} A_i$ and $\bar{b} \in A_{n+1}$ with $y = (\bar{a}, \bar{b})$. Thus, $\bar{a}$ is a family with index set $\{1, \ldots, n\}$ and terms satisfying

$$\forall i\, (i \in \{1, \ldots, n\} \Leftrightarrow \bar{a}_i \in A_i). \tag{4.689}$$

As $n+1$ is not an element of $\{1, \ldots, n\}$ in view of (4.250), we can form the function $g = \bar{a} \cup \{(n+1, \bar{b})\}$ by applying Proposition 3.177, whose domain is then given by $\{1, \ldots, n\} \cup \{n+1\} = \{1, \ldots, n+1\}$, recalling (4.241). Thus, g is a family with index set $\{1, \ldots, n+1\}$, and we show now that its terms satisfy

$$\forall i\, (i \in \{1, \ldots, n+1\} \Rightarrow g_i \in A_i). \tag{4.690}$$

Letting i be arbitrary and assuming $i \in \{1, \ldots, n+1\}$ $[= \{1, \ldots, n\} \cup \{n+1\}]$ to be true, we obtain with the definition of the union of two sets the true disjunction $i \in \{1, \ldots, n\} \vee i \in \{n+1\}$, which we use to prove $g_i \in A_i$ by cases. The case $i \in \{1, \ldots, n\}$ implies $\bar{a}_i \in A_i$ with (4.689), where we can write $\bar{a}_i = \bar{a}(i)$ and then also $(i, \bar{a}_i) \in \bar{a}$. Consequently, the ordered pair $(i, \bar{a}_i)$ is also an element of $g = \bar{a} \cup \{(n+1, \bar{b})\}$, by definition of the union of a set system. We can then write this finding $(i, \bar{a}_i) \in g$ in family notation as $\bar{a}_i = g_i$, as desired. The other case $i \in \{n+1\}$ gives us first $i = n+1$ with (2.169) and then $(n+1, \bar{b}) = (i, \bar{b})$ via substitution. This ordered pair is also an element of the union $g = \bar{a} \cup \{(n+1, \bar{b})\}$, that is, $(i, \bar{b}) \in g$, and we can write this in the form $\bar{b} = g_i$. Recalling the truth of $\bar{b} \in A_{n+1}$ and of $i = n+1$, we therefore obtain the desired $g_i \in A_i$ by means of substitutions, completing the proof by cases. Since i was arbitrary, we may infer from the truth of $g_i \in A_i$ the truth of the universal sentence (4.690), which demonstrates that the family g (with index set $\{1, \ldots, n+1\}$) is an element of the Cartesian product $\bigtimes_{i=1}^{n+1} A_i$, and thus of the domain of the function f. The value of f associated with g is then $f(g) = (g \restriction \{1, \ldots, n\}, g_{n+1})$. Within the preceding proof by cases, the first case gave $(i, \bar{a}_i) \in g$ and thus $\bar{a}_i = g_i$ for an arbitrary $i \in \{1, \ldots, n\}$, so that $\bar{a} = g \restriction \{1, \ldots, n\}$ follows to be true according to the Equality Criterion for functions. In addition, the second case gave $\bar{b} = g_{n+1}$, so that we find $f(g) = (\bar{a}, \bar{b})$ $[= y]$ to be true. Writing the resulting equation $y = f(g)$ in the form $(g, y) \in f$, we now see in light of the definition of a range that $y \in \mathrm{ran}(f)$ is true, which completes the proof of the equivalence in (4.684). As y was arbitrary, we can infer from the truth of that equivalence the truth of (4.684), which universal sentence in turn implies with the Equality Criterion for sets

$$\mathrm{ran}(f) = \left[\bigtimes_{i=1}^{n} A_i \right] \times A_{n+1}.$$

We thus proved that the function

$$f : \bigtimes_{i=1}^{n+1} A_i \to \left[\bigtimes_{i=1}^{n} A_i \right] \times A_{n+1} \tag{4.691}$$

constitutes a surjection.

Finally, we show that this surjection is also an injection, i.e. that f satisfies

$$\forall x, x' \left(\left[x, x' \in \bigtimes_{i=1}^{n+1} A_i \wedge f(x) = f(x') \right] \Rightarrow x = x' \right). \tag{4.692}$$

We take arbitrary x and x', assuming $x, x' \in \bigtimes_{i=1}^{n+1} A_i$ and $f(x) = f(x')$ to be both true. By definition of the function f, these two values are given by the ordered pairs

$$f(x) = (x \restriction \{1, \ldots, n\}, x_{n+1})$$
$$f(x') = (x' \restriction \{1, \ldots, n\}, x'_{n+1}),$$

so that their assumed equality can be expressed by

$$(x \restriction \{1, \ldots, n\}, x_{n+1}) = (x' \restriction \{1, \ldots, n\}, x'_{n+1}).$$

Consequently, the Equality Criterion for ordered pairs gives us the two equations

$$x \upharpoonright \{1, \ldots, n\} = x' \upharpoonright \{1, \ldots, n\} \tag{4.693}$$

and

$$x_{n+1} = x'_{n+1}, \tag{4.694}$$

which allow us in the following to prove $x = x'$ via the Equality Criterion for functions (noting that the assumption $x, x' \in \bigtimes_{i=1}^{n+1} A_i$ means that x and x' are both families with common domain $\{1, \ldots, n+1\}$). Thus, the task is to prove the universal sentence

$$\forall i \, (i \in \{1, \ldots, n+1\} \Rightarrow x_i = x'_i). \tag{4.695}$$

Letting $\bar{k}$ be arbitrary and assuming $\bar{k} \in \{1, \ldots, n+1\}$ to be true, we consider once again the two cases $\bar{k} \in \{1, \ldots, n\}$ and $\bar{k} \in \{n+1\}$. In the first case $\bar{k} \in \{1, \ldots, n\}$, we observe first that (4.693) yields with the Equality Criterion for functions

$$\forall i \, (i \in \{1, \ldots, n\} \Rightarrow [x \upharpoonright \{1, \ldots, n\}]_i = [x' \upharpoonright \{1, \ldots, n\}]_i). \tag{4.696}$$

We obtain the true equations

$$x_{\bar{k}} = [x \upharpoonright \{1, \ldots, n\}]_{\bar{k}} = [x' \upharpoonright \{1, \ldots, n\}]_{\bar{k}} = x'_{\bar{k}}$$

by using (3.567), (4.696), and then again (3.567). The second case $\bar{k} \in \{n+1\}$ yields $\bar{k} = n+1$ with (2.169) and then

$$x_{\bar{k}} = x_{n+1} = x'_{n+1} = x'_{\bar{k}}$$

by applying substitutions based on $\bar{k} = n+1$ and based on (4.694). Thus, we find $x_{\bar{k}} = x'_{\bar{k}}$ to be true in both cases, and since $\bar{k}$ is arbitrary, we may therefore conclude that the universal sentence (4.695) holds. This proves that the functions x and x' are indeed equal, so that the proof of the implication in (4.692) is now also complete. Here, x and x' were arbitrary, so that the universal sentence (4.692) is true. Consequently, the surjection (4.691) is also an injection and therefore a bijection. This finding demonstrates that there exists a bijection from the Cartesian product $\bigtimes_{i=1}^{n+1} A_i$ to the Cartesian $\left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}$, which sets are therefore equinumerous, by definition. Initially, the sets n and A were arbitrary, so that the proposed universal sentence follows finally to be true. □

Theorem 4.148 (Cantor-Schröder-Bernstein theorem). *It is true that any two sets X and Y are equinumerous if there are injections from X to Y and from Y to X, that is,*

$$\forall X, Y \, ([\exists f \, (f : X \hookrightarrow Y) \wedge \exists g \, (g : Y \hookrightarrow X)] \Rightarrow X \sim Y). \tag{4.697}$$

Proof. We take arbitrary sets X and Y, assuming that there are injections from X to Y and from Y to X, say $\bar{f} : X \hookrightarrow Y$ and $\bar{g} : Y \hookrightarrow X$. We obtain then the bijection $\bar{g} : Y \rightleftarrows \mathrm{ran}(\bar{g})$ with (3.664), and moreover the bijection $\bar{g}^{-1} : \mathrm{ran}(\bar{g}) \rightleftarrows Y$ with (3.683).

Next, we apply Function definition by replacement to establish the unique function F with domain $\mathbb{N} \times \mathcal{P}(X)$ such that

$$\forall z \, (z \in \mathbb{N} \times \mathcal{P}(X) \Rightarrow \exists n, U \, (z = (n, U) \, \wedge \, \bar{g}[\bar{f}[U]] = F(z))). \qquad (4.698)$$

To achieve this, we prove the universal sentence

$$\forall z \, (z \in \mathbb{N} \times \mathcal{P}(X) \Rightarrow \exists! Y \, (\exists n, U \, (z = (n, U) \, \wedge \, \bar{g}[\bar{f}[U]] = Y))), \qquad (4.699)$$

letting $\bar{z}$ be arbitrary and assuming $\bar{z} \in \mathbb{N} \times \mathcal{P}(X)$ to be true. By definition of the Cartesian product of two sets, there exist then particular elements $\bar{n} \in \mathbb{N}$ and $\bar{U} \in \mathcal{P}(X)$ with $(\bar{n}, \bar{U}) = \bar{z}$, which findings imply especially $\bar{U} \subseteq X$ with the definition of a power set. Therefore, the image $f[\bar{U}]$ under $\bar{f} : X \hookrightarrow Y$ is defined and constitutes a subset of the codomain Y of f, according to (3.720). Consequently, the image $\bar{Y} = \bar{g}[\bar{f}[\bar{U}]]$ under $\bar{g} : Y \hookrightarrow X$ is also a defined. These findings demonstrate the existence of constants n and U for which $\bar{z} = (n, U)$ and $\bar{g}[\bar{f}[U]] = \bar{Y}$ are both true. Having found the constant $\bar{Y}$ satisfying this existential sentence, we now see that the existential part

$$\exists! Y \, (\exists n, U \, (\bar{z} = (n, U) \, \wedge \, \bar{g}[\bar{f}[U]] = Y))$$

of the uniquely existential sentence to be proven holds (for the given element $\bar{z}$). Concerning the uniqueness part, we prove

$$\begin{aligned} \forall Y, Y' \, ([\exists n, U \, (\bar{z} = (n, U) \, \wedge \, \bar{g}[\bar{f}[U]] = Y) \\ \wedge \, \exists n, U \, (\bar{z} = (n, U) \, \wedge \, \bar{g}[\bar{f}[U]] = Y')] \Rightarrow Y = Y'), \end{aligned} \qquad (4.700)$$

taking arbitrary Y, Y and assuming the antecedent to be true. On the one hand, there exist then constants, say $\bar{n}$ and $\bar{U}$, such that $\bar{z} = (\bar{n}, \bar{U})$ and $\bar{g}[\bar{f}[U]] = Y$ holds. On the other hand, there are constants, say $\bar{n}'$ and $\bar{U}'$, satisfing both $\bar{z} = (\bar{n}', \bar{U}')$ and $\bar{g}[\bar{f}[\bar{U}']] = Y'$. Combining the two equations for $\bar{z}$ yields $(\bar{n}, \bar{U}) = (\bar{n}', \bar{U}')$, which equation in turn implies with the Equality Criterion for ordered pairs especially $\bar{U} = \bar{U}'$. Applying now substitution based on this equation, we obtain

$$Y = \bar{g}[\bar{f}[\bar{U}]] = \bar{g}[\bar{f}[\bar{U}']] = Y',$$

so that the resulting $Y = Y$ proves the implication in (4.700). Since Y and Y' are arbitrary, we may therefore conclude that the universal sentence

(4.700) is true, which completes the proof of the uniqueness part and thus the proof of the uniquely existential sentence in (4.699). Here, $\bar{z}$ was also arbitrary, so that the universal sentence (4.699) follows to be true as well. It then follows from this that there exists indeed a unique function F with domain $\mathbb{N} \times \mathcal{P}(X)$ such that (4.698) holds. Let us verify that its range is included in $\mathcal{P}(X)$, i.e. that F satisfies

$$\forall y\, (y \in \mathrm{ran}(F) \Rightarrow y \in \mathcal{P}(X)). \tag{4.701}$$

We let y be arbitrary and assume $y \in \mathrm{ran}(F)$ to be true, so that the definition of a range gives us a particular constant $\bar{z}$ such that $(\bar{z}, y) \in F$ holds. We therefore obtain $\bar{z} \in \mathbb{N} \times \mathcal{P}(X)$ $[= \mathrm{dom}(F)]$ with the definition of a domain. Due to (4.698), there are then particular constants $\bar{n}$ and $\bar{U}$ satisfying $\bar{z} = (\bar{n}, \bar{U})$ as well as $\bar{g}[\bar{f}[\bar{U}]] = F(\bar{z})$. Since we may write $(\bar{z}, y) \in F$ in function notation in the form $y = F(\bar{z})$, we may carry out substitution to obtain $\bar{g}[\bar{f}[\bar{U}]] = y$. As noted earlier, the image $\bar{f}[\bar{U}]$ is a defined subset of Y, so that the image $y = \bar{g}[\bar{f}[\bar{U}]]$ under $\bar{g} : Y \hookrightarrow X$ is also defined and furthermore a subset of X according to (3.720). The definition of a power set gives us then $y \in \mathcal{P}(X)$, as desired, where y was arbitrary, so that the universal sentence (4.701) follows to be true. As this implies the inclusion $\mathrm{ran}(F) \subseteq \mathcal{P}(X)$ with the definition of a subset, we see that $\mathcal{P}(X)$ is a codomain of F.

Based on this function $F : \mathbb{N} \times \mathcal{P}(X) \to \mathcal{P}(X)$, we may apply now Definition by recursion (for $\mathbb{N}$) to obtain the unique sequence $u : \mathbb{N} \to \mathcal{P}(X)$ with terms

$$(1) \quad u(0) \ = X \setminus \mathrm{ran}(\bar{g}), \tag{4.702}$$

$$(2) \quad u(n^+) = F((n, u(n))) \quad \text{for any } n \in \mathbb{N}, \tag{4.703}$$

noting that the inclusion $X \setminus \mathrm{ran}(\bar{g}) \subseteq X$ holds according to (2.125), so that $X \setminus \mathrm{ran}(\bar{g}) \in \mathcal{P}(X)$ is true by definition of a power set (as required). Thus, we may form the union $\bigcup_{n=0}^{\infty} u_n$ of the sequence u.

In the next step we apply once again Function definition by replacement to establish a new function h with domain X such that

$$\begin{aligned} &\forall x\, (x \in X \\ &\quad \Rightarrow ([x \in \bigcup_{n=0}^{\infty} u_n \Rightarrow h(x) = \bar{f}(x)] \wedge [x \notin \bigcup_{n=0}^{\infty} u_n \Rightarrow h(x) = \bar{g}^{-1}(x)])). \end{aligned} \tag{4.704}$$

To accomplish task, we prove accordingly

$$\forall x\, (x \in X \Rightarrow \exists! y\, ([x \in \bigcup_{n=0}^{\infty} u_n \Rightarrow y = \bar{f}(x)] \wedge [x \notin \bigcup_{n=0}^{\infty} u_n \Rightarrow y = \bar{g}^{-1}(x)])), \tag{4.705}$$

letting $\bar{x}$ be arbitrary and assuming $\bar{x} \in X$ to hold. Here, X is the domain of $\bar{f}$, so that the value $\bar{f}(\bar{x})$ is defined. Let us verify that the value $\bar{g}^{-1}(x)$ of the inverse function $\bar{g}^{-1} : \mathrm{ran}(\bar{g}) \to Y$ is defined under the additional assumption $\bar{x} \notin \bigcup_{n=0}^{\infty} u_n$. According to the Characterization of the union of a family of sets, the preceding negation implies the truth of the negation

$$\neg\exists n\,(n \in \mathbb{N} \wedge \bar{x} \in u_n),$$

which in turn implies with the Negation Law for existential conjunctions

$$\forall n\,(n \in \mathbb{N} \Rightarrow \neg\bar{x} \in u_n).$$

With this, the truth of $0 \in \mathbb{N}$ in (2.297) implies $\neg\bar{x} \in u_0$ [$= u(0)$], so that (4.702) yields $\neg\bar{x} \in X \setminus \mathrm{ran}(\bar{g})$ via substitution. According to the definition of a set difference, this means

$$\neg[\bar{x} \in X \wedge \bar{x} \notin \mathrm{ran}(\bar{g})],$$

so that the disjunction of the negations $\neg\bar{x} \in X$ and $\neg\bar{x} \notin \mathrm{ran}(\bar{g})$ follows to be true with De Morgan's Law for the conjunction. Here, the first part $\neg\bar{x} \in X$ of the disjunction is false in view of the initial assumption $\bar{x} \in X$. Consequently, the second part $\neg\bar{x} \notin \mathrm{ran}(\bar{g})$ must be true, so that $\bar{x} \in \mathrm{ran}(\bar{g})$ follows to be true with the Double Negation Law. Thus, $\bar{x}$ turns out to be indeed an element of the domain of $\bar{g}^{-1}$ if $\bar{x} \notin \bigcup_{n=0}^{\infty} u_n$ holds. We now make use of the fact that the Law of the Excluded Middle yields the true disjunction

$$\bar{x} \in \bigcup_{n=0}^{\infty} u_n \vee \bar{x} \notin \bigcup_{n=0}^{\infty} u_n, \tag{4.706}$$

which we use to prove the existential part by cases. In the first case $\bar{x} \in \bigcup_{n=0}^{\infty} u_n$, replacing the variable y by the constant $\bar{f}(\bar{x})$ gives the true implications

$$\left[\bar{x} \in \bigcup_{n=0}^{\infty} u_n \Rightarrow \bar{f}(\bar{x}) = \bar{f}(\bar{x})\right] \wedge \left[\bar{x} \notin \bigcup_{n=0}^{\infty} u_n \Rightarrow \bar{f}(\bar{x}) = \bar{g}^{-1}(\bar{x})\right],$$

because the antecedent and the consequent of the first implication are both true, and because the antecedent of the second implication is false. In the second case $\bar{x} \notin \bigcup_{n=0}^{\infty} u_n$, we may replace y by $\bar{g}^{-1}(\bar{x})$ to obtain the true implications

$$\left[\bar{x} \in \bigcup_{n=0}^{\infty} u_n \Rightarrow \bar{g}^{-1}(\bar{x}) = \bar{f}(\bar{x})\right] \wedge \left[\bar{x} \notin \bigcup_{n=0}^{\infty} u_n \Rightarrow \bar{g}^{-1}(\bar{x}) = \bar{g}^{-1}(\bar{x})\right],$$

since the antecedent of the first one is false and since the antecedent and the consequent of the second one are both true. Thus, the existential part of the uniquely existential sentence in (4.705) holds in any case. To prove the uniqueness part, we let y and y' be arbitrary, assume that the conjunctions

$$\left[x \in \bigcup_{n=0}^{\infty} u_n \Rightarrow y = \bar{f}(x)\right] \wedge \left[x \notin \bigcup_{n=0}^{\infty} u_n \Rightarrow y = \bar{g}^{-1}(x)\right]$$
$$\left[x \in \bigcup_{n=0}^{\infty} u_n \Rightarrow y' = \bar{f}(x)\right] \wedge \left[x \notin \bigcup_{n=0}^{\infty} u_n \Rightarrow y' = \bar{g}^{-1}(x)\right]$$

hold, and show that $y = y'$ is implied. We establish this equation by means of a proof by cases, based again on the previously established disjunction (4.706). Then, the first case $\bar{x} \in \bigcup_{n=0}^{\infty} u_n$ implies the equations $y = \bar{f}(x)$ and $y' = \bar{f}(x)$, whereas the second case implies the equations $y = \bar{g}^{-1}(x)$ and $y' = \bar{g}^{-1}(x)$, so that substitutions give us the desired equation $y = y'$ in both cases. Since y and y' are arbitrary, we may infer from the truth of this equation the truth of the uniqueness part and therefore the truth of the uniquely existential sentence in (4.705). Because $\bar{x}$ was also arbitrary, we may furthermore conclude that the universal sentence (4.705) is true, with the consequence that there exists a unique function h with domain X and values defined according to (4.704). We may now readily show that Y is a codomain of h, i.e. that the range of h is included in Y. According to the definition of a subset, we may show this by demonstrating the truth of the equivalent universal sentence

$$\forall y\,(y \in \mathrm{ran}(h) \Rightarrow y \in Y). \tag{4.707}$$

We let y be arbitrary and assume $y \in \mathrm{ran}(h)$ to be true, so that there exists – by definition of a range – a constant, say $\bar{\bar{x}}$, such that $(\bar{\bar{x}}, y) \in h$. On the one hand, we already established h as a function, so that we may write $y = h(\bar{\bar{x}})$. On the other hand, we see in light of the definition of a domain that $\bar{\bar{x}} \in X$ $[= \mathrm{dom}(h)]$ is true. Because we obtained the disjunction (4.706) for an arbitrary $\bar{x} \in X$, it holds also for the current constant $\bar{x}$, allowing us to prove $h(\bar{\bar{x}}) \in Y$ by corresponding cases. The first case $\bar{x} \in \bigcup_{n=0}^{\infty} u_n$ implies then with (4.704) $h(\bar{x}) = \bar{f}(\bar{x})$, which shows that $\bar{x}$ is in the domain of $\bar{f}$, so that the associated function value $h(\bar{x})$ is in the codomain Y of $\bar{f}$. Similarly, the second case $\bar{x} \in \bigcup_{n=0}^{\infty} u_n$ implies $h(\bar{x}) = \bar{g}^{-1}(\bar{x})$ by definition of the function h; here, $\bar{x}$ is in the domain of $\bar{g}^{-1}$, and therefore the associated function value $h(\bar{x})$ is in the codomain Y of $\bar{g}^{-1}$. We thus found $h(\bar{x}) \in Y$ to be true for both cases, which yields now after substitution based on the previously obtained equation $y = h(\bar{\bar{x}})$ the desired consequent $y \in Y$ of the implication in (4.707). Since y was arbitrary, we may infer from the truth of that implication the truth of the universal sentence (4.707), and consequently the truth of the inclusion $\mathrm{ran}(h) \subseteq Y$. Thus, Y is indeed a codomain of h, so that we may write $h : X \to Y$.

It remains for us to prove that h is a bijection from X to Y, i.e. that h is both an injection and a surjection from X to Y. We first check that h satisfies the definition of an injection, that is,

$$\forall x, x' \, ([x, x' \in X \, \wedge \, h(x) = h(x')] \Rightarrow x = x'). \tag{4.708}$$

For this purpose, we let x and x' be arbitrary, and we assume $x, x' \in X$ and $h(x) = h(x')$. Evidently, $x \in \bigcup_{n=0}^{\infty} u_n$ or $x \notin \bigcup_{n=0}^{\infty} u_n$ is true, which fact allows us in the following to prove $x = x'$ by cases.

In the first case $x \in \bigcup_{n=0}^{\infty} u_n$, which implies $x \in u_{\bar{n}}$ for some particular number $\bar{n} \in \mathbb{N}$ (according to the Characterization of the union of a family of sets), we obtain with (4.703) the equation $u(\bar{n}^+) = F((\bar{n}, u(\bar{n})))$. Clearly, $(\bar{n}, u(\bar{n}))$ is in the domain $\mathbb{N} \times \mathcal{P}(X)$ of the function F, so that there exist due to (4.698) particular constants $\bar{N}, \bar{U}$ satisfying both $(\bar{n}, u(\bar{n})) = (\bar{N}, \bar{U})$ and

$$\bar{g}[\bar{f}[\bar{U}]] = F((\bar{n}, u(\bar{n}))) \quad [= u(\bar{n}^+)].$$

Because the Equality Criterion for ordered pairs yields in particular $u(\bar{n}) = \bar{U}$, the two previously stated equations lead to $u(\bar{n}^+) = \bar{g}[\bar{f}[u(\bar{n})]]$ via substitutions. Now, as the previously found $x \in u_{\bar{n}}$ implies $\bar{g}(\bar{f}(x)) \in \bar{g}[\bar{f}[u(\bar{n})]]$ with (3.739), obtain after substitution based on the preceding equation

$$\bar{g}(\bar{f}(x)) \in u(\bar{n}^+). \tag{4.709}$$

Moreover, we may prove $x' \in \bigcup_{n=0}^{\infty} u_n$ by contradiction, assuming its negation $x' \notin \bigcup_{n=0}^{\infty} u_n$ to be true. Then, the preceding assumption imply with (4.704) $h(x) = \bar{f}(x)$ as well as $h(x') = \bar{g}^{-1}(x')$, so that $h(x) = h(x')$ yields $\bar{f}(x) = \bar{g}^{-1}(x')$ through substitutions. Here, $\bar{f}(x)$ is evidently an element of the codomain Y of $\bar{f}$, and thus an element of the domain of $\bar{g}$. Consequently, $\bar{f}(x)$ gives rise to the value $\bar{g}(\bar{f}(x))$, so that substitution based on the preceding equation gives us

$$\bar{g}(\bar{f}(x)) = \bar{g}(\bar{g}^{-1}(x')) = (\bar{g} \circ \bar{g}^{-1})(x') = \mathrm{id}_{\mathrm{ran}(\bar{g})}(x') = x',$$

using also the notation for compositions functions, (3.680) in connection with $\bar{g} : Y \rightleftarrows \mathrm{ran}(\bar{g})$, and the definition of the identity function. Combining the resulting $x' = \bar{g}(\bar{f}(x))$ with (4.709), we get $x' \in u(\bar{n}^+)$. Observing now that $\bar{n} \in \mathbb{N}$ implies $\bar{n}^+ \in \mathbb{N}$ with (2.297), we thus see that there exists a natural number n for which $x' \in u(n)$ $[= u_n]$ holds. Consequently, we have

$x' \in \bigcup_{n=0}^{\infty} u_n$ according to the Characterization of the union of a family of sets, in contradiction to the assumed negation $x' \notin \bigcup_{n=0}^{\infty} u_n$. Having completed the proof of $x' \in \bigcup_{n=0}^{\infty} u_n$, we recall the current case assumption $x \in \bigcup_{n=0}^{\infty} u_n$ and evaluate these element x, x' by means of (4.704), giving the values $h(x) = \bar{f}(x)$ as well as $h(x') = \bar{f}(x')$. Then, substitution yields $\bar{f}(x) = \bar{f}(x')$, which equation implies in conjunction with the assumed $x, x' \in X$ that $x = x'$ holds, because $\bar{f}$ is an injection with domain X.

In the second case $x \notin \bigcup_{n=0}^{\infty} u_n$, we may simply rearrange the arguments used in the proof of the first case. First, we carry out a proof by contradiction to establish $x' \notin \bigcup_{n=0}^{\infty} u_n$, assuming the negation's negation to be true, so that the Double Negation Law yields the true sentence $x' \in \bigcup_{n=0}^{\infty} u_n$ to be true. This gives us $x' \in u_{\bar{n}'}$ for some particular element $\bar{n}' \in \mathbb{N}$, giving rise to the term $u(\bar{n}'^{+}) = F((\bar{n}', u(\bar{n}')))$ of the recursively defined sequence u. Thus, the ordered pair $(\bar{n}', u(\bar{n}'))$ is in the domain of the function F, ensuring the existence of particular constants $\bar{N}', \bar{U}'$ such that $(\bar{n}', u(\bar{n}')) = (\bar{N}', \bar{U}')$ and

$$\bar{g}[\bar{f}[\bar{U}']] = F((\bar{n}', u(\bar{n}'))) \quad [= u(\bar{n}'^{+})].$$

hold. Here, the equation $(\bar{n}', u(\bar{n}')) = (\bar{N}', \bar{U}')$ yields especially $u(\bar{n}') = \bar{U}'$, which allows us to combine the previous equations to the single equation $u(\bar{n}'^{+}) = \bar{g}[\bar{f}[u(\bar{n}')]]$. Here, we note that $x' \in u_{\bar{n}'}$ implies $\bar{g}(\bar{f}(x')) \in \bar{g}[\bar{f}[u(\bar{n}')]]$, with the consequence that

$$\bar{g}(\bar{f}(x')) \in u(\bar{n}'^{+}). \tag{4.710}$$

In addition, $x \notin \bigcup_{n=0}^{\infty} u_n$ and $x' \in \bigcup_{n=0}^{\infty} u_n$ imply with the definition of the function h the equations $h(x) = \bar{g}^{-1}(x)$ and $h(x') = \bar{f}(x')$, so that the initial assumption $h(x) = h(x')$ yields $\bar{f}(x') = \bar{g}^{-1}(x)$. Here, the value of $\bar{f} : X \to Y$ at x' is in Y and thus in the domain of $\bar{g} : Y \to X$, so that the value $\bar{g}(\bar{f}(x'))$ is specified. Combining these findings and using the definition/property of the identity function, we obtain the equations

$$\bar{g}(\bar{f}(x')) = \bar{g}(\bar{g}^{-1}(x)) = (\bar{g} \circ \bar{g}^{-1})(x) = \mathrm{id}_{\mathrm{ran}(\bar{g})}(x) = x,$$

The resulting equation $x = \bar{g}(\bar{f}(x'))$ and (4.710) lead to $x \in u(\bar{n}'^{+})$, which proves the existence of some $n \in \mathbb{N}$ with $x \in u(n)$ $[= u_n]$, so that $x \in \bigcup_{n=0}^{\infty} u_n$ holds. This however contradicts the current case assumption, completing the proof of $x' \notin \bigcup_{n=0}^{\infty} u_n$. The definition of the function h gives us therefore the values $h(x) = \bar{g}^{-1}(x)$ and $h(x') = \bar{g}^{-1}(x')$, with the consequence that $\bar{g}^{-1}(x) = \bar{g}^{-1}(x')$. Because the bijection $\bar{g}^{-1} : \mathrm{ran}(\bar{g}) \rightleftarrows Y$ is an injection and since the preceding equation clearly shows that x and x' are both in the domain $\mathrm{ran}(\bar{g})$ of $\bar{g}^{-1}$, it follows from this and the previous equation that $x = x'$ is true.

We thus completed the proof by cases, and thus the proof of the implication in (4.708), in which x and x' were arbitrary, so that the universal sentence (4.708) follows to be true. This means that the function $h : X \to Y$ is an injection. Our final task is now to prove that this injection $h : X \hookrightarrow Y$ is a surjection from X to Y, i.e. that Y is the range of h. We already know that the range of h is included in Y, so that it suffices to prove that Y is included in the range of h, because the truth of these two inclusion will imply the desired equality $Y = \operatorname{ran}(h)$ by means of the Axiom of Extension. Now, to prove the required $Y \subseteq \operatorname{ran}(h)$, we apply once again the definition of a subset and establish the equivalent universal sentence

$$\forall y\, (y \in Y \Rightarrow y \in \operatorname{ran}(h)), \tag{4.711}$$

letting y be arbitrary and assuming that $y \in Y$ holds. Since Y is the domain of $\bar{g}$, it follows that the value $\bar{g}(y)$ is in the codomain X of $\bar{g}$. Because the disjunction $\bar{g}(y) \in \bigcup_{n=0}^{\infty} u_n \lor \bar{g}(y) \notin \bigcup_{n=0}^{\infty} u_n$ is true (according to the Law of the Excluded Middle), we may prove $y \in \operatorname{ran}(h)$ by cases.

The first case $\bar{g}(y) \in \bigcup_{n=0}^{\infty} u_n$ implies (according to the Characterization of the union of a family of sets) that $\bar{g}(y) \in u_{\bar{n}}$ holds for some particular $\bar{n} \in \mathbb{N}$. Clearly, the value $\bar{g}(y)$ is also in the range of $\bar{g}$, so that the negation $\neg \bar{g}(y) \in \operatorname{ran}(\bar{g})$ is false. Therefore, the conjunction $\bar{g}(y) \in X \land \bar{g}(y) \notin \operatorname{ran}(\bar{g})$ is false as well, which means by definition of a set difference that $\bar{g}(y) \in X \setminus \operatorname{ran}(\bar{g})$ is false. In view of (4.702), this means that $\bar{g}(y) \notin u(0)$ holds. We may now prove by contradiction that $\bar{n} \neq 0$ is then also true. Indeed, assuming $\neg \bar{n} \neq 0$ to be true, we obtain $\bar{n} = 0$ with the Double Negation Law, so that the previously found $\bar{g}(y) \in u_{\bar{n}}$ gives via substitution $\bar{g}(y) \in u_0$ $[= u(0)]$, in contradiction to the true sentence $\bar{g}(y) \notin u(0)$. Having thus proved $\bar{n} \neq 0$ (where $\bar{n} \in \mathbb{N}$), it follows with (4.39) that there exists a natural number, say $\bar{m}$, satisfying $\bar{m} \neq \bar{n}$ and $\bar{m}^+ = \bar{n}$. Here, $\bar{m} \in \mathbb{N}$ gives rise to the value $u(\bar{m}^+) = F((\bar{m}, u(\bar{m})))$ according to (4.703), which we may write also as $u(\bar{n}) = F((\bar{m}, u(\bar{m})))$ after substitution. This equation shows that the ordered pair $(\bar{m}, u(\bar{m}))$ is in the domain of F, so that we obtain with (4.698) particular constants $\bar{N}, \bar{U}$ satisfying $(\bar{m}, u(\bar{m})) = (\bar{N}, \bar{U})$ as well as $\bar{g}[\bar{f}[\bar{U}]] = F((\bar{m}, u(\bar{m})))$. We can apply here the Equality Criterion for ordered pairs to obtain first $u(\bar{m}) = \bar{U}$ and therefore

$$\bar{g}[\bar{f}[\bar{U}]] = F((\bar{m}, u(\bar{m}))) \quad [= u(\bar{n})]$$

via substitution, resulting in the equation $u(\bar{n}) = \bar{g}[\bar{f}[u(\bar{m})]]$. Recalling now the previous finding $\bar{g}(y) \in u_{\bar{n}}$, we obtain now (by means of substitution) also $\bar{g}(y) \in \bar{g}[\bar{f}[u(\bar{m})]]$. Noting the truth of the equations

$$\bar{g}[\bar{f}[u(\bar{m})]] = (\bar{g} \circ \bar{f})[u(\bar{m})] = \operatorname{ran}((\bar{g} \circ \bar{f}) \restriction u(\bar{m}))$$

in light of (3.724) and the definition of an image, we get

$$\bar{g}(y) \in \operatorname{ran}((\bar{g} \circ \bar{f}) \restriction u(\bar{m}))$$

via substitution. By definition of a range, there exists then a constant, say $\bar{x}$, for which

$$(\bar{x}, \bar{g}(y)) \in (\bar{g} \circ \bar{f}) \restriction u(\bar{m})$$

holds. This implies with the definition of a restriction that $(\bar{x}, \bar{g}(y)) \in \bar{g} \circ \bar{f}$ and $\bar{x} \in u(\bar{m})$ are true, where we may write the former finding also as $\bar{g}(y) = \bar{g}(\bar{f}(\bar{x}))$, using the notation for function compositions. Since we assumed $y \in Y$ and since the value $\bar{f}(\bar{x})$ is clearly an element of the codomain Y of $\bar{f}$, we thus have $y, \bar{f}(\bar{x}) \in Y$. In conjunction with the preceding equation, this implies now $y = \bar{f}(\bar{x})$ with the injectivity of $\bar{g}$. We now draw two conclusions from the previous finding $\bar{x} \in u(\bar{m})$. On the one, recalling that $\mathcal{P}(X)$ is a codomain of u, we clearly have $u(\bar{m}) \in \mathcal{P}(X)$ and therefore $u(\bar{m}) \subseteq X$ (by definition of a power set), so that $\bar{x} \in u(\bar{m})$ gives $\bar{x} \in X$ (by definition of a subset). On the other hand, $\bar{x} \in u(\bar{m})$ implies in conjunction with $\bar{m} \in \mathbb{N}$ the truth of $\bar{x} \in \bigcup_{n=0}^{\infty} u_n$ (according to the Characterization of the union of a family of sets). Because of $\bar{x} \in X$, the latter conclusion implies $h(\bar{x}) = \bar{f}(\bar{x})$ by definition of the function h in (4.704). Combining this with the previously found equation $y = \bar{f}(\bar{x})$, we obtain $y = h(\bar{x})$ by substitution. Writing this in the form $(\bar{x}, y) \in h$, it is now clear that $y \in \operatorname{ran}(h)$ holds (in view of the definition of a range). Thus, the desired consequent of the implication in (4.711) is true in the first case.

In the second case $\bar{g}(y) \notin \bigcup_{n=0}^{\infty} u_n$, we recall the truth of $\bar{g}(y) \in X$ and use the definition of the function h to obtain

$$h(\bar{g}(y)) = \bar{g}^{-1}(\bar{g}(y)) = (\bar{g}^{-1} \circ \bar{g})(y) = \mathrm{id}_Y(y) = y,$$

applying also (3.679) and the definition of the identity function based on $\bar{g} : Y \rightleftarrows \operatorname{ran}(\bar{g})$. We can write the resulting equation $y = h(\bar{g}(y))$ also in the form $(\bar{g}(y), y) \in h$, so that we see (in light of the definition of a range) that $y \in \operatorname{ran}(h)$ follows to be true also for the current second case.

Because y was arbitrary, we may therefore conclude that the universal sentence (4.711) holds, and this sentence gives us the equivalent inclusion $Y \subseteq \operatorname{ran}(h)$. In conjunction with the already established reversed inclusion $\operatorname{ran}(h) \subseteq Y$, this implies $Y = \operatorname{ran}(h)$, so that the injection $h : X \hookrightarrow Y$ is a surjection from X to Y, thus a bijection from X to Y. This proves the existential sentence $\exists f\,(f : X \rightleftarrows Y)$, which means that X and Y are equinumerous (by definition), symbolically $X \sim Y$. We thus completed the proof of the implication in (4.697), where the sets X and Y were initially arbitrary, so that the stated theorem is indeed true. $\square$

4.8. Matrices

Definition 4.19 (Matrix, dimensions of a matrix, number of rows, number of columns, entry, row, column, row vector, column vector, vector). For any positive natural numbers m, n and any nonempty set Y, we say that a set $\boldsymbol{A}$ is a *matrix* iff $\boldsymbol{A}$ is a function from the Cartesian product of the initial segment of $\mathbb{N}_+$ up to m and the initial segment of $\mathbb{N}_+$ up to n to Y, that is, iff

$$\boldsymbol{A} : \{1, \ldots, m\} \times \{1, \ldots, n\} \to Y. \tag{4.712}$$

Here, we call m and n the *dimensions of* $\boldsymbol{A}$, and we say that $\boldsymbol{A}$ is a matrix of dimensions m and n or an $(m \times n)$-matrix or an m-by-n matrix (with values in Y). Furthermore, we call m the *number of rows* and n the *number of columns* of $\boldsymbol{A}$.

Moreover, for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$, we call the function value

$$a_{i,j} = \boldsymbol{A}((i,j)) \tag{4.713}$$

an *entry* of $\boldsymbol{A}$, and we write

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}. \tag{4.714}$$

In addition, we call the restrictions

$$\boldsymbol{A}_i = \boldsymbol{A} \upharpoonright \{i\} \times \{1, \ldots, n\}, \tag{4.715}$$

$$\boldsymbol{A}^{(j)} = \boldsymbol{A} \upharpoonright \{1, \ldots, m\} \times \{j\}, \tag{4.716}$$

respectively, the ith *row* and the jth *column* of $\boldsymbol{A}$.

In case of $m = 1$, we symbolize a $(1 \times n)$-matrix $\boldsymbol{A}$ also by

$$\boldsymbol{a} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \end{bmatrix} \tag{4.717}$$

and call $\boldsymbol{a}$ a (n-dimensional) *row vector*; similarly, in case of $n = 1$, we write for an $(m \times 1)$-matrix $\boldsymbol{A}$ also

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{m,1} \end{bmatrix} \tag{4.718}$$

and speak of a (m-dimensional) *column vector*. In both cases, we call $\boldsymbol{a}$ also simply a *vector*.

Note 4.27. The vector concept (4.718) differs from that of the n-vector (4.383) (here with $n = m$) both in structure and notation. Whereas the former is a function

$$\boldsymbol{a} : \{1, \ldots, m\} \times \{1\} \to Y, \tag{4.719}$$

the latter is a function

$$\boldsymbol{a} : \{1, \ldots, m\} \to Y. \tag{4.720}$$

Notation 4.9. In view of Proposition 3.164, the restrictions (4.715) and (4.716) are the functions $\boldsymbol{A}_i : \{i\} \times \{1, \ldots, n\} \to Y$ and $\boldsymbol{A}^{(j)} = \{1, \ldots, m\} \times \{j\} \to Y$, respectively, which we symbolize by

$$\boldsymbol{A}_i = \begin{bmatrix} a_{i,1} & \cdots & a_{i,n} \end{bmatrix} \tag{4.721}$$

$$\boldsymbol{A}^{(j)} = \begin{bmatrix} a_{1,j} \\ \vdots \\ a_{m,j} \end{bmatrix}. \tag{4.722}$$

Notation 4.10. Applying the notion of a set of functions to matrices, we now define the set of m-by-n matrices with values in Y and use the convenient abbreviation

$$Y^{m \times n} = Y^{\{1,\ldots,m\} \times \{1,\ldots,n\}}. \tag{4.723}$$

Proposition 4.149. *It is true for any m-by-n matrix $\boldsymbol{A}$ with values in any set Y that*

a) there exists a unique function $\boldsymbol{A}^T$ with domain $\{1, \ldots, n\} \times \{1, \ldots, m\}$ such that

$$\forall z \, (z \in \{1, \ldots, n\} \times \{1, \ldots, m\} \Rightarrow \exists i, j \, (i \in \{1, \ldots, n\} \wedge j \in \{1, \ldots, m\} \wedge (i, j) = z \wedge \boldsymbol{A}^T((i, j)) = a_{j,i}). \tag{4.724}$$

b) the function $\boldsymbol{A}^T$ is an n-by-m matrix with values in Y whose entries are given by

$$\forall i, j \, ([i \in \{1, \ldots, n\} \wedge j \in \{1, \ldots, m\}] \Rightarrow a^T_{i,j} = a_{j,i}). \tag{4.725}$$

Proof. Letting $\boldsymbol{A}$ be an arbitrary m-by-n matrix in an arbitrary set Y, we apply Function definition by replacement to establish the desired matrix. For this purpose, we prove

$$\forall z \, (z \in \{1, \ldots, n\} \times \{1, \ldots, m\} \Rightarrow \exists! y \, (\exists i, j \, (i \in \{1, \ldots, n\} \wedge j \in \{1, \ldots, m\} \wedge (i, j) = z \wedge y = a_{j,i})). \tag{4.726}$$

Letting $z \in \{1,\ldots,n\} \times \{1,\ldots,m\}$ be arbitrary, there are particular elements $I \in \{1,\ldots,n\}$ and $J \in \{1,\ldots,m\}$ such that $(I,J) = z$, according to Exercise 3.4. Therefore, the ordered pair (J,I) is an element of the Cartesian product $\{1,\ldots,m\} \times \{1,\ldots,n\}$, thus an element of the domain of the matrix $\boldsymbol{A}$. The associated entry is given by $\boldsymbol{A}((J,I)) = a_{J,I}$. These findings demonstrate the truth of the existential sentence

$$\exists i,j\,(i \in \{1,\ldots,n\} \wedge j \in \{1,\ldots,m\} \wedge (i,j) = z \wedge \boldsymbol{A}((J,I)) = a_{j,i}),$$

which in turn demonstrates the truth of the existential part of the uniquely existential sentence in (4.726). To establish the uniqueness part, we let y and y' be arbitrary, assuming $y = a_{j,i}$ and $y' = a_{j',i'}$ to be satisfied by some particular elements $i,i' \in \{1,\ldots,n\}$ and $j,j' \in \{1,\ldots,m\}$ with $(i,j) = z$ and $(i',j') = z$. The last two equations imply $(i,j) = (i',j')$ via substitution, so that the Equality Criterion for ordered pairs yields the true equations $i = i'$ and $j = j'$. Consequently, the previous equations for y and y' become $y = a_{j,i} = a_{j',i'} = y'$ after carrying out substitutions. Since y and y' are arbitrary, the resulting equation $y = y'$ implies the truth of the uniqueness part of the uniquely existential sentence in (4.726). As z was also arbitrary, we may therefore conclude that the universal sentence (4.726) holds, which in turn implies the unique existence of a function $\boldsymbol{A}^T$ on $\{1,\ldots,n\} \times \{1,\ldots,m\}$ such that (4.724). Next, we prove that Y is a codomain of that function, i.e., that the range of $\boldsymbol{A}^T$ is included in Y. To do this, we apply the definition of a subset and let accordingly $\bar{y} \in \mathrm{ran}(\boldsymbol{A}^T)$ be arbitrary. By definition of a range, there exists then a particular constant $\bar{z} \in \{1,\ldots,n\} \times \{1,\ldots,m\}$ such that $(\bar{z},\bar{y}) \in \boldsymbol{A}^T$. Since $\boldsymbol{A}^T$ constitutes a function, we may write this finding as $\bar{y} = \boldsymbol{A}^T(\bar{z})$. By definition of a domain, we also find $\bar{z} \in \{1,\ldots,n\} \times \{1,\ldots,m\}$, so that (4.724) yields $\boldsymbol{A}^T((i,j)) = a_{j,i}$ for some particular elements $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$ satisfying $(i,j) = \bar{z}$. Substitution therefore gives us $[\bar{y} =\,]\ \boldsymbol{A}^T(\bar{z}) = a_{j,i}\ [= \boldsymbol{A}((j,i))]$. As $\boldsymbol{A}$ is a function from $\{1,\ldots,m\} \times \{1,\ldots,n\}$ to Y, the resulting equation $\bar{y} = \boldsymbol{A}((j,i))$ implies $\bar{y} \in Y$ with (3.517). Since $\bar{y}$ was arbitrary, we may therefore conclude that the inclusion $\mathrm{ran}(\boldsymbol{A}^T) \subseteq Y$ is true indeed, so that $\boldsymbol{A}^T$ is a function from $\{1,\ldots,n\} \times \{1,\ldots,m\}$ to Y. Thus, $\boldsymbol{A}^T$ constitutes an n-by-m matrix with values in Y.

It remains for us to establish (4.725). For this purpose, we let $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$ be arbitrary, with the consequence that (i,j) is an element of the Cartesian product $\{1,\ldots,n\} \times \{1,\ldots,m\}$, thus of the domain of $\boldsymbol{A}^T$. By definition of that function, there exist then particular elements $I \in \{1,\ldots,n\}$ and $J \in \{1,\ldots,m\}$ satisfying $(I,J) = (i,j)$ and $\boldsymbol{A}^T((I,J)) = a_{J,I}$. Clearly, the former equation implies $I = i$ and $J = j$, so that the other equation can be re-expressed as $\boldsymbol{A}^T((i,j)) = a_{j,i}$. Writing $a^T_{i,j}$ for $\boldsymbol{A}^T((i,j))$ as in (4.713), we therefore obtain $a^T_{i,j} = a_{j,i}$, which proves the implication in (4.725). Since i and j are arbitrary, we may infer from the truth of that implication the truth of the universal sentence (4.725). Since $\boldsymbol{A}$ was initially an arbitrary m-by-n matrix in an arbitrary set Y, we may now further conclude that the proposed universal sentence is true. □

Definition 4.20 (Transpose of a matrix). For any m-by-n matrix $\boldsymbol{A}$ with values in any set Y, we call the n-by-m matrix

$$\boldsymbol{A}^T \tag{4.727}$$

with values in Y the *transpose of* $\boldsymbol{A}$.

Theorem 4.150 (Double Transposition Law). *Every m-by-n matrix $\boldsymbol{A}$ with values in any set Y is identical with the transpose of its transpose, that is,*

$$(\boldsymbol{A}^T)^T = \boldsymbol{A}. \tag{4.728}$$

Exercise 4.51. Prove the Double Transposition Law.

(Hint: Apply the Equality Criterion for functions and (4.725).)

Besides vectors, the following types of matrices also play a prominent role in real matrix calculus.

Definition 4.21 (Square matrix, main diagonal). We say that an $(m \times n)$-matrix $\boldsymbol{A}$ is a *square matrix* (*of order* n) iff its dimensions are identical, that is, iff

$$m = n. \tag{4.729}$$

Then, we call for any square matrix of order n

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \tag{4.730}$$

the restriction

$$\boldsymbol{A} \upharpoonright \{(i,i) : i \in \{1,\ldots,n\}\} \tag{4.731}$$

the *main diagonal* of $\boldsymbol{A}$, and we call each value $a_{i,i}$ of the main diagonal a *main diagonal entry* of $\boldsymbol{A}$.

Definition 4.22 (Symmetric matrix). We say that a square matrix $\boldsymbol{A}$ of any order n is *symmetric* iff $\boldsymbol{A}$ and its transpose are identical, that is, iff

$$\boldsymbol{A}^T = \boldsymbol{A}. \tag{4.732}$$

Chapter 5.

The Ordered Elementary Domain of Natural Numbers

5.1. Binary Operations

In Part a) of the Strong Recursion Theorem we encountered a function f whose domain is given by a Cartesian product $N \times B^{<\mathbb{N}}$, which function we may therefore recognize as a ternary relation. In the current chapter, we will in particular define ternary relations for which the domain is a Cartesian product of two identical sets.

Definition 5.1 (Binary operation). For any set X we say that a set $*$ is a *binary operation* on X iff $*$ is a function from $X \times X$ to X, i.e. iff

$$* : X \times X \to X. \tag{5.1}$$

Notation 5.1. We then write $(a, b) \mapsto *(a, b) = a * b$. The symbol $*$ may be viewed as a placeholder for specific binary operations that typically arise in dealing with sets, numbers and functions. Instead of $*$ we will also use the symbol $\odot$ to denote a generic binary operation.

Definition 5.2 (Restricted binary operation). For any set X, any set $A \subseteq X$ and any binary operation $*$ on X, we say that the restriction of $*$ to $A \times A$ is a *restricted binary operation* iff it constitutes a binary operation on A, i.e., iff

$$* \restriction A \times A : A \times A \to A. \tag{5.2}$$

Note 5.1. The restriction of a binary operation on a set X to the Cartesian product of a subset $A \subseteq X$ with itself always constitutes a function

$$* \restriction A \times A : A \times A \to X. \tag{5.3}$$

in view of (3.566); however, A need not be a codomain of that restriction, so that the restriction is not necessarily a binary operation itself.

Proposition 5.1. *The following universal sentence holds for any set X, any set $A \subseteq X$ and any binary operation $*$ on X such that the restriction $*_A$ of $*$ to $A \times A$ is a binary operation on A:*

$$\forall a, b\,(a, b \in A \Rightarrow a * b = a *_A b). \tag{5.4}$$

Proof. We let $*$ be an arbitrary binary operation on an arbitrary set X and A an arbitrary subset of X such that the restriction $*_A = * \restriction A \times A$ is a binary operation on A. For arbitrary $a, b \in A$, we obtain

$$a *_A b = *_A(a, b) = * \restriction A \times A(a, b) = *(a, b) = a * b$$

by applying Notation 5.1, the definition of $*_A$, (3.567) with the evident fact that $a, b \in A$ implies $(a, b), (b, a)$ in $A \times A$ with the definition of the Cartesian product of two sets, and finally again Notation 5.1. Thus, $a *_A b = a * b$ follows to be true, and since X, A, $*$, a and b were initially all arbitrary, we may therefore conclude that the proposed universal sentence is true. □

We now give a first example of a binary operation.

Proposition 5.2. *For any lattice $(X, \leq_X)$ there exist the binary operations*

$$\sqcup : X \times X \to X, \quad (a, b) \mapsto a \sqcup b = \sup\{a, b\}, \tag{5.5}$$

$$\sqcap : X \times X \to X, \quad (a, b) \mapsto a \sqcap b = \inf\{a, b\}. \tag{5.6}$$

Proof. Letting X and $\leq_X$ be arbitrary such that $(X, \leq_X)$ is a lattice, we may apply Function definition by replacement to establish (5.5). For this purpose, we first verify the universal sentence

$$\forall Z\,(Z \in X \times X \Rightarrow \exists! y\,(\exists a, b\,(y = \sup\{a, b\} \,\wedge\, (a, b) = Z))). \tag{5.7}$$

Letting Z be arbitrary and assuming $Z \in X \times X$ to be true, it follows with the definition of the Cartesian product of two sets that there are constants, say $\bar{a}$ and $\bar{b}$, such that $\bar{a} \in X$, $\bar{b} \in X$ and $(\bar{a}, \bar{b}) = Z$ hold. Then, regarding the existential part, we observe that $\bar{a}, \bar{b} \in X$ implies the existence of the supremum $\sup\{\bar{a}, b\}$, because $(X, <_X)$ is a lattice; together with the preceding equation, this shows that there are constants y, a and b which satisfy $y = \sup\{a, b\} \,\wedge\, (a, b) = Z$. Regarding the uniqueness part, we let y and y' be arbitrary such that the existential sentences

$$\exists a, b\,(y = \sup\{a, b\} \,\wedge\, (a, b) = Z)$$
$$\exists a, b\,(y' = \sup\{a, b\} \,\wedge\, (a, b) = Z)$$

are both true. Thus, there are particular constants $\bar{\bar{a}}, \bar{\bar{b}}$ satisfying $y = \sup\{\bar{\bar{a}}, \bar{\bar{b}}\}$ and

$$(\bar{\bar{a}}, \bar{\bar{b}}) = Z\ [= (\bar{a}, \bar{b})],$$

as well as particular constants $\bar{\bar{a}}', \bar{\bar{b}}'$ such that $y' = \sup\{\bar{\bar{a}}', \bar{\bar{b}}'\}$ and

$$(\bar{\bar{a}}', \bar{\bar{b}}') = Z \quad [= (\bar{a}, \bar{b})]$$

hold. The equations involving the ordered pairs imply now with the Equality Criterion for ordered pairs $\bar{\bar{a}} = \bar{a}$ and $\bar{\bar{b}} = \bar{b}$ as well as $\bar{\bar{a}}' = \bar{a}$ and $\bar{\bar{b}}' = \bar{b}$. Then, the preceding equations for y and y' become (after carrying out substitutions) $y = \sup\{\bar{a}, \bar{b}\}$ and $y' = \sup\{\bar{a}, \bar{b}\}$ $[= y]$, so that we obtain $y = y'$. Since y and y' are arbitrary, we may therefore conclude that the uniqueness part also holds, completing the proof of the uniquely existential sentence in (5.7). As Z was also arbitrary, we may further conclude that the universal sentence (5.7) holds. which in turn implies that there exists a unique function $\sqcup$ with domain $X \times X$ such that

$$\forall Z \,(Z \in X \times X \Rightarrow \exists a, b \,(\sqcup(Z) = \sup\{a, b\} \,\wedge\, (a, b) = Z)) \tag{5.8}$$

holds. Let us now verify that $\sqcup$ maps indeed every ordered pair $(a, b) \in X \times X$ to $\sup\{a, b\}$, in the sense of

$$\forall a, b \,((a, b) \in X \times X \Rightarrow \sqcup(a, b) = \sup\{a, b\}). \tag{5.9}$$

Letting a and b be arbitrary and assuming $(a, b) \in X \times X$ to be true, it follows with (5.8) that there are constants, say $\bar{a}$ and $\bar{b}$, with $\sqcup(a, b) = \sup\{\bar{a}, \bar{b}\}$ and $(\bar{a}, \bar{b}) = (a, b)$ are true. The latter equation yields $\bar{a} = a$ and $\bar{b} = b$ with the Equality Criterion for ordered pairs, so that the former equation yields $\sqcup(a, b) = \sup\{a, b\}$ via substitution. Since a and b are arbitrary, we may infer from this the truth of (5.9), which shows that the function $\sqcup$ is indeed characterized by the mapping $(a, b) \mapsto \sup\{a, b\}$. According to Notation 3.2, we may then write for this mapping by $\sqcup$ also $(a, b) \mapsto a \sqcup b$.

It now remains for us to show that X is a codomain of $\sqcup$, i.e. that the range of $\sqcup$ is included in X. To do this, we verify

$$\forall y \,(y \in \operatorname{ran}(\sqcup) \Rightarrow y \in X). \tag{5.10}$$

We take an arbitrary $\bar{y}$ and assume that $\bar{y} \in \operatorname{ran}(\sqcup)$ is true. It then follows with the definition of a range that there is a constant, say $\bar{Z}$, with $(\bar{Z}, \bar{y}) \in \sqcup$. Therefore, we obtain $\bar{Z} \in X \times X$ $[= \operatorname{dom}(\sqcup)]$ with the definition of a domain, so that there are (by definition of the Cartesian product of two sets) constants, say $\bar{a}$ and $\bar{b}$, such that $\bar{a} \in X$, $\bar{b} \in X$ and $(\bar{a}, \bar{b}) = \bar{Z}$ hold. With the latter equation, $(\bar{Z}, \bar{y}) \in \sqcup$ evidently yields $((\bar{a}, \bar{b}), \bar{y}) \in \sqcup$ via substitution, which we may also write as $\bar{y} = \bar{a} \sqcup \bar{b} = \sup\{\bar{a}, \bar{b}\}$. As this supremum is taken with respect to the partial ordering $\leq_X$, it is an element of X according to Note 3.12, so that we obtain the desired $\bar{y} \in X$ with the preceding equations. Since $\bar{y}$ is arbitrary, we may therefore conclude that the universal sentence (5.10) is true, so that $\mathrm{ran}(\sqcup) \subseteq X$ follows to be true by definition of a subset. Thus, $\sqcup$ is a function from $X \times X$ to X, and the proof of the existence of the function (5.5) is complete. As X and $\leq_X$ were arbitrary, we may further conclude that this binary operation exists for any X and any $\leq_X$. $\square$

Exercise 5.1. Establish the existence of the function (5.6).

Notation 5.2. We will also write for a lattice $(X, \leq)$

$$(X, \sqcup, \sqcap, \leq). \tag{5.11}$$

Here, we call $\sqcup$ the *join* and $\sqcap$ the *meet*.

We now apply the preceding proposition to the lattice established in Corollary 3.115.

Proposition 5.3. *For any x it is true that the join with respect to the lattice $(\{x\}, \{(x,x)\})$ is given by*

$$\sqcup = \{((x,x),x)\}. \tag{5.12}$$

Proof. Letting x be arbitrary, we denote the set $\{((x,x),x)\}$ by f and apply (3.39) to obtain the equations

$$f = \{((x,x),x)\} \tag{5.13}$$
$$= \{(x,x)\} \times \{x\} \tag{5.14}$$
$$= (\{x\} \times \{x\}) \times \{x\}. \tag{5.15}$$

Therefore, f is a function from $\{x\} \times \{x\}$ to $\{x\}$ according to Proposition 3.153. Since $\sqcup$ is also a function from $\{x\} \times \{x\}$ to $\{x\}$ according to (5.5), we may apply the Equality Criterion for functions to prove $\sqcup = f$. For this purpose, we verify

$$\forall Z\,(Z \in \{x\} \times \{x\} \Rightarrow \sqcup(Z) = f(Z)). \tag{5.16}$$

We let Z be arbitrary and assume $Z \in \{x\} \times \{x\}$ $[= \{(x,x)\}]$ to be true, which yields $Z = (x,x)$ with (2.169). Because $\sqcup$ is a function (which thus satisfies Corollary 3.150), the preceding equation implies

$$\sqcup(Z) = \sqcup((x,x)) = x \sqcup x = \sup\{x,x\}$$

where we used also (5.5). Now, since the equation $\{x,x\} = \{x\}$ holds by definition of a singleton, we obtain

$$\sqcup(Z) = \sup\{x\} = x \tag{5.17}$$

by applying substitution and then Corollary 3.106 (based on the fact that the lattice $(\{x\},\{(x,x)\})$ is partially ordered). Furthermore, since $((x,x),x) \in \{((x,x),x)\}$ holds because of (2.153), equation (5.13) gives $((x,x),x) \in f$ by substitution, and therefore the previously established equation $Z = (x,x)$ yields $(Z,x) \in f$ again via substitution. We may write this also as $x = f(Z)$, which we may now substitute into (5.17) to arrive at the desired equation $\sqcup(Z) = f(Z)$. As Z was arbitrary, we may infer from this finding the truth of (5.16), so that $\sqcup = \{((x,x),x)\}$ $[= f]$ follows to be true (with the Equality Criterion for functions). Since x was also arbitrary, we may further conclude that the proposition holds. □

Exercise 5.2. Show for any x that the meet with respect to the lattice $(\{x\},\{(x,x)\})$ is given by

$$\sqcap = \{((x,x),x)\}. \tag{5.18}$$

Note 5.2. In view of Notation 5.2 and the equations (5.12) & (5.18), we may write the lattice $(\{x\},\{(x,x)\})$ also as

$$(\{x\},\{((x,x),x)\},\{((x,x),x)\},\{(x,x)\}). \tag{5.19}$$

Proposition 5.4. *For any x, y it is true that the meet with respect to the lattice $(\{x,y\},\{(x,x),(x,y),(y,y)\})$ is given by the quadruple*

$$\sqcap = \{((x,x),x),((x,y),x),((y,x),x),((y,y),y)\}. \tag{5.20}$$

Proof. We let x and y be arbitrary and introduce the notation

$$f = \{((x,x),x),((x,y),x),((y,x),x),((y,y),y)\},$$

so that we obtain the true sentences

$$((x,x),x) \in f \tag{5.21}$$
$$((x,y),x) \in f \tag{5.22}$$
$$((y,x),x) \in f \tag{5.23}$$
$$((y,y),y) \in f \tag{5.24}$$

with (2.244). Let us now observe that

$$(x,x),(x,y),(y,y) \in \{(x,x),(x,y),(y,y)\}$$

holds according to (2.233), where the triple is the total ordering $\leq_X$ of $X = \{x,y\}$ in the given lattice. Therefore, the inequalities $x \leq_X x$, $x \leq_X y$

and $y \leq_Y y$ are true, which then further imply the three corresponding equations

$$\inf\{x, x\} = x, \tag{5.25}$$
$$\inf\{x, y\} = x, \tag{5.26}$$
$$\inf\{y, y\} = y \tag{5.27}$$

with Proposition 3.109. Moreover, since $\{x, y\} = \{y, x\}$ is true according to (2.161), we see that (5.26) yields

$$\inf\{y, x\} = x \tag{5.28}$$

via substitution. By definition of the meet $\sqcap$, we may now write the equations for the four infima as

$$x \sqcap x = x, \tag{5.29}$$
$$x \sqcap y = x, \tag{5.30}$$
$$y \sqcap x = x, \tag{5.31}$$
$$y \sqcap y = y, \tag{5.32}$$

and then in binary relation notation as

$$((x, x), x) \in \sqcap, \tag{5.33}$$
$$((x, y), x) \in \sqcap, \tag{5.34}$$
$$((y, x), x) \in \sqcap, \tag{5.35}$$
$$((y, y), y) \in \sqcap. \tag{5.36}$$

We now apply the Equality Criterion for sets to establish equation (5.20) and prove the universal sentence

$$\forall Z\, (Z \in \sqcap \Leftrightarrow Z \in f). \tag{5.37}$$

We take an arbitrary Z and assume first $Z \in \sqcap$ to be true. Being a function from $\{x, y\} \times \{x, y\}$ to $\{x, y\}$, the binary operation $\sqcap$ is included in $(\{x, y\} \times \{x, y\}) \times \{x, y\}$ according to (3.514), so that the preceding assumption yields $Z \in (\{x, y\} \times \{x, y\}) \times \{x, y\}$ with the definition of a subset. This finding further implies with Exercise 3.4 that there are particular elements $\bar{Y} \in \{x, y\} \times \{x, y\}$ and $\bar{c} \in \{x, y\}$ with $(\bar{Y}, \bar{c}) = Z$. Here, $\bar{Y} \in \{x, y\} \times \{x, y\}$ implies again with Exercise 3.4 that there are particular elements $\bar{a} \in \{x, y\}$ and $\bar{b} \in \{x, y\}$ with $(\bar{a}, \bar{b}) = \bar{Y}$. Applying now substitution based on this equation, the previously established $Z = (\bar{Y}, \bar{c})$ gives $Z = ((\bar{a}, \bar{b}), \bar{c})$, and therefore the assumed $Z \in \sqcap$ yields $((\bar{a}, \bar{b}), \bar{c}) \in \sqcap$, which we may also write in function notation as

$$\begin{aligned} \bar{c} &= \sqcap((\bar{a}, \bar{b})) = \bar{a} \sqcap \bar{b} \\ &= \inf\{\bar{a}, \bar{b}\}, \end{aligned} \tag{5.38}$$

because $\sqcap$ is a function/binary operation according to (5.6). As the previously obtained $\bar{a} \in \{x, y\}$ implies the disjunction $\bar{a} = x \lor \bar{a} = y$ with the definition of a pair, we may now apply a proof by cases to establish the desired sentence $Z \in f$.

In the first case $\bar{a} = x$, we observe that the previously found $\bar{b} \in \{x, y\}$ gives the additional disjunction $\bar{b} = x \lor \bar{b} = y$, which allows us to consider two subcases to prove $Z \in f$. On the one hand, if $\bar{b} = x$ is true, then the equation (5.38) gives in connection with (5.25) $\bar{c} = \inf\{x, x\} = x$ and therefore $\bar{c} = x$. Furthermore, $\bar{a} = x$ and $\bar{b} = x$ imply with the Equality Criterion for ordered pairs $[\bar{Y} =]$ $(\bar{a}, \bar{b}) = (x, x)$, so that $Y = (x, x)$ and $\bar{c} = x$ imply (again with the Equality Criterion for ordered pairs) $[Z =]$ $(\bar{Y}, \bar{c}) = ((x, x), x)$. In view of these equations, we now see that (5.21) implies the desired $Z \in f$ for the first subcase. On the other hand, if $\bar{b} = y$ holds, then (5.38) yields together with (5.26) $\bar{c} = \inf\{x, y\} = x$. Applying now substitutions based on the true equations $\bar{a} = x$, $\bar{b} = y$ and $\bar{c} = x$, we obtain first $[\bar{Y} =]$ $(\bar{a}, \bar{b}) = (x, y)$ and then $[Z =]$ $(\bar{Y}, \bar{c}) = ((x, y), x)$ with the Equality Criterion for ordered pairs. Therefore, (5.22) yields $Z \in f$ also for the second subcase, so that the proof for the first case $(\bar{a} = y)$ is complete.

In the second case $\bar{a} = y$, we may consider the same two subcases and apply essentially the same arguments as in the first case to prove $Z \in f$. On the one hand, if $\bar{b} = x$ holds, then (5.38) and (5.28) imply $\bar{c} = \inf\{y, x\} = x$, thus $\bar{c} = x$. Together with $\bar{a} = y$ and $\bar{b} = x$, this evidently yields $[\bar{Y} =]$ $(\bar{a}, \bar{b}) = (y, x)$ and subsequently $[Z =]$ $(\bar{Y}, \bar{c}) = ((y, x), x)$. Consequently, (5.23) gives $Z \in f$, as desired. On the other hand, if $\bar{b} = y$ is true, we obtain with (5.38) and (5.27) the equations $\bar{c} = \inf\{y, y\} = y$ and therefore $\bar{c} = y$. This finding in turn implies (together with the case/subcase assumptions) $[\bar{Y} =]$ $(\bar{a}, \bar{b}) = (y, y)$ as well as $[Z =]$ $(\bar{Y}, \bar{c}) = ((y, y), y)$. Finally, (5.24) shows that $Z \in f$ is true once again, so that the nested proofs by cases are complete. We therefore have that the first part of the equivalence in (5.37) holds.

Regarding the second part ('$\Leftarrow$') of that equivalence, we assume conversely that $Z \in f$ holds, so that the disjunction

$$[Z = ((x, x), x) \lor Z = ((x, y), x)] \lor [Z = ((y, x), x) \lor Z = ((y, y), y)] \tag{5.39}$$

follows to be true with (2.243), where rearranged the brackets according to the Associative Law for the disjunction. We now use this disjunction to prove $Z \in \sqcap$ by cases. In the first case that $Z \in ((x,x),x) \vee Z \in ((x,y),x)$ is true, we apply another proof by cases based on that disjunction to establish the desired $Z \in \sqcap$. If $Z = ((x,x),x)$ holds, then (5.33) immediately gives $Z \in \sqcap$ via substitution, and if $Z = ((x,y),x)$ holds, then the desired $Z \in \sqcap$ follows from (5.34). In the second case that $Z = ((y,x),x) \vee Z = ((y,y),y)$ is true, we consider two further subcases based on that disjunction. If $Z = ((y,x),x)$ is true, then we evidently obtain $Z \in \sqcap$ from (5.35), and if $Z = ((y,y),y)$ holds, then we obtain $Z \in \sqcap$ from (5.36). Thus, the proof by cases is complete based on (5.39) is complete, so that we found $Z \in \sqcap$ to be true, which in turn proves the second part of the equivalence in (5.37).

Since Z is arbitrary, we may therefore conclude that the universal sentence (5.37) holds, so that the equation (5.20) follows to be true with the Equality Criterion for sets. As x and y were also arbitrary, we may then further conclude that the proposed sentence holds, as claimed. □

Exercise 5.3. Show for any x, y that the join with respect to the lattice $(\{x,y\}, \{(x,x),(x,y),(y,y)\})$ is given by the quadruple

$$\sqcup = \{((x,x),x),((x,y),y),((y,x),y),((y,y),y)\}. \tag{5.40}$$

Note 5.3. In light of the preceding proposition and exercise, we may write for the lattice $(\{x,y\}, \{(x,x),(x,y),(y,y)\})$ also (including the join and the meet)

$$\begin{aligned}&(\{x,y\},\\ &\ \{((x,x),x),((x,y),y),((y,x),y),((y,y),y)\},\\ &\ \{((x,x),x),((x,y),x),((y,x),x),((y,y),y)\},\\ &\ \{(x,x),(x,y),(y,y)\}).\end{aligned} \tag{5.41}$$

When the set that underlies a given lattice is a set of functions Y^X, we will use 'curly' versions of the previously introduced join and meet symbol. The unique existence of the join and meet on such function sets and their 'pointwise' definitions in terms of the join and meet on the codomain Y are immediate consequences of Proposition 5.2 and of Theorem 3.254.

Corollary 5.5. *For any set X and any lattice $(Y, \sqcup, \sqcap, \leq)$ there exist the binary operations*

$$\curlyvee_{Y^X} : Y^X \times Y^X \to Y^X, \quad (f,g) \mapsto f \curlyvee g = \overset{\preceq}{\sup}\{f,g\}, \tag{5.42}$$

$$\curlywedge_{Y^X} : Y^X \times Y^X \to Y^X, \quad (f,g) \mapsto f \curlywedge g = \overset{\preceq}{\inf}\{f,g\}, \tag{5.43}$$

where the functions $f \curlyvee g : X \to Y$ *and* $f \curlywedge g : X \to Y$ *satisfy*

$$\forall x \, (x \in X \Rightarrow (f \curlyvee g)(x) = f(x) \sqcup g(x)), \tag{5.44}$$

$$\forall x \, (x \in X \Rightarrow (f \curlywedge g)(x) = f(x) \sqcap g(x)). \tag{5.45}$$

Notation 5.3. Thus, we write for a lattice $(Y^X, \preceq)$ also

$$(Y^X, \curlyvee, \curlywedge, \preceq). \tag{5.46}$$

Furthermore, we call $\sup^{\preceq}\{f, g\}$ $(= f \curlyvee g)$ the *pointwise supremum* of $\{f, g\}$, and we call $\inf^{\preceq}\{f, g\}$ $(= f \curlywedge g)$ the *pointwise infimum* of $\{f, g\}$, which we then symbolize also by, respectively,

$$\sup\{f, g\} \text{ pointwise} \tag{5.47}$$

and

$$\inf\{f, g\} \text{ pointwise.} \tag{5.48}$$

We now introduce notations for two more types of binary operations, which we will define explicitly later on.

Notation 5.4. We use for a binary operation on X in the form of an *addition* the symbol $+$, and we write for the function value at any $(a, b) \in X \times X$

$$a + b, \tag{5.49}$$

which we call the *sum* of a and b; here, we call a and b the *terms* or the *summands* of the sum. Similarly, we symbolize a binary operation on X in the form of a *multiplication* by $\cdot$, in which case we write the function value at each $(a, b) \in X \times X$ as

$$a \cdot b \quad \text{ or shorter } \quad ab, \tag{5.50}$$

which we then call the *product* of a and b.

Additions and multiplications (and in fact any kind of binary operation) on sets, in a certain sense, induce additions and multiplications on associated sets of functions.

Theorem 5.6 (Pointwise addition of functions). *The following sentences are true for any sets* X, Y *and for any addition* $+_Y$ *on* Y*.*

a) For any functions $f : X \to Y$ *and* $g : X \to Y$ *there exists a unique function* $h : X \to Y$ *satisfying*

$$\forall x \, (x \in X \Rightarrow h(x) = f(x) +_Y g(x)). \tag{5.51}$$

b) *There exists a unique set* $+_{Y^X}$ *such that an element* Z *is in* $+_{Y^X}$ *iff* Z *is in* $(Y^X \times Y^X) \times Y^X$ *and moreover if there are functions* f, g, h *from* X *to* Y *satisfying (5.51) and* $((f,g),h) = Z$*, i.e. such that*

$$\forall Z\,(Z \in +_{Y^X} \Leftrightarrow [Z \in (Y^X \times Y^X) \times Y^X \wedge \exists f, g, h\,(f, g, h \in Y^X \wedge \forall x\,(x \in X \Rightarrow h(x) = f(x) +_Y g(x)) \wedge ((f,g),h) = Z)]). \qquad (5.52)$$

The set $+_{Y^X}$ *is a binary operation on* Y^X *satisfying*

$$\forall f, g, h\,(f, g, h \in Y^X \Rightarrow [h = f +_{Y^X} g \Leftrightarrow \forall x\,(x \in X \Rightarrow h(x) = f(x) +_Y g(x))]). \qquad (5.53)$$

Proof. We let X, Y and $\cdot_Y$ be arbitrary sets and assume that $\cdot_Y$ is a multiplication on Y. Concerning a), letting f and g be arbitrary functions from X to Y, we apply Function definition by replacement and verify for this purpose

$$\forall x\,(x \in X \Rightarrow \exists! y\,(y = f(x) +_Y g(x))), \qquad (5.54)$$

letting x be arbitrary and assuming $x \in X$ to be true. Evidently then, the initial assumptions $f : X \to Y$ and $g : X \to Y$ yield $f(x) \in Y$ as well as $g(x) \in Y$, so that $(f(x), g(x)) \in Y \times Y$ [$= \mathrm{dom}(+_Y)$] holds by definition of the Cartesian product of two sets. Because of $+_Y : Y \times Y \to Y$, the function value $y = f(x) +_Y g(x)$ exists then uniquely (in Y). Since x was arbitrary, we may therefore conclude that the universal sentence (5.54) is true, which implies now the unique existence of a function h with domain X such that (5.51) is satisfied. We already showed that any function value $[h(x) =]\ y = f(x) +_Y g(x)$ is an element of Y, so that we may take Y as a codomain of h. As X, Y, f, g and $+_Y$ are arbitrary, we may then further conclude that the proposed universal sentence a) holds.

We evidently obtain the stated uniquely existential sentence in b) by means of the Axiom of Specification and the Equality Criterion for sets. Thus, the uniquely specified set $+_{Y^X}$ satisfies (5.52). This sentence shows for any Z that $Z \in +_{Y^X}$ implies in particular $Z \in (Y^X \times Y^X) \times Y^X$, so that the inclusion $Z \subseteq (Y^X \times Y^X) \times Y^X$ holds. To prove that $+_{Y^X}$ is a binary operation on Y^X, i.e. to establish $+_{Y^X} : Y^X \times Y^X \to Y^X$, we may therefore apply the Function Criterion and verify accordingly

$$\forall z\,(z \in Y^X \times Y^X \Rightarrow \exists! h\,(h \in Y^X \wedge (z, h) \in +_{Y^X})). \qquad (5.55)$$

For this purpose, we let z be arbitrary and assume $z \in Y^X \times Y^X$ to be true. Consequently, there are evidently particular elements $\bar{f} \in Y^X$ and

$\bar{g} \in Y^X$ with $(\bar{f}, \bar{g}) = z$. Because of a), there exists then a (unique) function $\bar{h} : X \to Y$ (i.e., a unique element $\bar{h} \in Y^X$) with

$$\forall x\,(x \in X \Rightarrow \bar{h}(x) = \bar{f}(x) +_Y \bar{g}(x)). \tag{5.56}$$

Defining now the ordered triple $\bar{Z} = ((\bar{f}, \bar{g}), \bar{h})$ $[= (z, \bar{h})]$, we thus see that $\bar{f}, \bar{g}, \bar{h} \in Y^X$ and $\bar{Z} \in (Y^X \times Y^X) \times Y^X$ hold; together with (5.56), these findings give $[(z, \bar{h}) =]$ $\bar{Z} \in +_{Y^X}$ with (5.52). Since $\bar{h}$ satisfies both $\bar{h} \in Y^X$ and $(z, \bar{h}) \in +_{Y^X}$, the existential part in (5.55) holds. To establish the uniqueness part, we take arbitrary sets h and h', assume $h, h' \in Y^X$ and $(z, h), (z, h') \in +_{Y^X}$ to be both true, and show that $h = h'$ is implied. Since the former assumption implies that h and h' are both functions from X to Y, we may prove the preceding equation by means of the Equality Criterion for functions, i.e. by verifying

$$\forall x\,(x \in X \Rightarrow h(x) = h'(x)). \tag{5.57}$$

The other assumption $(z, h), (z, h') \in +_{Y^X}$ implies with (5.52) the existence of particular elements $\bar{f}, \bar{g}, \bar{h}$ and $\bar{f}', \bar{g}', \bar{h}'$ in Y^X satisfying

$$\forall x\,(x \in X \Rightarrow \bar{h}(x) = \bar{f}(x) +_Y \bar{g}(x)) \tag{5.58}$$

and $((\bar{f}, \bar{g}), \bar{h}) = (z, h)$, as well as

$$\forall x\,(x \in X \Rightarrow \bar{h}'(x) = \bar{f}'(x) +_Y \bar{g}'(x)) \tag{5.59}$$

and $((\bar{f}', \bar{g}'), \bar{h}') = (z, h')$. Then, the Equality Criterion for ordered pairs yields first the conjunction of $(\bar{f}, \bar{g}) = z$ and $\bar{h} = h$ as well as the conjunction of $(\bar{f}', \bar{g}') = z$ and $\bar{h}' = h'$; thus, substitution gives $(\bar{f}, \bar{g}) = (\bar{f}', \bar{g}')$, so that we obtain (again with the Equality Criterion for ordered pairs) $\bar{f} = \bar{f}'$ as well as $\bar{g} = \bar{g}'$. We are now in a position to verify the universal sentence (5.57), letting x be arbitrary in X, so that the previous two equations, alongside the previously established equations $\bar{h} = h$ and $\bar{h}' = h'$, imply with the Equality Criterion for functions $\bar{f}(x) = \bar{f}'(x)$, $\bar{g}(x) = \bar{g}'(x)$, $\bar{h}(x) = h(x)$ and $\bar{h}'(x) = h'(x)$. We therefore obtain with (5.58) and (5.59) via substitutions the equations

$$h(x) = \bar{h}(x) = \bar{f}(x) +_Y \bar{g}(x) = \bar{f}'(x) +_Y \bar{g}'(x) = \bar{h}'(x) = h'(x).$$

Since x is arbitrary, we may now conclude that the universal sentence (5.57) is true, so that $h = h'$ holds indeed. We thus proved the uniquely existential sentence in (5.55), and as z was arbitrary, we may infer from this the truth also of the universal sentence (5.55). This finding in turn establishes $+_{Y^X}$ as a function from $Y^X \times Y^X$ to Y^X, and thus as a binary operation on Y^X.

It now remains for us to prove that $+_{Y^X}$ satisfies (5.53). To do this, we take arbitrary $f, g, h \in Y^X$ and assume first $h = f +_{Y^X} g$ to be true, which we may write as $((f, g), h) \in +_{Y^X}$. Consequently, the universal sentence

$$\forall x\,(x \in X \Rightarrow h(x) = f(x) +_Y g(x))$$

follows to be true with (5.52). We now conversely assume the preceding universal sentence to be true and define the ordered triple $Z = ((f, g), h)$. Furthermore, because of the assumption $f, g, h \in Y^X$, we evidently obtain $Z \in (Y^X \times Y^X) \times Y^X$ by applying (twice) the definition of the Cartesian product of two sets. Clearly, these findings imply $((f, g), h) \in +_{Y^X}$ with (5.52), which we may write as $h = f +_{Y^X} g$. Thus, the proof of the equivalence in (5.53) is complete, and since f, g, h were arbitrary, we may therefore conclude that $+_{Y^X}$ satisfies indeed (5.53). This completes the proof of b); as X, Y and $+_Y$ were initially arbitrary sets, we may now finally conclude that the theorem holds, as claimed. $\square$

As a first application of the preceding proposition, we consider the addition of matrices.

Exercise 5.4 (Addition of matrices). Show for any positive natural numbers m, n, any set Y and any addition $+_Y$ on Y that the addition $+_{Y^{m \times n}}$ in the sense of (5.52) exists uniquely and satisfies

$$\begin{aligned}&\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\,(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in Y^{m \times n} \Rightarrow [\boldsymbol{C} = \boldsymbol{A} +_{Y^{m \times n}} \boldsymbol{B} \\ &\quad\Leftrightarrow \forall i, j\,([i \in \{1, \ldots, m\} \wedge j \in \{1, \ldots, n\}] \Rightarrow c_{i,j} = a_{i,j} +_Y b_{i,j})]).\end{aligned} \tag{5.60}$$

(Hint: Use Notation 4.10, Definition 3.4, Definition 4.19, and Exercise 3.4.)

Notation 5.5. The addition of two given matrices can be expressed conveniently by

$$\begin{aligned}\boldsymbol{A} +_{Y^{m \times n}} \boldsymbol{B} &= \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} +_{Y^{m \times n}} \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{m,1} & \cdots & b_{m,n} \end{bmatrix} \\ &= \begin{bmatrix} a_{1,1} +_Y b_{1,1} & \cdots & a_{1,n} +_Y b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} +_Y b_{m,1} & \cdots & a_{m,n} +_Y b_{m,n} \end{bmatrix}.\end{aligned} \tag{5.61}$$

Multiplications on sets of functions can be established in analogy to additions.

Theorem 5.7 (Pointwise multiplication of functions). *following sentences for any sets X, Y and for any multiplication $\cdot_Y$ on Y.*

a) For any functions $f : X \to Y$ and $g : X \to Y$ there exists a unique function $h : X \to Y$ satisfying

$$\forall x\, (x \in X \Rightarrow h(x) = f(x) \cdot_Y g(x)). \tag{5.62}$$

b) There exists a unique set $\cdot_{Y^X}$ such that an element Z is in $\cdot_{Y^X}$ iff Z is in $(Y^X \times Y^X) \times Y^X$ and moreover if there are functions f, g, h from X to Y satisfying (5.62) and $((f, g), h) = Z$, i.e. such that

$$\forall Z\, (Z \in \cdot_{Y^X} \Leftrightarrow [Z \in (Y^X \times Y^X) \times Y^X \wedge \exists f, g, h\, (f, g, h \in Y^X \wedge \\ \forall x\, (x \in X \Rightarrow h(x) = f(x) \cdot_Y g(x)) \wedge ((f, g), h) = Z)]). \tag{5.63}$$

The set $\cdot_{Y^X}$ is a binary operation on Y^X satisfying

$$\forall f, g, h\, (f, g, h \in Y^X \\ \Rightarrow [h = f \cdot_{Y^X} g \Leftrightarrow \forall x\, (x \in X \Rightarrow h(x) = f(x) \cdot_Y g(x))]). \tag{5.64}$$

Exercise 5.5. Establish the Pointwise multiplication of functions.

Theorem 5.8 (Addition of sets). *The following sentences are true for any set X and for any addition $+$ on X:*

a) For any sets $A \subseteq X$ and $B \subseteq X$ there exists a unique set $A \oplus B$ satisfying

$$\forall s\, (s \in A \oplus B \Leftrightarrow \exists a, b\, (a \in A \wedge b \in B \wedge s = a + b)), \tag{5.65}$$

and this set is a subset of X.

b) There exists a unique function $\bigoplus$ on $\mathcal{P}(X) \times \mathcal{P}(X)$ such that

$$\forall x\, (x \in \mathcal{P}(X) \times \mathcal{P}(X) \Rightarrow \exists A, B\, (A \subseteq X \wedge B \subseteq X \wedge (A, B) = x \\ \wedge \bigoplus(x) = A \oplus B)), \tag{5.66}$$

and this function is a binary operation on $\mathcal{P}(X)$.

Proof. We take an arbitrary set X and an arbitrary addition $+$ on X. Concerning a), we may evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of a set $A \oplus B$ such that

$$\forall s\, (s \in A \oplus B \Leftrightarrow [s \in X \wedge \exists a, b\, (a \in A \wedge b \in B \wedge s = a + b)]). \tag{5.67}$$

Letting now s be arbitrary, we see that $s \in A \oplus B$ implies especially the truth of the existential sentence, so that the first part ('$\Rightarrow$') of the equivalence in (5.65) is satisfied. Conversely, that existential sentence implies the truth of $s = \bar{a} + \bar{b}$ for some particular elements $\bar{a} \in A$ and $\bar{b} \in B$. Since $+$ is a binary operation on X, the value s of that addition clearly constitutes an element of X. This finding $s \in X$ implies now in conjunction with the assumed existential sentence, by virtue of (5.67), that $y \in A \oplus B$ holds. This establishes the second part ('$\Leftarrow$') of the equivalence in (5.65), and since s is arbitrary, we may therefore conclude that $A \oplus B$ satisfies indeed (5.65). Since $s \in A \oplus B$ implies $s \in X$ for an arbitrary s, the inclusion $A \oplus B \subseteq X$ follows to be true by definition of a subset. As A and B were arbitrary, we may now conclude that Part a) of the stated theorem holds.

To establish b), we first apply Function definition by replacement and prove

$$\forall x\,(x \in \mathcal{P}(X) \times \mathcal{P}(X) \Rightarrow \exists! y\,(\exists A, B\,(A \subseteq X \;\wedge\; B \subseteq X \;\wedge\; (A,B) = x \\ \wedge\; y = A \oplus B))). \tag{5.68}$$

Letting $x \in \mathcal{P}(X) \times \mathcal{P}(X)$ be arbitrary, there exist then particular sets $\bar{A}, \bar{B} \in \mathcal{P}(X)$ with $(\bar{A}, \bar{B}) = x$ in view of Exercise 3.4. By definition of a power set, $\bar{A}$ and $\bar{B}$ are then both subsets of X. Consequently, the set $\bar{y} = \bar{A} \oplus \bar{B}$ is uniquely specified by a). These findings demonstrate the truth of the existential sentence

$$\exists A, B\,(A \subseteq X \;\wedge\; B \subseteq X \;\wedge\; \bar{y} = A \oplus B),$$

which is satisfied by the specific set $\bar{A} \oplus \bar{B}$, so that the existential sentence with respect to y in (5.68) also holds. Regarding the uniqueness part, we now take arbitrary sets y and y' satisfying the corresponding existential sentences

$$\exists A, B\,(A \subseteq X \;\wedge\; B \subseteq X \;\wedge\; (A,B) = x \;\wedge\; y = A \oplus B),$$
$$\exists A, B\,(A \subseteq X \;\wedge\; B \subseteq X \;\wedge\; (A,B) = x \;\wedge\; y' = A \oplus B).$$

We therefore find $y = \bar{A} \oplus \bar{B}$ and $y' = \bar{A}' \oplus \bar{B}'$ for some particular subsets $\bar{A}, \bar{A}', \bar{B}, \bar{B}'$ of X satisfying $(\bar{A}, \bar{B}) = x = (\bar{A}', \bar{B}')$. The latter equation yields with the Equality Criterion for ordered pairs $\bar{A} = \bar{A}'$ as well as $\bar{B} = \bar{B}'$, so that the previous equation for y becomes after substitutions $y = \bar{A}' \oplus \bar{B}'$ $[= y']$; thus, $y = y'$. Since y and y' were arbitrary, we may therefore conclude that the uniquely existential sentence in (5.66) holds. As x was also arbitrary, we may infer from this the truth of the universal sentence (5.66), and therefore the existence of a unique function $\bigoplus$ on $\mathcal{P}(X) \times \mathcal{P}(X)$ satisfying (5.66). To prove that this function constitutes a binary operation on $\mathcal{P}(X)$, we need to show that this power set is a codomain of the function, i.e., that the inclusion $\mathrm{ran}(\bigoplus) \subseteq \mathcal{P}(X)$ holds. Letting for this purpose $y \in \mathrm{ran}(\bigoplus)$ be arbitrary, it follows by definition of a range that there exists a particular set x with $(x, y) \in \mathrm{ran}(\bigoplus)$. Since $\bigoplus$ is a function, we may write this finding also as $y = \bigoplus(x)$, and we also see in light of the definition of a domain that $x \in \mathcal{P}(X) \times \mathcal{P}(X)$ $[= \mathrm{dom}(\bigoplus)]$ is true. By virtue of (5.66), we then have $(\bar{A}, \bar{B}) = x$ and $[y =]$ $\bigoplus(x) = \bar{A} \oplus \bar{B}$ for particular subsets $\bar{A}$ and $\bar{B}$ of X. Since $\bar{A} \oplus \bar{B}$ is a subset of X, as shown in a), we find $y \subseteq X$ through substitution, and this clearly means that y is an element of the power set of X. As y was arbitrary, we may therefore conclude that the inclusion $\mathrm{ran}(\bigoplus) \subseteq \mathcal{P}(X)$ holds indeed. We thus showed that $\bigoplus$ is a binary operation on $\mathcal{P}(X)$, so that the proof of Part b) is now complete.

As X and $+$ were initially arbitrary, we conclude that the stated theorem is true. □

Exercise 5.6. Establish the truth of the following universal sentence for any set X and any addition $+$ on X:

$$\forall A, B\,([A \subseteq X \;\wedge\; B \subseteq X] \Rightarrow \bigoplus((A, B)) = A \oplus B). \tag{5.69}$$

(Hint: Use (5.66) in connection with the Equality Criterion for ordered pairs.)

Notation 5.6. On the one hand, since $\bigoplus$ is a binary operation, we may write $\bigoplus((A, B)) = A \bigoplus B$. In view of the preceding exercise, we may therefore also write $A \oplus B$ instead of $A \bigoplus B$ for the value of that binary operation.

Exercise 5.7. Show for any set X that there exists a unique function $\circ_X$ on the set X^X of transformations on X such that

$$\forall z\,(z \in X^X \times X^X \Rightarrow \exists g, h\,(g, h \in X^X \wedge (g, h) = x \wedge \circ_X(z) = g \circ h)) \tag{5.70}$$

and that this function constitutes a binary operation on X^X. Establish in addition the truth of

$$\forall g, h\,(g, h \in X^X \Rightarrow g \circ_X h = g \circ h). \tag{5.71}$$

(Hint: Use the fact (3.604).)

Proposition 5.9. *It is true for any set $T(X)$ of invertible transformations on any set X that the restriction*

$$\circ_{T(X)} = \circ_X \restriction T(X) \times T(X) \tag{5.72}$$

of the composition on the set X^X of transformations on X constitutes a restricted binary operation.

Proof. We let X be an arbitrary set and $T(X)$ an arbitrary set of invertible transformations on X. Thus, $T(X)$ is a subset of the set X^X of transformations on X (see Note 3.27). Consequently, the Cartesian product $T(X) \times T(X)$ is a subset of the Cartesian product $X^X \times X^X$ because of Proposition 3.8. Therefore, the restriction (5.72) of $\circ_X : X^X \times X^X \to X^X$ to $T(X) \times T(X)$ is a function from $T(X) \times T(X)$ to X^X due to (3.566). It remains for us to prove that $T(X)$ is also a codomain of that restriction, i.e., that the range of $\circ_{T(X)}$ is a subset of $T(X)$. For this purpose, we apply the definition of a subset and let accordingly $y \in \mathrm{ran}(\circ_{T(X)})$ be arbitrary. By definition of a range, there exists then a particular constant z such that $(z, y) \in \circ_{T(X)}$. Since $\circ_{T(X)}$ is a function with domain $T(X) \times T(X)$, we may on the one hand write $y = \circ_{T(X)}(z)$; on the other hand, we find that z is an element of that domain (by definition). These findings imply now on the one hand $[\circ_{T(X)}(z) =]\ y = \circ_X(z)$ with (3.567). On the other hand, $z \in T(X) \times T(X)$ implies by virtue of Exercise 3.4 that there exist particular elements $g, h \in T(X)$ with $(g, h) = z$. This allows us to write $y = \circ_X((g, h)) = g \circ_X h = g \circ h$ by using the fact that $\circ_X$ is a binary operation satisfying (5.71). Furthermore, $g, h \in T(X)$ yields $g : X \rightleftarrows X$ and $h : X \rightleftarrows X$ by means of (3.685). Due to the Bijectivity of the composition of two bijections (3.672), we obtain now $[y =]\ g \circ h : X \rightleftarrows X$, so that $y \in T(X)$ is clearly true. As y was arbitrary, we may therefore conclude that the range of $\circ_{T(X)}$ is indeed a subset of $T(X)$, i.e., that $T(X)$ is a codomain of the restriction (5.72). This means that this restriction, being a function on $T(X) \times T(X)$, is a binary operation on $T(X)$. Thus, $\circ_{T(X)}$ constitutes a restricted binary operation on $T(X)$, by definition. Since X and $T(X)$ were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Corollary 5.10. *It is true for any finite set X that the restriction*

$$\circ_{\Pi(X)} = \circ_X \restriction \Pi(X) \times \Pi(X) \tag{5.73}$$

of the composition on the set X^X of transformations on X constitutes a restricted binary operation on the set $\Pi(X)$ of permutations of X.

Corollary 5.11. *The composition of two invertible transformations g and h in any set $T(X)$ of invertible transformations on any set X is identical with the value of the binary composition operation on $T(X)$ associated with g and h, that is,*

$$\forall g, h\, (g, h \in T(X) \Rightarrow g \circ_{T(X)} h = g \circ h). \tag{5.74}$$

Proof. We let X be an arbitrary set, $T(X)$ an arbitrary set of invertible transformations on X, g, h arbitrary invertible transformations on X, and

x an arbitrary element of X. We first observe that $g \circ_{T(X)} h$ is an element of $T(X)$ since $\circ_{T(X)}$ is a binary operation on $T(X)$ (see Proposition 5.9). This means that $g \circ_{T(X)} h$ is an invertible transformation on X and thus a (bijective) function on X by definition. Furthermore, $g \circ h$ is also a function on X due to Proposition 3.178, using the fact that the invertible transformations g and h on X are themselves (bijective) functions on X. We now let $x \in X$ be arbitrary and observe the truth of the equations

$$\begin{aligned} g \circ_{T(X)} h(x) &= g \circ_X h \restriction T(X) \times T(X)(x) \\ &= g \circ_X h(x) \\ &= g \circ h(x) \end{aligned}$$

in light of (5.72), (3.567) and (5.71). Since x was arbitrary, we may now infer from the truth of the resulting equation $g \circ_{T(X)} h(x) = g \circ h(x)$ the truth of $g \circ_{T(X)} h = g \circ h$ by virtue of the Equality Criterion for functions. As X, $T(X)$, g and h were also arbitrary, we may therefore conclude that the stated universal sentence holds. □

Note 5.4. Since the set of permutations of a (finite) set X constitutes a set of invertible transformations, we may write in particular

$$\forall g, h \, (g, h \in \Pi(X) \Rightarrow g \circ_{\Pi(X)} h = g \circ h). \tag{5.75}$$

Exercise 5.8. Show for any function $f : X \to Y$ and any set $T_{fi}(X)$ under which f is invariant that the restriction

$$\circ_{T_{fi}(X)} = \circ_X \restriction T_{fi}(X) \times T_{fi}(X) \tag{5.76}$$

of the composition on the set X^X of transformations on X constitutes a restricted binary operation on $T_{fi}(X)$. Establish then the truth of

$$\forall g, h \, (g, h \in T_{fi}(X) \Rightarrow g \circ_{T_{fi}(X)} h = g \circ h). \tag{5.77}$$

(Hint: Proceed in analogy to the proofs of Proposition 5.9 and Corollary 5.11, using Proposition 3.183.)

Definition 5.3 (Translation, scaling). We say for any set X, any element $a \in X$ and

a) any addition $+$ on X that a transformation on X is a *translation* (*on X by a*), symbolically

$$g_{La} : X \to X, \tag{5.78}$$

iff every element of its domain is associated with the sum of x and a, i.e.

$$\forall x \, (x \in X \Rightarrow f(x) = x + a). \tag{5.79}$$

b) any multiplication $\cdot$ on X that a transformation on X is a *scaling* (*on X by a*), symbolically
$$g_{Sa} : X \to X, \tag{5.80}$$
iff every element of its domain is associated with the product of a and x, i.e.
$$\forall x\, (x \in X \Rightarrow f(x) = a \cdot x). \tag{5.81}$$

Exercise 5.9. Show for any set X, any element $a \in X$ and

a) any addition $+$ on X that there exists a unique function g_{La} on X with the definite property (5.79), and that this function is a translation on X by a.

b) any multiplication $\cdot$ on X that there exists a unique function g_{Sa} on X with the definite property (5.81), and that this function is a scaling on X by a.

(Hint: Apply Function definition by replacement and show then that the defined functions constitute transformations on X.)

Note 5.5. For any set X, any addition $+$ and any multiplication $\cdot$ on X, we may evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of sets $G_L(X)$ and $G_S(X)$ consisting, respectively, of all translations and all scalings on X, in the sense that
$$\forall f\, (f \in G_L(X) \Leftrightarrow [f \in X^X \wedge \exists a\, (a \in X \wedge \forall x\, (x \in X \Rightarrow f(x) = x + a))]), \tag{5.82}$$
$$\forall f\, (f \in G_S(X) \Leftrightarrow [f \in X^X \wedge \exists a\, (a \in X \wedge \forall x\, (x \in X \Rightarrow f(x) = a \cdot x))]). \tag{5.83}$$

Since $f \in G_L(X)$ and $f \in G_S(X)$ both imply $f \in X^X$ for an arbitrary f, we see in light of the definition of subset that the inclusions $G_L(X) \subseteq X^X$ and $G_S(X) \subseteq X^X$ hold.

Definition 5.4 (Set of translations, set of scalings). For any set X

a) and any addition $+$ on X, we call
$$G_L(X) \tag{5.84}$$
the *set of translations* on X.

b) and any multiplication $\cdot$ on X, we call
$$G_S(X) \tag{5.85}$$
the *set of scalings* on X.

Proposition 5.12. *For any set X and any binary operation $*$ on X it is true that, if there exists an element e in X such that $e*a=a$ and $a*e=a$ hold for any $a\in X$, then e is unique, that is,*

$$\forall X,*\,([*\in X^{X\times X}\ \wedge\ \exists e\,(e\in X\ \wedge\ \forall a\,(a\in X\Rightarrow[e*a=a\ \wedge\ a*e=a]))]$$
$$\Rightarrow\exists!e\,(e\in X\ \wedge\ \forall a\,(a\in X\Rightarrow[e*a=a\ \wedge\ a*e=a]))). \tag{5.86}$$

Proof. We let X and $*$ be arbitrary sets, assume that $*$ is a binary operation on X, and assume moreover that there is a constant, say $\bar{e}$, such that the conjunction of $\bar{e}\in X$ and

$$\forall a\,(a\in X\Rightarrow[\bar{e}*a=a\ \wedge\ a*\bar{e}=a]) \tag{5.87}$$

is true. Let us now apply Method 1.18 to establish the uniquely existential sentence. Since $\bar{e}$ is a particular constant which satisfies the conjunction within the uniquely existential sentence, it only remains for us to verify

$$\forall e'\,([e'\in X\ \wedge\ \forall a\,(a\in X\Rightarrow[e'*a=a\ \wedge\ a*e'=a])]\Rightarrow\bar{e}=e'). \tag{5.88}$$

To prove this universal sentence, we let e' be arbitrary, assume the conjunction of $e'\in X$ and

$$\forall a\,(a\in X\Rightarrow[e'*a=a\ \wedge\ a*e'=a]), \tag{5.89}$$

to be true, and show that $\bar{e}=e'$ is implied. On the one hand, the previously established $\bar{e}\in X$ implies with (5.89)

$$e'*\bar{e}=\bar{e}\ \wedge\ \bar{e}*e'=\bar{e}.$$

On the other hand, the assumed $e'\in X$ implies with (5.87)

$$\bar{e}*e'=e'\ \wedge\ e'*\bar{e}=e'.$$

Thus, the two equations

$$\bar{e}=e'*\bar{e}=e'$$

hold in particular, so that $\bar{e}=e'$ follows indeed to be true. Since e' is arbitrary, we may therefore conclude that the universal sentence (5.88) holds, completing the proof of the uniquely existential sentence in (5.86). This in turn proves the implication in (5.86), and as X and $*$ were initially arbitrary sets, we may finally conclude that the proposed universal sentence is true. □

Definition 5.5 (Identity element, neutral element). For any set X and any binary operation $*$ on X we call the unique element $e\in X$ that satisfies

$$\forall a\,(a\in X\Rightarrow[e*a=a\ \wedge\ a*e=a]) \tag{5.90}$$

the *identity element* or the *neutral element* of X with respect to $*$.

Note 5.6. In view of the Neutrality of identity functions under composition in connection with (5.71), we obtain for any set X from (3.610) the universal sentence

$$\forall f\,(f \in X^X \Rightarrow [\mathrm{id}_X \circ_X f = f \,\wedge\, f \circ_X \mathrm{id}_X = f]), \tag{5.91}$$

which demonstrates that the identity function on X constitutes the identity element of the set of transformations on X with respect to $\circ_X$.

Proposition 5.13. *It is true for any set X, any binary operation $*$ on X and any set $A \subseteq X$ such that the restriction $*_A$ of $*$ to $A \times A$ is a binary operation (on A) that a constant $e \in A$ is the identity element of A with respect to $*_A$ if e is the identity element of X with respect to $*$.*

Proof. We let X be an arbitrary set, $*$ an arbitrary binary operation on X, A an arbitrary subset of X, and e an arbitrary element of A. We now assume that the restriction $*_A = * \restriction A \times A$ is a binary operation on A, and furthermore that e is the identity element of X with respect to $*$. To prove that e is the identity element of A with respect to $*_A$, we establish

$$\forall a\,(a \in A \Rightarrow [e *_A a \,\wedge\, a *_A e = a]). \tag{5.92}$$

Letting $a \in A$ be arbitrary, we find

$$\begin{aligned} e *_A a &= e * a = a, \\ a *_A e &= a * e = a \end{aligned}$$

by means of (5.4) and the assumption that e is the identity element of X with respect to $*$. We thus obtained $e *_A a = a$ and $a *_A e = a$. As a was arbitrary, we may infer from these findings the truth of the universal sentence (5.92), so that e is by definition the identity element of A with respect to $*_A$. As X, $*$, e were initially arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Note 5.7. The facts

- that the set $\mathcal{T}(X)$ of invertible transformations on a set X is a subset of the set X^X of transformations on X (see Note 3.27),
- that the identity function id_X on any set X is an invertible transformation on that set (see Note 3.26) and thus an element of $\mathcal{T}(X)$, and
- that id_X is the identity element of X^X with respect to the binary composition operation $\circ_X$ on X^X (see Note 5.6)

immediately imply with Proposition 5.13 that id_X is also the identity element of $\mathcal{T}(X)$ with respect to the restricted binary composition operation $\circ_{\mathcal{T}(X)}$ on $\mathcal{T}(X)$ established in Proposition 5.9. In view of Proposition 5.13, id_X is then also the identity element of any subset $T(X) \subseteq \mathcal{T}(X)$ containing id_X with respect to the restricted binary operation $\circ_{T(X)}$.

Note 5.8. In particular, when X is any finite set, then id_X is the identity element of the set $\Pi(X)$ of permutations of X with respect to the restricted binary composition operation $\circ_{\Pi(X)}$ on $\Pi(X)$.

Note 5.9. In analogy to Note 5.7, the facts

- that the set $\mathcal{T}_{fi}(X)$ (of invertible transformations on a set X under which a function $f : X \to Y$ is invariant) is included in the set X^X of transformations on X (see Note 3.28),
- that such a function $f : X \to Y$ is invariant under the identity function id_X (see Note 3.22) and thus an element of $\mathcal{T}_{fi}(X)$, and
- that id_X is the identity element of X^X with respect to the binary composition operation $\circ_X$ on X^X

immediately imply with Proposition 5.13 that id_X is also the identity element of $\mathcal{T}_{fi}(X)$ with respect to the restricted binary composition operation $\circ_{\mathcal{T}_{fi}(X)}$ on $\mathcal{T}_{fi}(X)$ established in Exercise 5.8. Consequently, id_X is also the identity element of any subset $T_{fi}(X) \subseteq \mathcal{T}_{fi}(X)$ containing id_X with respect to the restricted binary operation $\circ_{T_{fi}(X)}$ (by Proposition 5.13).

Proposition 5.14. *For any lattice* $(X, \sqcup, \sqcap, \leq)$ *such that the bottom element* $\bot$ *of* X *exists, it is true that* $\bot$ *is the neutral element of* X *with respect to the join* $\sqcup$.

Proof. We let X, $\sqcup$, $\sqcap$ and $\leq$ be arbitrary sets, assume that $(X, \sqcup, \sqcap, \leq)$ is a lattice, and assume also that the least element $\bot$ of X exists. To prove that $\bot$ is the neutral element of X with respect to $\sqcup$, we verify

$$\forall a\,(a \in X \Rightarrow [\bot \sqcup\ a = a \ \wedge\ a \sqcup \bot = a]). \tag{5.93}$$

For this purpose, we take an arbitrary a in X and verify first

$$[\bot \sqcup\ a =]\quad \sup\{\bot, a\} = a \tag{5.94}$$

(where we used (5.5)). Since the least element $\bot$ of X is in particular a lower bound for X, it is true that

$$\forall b\,(b \in X \Rightarrow \bot \leq b).$$

Thus, the assumed $a \in X$ implies $\bot \leq a$, which inequality further implies with Proposition 3.109 that (5.94) is true. We thus proved the first part of the conjunction in (5.93). Regarding the second part, we observe that $\{a, \bot\} = \{\bot, a\}$ holds according to (2.161), so that (5.94) yields via substitution

$$[a \sqcup \bot =] \quad \sup\{a, \bot\} = a,$$

which equations give the desired $a \sqcup \bot = a$. Since a is arbitrary, we may therefore conclude that (5.93) is true, which shows that $\bot$ is indeed the neutral element of X with respect to $\sqcup$. As X, $\sqcup$, $\sqcap$ and $\leq$ were arbitrary, the proposed universal sentence follows to be true. □

Exercise 5.10. Show for any lattice $(X, \sqcup, \sqcap, \leq)$ for which the top element $\top$ of X exists that $\top$ is the neutral element of X with respect to the meet $\sqcap$.

The preceding proposition and exercise apply immediately to the lattice (5.19) because the bottom and the top element of the singleton underlying this lattice exist according to (3.289).

Corollary 5.15. *For any x it is true that x is the neutral element of $\{x\}$ with respect to both the join and the meet of the lattice $(\{x\}, \{(x,x)\})$.*

We may apply Proposition 5.14 also to the lattice (5.41).

Corollary 5.16. *For any x and any y satisfying $x \neq y$ it is true that*

a) x is the neutral element of $\{x, y\}$ with respect to the join and

b) y is the neutral element of $\{x, y\}$ with respect to the meet

of the lattice $(\{x, y\}, \{(x, x), (x, y), (y, y)\})$.

Proof. Letting x and y be arbitrary, (5.26) and the corresponding equation in Exercise 5.3 show that $\inf\{x, y\} = x$ and $\sup\{x, y\} = y$ hold. Then, Exercise 3.52 and Proposition 3.113 yield $\min\{x, y\} = \inf\{x, y\}$ $[= x]$ and $\max\{x, y\} = \sup\{x, y\}$ $[= y]$, so that x is the bottom element and y the top element of $\{x, y\}$. Consequently, x is the neutral element of $\{x, y\}$ with respect to the join and y the neutral element of $\{x, y\}$ with respect to the meet of the given lattice, according to Proposition 5.14 and Exercise 5.10. Since x and y were arbitrary, we may therefore conclude that the corollary is true. □

Proposition 5.17. *For any counting domain $(C, s, 0_C)$ it is true that 0_C is the neutral element for the join $\sqcup_C$ with respect to the lattice $(C, <_C)$.*

Proof. Letting C, s and 0_C be arbitrary such that $(C, s, 0_C)$ is a counting domain, we have in view of Proposition 4.28 that C – together with the standard linear ordering $<_C$ of C – forms the lattice $(C, <_C)$, including the join $\sqcup_C$. Since 0_C is the least element of C due to Proposition 4.36a), it follows with Proposition 5.14 that 0_C is the neutral element of C with respect to $\sqcup_C$. As C, s and 0_C are arbitrary, we may therefore conclude that the proposition holds. □

Corollary 5.18. *It is true that* 0 *is the neutral element for the join* $\sqcup_{\mathbb{N}}$ *with respect to the lattice* $(\mathbb{N}, <_{\mathbb{N}})$.

We now fix common notations for neutral elements with respect to addition and multiplication.

Notation 5.7. We write for the identity element of a set X with respect to an addition $+$ on X also

$$0_X, \tag{5.95}$$

in which case we speak of the *neutral element for addition* or of the *zero element* of X. Similarly, we write for the identity element of a set X with respect to a multiplication $\cdot$ on X

$$1_X, \tag{5.96}$$

and we call this the *neutral element for multiplication* or the *unity element* of X.

Proposition 5.19. *For any sets* X, Y *and any addition* $+_Y$ *such that the zero element* 0_Y *of* Y *exists, it is true that the set* $X \times \{0_Y\}$ *is*

a) a constant function from X *to* Y *with value* 0_Y

b) and furthermore the zero element of Y^X *with respect to the pointwise addition* $+_{Y^X}$ *of functions.*

Proof. Letting X, Y and $+_Y$, assuming the latter set to be an addition on Y, and assuming the zero element 0_Y to exist, we see that the Cartesian product $g_{0_Y} = X \times \{0_Y\}$ is the constant function on X with value 0_Y. Since $0_Y \in Y$ holds by definition of a neutral element, so that $\{0_Y\} \subseteq Y$ follows to be true with (2.184), we may choose Y as a codomain of the constant function g_{0_Y}. We thus proved a). Furthermore, we see that g_{0_Y} is an element of the set Y^X of functions from X to Y. To prove that g_{0_Y} is the zero element of Y^X, we verify

$$\forall f\,(f \in Y^X \Rightarrow [g_{0_Y} +_{Y^X} f = f \,\wedge\, f +_{Y^X} g_{0_Y} = f]), \tag{5.97}$$

letting f be arbitrary and assuming $f \in Y^X$ to hold. Let us observe here in light of (5.53) that the first of the two equations in (5.97) is equivalent to the universal sentence

$$\forall x\,(x \in X \Rightarrow g_{0_Y}(x) +_Y f(x) = f(x)), \tag{5.98}$$

which we now prove by letting x be arbitrary in X. Since $g_{0_Y}(x) = 0_Y$ holds according to (3.534), where 0_Y is the zero element of Y, we obtain the true equation $[g_{0_Y}(x) =]\ 0_Y +_Y f(x) = f(x)$. As x is arbitrary, we may therefore conclude that the universal sentence (5.98) is true, so that the first part $g_{0_Y} +_{Y^X} f = f$ of the conjunction in (5.97) holds. We may now apply exactly the same arguments to establish the second part $f +_{Y^X} g_{0_Y} = f$, by proving the equivalent universal sentence.

$$\forall x\,(x \in X \Rightarrow f(x) +_Y g_{0_Y}(x) = f(x)). \tag{5.99}$$

Letting $x \in X$ be arbitrary, we obtain $f(x) +_Y g_{0_Y}(x) = f(x) +_Y 0_Y = f(x)$, as desired, which is evidently true for any x. Because f is arbitrary, we may therefore infer from these findings the truth of (5.97), which shows that the set $g_{0_Y} = X \times \{0_Y\}$ is the zero element of Y^X. As X, Y and $+_{Y^X}$ were initially arbitrary, we may finally conclude that the proposition is true. □

As a first example, we may use the preceding proposition to obtain the neutral element with respect to the addition of matrices.

Corollary 5.20. *For any positive natural numbers m, n, any set Y and any addition $+_Y$ such that the zero element 0_Y exists, it is true that the Cartesian product*

$$0_{Y^{m\times n}} = (\{1,\ldots,m\} \times \{1,\ldots,n\}) \times \{0_Y\} \tag{5.100}$$

is the zero element of $Y^{m\times n}$ (with respect to the addition of matrices).

Notation 5.8. We symbolize the zero element (5.100) also by

$$\mathbf{0} = \mathbf{0}_{[m\times n]} = \begin{bmatrix} 0_Y & \cdots & 0_Y \\ \vdots & \ddots & \vdots \\ 0_Y & \cdots & 0_Y \end{bmatrix} \tag{5.101}$$

and call it the *zero matrix* (of $Y^{m\times n}$).

Definition 5.6 (Diagonal matrix). We say for any positive natural number n, any set Y and any addition $+_Y$ such that the zero element 0_Y exists that a square $(n \times n)$-matrix $\boldsymbol{A}$ with values in Y is a *diagonal matrix* iff

$$\forall i, j\,([i, j \in \{1,\ldots,n\} \wedge i \neq j] \Rightarrow a_{i,j} = 0_Y). \tag{5.102}$$

Notation 5.9. Any diagonal matrix $\boldsymbol{A}$ of order n evidently takes the form

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1} & \cdots & 0_Y \\ \vdots & \ddots & \vdots \\ 0_Y & \cdots & a_{n,n} \end{bmatrix}. \tag{5.103}$$

Exercise 5.11. Show for any sets X, Y and any multiplication $\cdot_Y$ such that the unity element 1_Y of Y exists that the set $X \times \{1_Y\}$ is

a) a constant function from X to Y with value 1_Y

b) and furthermore the unity element of Y^X with respect to the pointwise multiplication $\cdot_{Y^X}$ of functions.

We will occasionally inspect relationships between two binary operations (on possibly distinct sets) that behave similarly in the following sense.

Definition 5.7 (Isomorphism). For any sets X and Y, for any binary operations $*$ on X and $\odot$ on Y, and for any bijection $f : X \rightleftarrows Y$, we say that f is an *isomorphism* from $(X, *)$ to $(Y, \odot)$, symbolically

$$f : (X, *) \rightleftarrows (Y, \odot), \tag{5.104}$$

if, and only if,

$$\forall a, b\, (a, b \in X \Rightarrow f(a * b) = f(a) \odot f(b)). \tag{5.105}$$

We then say also that $(X, *)$ and $(Y, \odot)$ are *isomorphic* under f.

Proposition 5.21. *It is true for any sets X and Y, any binary operations $*$ on X and $\odot$ on Y, and any isomorphism $f : (X, *) \rightleftarrows (Y, \odot)$ that*

$$\forall u, v\, (u, v \in Y \Rightarrow u \odot v = f(f^{-1}(u) * f^{-1}(v))). \tag{5.106}$$

Proof. We let X, Y, $*$, $\odot$, f, u and v be arbitrary such that $*$ is a binary operation on X, such that $\odot$ is a binary operation on Y, such that f is an isomorphism from $(X, *)$ to $(Y, \odot)$, and such that u, v are elements of Y. This means in particular that f is a bijection from X to Y, which has the bijective inverse $f^{-1} : Y \rightleftarrows X$ (see Theorem 3.212). Consequently, the values $f^{-1}(u)$ and $f^{-1}(v)$ are elements of X, which fact implies

$$\begin{aligned} f(f^{-1}(u) * f^{-1}(v)) &= f(f^{-1}(u)) \odot f(f^{-1}(v)) \\ &= [f \circ f^{-1}](u) \odot [f \circ f^{-1}](v) \\ &= \mathrm{id}_Y(u) \odot \mathrm{id}_Y(v) \\ &= u \odot v \end{aligned}$$

with (5.105), the notation for function compositions, (3.680) and the definition of the identity function. Because X, Y, $*$, $\odot$, f, a and b are all arbitrary, we may infer from the resulting equation $u \odot v = f(f^{-1}(u) * f^{-1}(v))$ the truth of the proposed sentence. $\square$

Exercise 5.12. Show for any sets X and Y, any binary operations $*$ on X and $\odot$ on Y, and any isomorphism $f : (X, *) \rightleftarrows (Y, \odot)$ that

$$\forall a, b\,(a, b \in X \Rightarrow a * b = f^{-1}(f(a) \odot f(b))). \tag{5.107}$$

(Hint: Apply (5.105) and (3.678).)

Note 5.10. According to (5.107), we can operate on the elements of a set X through an operation on an 'isomorphic' set Y. The existence of an isomorphism and its inverse allows one to transform the elements from one of these sets into corresponding elements within the other set.

5.2. Semigroups $(X, *)$ and Semirings $(X, +, \cdot)$

Definition 5.8 (Idempotent binary operation). We say that a binary operation $*$ on a set X is *idempotent* iff

$$\forall a\,(a \in X \Rightarrow a * a = a). \tag{5.108}$$

Proposition 5.22. *It is true for any set X, any binary operation $*$ on X and any set $A \subseteq X$ such that the restriction of $*$ to $A \times A$ is a binary operation (on A) that this restricted binary operation is idempotent if $*$ is idempotent.*

Proof. We let X be an arbitrary set, $*$ an arbitrary binary operation on X, and A an arbitrary subset of X. We now assume that the restriction $*_A = * \upharpoonright A \times A$ is a restricted binary operation (on A), and moreover that $*$ is idempotent. To demonstrate that $*_A$ is idempotent itself, we establish the truth of the universal sentence

$$\forall a\,(a \in A \Rightarrow a *_A a = a). \tag{5.109}$$

Letting a be an arbitrary element of A, we obtain

$$a *_A a = a * a = a$$

by using (5.4) and the assumption that $*$ is idempotent. Thus, we obtain $a *_A a = a$, and since a was arbitrary, we therefore conclude that the universal sentence (5.109) holds. This means by definition that the restricted binary operation $* \upharpoonright A \times A$ is idempotent. As X, $*$ and A were initially arbitrary, we may therefore conclude that the proposition is true. $\square$

Proposition 5.23. *For any lattice $(X, \sqcup, \sqcap, \leq)$ the join $\sqcup$ is an idempotent binary operation.*

Proof. Letting X, $\sqcup$, $\sqcap$ and $\leq$ be arbitrary sets and assuming $(X, \sqcup, \sqcap, \leq)$ to be a lattice, we obtain for an arbitrary $a \in X$ the equation

$$a \sqcup a = \sup\{a, a\}. \tag{5.110}$$

with the definition of the join in $(X, \sqcup, \sqcap, <)$. Since $\{a, a\} = \{a\}$ holds by definition of a singleton, we may now apply substitution to (5.110), with the consequence that

$$a \sqcup a = \sup\{a\} = a, \tag{5.111}$$

where the second equation holds because of Corollary 3.106. The preceding equations give $a \sqcup a = a$, and since a is arbitrary, we may therefore conclude that the binary operation $\sqcup$ satisfies (5.108). As X, $\sqcup$, $\sqcap$ and $\leq$ were also arbitrary, the proposition follows to be true. $\square$

Exercise 5.13. Show for any lattice $(X, \sqcup, \sqcap, \leq)$ that the meet $\sqcap$ is an idempotent binary operation.

Definition 5.9 (Commutative binary operation). We say that a binary operation $*$ on a set X is *commutative* iff

$$\forall a, b\,(a, b \in X \Rightarrow a * b = b * a). \tag{5.112}$$

Exercise 5.14. Show for any set X, any binary operation $*$ on X and any set $A \subseteq X$ such that the restriction of $*$ to $A \times A$ is a binary operation (on A) that this restricted binary operation is commutative if $*$ is commutative. (Hint: Proceed similarly as in the proof of Proposition 5.22.)

Proposition 5.24. *For any lattice $(X, \sqcup, \sqcap, \leq)$ the join $\sqcup$ is a commutative binary operation.*

Proof. Letting X, $\sqcup$, $\sqcap$ and $\leq$ be arbitrary sets and assuming $(X, \sqcup, \sqcap, \leq)$ to be a lattice, we have for arbitrary $a, b \in X$ by definition of $\sqcup$ that

$$a \sqcup b = \sup\{a, b\} \tag{5.113}$$

holds. Then, substitution based on the fact that $\{a, b\}$ and $\{b, a\}$ are identical sets according to (2.161) yields

$$a \sqcup b = \sup\{b, a\} = b \sqcup a, \tag{5.114}$$

where the second equation holds according to the definition of $\sqcup$. These equations give $a \sqcup b = b \sqcup a$, and because a and b are arbitrary, we may infer from this that $\sqcup$ satisfies (5.112). Since X, $\sqcup$, $\sqcap$ and $\leq$ were initially arbitrary sets, we may then further conclude that the proposition holds, as claimed. □

Exercise 5.15. Show for any lattice $(X, \sqcup, \sqcap, \leq)$ that the meet $\sqcap$ is a commutative binary operation.

Proposition 5.25. *For any sets X, Y and any commutative addition $+_Y$ on Y, it is true that the pointwise addition $+_{Y^X}$ of functions in Y^X is also commutative.*

Proof. We take arbitrary sets X, Y and $+_Y$ such that $+_Y$ is a commutative addition on Y; thus, the addition $+_Y$ satisfies

$$\forall y, z\,(y, z \in Y \Rightarrow y +_Y z = z +_Y y). \tag{5.115}$$

To prove that the addition $+_{Y^X}$ is commutative, we show that this addition satisfies

$$\forall f, g\,(f, g \in Y^X \Rightarrow f +_{Y^X} g = g +_{Y^X} f). \tag{5.116}$$

We take arbitrary sets f and g, assuming $f, g \in Y^X$ to be true, and we use the notations $h = f +_{Y^X} g$ as well as $h' = g +_{Y^X} f$, so that the equation in (5.116) to be proven may evidently be written equivalently as $h = h'$. As elements of Y^X, the sums h and h' are both functions from X to Y. Therefore, we may apply the Equality Criterion for functions to establish $h = h'$. For this purpose, we verify

$$\forall x\,(x \in X \Rightarrow h(x) = h'(x)), \tag{5.117}$$

letting $x \in X$ be arbitrary. We then obtain the true equations

$$h(x) = f(x) +_Y g(x) = g(x) +_Y f(x) = h'(x)$$

by applying (5.53) to $h = f +_{Y^X} g$, then the property of commutativity (5.115), and finally (5.53) to $h' = g +_{Y^X} f$. Since x is arbitrary, we may infer from the resulting equation $h(x) = h'(x)$ the truth of the universal sentence (5.117), which gives $h = h'$ and thus $f +_{Y^X} g = g +_{Y^X} f$. As f and g are also arbitrary, we may now further conclude that the universal sentence (5.116) holds, which means that $+_{Y^X}$ is commutative, by definition. Because the sets X, Y and $+_{Y^X}$ were initially arbitrary, we may finally conclude that the proposed sentence is true. □

We can directly apply Proposition 5.25 to matrices.

Corollary 5.26. *It is true for any positive natural numbers* m, n*, any set* Y *and any commutative addition on* Y *that the corresponding addition* $+_{Y^{m \times n}}$ *of matrices is commutative.*

Exercise 5.16. Verify for any sets X, Y and any commutative multiplication $\cdot_Y$ on Y that the pointwise multiplication $\cdot_{Y^X}$ of functions in Y^X is also commutative.

Proposition 5.27. *For any set* X *and any commutative addition* $+$ *on* X*, it is true that the addition* $\bigoplus$ *of sets in* $\mathcal{P}(X)$ *is also commutative.*

Proof. Letting X and $+$ be arbitrary such that $+$ is a commutative addition on X, and letting moreover A and B be arbitrary elements of $\mathcal{P}(X)$, we prove the required equation $A \oplus B = B \oplus A$ by means of the Equality Criterion for sets. Letting for this purpose s be arbitrary and assuming first $s \in A \oplus B$ to be true, it follows by definition of the addition of sets according to (5.65) that there are constants, say $\bar{a}$ and $\bar{b}$, such that

$$\bar{a} \in A \wedge \bar{b} \in B \wedge s = \bar{a} + \bar{b}. \tag{5.118}$$

Since the conjunction of sentences and the addition on X are commutative, we obtain

$$\bar{b} \in B \wedge \bar{a} \in A \wedge s = \bar{b} + \bar{a}, \tag{5.119}$$

which evidently implies $s \in B \oplus A$. Assuming now conversely $s \in B \oplus A$ to be true, we obtain (5.119) and therefore also (5.118), so that $s \in A \oplus B$ follows to be true as well. We thus demonstrated the truth of the equivalence of $s \in A \oplus B$ and $s \in B \oplus A$, in which s is arbitrary, so that the sets $A \oplus B$ and $B \oplus A$ are indeed identical. As A and B were arbitrary elements of X, we may therefore conclude that the binary operation $\bigoplus$ is commutative. Since X and $+$ were initially also arbitrary, we may further conclude that the proposed universal sentence holds. $\square$

Definition 5.10 (Left-distributive & right-distributive & distributive binary operation). We say for any set X and any binary operations $*$ and $\odot$ on X that

(1) $*$ is *left-distributive* over $\odot$ iff

$$\forall a, b, c\, (a, b, c \in X \Rightarrow a * (b \odot c) = (a * b) \odot (a * c)). \tag{5.120}$$

(2) $*$ is *right-distributive* over $\odot$ iff

$$\forall a, b, c\, (a, b, c \in X \Rightarrow (b \odot c) * a = (b * a) \odot (c * a)). \tag{5.121}$$

(3) $*$ is *distributive* over $\odot$ iff $*$ is left- and right-distributive over $\odot$.

Proposition 5.28. *It is true for any set X and any binary operations $*$ and $\odot$ that $*$ is distributive over $\odot$ if $*$ is commutative and left-distributive over $\odot$.*

Proof. Letting X, $*$ and $\odot$ be arbitrary sets such that $*$ and $\odot$ are binary operations on X, and assuming $*$ to be commutative and left-distributive over $\odot$, so that $*$ and $\odot$ satisfy (5.120), we may show that these binary operations satisfy also (5.121). Letting for this purpose a, b and c be arbitrary, we obtain the equations

$$(b \odot c) * a = a * (b \odot c) = (a * b) \odot (a * c) = (b * a) \odot (c * a)$$

using the assumed commutativity of $*$, (5.120), and again the commutativity of $*$ (twice). Since a, b and c are arbitrary, we may therefore conclude that $*$ and $\odot$ satisfy indeed the universal sentence (5.121). Thus, $*$ is not only left-distributive but also right-distributive over $\odot$, and consequently by definition distributive over $\odot$. Here, X, $*$ and $\odot$ were initially arbitrary sets, so that the proposed universal sentence follows to be true. $\square$

Exercise 5.17. Show for any set X and any binary operations $*$ and $\odot$ that $*$ is distributive over $\odot$ if $*$ is commutative and right-distributive over $\odot$.

(Hint: Proceed in analogy to the proof of Proposition 5.28.)

Exercise 5.18. Prove for any set X, any binary operations $*$ and $\odot$ on X and any set $A \subseteq X$ such that the restrictions of $*$ and $\odot$ to $A \times A$ are binary operations (on A) that the restricted binary operation $*_A = * \restriction A \times A$ is left-distributive/right-distributive/distributive over the restricted binary operation $\odot_A = \odot \restriction A \times A$ if $*$ is correspondingly left-distributive/right-distributive/distributive over $\odot$.

(Hint: Proceed in analogy to Exercise 5.14.)

Proposition 5.29. *For any totally ordered lattice $(X, \sqcup, \sqcap, \leq)$ the meet $\sqcap$ is distributive over the join $\sqcup$.*

Proof. We let X, $\sqcup$, $\sqcap$ and $\leq$ be arbitrary such that $\leq$ is a total ordering of X and such that $(X, \sqcup, \sqcap, <)$ is a lattice. We begin with the verification that $\sqcap$ is left-distributive over $\sqcup$. For this purpose, we let $a, b, c \in X$ be arbitrary and show that this implies

$$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c), \tag{5.122}$$

which we may also write as

$$\inf\{a, \sup\{b, c\}\} = \sup\{\inf\{a, b\}, \inf\{a, c\}\} \tag{5.123}$$

by definition of the join and of the meet. Since $\leq$ is total, the three disjunctions

$$a \leq b \vee b \leq a \tag{5.124}$$
$$a \leq c \vee c \leq a \tag{5.125}$$
$$b \leq c \vee c \leq b \tag{5.126}$$

are true, which we now use to establish the preceding equation via nested proofs by cases.

We first consider the disjunction (5.124) and begin with first the case $a \leq b$, which implies

$$\inf\{a, b\} = a \tag{5.127}$$

with Proposition 3.109. We now consider the second disjunction (5.125), assuming first $a \leq c$ to be true, which yields (again with Proposition 3.109)

$$\inf\{a, c\} = a. \tag{5.128}$$

Let us now observe the truth of the equations

$$\sup\{a, a\} = \sup\{a\} = a \tag{5.129}$$

in light of the equations (5.110) and (5.111). Due to (5.127) and (5.128), we may apply substitutions and rewrite (5.129) as

$$\sup\{\inf\{a,b\},\inf\{a,c\}\} = a. \tag{5.130}$$

Next, we use the third true disjunction (5.126) to prove the proposed equation. On the one hand, if $b \leq c$ holds, then we evidently obtain

$$\sup\{b,c\} = c, \tag{5.131}$$

so that (5.128) becomes after substitution

$$\inf\{a,\sup\{b,c\}\} = a. \tag{5.132}$$

Because of the true equation (5.130), we may now apply substitution and write the preceding equation (5.132) as (5.123), which thus holds in the cases of $a \leq b$, $a \leq c$ and $b \leq c$. On the other hand, if the second part $c \leq b$ of the disjunction (5.126) holds, then

$$\sup\{b,c\} = b, \tag{5.133}$$

evidently follows to be true so that we may now rewrite (5.127) as (5.132). Consequently, the true equations (5.132) and (5.130) give the desired equation (5.123) also for the cases $a \leq b$, $a \leq c$ and $c \leq b$. We now consider the case that the second part $c \leq a$ of the disjunction (5.125) is true, which evidently implies

$$\inf\{a,c\} = c \tag{5.134}$$

and

$$\sup\{a,c\} = a. \tag{5.135}$$

Applying now substitutions based on (5.127) and (5.134), we may write (5.135) as

$$\sup\{\inf\{a,b\},\inf\{a,c\}\} = a. \tag{5.136}$$

Now, since $c \leq a$ and $a \leq b$ are currently both true, it follows with the transitivity of the total ordering $\leq$ that $c \leq b$ also holds. Then, we obtain again (5.133), so that (5.127) becomes (5.132). Thus, the latter implies together with (5.136) the desired equation for $c \leq a$, so that the proof of (5.123) for the first case $a \leq b$ is complete.

In the second case $b \leq a$, we evidently obtain

$$\inf\{a,b\} = b \tag{5.137}$$

instead of (5.127). Considering now the second true disjunction (5.125) again, we first assume that $a \leq c$ holds, so that (5.128) follows to be true.

Furthermore, the transitivity of $\leq$ allows to infer from the truth of $b \leq a$ and $a \leq c$ the truth of $b \leq c$, which inequality in turn implies (5.131). Substitution based on (5.131) allows us then to write (5.128) as (5.132). Moreover, the case assumption $b \leq a$ clearly implies

$$\sup\{b, a\} = a, \tag{5.138}$$

which equation we may write in view of the true (5.137) and (5.128) as (5.136). Therefore, we may combine by means of substitution (5.132) and (5.136) to obtain (5.123), as desired.

We now assume that the second part $c \leq a$ of the disjunction (5.125) holds, which gives (5.134). To complete the proof, we use once again the third true disjunction (5.126) and assume first $b \leq c$ to be true, which yields (5.131). We may now write the latter as

$$\sup\{\inf\{a, b\}, \inf\{a, c\}\} = c \tag{5.139}$$

using (5.137) and (5.134). In addition, we may write (5.134) as

$$\inf\{a, \sup\{b, c\}\} = c \tag{5.140}$$

by means of (5.131). Then, the preceding equation yields the desired (5.123) via substitution based on (5.139). Finally, we assume that $c \leq b$ holds, which implies (5.133), so that we may write (5.137) as

$$\inf\{a, \sup\{b, c\}\} = b. \tag{5.141}$$

Performing now substitutions based on (5.137) and (5.134), we may write instead of (5.133)

$$\sup\{\inf\{a, b\}, \inf\{a, c\}\} = b, \tag{5.142}$$

and therefore (5.141) may be written as (5.123), which equation thus holds both for $b \leq c$ and for $c \leq b$. This completes the proof for the second (sub)case $c \leq a$ and thus the proof for the second case $b \leq a$. Consequently, we may infer from these findings that $\sqcup$ and $\sqcap$ satisfy the implication in (5.120). Since a, b and c are arbitrary, we may therefore conclude that the universal sentence (5.120) holds, so that $\sqcap$ is indeed left-distributive over $\sqcup$. By virtue of Proposition 5.28, $\sqcap$ is then also distributive over $\sqcup$. As X, $\sqcup$, $\sqcap$ and $<$ were initially arbitrary sets, we may then further conclude that the proposition holds, as claimed. □

Exercise 5.19. Show for any totally ordered lattice $(X, \sqcup, \sqcap, \leq)$ that the join $\sqcup$ is distributive over the meet $\sqcap$.

(Hint: Apply (1.37) to switch the two parts in each of the disjunctions (5.124) – (5.126).)

Proposition 5.30. *For any sets X, Y, for any addition $+_Y$ on Y and for any multiplication $\cdot_Y$ on Y which is left-distributive over $+_Y$, it is true that the pointwise multiplication $\cdot_{Y^X}$ of functions in Y^X is left-distributive over the pointwise addition $+_{Y^X}$.*

Proof. We let X, Y, $+_Y$ and $\cdot_Y$ be arbitrary such that $+_Y$ is an addition on Y and such that $\cdot_Y$ is a multiplication on Y which is left-distributive over $+_Y$. To prove that the multiplication $\cdot_{Y^X}$ on Y^X is left-distributive over the addition $+_{Y^X}$ on Y^X, we verify

$$\forall e, f, g\,(e, f, g \in Y^X \Rightarrow e \cdot_{Y^X} (f +_{Y^X} g) = (e \cdot_{Y^X} f) +_{Y^X} (e \cdot_{Y^X} g)). \quad (5.143)$$

Letting e, f, g be arbitrary in Y^X and x be arbitrary in X, we prove the stated equation via the Equality Criterion for functions. To do this, we take an arbitrary x and assume $x \in X$ to be true. We then obtain the equations

$$\begin{aligned}
[e \cdot_{Y^X} (f +_{Y^X} g)](x) &= e(x) \cdot_Y (f +_{Y^X} g)(x) \\
&= e(x) \cdot_Y (f(x) +_Y g(x)) \\
&= (e(x) \cdot_Y f(x)) +_Y (e(x) \cdot_Y g(x)) \\
&= (e \cdot_{Y^X} f)(x) +_Y (e \cdot_{Y^X} g)(x) \\
&= [(e \cdot_{Y^X} f) +_{Y^X} (e \cdot_{Y^X} g)](x)
\end{aligned}$$

by applying (5.64), (5.53), the left-distributivity of $\cdot_Y$ over $+_Y$, again (5.64), and finally again (5.53). Because x is arbitrary, we may therefore conclude that the functions $[e \cdot_{Y^X} (f +_{Y^X} g)]$ and $[(e \cdot_{Y^X} f) +_{Y^X} (e \cdot_{Y^X} g)]$ are identical. Since e, f and g are arbitrary, we may further conclude that (5.143) holds, so that $\cdot_{Y^X}$ is indeed left-distributive over $+_{Y^X}$. Finally, as X, Y, $+_Y$ and $\cdot_Y$ were also arbitrary, the proposed universal sentence follows to be true. □

Exercise 5.20. Prove for any sets X, Y, for any addition $+_Y$ on Y and for any multiplication $\cdot_Y$ on Y which is right-distributive over $+_Y$ that the pointwise multiplication $\cdot_{Y^X}$ of functions in Y^X is right-distributive over the pointwise addition $+_{Y^X}$.

The following property allows us to change the order of evaluating the operation for three elements successively.

Definition 5.11 (Associative binary operation, semigroup, commutative/Abelian semigroup). We say for any set X and any binary operation $*$ on X that

1. $*$ is *associative* iff

$$\forall a,b,c\,(a,b,c\in X\Rightarrow (a*b)*c=a*(b*c)). \tag{5.144}$$

2. the ordered pair

$$(X,*) \tag{5.145}$$

is a *semigroup* iff $*$ is associative.

Furthermore, we say that a semigroup $(X,*)$ is *commutative* or *Abelian* iff $*$ is commutative.

Proposition 5.31. *It is true for any semigroup* $(X,+)$ *that the restriction*

$$\circ_{G_L(X)}=\circ_X\restriction G_L(X)\times G_L(X) \tag{5.146}$$

of the composition on the set X^X *of transformations on* X *constitutes a restricted binary operation on the set* $G_L(X)$ *of translations on* X.

Proof. We let X be an arbitrary and $+$ an arbitrary addition on X such that $(X,+)$ is a semigroup. As mentioned in Note 5.5, the set $G_L(X)$ of translations on X is a subset of the set X^X of transformations on X. Therefore, the Cartesian product $X^X\times X^X$ is included in the Cartesian product $G_L(X)\times G_L(X)$ due to Proposition 3.8. Consequently, the restriction (5.146) of $\circ_X : X^X\times X^X\to X^X$ to $G_L(X)\times G_L(X)$ is a function from $G_L(X)\times G_L(X)$ to X^X due to (3.566). It remains for us to prove that $G_L(X)$ is also a codomain of that restriction, i.e., that the range of $\circ_{G_L(X)}$ is a subset of $G_L(X)$. For this purpose, we apply the definition of a subset and let accordingly $y\in\operatorname{ran}(\circ_{G_L(X)})$ be arbitrary. By definition of a range, there exists then a particular constant z such that $(z,y)\in\circ_{G_L(X)}$. Since $\circ_{G_L(X)}$ is a function with domain $G_L(X)\times G_L(X)$, we may on the one hand write $y=\circ_{G_L(X)}(z)$; on the other hand, we find that z is an element of that domain (by definition). These findings imply now on the one hand $[\circ_{G_L(X)}(z)\ =]\ y=\circ_X(z)$ with (3.567). On the other hand, $z\in G_L(X)\times G_L(X)$ implies by virtue of Exercise 3.4 that there exist particular elements $g,h\in G_L(X)$ with $(g,h)=z$. This allows us to write

$$y=\circ_X((g,h))=g\circ_X h=g\circ h \tag{5.147}$$

by using the fact that $\circ_X$ is a binary operation satisfying (5.71). Furthermore, the translations $g,h\in G_L(X)$ satisfy

$$\forall x\,(x\in X\Rightarrow g(x)=x+\bar{a}), \tag{5.148}$$
$$\forall x\,(x\in X\Rightarrow h(x)=x+\bar{b}) \tag{5.149}$$

for some particular elements $\bar{a}, \bar{b} \in X$. We now prove

$$\forall x \, (x \in X \Rightarrow (g \circ h)(x) = x + (\bar{a} + \bar{b})). \tag{5.150}$$

Letting $x \in X$ be arbitrary, we obtain

$$(g \circ h)(x) = g(h(x)) = g(x + \bar{b}) = (x + \bar{b}) + \bar{a} = x + (\bar{b} + \bar{a})$$

using the notation for the composition of functions, (5.149), (5.148) with the fact that $x + \bar{b} \in X$ since $+$ is a binary operation on X, and finally the associativity of the addition $+$ on X. Since x is an arbitrary element of X, we may therefore conclude that the universal sentence (5.150) is indeed true. Denoting $\bar{c} = \bar{b} + \bar{a}$, which sum is evidently in X, we now see that the existential sentence

$$\exists c \, (c \in X \ \wedge \ \forall x \, (x \in X \Rightarrow (g \circ h)(x) = x + c)$$

is true. We also see in light of the previous equations (5.147) that $g \circ h$ is a value of the binary operation $\circ_X$ on X^X and therefore an element of X^X. These two findings imply now $[y =] \; g \circ h \in G_L(X)$ with (5.82), thus $y \in G_L(X)$. As y was arbitrary, we may therefore conclude that the range of $\circ_{G_L(X)}$ is indeed a subset of $G_L(X)$, i.e., that $G_L(X)$ is a codomain of the restriction (5.146). This means that this restriction with domain $G_L(X) \times G_L(X)$ is a binary operation on $G_L(X)$. Thus, $\circ_{G_L(X)}$ constitutes a restricted binary operation on $G_L(X)$, by definition. □

Exercise 5.21. Show for any semigroup $(X, \cdot)$ that the restriction

$$\circ_{G_S(X)} = \circ_X \upharpoonright G_S(X) \times G_S(X) \tag{5.151}$$

of the composition on the set X^X of transformations on X constitutes a restricted binary operation on the set $G_S(X)$ of scalings on X.

(Hint: Proceed as in the proof of Proposition 5.31.)

Exercise 5.22. Show that the composition of two translations/scalings g and h is identical with the value of the binary composition operation on the set of translations/scalings associated with g and h, that is,

$$\forall g, h \, (g, h \in G_L(X) \Rightarrow g \circ_{G_L(X)} h = g \circ h), \tag{5.152}$$

$$\forall g, h \, (g, h \in G_S(X) \Rightarrow g \circ_{G_S(X)} h = g \circ h). \tag{5.153}$$

(Hint: Recall the proof of Corollary 5.11.)

Proposition 5.32. *For any semigroup* $(X, +)$ *such that the neutral element* 0 *of* X *with respect to the addition exists, it is true that the identity function on any set* X *constitutes the neutral element of the set* $G_L(X)$ *of translations on* X *with respect to the (restricted) binary composition operation* $\circ_{G_L(X)}$.

Proof. We let $(X, +)$ be an arbitrary set, assuming $(X, +)$ to be a semigroup and assuming the neutral element 0 of X with respect to the addition to exist. Thus, 0 is an element of X by definition of a neutral element. We may now prove that the identity function on X satisfies

$$\forall x\,(x \in X \Rightarrow \mathrm{id}_X(x) = x + 0).$$

Letting $x \in X$ be arbitrary, we indeed obtain the equations

$$\mathrm{id}_X(x) = x = x + 0$$

using the definition of an identity function and the definition of a neutral element. Since x was arbitrary, we may therefore conclude that the preceding universal sentence is true, and this finding implies that id_X is a translation on X by 0 (symbolically: g_{L0}) and thus itself an element of $G_L(X)$. Then, as $G_L(X)$ is a subset of X^X (see Note 5.5) and as id_X is the neutral element of X^X with respect to the binary composition operation $\circ_X$ on X^X (see Note 5.6), we may apply Proposition 5.13 to infer that id_X is also the neutral element of $G_L(X)$ with respect to the restricted binary composition operation $\circ_{G_L(X)}$ on $G_L(X)$, given by (5.146). Since $(X, +)$ is arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Exercise 5.23. Prove for any semigroup $(X, \cdot)$ such that the neutral element 1 of X with respect to the multiplication exists that the identity function on any set X constitutes the neutral element of the set $G_S(X)$ of scalings on X with respect to the (restricted) binary composition operation $\circ_{G_S(X)}$.
(Hint: Show that id_X is a scaling on X by 1.)

Note 5.11. Recalling the Associative Law for function composition and (5.71), we obtain for any set X from (3.613), after some relabeling, the universal sentence

$$\forall f, g, h\,(f, g, h \in X^X \Rightarrow [f \circ_X g] \circ_X h = f \circ_X [g \circ_X h]), \tag{5.154}$$

which shows that the binary operation of composition on the set of transformations on X is associative. Thus, the ordered pair

$$(X^X, \circ_X) \tag{5.155}$$

constitutes a semigroup.

Exercise 5.24. Demonstrate for any set X, any binary operation $*$ on X and any set $A \subseteq X$ such that the restriction of $*$ to $A \times A$ is a binary operation (on A) that the restricted binary operation $*_A = * \upharpoonright A \times A$ is associative if $*$ is associative.

(Hint: Proceed in analogy to Exercise 5.14.)

Note 5.12. The Exercises 5.14 and (5.24) show that every (commutative) semigroup $(X, *)$ gives rise to a corresponding (commutative) semigroup

$$(X, *_A) \tag{5.156}$$

if the restriction $*_A$ of $*$ to $A \times A$ with $A \subseteq X$ is a binary operation on A.

Note 5.13. For any set X, the semigroup (5.155) in particular gives rise to

- the semigroup
$$(T(X), \circ) = (T(X), \circ_{T(X)}) \tag{5.157}$$
for any set $T(X)$ of invertible transformations on X, in particular to
- the semigroup
$$(\Pi(X), \circ) = (\Pi(X), \circ_{\Pi(X)}) \tag{5.158}$$
of permutations on X in case of a finite set X,
- the semigroup
$$(T_{fi}(X), \circ) = (T_{fi}(X), \circ_{T_{fi}(X)}), \tag{5.159}$$
for any set $T_{fi}(X)$ of invertible transformations on X under which a function $f : X \to Y$ is invariant, as well as to
- the semigroups
$$(G_L(X), \circ) = (G_L(X), \circ_{G_L(X)}), \tag{5.160}$$
$$(G_S(X), \circ) = (G_S(X), \circ_{G_S(X)}) \tag{5.161}$$
of translations and scalings on X.

Proposition 5.33. *For any lattice* $(X, \sqcup, \sqcap, \leq)$ *it is true that* $(X, \sqcup)$ *is a commutative semigroup.*

Proof. We let X, $\sqcup$, $\sqcap$ and $\leq$ be arbitrary such that $(X, \sqcup, \sqcap, \leq)$ is a lattice and let a, b, c be arbitrary elements of X. On the one hand, we obtain

$$(a \sqcup b) \sqcup c = \sup\{a, b\} \sqcup c = \sup\{\sup\{a, b\}, c\} = \sup\{a, b, c\}. \tag{5.162}$$

with the definition of the join (applied twice) and with Theorem 3.108. On the other hand, the equations

$$a \sqcup (b \sqcup c) = (b \sqcup c) \sqcup a = \sup\{\sup\{b, c\}, a\} = \sup\{b, c, a\} \tag{5.163}$$

are true in view of Proposition 5.24, the definition of the join, and Theorem 3.108. As the sets $\{b, c, a\}$ and $\{a, b, c\}$ are equal according to (2.232), we may now apply substitution to (5.163) to obtain

$$a \sqcup (b \sqcup c) = \sup\{a, b, c\}. \tag{5.164}$$

Finally, we arrive at the desired equation $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$ by combining (5.162) and (5.164). Since a, b, c are arbitrary, we may therefore conclude that $\sqcup$ satisfies (5.144) and is thus associative. Since $\sqcup$ is also commutative according to Proposition 5.24, we have that the ordered pair $(X, \sqcup)$ is a commutative semigroup by definition. As X, $\sqcup$, $\sqcap$ and $\leq$ were arbitrary, we may now conclude that the proposition is true. □

Exercise 5.25. Show for any lattice $(X, \sqcup, \sqcap, \leq)$ that $(X, \sqcap)$ is a commutative semigroup.

Notation 5.10. In light of the uniqueness of the evaluation of an associative binary operation for any three elements, we will oftentimes omit the brackets in such situations and write

$$a * b * c \tag{5.165}$$

instead of $(a * b) * c$ or $a * (b * c)$.

Proposition 5.34. *For any sets* X, Y *and any associative addition* $+_Y$ *on* Y*, it is true that the pointwise addition* $+_{Y^X}$ *of functions in* Y^X *is also associative.*

Proof. We let X, Y and $+_Y$ be arbitrary such that $+_Y$ is an associative addition on Y. To prove that $+_{Y^X}$ is associative, we verify

$$\forall e, f, g\, (e, f, g \in Y^X \Rightarrow (e +_{Y^X} f) +_{Y^X} g = e +_{Y^X} (f +_{Y^X} g)). \tag{5.166}$$

Letting e, f, g be arbitrary in Y^X and x be arbitrary in X, we observe the truth of the equations

$$\begin{aligned}
[(e +_{Y^X} f) +_{Y^X} g](x) &= (e +_{Y^X} f)(x) +_Y g(x) \\
&= (e(x) +_Y f(x)) +_Y g(x) \\
&= e(x) +_Y (f(x) +_Y g(x)) \\
&= e(x) +_Y (f +_{Y^X} g)(x) \\
&= [e +_{Y^X} (f +_{Y^X} g)](x)
\end{aligned}$$

in light of (5.53) and the associativity of $+_Y$. As x is arbitrary, we then obtain the desired equation in (5.166) with the Equality Criterion for functions. Since e, f and g are arbitrary as well, we may conclude that $+_{Y^X}$ is associative. This finding follows now to be true for any X, Y and $+_Y$, because these sets were initially arbitrary. □

Corollary 5.35. *It is true for any positive natural numbers m, n, any set Y and any associative addition on Y that the corresponding addition $+_{Y^{m\times n}}$ of matrices is associative.*

Exercise 5.26. Prove for any sets X, Y and any associative multiplication $\cdot_Y$ on Y that the pointwise multiplication $\cdot_{Y^X}$ of functions in Y^X is also associative.

Note 5.14. Any (commutative) semigroup $(Y, +_Y)$ or $(Y, \cdot_Y)$ induces for any set X a corresponding (commutative) semigroup $(Y^X, +_{Y^X})$ or $(Y^X, \cdot_{Y^X})$ involving the pointwise addition/multiplication of functions in Y^X. For instance, any (commutative) semigroup $(Y, +_Y)$ and any $m, n \in \mathbb{N}_+$ give rise to the (commutative) semigroup $(Y^{m\times n}, +_{Y^{m\times n}})$ with respect to the addition of m-by-n matrices with values in Y.

Proposition 5.36. *For any set X and any associative addition $+$ on X, it is true that the addition $\bigoplus$ of sets in $\mathcal{P}(X)$ is also associative.*

Proof. We let X be an arbitrary set and $+$ an arbitrary associative addition on X. Next, we take arbitrary sets A, B and C in $\mathcal{P}(X)$ and demonstrate the truth of the equation $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. For this purpose, we apply the Equality Criterion for sets, letting s be arbitrary and assuming first $s \in (A \oplus B) \oplus C$ to hold. According to the definition of the addition of sets in (5.65), we then have $s = \bar{f} + \bar{c}$ for some particular constants $\bar{f} \in A \oplus B$ and $\bar{c} \in C$. Here, $\bar{f} \in A \oplus B$ implies the existence of particular constants $\bar{a} \in A$ and $\bar{b} \in B$ with $\bar{f} = \bar{a} + \bar{b}$. Consequently, the previous equation for s becomes $s = (\bar{a} + \bar{b}) + \bar{c}$ after substitution. Since we assumed the addition on X to be associative, $(\bar{a} + \bar{b}) + \bar{c} = \bar{a} + (\bar{b} + \bar{c})$ holds, so that another substitution in the preceding equation for s gives us $s = \bar{a} + (\bar{b} + \bar{c})$. Here, the sum $\bar{g} = \bar{b} + \bar{c}$ is evidently an element of $B \oplus C$, and therefore $s = \bar{a} + \bar{g}$ an element of $A \oplus (B \oplus C)$, as desired. The converse assumption $s \in A \oplus (B \oplus C)$ yields $s = a^* + g^*$ for particular elements $a^* \in A$ and $g^* \in B \oplus C$, where the latter gives us $g^* = b^* + c^*$ for particular elements $b^* \in B$ and $c^* \in C$. We therefore find evidently $s = a^* + (b^* + c^*) = (a^* + b^*) + c^*$. Defining here $f^* = a^* + b^*$, we obtain $f^* \in A \oplus B$, with the consequence that $s = f^* + c^*$ is in the set $(A \oplus B) \oplus C$. We thus demonstrated that $s \in A \oplus (B \oplus C)$ and $s \in (A \oplus B) \oplus C$ are equivalent, where s is arbitrary, so that the sets $A \oplus (B \oplus C)$ and $(A \oplus B) \oplus C$ are identical. As A, B and C were arbitrary, it follows that the binary operation $\bigoplus$ is associative, by definition. Initially, the sets X and $+$ were also arbitrary, so that we may further conclude that the proposition holds. $\square$

In analogy to the formation of a semigroup we may combine a set with two binary operations to form the following kind of ordered triple.

Definition 5.12 (Semiring, commutative/Abelian semiring). For any set X, any addition $+$ on X and any multiplication $\cdot$ on X we say that the ordered triple

$$(X, +, \cdot) \tag{5.167}$$

is a *semiring* iff

1. $(X, +)$ is a commutative semigroup,

2. $(X, \cdot)$ is a semigroup and

3. the multiplication $\cdot$ is distributive over the addition $+$.

Moreover, we say that a semiring $(X, +, \cdot)$ is *commutative* or *Abelian* iff the multiplication $\cdot$ is commutative.

Notation 5.11*.* We also write $a \cdot b + c$ instead of $(a \cdot b) + c$ in view of the widely used convention that a multiplication is evaluated before an addition.

Note 5.15*.* According to Note 5.12 and Exercise 5.18, every (commutative) semiring $(X, +, -)$ induces a corresponding (commutative) semiring

$$(X, +_A, \cdot_A) \tag{5.168}$$

if the restrictions $+_A : + \restriction A \times A$ and $\cdot_A : \cdot \restriction A \times A$ with $A \subseteq X$ are binary operations on A.

We may apply Proposition 5.33, Exercise 5.25 and Proposition 5.29 in connection with Notation 5.4 to obtain the following first example of a semiring.

Corollary 5.37. *For any totally ordered lattice $(X, \leq)$ it is true that, if we view the join $\sqcup$ as an addition $(+)$ and the meet $\sqcap$ as a multiplication $(\cdot)$, then the ordered triple $(X, +, \cdot)$ constitutes a commutative semiring.*

Let us now apply the preceding corollary to the lattices (5.19) and (5.41).

Corollary 5.38. *For any x and any y it is true that the ordered triple*

$$(\{x\}, \{((x, x), x)\}, \{((x, x), x)\}) \tag{5.169}$$

and

$$\begin{aligned}&(\{x,y\},\\&\{((x,x),x),((x,y),y),((y,x),y),((y,y),y)\},\\&\{((x,x),x),((x,y),x),((y,x),x),((y,y),y)\})\end{aligned} \tag{5.170}$$

are commutative semirings.

Note 5.16. Since we established in Corollary 5.37 the join $\sqcup$ as the addition $(+)$ and the meet $\sqcap$ as the multiplication $(\cdot)$ in the semiring based on a totally ordered lattice $(X,\leq)$, we may view the neutral element $\perp$ with respect to the join as the zero element 0_X and the neutral element $\top$ with respect to the meet as the unity element 1_X.

The next corollary summarizes the findings of Proposition 5.25, Exercise 5.16 Proposition 5.34, Exercise 5.26, Proposition 5.30, and Exercise 5.20.

Corollary 5.39. *For any set X and any semiring $(Y,+_Y,\cdot_Y)$, it is true that the ordered triple $(Y^X,+_{Y^X},\cdot_{Y^X})$ containing the pointwise addition and multiplication of functions in Y^X constitutes a semiring as well. If a semiring $(Y,+_Y,\cdot_Y)$ is commutative, then the semiring $(Y^X,+_{Y^X},\cdot_{Y^X})$ is commutative, too.*

Definition 5.13 (Semiring of functions from X to Y). For any set X and any semiring $(Y,+_Y,\cdot_Y)$, we call the ordered triple

$$(Y^X,+_{Y^X},\cdot_{Y^X}) \tag{5.171}$$

(containing the pointwise addition and multiplication of functions in Y^X) the *semiring of functions from X to Y*.

Definition 5.14 (Trivial semiring, nontrivial semiring). We say that a semiring $(X,+,\cdot)$ is *trivial* iff $X=\{0_X\}$, and we say that a semiring $(X,+,\cdot)$ is *nontrivial* iff $X\neq\{0_X\}$.

Taking the semiring (5.169) and recalling from Corollary 5.15 that the singleton set has a neutral element with respect to the join/addition, which we may denote by 0_X according to Notation 5.7, we have an example of a trivial semiring.

Corollary 5.40. *For any x the semiring $(\{x\},\{((x,x),x)\},\{((x,x),x)\})$ is trivial.*

By contrast, the semiring (5.170) turns out to be nontrivial in case the underlying set is a genuine pair.

Corollary 5.41. *For any x and any y satisfying $x \neq y$ it is true that the semiring $(\{x, y\}, \sqcup, \sqcap)$ in (5.170) is nontrivial.*

Proof. Letting x and y be arbitrary and assuming $x \neq y$ to be true, we see in light of (2.167) that $\{x, y\} \neq \{x\}$ follows to be also true. Since x is the neutral element of $\{x, y\}$ with respect to the join/addition in the semiring (5.170), which element we may therefore denote by 0_X with $X = \{x, y\}$, we see that the given semiring is nontrivial by definition. □

It will also be useful for us to consider the following connection between multiplication and the neutral element for addition (given it exists in X).

Definition 5.15 (Zero divisor, zero-divisor-free semiring). For any semiring $(X, +, \cdot)$ for which the zero element 0_X of X with respect to the addition $+$ exists, we say that an element $a \in X$ is a *zero divisor* iff

1. a is not identical with the zero element, that is,

$$a \neq 0_X, \tag{5.172}$$

   and

2. there exists an element b in X also different from the zero element such that $a \cdot b = 0_X$ or $b \cdot a = 0_X$, that is,

$$\exists b\, (b \in X \;\wedge\; b \neq 0_X \;\wedge\; [a \cdot b = 0_X \;\vee\; b \cdot a = 0_X]). \tag{5.173}$$

Furthermore, we say that a semiring $(X, +, \cdot)$ with zero element 0_X is *zero-divisor free* iff there does not exist a zero divisor in X, i.e. iff

$$\neg\exists a\, (a \in X \;\wedge\; a \neq 0_X \;\wedge\; \exists b\, (b \in X \;\wedge\; b \neq 0_X \;\wedge\; (a \cdot b = 0_X \;\vee\; b \cdot a = 0_X))). \tag{5.174}$$

Based on this definition of a zero divisor, the following theorem gives two equivalent conditions for a set X to contain no zero divisors.

Theorem 5.42 (Criterion for zero-divisor freeness). *Any semiring $(X, +, \cdot)$ with zero element 0_X is zero-divisor free iff*

$$\forall a, b\, (a, b \in X \Rightarrow [(a \cdot b = 0_X \vee b \cdot a = 0_X) \Rightarrow (a = 0_X \vee b = 0_X)]), \tag{5.175}$$

or equivalently iff

$$\forall a, b\, (a, b \in X \Rightarrow [(a \neq 0_X \wedge b \neq 0_X) \Rightarrow (a \cdot b \neq 0_X \wedge b \cdot a \neq 0_X)]). \tag{5.176}$$

Proof. We let X, $+$ and $\cdot$ be arbitrary sets and assume that $(X,+,\cdot)$ is a semiring for which X contains the zero element 0_X with respect to the addition $+$. Then, zero-divisor freeness is by definition equivalent to (5.174), which negation we may write equivalently as

$$\forall a\,([a \in X \wedge a \neq 0_X] \Rightarrow \neg\exists b\,(b \in X \wedge b \neq 0_X \wedge [a \cdot b = 0_X \vee b \cdot a = 0_X])]). \tag{5.177}$$

by applying the Negation Law for existential conjunctions. To prove the first of the stated equivalences, we assume first that $(X,+,\cdot)$ is zero-divisor free, i.e. we assume that $(X,+,\cdot)$ satisfies (5.177), and we show that this implies (5.175). To do this, we take arbitrary $\bar{a}$ and $\bar{b}$, assume $\bar{a},\bar{b} \in X$ to be true, and prove the second implication by contradiction, assuming

$$\bar{a} \cdot \bar{b} = 0_X \vee \bar{b} \cdot \bar{a} = 0_X \tag{5.178}$$

and $\neg(\bar{a} = 0_X \vee \bar{b} = 0_X)$ to be true. The latter negation implies with De Morgan's Law for sentences (1.52) that $\bar{a} \neq 0_X$ and $\bar{b} \neq 0_X$ are both true. Then, the truth of $\bar{a} \in X$ and of $\bar{a} \neq 0_X$ implies with the assumed (5.177) the truth of

$$\neg\exists b\,(b \in X \wedge b \neq 0_X \wedge [\bar{a} \cdot b = 0_X \vee b \cdot \bar{a} = 0_X]),$$

which negation in turn implies

$$\forall b\,([b \in X \wedge b \neq 0_X] \Rightarrow \neg[\bar{a} \cdot b = 0_X \vee b \cdot \bar{a} = 0_X]),$$

the Negation Law for existential conjunctions. Since $\bar{b} \in X$ and $\bar{b} \neq 0_X$ are both true, we therefore obtain the true negation

$$\neg[\bar{a} \cdot \bar{b} = 0_X \vee \bar{b} \cdot \bar{a} = 0_X],$$

which contradicts the true disjunction (5.178). Thus, the proof by contradiction is complete, and since $\bar{a},\bar{b}$ were arbitrary, we may therefore conclude that the semiring satisfies (5.175), so that the first part ('$\Rightarrow$') of the first proposed equivalence holds.

To prove the second part ('$\Leftarrow$') of that equivalence, we now assume (5.175) to be true, and we show that (5.177) also holds. Letting $\bar{a}$ be arbitrary, we may now prove the implication in (5.177) by contradiction, assuming the conjunction of $\bar{a} \in X$ and $\bar{a} \neq 0_X$ as well as the negation of the negated existential sentence to be true. Because of the Double Negation Law, the latter assumption yields the existential sentence

$$\exists b\,(b \in X \wedge b \neq 0_X \wedge [\bar{a} \cdot b = 0_X \vee b \cdot \bar{a} = 0_X]),$$

so that there is a particular constant $\bar{b}$ satisfying $\bar{b} \in X$, $\bar{b} \neq 0_X$ and the disjunction $\bar{a} \cdot \bar{b} = 0_X \lor \bar{b} \cdot \bar{a} = 0_X$. Now, the truth of $\bar{a}, \bar{b} \in X$ and the truth of the preceding disjunction imply with the assumed (5.175) that $\bar{a} = 0_X \lor \bar{b} = 0_X$ holds, which disjunction further implies the negated conjunction $\neg(\bar{a} \neq 0_X \land \bar{b} \neq 0_X)$. Since we previously established the true negations $\bar{a} \neq 0_X$ and $\bar{b} \neq 0_X$, we have that the conjunction $\bar{a} \neq 0_X \land \bar{b} \neq 0_X$ also holds, so that we obtained a contradiction. Thus, the proof of the implication in (5.177) is complete, and as a was arbitrary, we may therefore conclude that (5.177) holds, which completes the proof of the first proposed equivalence. Since X, $+$ and $\cdot$ were initially arbitrary, this equivalence is then true for any such sets. □

Exercise 5.27. Establish the second proposed equivalence in Theorem 5.42.

Proposition 5.43. *It is true for any zero-divisor free semiring $(X, +, \cdot)$ and any set $A \subseteq X$, such that A contains the zero element 0_X of X with respect to the addition $+$ and such that the restrictions of the addition and the multiplication to $A \times A$ are binary operations (on A) that the induced semiring $(X, +_A, \cdot_A)$ is zero-divisor free.*

Proof. We first let $(X, +, \cdot)$ be an arbitrary zero-divisor-free semiring and A an arbitrary subset of X. Next, we assume that A contains the zero element 0_X of X with respect to the addition $+$ and furthermore that the restrictions $+_A$ and $\cdot_A$ of the addition $+$ and the multiplication $\cdot$ to $A \times A$ are binary operations on A. In view of Note 5.15, $(A, +_A, \cdot_A)$ is then indeed a semiring. In addition, as the zero element 0_X of X is contained in A, that element is also the zero element of A with respect to $+_A$, as shown by Proposition 5.13. To demonstrate that $(A, +_A, \cdot_A)$ is zero-divisor free, we apply the Criterion for zero-divisor freeness and verify accordingly

$$\forall a, b\,(a, b \in A \Rightarrow [(a \neq 0_X \land b \neq 0_X) \Rightarrow (a \cdot_A b \neq 0_X \land b \cdot_A a \neq 0_X)]). \tag{5.179}$$

For this purpose, we take two arbitrary elements $a, b \in A$ satisfying $a \neq 0_X$ and $b \neq 0_X$. Since the assumed inclusion $A \subseteq X$, $a, b \in A$ implies $a, b \in X$ with the definition of a subset. In conjunction with the preceding inequalities, this implies $a \cdot b \neq 0_X$ and $b \cdot a \neq 0_X$ in view of (5.176) and the initial assumption that $(X, +, \cdot)$ is a zero-divisor-free semiring. Since $a, b \in A$ implies the equalities $a \cdot b = a \cdot_A b$ and $b \cdot a = b \cdot_A a$, we obtain the desired inequalities $a \cdot_A b \neq 0_X$ and $b \cdot_A a \neq 0_X$ via substitutions. As a and b were arbitrary, we may therefore conclude that the universal sentence (5.179) holds, so that the semiring $(A, +_A, \cdot_A)$ is indeed zero-divisor free. Since $(X, +, \cdot)$ and A were initially also arbitrary, we may now further conclude that the stated proposition holds. □

Proposition 5.44. *For any x it is true that the semiring* $(\{x\}, \sqcup, \sqcap)$ *in (5.169) is zero-divisor free.*

Proof. We take an arbitrary x and demonstrate that (5.175) is true for the semiring (5.169). For this purpose, we let a and b be arbitrary and assume that $a, b \in \{x\}$ holds. This assumption implies $a = x$ and $b = x$ with (2.169), so that the disjunction $a = 0_X \lor b = 0_X$ is true, because $0_X = x$ is the zero element in $\{x\}$ with respect to the join/addition in the semiring (5.169). Therefore, the implications in (5.175) are clearly true for the given semiring. Since a and b are arbitrary, we may therefore conclude that this semiring satisfies the Criterion for zero-divisor freeness. As x was also arbitrary, we my then further conclude that the proposed universal sentence is true. □

Proposition 5.45. *For any x and any y satisfying $x \neq y$ it is true that the semiring* $(\{x, y\}, \sqcup, \sqcap)$ *in (5.170) is zero-divisor free.*

Proof. We let x and y be arbitrary such that $x \neq y$ holds and show that the semiring (5.170) satisfies (5.176). To do this, we let $a, b \in \{x, y\}$ be arbitrary and assume $a \neq x \land b \neq x$ to be true, recalling x is the zero element 0_X in $X = \{x, y\}$ with respect to the addition/join $\sqcup$. Now, $a \in \{x, y\}$ shows in light of the definition of a pair that $a = x \lor a = y$ holds; since $a = x$ is false according to the first part of the assumed conjunction, we have that the second part $a = y$ of the preceding disjunction is true. Similarly, $b \in \{x, y\}$ yields the disjunction $b = x \lor b = y$, where $b = x$ is false according to the second part of the assumed conjunction, so that $b = y$ is true. We may now apply substitutions based on the equations $a = y$ and $b = y$ to obtain

$$a \sqcap b = y \sqcap y = y,$$
$$b \sqcap a = y \sqcap y = y,$$

where we also exploited the idempotence of $\sqcap$ (recalling Exercise 5.13). Then, initial assumption $x \neq y$ gives $x \neq a \sqcap b$ as well as $x \neq b \sqcap a$. Since we defined the meet $\sqcap$ to be the multiplication in the semiring (5.170), we may write these findings as $a \cdot b \neq 0_X \land b \cdot a \neq 0_X$, and because a and b are arbitrary, we may therefore conclude that the given semiring satisfies indeed (5.176). As x and y were initially also arbitrary, it then follows that the proposition holds, as claimed. □

5.3. The Semiring $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$

In this section, we will define the two binary operations of addition and multiplication on the set of natural numbers, which will be seen to be symbolic representations of the way we could accomplish this task literally manually by merging (in case of addition) or duplicating (in case of multiplication) collections of objects. We begin with the addition on the set of natural numbers, whose definition will be based on counting the total number of elements that are in two given disjoint sets.

Theorem 5.46. *The following sentences are true.*

a) For any $m, n \in \mathbb{N}$, the Cartesian products

$$M = \{1, \ldots, m\} \times \{0\}, \tag{5.180}$$

$$N = \{1, \ldots, n\} \times \{1\} \tag{5.181}$$

are disjoint sets with respective cardinality m and n, i.e. the Cartesian products M and N satisfy

$$|M| = m \wedge |N| = n \wedge M \cap N = \emptyset. \tag{5.182}$$

b) For any $m, n \in \mathbb{N}$, the cardinality of the union of two disjoint sets with respective cardinalities m and n is unique in the sense that

$$\begin{aligned} \forall M, N, M', N' \, ([|M| = m \wedge |N| = n \wedge M \cap N = \emptyset \\ \wedge |M'| = m \wedge |N'| = n \wedge M' \cap N' = \emptyset] \\ \Rightarrow |M \cup N| = |M' \cup N'|). \end{aligned} \tag{5.183}$$

c) There exists a unique set $+_{\mathbb{N}}$ such that an element z is in $+_{\mathbb{N}}$ iff z is in $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ and moreover if z is some ordered triple $((m, n), s)$ such that m and n are the cardinalities of some disjoint sets M and N and such that s is the cardinality of the union of M and N, i.e.

$$\begin{aligned} \exists! +_{\mathbb{N}} \, \forall z \, (z \in +_{\mathbb{N}} \Leftrightarrow [z \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \wedge \exists m, n, M, N, s \, (m = |M| \\ \wedge n = |N| \wedge M \cap N = \emptyset \wedge s = |M \cup N| \wedge ((m, n), s) = z)]). \end{aligned}$$

d) This set $+_{\mathbb{N}}$ is an addition on $\mathbb{N}$ satisfying

$$(m, n) \mapsto m +_{\mathbb{N}} n = |[\{1, \ldots, m\} \times \{0\}] \cup [\{1, \ldots, n\} \times \{1\}]|. \tag{5.184}$$

e) Furthermore, the addition $+_{\mathbb{N}}$ on $\mathbb{N}$ satisfies for any $m, n \in \mathbb{N}$

$$\forall M, N \, (|M| = m \wedge |N| = n \wedge M \cap N = \emptyset] \Rightarrow m +_{\mathbb{N}} n = |M \cup N|). \tag{5.185}$$

Proof. We let m and n be arbitrary natural numbers. Concerning a), we observe that the Cartesian products (5.180) and (5.181) have the cardinalities $|M| = m$ and $|N| = n$ in view of (4.508). It remains to show that M and N are disjoint. For this purpose, we notice that $0 \neq 1$ holds according to (4.165), which inequality further implies $\{0\} \cap \{1\} = \emptyset$ with (2.174), and therefore we obtain

$$(\{1, \ldots, m\} \times \{0\}) \cap (\{1, \ldots, n\} \times \{1\}) = \emptyset$$

with (3.55), which gives the desired $M \cap N = \emptyset$ after substitution. We thus proved the multiple conjunction (5.182), and since m and n were arbitrary, we may therefore conclude that a) is true.

Concerning b), we let m and n be arbitrary natural numbers and let M, N, M', N' be arbitrary sets with $|M| = |M'| = m$, $|N| = |N'| = n$ and $M \cap N = M' \cap N' = \emptyset$. To show that these assumptions imply $|M \cup N| = |M' \cup N'|$, we will construct in the following a bijection from $M \cup N$ to $M' \cup N'$, using the fact that the finiteness of the sets M, N, M' and N' implies the finiteness of $M \cup N$ and $M' \cup N'$ with Theorem 4.114. To begin with, $m = |M|$ and $m = |M'|$ imply with the definition of a cardinality that there exist bijections from $\{1, \ldots, m\}$ to M and from $\{1, \ldots, m\}$ to M', say

$$\bar{c} : \{1, \ldots, m\} \rightleftarrows M, \tag{5.186}$$
$$\bar{c}' : \{1, \ldots, m\} \rightleftarrows M'. \tag{5.187}$$

Similarly, $n = |N|$ and $n = |N'|$ imply the existence of bijections from $\{1, \ldots, n\}$ to N and from $\{1, \ldots, n\}$ to N', say

$$\bar{d} : \{1, \ldots, n\} \rightleftarrows N, \tag{5.188}$$
$$\bar{d}' : \{1, \ldots, n\} \rightleftarrows N'. \tag{5.189}$$

Now, (5.186) and (5.188) give the bijections

$$\bar{c}^{-1} : M \rightleftarrows \{1, \ldots, m\}$$
$$\bar{d}^{-1} : N \rightleftarrows \{1, \ldots, n\}$$

with (3.683) and then in connection with (5.187) as well as (5.189)

$$\bar{c}' \circ \bar{c}^{-1} : M \rightleftarrows M', \tag{5.190}$$
$$\bar{d}' \circ \bar{d}^{-1} : N \rightleftarrows N'. \tag{5.191}$$

because (3.672). Thus, M is the domain of $\bar{c}' \circ \bar{c}^{-1}$ and N the domain of $\bar{d}' \circ \bar{d}^{-1}$. As we assumed $M \cap N = \emptyset$, it follows with Exercise 3.73 that $\bar{c}' \circ \bar{c}^{-1}$

and $\bar{d}' \circ \bar{d}^{-1}$ are compatible functions (having disjoint domains). Therefore, the pair $\{\bar{c}' \circ \bar{c}^{-1}, \bar{d}' \circ \bar{d}^{-1}\}$ is a set of compatible functions according to Proposition 3.174, so that

$$\bar{e} = \bigcup \{\bar{c}' \circ \bar{c}^{-1}, \bar{d}' \circ \bar{d}^{-1}\} = \bar{c}' \circ \bar{c}^{-1} \cup \bar{d}' \circ \bar{d}^{-1} \tag{5.192}$$

(using the definition of the union of a pair) is a function because of Theorem 3.175a). Moreover, the pair $\{M, N\}$ consists of the domains of $\bar{c}' \circ \bar{c}^{-1}$ and $\bar{d}' \circ \bar{d}^{-1}$, so that the domain of $\bar{e}$ is given by $M \cup N$ due to Theorem 3.175b).

We now show that $\bar{e}$ is a surjection from $M \cup N$ to $M' \cup N'$, by verifying

$$\forall y\, (y \in M' \cup N' \Leftrightarrow \exists x\, (\bar{e}(x) = y)) \tag{5.193}$$

according to Corollary 3.191. For this purpose, we let y be arbitrary and prove the implication directly, assuming $y \in M' \cup N'$ to be true. By definition of the union of two sets, the disjunction $y \in M' \vee y \in N'$ is true, which we now use to prove the existential sentence in (5.193) by cases. Let us observe for this purpose that the ranges of the bijections (5.190) and (5.191) are M' and N', respectively.

If $y \in M'$ $[= \mathrm{ran}(\bar{c}' \circ \bar{c}^{-1})]$ holds, then there exists by definition of a range an element, say $\bar{x}$, with

$$(\bar{x}, y) \in \bar{c}' \circ \bar{c}^{-1}. \tag{5.194}$$

Since

$$\bar{c}' \circ \bar{c}^{-1} \subseteq \bar{e} \quad [= \bar{c}' \circ \bar{c}^{-1} \cup \bar{d}' \circ \bar{d}^{-1}]$$

holds with (2.201), it follows from (5.194) that $(\bar{x}, y) \in \bar{e}$is true , which we may also write as $y = \bar{e}(\bar{x})$ since $\bar{e}$ is a function. Thus, the truth of $\bar{e}(\bar{x}) = y$ shows that the existential sentence in (5.193) holds in case of $y \in M'$.

If $y \in N'$ $[= \mathrm{ran}(\bar{d}' \circ \bar{d}^{-1})]$ holds, then we may apply the same arguments as in the first case to infer the truth of the existential sentence to be proven. There now exists an element, say $\bar{x}'$, such that

$$(\bar{x}', y) \in \bar{d}' \circ \bar{d}^{-1}. \tag{5.195}$$

Because of

$$\bar{d}' \circ \bar{d}^{-1} \subseteq \bar{e} \quad [= \bar{c}' \circ \bar{c}^{-1} \cup \bar{d}' \circ \bar{d}^{-1}],$$

we have that (5.195) implies $(\bar{x}', y) \in \bar{e}$ and therefore $y = \bar{e}(\bar{x}')$. Thus, the existential sentence in (5.193) holds also in case of $y \in N'$, completing the proof of the first part of the equivalence in (5.193).

To prove the second part ('$\Leftarrow$'), we now assume that there exists a constant, say $\bar{\bar{x}}$, satisfying $\bar{e}(\bar{\bar{x}}) = y$. Thus,

$$(\bar{\bar{x}}, y) \in \bar{e} \quad [= \bar{c}' \circ \bar{c}^{-1} \cup \bar{d}' \circ \bar{d}^{-1}]$$

holds, so that the disjunction of $(\bar{\bar{x}}, y) \in \bar{c}' \circ \bar{c}^{-1}$ and $(\bar{\bar{x}}, y) \in \bar{d}' \circ \bar{d}^{-1}$ follows to be true by definition of the union of two sets. We may therefore apply a proof by cases to establish the desired consequent $y \in M' \cup N'$.

In case of $(\bar{\bar{x}}, y) \in \bar{c}' \circ \bar{c}^{-1}$, it follows (by definition of a range) that

$$y \in M' \; [= \mathrm{ran}(\bar{c}' \circ \bar{c}^{-1})]$$

is true. Here, $M' \subseteq M' \cup N'$ holds with (2.201), so that $y \in M'$ implies the desired $y \in M' \cup N'$ (using the definition of a subset).

In the other case of $(\bar{\bar{x}}, y) \in \bar{d}' \circ \bar{d}^{-1}$, we obtain

$$y \in N' \; [= \mathrm{ran}(\bar{d}' \circ \bar{d}^{-1})]$$

by applying the definition of a range. Now, $N' \subseteq M' \cup N'$ holds (again with (2.201)), and then $y \in N'$ implies $y \in M' \cup N'$ (again by definition of a subset), as desired.

Thus, the existential sentence in (5.193) implies $y \in M' \cup N'$ in any case, completing the proof of the equivalence in (5.193). As y is arbitrary, we may therefore conclude that the universal sentence (5.193) is true, so that $\bar{e}$ is indeed a surjection from $M \cup N$ to $M' \cup N'$.

Next, we prove that $\bar{e}$ is also an injection from $M \cup N$ to $M' \cup N'$ by verifying

$$\forall x, x' \, ([x, x' \in M \cup N \, \wedge \, \bar{e}(x) = \bar{e}(x')] \Rightarrow x = x'). \tag{5.196}$$

We let x and x' be arbitrary and assume both $x, x' \in M \cup N$ and the equation(s) $\bar{e}(x) = \bar{e}(x')$ $[= y]$ to be true, which we may write as

$$(x, y), (x', y) \in \bar{e} \quad [= \bar{c}' \circ \bar{c}^{-1} \, \cup \, \bar{d}' \circ \bar{d}^{-1}] \tag{5.197}$$

Since we already established $\bar{e}$ as a surjection with codomain/range $M' \cup N'$, we may infer from (5.197) the truth of $y \in M' \cup N'$ $[= \mathrm{ran}(\bar{e})]$ with the definition of a range. Consequently, the definition of the union of two sets yields the true disjunction $y \in M' \vee y \in N'$, which we now employ to prove $x = x'$ by cases.

Regarding the first case, if $y \in M'$ holds, then we may prove the sentence $(x, y) \notin \bar{d}' \circ \bar{d}^{-1}$ by contradiction. To do this, we assume the negation of that sentence to be true, which gives $(x, y) \in \bar{d}' \circ \bar{d}^{-1}$ with the Double Negation Law, which implies

$$y \in N' \; [= \mathrm{ran}(\bar{d}' \circ \bar{d}^{-1})] \tag{5.198}$$

with the definition of a range. Thus, the conjunction $y \in M' \, \wedge \, y \in N'$ is true, which yields $y \in M' \cap N'$ by definition of the intersection of two sets, so that there exists an element in $M' \in N'$. Consequently, $M' \cap N' \neq \emptyset$ holds according to (2.42), which is in contradiction to the initial assumption $M' \cap N' = \emptyset$ and which thus proves $(x, y) \notin \bar{d}' \circ \bar{d}^{-1}$. Now, (5.197) shows in light of the definition of the union of two sets that the disjunction of $(x, y) \in \bar{c}' \circ \bar{c}^{-1}$ and $(x, y) \in \bar{d}' \circ \bar{d}^{-1}$ is true, where $(x, y) \in \bar{d}' \circ \bar{d}^{-1}$ is false, as shown before. Therefore, $(x, y) \in \bar{c}' \circ \bar{c}^{-1}$ is true, which we may write on the one hand in function notation as

$$y = \bar{c}' \circ \bar{c}^{-1}(x), \tag{5.199}$$

and which yields on the other hand $x \in M$ $[= \mathrm{dom}(\bar{c}' \circ \bar{c}^{-1})]$ by definition of a domain.

In the same way, we may now establish $y = \bar{c}' \circ \bar{c}^{-1}(x')$ as well as $x' \in M$, by proving first via contradiction that the sentence $(x', y) \notin \bar{d}' \circ \bar{d}^{-1}$ holds. Assuming for this purpose the negation of that sentence to hold, we obtain $(x', y) \in \bar{d}' \circ \bar{d}^{-1}$, and this evidently gives (5.198) again. Thus, $y \in M' \wedge y \in N'$ is also true, which yields $y \in M' \cap N'$, so that $M' \cap N' \neq \emptyset$ holds, in contradiction to the initial assumption $M' \cap N' = \emptyset$. This completes the proof of $(x', y) \notin \bar{d}' \circ \bar{d}^{-1}$, and since (5.197) shows that $(x', y) \in \bar{c}' \circ \bar{c}^{-1}$ or $(x', y) \in \bar{d}' \circ \bar{d}^{-1}$ is true, we may infer from this disjunction and the preceding negation that $(x', y) \in \bar{c}' \circ \bar{c}^{-1}$ holds indeed. Then, we may write this as

$$y = \bar{c}' \circ \bar{c}^{-1}(x'), \tag{5.200}$$

and we obtain moreover $x' \in M$ $[= \mathrm{dom}(\bar{c}' \circ \bar{c}^{-1})]$.

We may now combine the equations (5.199) and (5.200) to obtain the new equation $\bar{c}' \circ \bar{c}^{-1}(x) = \bar{c}' \circ \bar{c}^{-1}(x')$. Together with the already established $x, x' \in M$, this further implies the desired $x = x'$, because the bijection $\bar{c}' \circ \bar{c}^{-1}$ is in particular an injection (with domain M).

Regarding the other case $y \in N'$, we may apply similar arguments to infer from this assumption the truth of $x = x'$. First, we prove by contradiction that $(x, y) \notin \bar{c}' \circ \bar{c}^{-1}$ is true. Assuming the negation of the preceding negation to holds, so that $(x, y) \in \bar{c}' \circ \bar{c}^{-1}$ follows to be true, we obtain

$$y \in M' \ [= \mathrm{ran}(\bar{c}' \circ \bar{c}^{-1})], \tag{5.201}$$

which shows in view of the case assumption $y \in N'$ that the intersection $M' \cap N'$ is not empty. As this contradicts the initial assumption $M' \cap N' = \emptyset$, we may infer from this that $(x, y) \notin \bar{c}' \circ \bar{c}^{-1}$ is indeed true. As mentioned before, the disjunction of $(x, y) \in \bar{c}' \circ \bar{c}^{-1}$ and $(x, y) \in \bar{d}' \circ \bar{d}^{-1}$ holds, where we just established the first part as false, so that $(x, y) \in \bar{d}' \circ \bar{d}^{-1}$ is true. On the one hand, we may write this as

$$y = \bar{d}' \circ \bar{d}^{-1}(x), \tag{5.202}$$

and on the other hand we see that $x \in N$ $[= \mathrm{dom}(\bar{d}' \circ \bar{d}^{-1})]$ holds.

Next, we establish via contradiction the truth of the sentence $(x', y) \notin \bar{c}' \circ \bar{c}^{-1}$, assuming that its negation is true, so that $(x', y) \in \bar{c}' \circ \bar{c}^{-1}$ holds. This yields again (5.201), so that y is both in M' and in N', with the consequence that $M' \cap N' \neq \emptyset$, in contradiction to the assumed $M' \cap N' = \emptyset$. This proves $(x', y) \notin \bar{c}' \circ \bar{c}^{-1}$, and as the disjunction of $(x', y) \in \bar{c}' \circ \bar{c}^{-1}$ and $(x', y) \in \bar{d}' \circ \bar{d}^{-1}$ is true, we may therefore infer from this the truth of $(x', y) \in \bar{d}' \circ \bar{d}^{-1}$. We may then on the one hand write this as

$$y = \bar{d}' \circ \bar{d}^{-1}(x'), \tag{5.203}$$

and we furthermore see that $x' \in N$ $[= \mathrm{dom}(\bar{d}' \circ \bar{d}^{-1})]$ holds. Let us now combine the equations (5.202) and (5.203) to form $\bar{d}' \circ \bar{d}^{-1}(x) = \bar{d}' \circ \bar{d}^{-1}(x')$ and notice that $x, x' \in N$ is also true. Then, as the bijection $\bar{d}' \circ \bar{d}^{-1}$ is especially an injection (with domain N), it follows that $x = x'$ holds, as in the first case.

Now, since x and x' were arbitrary, we may therefore conclude that (5.196) is true, which means that the function $\bar{e} : M \cup N \to M' \cup N'$ is an injection. Together with the previous finding that $\bar{e}$ is a surjection from $M \cup N$ to $M' \cup N'$, this shows that $\bar{e} : M \cup N \to M' \cup N'$ is a bijection. We thus proved that there exists a bijection from $M \cup N$ to $M' \cup N'$, which existential sentence then implies with Proposition 4.107 the equinumerosity of the (finite) sets $M \cup N$ and $M' \cup N'$, i.e. the equation $|M \cup N| = |M' \cup N'|$. Since M, N, M' and N' were initially arbitrary sets, we may further conclude that the proposed universal sentence (5.183) is true. Then, as m and n were also arbitrary, we may finally conclude that b) holds.

Concerning c), we apply the Axiom of Specification and the Axiom of Extension to obtain the stated uniquely existential sentence. Thus, the set $+_{\mathbb{N}}$ is uniquely characterized by

$$\begin{aligned} &\forall z\, (z \in +_{\mathbb{N}} \Leftrightarrow [z \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \wedge \exists m, n, M, N, s\, (m = |M| \\ &\quad \wedge n = |N| \wedge M \cap N = \emptyset \wedge s = |M \cup N| \wedge ((m, n), s) = z)]). \end{aligned} \tag{5.204}$$

Concerning d), since $z \in +_{\mathbb{N}}$ implies in particular $z \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ for any z, it follows by definition of a subset that $+_{\mathbb{N}} \subseteq (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ holds. We now apply the Function Criterion to prove that $+_{\mathbb{N}}$ is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For this purpose, we verify

$$\forall x\, (x \in \mathbb{N} \times \mathbb{N} \Rightarrow \exists! s\, (s \in \mathbb{N} \wedge (x, s) \in +_{\mathbb{N}})). \tag{5.205}$$

We let $\bar{x} \in \mathbb{N} \times \mathbb{N}$ be arbitrary, which implies by definition of the Cartesian product of two sets that there exist elements of $\mathbb{N}$, say $\bar{m}$ and $\bar{n}$, with

$(\bar{m}, \bar{n}) = \bar{x}$. In view of a), the Cartesian products

$$\bar{M} = \{1, \ldots, \bar{m}\} \times \{0\},$$
$$\bar{N} = \{1, \ldots, \bar{n}\} \times \{1\}$$

are disjoint sets with cardinalities $|\bar{M}| = \bar{m}$ and $|\bar{N}| = \bar{n}$. As $\bar{M}$ and $\bar{N}$ are finite sets, their union $\bar{M} \cup \bar{N}$ is also finite due to Theorem 4.114, so that there exist by definition a natural number, say $\bar{s}$, and a bijection from $\{1, \ldots, \bar{s}\}$ to $\bar{M} \cup \bar{N}$. This bijection shows that the set $\bar{M} \cup \bar{N}$ has the cardinality $\bar{s} = |\bar{M} \cup \bar{N}|$ in view of the definition of a cardinality. Let us now observe that the true equations $(\bar{m}, \bar{n}) = \bar{x}$ and $\bar{s} = \bar{s}$ imply with the Equality Criterion for ordered pairs $((\bar{m}, \bar{n}), \bar{s}) = (\bar{x}, \bar{s})$. We thus showed that the multiple conjunction

$$\bar{m} = |\bar{M}| \wedge \bar{n} = |\bar{N}| \wedge \bar{M} \cap \bar{N} = \emptyset \wedge \bar{s} = |\bar{M} \cup \bar{N}| \wedge ((\bar{m}, \bar{n}), \bar{s}) = (\bar{x}, \bar{s})$$

holds, so that $\bar{z} = (\bar{x}, \bar{s})$ satisfies the existential sentence in (5.204). Because $\bar{x} \in \mathbb{N} \times \mathbb{N}$ and $\bar{s} \in \mathbb{N}$ imply (by definition of the Cartesian product of two sets) that

$$(\bar{x}, \bar{s}) \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$$

also holds, we obtain now $(\bar{x}, \bar{s}) \in +_{\mathbb{N}}$ with (5.204). Observing now that $\bar{s}$, as a cardinality, is by definition a natural number, we thus have that the existential sentence

$$\exists s \, (s \in \mathbb{N} \wedge (\bar{x}, s) \in +_{\mathbb{N}})$$

is true, so that the proof of the existential part of the uniquely existential sentence in (5.205) is complete.

To prove the uniqueness part, we verify

$$\forall s, s' \, ([s \in \mathbb{N} \wedge (\bar{x}, s) \in +_{\mathbb{N}} \wedge s' \in \mathbb{N} \wedge (\bar{x}, s') \in +_{\mathbb{N}}] \Rightarrow s = s'). \quad (5.206)$$

We let $\bar{s}$ and $\bar{s}'$ be arbitrary natural numbers with $(\bar{x}, \bar{s}) \in +_{\mathbb{N}}$ and $(\bar{x}, \bar{s}') \in +_{\mathbb{N}}$. It now follows from these assumptions with (5.204) in particular that there exist particular sets $\bar{m}$, $\bar{n}$, $\bar{M}$, $\bar{N}$ and $\bar{s}$, as well as particular sets $\bar{m}'$, $\bar{n}'$, $\bar{M}'$, $\bar{N}'$ and $\bar{s}'$ satisfying the multiple conjunctions

$$\begin{aligned} &\bar{m} = |\bar{M}| \wedge \bar{n} = |\bar{N}| \wedge \bar{M} \cap \bar{N} = \emptyset \wedge \bar{s} = |\bar{M} \cup \bar{N}| && (5.207) \\ &\quad \wedge ((\bar{m}, \bar{n}), \bar{s}) = (\bar{x}, \bar{s}), \\ &\bar{m}' = |\bar{M}'| \wedge \bar{n}' = |\bar{N}'| \wedge \bar{M}' \cap \bar{N}' = \emptyset \wedge \bar{s}' = |\bar{M}' \cup \bar{N}'| && (5.208) \\ &\quad \wedge ((\bar{m}', \bar{n}'), \bar{s}') = (\bar{x}, \bar{s}'), \end{aligned}$$

which give in particular $((\bar{m}, \bar{n}), \bar{s}) = (\bar{x}, \bar{s})$ and $((\bar{m}', \bar{n}'), \bar{s}') = (\bar{x}, \bar{s})$. The former equation further implies in particular $(\bar{m}, \bar{n}) = \bar{x}$ and the latter in particular $(\bar{m}', \bar{n}') = \bar{x}$ with the Equality Criterion for ordered pairs. These equations then yield after substitution $(\bar{m}, \bar{n}) = (\bar{m}', \bar{n}')$, which gives $\bar{m} = \bar{m}'$ and $\bar{n} = \bar{n}'$, using again the Equality Criterion for ordered pairs. Applying now substitutions based on these equations, (5.207) and (5.208) give the multiple conjunction

$$\begin{gathered}|\bar{M}| = \bar{m} \,\wedge\, |\bar{N}| = \bar{n} \,\wedge\, \bar{M} \cap \bar{N} = \emptyset \\ \wedge\, |\bar{M}'| = \bar{m} \,\wedge\, |\bar{N}'| = \bar{n} \,\wedge\, \bar{M}' \cap \bar{N}' = \emptyset,\end{gathered}$$

which implies $|\bar{M} \cup \bar{N}| = |\bar{M}' \cup \bar{N}'|$ with (5.183). Since (5.207) and (5.208) imply in particular $\bar{s} = |\bar{M} \cup \bar{N}|$ and $\bar{s}' = |\bar{M}' \cup \bar{N}'|$, we then obtain with the preceding finding the desired $\bar{s} = \bar{s}'$. Because $\bar{s}$ and $\bar{s}'$ are arbitrary, we may therefore conclude that (5.206) holds, which universal sentence proves the uniqueness part. Thus, the proof of the uniquely existential sentence in (5.205) is complete, and since $\bar{x}$ is arbitrary, we may now further conclude that the universal sentence (5.205) is true. It then follows with the Function Criterion that $+_{\mathbb{N}}$ is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$; thus, $+_{\mathbb{N}}$ is a uniquely specified binary operation on $\mathbb{N}$, which we may view as an addition. To prove that an ordered pair formed by any natural numbers is mapped according to (5.184), we now establish the universal sentence

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow m +_{\mathbb{N}} n = |[\{1, \ldots, m\} \times \{0\}] \cup [\{1, \ldots, n\} \times \{1\}]|). \tag{5.209}$$

We let $\bar{m}$ and $\bar{n}$ be arbitrary and assume $\bar{m}, \bar{n} \in \mathbb{N}$ to be true, which yields $(\bar{m}, \bar{n}) \in \mathbb{N} \times \mathbb{N}$ with the definition of the Cartesian product of two sets. According to a), we may define the Cartesian products

$$\begin{aligned}\bar{M} &= \{1, \ldots, \bar{m}\} \times \{0\}, \\ \bar{N} &= \{1, \ldots, \bar{n}\} \times \{1\},\end{aligned}$$

which satisfy

$$|\bar{M}| = \bar{m} \,\wedge\, |\bar{N}| = \bar{n} \,\wedge\, \bar{M} \cap \bar{N} = \emptyset.$$

According to b), the cardinality $\bar{s} = |\bar{M} \cup \bar{N}|$ is determined and constitutes thus a natural number, so that we obtain (again by definition of the Cartesian product of two sets)

$$((\bar{m}, \bar{n}), \bar{s}) \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}. \tag{5.210}$$

Moreover, the preceding equations show that the existential sentence

$$\begin{aligned}\exists m, n, M, N, s\, (&m = |M| \,\wedge\, n = |N| \,\wedge\, M \cap N = \emptyset \,\wedge\, s = |M \cup N| \\ &\wedge\, ((m, n), s) = ((\bar{m}, \bar{n}), |\bar{M} \cup \bar{N}|))\end{aligned}$$

holds. Then, the conjunction of (5.210) and the preceding existential sentence implies in view of (5.204) that $((\bar{m}, \bar{n}), |\bar{M} \cup \bar{N}|) \in +_{\mathbb{N}}$, which we may write in addition notation with the definitions of the sets $\bar{M}$ and $\bar{N}$ as

$$\bar{m} +_{\mathbb{N}} \bar{n} = |[\{1, \ldots, \bar{m}\} \times \{0\}] \cup [\{1, \ldots, \bar{n}\} \times \{1\}]|. \tag{5.211}$$

Since $\bar{m}$ and $\bar{n}$ were arbitrary, we may therefore conclude that (5.209) holds, so that the addition $+_{\mathbb{N}}$ on $\mathbb{N}$ is indeed characterized by the mapping (5.184).

Concerning e), we let m and n be arbitrary, assume $m, n \in \mathbb{N}$, let $\bar{M}$ and $\bar{N}$ be arbitrary sets, and assume

$$|\bar{M}| = m \wedge |\bar{N}| = n \wedge \bar{M} \cap \bar{N} = \emptyset \tag{5.212}$$

to be true. Let us now recall for the sets (5.180) and (5.181) the truth of

$$|M| = m \wedge |N| = n \wedge M \cap N = \emptyset \tag{5.213}$$

according to (5.182). We then obtain from the conjunction of (5.212) and (5.213) the equation $|\bar{M} \cup \bar{N}| = |M \cup N|$ with (5.183), and therefore with (5.184)

$$\begin{aligned} \bar{m} +_{\mathbb{N}} \bar{n} &= |[\{1, \ldots, \bar{m}\} \times \{0\}] \cup [\{1, \ldots, \bar{n}\} \times \{1\}]| \\ &= |M \cup N| \\ &= |\bar{M} \cup \bar{N}|, \end{aligned}$$

proving the implication in (5.185). Since $\bar{M}$ and $\bar{N}$ are arbitrary, we may therefore conclude that the universal sentence (5.185) holds. Because m and n were initially also arbitrary, we may further conclude that e) is true. □

Definition 5.16 (Addition on the set of natural numbers). We call

$$+_{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m, n) \mapsto m +_{\mathbb{N}} n \tag{5.214}$$

the *addition on the set of natural numbers*.

Proposition 5.47. *It is true that* 0 *is the neutral element of* $\mathbb{N}$ *with respect to the addition* $+_{\mathbb{N}}$ *on* $\mathbb{N}$.

Proof. We verify

$$\forall n \, (n \in \mathbb{N} \Rightarrow [0 +_{\mathbb{N}} n = n \wedge n +_{\mathbb{N}} 0 = n]), \tag{5.215}$$

letting $n \in \mathbb{N}$ be arbitrary. Now, the set $M = \{1, \ldots, n\}$ satisfies

$$|M| = n \tag{5.216}$$

with (4.506), and the set $N = \emptyset$ is finite (by definition) with cardinality $|N| = 0$ according to (4.527). Furthermore,

$$M \cap N = M \cap \emptyset = \emptyset$$

holds due to (2.62), which we may write equivalently as

$$N \cap M = \emptyset$$

by using (2.58). Thus, the multiple conjunctions

$$\begin{aligned}&|M| = n \wedge |N| = \emptyset \wedge M \cap N = \emptyset \\ &|N| = \emptyset \wedge |M| = n \wedge N \cap M = \emptyset\end{aligned}$$

hold, so that we may form the sums

$$\begin{aligned}n +_{\mathbb{N}} 0 &= |M \cup N| = |M \cup \emptyset| = |M| = n \\ 0 +_{\mathbb{N}} n &= |N \cup M| = |\emptyset \cup M| = |M| = n\end{aligned}$$

by applying (5.185), substitution based on $N = \emptyset$, (2.216), and (5.216). These equations give the desired conjunction of $0 +_{\mathbb{N}} n = n$ and $n +_{\mathbb{N}} 0 = n$, which proves the implication (5.215). Since n was arbitrary, it then follows that (5.215) holds, so that 0 is by definition the identity in $\mathbb{N}$ with respect to $+_{\mathbb{N}}$. □

Proposition 5.48. *The successor of a natural number n is identical with the sum of n and 1, that is,*

$$\forall n \, (n \in \mathbb{N} \Rightarrow n^+ = n +_{\mathbb{N}} 1). \tag{5.217}$$

Proof. We let n be an arbitrary natural number and observe first the truth of the multiple conjunction

$$|\{1, \ldots, n\}| = n \wedge |\{n^+\}| = 1 \wedge \{1, \ldots, n\} \cap \{n^+\} = \emptyset \tag{5.218}$$

in light of (4.506), (4.528) and (4.243). We now obtain the true equations

$$\begin{aligned}n +_{\mathbb{N}} 1 &= |\{1, \ldots, n\} \cup \{n^+\}| \\ &= |\{1, \ldots, n^+\}| \\ &= n^+\end{aligned}$$

with Theorem 5.46e), (4.241) and (4.506), as desired. Since n was arbitrary, it then follows that the proposed universal sentence (5.217) is true. □

Note 5.17. Since we established in Proposition 2.102 the natural number 2 as the successor of the natural number 1, we now obtain in view of (5.217) the set-theoretical justification of

$$1 +_{\mathbb{N}} 1 = 2.$$

Exercise 5.28. Establish the following equations for any $m, n \in \mathbb{N}$.

$$\begin{aligned} m +_{\mathbb{N}} n &= |[\{1, \ldots, m\} \times \{0\}] \cup [\{1, \ldots, n\} \times \{2\}]| \qquad (5.219) \\ &= |[\{1, \ldots, m\} \times \{1\}] \cup [\{1, \ldots, n\} \times \{2\}]|. \qquad (5.220) \end{aligned}$$

(Hint: Use Theorem 5.46a,b), (5.211), (4.508), (4.169), (4.167), (2.174), and (3.55).)

Theorem 5.49 (Associative Law for the addition on $\mathbb{N}$). *The addition $+_{\mathbb{N}}$ on $\mathbb{N}$ is associative.*

Proof. To prove that the binary operation $+_{\mathbb{N}}$ on $\mathbb{N}$ is associative, we verify

$$\forall m, n, p \, (m, n, p \in \mathbb{N} \Rightarrow (m +_{\mathbb{N}} n) +_{\mathbb{N}} p = m +_{\mathbb{N}} (n +_{\mathbb{N}} p)). \qquad (5.221)$$

Letting m, n and p be arbitrary natural numbers, we first define the sets

$$\begin{aligned} M &= \{1, \ldots, m\} \times \{0\}, \\ N &= \{1, \ldots, n\} \times \{1\}, \\ P &= \{1, \ldots, p\} \times \{2\}, \end{aligned}$$

which have the cardinalities

$$\begin{aligned} |M| &= m, \qquad (5.222) \\ |N| &= n, \qquad (5.223) \\ |P| &= p \qquad (5.224) \end{aligned}$$

due to (4.508). We now apply (5.211) as well as (5.220) to form the sums

$$\begin{aligned} m +_{\mathbb{N}} n &= |M \cup N|, \qquad (5.225) \\ n +_{\mathbb{N}} p &= |N \cup P|. \qquad (5.226) \end{aligned}$$

Next, we recall that $0 \neq 1$, $1 \neq 2$ and $0 \neq 2$ hold according to (4.165) – (4.169), which inequalities imply with (2.174)

$$\begin{aligned} \{0\} \cap \{1\} &= \emptyset, \\ \{1\} \cap \{2\} &= \emptyset, \\ \{0\} \cap \{2\} &= \emptyset, \end{aligned}$$

and therefore with (3.55)

$$\begin{aligned}(\{1,\ldots,m\}\times\{0\})\cap(\{1,\ldots,n\}\times\{1\})&=\emptyset, \qquad (5.227)\\ (\{1,\ldots,n\}\times\{1\})\cap(\{1,\ldots,p\}\times\{2\})&=\emptyset,\\ (\{1,\ldots,m\}\times\{0\})\cap(\{1,\ldots,p\}\times\{2\})&=\emptyset,\end{aligned}$$

which equations yield after substitution

$$M\cap N=\emptyset, \tag{5.228}$$

$$N\cap P=\emptyset, \tag{5.229}$$

$$M\cap P=\emptyset. \tag{5.230}$$

On the one hand, we obtain with the previous findings the true equations

$$\begin{aligned}M\cap(N\cup P)&=(M\cap N)\cup(M\cap P)\\ &=\emptyset\cup\emptyset\\ &=\emptyset\end{aligned} \tag{5.231}$$

by using the Distributive Law for the intersection of two sets, (5.223) together with (5.225), and (2.216). Thus, the multiple conjunction

$$|M|=m\;\wedge\;|N\cup P|=n+_{\mathbb{N}}p\;\wedge\;M\cap(N\cup P)=\emptyset$$

holds in view of (5.222), (5.226) and (5.231), which conjunction yields with (5.185) the sum

$$m+_{\mathbb{N}}(n+_{\mathbb{N}}p)=|M\cup(N\cup P)|. \tag{5.232}$$

On the other hand, we obtain

$$\begin{aligned}(M\cup N)\cap P&=P\cap(M\cup N)\\ &=(P\cap M)\cup(P\cap N)\\ &=(M\cap P)\cup(N\cap P)\\ &=\emptyset\cup\emptyset\\ &=\emptyset\end{aligned} \tag{5.233}$$

by applying the Commutative Law for the intersection of two sets, the Distributive Law for the intersection of two sets, again the preceding Commutative Law, then (5.230) alongside (5.229), and finally (2.216). Thus, we see that the multiple conjunction

$$|M\cup N|=m+_{\mathbb{N}}n\;\wedge\;|P|=p\;\wedge\;(M\cup N)\cap P=\emptyset \tag{5.234}$$

holds in light of (5.225), (5.224) and (5.233), and this conjunction gives

$$\begin{aligned}(m +_{\mathbb{N}} n) +_{\mathbb{N}} p &= |(M \cup N) \cup P| \\ &= |M \cup (N \cup P)| \\ &= m +_{\mathbb{N}} (n +_{\mathbb{N}} p),\end{aligned}$$

where we applied also the Associative Law for the union of two sets and the previously established sume (5.232).

This finding proves the implication in (5.222), and since m, n and p were arbitrary, we may therefore conclude that the universal sentence (5.221) is true. Thus, the addition $+_{\mathbb{N}}$ is an associative binary operation by definition. □

Note 5.18. In view of the Associative Law for the addition on $\mathbb{N}$, the ordered pair $(\mathbb{N}, +_{\mathbb{N}})$ is a semigroup.

Theorem 5.50 (Commutative Law for the addition on $\mathbb{N}$). *The addition $+_{\mathbb{N}}$ on the set of natural numbers is a commutative binary operation.*

Exercise 5.29. Prove the Commutative Law for the addition on $\mathbb{N}$.

(Hint: Proceed similarly as in the proof of Theorem 5.49, applying now (2.214) instead of (2.219).)

Note 5.19. The ordered pair $(\mathbb{N}, +_{\mathbb{N}})$ is a commutative semigroup.

The *product of two natural numbers* m and n can be defined through a 'duplication' process via the Cartesian product of sets M and N with cardinalities m and n, respectively.

Theorem 5.51 (Multiplication on the set of natural numbers). *The following sentences are true.*

a) For any $m, n \in \mathbb{N}$, the initial segments of $\mathbb{N}_+$

$$M = \{1, \ldots, m\} \tag{5.235}$$

$$N = \{1, \ldots, n\} \tag{5.236}$$

are sets with respective cardinality m and n, i.e. the sets M and N satisfy

$$|M| = m \wedge |N| = n. \tag{5.237}$$

b) For any $m, n \in \mathbb{N}$, the cardinality of the Cartesian product of two sets with respective cardinalities m and n is unique in the sense that

$$\begin{aligned}&\forall M, N, M', N' \, ([|M| = m \wedge |N| = n \wedge |M'| = m \wedge |N'| = n] \\ &\quad \Rightarrow |M \times N| = |M' \times N'|).\end{aligned} \tag{5.238}$$

c) There exists a unique set $\cdot_{\mathbb{N}}$ *such that an element* z *is in* $\cdot_{\mathbb{N}}$ *iff* z *is in* $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ *and moreover if* z *is some ordered triple* $((m,n),p)$ *such that* m *and* n *are the cardinalities of some sets* M *and* N *and such that* p *is the cardinality of the Cartesian product of* M *and* N*, i.e.*

$$\exists! \cdot_{\mathbb{N}} \forall z\,(z \in \cdot_{\mathbb{N}} \Leftrightarrow [z \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \wedge \exists m,n,M,N,s\,(m = |M| \\ \wedge\ n = |N| \wedge p = |M \times N| \wedge ((m,n),p) = z)]).$$

d) This set $\cdot_{\mathbb{N}}$ *is a multiplication on* $\mathbb{N}$ *satisfying*

$$(m,n) \mapsto m \cdot_{\mathbb{N}} n = |\{1,\ldots,m\} \times \{1,\ldots,n\}|. \tag{5.239}$$

e) Furthermore, the multiplication $\cdot_{\mathbb{N}}$ *on* $\mathbb{N}$ *satisfies for any* $m,n \in \mathbb{N}$

$$\forall M,N\,(|M| = m \wedge |N| = n] \Rightarrow m \cdot_{\mathbb{N}} n = |M \times N|). \tag{5.240}$$

Proof. Concerning b), we take arbitrary sets m, n, M, N, M', and N', and we assume $m,n \in \mathbb{N}$, $|M| = |M'| = m$ as well as $|N| = |N'| = n$ to be true. Let us now observe in light of Theorem 4.121 that the finiteness of the sets M and N implies the finiteness of the Cartesian product $M \times N$, and that for the same reason the finiteness of M' and N' implies the finiteness of the Cartesian product $M' \times N'$. To prove that these two Cartesian products satisfy the stated equation $|M \times N| = |M' \times N'|$, we establish now a bijection from $M \times N$ to $M' \times N'$.

On the one hand, the two assumed equations $m = |M|$ and $m = |M'|$ imply by definition of a cardinality that there are two particular corresponding bijections

$$\bar{c} : \{1,\ldots,m\} \rightleftarrows M, \tag{5.241}$$
$$\bar{c}' : \{1,\ldots,m\} \rightleftarrows M'. \tag{5.242}$$

On the other hand, the other two assumed equations $n = |N|$ and $n = |N'|$ imply (again with the definition of a cardinality) the existence of particular bijections

$$\bar{d} : \{1,\ldots,n\} \rightleftarrows N, \tag{5.243}$$
$$\bar{d}' : \{1,\ldots,n\} \rightleftarrows N'. \tag{5.244}$$

Let us observe here that the values $\bar{c}(i)$, $\bar{d}(j)$, $\bar{c}'(i)$ and $\bar{d}'(j)$ are specified for all $i \in \{1,\ldots,m\}$ and all $j \in \{1,\ldots,n\}$, since $\{1,\ldots,m\}$ is the domain of $\bar{c}$ as well as of $\bar{c}'$ and since $\{1,\ldots,n\}$ is the domain of $\bar{d}$ as well as of $\bar{d}'$.

We may therefore apply the Axiom of Specification in connection with the Equality Criterion for sets to obtain the true uniquely existential sentence

$$\exists! \bar{e} \forall z \, (z \in \bar{e} \Leftrightarrow [z \in (M \times N) \times (M' \times N') \, \wedge \, \exists i, j \, (i \in \{1, \ldots, m\} \\ \wedge \, j \in \{1, \ldots, n\} \, \wedge \, ((\bar{c}(i), \bar{d}(j)), (\bar{c}'(i), \bar{d}'(j))) = z)]).$$

Thus, the set $\bar{e}$ is uniquely characterized by

$$\forall z \, (z \in \bar{e} \Leftrightarrow [z \in (M \times N) \times (M' \times N') \, \wedge \, \exists i, j \, (i \in \{1, \ldots, m\} \\ \wedge \, j \in \{1, \ldots, n\} \, \wedge \, ((\bar{c}(i), \bar{d}(j)), (\bar{c}'(i), \bar{d}'(j))) = z)]). \tag{5.245}$$

Then, $z \in \bar{e}$ implies in particular $z \in (M \times N) \times (M' \times N')$ for any z, which means that $\bar{e}$ is included in $(M \times N) \times (M' \times N')$ by definition of a subset. Applying the notation $A = M \times N$ and $B = M' \times N'$, we may therefore treat $\bar{e}$ as a binary relation relation with $\bar{e} \subseteq A \times B$.

We now apply the Function Criterion to prove that $\bar{e}$ is a function from $M \times N$ to $M' \times N'$, and verify accordingly

$$\forall x \, (x \in M \times N \Rightarrow \exists! y \, (y \in M' \times N' \, \wedge \, (x, y) \in \bar{e})). \tag{5.246}$$

Letting $\bar{x} \in M \times N$ be arbitrary, we first establish the existential part. To begin with, it follows by definition of the Cartesian product of two sets that there exist elements, say $\bar{m}$ and $\bar{n}$, such that $\bar{m} \in M$, $\bar{n} \in N$ and $(\bar{m}, \bar{n}) = \bar{x}$ hold. Since the bijections $\bar{c}$ and $\bar{d}$ are in particular surjections with codomain M and N, respectively, we have that M is the range of $\bar{c}$ and that N is the range of $\bar{d}$. Consequently, $\bar{m} \in M$, $\bar{n} \in N$ imply that there exist elements, say $\bar{i}$ and $\bar{j}$, with $(\bar{i}, \bar{m}) \in \bar{c}$ and with $(\bar{j}, \bar{n}) \in \bar{d}$. On the one hand, we may write these findings in function notation as

$$\bar{c}(\bar{i}) = \bar{m}, \tag{5.247}$$

$$\bar{d}(\bar{j}) = \bar{n}. \tag{5.248}$$

On the other hand, it follows by definition of a domain that $\bar{i} \in \{1, \ldots, m\}$ [$=$ $\mathrm{dom}(\bar{c})$] and $\bar{j} \in \{1, \ldots, n\}$ [$= \mathrm{dom}(\bar{d})$] hold with (5.241) and (5.243). Since $\{1, \ldots, m\} = \mathrm{dom}(\bar{c}')$ and $\{1, \ldots, n\} = \mathrm{dom}(\bar{d}')$ are also true according to (5.242) and (5.244), it follows (again by definition of a domain) that there exist elements, say $\bar{\bar{m}}$ and $\bar{\bar{n}}$, such that $(\bar{i}, \bar{\bar{m}}) \in \bar{c}'$ and $(\bar{j}, \bar{\bar{n}}) \in \bar{d}'$. We therefore obtain on the one hand

$$\bar{c}'(\bar{i}) = \bar{\bar{m}}, \tag{5.249}$$

$$\bar{d}'(\bar{j}) = \bar{\bar{n}}, \tag{5.250}$$

and on the other hand it follows (by definition of a range) that $\bar{\bar{m}} \in M'$ [$=$ $\mathrm{ran}(\bar{c}')$] and $\bar{\bar{n}} \in N'$ [$= \mathrm{ran}(\bar{d}')$]. In view of the existence of the values $\bar{c}(\bar{i})$,

$\bar{d}(\bar{j})$, $\bar{c}'(\bar{i})$ and $\bar{d}'(\bar{j})$ in (5.247) – (5.250), the ordered pairs $(\bar{c}(\bar{i}), \bar{d}(\bar{j}))$ $[= \bar{x}]$ and $\bar{y} = (\bar{c}'(\bar{i}), \bar{d}'(\bar{j}))$ are uniquely specified, and then the ordered pair

$$\bar{z} = ((\bar{c}(\bar{i}), \bar{d}(\bar{j})), (\bar{c}'(\bar{i}), \bar{d}'(\bar{j}))) = ((\bar{m}, \bar{n}), (\bar{\bar{m}}, \bar{\bar{n}})) = (\bar{x}, \bar{y})$$

also exists. Together with the previously established $\bar{i} \in \{1, \ldots, m\}$ and $\bar{j} \in \{1, \ldots, n\}$, this proves that $\bar{z}$ satisfies the existential sentence in (5.245). We now verify that $\bar{z}$ satisfies also the first part of the conjunction in (5.245). Besides the initially chosen element $\bar{x} \in M \times N$, we also have $\bar{y} = (\bar{\bar{m}}, \bar{\bar{n}})$, which equation implies together with $\bar{\bar{m}} \in M'$ and $\bar{\bar{n}} \in N'$ (by definition of the Cartesian product of two sets) that $\bar{y} \in M' \times N'$ holds. Then, the conjunction of $\bar{x} \in M \times N$, $\bar{y} \in M' \times N'$ and $\bar{z} = (\bar{x}, \bar{y})$ further implies (using again the definition of the Cartesian product of two sets) the truth of $\bar{z} \in (M \times N) \times (M' \times N')$. Thus, $\bar{z}$ satisfies the conjunction on the right-hand side of the equivalence in (5.245), so that $\bar{z} = (\bar{x}, \bar{y})$ follows to be an element of $\bar{e}$, where $\bar{y} \in M' \times N'$. Therefore, $\bar{x}$ satisfies the existential part of the uniquely existential sentence in (5.246).

Regarding the uniqueness part, we prove

$$\forall y, y' \, ([y \in M' \times N' \;\wedge\; (\bar{x}, y) \in \bar{e} \;\wedge\; y' \in M' \times N' \;\wedge\; (\bar{x}, y') \in \bar{e}] \Rightarrow y = y'). \tag{5.251}$$

We let $\bar{y}$ and $\bar{y}'$ be arbitrary elements of $M' \times N'$ such that $(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{y}')$ are in $\bar{e}$. On the one hand, $(\bar{x}, \bar{y})$ implies with (5.245) in particular that there exist an element of $\{1, \ldots, m\}$, say $\bar{i}$, and an element of $\{1, \ldots, n\}$, say $\bar{j}$, with

$$((\bar{c}(\bar{i}), \bar{d}(\bar{j})), (\bar{c}'(\bar{i}), \bar{d}'(\bar{j}))) = (\bar{x}, \bar{y}).$$

This equation then yields $(\bar{c}(\bar{i}), \bar{d}(\bar{j})) = \bar{x}$ and $(\bar{c}'(\bar{i}), \bar{d}'(\bar{j})) = \bar{y}$ with the Equality Criterion for ordered pairs. On the other hand, $(\bar{x}, \bar{y}') \in \bar{e}$ implies again with (5.245) in particular that there are an element of $\{1, \ldots, m\}$, say $\bar{i}'$, and an element of $\{1, \ldots, n\}$, say $\bar{j}'$, such that

$$((\bar{c}(\bar{i}'), \bar{d}(\bar{j}')), (\bar{c}'(\bar{i}'), \bar{d}'(\bar{j}'))) = (\bar{x}, \bar{y}').$$

This equation gives $(\bar{c}(\bar{i}'), \bar{d}(\bar{j}')) = \bar{x}$ and $(\bar{c}'(\bar{i}'), \bar{d}'(\bar{j}')) = \bar{y}'$, utilizing again the Equality Criterion for ordered pairs. Combining the previous two expressions for $\bar{x}$, we arrive at

$$[\bar{x} =] \quad (\bar{c}(\bar{i}), \bar{d}(\bar{j})) = (\bar{c}(\bar{i}'), \bar{d}(\bar{j}')),$$

which in turn implies $\bar{c}(\bar{i}) = \bar{c}(\bar{i}')$ and $\bar{d}(\bar{j}) = \bar{d}(\bar{j}')$, applying the Equality Criterion for ordered pairs once again. Since the bijection $\bar{c}$ is in particular an injection and since both $\bar{i}$ and $\bar{i}'$ are elements of the domain $\{1, \ldots, m\}$ of $\bar{c}$, it follows from $\bar{c}(\bar{i}) = \bar{c}(\bar{i}')$ that $\bar{i} = \bar{i}'$ holds. Similarly, as the bijection $\bar{d}$ is in particular an injection and as both $\bar{j}$ and $\bar{j}'$ are in the domain $\{1, \ldots, n\}$ of $\bar{d}$, it follows from $\bar{d}(\bar{j}) = \bar{d}(\bar{j}')$ that $\bar{j} = \bar{j}'$ is true. Consequently, we obtain via substitution

$$\bar{y} = (\bar{c}'(\bar{i}), \bar{d}'(\bar{j})) = (\bar{c}'(\bar{i}'), \bar{d}'(\bar{j}')) = \bar{y}',$$

which yields the desired $\bar{y} = \bar{y}'$. Since $\bar{y}$ and $\bar{y}'$ are arbitrary, we may therefore conclude that the universal sentence (5.251) holds, so that the proof of the uniqueness part of the uniquely existential sentence in (5.246) is complete. As $\bar{x}$ was also arbitrary, we may then further conclude that the universal sentence (5.246) is true, which implies with the Function Criterion that $\bar{e}$ is a function from $M \times N$ to $M' \times N'$, i.e.

$$\bar{e} : M \times N \to M' \times N'.$$

Let us now verify that $\bar{e}$ is a surjection. Since $M' \times N'$ is a codomain of $\bar{e}$, we see that $\operatorname{ran}(\bar{e}) \subseteq M' \times N'$ holds. We now establish the truth of $M' \times N' \subseteq \operatorname{ran}(\bar{e})$, i.e. of

$$\forall y \, (y \in M' \times N' \Rightarrow y \in \operatorname{ran}(\bar{e})). \tag{5.252}$$

For this purpose, we let $\bar{y} \in M' \times N'$ be arbitrary, which implies (by definition of the Cartesian product of two sets) that there exist an element in M', say $\bar{m}'$, and an element in N', say $\bar{n}'$, such that $(\bar{m}', \bar{n}') = \bar{y}$ holds. Recalling that the one-to-one correspondence $\bar{c}'$ is in particular an onto function with codomain/range M' and that the bijection $\bar{d}'$ is in particular a surjection with codomain/range N', it follows on the one hand from $\bar{m}' \in M'$ by definition of a range that there exists an element, say $\bar{i}$, with $(\bar{i}, \bar{m}') \in \bar{c}'$, which we may also write as $\bar{m}' = \bar{c}'(\bar{i})$. On the other hand, it follows from $\bar{n}' \in N'$ (again by definition of range) that there exists an element, say $\bar{j}$, such that $(\bar{j}, \bar{n}') \in \bar{d}'$, i.e. such that $\bar{n}' = \bar{d}'(\bar{j})$. Therefore, the previously established equation $\bar{y} = (\bar{m}', \bar{n}')$ becomes

$$\bar{y} = (\bar{c}'(\bar{i}), \bar{d}'(\bar{j})).$$

Let us also recall that $\{1, \ldots, m\}$ is the domain of $\bar{c}'$ and that $\{1, \ldots, n\}$ is the domain of $\bar{d}'$. Then, the previously obtained $(\bar{i}, \bar{m}') \in \bar{c}'$ and $(\bar{j}, \bar{n}') \in \bar{d}'$ imply by definition of a domain that $\bar{i} \in \{1, \ldots, m\}$ and that $\bar{j} \in \{1, \ldots, n\}$ hold. Since $\{1, \ldots, m\}$ is the domain also of $\bar{c}$ and $\{1, \ldots, n\}$ the domain of $\bar{d}$, it follows on the one hand from $\bar{i} \in \{1, \ldots, m\}$ that there exists an element, say $\bar{\bar{m}}$, with $(\bar{i}, \bar{\bar{m}}) \in \bar{c}$, i.e. with $\bar{\bar{m}} = \bar{c}(\bar{i})$. On the other hand, it follows from $\bar{j} \in \{1, \ldots, n\}$ that there is an element, say $\bar{\bar{n}}$, with $(\bar{j}, \bar{\bar{n}}) \in \bar{d}$, i.e. with $\bar{\bar{n}} = \bar{d}(\bar{j})$. Recaling now that M is the codomain/range of the bijection/surjection $\bar{c}$ and N the codomain/range of the bijection/surjection

$\bar{d}$, we have that $(\bar{i}, \bar{\bar{m}}) \in \bar{c}$ implies $\bar{\bar{m}} \in M$ and that $(\bar{j}, \bar{\bar{n}}) \in \bar{d}$ implies $\bar{\bar{n}} \in N$. Then, the ordered pair

$$\bar{x} = (\bar{\bar{m}}, \bar{\bar{n}}) = (\bar{c}(\bar{i}), \bar{d}(\bar{j}))$$

is specified and implies together with $\bar{\bar{m}} \in M$ and $\bar{\bar{n}} \in N$, by definition of the Cartesian product of two sets, that $\bar{x} \in M \times N$ is true. The ordered pair

$$\bar{z} = (\bar{x}, \bar{y}) = ((\bar{\bar{m}}, \bar{\bar{n}}), (\bar{m}', \bar{n}')) = ((\bar{c}(\bar{i}), \bar{d}(\bar{j})), (\bar{c}'(\bar{i}), \bar{d}'(\bar{j})))$$

is then also specified, so that the first of these equations implies with $\bar{x} \in M \times N$ and the initially assumed $\bar{y} \in M' \times N'$ (again by definition of the Cartesian product of two sets) that $\bar{z} \in (M \times N) \times (M' \times N')$ holds. The preceding equations also show that there exist i and j with $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ and $((\bar{c}(i), \bar{d}(j)), (\bar{c}'(i), \bar{d}'(j))) = \bar{z}$. The conjunction of the previous two findings therefore implies with (5.245) that $[(\bar{x}, \bar{y}) =]\, \bar{z} \in \bar{e}$ holds. Thus, $(\bar{x}, \bar{y}) \in \bar{e}$ is true, which further implies $\bar{y} \in \mathrm{ran}(\bar{e})$ (applying the definition of a range). This completes the proof of the implication in (5.252), and since $\bar{y}$ was arbitrary, we may therefore conclude that (5.252) holds. It then follows from this with the definition of a subset that $M' \times N' \subseteq \mathrm{ran}(\bar{e})$ is indeed true. Together with the already mentioned fact $\mathrm{ran}(\bar{e}) \subseteq M' \times N'$, this gives then the equation $\mathrm{ran}(\bar{e}) = M' \times N'$ with the Axiom of Extension. Consequently, the function $\bar{e} : M \times N \to M' \times N'$ is a surjection, i.e.

$$\bar{e} : M \times N \twoheadrightarrow M' \times N'.$$

Next, we demonstrate that this surjection is also an injection from $M \times N$ to $M' \times N'$. To do this, we verify

$$\forall x, x' \,([x, x' \in M \times N \,\wedge\, \bar{e}(x) = \bar{e}(x')] \Rightarrow x = x'), \tag{5.253}$$

letting $\bar{x}$ and $\bar{x}'$ be arbitrary elements of $M \times N$ such that $\bar{e}(\bar{x}) = \bar{e}(\bar{x}')$ holds. Denoting this value of the function $\bar{e}$ by $\bar{y}$, we may then also write the preceding equations as $(\bar{x}, \bar{y}) \in \bar{e}$ and $(\bar{x}', \bar{y}) \in \bar{e}$. On the one hand, $(\bar{x}, \bar{y}) \in \bar{e}$ implies with (5.245) in particular that there exists an element of $\{1, \ldots, m\}$, say $\bar{i}$, and that there exists an element of $\{1, \ldots, n\}$, say $\bar{j}$, such that

$$((\bar{c}(\bar{i}), \bar{d}(\bar{j})), (\bar{c}'(\bar{i}), \bar{d}'(\bar{j}))) = (\bar{x}, \bar{y}).$$

Let us here notice that this equation yields $(\bar{c}(\bar{i}), \bar{d}(\bar{j})) = \bar{x}$ as well as $(\bar{c}'(\bar{i}), \bar{d}'(\bar{j})) = \bar{y}$ with the Equality Criterion for ordered pairs. On the other hand, $(\bar{x}', \bar{y}) \in \bar{e}$ implies with (5.245) in particular that there is an element in $\{1, \ldots, m\}$, say $\bar{i}'$, and that there is an element in $\{1, \ldots, n\}$, say $\bar{j}'$, satisfying

$$((\bar{c}(\bar{i}'), \bar{d}(\bar{j}')), (\bar{c}'(\bar{i}'), \bar{d}'(\bar{j}'))) = (\bar{x}', \bar{y}).$$

Then, this equation gives $(\bar{c}(\bar{i}'), \bar{d}(\bar{j}')) = \bar{x}'$ as well as $(\bar{c}'(\bar{i}'), \bar{d}'(\bar{j}')) = \bar{y}$ again with the Equality Criterion for ordered pairs. We thus obtained two expressions for $\bar{y}$, so that substitution yields

$$[\bar{y} =] \quad (\bar{c}'(\bar{i}), \bar{d}'(\bar{j})) = (\bar{c}'(\bar{i}'), \bar{d}'(\bar{j}')).$$

We may then evidently infer from these identical ordered pairs the truth of $\bar{c}'(\bar{i}) = \bar{c}'(\bar{i}')$ and $\bar{d}'(\bar{j}) = \bar{d}'(\bar{j}')$. Since $\bar{c}'$ and $\bar{d}'$ are one-to-one functions and since $\bar{i}, \bar{i}' \in \{1, \ldots, m\}$ as well as $\bar{j}, \bar{j}' \in \{1, \ldots, n\}$ holds, the previous two equations imply $\bar{i} = \bar{i}'$ and $\bar{j} = \bar{j}'$, respectively. Applying now substitutions based on these equations, we obtain

$$\bar{x} = (\bar{c}(\bar{i}), \bar{d}(\bar{j})) = (\bar{c}(\bar{i}'), \bar{d}(\bar{j}')) = \bar{x}',$$

so that $\bar{x} = \bar{x}'$ follows to be true, as desired. Because $\bar{x}$ and $\bar{x}'$ are arbitrary, we may therefore conclude that the universal sentence (5.253) holds, which shows that $\bar{e} : M \times N \to M' \times N'$ is an injection, that is,

$$\bar{e} : M \times N \hookrightarrow M' \times N'.$$

As the function $\bar{e}$ is both an injection and a surjection, this means that $\bar{e}$ is a bijection, i.e.

$$\bar{e} : M \times N \rightleftarrows M' \times N'.$$

We thus demonstrated the existence of a bijection from the finite set $M \times N$ to the finite set $M' \times N'$, so that $|M \times N| = |M' \times N'|$ follows to be true with Proposition 4.107. As M, N, M' and N' were arbitrary sets, we may therefore conclude that the universal sentence (5.238) is true, so that the proof of b) is complete.

Concerning c), we apply the Axiom of Specification and the Equality Criterion for sets to obtain the stated uniquely existential sentence, so that the set $\cdot_{\mathbb{N}}$ is uniquely characterized by

$$\begin{aligned} \forall z\, (z \in \cdot_{\mathbb{N}} \Leftrightarrow [z \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \,\wedge\, \exists m, n, M, N, p\, (m = |M| \\ \wedge\, n = |N| \,\wedge\, p = |M \times N| \,\wedge\, ((m, n), p) = z)]). \end{aligned} \tag{5.254}$$

Concerning d), we notice in light of (5.254) that $\cdot_{\mathbb{N}} \subseteq (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ holds by definition of a subset because $z \in \cdot_{\mathbb{N}}$ implies in particular $z \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ for all z. Next, we use the Function Criterion to show that $\cdot_{\mathbb{N}}$ is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. To do this, we prove

$$\forall x\, (x \in \mathbb{N} \times \mathbb{N} \Rightarrow \exists! p\, (p \in \mathbb{N} \,\wedge\, (x, p) \in \cdot_{\mathbb{N}})), \tag{5.255}$$

letting $\bar{x} \in \mathbb{N} \times \mathbb{N}$ be arbitrary. Consequently, there exist (by definition of the Cartesian product of two sets) elements of $\mathbb{N}$, say $\bar{m}$ and $\bar{n}$, such that $(\bar{m}, \bar{n}) = \bar{x}$ holds. Therefore, the initial segments of $\mathbb{N}_+$

$$\bar{M} = \{1, \ldots, \bar{m}\},$$
$$\bar{N} = \{1, \ldots, \bar{n}\}$$

are specified, where

$$|\bar{M}| = |\{1, \ldots, \bar{m}\}| = \bar{m},$$
$$|\bar{N}| = |\{1, \ldots, \bar{n}\}| = \bar{n}$$

are true because of Corollary 4.104. The Cartesian product $\bar{M} \times \bar{N}$ of the finite sets $\bar{M}$ and $\bar{N}$ is itself finite in view of Theorem 4.121, which fact then implies the existence of particular natural number $\bar{p}$ and of a particular bijection $\bar{c} : \{1, \ldots, \bar{p}\} \rightleftarrows \bar{M} \times \bar{N}$; thus, the definition of a cardinality yields $\bar{p} = |\bar{M} \times \bar{N}|$ $[\in \mathbb{N}]$. Consequently, the truth of $\bar{x} \in \mathbb{N} \times \mathbb{N}$ and of $\bar{p} \in \mathbb{N}$ shows that the ordered pair $\bar{z} = (\bar{x}, \bar{p})$ is an element of $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$. Furthermore, since the previous findings demonstrate the truth of the multiple conjunction

$$\bar{m} = |\bar{M}| \wedge \bar{n} = |\bar{N}| \wedge \bar{p} = |\bar{M} \times \bar{N}| \wedge ((\bar{m}, \bar{n}), \bar{p}) = \bar{z},$$

we see that $\bar{z}$ satisfies the right-hand side of the equivalence in (5.254), so that $\bar{z}$ follows to be in $\cdot_{\mathbb{N}}$. This means that $(\bar{x}, \bar{p}) \in \cdot_{\mathbb{N}}$ holds (alongside $\bar{p} \in \mathbb{N}$), so that there exists a set p such that the conjunction $p \in \mathbb{N} \wedge (x, p) \in \cdot_{\mathbb{N}}$ is true. This completes the proof of the existential part of the uniquely existential sentence in (5.255).

It remains for us to prove the uniqueness part, for which we verify

$$\forall p, p' \left([p \in \mathbb{N} \wedge (\bar{x}, p) \in \cdot_{\mathbb{N}} \wedge p' \in \mathbb{N} \wedge (\bar{x}, p') \in \cdot_{\mathbb{N}}] \Rightarrow p = p' \right). \quad (5.256)$$

For this purpose, we let $\bar{p}$ and $\bar{p}'$ be arbitrary in $\mathbb{N}$ such that $(\bar{x}, \bar{p})$ and $(\bar{x}, \bar{p}')$ are elements of $\cdot_{\mathbb{N}}$. Then, $(\bar{x}, \bar{p}) \in \cdot_{\mathbb{N}}$ implies with (5.254) that $(\bar{x}, \bar{p})$ is in $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ and that there are sets, say $\bar{m}$, $\bar{n}$, $\bar{M}$, $\bar{N}$ and $\bar{p}$, satisfying the multiple conjunction

$$\bar{m} = |\bar{M}| \wedge \bar{n} = |\bar{N}| \wedge \bar{p} = |\bar{M} \times \bar{N}| \wedge ((\bar{m}, \bar{n}), \bar{p}) = (\bar{x}, \bar{p}). \quad (5.257)$$

Similarly, $(\bar{x}, \bar{p}') \in \cdot_{\mathbb{N}}$ implies that $(\bar{x}, \bar{p}')$ is in $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ and that there exist sets, say $\bar{m}'$, $\bar{n}'$, $\bar{M}'$, $\bar{N}'$ and $\bar{p}'$ such that

$$\bar{m}' = |\bar{M}'| \wedge \bar{n}' = |\bar{N}'| \wedge \bar{p}' = |\bar{M}' \times \bar{N}'| \wedge ((\bar{m}', \bar{n}'), \bar{p}') = (\bar{x}, \bar{p}'). \quad (5.258)$$

Now, (5.257) and (5.258) show in particular that the equations $((\bar{m}, \bar{n}), \bar{p}) = (\bar{x}, \bar{p})$ and $((\bar{m}', \bar{n}'), \bar{p}') = (\bar{x}, \bar{p}')$ are true. Here, the former equation gives in particular $(\bar{m}, \bar{n}) = \bar{x}$ and the latter in particular $(\bar{m}', \bar{n}') = \bar{x}$ because of the Equality Criterion for ordered pairs. Combining these equations via substitution, we obtain $(\bar{m}, \bar{n}) = (\bar{m}', \bar{n}')$, which then further implies $\bar{m} = \bar{m}'$ and $\bar{n} = \bar{n}'$ again with the Equality Criterion for ordered pairs. We may now apply further substitutions based on these equations in (5.257) and (5.258) to obtain the conjunction

$$|\bar{M}| = \bar{m} \;\wedge\; |\bar{N}| = \bar{n} \;\wedge\; |\bar{M}'| = \bar{m} \;\wedge\; |\bar{N}'| = \bar{n},$$

which then yields $|\bar{M} \times \bar{N}| = |\bar{M}' \times \bar{N}'|$ with (5.238). As the multiple conjunctions (5.257) and (5.258) show in particular that $\bar{p} = |\bar{M} \times \bar{N}|$ and $\bar{p}' = |\bar{M}' \times \bar{N}'|$ are true, we find after substitutions based on the preceding three equations $\bar{p} = \bar{p}'$, as desired. Since $\bar{p}$ and $\bar{p}'$ are arbitrary, (5.256) follows then to be true, which completes the proof of the uniqueness part and thus the proof of the uniquely existential sentence in (5.255). As $\bar{x}$ was also arbitrary, we may now further conclude that (5.255) holds, from which it follows with the Function Criterion that $\cdot_{\mathbb{N}}$ is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. We thus proved that $\cdot_{\mathbb{N}}$ is a uniquely specified binary operation on $\mathbb{N}$, which we may consider to be a multiplication. To establish the mapping (5.239), we verify

$$\forall m, n \,(m, n \in \mathbb{N} \Rightarrow m \cdot_{\mathbb{N}} n = |\{1, \ldots, m\} \times \{1, \ldots, n\}). \tag{5.259}$$

We let $\bar{m}, \bar{n} \in \mathbb{N}$ be arbitrary, so that we obtain $(\bar{m}, \bar{n}) \in \mathbb{N} \times \mathbb{N}$ by definition of the Cartesian product of two sets. According to a), we may define the sets

$$\bar{M} = \{1, \ldots, \bar{m}\}$$
$$\bar{N} = \{1, \ldots, \bar{n}\}$$

with cardinalities

$$|\bar{M}| = \bar{m} \;\wedge\; |\bar{N}| = \bar{n}.$$

Since b) shows that the cardinality $\bar{p} = |\bar{M} \times \bar{N}|$ is a specified natural number, we furthermore obtain (again by definition of the Cartesian product of two sets)

$$((\bar{m}, \bar{n}), \bar{p}) \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}. \tag{5.260}$$

Evidently, the existential sentence

$$\exists m, n, M, N, s \,(m = |M| \;\wedge\; n = |N| \;\wedge\; p = |M \times N|$$
$$\wedge\; ((m, n), p) = ((\bar{m}, \bar{n}), |\bar{M} \times \bar{N}|))$$

is true, which implies together with (5.260) $((\bar{m},\bar{n}),|\bar{M}\times\bar{N}|)\in\cdot_{\mathbb{N}}$ according to (5.254); using the notation for a multiplication and the definitions of the sets $\bar{M}$ and $\bar{N}$, we may therefore write

$$\bar{m}\cdot_{\mathbb{N}}\bar{n}=|\{1,\ldots,\bar{m}\}\times\{1,\ldots,\bar{n}\}|. \tag{5.261}$$

As $\bar{m}$ and $\bar{n}$ were arbitrary, we may therefore conclude that (5.259) is true, which shows that the multiplication $\cdot_{\mathbb{N}}$ on $\mathbb{N}$ is characterized by (5.239). □

Exercise 5.30. Establish Theorem 5.51a,e).

Definition 5.17 (Multiplication on the set of natural numbers). We call

$$\cdot_{\mathbb{N}}:\mathbb{N}\times\mathbb{N}\to\mathbb{N},\quad (m,n)\mapsto m\cdot_{\mathbb{N}}n \tag{5.262}$$

the *multiplication on the set of natural numbers*.

Theorem 5.52 (Associative Law for the multiplication on $\mathbb{N}$). *The multiplication $\cdot_{\mathbb{N}}$ on the set of natural numbers is an associative binary operation.*

Proof. To demonstrate the associativity of the binary operation $\cdot_{\mathbb{N}}$ on $\mathbb{N}$, we prove the universal sentence

$$\forall m,n,p\,(m,n,p\in\mathbb{N}\Rightarrow(m\cdot_{\mathbb{N}}n)\cdot_{\mathbb{N}}p=m\cdot_{\mathbb{N}}(n\cdot_{\mathbb{N}}p)). \tag{5.263}$$

For this purpose, we let m, n and p be arbitrary natural numbers, so that the initial segments

$$\begin{aligned} M &= \{1,\ldots,m\},\\ N &= \{1,\ldots,n\},\\ P &= \{1,\ldots,p\}\end{aligned}$$

have the required cardinalities $|M|=m$, $|N|=n$ and $|P|=p$, as shown by (4.506). Consequently, we may apply (5.239) to form the products

$$m\cdot_{\mathbb{N}}n = |M\times N|, \tag{5.264}$$

$$n\cdot_{\mathbb{N}}p = |N\times P|. \tag{5.265}$$

Since the conjunction

$$|M|=m\wedge|N\times P|=n\cdot_{\mathbb{N}}p$$

is then evidently true, we may use (5.240) to form the product

$$m\cdot_{\mathbb{N}}(n\cdot_{\mathbb{N}}p)=|M\times(N\times P)|. \tag{5.266}$$

Moreover, the conjunction

$$|M \times N| = m \cdot_{\mathbb{N}} n \, \wedge \, |P| = p$$

holds as well, so that we obtain the equations

$$\begin{aligned}(m \cdot_{\mathbb{N}} n) \cdot_{\mathbb{N}} p &= |(M \times N) \times P| \\ &= |M \times (N \times P)| \\ &= m \cdot_{\mathbb{N}} (n \cdot_{\mathbb{N}} p)\end{aligned}$$

applying (5.240), then Proposition 4.122d), and finally (5.266).

Because m, n and p were arbitrary, it then follows that (5.263) is true, and therefore the multiplication $\cdot_{\mathbb{N}}$ is an associative binary operation. □

Note 5.20. We now see that the ordered pair $(\mathbb{N}, \cdot_{\mathbb{N}})$ is a semigroup.

Theorem 5.53 (Commutative Law for the multiplication on $\mathbb{N}$). *The addition $\cdot_{\mathbb{N}}$ on the set of natural numbers is a commutative binary operation.*

Exercise 5.31. Prove the Commutative Law for the multiplication on $\mathbb{N}$.

(Hint: Proceed similarly as in the proof of Theorem 5.52, applying Exercise 4.37.)

Note 5.21. The ordered pair $(\mathbb{N}, \cdot_{\mathbb{N}})$ is a commutative semigroup.

Proposition 5.54. *It is true that* 1 *is the neutral element of* $\mathbb{N}$ *with respect to the multiplication* $\cdot_{\mathbb{N}}$ *on* $\mathbb{N}$.

Proof. We prove the universal sentence

$$\forall n \, (n \in \mathbb{N} \Rightarrow [1 \cdot_{\mathbb{N}} n = n \, \wedge \, n \cdot_{\mathbb{N}} 1 = n]), \tag{5.267}$$

letting n be arbitrary in $\mathbb{N}$. We observe that the set $M = \{1, \ldots, n\}$ has the cardinality $|M| = n$ according to (4.506), and that the singleton $N = \{1\}$ is finite with cardinality $|N| = 1$ due to (4.470) and (4.529). Since the conjunctions $|M| = n \, \wedge \, |N| = 1$ and $|N| = 1 \, \wedge \, |M| = n$ are thus true, we may form the products $n \cdot_{\mathbb{N}} 1$ and $1 \cdot_{\mathbb{N}} n$, and we obtain

$$1 \cdot_{\mathbb{N}} n = n \cdot_{\mathbb{N}} 1 = |M \times N| = |\{1, \ldots, n\} \times \{1\}| = n,$$

by applying the Commutative Law for the multiplication on $\mathbb{N}$, 5.240, substitutions based on the equations $M = \{1, \ldots, n\}$ as well as $N = \{1\}$, and (4.508). These equations yield $1 \cdot_{\mathbb{N}} n = n$ and $n \cdot_{\mathbb{N}} 1 = n$, as desired. As n was arbitrary, (5.267) therefore follows to be true, so that 1 is by definition the identity in $\mathbb{N}$ with respect to $\cdot_{\mathbb{N}}$. □

Theorem 5.55 (Distributive Law for $\mathbb{N}$). *The multiplication $\cdot_{\mathbb{N}}$ on the set of natural numbers is distributive over the addition $+_{\mathbb{N}}$ on $\mathbb{N}$.*

Proof. To demonstrate the distributivity of the binary operation $\cdot_{\mathbb{N}}$ over $+_{\mathbb{N}}$, we establish left-distributivity by verifying the universal sentence

$$\forall m, n, p\,(m, n, p \in \mathbb{N} \Rightarrow p \cdot_{\mathbb{N}} (m +_{\mathbb{N}} n) = (p \cdot_{\mathbb{N}} m) +_{\mathbb{N}} (p \cdot_{\mathbb{N}} n)). \tag{5.268}$$

We let m, n and p be arbitrary natural numbers, so that the sets

$$\begin{aligned} M &= \{1, \ldots, m\} \times \{0\}, \\ N &= \{1, \ldots, n\} \times \{1\}, \\ P &= \{1, \ldots, p\} \end{aligned}$$

have the cardinalities $|M| = m$, $|N| = n$ and $|P| = p$, according to (4.508) and (4.506). Furthermore, observing that (5.227) holds for the sets M and N, we may infer from this the truth of $M \cap N = \emptyset$. Next, we notice that the sum of m and n is determined by

$$m +_{\mathbb{N}} n = |M \cup N|,$$

according to (5.184). Thus, the conjunction

$$|P| = p \wedge |M \cup N| = m +_{\mathbb{N}} n$$

holds, which gives the product

$$\begin{aligned} p \cdot_{\mathbb{N}} (m +_{\mathbb{N}} n) &= |P \times (M \cup N)| \\ &= |(P \times M) \cup (P \times N)|, \end{aligned} \tag{5.269}$$

according to (5.240) and (3.56). As the conjunctions $|P| = p \wedge |M| = m$ and $|P| = p \wedge |N| = n$ are clearly true, the product of p and m and the product of p and n are determined by

$$\begin{aligned} p \cdot_{\mathbb{N}} m &= |P \times M|, \\ p \cdot_{\mathbb{N}} n &= |P \times N|, \end{aligned}$$

using again (5.240); regarding the Cartesian products occurring in these equations, the previously mentioned $M \cap N = \emptyset$ implies the disjointness of $(P \times M)$ and $(P \times N)$ with (3.55), so that the multiple conjunction

$$|P \times M| = p \cdot_{\mathbb{N}} m \wedge |P \times N| = p \cdot_{\mathbb{N}} n \wedge (P \times M) \cap (P \times N) = \emptyset$$

holds. Therefore, the sum of $p \cdot_{\mathbb{N}} m$ and $p \cdot_{\mathbb{N}} n$ is determined by

$$\begin{aligned} (p \cdot_{\mathbb{N}} m) +_{\mathbb{N}} (p \cdot_{\mathbb{N}} n) &= |(P \times M) \cup (P \times N)| \\ &= p \cdot_{\mathbb{N}} (m +_{\mathbb{N}} n), \end{aligned}$$

using (5.185) and then (5.269), proving the implication in (5.268). Since m, n and p were arbitrary, we may therefore conclude that (5.268) holds, so that $\cdot_{\mathbb{N}}$ is by definition left-distributive over $+_{\mathbb{N}}$. Since we already know that $\cdot_{\mathbb{N}}$ is commutative, it follows with Proposition 5.28 that $\cdot_{\mathbb{N}}$ is distributive over $+_{\mathbb{N}}$. □

Corollary 5.56. *The ordered triple* $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$ *is a commutative, nontrivial semiring.*

Proof. In light of Note 5.19, Note 5.21 and Theorem 5.55, we wee that $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$ is a commutative semiring. To show that this semiring is nontrivial, we recall Proposition 5.47 and prove $\mathbb{N} \neq \{0\}$ by contradiction, assuming the negation of that inequality to be true, so that $\mathbb{N} = \{0\}$ follows to be true with the Double Negation Law. Since $\mathbb{N}$ is a singleton, we obtain with (2.180) the uniquely existential sentence $\exists! n\,(n \in \mathbb{N})$, whose uniqueness part is given by

$$\forall n, n'([n \in \mathbb{N} \wedge n' \in \mathbb{N}] \Rightarrow n = n').$$

Therefore, $0 \in \mathbb{N}$ and $1 \in \mathbb{N}$ imply $0 = 1$, which contradicts the fact that $0 \neq 1$ holds according to (4.165). This completes the proof of $\mathbb{N} \neq \{0\}$, so that the semiring $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$ is indeed nontrivial. □

Theorem 5.57. *The semiring* $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$ *is zero-divisor free.*

Proof. Recalling from Proposition 5.47 that 0 is the zero element in $\mathbb{N}$, we apply the Criterion for zero-divisor freeness (5.175) and prove accordingly

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [(m \cdot_{\mathbb{N}} n = 0 \vee n \cdot_{\mathbb{N}} m = 0) \Rightarrow (m = 0 \vee n = 0)]). \quad (5.270)$$

For this purpose, we let $m, n \in \mathbb{N}$ be arbitrary and assume the disjunction $m \cdot_{\mathbb{N}} n = 0 \vee n \cdot_{\mathbb{N}} m = 0$ to be true. Based on this disjunction, we now prove the desired consequent $m = 0 \vee n = 0$ by cases.

In the first case $m \cdot_{\mathbb{N}} n = 0$, we observe in light of the Law of the Excluded Middle that the disjunction $m = 0 \vee m \neq 0$ holds and consider accordingly the two further subcases $m = 0$ and $m \neq 0$. On the one hand, if $m = 0$ holds, then the disjunction $m = 0 \vee n = 0$ to be proven is also true. On the other hand, if $m \neq 0$ holds, then the product of m and n (which is identical with 0 according to the current case assumption) is determined by

$$0 = m \cdot_{\mathbb{N}} n = |\{1, \ldots, m\} \times \{1, \ldots, n\}|,$$

applying (5.239). These equations imply

$$\{1, \ldots, m\} \times \{1, \ldots, n\} = \emptyset$$

with (4.527), using the evident fact that Cartesian product is a finite set. Then, the preceding equation implies with (3.27) that the disjunction

$$\{1, \ldots, m\} = \emptyset \vee \{1, \ldots, n\} = \emptyset$$

holds. Here, the subcase assumption $m \neq 0$ implies $\{1, \ldots, m\} \neq \emptyset$ by definition of an initial segment, so that the second part $\{1, \ldots, n\} = \emptyset$ of the preceding disjunction is true. This means in view of (4.239) that $n = 0$ holds, and therefore the desired disjunction $m = 0 \vee n = 0$ follows again to be true. We thus showed in the first case that $m \cdot_{\mathbb{N}} n = 0$ implies the truth of $m = 0 \vee n = 0$.

In the other case that $n \cdot_{\mathbb{N}} m = 0$ is true, we obtain

$$0 = n \cdot_{\mathbb{N}} m = m \cdot_{\mathbb{N}} n$$

with the Commutative Law for the multiplication on $\mathbb{N}$. These equations give $m \cdot_{\mathbb{N}} n = 0$, which then implies $m = 0 \vee n = 0$ as shown in the proof for the first case.

This completes the proof of the implication in (5.270), and since m and n were arbitrary, we may therefore conclude that (5.270) is true. This universal sentence in turn implies with the Criterion for zero-divisor freeness that the semiring $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$ has no zero divisors, as claimed. $\square$

5.4. Ordered Elementary Domains $(X, +, \cdot, <)$

Definition 5.18 (Ordered elementary domain). For any set X, any addition $+$ on X, any multiplication $\cdot$ on X and any binary relation $<$, we say that the ordered quadruple

$$(X, +, \cdot, <) \tag{5.271}$$

is an *ordered elementary domain* iff

1. the zero element 0_X exists,
2. the unity element 1_X exists,
3. $(X, +, \cdot)$ is a nontrivial semiring,
4. $(X, +, \cdot)$ is a commutative semiring,
5. $(X, +, \cdot)$ is zero-divisor free,
6. $<$ is comparable, and
7. $<$ satisfies

$$\forall a, b\, (a, b \in X \Rightarrow [a < b \Leftrightarrow \exists d\, (d \in X \,\wedge\, d \neq 0_X \,\wedge\, a + d = b)]), \tag{5.272}$$

The following example shows that it is not a trivial task to find an example of an ordered elementary domain. It will in fact be the purpose of the subsequent Chapter 5 to accomplish this task.

Proposition 5.58. *For any x and any y satisfying $x \neq y$ it is true that*

$$\begin{aligned}&(\{x, y\},\\ &\ \{((x, x), x), ((x, y), y), ((y, x), y), ((y, y), y)\},\\ &\ \{((x, x), x), ((x, y), x), ((y, x), x), ((y, y), y)\},\\ &\ \{(x, y)\})\end{aligned} \tag{5.273}$$

satisfies all of the properties of an ordered elementary domain except for Property 7.

Proof. Letting x and y be arbitrary and assuming $x \neq y$ to be true, we have that

1. the zero element 0_X of $X = \{x, y\}$ with respect to the join/addition $\sqcup$ on $\{x, y\}$ exists according to Corollary 5.16a),

2. the unity element 1_X of $X = \{x, y\}$ with respect to the meet/multiplication $\sqcap$ on $\{x, y\}$ exists according to Corollary 5.16b),
3. $(X, \sqcup, \sqcap)$ is a nontrivial semiring in view of Corollary 5.41,
4. $(X, \sqcup, \sqcap)$ is a commutative semiring because of Corollary 5.38,
5. the semiring $(X, \sqcup, \sqcap)$ is zero-divisor free due to Proposition 5.45,
6. $<= \{(x, y)\}$ is a comparable binary relation on $X = \{x, y\}$ as shown in Proposition 3.50.

To verify that (5.273) does not satisfy Property 7 of an ordered elementary domain, we prove the negation of (5.272) by contradiction, assuming the double negation of (5.272) to be true. To do this, we apply the Double Negation Law, so that (5.272) follows to be true. We therefore have in particular for the elements $a = y$ and $b = y$ in $\{x, y\}$ that the equivalence

$$y < y \Leftrightarrow \exists d\,(d \in \{x, y\} \,\wedge\, d \neq 0_X \,\wedge\, y \sqcup d = y) \tag{5.274}$$

holds, so that the inequality $y < y$ and the existential sentence take identical truth values. Let us now observe on the one hand that the initial assumption $y \neq x$ implies with the Equality Criterion for ordered pairs evidently $(y, y) \neq (x, y)$, which in turn implies $(y, y) \notin \{(x, y)\}$ $[=<]$ with (2.169), so that $\neg(y, y) \in <$ is true, which negation we may also write as $\neg y < y$. Thus, the left-hand side of the equivalence (5.274) is true. On the other hand, we may choose the element $d = y$ in $\{x, y\}$, which satisfies $d \neq 0_X$ $[= x]$ because of the initial assumption $y \neq x$. Furthermore, the definition of the join

$$\sqcup = \{((x, x), x), ((x, y), y), ((y, x), y), ((y, y), y)\}$$

shows in light of (2.244) that $((y, y), y) \in \sqcup$ holds, which we may also write as $y \sqcup y = y$, so that substitution yields the true equation $y \sqcup d = y$. Thus, the existential sentence in (5.274) is true, so that the two parts of the equivalence take different truth values. This finding contradicts the previously established statement that $y < y$ and the existential sentence take the same truth value. Consequently, we may infer from this that the equivalence in (5.272) does not hold for all elements $a, b \in \{x, y\}$, as required by Property 7 of an ordered elementary domain. $\square$

In the remainder of the current chapter, we establish some useful properties of ordered elementary domains, which are derived from the Properties 1 – 7.

Proposition 5.59. *For any ordered elementary domain* $(X, +, \cdot, <)$ *it is true that*

a) *every nonzero element of* X *is greater than the zero element, i.e.*

$$\forall a\,(a \in X \Rightarrow [a \neq 0_X \Leftrightarrow 0_X < a]). \tag{5.275}$$

b) *every element of* X *is greater than or equal to the zero element, i.e.*

$$\forall a\,(a \in X \Rightarrow [0_X < a \lor a = 0_X). \tag{5.276}$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary sets and assume that $(X, +, \cdot, <)$ is an ordered elementary domain.

Concerning a), we let $a \in X$ be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $a \neq 0_X$ holds. Then, the assumption $a \in X$ implies the truth of the equation $0_X + a = a$ with the definition of the zero element. In view of the preceding assumptions and this equation, we thus see that the existential sentence

$$\exists d\,(d \in X \land d \neq 0_X \land 0_X + d = a)$$

holds, so that $0_X < a$ follows to be true with Property 7 of an ordered elementary domain. This completes the proof of the first part of the equivalence in (5.275).

To prove the second part ('$\Leftarrow$'), we now assume $0_X < a$, which implies that $0_X = a$ is false because of Property 6 of an ordered elementary domain in connection with the Characterization of comparability. Thus, $a \neq 0_X$ is true, which completes the proof of the equivalence, and therefore also the proof of the implication in (5.275).

As a is arbitrary, we may therefore conclude that the universal sentence (5.275) holds.

Concerning b), we let a again be arbitrary, and we now prove the implication by cases, considering the two exhaustive cases $a = 0_X$ and $a \neq 0_X$. In the first case, we assume accordingly that $a \in X$ and $a = 0_X$ are true. The latter equation then evidently implies the truth of the disjunction $0_X < a \lor a = 0_X$. In the second case, we now assume the conjunction of $a \in X$ and $a \neq 0_X$ to be true, which assumptions imply $0_X < a$ with (5.275). Consequently, the disjunction $0_X < a \lor 0_X = a$ is also true, so that the proof of the implication in (5.276) by cases is complete. Since a is arbitrary, the universal sentence (5.276) follows then to be true.

As X, $+$, $\cdot$ and $<$ were arbitrary sets, we may now conclude that the proposition holds. $\square$

Corollary 5.60. *For any ordered elementary domain* $(X, +, \cdot, <)$ *and any element* a *in* X*, it is false that* a *is less than* 0_X*, i.e.*

$$\forall a\,(a \in X \Rightarrow \neg a < 0_X). \tag{5.277}$$

Proof. Letting X, $+$, $\cdot$ and $<$ be arbitrary such that $(X, +, \cdot, <)$ is an ordered elementary domain and letting a be an arbitrary element of X, it follows with (5.276) that the disjunction $0_X < a \vee a = 0_X$ is true, which we may now use to prove the desired negation $\neg a < 0_X$ by cases. In case that $0_X < a$ holds, it follows with the comparability of $<$ (according to Property 6 of an ordered elementary domain) that $a < 0_X$ is false, so that the negation $\neg a < 0_X$ is true. In case that the second part $a = 0_X$ of the preceding disjunction holds, it follows with the comparability of $<$ that $a < 0_X$ is false, and therefore $\neg a < 0_X$ is true again. We thus proved that $a \in X$ implies $\neg a < 0_X$, and as a is arbitrary, we may therefore conclude that (5.277) holds. Since X, $+$, $\cdot$ and $<$ were also arbitrary, we may then further conclude that the corollary is true. □

Note 5.22. We will refer to the property (5.277) of an ordered elementary domain $(X, +, \cdot, <)$ also as the *nonnegativity* of the elements of X, and we say accordingly that the elements of X are *nonnegative*. In doing so, we anticipate the existence of 'negative' elements, which will be defined in the second part on numerical measurement data.

Corollary 5.61. *For any ordered elementary domain* $(X, +, \cdot, <)$ *it is true that the sum of a nonzero element of* X *and any element of* X *is nonzero, i.e.*

$$\forall a, b\,(a, b \in X \Rightarrow [a \neq 0_X \Rightarrow a + b \neq 0_X]). \tag{5.278}$$

Proof. Letting X, $+$, $\cdot$ and $<$ be arbitrary sets such that $(X, +, \cdot, <)$ is an ordered elementary domain (with zero element 0_X) and letting then a and b be arbitrary elements of X, we may prove the implication by contradiction, assuming $a \neq 0_X$ and $\neg a + b \neq 0_X$ to be true. To begin with, the latter negation implies $a + b = 0_X$ with the Double Negation Law. Since $b + a = a + b\ (= 0_X)$ holds due to the commutativity of the addition $+$ (according to Property 4 of an ordered elementary domain), we obtain the equation $b + a = 0_X$. Together with the initial assumptions $a \in X$ and $a \neq 0_X$, this shows that there exists a constant d which satisfies $d \in X$, $d \neq 0_X$ and $b + d = 0_X$. Consequently, Property 7 of an ordered elementary domain yields $b < 0_X$. Since $b \in X$ implies also the truth of $\neg b < 0_X$ with (5.277), we evidently obtained a contradiction, which finding proves the implication in (5.278). Since a and b are arbitrary, we may therefore conclude that the universal sentence (5.278) holds. Then, as the sets X, $+$, $\cdot$ and $<$ were also arbitrary, the corollary follows to be true. □

Theorem 5.62 (Linear ordering of ordered elementary domains). *For any ordered elementary domain $(X, +, \cdot, <)$ it is true that $<$ is a linear ordering of X.*

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary sets and assume that $(X, +, \cdot, <)$ is an ordered elementary domain. Since $<$ is a comparable binary relation according to Property 6 of an ordered elementary domain, it suffices in view of the Characterization of a linearly ordered set to prove that $<$ is transitive. For this purpose, we verify

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [(a < b \,\wedge\, b < c) \Rightarrow a < c]), \tag{5.279}$$

letting $a, b, c \in X$ be arbitrary and assuming $a < b \,\wedge\, b < c$ to be true. On the one hand, $a < b$ implies with (5.272) that there exists an element of X, say $\bar{d}$, with $\bar{d} \neq 0_X$ and $a + \bar{d} = b$. On the other hand, $b < c$ implies – again with (5.272) – that there is an element of X, say $\bar{\bar{d}}$, such that $\bar{\bar{d}} \neq 0_X$ and $b + \bar{\bar{d}} = c$ hold. The previous two equations then yield (by substitution)

$$c = b + \bar{\bar{d}} = (a + \bar{d}) + \bar{\bar{d}},$$

and therefore

$$a + (\bar{d} + \bar{\bar{d}}) = c, \tag{5.280}$$

utilizing the associativity of the addition $+$. Here, we notice that

$$\bar{d} + \bar{\bar{d}} \in X \tag{5.281}$$

holds as the addition $+$ is a binary operation on X. Furthermore, since $\bar{d}, \bar{\bar{d}} \in X$ and $\bar{d} \neq 0_X$ are both true, we obtain

$$\bar{d} + \bar{\bar{d}} \neq 0_X \tag{5.282}$$

with (5.278). The findings (5.280) – (5.282) show that the existential sentence

$$\exists d\,(d \in X \,\wedge\, d \neq 0_X \,\wedge\, a + d = c)$$

holds, so that the desired inequality $a < c$ follows to be true with (5.272). Since a, b and c are arbitrary, we may therefore conclude that (5.279) is true, which means that $<$ is indeed a transitive binary relation. Consequently, $<$ is a linear ordering of X according to Theorem 3.75, and since X, $+$, $\cdot$ and $<$ were initially arbitrary sets, we may infer from this that the theorem is indeed true. □

Note 5.23. From now on, we will treat $\leq$ as the total ordering of X induced by the linear ordering $<$ of X

Lemma 5.63. *The following sentences hold for any ordered elementary domain* $(X, +, \cdot, <)$.

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a < b \Rightarrow a + c < b + c]), \tag{5.283}$$
$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a \leq b \Rightarrow a + c \leq b + c]), \tag{5.284}$$
$$\forall a, b, c\,([a, b, c \in X \,\wedge\, 0_X < c] \Rightarrow [a < b \Rightarrow a \cdot c < b \cdot c]), \tag{5.285}$$
$$\forall a, b, c\,([a, b, c \in X \,\wedge\, 0_X < c] \Rightarrow [a \leq b \Rightarrow a \cdot c \leq b \cdot c]) \tag{5.286}$$

Proof. First we let X, $+$, $\cdot$ and $<$ be arbitrary sets and assume $(X, +, \cdot, <)$ to be an ordered elementary domain.

Regarding (5.283), we let a, b, c arbitrary elements of X and prove the implication $a < b \Rightarrow a + c < b + c$ directly, assuming that $a < b$ holds. This inequality implies with Property 7 of an ordered elementary domain that there exists a nonzero element in X, say $\bar{d} \neq 0_X$, satisfying $a + \bar{d} = b$. We then obtain the true equations

$$b + c = (a + \bar{d}) + c = a + (\bar{d} + c) = a + (c + \bar{d}) = (a + c) + \bar{d},$$

applying substitution, the associativity of the addition $+$ on X, the commutativity of the addition $+$ on X, and finally again the associativity of $+$. These equations yield $(a + c) + \bar{d} = b + c$, where $\bar{d} \in X$ and $\bar{d} \neq 0_X$ are also true, as stated before. We thus showed that there exists a constant d such that $d \in X$, $d \neq 0_X$ and $(a + c) + d = b + c$ hold, which existential sentence then implies the desired inequality $a + c < b + c$ with (5.272). Thus, the two implications in (5.283) are true, and as a, b, c were arbitrary, we may further conclude that the universal sentence (5.283) is true.

Regarding (5.284), we let a, b and c be arbitrary elements of X and assume $a \leq b$ to be true. According to Theorem 3.81b), the induced reflexive partial ordering $\leq$ satisfies the disjunction $a < b \,\vee\, a = b$, which we may now use to prove the sentence $a + c \leq b + c$ by cases. On the one hand, if $a < b$ holds, we obtain $a + c < b + c$ with (5.283); therefore, the disjunction $a + c < b + c \,\vee\, a + c = b + c$ is also true, so that $a + c \leq b + c$ holds by definition of the induced partial ordering $\leq$. On the other hand, if $a = b$ holds, then substitution evidently yields the equation $a + c = b + c$; consequently, the disjunction $a + c < b + c \,\vee\, a + c = b + c$ is again true, so that $a + c \leq b + c$ follows to be true as for the first case. This completes the proof by cases of $a + c \leq b + c$, and since a, b and c are arbitrary, we may therefore conclude that (5.284) holds.

Regarding (5.285), we let again $a, b, c \in X$ be arbitrary and assume first $0_X < c$ and then $a < b$ to be true. Let us observe here that the former inequality implies $0_X \neq c$ because of the comparability of the linear ordering

$<$ of X. Furthermore, the assumed inequality $a < b$ implies with (5.272) that there exists a constant, say $\bar{d}$, such that $\bar{d} \in X$, $\bar{d} \neq 0_X$ and $a + \bar{d} = b$ are true. We now obtain the equations

$$c \cdot b = c \cdot (a + \bar{d}) = (c \cdot a) + (c \cdot \bar{d})$$

by applying substitution and then the distributivity of $\cdot_X$ over $+_X$. In view of the commutativity of the multiplication $\cdot$, the preceding equations then give

$$(a \cdot c) + (\bar{d} \cdot c) = b \cdot c. \tag{5.287}$$

Here, we notice that

$$\bar{d} \cdot c \in X \tag{5.288}$$

holds because the multiplication is a binary operation on X. Now, since $\bar{d}, c \in X$ also holds and since the conjunction $\bar{d} \neq 0_X \wedge c \neq 0_X$ is also true, it follows with the Criterion for zero-divisor freeness (5.176) in particular that

$$\bar{d} \cdot c \neq 0_X \tag{5.289}$$

holds. Thus, (5.287) – (5.289) show that there is a d which satisfies $d \in X$, $d \neq 0_X$ and $(a \cdot c) + d = b \cdot c$, so that the desired inequality $a \cdot c < b \cdot c$ follows to be true with (5.272). As a, b and c are arbitrary, we may therefore conclude that (5.285) holds.

Since X, $+$, $\cdot$ and $<$ were arbitrary sets, we may finally conclude that the stated sentences are true for any such sets. $\square$

Exercise 5.32. Establish the universal sentence (5.286) for any ordered elementary domain $(X, +, \cdot, <)$.

(Hint: Proceed in analogy to the proof of (5.284).)

The implications in (5.283) – (5.286) linking the inequalities can be turned into equivalences.

Theorem 5.64 (Monotony Laws for ordered elementary domains). *The following laws hold for any ordered elementary domain* $(X, +, \cdot, <)$.

a) **Monotony Law for $+$ and $<$:**

$$\forall a, b, c\, (a, b, c \in X \Rightarrow [a < b \Leftrightarrow a + c < b + c]). \tag{5.290}$$

b) **Monotony Law for $+$ and $\leq$:**

$$\forall a, b, c\, (a, b, c \in X \Rightarrow [a \leq b \Leftrightarrow a + c \leq b + c]). \tag{5.291}$$

c) **Monotony Law for $\cdot$ and $<$***:*

$$\forall a, b, c\,([a, b, c \in X \,\wedge\, 0_X < c] \Rightarrow [a < b \Leftrightarrow a \cdot c < b \cdot c]). \quad (5.292)$$

d) **Monotony Law for $\cdot$ and $\leq$***:*

$$\forall a, b, c\,([a, b, c \in X \,\wedge\, 0_X < c] \Rightarrow [a \leq b \Leftrightarrow a \cdot c \leq b \cdot c]). \quad (5.293)$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary such that $(X, +, \cdot, <)$ is an ordered elementary domain.

Concerning a), we let a, b and c be arbitrary elements of X and notice that the first part ('$\Rightarrow$') of the equivalence holds in view of (5.283). We now prove the second part ('$\Leftarrow$') by contradiction, assuming $a + c < b + c$ and $\neg a < b$ to be both true. Here, the negation $\neg b > a$ implies $b \leq a$ with the Negation Formula for $<$, using the fact that $<$ is a linear ordering of X according to Theorem 5.62. It then follows from this with (5.284) that $b + c \leq a + c$ holds, which we may write equivalently as $\neg a + c < b + c$ by using the Negation Formula for $<$. This finding contradicts the initially assumed $a+c < b+c$, so that the proof of the second part of the equivalence in (5.290) is complete. This in turn proves the implication in (5.290), and since a, b, c, X, $+$, $\cdot$ and $<$ were arbitrary, we may therefore conclude that the Monotony Law for $+$ and $<$ is true. □

Exercise 5.33. Prove for any ordered elementary domain $(X, +, \cdot, <)$

- the Monotony Law for $+$ and $\leq$,
- the Monotony Law for $\cdot$ and $<$ and
- the Monotony Law for $\cdot$ and $\leq$.

(Hint: Apply proofs by contradiction.)

Theorem 5.65 (Additivity of inequalities). *The following laws hold for any ordered elementary domain $(X, +, \cdot, <)$.*

a) **Additivity of $<$-inequalities***:*

$$\forall a, b, c, d\,(a, b, c, d \in X \Rightarrow [(a < b \wedge c < d) \Rightarrow a+c < b+d]). \quad (5.294)$$

b) **Additivity of $\leq$-inequalities***:*

$$\forall a, b, c, d\,(a, b, c, d \in X \Rightarrow [(a \leq b \wedge b \leq d) \Rightarrow a+c \leq b+d]). \quad (5.295)$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary sets such that $(X, +, \cdot, <)$ is an ordered elementary domain. We then let a, b, c and d be arbitrary and assume $a, b, c, d \in X$ to be true. Next, we assume the inequalities $a < b$ and $c < d$ to be both true, which imply respectively

$$\begin{aligned} a + c &< b + c, \\ c + b &< d + b \end{aligned} \tag{5.296}$$

with the Monotony Law for $+$ and $<$. We may write the latter also as

$$b + c < b + d \tag{5.297}$$

by applying the commutativity of the addition on X. Thus, the conjunction of (5.296) and (5.297) holds, which now implies the desired $a + c < b + d$ with the transitivity of the linear ordering $<$ of X. Thus, the proof of the second implication in (5.294) is complete, which in turn proves the implication based on the antecedent $a, b, c, d \in X$. Since a, b, c and d are arbitrary, we may therefore conclude that the universal sentence (5.294) is true. Then, as X, $+$, $\cdot$ and $<$ were initially arbitrary sets, we may infer from this finding the truth of Part a) of the stated theorem. □

Exercise 5.34. Prove Part b) of Theorem 5.65.

Theorem 5.66 (Cancellation Laws for ordered elementary domains). *The following laws are true for any ordered elementary domain* $(X, +, \cdot, <)$.

a) **Cancellation Law for $+$***:*

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a + b = a + c \Rightarrow b = c]). \tag{5.298}$$

b) **Cancellation Law for $\cdot$***:*

$$\forall a, b, c\,([a, b, c \in X \,\wedge\, a \neq 0_X] \Rightarrow [a \cdot b = a \cdot c \Rightarrow b = c]). \tag{5.299}$$

c) **Cancellation Law for $\mathbf{0_X}$***:*

$$\forall a\,(a \in X \Rightarrow [a \cdot 0_X = 0_X \,\wedge\, 0_X \cdot a = 0_X]). \tag{5.300}$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary and assume $(X, +, \cdot, <)$ to be an ordered elementary domain.

Concerning b), we take arbitrary $a, b, c \in X$ such that $a \neq 0_X$ holds, and we prove the implication $a \cdot b = a \cdot c \Rightarrow b = c$ by contradiction, assuming $a \cdot b = a \cdot c$ and $b \neq c$ to be both true. Let us now first observe that the assumption $a \neq 0_X$ implies $0_X < a$ with (5.275). Since $<$ is comparable, the assumed inequality $b \neq c$ implies that $b < c$ or $b > c$ is true. In case $b < c$ holds, the Monotony Law for $\cdot$ and $<$ gives in view of the assumption $a \neq 0$ the inequality $b \cdot a < c \cdot a$, and therefore $a \cdot b < a \cdot c$ with the commutativity of $\cdot$. Thus, $a \cdot b < a \cdot c$ and $a \cdot b = a \cdot c$ are both true, which contradicts the fact that the conjunction of $a \cdot b < a \cdot c$ and $a \cdot b = a \cdot c$ is false because of the comparability of $<$. Similarly, the case of $b > c$ implies $c \cdot a < b \cdot a$ and therefore $a \cdot c < a \cdot b$, which yields again a contradiction with the initial assumption $a \cdot c = a \cdot b$. Thus, the proof of the implications is complete, and since a, b and c are arbitrary, the proposed law (5.299) follows then to be true.

Concerning c), we let a be arbitrary in X and observe the truth of the equations

$$\begin{aligned}(a \cdot 0_X) + (a \cdot 0_X) &= a \cdot (0_X + 0_X) \\ &= a \cdot 0_X \\ &= (a \cdot 0_X) + 0_X,\end{aligned}$$

using the distributivity of $\cdot_X$ over $+_X$ and then the definition of the zero element (which is applied twice). We now see in light of the Cancellation Law for $+$ that the resulting equation

$$(a \cdot 0_X) + (a \cdot 0_X) = (a \cdot 0_X) + 0_X$$

implies $a \cdot 0_X = 0_X$, which proves the first part of the conjunction in (5.300). Using the same arguments, we obtain also

$$\begin{aligned}(0_X \cdot a) + (0_X \cdot a) &= (0_X + 0_X) \cdot a \\ &= 0_X \cdot a \\ &= (0_X \cdot a) + 0_X,\end{aligned}$$

and therefore $0_X \cdot a = 0_X$, so that the second part of the conjunction to be proven also holds. Thus, the proof of the implication in (5.300) is complete, and as a is arbitrary, the Cancellation Law for 0_X follows also to be true.

Since X, $+$, $\cdot$ and $<$ were arbitrary sets, we may finally conclude that the stated sentences are true for any such sets. □

Exercise 5.35. Prove for any ordered elementary domain $(X, +, \cdot, <)$ the Cancellation Law for $+$.

(Hint: Proceed in analogy to the proof of the Cancellation Law for $\cdot$.)

We now investigate a few useful consequences of the Cancellation Laws.

Proposition 5.67. *For any ordered elementary domain $(X,+,\cdot,<)$ it is true that, if an element b of X is the sum of two elements a and d of X, then b is greater than or equal to a, i.e.*

$$\forall a,b,d\,([a,b,d \in X \,\wedge\, a+d=b] \Rightarrow a \leq b). \tag{5.301}$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary sets such that $(X,+,\cdot,<)$ is an ordered elementary domain, we let a, b and d also be arbitrary, and we prove the implication by contradiction, assuming $a,b,d \in X$, the equation $a+d=b$ and the negation $\neg a \leq b$ to be true. Since $<$ is a linear ordering of X according to Theorem 5.62 and since $\leq$ is the induced total ordering of X, we may apply the Negation Formula for $\leq$ to infer from $\neg a \leq b$ the truth of $b < a$. In view of Property 7 of an ordered elementary domain, there then exists a constant, say $\bar{e}$, such that $\bar{e} \in X$, $\bar{e} \neq 0_X$ and $b+\bar{e}=a$ hold. Next, we may apply substitution based on this equation to obtain from $a+d=b$ the equation $(b+\bar{e})+d=b$, which we may also write as

$$b+(\bar{e}+d)=b+0$$

utilizing the associativity of the addition and the definition of a neutral element. Then, the Cancellation Law for $+$ yields $\bar{e}+d=0$. Since $\bar{e} \neq 0_X$ implies also $\bar{e}+d \neq 0$ with (5.278), we evidently obtained a contradiction, so that the proof of the implication in (5.301) is complete. Because a, b, d, X, $+$, $\cdot$ and $<$ were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Theorem 5.68 (Distinctness of the zero element and the unity element in ordered elementary domains). *For any ordered elementary domain $(X,+,\cdot,<)$ it is true that the zero and the unity element are distinct elements of X, i.e.*

$$0_X \neq 1_X. \tag{5.302}$$

Proof. Letting X, $+$, $\cdot$ and $<$ be arbitrary sets such that $(X,+,\cdot,<)$ is an ordered elementary domain, we prove the proposed inequality $0_X \neq 1_X$ by contradiction, assuming $\neg 0_X \neq 1_X$ to be true, which assumption we may write equivalently as $0_X = 1_X$ by applying the Double Negation Law. We now show that these assumptions imply $X = \{0_X\}$, which we may write equivalently as

$$\forall a\,(a \in X \Leftrightarrow a \in \{0_X\}), \tag{5.303}$$

using (2.18). We let a be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $a \in X$ to be true. We then obtain the true equations

$$a = a \cdot 1_X = a \cdot 0_X = 0_X$$

by applying the definition of the unity element, then substitution based on the initial assumption $0_X = 1_X$, and finally the Cancellation Law for 0_X. These equations give $a = 0_X$ and therefore $a \in \{0_X\}$ due to (2.169), proving the first part of the equivalence. To prove the second part ('$\Leftarrow$'), we now assume that $a \in \{0_X\}$ holds, so that evidently $a = 0_X$. Since $0_X \in X$ is also true by definition of the zero element, we obtain the desired $a \in X$ via substitution. This completes the proof of the equivalence in (5.303), and since a is arbitrary, we may therefore conclude that the universal sentence (5.303) holds. It then follows with (2.18) that $X = \{0_X\}$ is true, in contradiction to the (nontriviality) Property 3 of an ordered elementary domain, which states that $X \neq \{0_X\}$ holds. This completes the proof of the inequality $0_X \neq 1_X$, and as X, $+$, $\cdot$ and $<$ were arbitrary, we may therefore conclude that the theorem holds. □

Thus, Proposition 5.59a) gives then immediately the following inequality between 0_X and 1_X.

Corollary 5.69. *For any ordered elementary domain* $(X, +, \cdot, <)$, *it is true that the zero element is less than the unity element of* X, *i.e.*

$$0_X < 1_X. \tag{5.304}$$

The Cancellation Law for 0_X also allows us to drop the condition imposed on the element c in the law (5.286).

Proposition 5.70. *The following sentence is true for any ordered elementary domain* $(X, +, \cdot, <)$.

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a \leq b \Rightarrow a \cdot c \leq b \cdot c]). \tag{5.305}$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary sets and assume $(X, +, \cdot, <)$ to be an ordered elementary domain. Furthermore, we let a, b, c be arbitrary elements of X. Here, $c \in X$ implies the truth of the disjunction $0_X < c \vee c = 0_X$ with (5.276), which we now apply to prove $a \cdot c \leq b \cdot c$ by cases. If the first part $0_X < c$ holds, then the implication in (5.305) follows to be true with (5.286). If the second part $c = 0_X$ holds, then we may prove the second implication in (5.305) directly. Assuming $a \leq b$, we obtain

$$a \cdot c = a \cdot 0_X = 0_X,$$
$$b \cdot c = b \cdot 0_X = 0_X$$

by applying substitution based on the case assumption $c = 0_X$ and then the Cancellation Law for 0_X. Consequently, these equations yield $a \cdot c = b \cdot c$ again by substitution. Then, the disjunction

$$a \cdot c < b \cdot c \vee a \cdot c = b \cdot c$$

is also true. Since the induced total ordering $\leq$ satisfies (3.238), the inequality $a \cdot c \leq b \cdot c$ is true, so that the proof of the second implication in (5.305) is complete. This in turn proves the implication based on the antecedent $a, b, c \in X$, and since a, b and c are arbitrary, we may further conclude that (5.305) holds. Then, as the sets X, $+$, $\cdot$ and $<$ were also arbitrary, the proposed sentence follows to be true. □

Property 7 (5.272) of an ordered elementary domain can be strengthened to guarantee unique existence of the element d satisfying $a + d = b$ in case of $a \leq b$.

Proposition 5.71. *For any ordered elementary domain $(X, +, \cdot, <)$ and any elements a, b in X such that a is less than or equal to b, it is true that there exists a unique d in X which satisfies $a + d = b$, i.e.*

$$\forall a, b\, (a, b \in X \Rightarrow [a \leq b \Rightarrow \exists! d\, (d \in X \,\wedge\, a + d = b)]). \tag{5.306}$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary sets such that $(X, +, \cdot, <)$ is an ordered elementary domain, we let a, b be arbitrary elements of X, and we assume $a \leq b$ to be true. Because the induced reflexive partial ordering $\leq$ of X satisfies (3.238), we have that the disjunction $a < b \vee a = b$ is true, which we now use to prove the existential part by cases. In case of $a < b$, there is some d satisfying $d \in X$ and $a + d = b$ due to Property 7 of an ordered elementary domain. In the other case of $a = b$, the choice $d = 0$ yields the true equation $a + 0_X = a$ with the definition of the zero element. To prove the uniqueness part, let d and d' be arbitrary such that $d \in X$, $a + d = b$, $d' \in X$ and $a + d' = b$ hold. The two equations give $a + d = a + d'$ via substitution, and the Cancellation Law for $+$ then yields $d = d'$, which finding establishes the uniqueness part. Thus, the uniquely existential sentence in (5.306), and as a and b are arbitrary, we may therefore conclude that the universal sentence (5.306) holds. Since X, $+$, $\cdot$ and $<$ were initially arbitrary sets, we may then further conclude that the proposition holds, as claimed. □

Definition 5.19 (Difference). We say for any ordered elementary domain $(X, +, \cdot, <)$ and any $a, b \in X$ with $a \leq b$ that an element d in X is the *difference* of b and a, symbolically

$$d = b - a, \tag{5.307}$$

iff $a + d = b$ holds.

Note 5.24. Whenever we are given an equation $a + d = b$, the difference $d = b - a$ is defined, because the given equation implies the required inequality $a \leq b$ with Proposition 5.67.

This definition yields because of the commutativity of + immediately the following result.

Corollary 5.72. *The following sentence holds for any ordered elementary domain* $(X, +, \cdot, <)$.

$$\forall a, b\,([a, b \in X \ \wedge\ a \leq b] \Rightarrow [b - a] + a = b). \tag{5.308}$$

Note 5.25. As the difference is not defined for all a and b in X (the difference is not defined for constants a, b with $a > b$), we cannot define a corresponding binary operation on X. To do this, we will extend the ordered elementary domain to an 'ordered integral domain' in the part on numerical measurement data.

Since the neutral element 0_X with respect to the addition in an ordered elementary domain satisfies $a + 0_X = a$ uniquely for any element a, we see that 0_X is the difference of identical elements.

Corollary 5.73. *For any ordered elementary domain* $(X, +, \cdot, <)$, *it is true that the difference of any element* $a \in X$ *and itself is identical with the zero element* 0_X, *i.e.*

$$\forall a\,(a \in X \Rightarrow a - a = 0_X). \tag{5.309}$$

Exercise 5.36. Verify the following sentence for any ordered elementary domain $(X, +, \cdot, <)$.

$$\forall a, b\,(a, b \in X \Rightarrow [b + a] - a = b). \tag{5.310}$$

(Hint: Establish first $a \leq b + a$ and $a + b = b + a$ for arbitrary $a, b \in X$.)

Returning to the 'restricted' Definition 5.19 of the difference, we now see with the help of the following lemma that the difference is distributive in a certain sense.

Lemma 5.74. *The following law holds for any ordered elementary domain* $(X, +, \cdot, <)$.

$$\forall a, b, c\,([a, b, c \in X \ \wedge\ c \leq a \ \wedge\ b \leq c] \Rightarrow [a - b] + [b - c] = a - c) \tag{5.311}$$

Proof. We take arbitrary sets X, $+$, $\cdot$, $<$ and assume that $(X, +, \cdot, <)$ is an ordered elementary domain. Then, we take arbitrary constants a, b, c and assume $a, b, c \in X$, $b \leq a$ and $c \leq b$ to hold. Because of Proposition 5.71, there exist a unique d satisfying $b + d = a$ (so that $d = a - b$) and a unique e satisfying $c + e = b$ (so that $e = b - c$). Substitutions now give the equations

$$a = b + d = (c + e) + [a - b] = (c + [b - c]) + [a - b],$$

which yield with the associativity and commutativity of the addition $+$

$$a = c + ([b - c] + [a - b]) = c + ([a - b] + [b - c]),$$

and therefore

$$c + ([a - b] + [b - c]) = a.$$

Since the assumed $c \leq b \wedge b \leq a$ implies $c \leq a$ with the transitivity of the induced total ordering $\leq$ of X, we may apply the definition of a difference and write $[a - b] + [b - c] = a - c$, as desired. As a, b and c are arbitrary, we may therefore conclude that (5.311) is true. Because X, $+$, $\cdot$ and $<$ were initially also arbitrary, we may now infer from this the truth of the lemma. $\square$

Corollary 5.75. *The following law holds for any ordered elementary domain* $(X, +, \cdot, <)$*.*

$$\forall a, b, c\,([a, b, c \in X \;\wedge\; c \leq a \;\wedge\; b \leq c] \Rightarrow [a - c] - [a - b] = b - c) \quad (5.312)$$

Proof. Letting X, $+$, $\cdot$, $<$, a, b and c be arbitrary such that $(X, +, \cdot, <)$ is an ordered elementary domain and such that $a, b, c \in X$ as well as the inequalities $c \leq a$ and $b \leq c$ hold, we first obtain the equation $[a - b] + [b - c] = a - c$ with (5.311). In view of Note 5.24, the difference $b - c = [a - c] - [a - b]$ is then defined, which equation proves the implication in (5.312). Since X, $+$, $\cdot$, $<$, a, b and c are arbitrary, we may therefore conclude that the corollary holds. $\square$

Exercise 5.37. Show for any ordered elementary domain $(X, +, \cdot, <)$ that any element in X is identical with the difference of itself and the zero element of X, i.e.

$$\forall a\,(a \in X \Rightarrow a = a - 0_X). \quad (5.313)$$

(Hint: Use (5.276), (3.238) and (5.90) with respect to $+$.)

Theorem 5.76 (Distributive Law for differences in ordered elementary domains). *The following sentence is true for any ordered elementary domain* $(X, +, \cdot, <)$*.*

$$\forall a, b, f\;([a, b, f \in X \;\wedge\; b \leq a] \Rightarrow f \cdot [a - b] = f \cdot a - f \cdot b) \quad (5.314)$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary, assume that $(X, +, \cdot, <)$ is an ordered elementary domain, let a, b, c also be arbitrary, and assume moreover that $a, b, f \in X$ and $b \leq a$ are true. Then, we obtain the true equations

$$\begin{aligned} f \cdot b + f \cdot [a - b] &= f \cdot (b + [a - b]) \\ &= f \cdot ([a - b] + b) \\ &= f \cdot ([a - b] + [b - 0_X]) \\ &= f \cdot (a - 0_X) \\ &= f \cdot a. \end{aligned} \tag{5.315}$$

by using the distributivity of $\cdot$ over $+$, the commutativity of $+$, (5.313), then Lemma (5.74), and finally again (5.313). Let us observe now that the assume $b \leq a$ implies $b \cdot f \leq a \cdot f$ with (5.305), which inequality we may also write as $f \cdot b \leq f \cdot a$ because of the commutativity of $\cdot$. Consequently, we may apply the definition of a difference to the equation (5.315) to obtain the desired equation in (5.314). Since a, b, f, X, $+$, $\cdot$ and $<$ were arbitrary, it follows from this finding that the theorem holds indeed. □

Theorem 5.77 (Monotony Laws for the difference in ordered elementary domains). *The following laws hold for any ordered elementary domain* $(X, +, \cdot, <)$.

a) **Monotony Law for $-$ and $<$:**

$$\forall a, b, c\,([a, b, c \in X \;\wedge\; c \leq a] \Rightarrow [a < b \Rightarrow a - c < b - c]). \tag{5.316}$$

b) **Monotony Law for $-$ and $\leq$:**

$$\forall a, b, c\,([a, b, c \in X \;\wedge\; c \leq a] \Rightarrow [a \leq b \Rightarrow a - c \leq b - c]). \tag{5.317}$$

Proof. We let X, $+$, $\cdot$, $<$, a, b and c be arbitrary such that $(X, +, \cdot, <)$ is an ordered elementary domain and such that the conjunction of $a, b, c \in X$ and $c \leq a$ is true; thus, the difference $a - c$ is defined. We now prove the implication

$$a < b \Rightarrow a - c < b - c \tag{5.318}$$

by contradiction, assuming $a < b$ and the negation $\neg a - c < b - c$ to be both true. Here, $a < b$ implies the truth of the disjunction $a < b \vee a = b$, which we may also write as $a \leq b$, because the induced total ordering $\leq$ of X satisfies (3.238). Then, since $c \leq a$ and $a \leq b$ are both true, we obtain $c \leq b$ with the transitivity of $\leq$, so that the difference $b - c$ is indeed defined. Furthermore, the assumed negation $\neg a - c < b - c$ implies $b - c \leq a - c$ with the Negation Formula for $<$, and this inequality further implies $(b - c) + c \leq (a - c) + c$ with the Monotony Law for $+$ and $\leq$. In view of (5.308), we have the equations $(b - c) + c = b$ and $(a - c) + c = a$, so that the preceding inequality becomes $b \leq a$, which in turn implies $\neg a < b$ with the Negation Formula for $<$. This finding is in contradiction to the assumption $a < b$, so that the proof of the implication (5.318) is complete. Since X, $+$, $\cdot$, $<$, a, b and c were initially arbitrary, we may therefore conclude that the Monotony Law for $-$ and $<$ holds indeed. □

Exercise 5.38. Prove the Monotony Law for $-$ and $\leq$.

(Hint: Proceed in analogy to the proof of the Monotony Law for $-$ and $<$, using Theorem 3.77a) and Theorem 5.64a).)

Corollary 5.78. *For any ordered elementary domain* $(X, +, \cdot, <)$ *and any elements* a, b *in* X*, it is true that, if* a *is less than* b*, then the difference of* b *and* a *is greater than the zero element, i.e.*

$$\forall a, b\,(a, b \in X \Rightarrow [a < b \Rightarrow b - a > 0_X]). \tag{5.319}$$

Proof. Letting X, $+$, $\cdot$, $<$, a and b be arbitrary such that $(X, +, \cdot, <)$ is an ordered elementary domain and such that a and b are elements of X, we now assume $a < b$ to be true. Since the induced total ordering $\leq$ of X is reflexive, the inequality $a \leq a$ also holds, so that we may apply the Monotony Law for $-$ and $<$ to infer from $a < b$ the truth of $a - a < b - a$. Here, $a - a = 0_X$ is true because of (5.309), so that substitution yields the inequality $0_X < b - a$, and thus the desired consequent $b - a > 0_X$. As X, $+$, $\cdot$, $<$, a and b were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Exercise 5.39. Establish the following laws for any ordered elementary domain $(X, +, \cdot, <)$.

$$\forall a, b, c\,([a, b, c \in X \wedge a \leq b] \Rightarrow [c \leq a \Rightarrow b - a \leq b - c]), \tag{5.320}$$

$$\forall a, b, c\,([a, b, c \in X \wedge a \leq b] \Rightarrow [c < a \Rightarrow b - a < b - c]). \tag{5.321}$$

(Hint: Apply similar arguments as in the proofs of the preceding Monotony Laws for the difference and apply in addition the commutativity and associativity of $+$.)

Proposition 5.79. *For any ordered elementary domain* $(X, +, \cdot, <)$*, it is true that subtracting two elements* b *and* c *of* X *successively from an element* a *of* X *gives the same result as subtracting the sum of* b *and* c *from* a*, given that this sum is less than or equal to* a*, i.e.*

$$\forall a, b, c\,([a, b, c \in X \wedge b + c \leq a] \Rightarrow (a - b) - c = a - (b + c)). \tag{5.322}$$

Proof. We take arbitrary sets X, $+$, $\cdot$, $<$, a, b and c, assuming $(X,+,\cdot,<)$ to be an ordered elementary domain, assuming a ,b, and c to be elements in X, and assuming the inequality $b+c \leq a$ to hold. Thus, the difference $a-(b+c)$ is defined, and we may now prove that the differences $a-b$ and $(a-b)-c$ are also specified.

For this purpose, we prove first the inequality $b \leq a$ by contradiction, assuming its negation to be true, so that $a < b$ holds according to the Negation Formula for $\leq$. Since $0_X \leq c$ follows also to be true with (5.276) and the Characterization of an induced reflexive partial ordering, we obtain with the Monotony Law for $+$ and $\leq$ the inequality $0_X + b \leq c + b$, which implies $b \leq b + c$ with the definition of the zero element and with the commutativity of the addition. In conjunction with the previously established $a < b$, this further implies $a < b + c$ with the Transitivity Formula for $<$ and $\leq$, and the Negation Formula for $\leq$ gives us then $\neg b + c \leq a$. Because $b+c \leq a$ is also true (by assumption), we arrived at a contradiction, so that the proof of $b \leq a$ is now complete; thus, the difference $a-b$ is defined.

Next, we may establish also the inequality $c \leq a - b$ by contradiction, assuming that its negation holds, so that $a - b < c$ follows to be true with the Negation Formula for $\leq$. Then, the Monotony Law for $+$ and $<$ yields $(a-b)+b < c+b$, which in turn implies $a < b + c$ with (5.308) and the commutativity of the addition. Consequently, $\neg b + c \leq a$ turns out to be true because of the Negation Formula for $\leq$, which contradicts again the initial assumption $b+c \leq a$. Having thus completed the proof of $c \leq a - b$, we may infer from this that the difference $(a-b)-c$ is also defined.

Let us observe now the truth of the equations

$$(a-b)-c = (a-b)-c \quad \wedge \quad a-(b+c) = a-(b+c),$$

which imply by definition of a difference

$$c + [(a-b)-c] = a-b, \tag{5.323}$$

$$(b+c) + [a-(b+c)] = a. \tag{5.324}$$

We obtain then the true equations

$$\begin{aligned}(b+c) + [(a-b)-c] &= (c + [(a-b)-c]) + b \\ &= (a-b)+b \\ &= a \\ &= (b+c) + [a-(b+c)]\end{aligned}$$

using the associativity and commutativity of the addition, (5.323), (5.308) and (5.324). Now, the Cancellation Law for $+$ gives us the equation $(a-b)-c = a-(b+c)$, so that the implication in (5.322) is true. As X, $+$, $\cdot$, $<$, a, b and c were arbitrary, we conclude that the proposition holds. $\square$

5.5. The Ordered Elementary Domain $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$

We finally arrive at the main result of the first part of the current exposition.

Theorem 5.80. *The ordered quadruple* $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$ *constitutes an ordered elementary domain with zero element* 0 *and unity element* 1*.*

Proof. We begin with the observation that the Properties 1 – 5 of an ordered elementary domain are satisfied by the semiring $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$in view of Proposition 5.47, Proposition 5.54, Corollary 5.56, and Theorem 5.57. Furthermore, Property 6 is also satisfied by the standard linear ordering $<_{\mathbb{N}}$ according to Corollary 4.21 and Theorem 3.75.

It therefore only remains fur us to verify Property 7 of an ordered elementary domain. For this purpose, we prove

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m <_{\mathbb{N}} n \Leftrightarrow \exists d\,(d \in \mathbb{N} \wedge d \neq 0 \wedge m +_{\mathbb{N}} d = n)]). \quad (5.325)$$

We let m and n be arbitrary natural numbers and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $m <_{\mathbb{N}} n$ to be true. Here, the assumed $n \in \mathbb{N}$ implies $n = 0 \vee n \in \mathbb{N}_+$ with (2.310). We may now prove by contradiction that the first part $n = 0$ of this disjunction is false. Indeed, assuming $n = 0$ to be true, the initial assumption $m <_{\mathbb{N}} n$ gives $m <_{\mathbb{N}} 0$, in contradiction to the fact that $m \in \mathbb{N}$ implies $\neg m <_{\mathbb{N}} 0$ with (4.188). Thus, $n = 0$ is false, and therefore the second part $n \in \mathbb{N}_+$ of the preceding disjunction is true. Now, $m \in \mathbb{N}$ implies $m = 0 \vee m \in \mathbb{N}_+$ again with (2.310), which disjunction we now use to prove the existential sentence in (5.325) by cases.

In case of $m = 0$, we obtain

$$m +_{\mathbb{N}} n = 0 +_{\mathbb{N}} n = n$$

by applying substitution and the fact that 0 is the neutral element for the addition on $\mathbb{N}$ (see Proposition 5.47). These equations give $m +_{\mathbb{N}} n = n$, where $n \in \mathbb{N}$ and $n \neq 0$ are also true, as shown before. This proves the existential sentence in (5.325) for the case $m = 0$.

In case of $m \in \mathbb{N}_+$, we have that m and n are both positive natural numbers, so that the corresponding initial segments $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ of $\mathbb{N}_+$ are defined to be nonempty sets. Furthermore, the assumed inequality $m <_{\mathbb{N}} n$ implies $\{1, \ldots, m\} \subset \{1, \ldots, n\}$ with Theorem 4.20 in connection with the definition of an initial segment of $\mathbb{N}_+$. Then, it follows with (2.127) and with (2.129) that the set difference

$$D = \{1, \ldots, n\} \setminus \{1, \ldots, m\}. \quad (5.326)$$

is a nonempty and proper subset of $\{1, \ldots, n\}$. We also notice that the initial segment $\{1, \ldots, n\}$ is a finite set with cardinality $|\{1, \ldots, n\}| = n$ according to Corollary 4.104. We may now apply Proposition 4.115 to infer from $n \in \mathbb{N}$, from the finiteness of $\{1, \ldots, n\}$, from the equation $|\{1, \ldots, n\}| = n$ and from the proper inclusion $D \subset \{1, \ldots, n\}$ that D is a finite set with cardinality $|D| <_{\mathbb{N}} n$. Since we showed that $D \neq \emptyset$ holds, we may further infer from this with (4.527) that $|D| \neq 0$ is true. Now, the initial segment $\{1, \ldots, m\}$ is also a finite set and has the cardinality $|\{1, \ldots, m\}| = m$ in view of Corollary 4.104. Moreover, (5.326) shows in light of (2.111) that $\{1, \ldots, m\} \cap D = \emptyset$ holds. Introducing the notation $\bar{d} = |D|$, we then see that the sum of the natural numbers m and $\bar{d}$ is determined by

$$\begin{aligned} m +_{\mathbb{N}} \bar{d} &= |\{1, \ldots, m\} \cup D| \\ &= |\{1, \ldots, m\} \cup (\{1, \ldots, n\} \setminus \{1, \ldots, m\})| \\ &= |(\{1, \ldots, n\} \setminus \{1, \ldots, m\}) \cup \{1, \ldots, m\}| \\ &= |\{1, \ldots, n\}| \\ &= n, \end{aligned}$$

using 5.185, (5.326), (2.214), (2.263) in connection with the fact that the previously established proper inclusion $\{1, \ldots, m\} \subset \{1, \ldots, n\}$ implies the inclusion $\{1, \ldots, m\} \subseteq \{1, \ldots, n\}$ due to (2.26), and finally the previously obtained equation $|\{1, \ldots, n\}| = n$. These equations then yield $m +_{\mathbb{N}} \bar{d} = n$, where $(|D| =)\ \bar{d} \neq 0$ holds, as shown earlier. Together with the evident fact that $\bar{d} \in \mathbb{N}$, we thus proved the existential sentence in (5.325) also for the case $m \in \mathbb{N}_+$, so that the proof of the first part of the equivalence in (5.325) is complete.

To prove the second part ('$\Leftarrow$'), we now assume that there exists a nonzero natural number, say $\bar{d}$, satisfying the equation $m +_{\mathbb{N}} \bar{d} = n$. Then, the Cartesian products

$$\begin{aligned} M &= \{1, \ldots, m\} \times \{0\}, \\ D &= \{1, \ldots, \bar{d}\} \times \{1\} \end{aligned}$$

have the cardinalities $|M| = m$ and $|D| = \bar{d}$ according to (4.508). Then, the sum of m and $\bar{d}$ may be expressed by

$$|M \cup D| = m +_{\mathbb{N}} \bar{d} = n,$$

according to (5.184. Let us here observe that $M \cup D$ is a finite set and that $M \cap D = \emptyset$ implies also $D \cap M = \emptyset$ with the Commutative Law for the intersection. The conjunction of the aforementioned inequality $D \neq \emptyset$

and the preceding equation then gives $M \subset M \cup D$ with (2.256). Thus, the previous findings show that the multiple conjunction

$$n \in \mathbb{N} \;\wedge\; M \cup D \text{ is finite } \wedge\; |M \cup D| = n \;\wedge\; M \subset M \cup D$$

is true, so that M is a finite set with cardinality $(m =)\ |M| <_{\mathbb{N}} n$, i.e. with $m <_{\mathbb{N}} n$, as desired. This completes the proof of the second part of the equivalence in (5.325), and the resultant truth of this equivalence proves then also the implication in (5.325). Moreover, as m and n were arbitrary, we may then further conclude that the universal sentence (5.325) holds. Thus, $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$ satisfies also Property 7 of an ordered elementary domain. □

Notation 5.12. We call

$$(\mathbb{N}, +, \cdot, <) = (\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}}) \tag{5.327}$$

the *ordered elementary domain of natural numbers*.

5.5.1. Basic laws for $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$

In the following, we restate for future reference the main Propositions, Corollaries and Theorems of Section 5.4 now for the specific ordered elementary domain $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$.

Corollary 5.81. *It is true for the ordered elementary domain of natural numbers that*

a) *every nonzero natural number is greater than* 0*, i.e.*

$$\forall n\,(n \in \mathbb{N} \Rightarrow [n \neq 0 \Leftrightarrow 0 < n]). \tag{5.328}$$

b) *every natural number is greater than or equal to the zero element, i.e.*

$$\forall n\,(n \in \mathbb{N} \Rightarrow [0 < n \,\vee\, n = 0). \tag{5.329}$$

Note 5.26. We may establish these properties also directly from the properties of the standard linear ordering $<_{\mathbb{N}}$ of $\mathbb{N}$, viz. by applying Corollary 4.37. Similarly, applying Corollary 5.60 to the ordered elementary domain of natural numbers yields the previously obtained sentence (4.188).

Corollary 5.82. *The sum of a nonzero natural number and any natural number is nonzero, i.e.*

$$\forall m, n\,([m, n \in \mathbb{N} \wedge m \neq 0] \Rightarrow m + n \neq 0). \tag{5.330}$$

Corollary 5.83 (Monotony Laws for natural numbers). *The following laws hold for the ordered elementary domain* $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$.

a) **Monotony Law for $+_{\mathbb{N}}$ and $<_{\mathbb{N}}$**:

$$\forall k, m, n\,(k, m, n \in \mathbb{N} \Rightarrow [k < m \Leftrightarrow k + n < m + n]). \tag{5.331}$$

b) **Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$**:

$$\forall k, m, n\,(k, m, n \in \mathbb{N} \Rightarrow [k \leq m \Leftrightarrow k + n \leq m + n]). \tag{5.332}$$

c) **Monotony Law for $\cdot_{\mathbb{N}}$ and $<_{\mathbb{N}}$**:

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge 0 < n] \Rightarrow [k < m \Leftrightarrow k \cdot n < m \cdot n]). \tag{5.333}$$

d) **Monotony Law for $\cdot_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$**:

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge 0 < n] \Rightarrow [k \leq m \Leftrightarrow k \cdot n \leq m \cdot n]). \tag{5.334}$$

Corollary 5.84 (Additivity of inequalities for natural numbers). *The following laws hold for the ordered elementary domain* $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$.

a) **Additivity of $<_{\mathbb{N}}$-inequalities**:

$$\forall j, k, m, n\,(j, k, m, n \in \mathbb{N} \Rightarrow [(j < k \wedge m < n) \Rightarrow j + m < k + n]). \tag{5.335}$$

b) **Additivity of $\leq_{\mathbb{N}}$-inequalities**:

$$\forall j, k, m, n\,(j, k, m, n \in \mathbb{N} \Rightarrow [(j \leq k \wedge m \leq n) \Rightarrow j + m \leq k + n]). \tag{5.336}$$

Corollary 5.85 (Cancellation Laws for natural numbers). *The following laws are true for the ordered elementary domain* $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$.

a) **Cancellation Law for $+_{\mathbb{N}}$**:

$$\forall k, m, n\,(k, m, n \in \mathbb{N} \Rightarrow [k + m = k + n \Rightarrow m = n]). \tag{5.337}$$

b) **Cancellation Law for $\cdot_{\mathbb{N}}$**:

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge k \neq 0] \Rightarrow [k \cdot m = k \cdot n \Rightarrow m = n]). \tag{5.338}$$

c) **Cancellation Law for 0**:

$$\forall n\,(n \in \mathbb{N} \Rightarrow [n \cdot 0 = 0 \wedge 0 \cdot n = 0]). \tag{5.339}$$

Corollary 5.86. *It is true that, if a natural number n is the sum of two natural numbers m and d, then n is greater than or equal to m, i.e.*

$$\forall m, n, d\,([m, n, d \in \mathbb{N} \,\wedge\, m + d = n] \Rightarrow m \leq n). \tag{5.340}$$

Note 5.27. The Distinctness of the zero element and the unity element in the ordered elementary domain of natural numbers was already established in (4.165). Furthermore, the fact that the zero element 0 is less than the unity element 1 was indicated by (4.164).

Corollary 5.87. *The following sentence is true for the ordered elementary domain of natural numbers.*

$$\forall k, m, n\,(k, m, n \in \mathbb{N} \Rightarrow [k \leq m \Rightarrow k \cdot n \leq m \cdot n]). \tag{5.341}$$

5.5.2. Differences of natural numbers

Corollary 5.88. *For any natural numbers m and n such that m is less than or equal to n, it is true that there exists a unique natural number d such that n is the sum of m and d, that is,*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \leq n \Rightarrow \exists! d\,(d \in \mathbb{N} \,\wedge\, m + d = n)]). \tag{5.342}$$

Thus, the difference of two natural numbers n and m satisfying $m \leq n$ is defined as the unique natural number $d = n - m$ which satisfies $m + d = n$.

Note 5.28. Whenever we encounter an equation $m + d = n$, the difference $d = n - m$ exists since the given equation implies the required inequality $m \leq n$ with Corollary 5.86.

Corollary 5.89. *The following sentences holds for the ordered elementary domain of natural numbers.*

$$\forall m, n\,([m, n \in \mathbb{N} \,\wedge\, m \leq n] \Rightarrow [n - m] + m = n). \tag{5.343}$$

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [n + m] - m = n). \tag{5.344}$$

Corollary 5.90. *The difference of any natural number n and itself is zero, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow n - n = 0). \tag{5.345}$$

Furthermore, any natural number is identical with the difference of itself and 0, that is,

$$\forall n\,(n \in \mathbb{N} \Rightarrow n = n - 0). \tag{5.346}$$

Corollary 5.91. *The following laws hold for the ordered elementary domain of natural numbers.*

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge n \leq k \wedge m \leq n] \Rightarrow [k-m]+[m-n]=k-n), \tag{5.347}$$

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge n \leq k \wedge m \leq n] \Rightarrow [k-n]-[k-m]=m-n) \tag{5.348}$$

Corollary 5.92 (Distributive law for the difference of natural numbers). *The following sentence is true for the ordered elementary domain of natural numbers.*

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge n \leq m] \Rightarrow k \cdot [m-n]=k \cdot m-k \cdot n). \tag{5.349}$$

Corollary 5.93 (Monotony Laws for the difference of natural numbers). *The following laws hold for the ordered elementary domain of natural numbers.*

a) **Monotony Law for $-_{\mathbb{N}}$ and $<_{\mathbb{N}}$***:*

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge n \leq k] \Rightarrow [k<m \Rightarrow k-n<m-n]). \tag{5.350}$$

b) **Monotony Law for $-_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$***:*

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge n \leq k] \Rightarrow [k \leq m \Rightarrow k-n \leq m-n]). \tag{5.351}$$

Corollary 5.94. *For any natural numbers m and n, it is true that, if m is less than n, then the difference of n and m is greater than zero, i.e.*

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m<n \Rightarrow n-m>0]). \tag{5.352}$$

Corollary 5.95. *The following laws are true for the ordered elementary domain of natural numbers.*

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge k \leq m] \Rightarrow [n \leq k \Rightarrow m-k \leq m-n]), \tag{5.353}$$

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge k \leq m] \Rightarrow [n<k \Rightarrow m-k<m-n]). \tag{5.354}$$

Corollary 5.96. *Subtracting two natural numbers m and n successively from a natural number k gives the same result as subtracting the sum of m and n from k, given that this sum is less than or equal to k, i.e.*

$$\forall k, m, n\,([k, m, n \in \mathbb{N} \wedge m+n \leq k] \Rightarrow (k-m)-n=k-(m+n)). \tag{5.355}$$

5.5.3. Tails of a sequence

Proposition 5.97. *The following sequences are true for any set Y, any sequence $f = (a_n)_{n\in\mathbb{N}_+}$ in Y and any $N \in \mathbb{N}_+$.*

a) There exists the unique sequence $t_N = (m_n)_{n\in\mathbb{N}_+}$ such that

$$t_N : \mathbb{N}_+ \to \mathbb{N}_+, \quad n \mapsto m_n = (N + n) - 1. \tag{5.356}$$

b) Then, the composition $T_N^f = f \circ t_N$ is a subsequence of $(a_n)_{n\in\mathbb{N}_+}$.

Proof. We let Y, f and N be an arbitrary sets such that $f = (a_n)_{n\in\mathbb{N}_+}$ is a sequence in Y and such that $N \in \mathbb{N}_+$ holds.

Concerning a), we may apply Function definition by replacement to establish the sequence t_N with domain $\mathbb{N}_+$ such that

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow t_N(n) = [N + n] - 1) \tag{5.357}$$

holds. For this purpose, we verify

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \exists! m\,(m = [N + n] - 1)). \tag{5.358}$$

We let $n \in \mathbb{N}_+$ be arbitrary, so that $n \in \mathbb{N}$ follows to be true by definition of the set of positive natural numbers (and the definition of a subset). Let us now observe that the assumed $N \in \mathbb{N}_+$ implies for the same reason that $N \in \mathbb{N}$ and $N \notin \{0\}$ holds, where the latter gives $N \neq 0$ with (2.169). Because of these findings, we may use (5.330) to obtain $N + n \neq 0$, which evidently gives $N + n \in \mathbb{N}$ and $N + n \notin \{0\}$, so that $N + n \in \mathbb{N}_+$. Consequently, the inequality $1 \leq N + n$ holds according to (4.278), which shows that the difference $[N + n] - 1$ exists uniquely (in $\mathbb{N}$). Then, the uniquely existential sentence in (5.358) follows to be true with (1.109). As n is arbitrary, we may therefore conclude that the universal sentence (5.358) holds, so that there exists a unique function t_N with domain $\mathbb{N}_+$ satisfying (5.357). Thus, we may view t_N as the sequence $(m_n)_{n\in\mathbb{N}_+}$ with $m_n = t_N(n) = (N + n) - 1$ for any $n \in \mathbb{N}_+$.

To prove that $\mathbb{N}_+$ is a codomain of t_N, we verify $\mathrm{ran}(t_N) \subseteq \mathbb{N}_+$. To do this, we apply the definition of a subset and prove the equivalent

$$\forall m\,(m \in \mathrm{ran}(t_N) \Rightarrow m \in \mathbb{N}_+). \tag{5.359}$$

Letting m be an arbitrary element of $\mathrm{ran}(t_N)$, there exists by definition of a range an element, say $\bar{n}$, with $(\bar{n}, m) \in t_N$. As t_N is a function, we may write this also as $m = t_N(\bar{n})$. By definition of t_N, we obtain $[m =]\ t_N(\bar{n}) = m_{\bar{n}} = (N+\bar{n})-1$, which gives $m = (N+\bar{n})-1$. Furthermore, since the domain of t_N is $\mathbb{N}_+$, it follows from $(\bar{n}, m) \in t_N$ that $\bar{n} \in \mathbb{N}_+$ holds. The latter finding and $N \in \mathbb{N}_+$ evidently imply $1 \leq \bar{n}$ as well as $1 \leq N$. Applying now the Monotony Law for $+_\mathbb{N}$ and $\leq_\mathbb{N}$, we obtain the inequalities

$$\begin{aligned} 1 + 1 &\leq \bar{n} + 1, \\ 1 + \bar{n} &\leq N + \bar{n}, \end{aligned}$$

which further imply $1 + 1 \leq N + \bar{n}$ with the commutativity of $+_\mathbb{N}$ and the transitivity of $\leq_\mathbb{N}$. Since $1 < 2$ holds according to (4.166), the disjunction $1 < 2 \vee 1 = 2$ is also true, so that we obtain $1 \leq 2$ by definition of an induced reflexive partial ordering; here we have $2 = 1^+ = 1+1$ due to (2.292) and (5.217), so that we may write the preceding inequality also as $1 \leq 1+1$. We may therefore apply the Monotony Law for $-_\mathbb{N}$ and $\leq_\mathbb{N}$ to infer from $1 + 1 \leq N + \bar{n}$ the truth of the inequality $(1 + 1) - 1 \leq (N + \bar{n}) - 1$ $[= m]$, which yields $1 \leq m$ with (5.344). Since $0 < 1$ also holds in view of (4.164), we obtain with the Transitivity Formula for $<$ and $\leq$ the inequality $0 < m$, and then $m \neq 0$ with the Characterization of comparability (applied to the standard linear ordering of $\mathbb{N}$). Thus, $m \in \mathbb{N}$ and $m \notin \{0\}$ are evidently true, so that the desired consequent $m \in \mathbb{N}_+$ follows to be true. As m was arbitrary, we may therefore conclude that the universal sentence (5.359) holds, which gives the $\mathrm{ran}(t_N) \subseteq \mathbb{N}_+$ with the definition of a subset. This finding shows that $\mathbb{N}_+$ is indeed a codomain of t_N, which is thus characterized by (5.356).

Concerning b), we first verify that the sequence $t_N = (m_n)_{n \in \mathbb{N}_+}$ is strictly increasing, by applying the Monotony Criterion for strictly increasing sequences. For this purpose, we let $n \in \mathbb{N}_+$ be arbitrary and show that this implies $m_n < m_{n+1}$. Let us observe the truth of $N + n < (N + n) + 1$ in light of (4.153) and (5.217). This inequality then implies

$$(N + n) - 1 < ([N + n] + 1) - 1$$

with the Monotony Law for $-_\mathbb{R}$ and $<_\mathbb{R}$, which we may apply because the assumed $n, N \in \mathbb{N}_+$ implies $1 \leq 2 \leq N + n$ (as mentioned in the proof of a)) and therefore $1 \leq N + n$ (applying the transitivity of the total ordering $\leq_\mathbb{N}$). By definition of t_N, we have the terms $m_n = (N + n) - 1$ and

$$m_{n+1} = (N + [n + 1]) - 1 = ([N + n] + 1) - 1$$

(where we applied the commutativity of $+_\mathbb{N}$), so that substitution yields the desired inequality $m_n < m_{n+1}$. As n was arbitrary, we may therefore conclude that $t_N = (m_n)_{n \in \mathbb{N}_+}$ satisfies the Monotony Criterion for strictly increasing sequences. Based on this finding, Exercise 4.26a) shows that the composition $f \circ t_N$ yields the sequence $(a'_n)_{n \in \mathbb{N}_+} = (a_{m_n})_{n \in \mathbb{N}_+}$ in Y, which is then by definition a subsequence of $(a_n)_{n \in \mathbb{N}_+}$. $\square$

Definition 5.20 (Tail). For any set Y, any sequence $f = (a_n)_{n\in\mathbb{N}_+}$ in Y and any $N \in \mathbb{N}_+$, we call

$$T_N^f = f \circ t_N = (a'_n)_{n\in\mathbb{N}_+} = (a_{m_n})_{n\in\mathbb{N}_+} = (a_{[N+n]-1})_{n\in\mathbb{N}_+} \tag{5.360}$$

the *N-th tail* of $(a_n)_{n\in\mathbb{N}_+}$.

Exercise 5.40. Show

a) for any set Y and any addition $+_Y$ on Y such that Y contains the zero element 0_Y, as well as for any elements $\bar{a}, \bar{b} \in Y$, that there exists a unique sequence $f = (a_n)_{n\in\mathbb{N}_+}$ in Y such that

$$a_n = \begin{cases} \bar{a} & \text{if } n = 1 \\ \bar{b} & \text{if } n = 2 \\ 0_Y & \text{if } n > 2 \end{cases} \tag{5.361}$$

b) for any set system $\mathcal{K}$ with $\emptyset \in \mathcal{K}$ as well as any sets $\bar{A}, \bar{B} \in \mathcal{K}$ that there is a unique sequence of sets $(A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{K}$ such that

$$A_n = \begin{cases} \bar{A} & \text{if } n = 1 \\ \bar{B} & \text{if } n = 2 \\ \emptyset & \text{if } n > 2 \end{cases}, \tag{5.362}$$

and that the union of this sequence equals the union of $\bar{A}$ and $\bar{B}$, i.e.

$$\bigcup_{n=1}^{\infty} A_n = \bar{A} \cup \bar{B}. \tag{5.363}$$

(Hint: Define the sequences by replacement and apply then the definition of a codomain; concerning b), use the Characterization of the union of a family of sets.)

Proposition 5.98. *For any set Y and any addition $+_Y$ on Y such that Y contains the zero element 0_Y, as well as for any elements $\bar{a}, \bar{b} \in Y$, it is true that the third tail of the sequence $f = (a_n)_{n\in\mathbb{N}_+}$ defined by (5.361) is identical with the constant function on $\mathbb{N}_+$ with value 0_Y, i.e.*

$$T_3^f = \mathbb{N}_+ \times \{0_Y\} \quad [= g_{0_Y}]. \tag{5.364}$$

Proof. We let Y and $+_Y$ be arbitrary sets, assume that $+$ is a addition on Y and that Y contains the neutral element 0_Y with respect to that addition, and we let $\bar{a}, \bar{b}$ be arbitrary elements of Y. Let us first observe that the tail T_3^f of the function f defined in Exercise 5.40 and the constant function $g_{0_Y} = \mathbb{N}_+ \times \{0_Y\}$ are both functions with domain $\mathbb{N}_+$, so that we may apply the Equality Criterion for functions to establish their identity. Letting $n \in \mathbb{N}_+$ be arbitrary, we notice the truth of $0 < 1 \leq n$ in view of (4.164) as well as (4.278), which inequalities give $0 < n$ with the Transitivity Formula for $<$ and $\leq$, and furthermore $0 + 2 < n + 2$ with the Monotony Law for $+_{\mathbb{N}}$ and $<_{\mathbb{N}}$. We therefore obtain $n + 2 > 2$ with the definition of a neutral element, which in turn yields $a_{n+2} = 0_Y$ with (5.361). Let us also observe the truth of the equations

$$\begin{aligned}[3+n]-1 &= [n+3]-1 \\ &= [n+2^+]-1 \\ &= [n+(2+1)]-1 \\ &= [(n+2)+1]-1 \\ &= n+2\end{aligned}$$

in view of the commutativity of $+_{\mathbb{N}}$, (2.293), (5.217), the associativity of $+_{\mathbb{N}}$, and (5.344). These findings, alongside the definition of a tail and Corollary 3.154 then yield the true equations

$$T_3^f(n) = a_{[3+n]-1} = a_{n+2} = 0_Y = g_{0_Y}(n).$$

As n is arbitrary, we may therefore infer from the resulting equation $T_3^f(n) = g_{0_Y}(n)$ the truth of the proposed identity $T_3^f = g_{0_Y}$.

Since Y, $+_Y$, $\bar{a}$ and $\bar{b}$ were initially arbitrary, we may further conclude that the proposition is true. $\square$

Exercise 5.41. Show for any set system $\mathcal{K}$ with $\emptyset \in \mathcal{K}$ and for any sets $\bar{A}, \bar{B} \in \mathcal{K}$ that the third tail of the sequence of sets $f = (A_n)_{n \in \mathbb{N}_+}$ defined by (5.362) equals both the constant function on $\mathbb{N}_+$ with value $\emptyset$ and the Cartesian product of the set of positive natural numbers and the natural number 1, i.e.

$$T_3^f = g_\emptyset = \mathbb{N}_+ \times 1. \tag{5.365}$$

(Hint: Recall (2.155).)

The range of a tail may be characterized as the range of a restriction of the original sequence to a subset of $\mathbb{N}_+$ without an initial segment.

Theorem 5.99 (Characterization of the range of a tail). *For any set Y, any sequence $f = (a_n)_{n \in \mathbb{N}_+}$ in Y and any $N \in \mathbb{N}_+$, it is true that the range of the N-th tail of f is identical with the range of the restriction of f to $\mathbb{N}_+ \setminus \{1, \ldots, N-1\}$, i.e.*

$$\mathrm{ran}(T_N^f) = \mathrm{ran}(f \upharpoonright \mathbb{N}_+ \setminus \{1, \ldots, N-1\}). \tag{5.366}$$

Proof. We let Y, f and N be arbitrary sets, we assume that $f = (a_n)_{n\in\mathbb{N}_+}$ is a sequence in Y, and we assume $N \in \mathbb{N}_+$ to be true, so that $T^f_N = (a'_n)_{n\in\mathbb{N}_+}$ is the N-th tail of f, whose terms thus satisfy $a'_n = a_{[N+n]-1}$ for any $n \in \mathbb{N}_+$. We also see that $N \in \mathbb{N}_+$ implies $1 \leq N$ with (4.278) and evidently $N \in \mathbb{N}$ with the definition of the set of positive natural numbers, so that the difference $N - 1$ exists indeed (in $\mathbb{N}$). Now, to establish the proposed equation, we verify

$$\forall y\,(y \in \mathrm{ran}(f \restriction \mathbb{N}_+ \setminus \{1, \ldots, N-1\}) \Leftrightarrow y \in \mathrm{ran}(T^f_N)). \tag{5.367}$$

For this purpose, we let y be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that y is in the range of the restriction $f \restriction \mathbb{N}_+ \setminus \{1, \ldots, N-1\}$). Then, there exists (by definition of a range) a constant, say $\bar{n}$, with

$$(\bar{n}, y) \in f \restriction \mathbb{N}_+ \setminus \{1, \ldots, N-1\},$$

which implies $\bar{n} \in \mathbb{N}_+ \setminus \{1, \ldots, N-1\}$ and $(\bar{n}, y) \in f$ by definition of a restriction. Here, we may write the latter also as $y = f(\bar{n}) = a_{\bar{n}}$ (since f is a function/sequence), whereas the former means by definition of a set difference that $\bar{n} \in \mathbb{N}_+$ and $\bar{n} \notin \{1, \ldots, N-1\}$ hold. Here, $\bar{n} \in \mathbb{N}_+$ gives the inequality $1 \leq \bar{n}$ with (4.278). Let us now observe that the previously established $1 \leq N$ implies $1 < N \vee 1 = N$ with the definition of an induced reflexive partial ordering; we use this disjunction to prove the sentence $N \leq \bar{n}$ by cases.

In the first case of $1 < N$, we notice that $1 \leq 1$ holds because the standard total ordering of $\mathbb{N}$ is reflexive. Then, the previous two inequalities imply $1 - 1 < N - 1$ with the Monotony Law of $-_{\mathbb{N}}$ and $<_{\mathbb{N}}$, so that we obtain $0 < N - 1$ with (5.345). An application of (5.328) therefore gives $N - 1 \neq 0$, which in turn implies $N - 1 \notin \{0\}$ with (2.169). Together with $N - 1 \in \mathbb{N}$, this further implies $N - 1 \in \mathbb{N}_+$, so that we may use (4.275) and the Law of Contraposition to infer from the previously found negation $\neg \bar{n} \in \{1, \ldots, N-1\}$ the truth of the negation $\neg \bar{n} \leq N - 1$, which yields $N - 1 < \bar{n}$ with the Negation Formula for $\leq$. Consequently, we obtain $(N-1)+1 \leq \bar{n}$ with (5.217) and (4.270), and we may write this inequality as $N \leq \bar{n}$ according to (5.343).

In the second case of $1 = N$, we may apply substitution to the previously established inequality $1 \leq \bar{n}$ to obtain the desired $N \leq \bar{n}$, so that the proof by cases is complete.

Thus, the difference $\bar{n} - N$ exists in $\mathbb{N}$, and its successor, which we write as $(\bar{n} - N) + 1$ due to (5.217), is then an element of $\mathbb{N}_+$ in view of (4.40).

We then obtain the true equations

$$\begin{aligned} T_N^f([\bar{n} - N] + 1) &= a'_{[\bar{n}-N]+1} = a_{[N+([\bar{n}-N]+1)]-1} \\ &= a_{[(N+[\bar{n}-N])+1]-1} \\ &= a_{N+[\bar{n}-N]} \\ &= a_{[\bar{n}-N]+N} \\ &= a_{\bar{n}} \\ &= y, \end{aligned}$$

by using the definition of a tail, the associativity of $+_{\mathbb{N}}$, (5.344), the commutativity of $+_{\mathbb{N}}$, (5.343), and finally the previously obtained equation $y = a_{\bar{n}}$. As the tail is a sequence/function, we may write the resulting equation $y = T_N^f([\bar{n} - N] + 1)$ also as $([\bar{n} - N] + 1, y) \in T_N^f$, which implies the desired $y \in \mathrm{ran}(T_N^f)$ by definition of a range.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume that $y \in \mathrm{ran}(T_N^f)$ holds, and we may use some of the arguments of the first part to establish the desired consequent. To begin with, the preceding assumption implies that there exists a constant, say $\bar{m}$, such that $(\bar{m}, y) \in T_N^f$ holds. On the one hand, this gives

$$y = T_N^f(\bar{m}) = a'_{\bar{m}} = a_{[N+\bar{m}]-1} = f([N + \bar{m}] - 1), \tag{5.368}$$

and on the other hand $\bar{m} \in \mathbb{N}_+$ $(= \mathrm{dom}(T_N^f))$ with the definition of a domain. Then, $\bar{m} \in \mathbb{N}_+$ and $N \in \mathbb{N}_+$ evidently imply $1 \leq \bar{m}$ and $1 \leq N$, which inequalities we may add to obtain $1+1 \leq N+\bar{m}$. Because of $1 \leq 1+1$, we may now form the inequality $(1 + 1) - 1 \leq (N + \bar{m}) - 1$, which we may write also as $[0 <]\ 1 \leq (N + \bar{m}) - 1$. Therefore, $0 < (N + \bar{m}) - 1$ is true, so that $(N + \bar{m}) - 1 \neq 0$ holds, with the consequence that

$$(N + \bar{m}) - 1 \in \mathbb{N}_+. \tag{5.369}$$

Furthermore, $[0 <]\ 1 \leq \bar{m}$ gives $0 < \bar{m}$ and then $0 + N < \bar{m} + N$, which results in $N < N + \bar{m}$. Because of $1 \leq N$, we may therefore form the inequality $N - 1 < (N + \bar{m}) - 1$, which yields the negation

$$\neg (N + \bar{m}) - 1 \leq N - 1. \tag{5.370}$$

Let us consider now again the two cases $1 < N$ and $1 = N$ in order to prove the sentence

$$(N + \bar{m}) - 1 \in \mathbb{N}_+ \setminus \{1, \ldots, N - 1\}. \tag{5.371}$$

We already showed that the first case gives $N - 1 \in \mathbb{N}_+$; since (5.369) also holds, we may infer from (5.370) the truth of the negation $\neg (N + \bar{m}) - 1 \in$

$\{1, \ldots, N-1\}$ because of (4.275). The conjunction of (5.369) and the preceding finding yields then the desired sentence (5.371).

The second case $N = 1$ gives

$$\begin{aligned}\mathbb{N}_+ \setminus \{1, \ldots, N-1\} = \mathbb{N}_+ \setminus \{1, \ldots, 1-1\} = \mathbb{N}_+ \setminus \{1, \ldots, 0\} = \mathbb{N}_+ \setminus \emptyset \\ = \mathbb{N}_+.\end{aligned}$$

by applying substitution, (5.345), the notation for initial segments of $\mathbb{N}_+$, and (2.102). Therefore, (5.369) implies via substitution (5.371), completing the proof by cases.

The equations in (5.368) give $y = f([N+\bar{m}]-1)$, which we may write as $([N+\bar{m}]-1, y) \in f$. Together with (5.371), this implies with the definition of a restriction

$$([N+\bar{m}]-1, y) \in f \upharpoonright \mathbb{N}_+ \setminus \{1, \ldots, N-1\},$$

so that y follows to be an element of the range of that restriction. Thus, the proof of the second part of the equivalence (5.367) is complete. Since y is arbitrary, we may therefore conclude that the universal sentence (5.367) is true, which then implies the proposed equation (5.366) with the Equality Criterion for sets. As Y, f and N were initially arbitrary sets, we may finally infer from this the truth of the stated theorem. □

Proposition 5.100. *For any set Y, any sequence $f = (a_n)_{n \in \mathbb{N}_+}$ in Y and any $M, N \in \mathbb{N}_+$ with $M \leq_{\mathbb{N}} N$, it is true that the N-th tail of f is the $(N-M+1)$-th tail of the M-th tail of f, i.e.*

$$\forall Y, f, M, N\, ([f \in Y^{\mathbb{N}_+} \wedge M, N \in \mathbb{N}_+ \wedge M \leq_{\mathbb{N}} N] \Rightarrow T_N^f = T_{[N-M]+1}^{T_M^f}). \tag{5.372}$$

Proof. We let Y, f, M and N be arbitrary sets such that $f = (a_n)_{n \in \mathbb{N}_+}$ is a sequence in Y and such that $M, N \in \mathbb{N}_+$ and $M \leq_{\mathbb{N}} N$ are true. By definition of a tail, we may write

$$T_M^f = (a'_n)_{n \in \mathbb{N}_+} = (a_{[M+n]-1})_{n \in \mathbb{N}_+}, \tag{5.373}$$

$$T_N^f = (a''_n)_{n \in \mathbb{N}_+} = (a_{[N+n]-1})_{n \in \mathbb{N}_+}, \tag{5.374}$$

$$\tag{5.375}$$

Next, we notice that $M \leq_{\mathbb{N}} N$ implies $N - M \in \mathbb{N}$ with the definition of a difference and then $(N-M)+1 \in \mathbb{N}_+$ with the fact that the range of the successor function on $\mathbb{N}$ equals $\mathbb{N}_+$. With this finding, we see that the

$([N-M]+1)$-th tail of T_M^f exists, which we may then write as

$$\begin{aligned} T_{[N-M]+1}^{T_M^f} &= (a'_{[([N-M]+1)+n]-1})_{n\in\mathbb{N}_+} \\ &= (a'_{[([N-M]+n)+1]-1})_{n\in\mathbb{N}_+} \\ &= (a'_{[N-M]+n})_{n\in\mathbb{N}_+} \\ &= (a_{[M+([N-M]+n)]-1})_{n\in\mathbb{N}_+} \\ &= (a_{[([N-M]+M)+n]-1})_{n\in\mathbb{N}_+} \\ &= (a_{[N+n]-1})_{n\in\mathbb{N}_+} \\ &= T_N^f \end{aligned}$$

using the definition of a tail (of T_M^f), the associativity as well as the commutativity of $\mathbb{N}_+$, (5.344), then (5.373), the associativity as well as the commutativity of $\mathbb{N}_+$, subsequently (5.343), and finally (5.374); thus the equation in (5.372) is true. Since Y, f, M and N were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Since tails are subsequences according to Proposition 5.97b), the following tail characteristics follow immediately from Proposition 4.80, Exercise 4.27 and Corollary 4.81.

Corollary 5.101. *For any partially ordered set $(Y, \leq_Y)$ it is true that*

a) any tail of any bounded-from-below sequence in Y is itself bounded from below (by the same lower bound).

b) any tail of any bounded-from-above sequence in Y is itself bounded from above (by the same upper bound).

c) any tail of any bounded sequence in Y is itself bounded.

Furthermore, we see that Proposition 4.82, Exercise 4.28 and Corollary 4.83 apply in particular to tails.

Corollary 5.102. *For any partially ordered set $(Y, \leq_Y)$ it is true that*

a) any tail of any increasing sequence in Y is itself increasing.

b) any tail of any decreasing sequence in Y is itself decreasing.

c) any tail of any monotone sequence in Y is itself monotone.

Proposition 5.103. *For any partially ordered set $(Y, \leq_Y)$, any decreasing sequence $f = (a_n)_{n\in\mathbb{N}_+}$ in Y, any $N \in \mathbb{N}_+$ and any $a \in X$, it is true that a is a lower bound for f iff a is a lower bound for the N-th tail of f.*

Proof. We let Y, $\leq_Y$, f, N and a be arbitrary and assume then $(Y, \leq_Y)$ to be a partially ordered set, $f = (a_n)_{n \in \mathbb{N}_+}$ to be a decreasing sequence in Y, N to be a positive natural number and a to be an element of Y. To prove the first part ('$\Rightarrow$') of the proposed equivalence, we assume that a is a lower bound for f, which already implies with Corollary 5.101a) that a is a lower bound for the tail T_N^f of f. To prove the second part ('$\Leftarrow$'), we now assume that a is a lower bound for T_N^f, i.e. that

$$\forall y\,(y \in \operatorname{ran}(T_N^f) \Rightarrow a \leq_Y y) \tag{5.376}$$

holds. To prove that a is also a lower bound for f, we verify accordingly

$$\forall y\,(y \in \operatorname{ran}(f) \Rightarrow a \leq_Y y). \tag{5.377}$$

To do this, we let $\bar{y}$ be arbitrary and assume $\bar{y} \in \operatorname{ran}(f)$, so that there exists by definition of a range a constant, say $\bar{n}$, such that $(\bar{n}, \bar{y}) \in f$ holds. This implies on the one hand $\bar{y} = f(\bar{n}) = a_{\bar{n}}$, and on the other hand $\bar{n} \in \mathbb{N}_+$ $[= \operatorname{dom}(f)]$ by definition of a domain. Since the disjunction $N \leq_{\mathbb{N}} \bar{n} \vee \neg N \leq_{\mathbb{N}} \bar{n}$ is true according to the Law of the Excluded Middle, we may now prove the desired consequent $a \leq_Y \bar{y}$ by cases.

The first case $N \leq_{\mathbb{N}} \bar{n}$ implies $\bar{n} - N \in \mathbb{N}$ with the definition of a difference and then $[\bar{n} - N] + 1 \in \mathbb{N}_+$ by using (5.217) together with the fact that the successor function on $\mathbb{N}$ is a bijection with range $\mathbb{N}_+$. Therefore, the $([\bar{n} - N] + 1)$-th term of the tail T_N^f is defined, and we obtain the equations

$$\begin{aligned} T_N^f([\bar{n} - N] + 1) &= a'_{[\bar{n}-N]+1} = a_{[N+([\bar{n}-N]+1)]-1} \\ &= a_{[([\bar{n}-N]+N)+1]-1} \\ &= a_{[(\bar{n})+1]-1} \\ &= a_{\bar{n}} \\ &= \bar{y} \end{aligned}$$

applying the definition of a tail, the associativity and commutativity of $+_{\mathbb{N}}$, then (5.343), subsequently (5.344), and finally substitution. We may write the resulting equation $\bar{y} = T_N^f([\bar{n} - N] + 1)$ also as $([\bar{n} - N] + 1, \bar{y}) \in T_N^f$, which yields (by definition of a range) $\bar{y} \in \operatorname{ran}(T_N^f)$, so that the desired inequality $a \leq_Y \bar{y}$ follows to be true with (5.376).

In the other case of $\neg N \leq_{\mathbb{N}} \bar{n}$, which implies $\bar{n} <_{\mathbb{N}} N$ with the Negation Formula for $\leq$, it follows that $[\bar{y} =]\ a_{\bar{n}} \geq_Y a_N$ holds because the sequence $(a_n)_{n \in \mathbb{N}_+}$ was assumed to be decreasing. Due to $1 \in \mathbb{N}_+$, we evidently obtain

$$T_N^f(1) = a'_1 = a_{[N+1]-1} = a_N,$$

so that the preceding inequality becomes $a_1' \leq_Y \bar{y}$. Furthermore, we may write the first of the previous equations as $(1, a_1') \in T_N^f$, which gives (by definition of a range) $a_1' \in \text{ran}(T_N^f)$, and this finding implies now $a \leq_Y a_1'$ with (5.376). Having found the two inequalities $a \leq_Y a_1' \leq_Y \bar{y}$, we now see in light of the transitivity of the reflexive partial ordering $\leq_Y$ that $a \leq_Y \bar{y}$ is also true in the second case.

Thus, the proof by cases is complete, and as $\bar{y}$ is arbitrary, we may therefore conclude that (5.377) holds, so that a is indeed a lower bound for the sequence f. This completes the proof of the stated equivalence, and since Y, $\leq_Y$, f, N and a were initially arbitrary, we may now finally conclude that the proposition is true. □

Exercise 5.42. Show for any partially ordered set $(X, \leq_X)$, any increasing sequence $f = (a_n)_{n \in \mathbb{N}_+}$ in X, any $N \in \mathbb{N}_+$ and any $u \in X$ that u is an upper bound for f iff u is an upper bound for the N-th tail of f.

Theorem 5.104 (Tail Criterion for convergence of an increasing & of a decreasing sequence). *The following sentences are true for any partially ordered set $(Y, \leq_Y)$, any $N \in \mathbb{N}_+$, and any $L \in Y$.*

a) For any increasing sequence $s = (a_n)_{n \in \mathbb{N}_+}$ in Y it is true that s converges increasingly to L iff the N-th tail $T_N^s = (a_n')_{n \in \mathbb{N}_+}$ of s converges increasingly to L, i.e.

$$L = \lim_{n \to \infty} a_n = \sup \text{ran}(s) \;\Leftrightarrow\; L = \lim_{n \to \infty} a_n' = \sup \text{ran}(T_N^s). \quad (5.378)$$

b) For any decreasing sequence $s = (a_n)_{n \in \mathbb{N}_+}$ in Y it is true that s converges decreasingly to L iff the N-th tail $T_N^s = (a_n')_{n \in \mathbb{N}_+}$ of s converges decreasingly to L, i.e.

$$L = \lim_{n \to \infty} a_n = \inf \text{ran}(s) \;\Leftrightarrow\; L = \lim_{n \to \infty} a_n' = \inf \text{ran}(T_N^s). \quad (5.379)$$

Proof. We let Y, $\leq_Y$, f, N and L be arbitrary and assume $(Y, \leq_Y)$ to be a partially ordered set, $f = (a_n)_{n \in \mathbb{N}_+}$ to be an increasing sequence in Y, N to be a positive natural number, and L to be an element of Y.

To prove the first part ('⇒') of the equivalence, we assume that the sequence s converges increasingly to L, i.e.

$$L = \lim_{n \to \infty} a_n = \sup \text{ran}(s). \quad (5.380)$$

Since s is by assumption increasing and since the N-th tail $T_N^s = (a_n')_{n \in \mathbb{N}_+}$ of s is a subsequence of s according to Proposition 5.97, it follows that T_N^s

is also increasing in view Proposition 4.82. Furthermore, as the supremum $L = \sup \operatorname{ran}(s)$ is an upper bound for s, it follows that L is also an upper bound for the tail T_N^s due to Exercise 4.27. To prove that L is the supremum of the range of T_N^s, we let S' be an arbitrary upper bound for $\operatorname{ran}(T_N^s)$ and demonstrate that this implies $L \leq_Y S'$. Since s is increasing and since T_N^s is a tail of s, it then follows with Exercise 5.42 that S' is also an upper bound for (the range of) s. Since L was assumed to be the supremum of the range of s, we obtain $L \leq_Y S'$, as desired, completing the proof that L is the supremum of the range of $T_N^s = (a'_n)_{n \in \mathbb{N}_+}$. By definition of the limit of an increasing sequence, we thus have that

$$L = \sup \operatorname{ran}(T_N^s) = \lim_{n \to \infty} a'_n. \tag{5.381}$$

To prove the second part ('$\Leftarrow$') of the equivalence (5.378), we now assume that the tail T_N^s converges increasingly to L, i.e. we assume (5.381) to be true. Now, since s is increasing and since the supremum L of $\operatorname{ran}(T_N^s)$ is an upper bound for the N-the tail T_N^s of s, it follows with Exercise 5.42 that L is an upper bound also for s. Next, we demonstrate that L is the least upper bound for s, by letting S' be an arbitrary upper bound for s and by showing that this implies $L \leq_Y S'$. Clearly, S' is then also an upper bound for the tail T_N^s of s, which gives the desired $L \leq_Y S'$ because $L = \sup \operatorname{ran}(T_N^s)$ is the least upper bound for T_N^s. Thus, the proof that L is the least upper bound also for s is complete, so that $L = \sup \operatorname{ran}(s)$ holds. By definition of an increasingly convergent sequence, this yields (5.380), which completes the proof of the equivalence (5.378).

Since Y, $\leq_Y$, f, N and L were arbitrary, it follows then that Part a) of the theorem holds. $\square$

Exercise 5.43. Prove the Tail Criterion for convergence of a decreasing sequence, i.e. Part b) of Theorem 5.104.

5.6. n-Fold Binary Operations

Proposition 5.105. *The following sentences are true for any set X and any binary operation $\odot$ on X such that X contains the neutral element e with respect to $\odot$.*

a) For any $n \in \mathbb{N}$ and any sequence $s = (a_i \mid i \in \{1, \ldots, n^+\})$ in X, there exists the unique function

$$f : \{0, \ldots, n\} \times X \to X, \quad (i, x) \mapsto x \odot a_{i^+}. \tag{5.382}$$

b) Then, there exists the unique sequence $u = (u_i \mid i \in \{0, \ldots, n^+\})$ in X whose terms are defined by

$$(1)\ \ u_0 \ \ = e, \tag{5.383}$$

$$(2)\ \ u_{i^+} = f(i, u_i) = u_i \odot a_{i^+} \quad \text{for any } i \in \{0, \ldots, n\}. \tag{5.384}$$

Proof. We let X, $\odot$, n and s be arbitrary, assume that $\odot$ is a binary operation on X such that the identity element e in X with respect to $\odot$ exists, assume that n is a natural number, and assume moreover that $s = (a_i \mid i \in \{1, \ldots, n^+\})$ is a sequence in X with domain $\{1, \ldots, n^+\}$.

Concerning a), let us first verify

$$\forall z\, (z \in \{0, \ldots, n\} \times X \Rightarrow \exists! y\, (\exists i, x\ (z = (i, x)\ \wedge\ y = x \odot a_{i^+}))). \tag{5.385}$$

We take an arbitrary element z in $\{0, \ldots, n\} \times X$. Regarding the existential part, we notice that $z \in \{0, \ldots, n\} \times X$ implies by definition of the Cartesian product of two sets that there exist an element in $\{0, \ldots, n\}$, say $\bar{i}$, and an element in X, say $\bar{x}$, with $z = (\bar{i}, \bar{x})$. Here, $\bar{i} \in \{0, \ldots, n\}$ implies $\bar{i} \leq_{\mathbb{N}} n$ with (4.180), which in turn gives $\bar{i}^+ \leq_{\mathbb{N}} n^+$ due to (4.163), and this finding yields $\bar{i}^+ \in \{1, \ldots, n^+\}$ $[= \mathrm{dom}(s)]$ according to (4.275) and the assumption that s is a sequence with domain $\{1, \ldots, n^+\}$. The latter finding implies now with the Function Criterion that there exists the unique value/term $\bar{a} = a_{\bar{i}^+}$ of that sequence in X. Together with the previously established $\bar{x} \in X$, this gives $(\bar{x}, a_{\bar{i}^+}) \in X \times X$ $[= \mathrm{dom}(\odot)]$, so that there exists the unique value $\bar{y} = \bar{x} \odot a_{\bar{i}^+}$ in X (using the Function Criterion with respect to the binary operation $\odot$), proving the existential part of the uniquely existential sentence in (5.385). Now, it may not be obvious that this value $\bar{y}$ is uniquely determined by z. To verify its uniqueness, we let y and y' be arbitrary and assume

$$\exists i, x\ (z = (i, x)\ \wedge\ y = x \odot a_{i^+})\ \wedge\ \exists i, x\ (z = (i, x)\ \wedge\ y' = x \odot a_{i^+})$$

to be true. Thus there exist elements, say $\bar{i}$ and $\bar{x}$, with $z = (\bar{i}, \bar{x})$ and $y = \bar{x} \odot a_{\bar{i}^+}$, and there exist elements, say $\bar{i}'$ and $\bar{x}'$, with $z = (\bar{i}', \bar{x}')$ and $y' = \bar{x}' \odot a_{\bar{i}'^+}$. Then, substitution yields $(\bar{i}, \bar{x}) = (\bar{i}', \bar{x}')$ and consequently $\bar{i} = \bar{i}'$ as well as $\bar{x} = \bar{x}'$ with (3.3). Substitution based on these equations now gives

$$(y =)\ \bar{x} \odot a_{\bar{i}^+} = \bar{x}' \odot a_{\bar{i}'^+}\ (= y'),$$

so that we obtain indeed $y = y'$, which proves the uniqueness part of the uniquely existential sentence in (5.385). As z is arbitrary, we may therefore conclude that (5.385) holds. Then, there exists because of Theorem 3.160 a unique function f with domain $\{0, \ldots, n\} \times X$ such that

$$\forall z\, (z \in \{0, \ldots, n\} \times X \Rightarrow \exists i, x\ (z = (i, x) \wedge f(i, x) = x \odot a_{i^+})).$$

This shows that every ordered pair $(i, x) \in \{0, \ldots, n\} \times X$ is mapped by f to the value $x \odot a_{i^+}$. To verify that X is a codomain of f, we demonstrate that the range of f is included in X. For this purpose, we let $y \in \mathrm{ran}(f)$ be arbitrary, so that there exists (by definition of a range) an element, say $\bar{z}$, with $(\bar{z}, y) \in f$. On the one hand, we may write this as $y = f(\bar{z})$. On the other hand, this implies by definition of a domain that $\bar{z} \in \mathrm{dom}(f)$ holds, so that $\bar{z} \in \{0, \ldots, n\} \times X$. By definition of a Cartesian product, there then exists an element of $\{0, \ldots, n\}$, say $\bar{i}$, and an element of X, say $\bar{x}$, such that $\bar{z} = (\bar{i}, \bar{x})$. With this equation, it follows from the previously obtained $y = f(\bar{z})$ with the definition of the functions f and $\odot$ that

$$y = f(\bar{z}) = f(\bar{i}, \bar{x}) = \bar{x} \odot a_{\bar{i}^+}\ (\in X),$$

is true. We thus showed that $y \in \mathrm{ran}(f)$ implies $y \in X$; since y is arbitrary, it then follows with the definition of a subset that the range of f is indeed included in X, so that X is a codomain of f. This completes the proof that the function f in (5.382) exists uniquely.

Concerning b), we may apply the Recursion Theorem for initial segments with respect to the counting domain $(\mathbb{N}, s^+, 0)$ to infer from a) the unique existence of the sequence $u = (u_i \,|\, i \in \{0, \ldots, n^+\})$ in X satisfying (5.383) – (5.384), where we use the fact that the identity e is an element of X. Since X and $\odot$ were arbitrary, we therefore conclude that the proposition is true. □

Notation 5.13. For any set X, for any binary operation $\odot$ on X such that X contains the neutral element e with respect to $\odot$, for any $n \in \mathbb{N}$ and for any sequence $s = (a_i \,|\, i \in \{1, \ldots, n^+\})$ in X, we will write for the recursively defined terms of the sequence $u = (u_i \,|\, i \in \{0, \ldots, n^+\})$ also

$$\bigodot_{j=1}^{i} a_j = u_i. \tag{5.386}$$

Thus, we may write in particular

$$\bigodot_{j=1}^{0} a_j = u_0 \;=\; e \tag{5.387}$$

$$\bigodot_{j=1}^{n^+} a_j = u_{n^+} \;=\; u_n \odot a_{n^+} \;=\; \left(\bigodot_{j=1}^{n} a_j \right) \odot a_{n^+}. \tag{5.388}$$

Notation 5.14. For any set X, for any binary operation $\odot$ on X such that X contains the neutral element e with respect to $\odot$, for any $m \in \mathbb{N}$ and for any sequence $s = (a_i \,|\, i \in \{1, \ldots, m\})$ in X, we write in case of $m = 0$

$$\bigodot_{i=1}^{0} a_i = e. \tag{5.389}$$

In case of $m \neq 0$, there exists because of (4.39) a particular $n \in \mathbb{N}$ with $m = n^+$, so that we may write $s = (a_i \,|\, i \in \{1, \ldots, n^+\})$ and then (because of Proposition 5.105)

$$\bigodot_{i=1}^{m} a_i = \bigodot_{j=1}^{n^+} a_j. \tag{5.390}$$

Furthermore, we will abbreviate "for any sequence $s = (a_i \,|\, i \in \{1, \ldots, m\})$ in X" by "for any $a_1, \ldots, a_m \in X$".

Exercise 5.44. Verify the following sentences for any set X and any binary operation $\odot$ on X such that X contains the neutral element e w.r.t. $\odot$.

a) For any $a_1 \in X$, it is true that

$$\bigodot_{i=1}^{1} a_i = a_1. \tag{5.391}$$

b) For any $a_1, a_2 \in X$, it is true that

$$\bigodot_{i=1}^{2} a_i = a_1 \odot a_2. \tag{5.392}$$

c) For any $a_1, a_2, a_3 \in X$, it is true that

$$\bigodot_{i=1}^{3} a_i = (a_1 \odot a_2) \odot a_3. \tag{5.393}$$

d) For any $a_1, \ldots, a_{n+1} \in X$, it is true that

$$\bigodot_{i=1}^{n+1} a_i = \left(\bigodot_{i=1}^{n} a_i \right) \odot a_{n+1}. \tag{5.394}$$

(Hint: Use (2.291), (2.292), (2.293), (5.388), (5.387), (5.384), (5.90), and (5.217).)

Proposition 5.106. *For any set X, for any binary operation $\odot$ on X such that X contains the neutral element e with respect to $\odot$, and for any $m \in \mathbb{N}$, it is true that there exists the unique function*

$$\bigodot_{i=1}^{m} : X^{\{1,\ldots,m\}} \to X, \quad (a_i \,|\, i \in \{1, \ldots, m\}) \mapsto \bigodot_{i=1}^{m} a_i. \tag{5.395}$$

Proof. We let X, $\odot$ and m be arbitrary such that $\odot$ is a binary operation on X and such that $m \in \mathbb{N}$ holds. We now apply Function definition by replacement and verify

$$\forall s \, (s = (a_i \,|\, i \in \{1, \ldots, m\}) \in X^{\{1,\ldots,m\}} \Rightarrow \exists! y \, (y = \bigodot_{i=1}^{m} a_i)). \tag{5.396}$$

To do this, we let $s = (a_i \,|\, i \in \{1, \ldots, m\})$ be an arbitrary sequence in X. In case of $m = 0$, we obtain in view of Notation 5.14 the element $\bigodot_{i=1}^{0} a_i = e$ (in X), so that the uniquely existential sentence follows to be true with (1.109). In case of $m \neq 0$, the notation (5.390) shows that $\bigodot_{i=1}^{m} a_i$ is also a uniquely specified element (of X), and therefore the uniquely existential sentence is again true. Since s is arbitrary, we may therefore conclude that the universal sentence (5.396) is true, which implies the existence of a unique function $\bigodot_{i=1}^{m}$ with domain $X^{\{1,\ldots,m\}}$ such that

$$\forall s \, (s = (a_i \,|\, i \in \{1, \ldots, m\}) \in X^{\{1,\ldots,m\}} \Rightarrow \bigodot_{i=1}^{m}(s) = \bigodot_{i=1}^{m} a_i)$$

holds. We already showed that any value of this function is in X, so that we established the mapping (5.395). As X, $\odot$ and m were initially arbitrary, we may therefore infer from this the truth of the proposition. □

Definition 5.21 (n-fold binary operations). For any set X, for any binary operation $\odot$ on X such that the neutral element e of X with respect to $\odot$ exists, and for any $n \in \mathbb{N}$, we call the function

$$\bigodot_{i=1}^{n} : X^{\{1,\ldots,n\}} \to X, \quad (a_i \,|\, i \in \{1, \ldots, n\}) \mapsto \bigodot_{i=1}^{n} a_i. \tag{5.397}$$

the *n-fold repeated binary operation* on X.

The concept of an n-fold binary operation may be applied to the join and meet of a lattice.

Definition 5.22 (n-fold repeated join & meet). For any set X, for any $n \in \mathbb{N}$, and

(1) for any lattice $(X, \sqcup, \sqcap, \leq)$ such that the neutral element of X with respect to $\sqcup$ exists, we call

$$\sqcup_{i=1}^{n} \tag{5.398}$$

the *n-fold repeated join* on X.

(2) for any lattice $(X, \sqcup, \sqcap, \leq)$ such that the neutral element of X with respect to $\sqcap$ exists, we call

$$\sqcap_{i=1}^{n} \tag{5.399}$$

the *n-fold repeated meet* on X.

Proposition 5.107. *For any lattice $(X, \sqcup, \sqcap, \leq)$ such that the neutral elements of X both with respect to $\sqcup$ and $\sqcap$ exist, for any positive natural number n and for any sequence $s = (a_i \,|\, i \in \{1, \ldots, n\})$ it is true that*

a) the n-fold join is identical with the supremum of the range of $(a_i \,|\, i \in \{1, \ldots, n\})$,

b) the n-fold meet is identical with the infimum of the range of $(a_i \,|\, i \in \{1, \ldots, n\})$.

Proof. We let X, $\sqcup$, $\sqcap$ and $\leq$ be arbitrary sets such that $\leq$ is a reflexive partial ordering of X, such that $\sqcup$ is the join and $\sqcap$ the meet on X (with respect to $\sqcup$ and $\sqcap$, respectively), and such that $(X, \sqcup, \sqcap, \leq)$ is a lattice containing the neutral elements with respect to $\sqcup$ and $\sqcap$. To prove a), we verify

$$\forall n\, (n \in \mathbb{N}_+ \Rightarrow \forall s\, (s \in X^{\{1,\ldots,n\}} \Rightarrow \sqcup_{i=1}^{n} s(i) = \sup \operatorname{ran}(s))) \tag{5.400}$$

where we use the usual sequence notation $s = (a_i \,|\, i \in \{1, \ldots, n\})$, so that we write $s(i) = a_i$ for any $i \in \{1, \ldots, n\}$. For this purpose, we carry out a proof by mathematical induction. Concerning the base case ($n = 1$), we let s be arbitrary in $X^{\{1,\ldots,1\}}$, so that the domain of s equals the singleton $\{1\} = \{1, \ldots, 1\}$ (according to Notation 4.4). Therefore, we see in light of Proposition 3.159 that the sequence/function s is the singleton $\{(1, s(1))\} = \{(1, a_1)\}$. Furthermore, s is a bijection with codomain/range $\{a_1\}$ due to Corollary 3.202, so that $\operatorname{ran}(s) = \{a_1\}$ holds. We then obtain the equations

$$\sqcup_{i=1}^{n} s(i) = \sqcup_{i=1}^{1} a_i = a_1 = \sup\{a_1\} = \sup \operatorname{ran}(s)$$

by applying substitution, Exercise 5.44, Corollary 3.106 and finally again substitution. Thus, the proof of the base case is complete; regarding the induction step, we let n be arbitrary in $\mathbb{N}_+$, assume the universal sentence

$$\forall s\,(s \in X^{\{1,\ldots,n\}} \Rightarrow \sqcup_{i=1}^{n} s(i) = \sup \operatorname{ran}(s))) \tag{5.401}$$

to be true, and we show that this induction assumption implies the truth of

$$\forall s\,(s \in X^{\{1,\ldots,n^+\}} \Rightarrow \sqcup_{i=1}^{n^+} s(i) = \sup \operatorname{ran}(s))). \tag{5.402}$$

To prove this universal sentence, we take an arbitrary set s and assume that $s \in X^{\{1,\ldots,n^+\}}$ holds, so that we may write s in sequence notation as $s = (a_i \,|\, i \in \{1,\ldots,n^+\})$. To show that $\sqcup_{i=1}^{n^+} s(i)$ is the supremum of $\operatorname{ran}(s)$, we utilize the Characterization of the supremum and establish first $\sqcup_{i=1}^{n^+} s(i)$ as an upper bound for $\operatorname{ran}(s)$. Thus, our task consists in the verification of the universal sentence

$$\forall y\,(y \in \operatorname{ran}(s) \Rightarrow y \leq \sqcup_{i=1}^{n^+} s(i)). \tag{5.403}$$

We let y be arbitrary such that $y \in \operatorname{ran}(s)$ holds. By definition of a range, there exists then a constant, say $\bar{k}$, satisfying $(\bar{k}, y) \in s$, which we may write also as $y = s(\bar{k}) = a_{\bar{k}}$. Moreover, we obtain $\bar{k} \in \{1,\ldots,n^+\}$ $[= \operatorname{dom}(s)]$ with the definition of a domain, and this finding gives $\bar{k} \in \{1,\ldots,n\} \cup \{n^+\}$ with (4.241). Therefore, the disjunction of $\bar{k} \in \{1,\ldots,n\}$ and $\bar{k} \in \{n^+\}$ is true by definition of the intersection of two sets. Based on this disjunction, we now prove the inequality $y \leq \sqcup_{i=1}^{n^+} s(i)$ by cases. Let us rewrite this inequality as

$$y \leq \sup\{\sqcup_{i=1}^{n} a_i, a_{n^+}\} \quad [= (\sqcup_{i=1}^{n} a_i) \sqcup a_{n^+}],$$

according to the definition of the join (and by using Exercise 5.44d)). Here, the supremum $\sup\{\sqcup_{i=1}^{n} a_i, a_{n^+}\}$ is evidently an upper bound for the pair $\{\sqcup_{i=1}^{n} a_i, a_{n^+}\}$, so that we may apply the Characterization of upper bounds for pairs to obtain the two inequalities

$$\sqcup_{i=1}^{n} a_i \leq \sup\{\sqcup_{i=1}^{n} a_i, a_{n^+}\}, \tag{5.404}$$

$$a_{n^+} \leq \sup\{\sqcup_{i=1}^{n} a_i, a_{n^+}\}. \tag{5.405}$$

Let us return to the proof by cases and let us consider first the case of $\bar{k} \in \{1,\ldots,n\}$. Here, we may view $\{1,\ldots,n\}$ as the domain of the sequence $(a_i \,|\, i \in \{1,\ldots,n\})$ in X, so that $y = a_{\bar{k}}$ is evidently a term and thus an element of the range of that sequence. Consequently, the n-ary join $\sqcup_{i=1}^{n} a_i$ follows to be the supremum and thus to be an upper bound for the range of $(a_i \,|\, i \in \{1,\ldots,n\})$ because of the induction assumption (5.401), so that $y \leq \sqcup_{i=1}^{n} a_i$ holds. Together with (5.404), this inequality implies $y \leq \sup\{\sqcup_{i=1}^{n} a_i, a_{n^+}\}$ since the partial ordering $\leq$ is transitive. In the other case of $\bar{k} \in \{n^+\}$, we obtain first $\bar{k} = n^+$ with (2.169) and therefore $y = a_{\bar{k}} = a_{n^+}$. Consequently, the inequality (5.405) yields via substitution again $y \leq \sup\{\sqcup_{i=1}^{n} a_i, a_{n^+}\}$, completing the proof by cases of $y \leq \sqcup_{i=1}^{n^+} s(i)$. As y was arbitrary, we may therefore conclude that the universal sentence (5.403) is true, which means that we indeed established $\sqcup_{i=1}^{n^+} s(i)$ as an upper bound for $\mathrm{ran}(s)$.

To establish the second part of the Characterization of the supremum, we take an arbitrary S' such that S' is an upper bound for $\mathrm{ran}(s)$, i.e. such that

$$\forall y\,(y \in \mathrm{ran}(s) \Rightarrow y \leq S'), \tag{5.406}$$

and we show that $\sqcup_{i=1}^{n^+} s(i) \leq S'$ follows to be true. Using again the definition of a join and Exercise 5.44d)), we may write the preceding inequality as $\sup\{\sqcup_{i=1}^{n} a_i, a_{n^+}\} \leq S'$, which is also equivalent to the conjunction of $\sqcup_{i=1}^{n} a_i \leq S'$ and $a_{n^+} \leq S'$ in view of Exercise 3.45. To demonstrate the truth of this conjunction, we consider again the sequence $(a_i \,|\, i \in \{1, \ldots, n\})$ (in X). Since $n < n^+$ holds with (4.153), which inequality evidently implies $n \leq n^+$, we see in light of Corollary 4.87 that the range of the sequence $(a_i \,|\, i \in \{1, \ldots, n\})$ is included in the range of the sequence $(a_i \,|\, i \in \{1, \ldots, n^+\})$. Therefore, since the range of the latter sequence is bounded from above by S', we may apply Proposition 3.94 to infer that S' is also an upper bound for the range of the sequence $(a_i \,|\, i \in \{1, \ldots, n\})$. According to the induction assumption, $\sqcup_{i=1}^{n} a_i$ is the least upper bound, so that $\sqcup_{i=1}^{n} a_i \leq S'$ is true, as desired. Furthermore, a_{n^+} is clearly an element of the range of the sequence $s = (a_i \,|\, i \in \{1, \ldots, n^+\})$, so that (5.406) yields also the desired second part $a_{n^+} \leq S'$. We thus proved $\sqcup_{i=1}^{n^+} s(i) \leq S'$ and established therefore $\sqcup_{i=1}^{n^+} s(i)$ as the supremum of $\mathrm{ran}(s)$. Since s was arbitrary, we may infer from this the truth of the universal sentence (5.402); as n is also arbitrary, we may further conclude that the induction step holds, alongside the base case. Finally, because the sets X, $\sqcup$, $\sqcap$ and $\leq$ were initially arbitrary, Part a) of the proposition follows to be true. □

Exercise 5.45. Establish Part b) of Proposition 5.107.

Corollary 5.108. *For any lattice $(X, \sqcup, \sqcap, \leq)$ such that the neutral elements of X both with respect to the join and meet exist, it is true that the supremum and the infimum of any finite subset of X exist.*

Proof. We take arbitrary sets X, $\sqcup$, $\sqcap$, $\leq$ and A be arbitrary sets, assume that $(X, \sqcup, \sqcap, \leq)$ is a lattice such that X contains the neutral elements with respect to $\sqcup$ and $\sqcap$, assume that $A \subseteq X$ holds, and assume moreover that A

is finite. Consequently, there exist a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}\}$ to A, say $\bar{c}$. Furthermore, the bijection $\bar{c}$ is in particular a surjection, so that $A = \operatorname{ran}(\bar{c})$ holds. Here, we see in view of the initial assumption $A \subseteq X$ that X is a codomain of $\bar{c}$. Thus, we may write the function $\bar{c}$ as the sequence $(a_i \,|\, i \in \{1, \ldots, \bar{n}\})$ in X, so that the preceding proposition gives for the supremum and the infimum of the range of that sequence, that is, for the supremum and the infimum of the set A

$$\begin{aligned} \sup A = \sup \operatorname{ran}(\bar{c}) &= \sqcup_{i=1}^{\bar{n}} a_i, \\ \inf A = \inf \operatorname{ran}(\bar{c}) \ &= \sqcap_{i=1}^{\bar{n}} a_i. \end{aligned}$$

Thus, the supremum and the infimum of A exist, and since X, $\sqcup$, $\sqcap$, $\leq$ and A were arbitrary, we may therefore conclude that the proposition holds, as claimed. $\square$

Definition 5.23 ($\boldsymbol{n}$-fold addition & multiplication, $\boldsymbol{n}$-fold sum & product). For any set X, any $n \in \mathbb{N}$, and

(1) for any addition $+$ on X such that the zero element 0_X exists, we call

$$\sum_{i=1}^{n} \tag{5.407}$$

the *n-fold addition* on X, and any of its values an *n-fold sum*.

(2) for any multiplication $\cdot$ on X such that the unity element 1_X exists, we call

$$\prod_{i=1}^{n} \tag{5.408}$$

the *n-fold multiplication* on X, and any of its values an *n-fold product*.

We now apply these notations to the findings of Exercise 5.44.

Corollary 5.109. *The following sentences hold for any set X, for any addition $+$ on X such that X contains the zero element 0_X, and for any multiplication on X such that X contains the unity element 1_X.*

a) It is true that

$$\sum_{i=1}^{0} a_i = 0_X, \tag{5.409}$$

$$\prod_{i=1}^{0} a_i = 1_X. \tag{5.410}$$

b) For any $a_1 \in X$*, it is true that*

$$\sum_{i=1}^{1} a_i = a_1, \tag{5.411}$$

$$\prod_{i=1}^{1} a_i = a_1. \tag{5.412}$$

c) For any $a_1, a_2 \in X$*, it is true that*

$$\sum_{i=1}^{2} a_i = a_1 + a_2, \tag{5.413}$$

$$\prod_{i=1}^{2} a_i = a_1 \cdot a_2. \tag{5.414}$$

d) For any $a_1, a_2, a_3 \in X$*, it is true that*

$$\sum_{i=1}^{3} a_i = (a_1 + a_2) + a_3, \tag{5.415}$$

$$\prod_{i=1}^{3} a_i = (a_1 \cdot a_2) \cdot a_3. \tag{5.416}$$

e) For any $n \in \mathbb{N}$ *and any* $a_1, \ldots, a_{n+1} \in X$*, it is true that*

$$\sum_{i=1}^{n+1} a_i = \left(\sum_{i=1}^{n} a_i\right) + a_{n+1}, \tag{5.417}$$

$$\prod_{i=1}^{n+1} a_i = \left(\prod_{i=1}^{n} a_i\right) \cdot a_{n+1}. \tag{5.418}$$

As a first important application of n-fold addition, we establish the following kind of function associated with a given matrix and vector.

Theorem 5.110 (Specification of the standard matrix-vector product). *The following sentences are true for any positive natural numbers* m *and* n*, any set* Y*, any addition* $+_Y$ *on* Y *such that the zero element* 0_Y *exists, any multiplication* $\cdot_Y$ *on* Y*, any* m*-by-*n *matrix* $\boldsymbol{A}$ *with values in* Y*, and any* n*-vector* $\boldsymbol{x}$ *in* Y*:*

a) For any $i \in \{1, \ldots, m\}$ the n-tuple

$$h_i = (\boldsymbol{A}((i,k)) \cdot_Y x_k \,|\, k \in \{1, \ldots, n\}) \tag{5.419}$$

exists uniquely in Y.

b) There exists a unique m-vector $\boldsymbol{y}$ in Y whose values are defined by

$$\forall i \, (i \in \{1, \ldots, m\} \Rightarrow y_i = \sum_{k=1}^{n} (\boldsymbol{A}((i,k)) \cdot_Y x_k)). \tag{5.420}$$

Exercise 5.46. Establish Theorem 5.110.
(Hint: Apply Function definition by replacement twice.)

Definition 5.24 (Standard matrix-vector product). For any positive natural numbers m and n, any set Y, any addition $+_Y$ on Y such that the zero element 0_Y exists, any multiplication $\cdot_Y$ on Y, any m-by-n matrix $\boldsymbol{A}$ with values in Y, and any n-vector $\boldsymbol{x}$ in Y, we call the m-vector $\boldsymbol{y}$ defined by (5.420) the *standard matrix-vector product* of $\boldsymbol{A}$ and $\boldsymbol{x}$, symbolically

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A} \cdot \boldsymbol{x}. \tag{5.421}$$

Theorem 5.111 (Matrix-vector notation for n-fold sums). *The following equations hold for any positive natural number n, any set Y, any addition $+_Y$ with zero element 0_Y, any multiplication $\cdot_Y$ with identity element 1_Y, and any n-vector $\boldsymbol{x}$ in Y.*

$$\begin{bmatrix} 1_Y & \cdots & 1_Y \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left(\sum_{k=1}^{n} x_k \right). \tag{5.422}$$

Proof. We take an arbitrary positive natural number n, an arbitrary set Y as well as an arbitrary addition $+_Y$ on Y such that the zero element 0_Y in Y exists, an arbitrary multiplication $\cdot_Y$ on Y, and an arbitrary n-vector $\boldsymbol{x}$ in Y. Defining the 1-by-n matrix (i.e., the n-dimensional row vector) $\boldsymbol{a} = \begin{bmatrix} 1_Y & \cdots & 1_Y \end{bmatrix} = g_{1_Y}$ to be the constant function on $\{1\} \times \{1, \ldots, n\}$ with value 1_Y, we obtain for the product of $\boldsymbol{a}$ and $\boldsymbol{x}$ the 1-vector $\boldsymbol{y}$ with the single value

$$y_1 = \sum_{k=1}^{n} (\boldsymbol{a}((1,k)) \cdot_Y x_k) = \sum_{k=1}^{n} (1_Y \cdot_Y x_k) = \sum_{k=1}^{n} x_k$$

using the definition of a unity element. Since n, Y, $+_Y$, $\cdot_Y$ and $\boldsymbol{x}$ were initially all arbitrary, we may therefore conclude that the stated universal sentence holds. $\square$

Recalling the facts (1) that any addition on a set Y with zero element gives rise to a corresponding pointwise addition of functions in a set Y^X also with zero element, we may establish the n-fold addition on such sets of functions, and moreover establish the following compatibility property.

Proposition 5.112. *The n-fold addition of functions in a set Y^X is compatible with the n-fold addition of the corresponding function values, that is,*

$$\forall x \,(x \in X \Rightarrow \left(\sum_{i=1}^{n} f_i\right)(x) = \sum_{i=1}^{n} f_i(x)) \tag{5.423}$$

holds for any $n \in \mathbb{N}$ and any n-tuple $f = (f_i \,|\, i \in \{1,\ldots,n\})$ of functions in Y^X.

Proof. We let X, Y and $+_Y$ be arbitrary sets such that $+_Y$ is an addition on Y and such that the zero element (0_Y) of Y exists. We carry out a proof by mathematical induction. Concerning the base case $(n = 0)$, we let f be an arbitrary 0-tuple $(f_i\, i \in \{1,\ldots,0\})$ and x an arbitrary element of X. We obtain then the equations

$$\left(\sum_{i=1}^{0} f_i\right)(x) = g_{0_Y}(x) = 0_Y = \sum_{i=1}^{0} f_i(x),$$

using (5.409) in connection with the fact that the zero element 0_Y defines the zero element of Y^X with respect to $+_{Y^X}$, which is the constant function g_{0_Y} on X with value 0_Y, according to Proposition 5.19). Since x was arbitrary, we may therefore conclude that the universal sentence (5.423) is true in the base case. To establish the induction step, we let n be an arbitrary natural number, make the induction assumption that (5.423) holds for any any n-tuple $f = (f_i \,|\, i \in \{1,\ldots,n\})$ of functions in Y^X, and we show that

$$\forall x \,(x \in X \Rightarrow \left(\sum_{i=1}^{n+1} f_i\right)(x) = \sum_{i=1}^{n+1} f_i(x))$$

follows to be true for any $(n+1)$-tuple $f = (f_i\, i \in \{1,\ldots,n+1\})$ of functions in Y^X. To do this, we let f be an arbitrary $(n+1)$-tuple of that kind, and x an arbitrary element of X. We note that the f gives rise to the $(n+1)$-tuple of function values $f^{(x)} = (f_i(x)\, i \in \{1,\ldots,n+1\})$ according to Proposition

3.231. We then obtain the equations

$$\left(\sum_{i=1}^{n+1} f_i\right)(x) = \left(\sum_{i=1}^{n} f_i + f_{n+1}\right)(x) = \left(\sum_{i=1}^{n} f_i\right)(x) + f_{n+1}(x)$$
$$= \sum_{i=1}^{n} f_i(x) + f_{n+1}(x) = \sum_{i=1}^{n+1} f_i(x)$$

with (5.417), the definition of the pointwise addition for functions, the induction assumption, and finally again (5.417). As x, f and n were arbitrary, we may therefore conclude that the induction step holds, besides the base case. Then, since X, Y and $+_Y$ were also arbitrary, the proposed universal sentence follows to be true. □

5.6.1. Basic laws for n-fold binary operations

Theorem 5.113 ($\boldsymbol{n}$-fold repeated addition & multiplication of the sum & product of two sequences). *It is true*

a) *for any commutative semigroup $(X, +)$ containing the zero element 0_X, for any $n \in \mathbb{N}$, any $a_1, \ldots, a_n \in X$ and any $b_1, \ldots, b_n \in X$ that*

$$\sum_{i=1}^{n}(a_i + b_i) = \left(\sum_{i=1}^{n} a_i\right) + \left(\sum_{i=1}^{n} b_i\right). \tag{5.424}$$

b) *for any commutative semigroup $(X, \cdot)$ containing the unity element 1_X, for any $n \in \mathbb{N}$, any $a_1, \ldots, a_n \in X$ and any $b_1, \ldots, b_n \in X$ that*

$$\prod_{i=1}^{n}(a_i \cdot b_i) = \left(\prod_{i=1}^{n} a_i\right) \cdot \left(\prod_{i=1}^{n} b_i\right). \tag{5.425}$$

Proof. Concerning a), we let X and $+$ be arbitrary sets and assume that $(X, +)$ is a commutative semigroup which contains the neutral element 0_X with respect to the addition $+$ on X. To establish the equation (5.424) for any $n \in \mathbb{N}$, any $a_1, \ldots, a_n \in X$ and any $b_1, \ldots, b_n \in X$, we prove

$$\begin{aligned} &\forall n\,(n \in \mathbb{N} \Rightarrow \forall s, t\,(s, t \in X^{\{1,\ldots,n\}} \\ &\quad \Rightarrow \sum_{i=1}^{n}(s(i) + t(i)) = \left[\sum_{i=1}^{n} s(i)\right] + \left[\sum_{i=1}^{n} t(i)\right])) \end{aligned} \tag{5.426}$$

by means of mathematical induction. In the base case ($n = 0$), we let s and t be arbitrary sets and assume $s, t \in X^{\{1,\ldots,0\}}$, so that we may write s and t

in sequence notation as $s = (a_i \,|\, i \in \{1,\ldots,0\})$ and $t = (b_i \,|\, i \in \{1,\ldots,0\})$. Furthermore, the preceding assumption allows us to form the new sequence $h = s +_{X^{\{1,\ldots,0\}}} t$ because of Theorem 5.6, which sequence satisfies $h(i) = s(i) + t(i)$ for any $i \in \{1,\ldots,0\}$. Thus, the 0-fold repeated additions are specified for the sequences h, s and t, and we obtain

$$\begin{aligned}\sum_{i=1}^{n}(s(i)+t(i)) &= \sum_{i=1}^{0}[s(i)+t(i)] = 0_X = 0_X + 0_X \\ &= \left[\sum_{i=1}^{0} s(i)\right] + \left[\sum_{i=1}^{0} t(i)\right] = \left[\sum_{i=1}^{n} s(i)\right] + \left[\sum_{i=1}^{n} t(i)\right]\end{aligned}$$

by applying substitution based on the assumed equation $n = 0$, then Corollary 5.109a), the definition of a neutral element, again Corollary 5.109a), and finally again substitution based on $n = 0$. Consequently, the base case holds. To establish the induction step, we take an arbitrary $n \in \mathbb{N}$, make the induction assumption

$$\forall s,t\,(s,t \in X^{\{1,\ldots,n\}} \Rightarrow \sum_{i=1}^{n}(s(i)+t(i)) = \left[\sum_{i=1}^{n} s(i)\right] + \left[\sum_{i=1}^{n} t(i)\right]), \tag{5.427}$$

and show that the universal sentence

$$\forall s,t\,(s,t \in X^{\{1,\ldots,n^+\}} \Rightarrow \sum_{i=1}^{n^+}(s(i)+t(i)) = \left[\sum_{i=1}^{n^+} s(i)\right] + \left[\sum_{i=1}^{n^+} t(i)\right]) \tag{5.428}$$

follows to be true. For this purpose, we take arbitrary sequences s, t in $X^{\{1,\ldots,n^+\}}$, which we may then write also as $s = (a_i \,|\, i \in \{1,\ldots,n^+\})$ and $t = (b_i \,|\, i \in \{1,\ldots,n^+\})$. Therefore, the sequence $h = s +_{X^{\{1,\ldots,n^+\}}} t$ which satisfies $h(i) = s(i) + t(i)$ for any $i \in \{1,\ldots,n^+\}$ is uniquely determined according to Theorem 5.6, so that we obtain

$$\sum_{i=1}^{n^+}(s(i)+t(i)) = \left[\sum_{i=1}^{n}(s(i)+t(i))\right] + (s(n^+)+t(n^+)) \tag{5.429}$$

with Corollary 5.109e). Here, the n-fold addition may be viewed as being applied to the sequences $(a_i + b_i \,|\, i \in \{1,\ldots,n\})$, $(a_i \,|\, i \in \{1,\ldots,n\})$ and $(b_i \,|\, i \in \{1,\ldots,n\})$, so that we may apply substitution in connection with the induction assumption (5.427) to write equation (5.429) as

$$\sum_{i=1}^{n^+}(s(i)+t(i)) = \left(\left[\sum_{i=1}^{n} s(i)\right] + \left[\sum_{i=1}^{n} t(i)\right]\right) + (s(n^+)+t(n^+)).$$

As we assumed $(X, +)$ to be a commutative semigroup, which means that the addition on X is both commutative and associative, we may rewrite the preceding equation also as

$$\begin{aligned}
\sum_{i=1}^{n^+}(s(i)+t(i)) &= \left[\sum_{i=1}^{n} s(i)\right] + \left(\left[\sum_{i=1}^{n} t(i)\right] + (s(n^+)+t(n^+))\right) \\
&= \left[\sum_{i=1}^{n} s(i)\right] + \left(\left(\left[\sum_{i=1}^{n} t(i)\right] + s(n^+)\right) + t(n^+)\right) \\
&= \left[\sum_{i=1}^{n} s(i)\right] + \left(\left(s(n^+) + \left[\sum_{i=1}^{n} t(i)\right]\right) + t(n^+)\right) \\
&= \left[\sum_{i=1}^{n} s(i)\right] + \left(s(n^+) + \left(\left[\sum_{i=1}^{n} t(i)\right] + t(n^+)\right)\right) \\
&= \left(\left[\sum_{i=1}^{n} s(i)\right] + s(n^+)\right) + \left(\left[\sum_{i=1}^{n} t(i)\right] + t(n^+)\right)
\end{aligned}$$

Applying now Corollary 5.109e) to the two summands on the right-hand side of the last equation, we evidently obtain the equation stated in (5.428). Since s and t are arbitrary, we may therefore conclude that the universal sentence (5.428) is true; consequently, because n was also arbitrary, we may further conclude that the induction step also holds, so that the proof of the universal sentence (5.426) via mathematical induction is complete. Finally, as the sets X and $+$ were initially arbitrary as well, we may infer from this finding the truth of Part a) of the stated theorem. □

Exercise 5.47. Establish Part b) of Theorem 5.113.

Our next task is to prepare ourselves for generalizing the associativity of binary operations in semigroups to n-fold binary operations.

Notation 5.15. For any $m, n \in \mathbb{N}$ we write for the intermediate segment of $\mathbb{N}_+$ from the successor of m to n also

$$\{m+1, \ldots, n\} = \{m^+, \ldots, n\}. \tag{5.430}$$

Corollary 5.114. *For any set X, any $m, n \in \mathbb{N}$ and any sequence $s = (a_i \,|\, i \in \{1, \ldots, m+n\})$ in X, it is true that the restriction of s to the intermediate segment of $\mathbb{N}_+$ from $m+1$ to $m+n$ is a function from that intermediate segment to X, i.e.*

$$s \upharpoonright \{m+1, \ldots, m+n\} : \{m+1, \ldots, m+n\} \to X. \tag{5.431}$$

Proof. We let X, m, n and s be arbitrary sets, we assume $m, n \in \mathbb{N}$, and we assume that s is a sequence $(a_i \,|\, i \in \{1, \ldots, m+n\})$ in X. Here, $m, n \in \mathbb{N}$ yields $m + n \in \mathbb{N}$ because $+$ is a binary operation on $\mathbb{N}$. Therefore, the intermediate segment $\{m+1, \ldots, m+n\}$ is included in the initial segment $\{1, \ldots, m+n\}$ according to (4.298). As the sequence s is an element of $X^{\{1,\ldots,m+n\}}$, we may now apply Proposition 3.164 to see that the restriction $s \upharpoonright \{m+1, \ldots, m+n\}$ is an element of $X^{\{m+1,\ldots,m+n\}}$. Consequently, this restriction is a function as stated in (5.431); since X, m, n and s are arbitrary sets, we may therefore conclude that the proposed universal sentence is true. □

Notation 5.16. For any set X, for any $m, n \in \mathbb{N}$ and for any sequence $s = (a_i \,|\, i \in \{1, \ldots, m+n\})$ in X we also write for the restriction of s to the intermediate segment of $\mathbb{N}_+$ from $m+1$ to $m+n$

$$(a_i \,|\, i \in \{m+1, \ldots, m+n\}) = s \upharpoonright \{m+1, \ldots, m+n\}. \tag{5.432}$$

We now come to the important idea of 'translating' the domain of a sequence.

Proposition 5.115. *The following sentences are true for any $m, n \in \mathbb{N}$.*

a) There exists a unique function t with domain $\{1, \ldots, n\}$ such that

$$\forall j\,(j \in \{1, \ldots, n\} \;\Rightarrow\; t(j) = m +_{\mathbb{N}} j) \tag{5.433}$$

b) This function t is a bijection from $\{1, \ldots, n\}$ to $\{m+1, \ldots, m+n\}$, i.e.

$$t : \{1, \ldots, n\} \rightleftarrows \{m+1, \ldots, m+n\}. \tag{5.434}$$

c) Furthermore, the inverse function of t is a bijection

$$t^{-1} : \{m+1, \ldots, m+n\} \rightleftarrows \{1, \ldots, n\} \tag{5.435}$$

satisfying

$$\forall k\,(k \in \{m+1, \ldots, m+n\} \;\Rightarrow\; t^{-1}(k) = k -_{\mathbb{N}} m). \tag{5.436}$$

Proof. We let $m, n \in \mathbb{N}$ be arbitrary and apply within the proof of a) Function definition by replacement, by verifying

$$\forall j\,(j\{1, \ldots, n\} \Rightarrow \exists! y\,(y = m +_{\mathbb{N}} j)). \tag{5.437}$$

To do this, we let j be arbitrary and assume $j \in \{1, \ldots, n\}$ to be true. Since the addition on $\mathbb{N}$ is a binary operation on $\mathbb{N}$, we have that the sum

$m +_{\mathbb{N}} j$ is a natural number, and therefore the uniquely existential sentence follows to be true with (1.109). Because j is arbitrary, we may now conclude that the universal sentence (5.396) is true, which implies the existence of a unique function t with domain $\{1, \ldots, n\}$ such that (5.433) holds. To show that $\{m+1, \ldots, m+n\}$ is a codomain of t, we verify that the range of t is included in that intermediate segment, i.e.

$$\forall y\, (y \in \mathrm{ran}(t) \Rightarrow y \in \{m+1, \ldots, m+n\}). \tag{5.438}$$

For this purpose, we let y be arbitrary and assume $y \in \mathrm{ran}(t)$ to be true. Then, there exists by definition of a range a constant, say $\bar{k}$, such that $(\bar{k}, y) \in t$ holds, which we may write in function notation also as $y = t(\bar{k})$. Consequently, we obtain with the definition of a domain $\bar{k} \in \{1, \ldots, n\}$ [$=$ $\mathrm{dom}(t)$], which yields with (5.433) [$y =$] $t(\bar{k}) = m +_{\mathbb{N}} \bar{k}$. Furthermore, we see that there exists an element in $\{1, \ldots, n\}$, so that this initial segment of $\mathbb{N}_+$ is evidently nonempty. Thus, $\bar{k}, n \in \mathbb{N}_+$ holds by definition of an initial segment of $\mathbb{N}_+$, which yields the inequality $1 \leq_{\mathbb{N}} \bar{k}$ with (4.278). Moreover, the previously established $\bar{k} \in \{1, \ldots, n\}$ yields $\bar{k} \leq_{\mathbb{N}} n$ with (4.275). We may now apply the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$ to the inequalities $1 \leq_{\mathbb{N}} \bar{k}$ and $\bar{k} \leq_{\mathbb{N}} n$, with the consequence that $1 + m \leq_{\mathbb{N}} \bar{k} + m$ and $\bar{k} + m \leq_{\mathbb{N}} n + m$ hold. In view of the Commutative Law for the addition on $\mathbb{N}$, we may write these inequalities also as $m + 1 \leq_{\mathbb{N}} m + \bar{k} \leq_{\mathbb{N}} m + n$. Thus, [$m + \bar{k} =$] $y \in \{m+1, m+n\}$ holds according to Proposition 4.62, so that the proof of the implication in (5.438) is complete. Since y is arbitrary, we may therefore conclude that (5.438) holds, which universal sentence then implies the truth of the desired inclusion $\mathrm{ran}(t) \subseteq \{m+1, \ldots, m+n\}$ with the definition of a subset. We thus showed that t is a function from $\{1, \ldots, n\}$ to $\{m+1, \ldots, m+n\}$, and because t satisfies (5.433), we now see that t is indeed the mapping (5.434).

Regarding b), we show first that t is an injection by applying the Injection Criterion, i.e. by verifying

$$\forall j, j'\, ([j, j'\{1, \ldots, n\} \wedge j \neq j'] \Rightarrow t(j) \neq t(j')). \tag{5.439}$$

We take arbitrary j and j' and assume $j, j'\{1, \ldots, n\}$ as well as $j \neq j'$ to be true. In view of the connexity of the standard linear ordering $<_{\mathbb{N}}$, the preceding inequality implies the truth of the disjunction $j <_{\mathbb{N}} j' \vee j' <_{\mathbb{N}} j$, which we use to prove $m +_{\mathbb{N}} j \neq m +_{\mathbb{N}} j'$ by cases. The first case $j <_{\mathbb{N}} j'$ implies $j +_{\mathbb{N}} m <_{\mathbb{N}} j' +_{\mathbb{N}} m$ with the Monotony Law for $+_{\mathbb{N}}$ and $<_{\mathbb{N}}$, so that we obtain $m +_{\mathbb{N}} j <_{\mathbb{N}} m +_{\mathbb{N}} j'$ by means of the commutativity of $+_{\mathbb{N}}$. In light of the Characterization of comparability with respect to the linear ordering $<_{\mathbb{N}}$, we now see that the truth of the preceding inequality implies that $m +_{\mathbb{N}} j = m +_{\mathbb{N}} j'$ is false, so that the desired $m +_{\mathbb{N}} j \neq m +_{\mathbb{N}} j'$

is true. Using the same arguments as in the first case, we obtain in the second case $j' <_{\mathbb{N}} j$ first $m +_{\mathbb{N}} j' <_{\mathbb{N}} m +_{\mathbb{N}} j$ and therefore $m +_{\mathbb{N}} j' \neq m +_{\mathbb{N}} j$, which we may write also in the desired form $m +_{\mathbb{N}} j \neq m +_{\mathbb{N}} j'$. Having thus completed the proof by cases, we observe now that the assumed $j, j'\{1, \ldots, n\}$ implies $t(j) = m +_{\mathbb{N}} j$ as well as $t(j') = m +_{\mathbb{N}} j'$ with (5.433), so that the preceding inequality becomes after substitutions $t(j) \neq t(j')$. As j and j' are arbitrary, we may therefore conclude that (5.439) is true, which means that t is indeed an injection.

Next, we show that $\{m + 1, \ldots, m + n\}$ is a codomain of the injection t, i.e. that the inclusion $\mathrm{ran}(t) \subseteq \{m + 1, \ldots, m + n\}$ holds. To do this, we utilize the definition of a subset and establish the equivalent universal sentence

$$\forall k\, (k \in \mathrm{ran}(t) \Rightarrow k \in \{m + 1, \ldots, m + n\}), \tag{5.440}$$

letting k be arbitrary and assuming $k \in \mathrm{ran}(t)$ to be true. By definition of a range, there exists then a particular constant $\bar{J}$ with $(\bar{J}, k) \in t$. On the one hand, this implies with the definition of a domain $\bar{J} \in \{1, \ldots, n\}$ $[= \mathrm{dom}(t)]$; observing that $\{1, \ldots, n\} = [0^+, n]$ holds according to (4.295) and (2.291), we thus have $1 \leq_{\mathbb{N}} \bar{J} \leq_{\mathbb{N}} n$, so that an application of the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$ in connection with the commutativity of $+_{\mathbb{N}}$ yields

$$m +_{\mathbb{N}} 1 \;\leq_{\mathbb{N}}\; m +_{\mathbb{N}} \bar{J} \;\leq_{\mathbb{N}}\; m +_{\mathbb{N}} n.$$

Since we may write $(\bar{J}, k) \in t$ in function notation as $k = t(\bar{J})$ $[= m +_{\mathbb{N}} \bar{J}]$, the preceding inequalities imply via substitution

$$m +_{\mathbb{N}} 1 \;\leq_{\mathbb{N}}\; t(\bar{J}) \;\leq_{\mathbb{N}}\; m +_{\mathbb{N}} n$$

and therefore $t(\bar{J}) \in \{m +_{\mathbb{N}} 1, \ldots, m +_{\mathbb{N}} n\}$ with (4.290).. This finding proves the implication in (5.440), and as k was arbitrary, we may infer from this the truth of the universal sentence (5.440). Consequently, the inclusion $\mathrm{ran}(t) \subseteq \{m + 1, \ldots, m + n\}$ holds indeed, which shows that t is an injection with codomain $\{m + 1, \ldots, m + n\}$.

To establish the injection $t : \{1, \ldots, n\} \hookrightarrow \{m + 1, \ldots, m + n\}$ as a bijection, we may now apply the Surjection Criterion and verify accordingly

$$\forall k\, (k \in \{m + 1, \ldots, m + n\} \Rightarrow \exists j\, (t(j) = k)). \tag{5.441}$$

For this purpose, we let k be arbitrary and assume $k \in \{m+1, \ldots, m+n\}$, so that $k \in \mathbb{N}$ and

$$m +_{\mathbb{N}} 1 \;\leq_{\mathbb{N}}\; k \;\leq_{\mathbb{N}}\; m +_{\mathbb{N}} n \tag{5.442}$$

are true because of (4.290). Since $m <_{\mathbb{N}} m^+$ is true according to (4.153), we clearly have that $m \leq_{\mathbb{N}} m +_{\mathbb{N}} 1$ $[\leq_{\mathbb{N}} k]$ holds; the transitivity of the standard total ordering $\leq_{\mathbb{N}}$ gives then also $m \leq_{\mathbb{N}} k$. We may therefore apply the Monotony Law for $-_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$ to infer from (5.442) the truth of the inequalities

$$(m +_{\mathbb{N}} 1) -_{\mathbb{N}} m \ \leq_{\mathbb{N}} \ k -_{\mathbb{N}} m \ \leq_{\mathbb{N}} \ (m +_{\mathbb{N}} n) -_{\mathbb{N}} m.$$

Then, twofold application of (5.344) in connection with the commutativity of $+_{\mathbb{N}}$ yields the inequalities $1 \leq_{\mathbb{N}} k -_{\mathbb{N}} m \leq_{\mathbb{N}} n$, so that $k -_{\mathbb{N}} m \in \{1, \ldots, n\}$ [$= \mathrm{dom}(t)$] is evidently true. The corresponding function value of t follows to be

$$t(k -_{\mathbb{N}} m) = m +_{\mathbb{N}} (k -_{\mathbb{N}} m) = k, \tag{5.443}$$

which equations prove the desired existential sentence $\exists j\,(t(j) = k)$. As k was arbitrary, we may therefore conclude that the universal sentence (5.441) holds, completing the proof of the surjectivity of the injection $t : \{1, \ldots, n\} \hookrightarrow \{m+1, \ldots, m+n\}$. Thus, t is a bijection from $\{1, \ldots, n\}$ to $\{m+1, \ldots, m+n\}$, by definition.

Regarding c), we obtain the bijective inverse (5.435) of t immediately with the Bijectivity of inverse functions. To demonstrate that t^{-1} satisfies (5.436), we let $k \in \{m+1, \ldots, m+n\}$ be arbitrary, with the consequence that $k = t(k -_{\mathbb{N}} m)$ holds, according to (5.443). This equation in turn implies $k -_{\mathbb{N}} m = t^{-1}(k)$ with the Characterization of the function values of an inverse function, as desired. Because k was arbitrary, we may now further conclude that t^{-1} satisfies indeed the universal sentence (5.436). As m and n were initially also arbitrary, we may therefore infer from the truth of a) – c) the truth of the proposition. □

Corollary 5.116. *For any set X, for any $m, n \in \mathbb{N}$ and for any sequence $s = (a_i \,|\, i \in \{1, \ldots, m+n\})$ in X, it is true that the composition s' of the restricted sequence $(a_i \,|\, i \in \{m+1, \ldots, m+n\})$ and the function t in (5.434) constitutes the function*

$$[s' =]\ s \restriction \{m+1, \ldots, m+n\} \circ t : \{1, \ldots, n\} \to X, \quad i \mapsto a_{m+i}. \tag{5.444}$$

Proof. Letting X, m, n and s be arbitrary such that m and n are natural numbers and such that s is a sequence $(a_i \,|\, i \in \{1, \ldots, m+n\})$ in X, we may apply Proposition 3.178 to the functions (5.434) and (5.431) to see that the composition s' of $s \restriction \{m+1, \ldots, m+n\}$ and t is a function from $\{1, \ldots, n\}$ to X. Thus, we may consider this function s' to be the sequence $(a'_i \,|\, i \in \{1, \ldots, n\})$ (in X). Next, we verify

$$\forall i\,(i \in \{1, \ldots, n\} \Rightarrow a'_i = a_{m+i}), \tag{5.445}$$

letting i be arbitrary in $\{1, \ldots, n\}$ and observing the truth of the equations

$$\begin{aligned} a'_i &= s'(i) \\ &= [s \restriction \{m+1, \ldots, m+n\} \circ t](i) \\ &= s \restriction \{m+1, \ldots, m+n\}(t(i)) \\ &= s \restriction \{m+1, \ldots, m+n\}(m+i) \\ &= s(m+i) \\ &= a_{m+i} \end{aligned}$$

in light of the definitions of the sequences s' and s, the notation for compositions, and (3.567). As i is arbitrary, we may therefore conclude that the composition s' is indeed the function (5.444); since X, m, n and s were arbitrary, we may then further conclude that the proposed sentence is true. □

Notation 5.17. For any set X, for any binary operation $\odot$ on X such that X contains the neutral element e with respect to $\odot$, for any $m, n \in \mathbb{N}$ and for any sequence $s = (a_i \,|\, i \in \{1, \ldots, m+n\})$ in X, we also write

$$\bigodot_{i=m+1}^{m+n} a_i = \bigodot_{j=1}^{n} a'_j \quad \left[= \bigodot_{j=1}^{n} a_{m+j} \right], \tag{5.446}$$

where $s' = (a'_i \,|\, i \in \{1, \ldots, n\}) = (a_{m+i} \,|\, i \in \{1, \ldots, n\})$.

Lemma 5.117. *For any semigroup $(X, \odot)$ such that X contains the neutral element e with respect to $\odot$ and for any $m \in \mathbb{N}$, it is true that*

$$\forall n \, (n \in \mathbb{N} \Rightarrow \forall s \, (s \in X^{\{1,\ldots,m+n\}} \Rightarrow \left(\bigodot_{i=1}^{m} a_i \right) \odot \left(\bigodot_{i=m+1}^{m+n} a_i \right) = \bigodot_{i=1}^{m+n} a_i)). \tag{5.447}$$

Proof. We let X, $\odot$ and m be arbitrary sets, assume that $(X, \odot)$ is semigroup such that X contains the neutral element e with respect to $\odot$, and we assume that m is a natural number. We now apply a proof by mathematical induction with respect to n. Regarding the base case $(n = 0)$, we let s be an arbitrary set, assume that $s = (a_i \,|\, i \in \{1, \ldots, m+0\})$ is a sequence in X, and observe the truth of the equations

$$\left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{m+0} a_i\right) = \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{j=1}^{0} a'_j\right) = \left(\bigodot_{i=1}^{m} a_i\right) \odot e$$

$$= \bigodot_{i=1}^{m} a_i = \bigodot_{i=1}^{m+0} a_i,$$

applying (5.446) in connection with (5.432), (5.387), the definition of a neutral element, and substitution based on the equation $m = m + 0$ and the fact that 0 is the neutral element in $\mathbb{N}$ with respect to $+_{\mathbb{N}}$ according to Theorem 5.80. Consequently, we obtain after substitution based on $n = 0$ the desired equation (5.447); since s was initially an arbitrary set, we may therefore conclude that the base case holds.

Regarding the induction step, we let $n \in \mathbb{N}$ be arbitrary, make the induction assumption

$$\forall s\,(s \in X^{\{1,\ldots,m+n\}} \Rightarrow \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{m+n} a_i\right) = \bigodot_{i=1}^{m+n} a_i), \tag{5.448}$$

and show that this implies

$$\forall s\,(s \in X^{\{1,\ldots,m+(n+1)\}} \Rightarrow \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{m+(n+1)} a_i\right) = \bigodot_{i=1}^{m+(n+1)} a_i). \tag{5.449}$$

To prove the latter universal sentence, we let s be arbitrary and assume that $s = (a_i \,|\, i \in \{1,\ldots,m+(n+1)\})$ is a sequence in X, so that we obtain the sequence $s' = (a'_i \,|\, i \in \{1,\ldots,n+1\}) = (a_{m+i} \,|\, i \in \{1,\ldots,n+1\})$ and then the equations

$$
\begin{aligned}
\left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{m+(n+1)} a_i\right) &= \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{j=1}^{n+1} a'_j\right) \\
&= \left(\bigodot_{i=1}^{m} a_i\right) \odot \left[\left(\bigodot_{j=1}^{n} a'_j\right) \odot a'_{n+1}\right] \\
&= \left[\left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{j=1}^{n} a'_j\right)\right] \odot a'_{n+1} \\
&= \left[\left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{m+n} a_i\right)\right] \odot a_{m+(n+1)} \\
&= \left(\bigodot_{i=1}^{m+n} a_i\right) \odot a_{m+(n+1)} \\
&= \left(\bigodot_{i=1}^{m+n} a_i\right) \odot a_{(m+n)+1} \\
&= \bigodot_{i=1}^{(m+n)+1} a_i \\
&= \bigodot_{i=1}^{m+(n+1)} a_i
\end{aligned}
$$

by applying (5.446), (5.388), the associativity of $\odot$, (5.446) together with (5.445), the induction assumption (5.448) based on the evident fact that the restriction $s \upharpoonright \{1, \ldots, m+n\}$ is the sequence $(a_i \,|\, i \in \{1, \ldots, m+n\})$ and thus an element of $X^{\{1,\ldots,m+n\}}$, then the associativity of $+_{\mathbb{N}}$, again (5.388), and finally again the associativity of $+_{\mathbb{N}}$. This proves the implication in (5.449), and since s is arbitrary, we may therefore conclude that the universal sentence (5.449) is true. As n is also arbitrary, we may then further conclude that the induction step holds, so that the proof by mathematical induction is complete. Moreover, X, $\odot$ and m were initially arbitrary sets, so that the proposed universal sentence follows to true. $\square$

Theorem 5.118 (Generalized Associative Law for semigroups). *For any semigroup $(X, \odot)$ such that X contains the neutral element with respect to $\odot$, for any $n \in \mathbb{N}_+$, for any sequence $s = (a_i \,|\, i \in \{1, \ldots, n\})$ in X and for any $m \in \mathbb{N}$ with $m \leq n$, it is true that*

$$\bigodot_{i=1}^{n} a_i = \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{n} a_i\right) \tag{5.450}$$

$$= \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{j=1}^{n-m} a_{m+j}\right) \tag{5.451}$$

Proof. We let X, $\odot$, m, n and s be arbitrary sets such that $(X, \odot)$ is a semigroup where X contains the neutral element e with respect to $\odot$, such that m and n are natural number satisfying $m \leq n$, and such that $s = (a_i \,|\, i \in \{1, \ldots, n\})$ is a sequence in X. Here, the assumed inequality implies with Corollary 5.88 that there exists a unique natural number N satisfying $m + N = n$. With this equation and Lemma 5.117 we then obtain

$$\begin{aligned}\bigodot_{i=1}^{n} a_i &= \bigodot_{i=1}^{m+N} a_i \\ &= \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{m+N} a_i\right) \\ &= \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{n} a_i\right)\end{aligned}$$

which equations yield the desired equations (5.450). Moreover, the inequality $m \leq n$ and the equation $m + N = n$ imply $N = n - m$ by definition of a difference, so that Lemma 5.117 gives now in connection with (5.446)

$$\begin{aligned}\bigodot_{i=1}^{n} a_i = \bigodot_{i=1}^{m+N} a_i &= \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{i=m+1}^{m+N} a_i\right) \\ &= \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{j=1}^{N} a_{m+j}\right) \\ &= \left(\bigodot_{i=1}^{m} a_i\right) \odot \left(\bigodot_{j=1}^{n-m} a_{m+j}\right),\end{aligned}$$

so that (5.451) also holds. Since X, $\odot$, m, n and s were initially arbitrary sets, we may therefore infer from this the truth of the stated theorem. □

Corollary 5.119. *It is true for any ordered elementary domain* $(X, +, \cdot, <)$ *that*

$$\forall m, n, s\,([m, n \in \mathbb{N} \,\wedge\, s : \{1, \ldots, n\} \to X] \Rightarrow [m \leq_{\mathbb{N}} n \Rightarrow \sum_{i=1}^{m} s_i \leq \sum_{i=1}^{n} s_i]). \tag{5.452}$$

Proof. We let X, $+$, $\cdot$, $<$, m, n and s be arbitrary sets such that $(X, +, \cdot, <)$ is an ordered elementary domain, such that m and n are natural numbers, and such that s is a function from $\{1, \ldots, n\}$ to X; thus, we may write s in sequence notation as $(s_i \,|\, i \in \{1, \ldots, n\})$. Next, we assume $m \leq_{\mathbb{N}} n$ to be true, so that the Generalized Associative Law for semigroups yields the equation

$$\sum_{i=1}^{n} s_i = \left(\sum_{i=1}^{m} s_i\right) + \left(\sum_{i=m+1}^{n} s_i\right). \tag{5.453}$$

Let us now observe in light of Proposition 5.59b), Theorem 5.62 and the definition of an induced reflexive partial ordering that $0 \leq \sum_{i=m+1}^{n} s_i$ holds. An application of the Monotony Law for $+$ and $\leq$ gives us then

$$0 + \sum_{i=1}^{m} s_i \;\leq\; \sum_{i=m+1}^{n} s_i + \sum_{i=1}^{m} s_i,$$

which inequality we may write also as

$$\sum_{i=1}^{m} s_i \;\leq\; \left(\sum_{i=1}^{m} s_i\right) + \left(\sum_{i=m+1}^{n} s_i\right)$$

by applying the commutativity of the addition on X and the fact that 0 is the zero element of X. Combining the preceding inequality with the previously established equation (5.453) via substitution yields then the desired inequality $\sum_{i=1}^{m} s_i \leq \sum_{i=1}^{n} s_i$. Since X, $+$, $\cdot$, $<$, m, n and s were initially arbitrary, we may therefore conclude that the corollary holds. □

Proposition 5.120. *For any ordered elementary domain* $(X, +, \cdot, <)$*, it is true that*

$$\begin{aligned}&\forall n\,(n \in \mathbb{N}_+ \Rightarrow \forall s, j, k\,([s : \{1, \ldots, n\} \to X \,\wedge\, j, k \in \{1, \ldots, n\}]\\ &\quad \Rightarrow [j \neq k \Rightarrow s_j + s_k \leq \sum_{i=1}^{n} s_i]))\end{aligned} \tag{5.454}$$

Proof. We let $(X, +, \cdot, <)$ be an arbitrary ordered elementary domain, and we prove the universal sentence with respect to n by mathematical induction. In the base case $n = 1$, we let s, j, k be arbitrary such that s is a 1-tuple in X and $j, k \in \{1\}$ (using the definition of an initial segment of $\mathbb{N}_+$). The latter implies $j = k = 1$ with (2.169), so that $j \neq k$ is false. This in turn implies the truth of the implication having that false inequality as the antecedent. Since s, j and k were arbitrary, we may therefore conclude that the base case holds. Regarding the induction step, we let $n \in \mathbb{N}_+$ be arbitrary, we make the induction assumption

$$\begin{aligned} &\forall s, j, k\,([s : \{1, \ldots, n\} \to X \ \wedge\ j, k \in \{1, \ldots, n\}] \\ &\quad \Rightarrow [j \neq k \Rightarrow s_j + s_k \leq \sum_{i=1}^{n} s_i]), \end{aligned} \tag{5.455}$$

and we show that

$$\begin{aligned} &\forall s, j, k\,([s : \{1, \ldots, n+1\} \to X \ \wedge\ j, k \in \{1, \ldots, n+1\}] \\ &\quad \Rightarrow [j \neq k \Rightarrow s_j + s_k \leq \sum_{i=1}^{n+1} s_i]), \end{aligned} \tag{5.456}$$

follows to be true as well. For this purpose, we let s be an arbitrary $(n+1)$-tuple in X and j, k arbitrary elements of the initial segment of $\mathbb{N}_+$ up to $n + 1$. Furthermore, we assume $j \neq k$ to be true. Let us observe here that $n \in \mathbb{N}_+$ implies $1 \leq_{\mathbb{N}} n$ with (4.278), so that the disjunction $1 <_{\mathbb{N}} n \vee 1 = n$ holds in view of the Characterization of induced irreflexive partial orderings. We prove the desired inequality $s_j + s_k \leq \sum_{i=1}^{n+1} s_i$ by two cases, based on this disjunction.

In the first case $1 <_{\mathbb{N}} n$, we note that $n \in \mathbb{N}_+$ implies $i, j \in \{1, \ldots, n\} \cup \{n + 1\}$ with (4.241), so that the disjunctions

$$\begin{aligned} j \in \{1, \ldots, n\} \ \vee\ j \in \{n + 1\}, \\ k \in \{1, \ldots, n\} \ \vee\ k \in \{n + 1\} \end{aligned}$$

follow to be true by definition of the union of two sets. We may prove here by contradiction that $j \in \{n+1\}$ implies $k \notin \{n+1\}$. Assuming $j \in \{n+1\}$ (so that $j = n+1$) and the negation of $k \notin \{n+1\}$ to be true, we find with the Double Negation Law $k \in \{n + 1\}$ (so that $k = n + 1$); consequently, $j = k$, in contradiction to the previous assumption that $j \neq k$ is true. We now use these disjunctions to prove the inequality $s_j + s_k \leq \sum_{i=1}^{n+1} s_i$ by two cases and by two subcases within the first case.

In the first case $j \in \{1, \ldots, n\}$ and the first subcase $k \in \{1, \ldots, n\}$, we evidently have that the restriction of s to $\{1, \ldots, n\}$ is an n-tuple in X_+^0.

Consequently, we obtain the inequality $s_j + s_k \leq \sum_{i=1}^{n} s_i$ with the induction assumption (5.455). Since the term s_{n+1} of the $(n+1)$-tuple s is clearly an element of the codomain X, the inequality $0 \leq s_{n+1}$ holds according to (5.276). Due to the Additivity of $\leq$-inequalities, we may combine the preceding two inequalities to

$$s_j + s_k + 0 \ \leq \ \sum_{i=1}^{n} s_i + s_{n+1},$$

(omitting brackets in view of the associativity of the addition) which yields the desired inequality with the property of a neutral element and (5.417). The second subcase $k \in \{n+1\}$ evidently implies $k = n+1$, so that $s_k = s_{n+1}$. Let us now consider the two further cases $j = n$ and $j \neq n$. If $j = n$ holds, we observe that $n \in \mathbb{N}_+$ implies $1 \in \{1, \ldots, n\}$ with (4.248). Furthermore, the previous case assumption $1 <_{\mathbb{N}} n$ implies $n \neq 1$ with the Characterization of comparability with respect to the linear ordering $<_{\mathbb{N}}$, so that substitution gives $j \neq 1$. These findings and the case assumption $j \in \{1, \ldots, n\}$ imply $s_1 + s_j \leq \sum_{i=1}^{n} s_i$ with the induction assumption. Clearly, the term s_1 is an element of the codomain X, so that $0 \leq s_1$ holds. Recalling now the truth of $s_k = s_{n+1}$, we may apply the Additivity of $\leq$-inequalities and the Monotony Law for $+$ and $\leq$ to the preceding inequality to obtain

$$s_1 + s_j + s_k + 0 \leq \sum_{i=1}^{n} s_i + s_{n+1} + s_1,$$

which inequality we can simplify by means of the Cancellation Law for $+$ (and the commutativity of the addition) to $s_j + s_k \leq \sum_{i=1}^{n+1} s_i$, as desired. If $j \neq n$, then we observe that $n \in \mathbb{N}_+$ implies $n \in \{1, \ldots, n\}$ with (4.247), so that the induction assumption yields $s_n + s_j \leq \sum_{i=1}^{n} s_i$. Using now facts $0 \leq s_n$ and $s_k = s_{n+1}$, we evidently obtain from that inequality

$$s_n + s_j + s_k + 0 \leq \sum_{i=1}^{n} s_i + s_{n+1} + s_n$$

and therefore the desired inequality $s_j + s_k \leq \sum_{i=1}^{n+1} s_i$.

In the second case $j \in \{n+1\}$, which implies $k \notin \{n+1\}$ as shown before and therefore $k \in \{1, \ldots, n\}$, we may evidently proceed in analogy to the preceding second subcase $k \in \{n+1\}$ of the first case $j \in \{1, \ldots, n\}$, if we consider the two possibilities $k = n$ and $k \neq n$. Here $j \in \{n+1\}$ evidently implies $j = n+1$, so that $s_j = s_{n+1}$. Let us now consider the two further cases $k = n$ and $k \neq n$. If $k = n$ holds, we recall the previous findings $1 \in \{1, \ldots, n\}$ and $n \neq 1$, so that $k \neq 1$. These findings and the subcase assumption $k \in \{1,\ldots,n\}$ imply $s_1 + s_k \leq \sum_{i=1}^n s_i$ with the induction assumption. Recalling the facts $0 \leq s_1$ and $s_j = s_{n+1}$, we obtain

$$s_1 + s_k + s_j + 0 \leq \sum_{i=1}^{n} s_i + s_{n+1} + s_1,$$

and therefore $s_j + s_k \leq \sum_{i=1}^{n+1} s_i$, as desired. If $k \neq n$, then we recall $n \in \{1,\ldots,n\}$, so that the induction assumption yields $s_n + s_k \leq \sum_{i=1}^n s_i$. Using now facts $0 \leq s_n$ and $s_j = s_{n+1}$, we evidently obtain from that inequality

$$s_n + s_k + s_j + 0 \leq \sum_{i=1}^{n} s_i + s_{n+1} + s_n$$

and therefore the desired inequality $s_j + s_k \leq \sum_{i=1}^{n+1} s_i$.

We now switch to the other case $1 = n$, which evidently gives $n + 1 = 2$ and therefore $\{1,\ldots,n+1\} = \{1,2\}$ with (4.242). By definition of a pair, the previous assumption $j,k \in \{1,\ldots,n+1\}$ therefore means that the disjunctions $j = 1 \vee j = 2$ and $k = 1 \vee k = 2$ are true. We use the first one to prove the inequality $s_j + s_k \leq \sum_{i=1}^2 s_i$ by cases. On the one hand, $j = 1$ evidently implies with the previous assumption $j \neq k$ that $k = 2$ holds. We then obtain

$$s_j + s_k = s_1 + s_2 = \sum_{i=1}^{2} s_i \quad \left[\leq \sum_{i=1}^{2} s_i\right]$$

applying substitutions, (5.413) and the reflexivity of the total ordering $\leq$; thus, $s_j + s_k \leq \sum_{i=1}^2 s_i$, as desired. On the other hand, $j = 2$ clearly implies $k = 1$ and therefore

$$s_j + s_k = s_2 + s_1 = s_1 + s_2 = \sum_{i=1}^{2} s_i \quad \left[\leq \sum_{i=1}^{2} s_i\right],$$

resulting in the same inequality again. Another substitution based on the equation $n + 1 = 2$ gives us then $s_j + s_k \leq \sum_{i=1}^{n+1} s_i$, completing the proof by cases. Since n was arbitrary, we may infer from the previous findings the truth of the induction step, and thus the truth of the universal sentence (5.454). As $(X,+,\cdot,<)$ was initially also arbitrary, the proposition finally follows to be true. $\square$

Proposition 5.121. *For any ordered integral domain* $(X,+,\cdot,-,<)$ *an* n*-fold sum is zero iff all terms of the* n*-tuple are zero, that is,*

$$\forall n\,(n\in\mathbb{N}\Rightarrow\forall s\,(s\in X^{\{1,\ldots,n\}}$$
$$\Rightarrow\left[\sum_{i=1}^{n}s_i=0_X\Leftrightarrow\forall i\,(i\in\{1,\ldots,n\}\Rightarrow s_i=0_X)\right])). \tag{5.457}$$

Proof. Letting $(X,+,\cdot,<)$ be an arbitrary ordered elementary domain, we apply a proof by mathematical induction. Regarding the base case ($n=0$), we let $s\in X^{\{1,\ldots,n\}}$ be arbitrary, so that $s=\emptyset$ due to (4.239) and (3.120). Let us observe on the one hand that $\sum_{i=1}^{0}s_i=0_X$ holds according to (5.409). On the other hand, letting i be arbitrary, we see that $i\in\{1,\ldots,0\}$ $[=\emptyset]$ is false, so that the implication $i\in\{1,\ldots,n\}\Rightarrow s_i=0_X$ is true. As i is arbitrary, we conclude that the universal sentence $\forall i\,(i\in\{1,\ldots,n\}\Rightarrow s_i=0_X)$ holds, too. Thus, the equivalence in (5.457) is true for $n=0$. Then, because s is arbitrary, we may infer from this finding the truth of the base case. Regarding the induction step, we let $n\in\mathbb{N}$ be arbitrary, make the induction assumption

$$\forall s\,(s\in X^{\{1,\ldots,n\}}\Rightarrow\left[\sum_{i=1}^{n}s_i=0_X\Leftrightarrow\forall i\,(i\in\{1,\ldots,n\}\Rightarrow s_i=0_X)\right]),$$

and show that

$$\forall s\,(s\in X^{\{1,\ldots,n+1\}}\Rightarrow\left[\sum_{i=1}^{n+1}s_i=0_X\Leftrightarrow\forall i\,(i\in\{1,\ldots,n+1\}\Rightarrow s_i=0_X)\right]) \tag{5.458}$$

follows to be true. Letting $s\in X^{\{1,\ldots,n+1\}}$ be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\sum_{i=1}^{n+1}s_i=0_X$ to be true and demonstrating the truth of

$$\forall i\,(i\in\{1,\ldots,n+1\}\Rightarrow s_i=0_X). \tag{5.459}$$

Due to the assumed equation, we thus have

$$\sum_{i=1}^{n}s_i+s_{n+1}=0_X. \tag{5.460}$$

in view of (5.417). We may now apply the Law of Contraposition with (5.278) to infer from that equation the truth of $\sum_{i=1}^{n}s_i=0_X$. Rewriting (5.460) by means of the commutativity of the addition as $s_{n+1}+\sum_{i=1}^{n}s_i=$

0_X, we obtain with the same arguments also $s_{n+1} = 0_X$. Here, the former gives us with the induction assumption the true universal sentence

$$\forall i\,(i \in \{1,\ldots,n\} \Rightarrow s_i = 0_X). \tag{5.461}$$

We are now in a position to prove the universal sentence (5.459). To do this, we let $i \in \{1,\ldots,n+1\}$ be arbitrary, so that

$$i \in \{1,\ldots,n\} \cup \{n+1\} \tag{5.462}$$

holds by virtue of (4.241). By definition of the union of two sets, this means that the disjunction

$$i \in \{1,\ldots,n\} \;\vee\; i = n+1 \tag{5.463}$$

is true (using also (2.169)). We now use this true disjunction to prove $s_i = 0_X$ by cases. The first case $i \in \{1,\ldots,n\}$ immediately gives the desired equation with (5.461). The second case $i = n+1$ yields $s_i = 0_X$ via substitution into the previously established equation $s_{n+1} = 0_X$. Thus, $s_i = 0_X$ holds in any case, and since i was arbitrary, we therefore conclude that the universal sentence (5.459) is indeed true. This finding completes the proof of the first part of the equivalence in (5.458) is complete.

To establish the second part ('$\Leftarrow$'), we assume now (5.459) to be true, and we show that $\sum_{i=1}^{n+1} s_i = 0_X$ holds, too. For this purpose, we prove (5.461), letting i be arbitrary and assuming $i \in \{1,\ldots,n\}$ to hold. Then, the disjunction (5.463) is also true, which evidently implies (5.462) and then $i \in \{1,\ldots,n+1\}$. This in turn implies $s_i = 0_X$ with the assumption (5.459). As i was arbitrary, we therefore conclude that (5.461) is true, and this universal sentence gives us now the true equation $\sum_{i=1}^{n} s_i = 0_X$ with the induction assumption. Since the evident fact $n+1 \in \{1,\ldots,n+1\}$ implies also $s_{n+1} = 0_X$ with the assumption (5.459), we evidently obtain the true equation (5.460), which finding completes the proof of the equivalence.

Since s was arbitrary, we may therefore conclude that the universal sentence (5.458) holds. Then, as n was also arbitrary, we may further conclude that the induction step is true. Thus, the proof of (5.121) by mathematical induction is complete. Here, $(X, +, \cdot, <)$ was initially also arbitrary, so that the proposition follows to be true. □

Exercise 5.48. Show for any set X, for any addition $+_X$ such that the zero element of X with respect to $+_X$ exists, and for any element c of X that an n-fold sum equals c if one term of the n-tuple equals c and the other terms are zero, i.e. that

$$\forall n\,(n \in \mathbb{N} \Rightarrow \forall s\,(s \in X^{\{1,\ldots,n\}} \Rightarrow [\exists j\,(j \in \{1,\ldots,n\} \wedge s_j = c$$
$$\wedge\ \forall k\,(k \in \{1,\ldots,n\} \setminus \{j\} \Rightarrow s_k = 0)) \Rightarrow \sum_{i=1}^{n} s_i = c]). \tag{5.464}$$

Proof. To do!

□

Notation 5.18. Equations (5.450) – (5.450) show that the value $\bigodot_{i=1}^{n} a_i$ of the n-fold repeated binary operation is independent of the 'order' (indicated by the brackets) in which the terms of the sequences are processed. We therefore write

$$\bigodot_{i=1}^{n} a_i = a_1 \odot \cdots \odot a_n. \tag{5.465}$$

To give a first example, we recall the n-fold join, based on a lattice $(X, \sqcup, \sqcap, \leq)$ with the semigroup $(X, \sqcup)$ (as established in Proposition 5.33).

Notation 5.19. For any lattice $(X, \sqcup, \sqcap, <)$ such that X contains the neutral element with respect to $\sqcup$, for any $n \in \mathbb{N}$ and for any sequence $(a_i \,|\, i \in \{1, \ldots, n\})$ in X, we also write

$$\sqcup_{i=1}^{n} a_i = a_1 \sqcup \cdots \sqcup a_n = \sup\{a_1, \ldots, a_n\} \tag{5.466}$$

and call this the supremum of $a_1, \ldots, a_n$. We then also say that the n-fold join $\sqcup_{i=1}^{n}$ is the *n-ary join* on X.

Notation 5.20. For any semigroup $(X, +)$ such that X contains the neutral element for the addition $+$, any $n \in \mathbb{N}$ and any sequence $(a_i \,|\, i \in \{1, \ldots, n\})$ in X, we also write

$$\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n \tag{5.467}$$

(read as "the sum as i goes from 1 to n of a_i") and call this the *sum* of $a_1, \ldots, a_n$. We then also say that the n-fold addition $\sum_{i=1}^{n}$ is the *n-ary addition* on X.

In case of $a_i = a$ for all $i \in \{1, \ldots, n\}$ (with $n \in \mathbb{N}$) we also write

$$na \tag{5.468}$$

instead of $\sum_{i=1}^{n} a_i$, which we call a *multiple* of a; here, we call n the *multiplier*.

Notation 5.21. For any semigroup $(X, \cdot)$ such that X contains the neutral element for the multiplication $\cdot$, for any $n \in \mathbb{N}$ and for any sequence $(a_i \mid i \in \{1, \ldots, n\})$ in X, we also write

$$\prod_{i=1}^{n} a_i = a_1 \cdot \ldots \cdot a_n \tag{5.469}$$

(read as "the product as i goes from 1 to n of a_i") and call this the *product* of $a_1, \ldots, a_n$. We then also say that the n-fold multiplication $\prod_{i=1}^{n}$ is the *n-ary multiplication* on X.

In case of $a_i = a$ for all $i \in \{1, \ldots, n\}$ (with $n \in \mathbb{N}$) we also write

$$a^n \tag{5.470}$$

instead of $\prod_{i=1}^{n} a_i$ and speak of the *n-th power* of a; here, we call a the *base* and n the *exponent* (semigroup). We call in particular a^2 the *square* of a.

Let us apply the preceding notations now to the 1-, 2-, 3-, and $(n+1)$-fold iterated additions and multiplications established in Corollary 5.109.

Corollary 5.122. *The following sentences are true.*

a) *For any semigroup $(X, +)$ such that X contains the zero element 0_X and for any $a \in X$, it is true that*

$$0a = 0_X, \tag{5.471}$$

$$1a = a, \tag{5.472}$$

$$2a = a + a, \tag{5.473}$$

$$3a = a + a + a, \tag{5.474}$$

$$(n+1)a = na + a \quad \text{for any } n \in \mathbb{N}. \tag{5.475}$$

b) *For any semigroup $(X, \cdot)$ such that X contains the unity element 1_X and for any $a \in X$, it is true that*

$$a^0 = 1_X, \tag{5.476}$$

$$a^1 = a, \tag{5.477}$$

$$a^2 = a \cdot a, \tag{5.478}$$

$$a^3 = a \cdot a \cdot a, \tag{5.479}$$

$$a^{n+1} = a^n \cdot a \quad \text{for any } n \in \mathbb{N}. \tag{5.480}$$

Proposition 5.123. *It is true for any semigroup $(X, +)$ with zero element 0_X that*

$$\forall a, b\,([a, b \in X \,\wedge\, a + b = b + a] \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow a + nb = nb + a)). \quad (5.481)$$

Proof. We let X, $+$, a and b be arbitrary such that $(X, +)$ is a semigroup with zero element 0_X, such that a and b are elements of X, and such that $a + b = b + a$ holds. We now prove the universal sentence with respect to n by mathematical induction. In the base case ($n = 0$), we obtain the equations

$$\begin{aligned} a + nb &= a + 0b \\ &= a + 0^X \\ &= a \\ &= 0_X + a \\ &= 0b + a \\ &= nb + a \end{aligned}$$

by applying substitution, (5.471), the definition of the zero element, again the definition of the zero element, again (5.471), and finally again substitution. In the induction step, we let $n \in \mathbb{N}$ be arbitrary, make the induction assumption $a + nb = nb + a$, and observe the truth of the equations

$$\begin{aligned} a + (n+1)b &= a + (nb + b) \\ &= (a + nb) + b \\ &= (nb + a) + b \\ &= nb + (a + b) \\ &= nb + (b + a) \\ &= (nb + b) + a \\ &= (n+1)b + a \end{aligned}$$

in light of (5.475), the associativity of $+$, the induction assumption, the associativity of $+$, the assumption $a + b = b + a$, the associativity of $+$, and (5.475). Since n was arbitrary, we may therefore conclude that the induction step holds, so that the proof by mathematical induction is complete. As X, $+$, a and b were also arbitrary, we may then further conclude that the proposed universal sentence is true. □

Exercise 5.49. Prove for any semigroup $(X, \cdot)$ with unity element 1_X that

$$\forall a, b\,([a, b \in X \,\wedge\, a \cdot b = b \cdot a] \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow a \cdot b^n = b^n \cdot a)). \quad (5.482)$$

(Hint: Proceed similarly as in the proof of Proposition 5.481.)

Proposition 5.124. *It is true for any semigroup $(X,+)$ with zero element 0_X that*

$$\forall a,b\,([a,b\in X\wedge a+b=b+a]\Rightarrow \forall n\,(n\in\mathbb{N}\Rightarrow n(a+b)=na+nb)). \quad (5.483)$$

Proof. We let X, $+$, a and b be arbitrary such that $(X,+)$ is a semigroup with zero element 0_X, such that a and b are elements of X, and such that $a+b=b+a$ holds. We now prove the universal sentence with respect to n by mathematical induction. In the base case ($n=0$), we have

$$n(a+b)^n = 0(a+b) = 0_X = 0_X + 0_X = 0a + 0b = na + nb$$

by applying substitution, (5.471), the definition of the identity element, again (5.471), and then two more substitutions. To prove the induction step, we let $n\in\mathbb{N}$ be arbitrary, make the induction assumption $n(a+b)=na+nb$, and derive the equations

$$\begin{aligned}
(n+1)(a+b) &= n(a+b)+(a+b)\\
&= (na+nb)+(a+b)\\
&= ((na+nb)+a)+b\\
&= (na+(nb+a))+b\\
&= (na+(a+nb))+b\\
&= ((na+a)+nb)+b\\
&= ((n+1)a+nb)+b\\
&= (n+1)a+(nb+b)\\
&= (n+1)a+(n+1)b
\end{aligned}$$

by means of (5.475), the induction assumption, the associativity of $+$, again the associativity of $+$, (5.481), again the associativity of $+$, again (5.475), again the associativity of $+$, and finally again (5.475). Since n is arbitrary, we may infer from the resulting equation $(n+1)(a+b)=(n+1)a+(n+1)b$ the truth of the induction step. As X, $+$, a and b were also arbitrary, we may then further conclude that the proposed universal sentence is true. □

Exercise 5.50. Show for any semigroup $(X,\cdot)$ with unity element 1_X that multiplying two powers with commuting bases and identical exponents corresponds to multiplying the bases, that is,

$$\forall a,b\,([a,b\in X\ \wedge\ a\cdot b=b\cdot a]\Rightarrow \forall n\,(n\in\mathbb{N}\Rightarrow (a\cdot b)^n=a^n\cdot b^n)). \quad (5.484)$$

(Hint: Apply a similar proof as for Proposition 5.124.)

Proposition 5.125. *It is true for any semigroup* $(X, +)$ *with zero element* 0_X *that any multiple of the zero element equals that element, i.e.*

$$\forall n\,(n \in \mathbb{N} \Rightarrow n0_X = 0_X). \tag{5.485}$$

Proof. We let X and $+$ be arbitrary sets and assume $(X, +)$ to be a semigroup such that the zero element of X exists. In the base case ($n = 0$), we have

$$n0_X = 00_X = 0_X$$

because of (5.471), as desired. Concerning the induction step, we take an arbitrary natural number n and make the induction assumption $n0_X = 0_X$. Then, we obtain

$$(n+1)0_X = n0_X + 0_X = 0_X + 0_X = 0_X$$

with (5.475), the induction assumption and the definition of the zero element. Since n is arbitrary, we may therefore conclude that the induction step holds (besides the base case), so that the proof of (5.485) is complete. As X and $+$ were also arbitrary, the proposition follows then to be true. □

Exercise 5.51. It is true for any semigroup $(X, \cdot)$ with unity element 1_X that any power of the unity element equals that element, i.e.

$$\forall n\,(n \in \mathbb{N} \Rightarrow [1_X]^n = 1_X). \tag{5.486}$$

(Hint: The proof is similar to that of Proposition 5.125.)

Theorem 5.126 (Addition & Multiplication Rules for multiples & powers in a semigroup). *The following laws hold, respectively, for any semigroup* $(X, +_X)$ *with zero element* 0_X *and for any semigroup* $(X, \cdot_X)$ *with identity element* 1_X*.*

a) **Addition Rule for multiples in a semigroup***:*

$$\forall a, m, n\,([a \in X \wedge m, n \in \mathbb{N}] \Rightarrow (m+n)a = ma +_X na). \tag{5.487}$$

b) **Multiplication Rule for multiples in a semigroup***:*

$$\forall a, m, n\,([a \in X \wedge m, n \in \mathbb{N}] \Rightarrow (m \cdot n)a = n[ma]). \tag{5.488}$$

c) **Addition Rule for powers in a semigroup***:*

$$\forall a, m, n\,([a \in X \wedge m, n \in \mathbb{N}] \Rightarrow a^{m+n} = a^m \cdot_X a^n). \tag{5.489}$$

d) **Multiplication Rule for powers in a semigroup***:*

$$\forall a, m, n\,([a \in X \,\wedge\, m, n \in \mathbb{N}] \Rightarrow a^{m \cdot n} = [a^m]^n). \tag{5.490}$$

Proof. We let X and $\cdot_X$ be arbitrary and $(X, +_X)$ to be a semigroup for which the zero element 0_X exists.

Concerning a), we take an arbitrary constant a and prove

$$\forall m\, \forall n\,([(a \in X \,\wedge\, m \in \mathbb{N}) \,\wedge\, n \in \mathbb{N}] \Rightarrow (m+n)a = ma +_X na), \tag{5.491}$$

which sentence is equivalent to

$$\forall m\,([a \in X \,\wedge\, m \in \mathbb{N}] \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow (m+n)a = ma +_X na)), \tag{5.492}$$

because of (1.90). Letting here m also be arbitrary and assuming $a \in X$ as well as $m \in \mathbb{N}$ to be both true, we can now apply a proof by mathematical induction with respect to n. In the base case ($n = 0$), we observe the truth of the equations

$$(m+n)a = (m+0)a = ma = ma +_X 0_X = ma +_X 0a = ma +_X na$$

in light of the equation $n = 0$, the definition of the zero element of $\mathbb{N}$, the definition of the zero element 0_X and (5.471). Thus, the resulting equation $(m+n)a = ma +_X na$ proves the base case. In the induction step, we take an arbitrary natural number n, we make the induction assumption $(m+n)a = ma +_X na$, and we derive the equations

$$\begin{aligned}(m+[n+1])a &= ([m+n]+1)a \\ &= (m+n)a +_X a \\ &= (ma +_X na) +_X a \\ &= ma +_X (na +_X a) \\ &= ma +_X (n+1)a\end{aligned}$$

by using the Associative Law for the addition on $\mathbb{N}$, (5.472), the induction assumption, the associativity of $+_X$, and again (5.472). Since n is arbitrary, the resulting equation $(m+[n+1])a = ma +_X (n+1)a$ evidently proves the induction step, completing the proof by mathematical induction. As m was also arbitrary, we may further conclude that (5.492) and thus the equivalent universal sentence (5.491) hold. Because a was initially arbitrary, the universal sentence (5.487) follows then to be true.

Concerning b), we let a be arbitrary, so that we are required to prove

$$\forall m\, \forall n\,([(a \in X \,\wedge\, m \in \mathbb{N}) \,\wedge\, n \in \mathbb{N}] \Rightarrow (m \cdot n)a = n[ma]), \tag{5.493}$$

which we may write equivalently as

$$\forall m\,([a \in X \,\wedge\, m \in \mathbb{N}] \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow (m \cdot n)a = n[ma])) \tag{5.494}$$

by utilizing (1.90). Next, we let m be arbitrary and assume $a \in X$ and $m \in \mathbb{N}$ to be both true, and we prove the universal sentence with respect to n by mathematical induction. The base case ($n = 0$) holds because we obtain the equations

$$(m \cdot n)a = (m \cdot 0)a = 0a = 0 = 0[ma] = n[ma]$$

by applying substitution based on $n = 0$, then the Cancellation Law for 0, (5.471), again (5.471), and finally again substitution based on $n = 0$. Regarding the induction step, we let $n \in \mathbb{N}$ be arbitrary, make the induction assumption $(m \cdot n)a = n[ma]$, and observe the truth of the equations

$$\begin{aligned}
[m \cdot (n+1)]a &= (m \cdot n + m \cdot 1)a \\
&= (m \cdot n)a +_X (m \cdot 1)a \\
&= n[ma] +_X ma \\
&= n[ma] +_X 1[ma] \\
&= (n+1)[ma]
\end{aligned}$$

in view of the distributivity of $\cdot_{\mathbb{N}}$ over $+_{\mathbb{N}}$, the previously established Addition Rule for multiples, the induction assumption as well as the definition of a neutral element, (5.472), and considering the Addition Rule for multiples once again. As n is arbitrary, we may therefore conclude that the induction step holds, so that the proof by mathematical induction is complete. Since m is also arbitrary, we may now further conclude that the universal sentence (5.494) is true. Thus, the equivalent sentence (5.493) also holds, and because a was also arbitrary, we may infer from this finding the truth of b).

Since X and $+_X$ were initially arbitrary, Part a) and Part b) of the stated theorem follow to be true. □

Exercise 5.52. Establish the Addition Rule and the Multiplication Rule for powers.

(Hint: Proceed in analogy to the proofs of the Addition Rule and the Multiplication Rule for multiples.)

Proposition 5.127. *For any ordered elementary domain $(X, +, \cdot, <)$, it is true that any power of an element $a \in X$ is different from the zero element if a is different from the zero element, i.e.*

$$\forall a\,([a \in X \,\wedge\, a \neq 0_X] \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow a^n \neq 0_X)). \tag{5.495}$$

Proof. We let X, $+$, $\cdot$, $<$ and a be arbitrary such that $(X, +, \cdot, <)$ is an ordered elementary domain and such that the conjunction of $a \in X$ and $a \neq 0_X$ holds. Next, we apply a proof by mathematical induction, observing first that the base case ($n = 0$) yields

$$a^n = a^0 = 1_X \ (\neq 0_X),$$

where we applied substitution, then (5.476) and Theorem 5.68, so that we obtain $a^n = 0_X$ as desired. Regarding the induction step, we take an arbitrary $n \in \mathbb{N}$, assume $a^n \neq 0_X$, and verify $a^{n+1} \neq 0_X$. Since $a^{n+1} = a^n \cdot a$ is true according to (5.480), where $a^n \neq 0_X$ and $a \neq 0_X$ hold because of the previously made assumptions, it follows with the Criterion for zero-divisor freeness in particular that $a^n \cdot a \neq 0_X$ is true, so that substitution based on the preceding equation yields the desired $a^{n+1} \neq 0_X$. As n is arbitrary, we may therefore conclude that the induction step also holds, which finding completes the proof of the universal sentence with respect to n in (5.495). Since X, $+$, $\cdot$, $<$ and a were initially arbitrary, we may further conclude that the proposition holds, as claimed. □

Corollary 5.128. *It is true for the ordered elementary domain* $(\mathbb{N}, +, \cdot, <)$ *that any power of a natural number k is different from zero if k is different from zero, i.e.*

$$\forall k\,([k \in \mathbb{N} \wedge k \neq 0] \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow k^n \neq 0)). \tag{5.496}$$

We now generalize the Distributive Law from binary operations to n-fold binary operations, which task we prepare by means of the following sentence.

Proposition 5.129. *For any set X, any multiplication $\cdot_X$ (on X), any element $c \in X$, any $n \in \mathbb{N}$ and any sequence $s = (a_i \,|\, i \in \{1, \ldots, n\})$ in X, it is true that there exists a unique sequence $t = (b_i \,|\, i \in \{1, \ldots, n\})$ such that every term b_i is the product of c and the term a_i, i.e. such that*

$$\forall i\,(i \in \{1, \ldots, n\} \Rightarrow b_i = c \cdot_X a_i) \tag{5.497}$$

holds. Furthermore, t is a sequence in X.

Proof. Letting X, $\cdot_X$, c, n and s be arbitrary, we assume that $\cdot_X$ is a binary (multiplication) operation on X, that $c \in X$ and $n \in \mathbb{N}$ are true, and moreover that $s = (a_i \,|\, i \in \{1, \ldots, n\})$ is a sequence in X. We now prove the universal sentence

$$\forall i\,(i \in \{1, \ldots, n\} \Rightarrow \exists! y\,(y = c \cdot_X a_i)), \tag{5.498}$$

letting i be arbitrary and assuming $i \in \{1, \ldots, n\}$ to be true. Since $\{1, \ldots, n\}$ is the domain of the sequence $s = (a_i \,|\, i \in \{1, \ldots, n\})$ in X, it follows with the Function Criterion that the term a_i is an element of X. As we assumed c to be an element of X as well, we find the product $c \cdot_X a_i$ to be an element of X, too. Then, the uniquely existential sentence in (5.498) follows to be true with Proposition 1.21, and because i is arbitrary, we may therefore conclude that the universal sentence (5.498) holds. According to Theorem 3.160, there then exists a unique function t with domain $\{1, \ldots, n\}$ such that $t(i) = c \cdot_X a_i$ holds for any $i \in \{1, \ldots, n\}$. Thus, we may write t as the sequence $t = (b_i \,|\, i \in \{1, \ldots, n\})$ whose terms evidently satisfy (5.497). In view of the already established fact that the term $[b_i =]\, c \cdot_X a_i$ is an element of X for any $i \in \{1, \ldots, n\}$, we see in light of the Function Criterion that t is a sequence in X. Since X, $\cdot_X$, c, n and s were initially arbitrary, we may infer from this the truth of the proposed universal sentence. □

Note 5.29. This proposition shows that a sequence $(a_i \,|\, i \in \{1, \ldots, n\})$ in X induces a sequence $(c \cdot a_i \,|\, i \in \{1, \ldots, n\})$ in X. Therefore, if X contains the neutral element with respect to an addition on X, we may form both $\sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{n} (c \cdot a_i)$.

Theorem 5.130 (Generalized Distributive Law for semirings). *The following law holds for any semiring $(X, +, \cdot)$, such that X contains the zero element 0_X, and for any element c in X.*

$$\forall n (n \in \mathbb{N} \Rightarrow \forall s \,(s \in X^{\{1, \ldots, n\}} \Rightarrow c \cdot \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (c \cdot a_i))). \tag{5.499}$$

Proof. We let X, $+_X$, $\cdot_X$ and c be arbitrary, assume that the ordered triple $(X, +, \cdot)$ is a semiring such that X contains the zero element 0_X (i.e., the neutral element of X with respect to $+_X$), and we assume that $c \in X$ holds. Now we apply a proof by mathematical induction. Regarding the base case $(n = 0)$, we let s be the empty sequence, and we obtain the equations

$$c \cdot \sum_{i=1}^{n} a_i = c \cdot \sum_{i=1}^{0} a_i = c \cdot 0_X = 0_X = \sum_{i=1}^{0} (c \cdot a_i) = \sum_{i=1}^{n} (c \cdot a_i)$$

by applying substitution based on $n = 0$, (5.409), the Cancellation Law for 0_X, again (5.409), and finally again substitution based on $n = 0$. These equations then yield the desired equation in (5.499).

Regarding the induction step, we let $n \in \mathbb{N}$ be arbitrary, make the induction assumption

$$\forall s\,(s \in X^{\{1,\ldots,n\}} \Rightarrow c \cdot \sum_{i=1}^{n} a_i = \sum_{i=1}^{n}(c \cdot a_i)), \tag{5.500}$$

and show that this implies

$$\forall s\,(s \in X^{\{1,\ldots,n^+\}} \Rightarrow c \cdot \sum_{i=1}^{n^+} a_i = \sum_{i=1}^{n^+}(c \cdot a_i)). \tag{5.501}$$

For this purpose, we let s be an arbitrary set and assume that $s = (a_i \,|\, i \in \{1,\ldots,n^+\})$ is a sequence in X. We then obtain the equations

$$\begin{aligned} c \cdot \sum_{i=1}^{n^+} a_i &= c \cdot \left(\sum_{i=1}^{n} a_i + a_{n^+}\right) \\ &= \left(c \cdot \sum_{i=1}^{n} a_i\right) + (c \cdot a_{n^+}) \\ &= \sum_{i=1}^{n}(c \cdot a_i) + (c \cdot a_{n^+}) \\ &= \sum_{i=1}^{n^+}(c \cdot a_i) \end{aligned}$$

applying (5.388) to the sequence $(a_i \,|\, i \in \{1,\ldots,n^+\})$, the distributivity of $\cdot$ over $+$ (according to Property 3 of a semiring), the induction assumption, (5.388) based on the sequence $s \restriction \{1,\ldots,n\}$, and applying (5.388) now to the sequence $(c \cdot a_i \,|\, i \in \{1,\ldots,n^+\})$. Since s is arbitrary, we may therefore conclude that the universal sentence (5.501) is true, and as n was also arbitrary, we may further conclude that the induction step is true (besides the base case). Thus, the proof of (5.499) is complete, and because X, $+_X$, $\cdot_X$ and c were initially arbitrary, we may infer from this the truth of the stated theorem. □

5.6.2. Factorials

Proposition 5.131. *There exists a unique function f with domain $\mathbb{N}$ with*

$$\forall n\,(n \in \mathbb{N} \Rightarrow f(n) = \prod_{i=1}^{n} i) \tag{5.502}$$

holds, and this function is a sequence in $\mathbb{N}$.

Proof. We first apply Function definition by replacement and verify for this purpose

$$\forall n\,(n \in \mathbb{N} \Rightarrow \exists! y\,(y = \prod_{i=1}^{n} i)). \tag{5.503}$$

Letting $n \in \mathbb{N}$ be arbitrary, we may evidently use the semigroup $(\mathbb{N}, \cdot_{\mathbb{N}})$ and the sequence

$$(i \,|\, i \in \{1, \ldots, n\}) = \mathrm{id}_{\mathbb{N}} \upharpoonright \{1, \ldots, n\}$$

to form the natural number $\prod_{i=1}^{n} i$. Then, the uniquely existential sentence follows immediately to be true with (1.109). Since n is arbitrary, we may therefore conclude that the universal sentence (5.503) is true, so that there exists a unique function f with domain $\mathbb{N}$ such that (5.502) holds. Thus, we may write f as the sequence $(f_n)_{n \in \mathbb{N}}$. Furthermore, because $\prod_{i=1}^{n} i$ is an element of $\mathbb{N}$ for any $n \in \mathbb{N}$, we also see in light of the Function Criterion that f is a function/sequence from $\mathbb{N}$ to $\mathbb{N}$. □

Definition 5.25 (Factorial function, factorial). We call the function

$$f : \mathbb{N} \to \mathbb{N}, \quad n \mapsto f(n) = \prod_{i=1}^{n} i \tag{5.504}$$

the *factorial function*, and we call for any $n \in \mathbb{N}$ the corresponding function value $f(n)$ the *factorial* of n, which we symbolize by

$$n! \tag{5.505}$$

Theorem 5.132 (Recurrence Relation for factorials). *The following law applies to factorials.*

$$0! = 1 \,\wedge\, \forall n\,(n \in \mathbb{N}_+ \Rightarrow n! = (n-1)! \cdot n). \tag{5.506}$$

Proof. The first part of the conjunction results from the equations

$$0! = \prod_{i=1}^{0} i = 1, \tag{5.507}$$

using (5.504) and (5.410). Next, we apply a proof by mathematical induction to establish the second part of the proposed conjunction. Regarding the base case $(n = 1)$, we obtain the equations

$$n! \,=\, 1! \,=\, \prod_{i=1}^{1} i \,=\, 1 \,=\, 1 \cdot 1 \,=\, 0! \cdot 1 \,=\, (1-1)! \cdot 1 \,=\, (n-1)! \cdot n$$

by applying substitution, (5.504), (5.412), the definition of a neutral element, then the previously established (5.507), (5.345), and finally again substitution.

Regarding the induction step, we let $n \in \mathbb{N}_+$ be arbitrary, assume that $n! = (n-1)! \cdot n$ holds, and observe the truth of the equations

$$\begin{aligned}(n+1)! &= \prod_{i=1}^{n+1} i \\ &= \left(\prod_{i=1}^{n} i\right) \cdot (n+1) \\ &= \left(\prod_{i=1}^{(n+1)-1} i\right) \cdot (n+1) \\ &= ([n+1]-1)! \cdot (n+1)\end{aligned}$$

in light of (5.504), (5.418) and (5.344). Because n is arbitrary, we may therefore conclude that the induction step holds (which we proved here without using the induction assumption!). Thus, the proof of the universal sentence in (5.506) by mathematical induction is complete, so that the stated conjunction is indeed true. □

Exercise 5.53. Establish the following law for factorials.

$$\forall n\,(n \in \mathbb{N} \Rightarrow (n+1)! = n! \cdot (n+1)). \tag{5.508}$$

5.6.3. Series

Exercise 5.54. Show for any set X, any addition $+$ on X such that X contains the zero element 0_X and any sequence $f = (a_n)_{n \in \mathbb{N}_+}$ in X that there exists a unique sequence $s = (s_n)_{n \in \mathbb{N}_+}$ in X satisfying

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow s_n = \sum_{i=1}^{n} a_i). \tag{5.509}$$

(Hint: Apply Theorem 3.160 in connection with Proposition 5.106 and the restrictions $(a_i \,|\, i \in \{1, \ldots, n\}) = f \upharpoonright \{1, \ldots, n\}$ with $n \in \mathbb{N}_+$.)

Definition 5.26 ((Generated) series, partial sum). For any set X, any addition $+$ on X such that X contains the zero element 0_X and any sequence $f = (a_n)_{n \in \mathbb{N}_+}$ in X, we call the sequence

$$(s_n)_{n \in \mathbb{N}_+} = (\sum_{i=1}^{n} a_i)_{n \in \mathbb{N}_+} \tag{5.510}$$

the *series generated* by $(a_n)_{n\in\mathbb{N}_+}$, and we call for any $n \in \mathbb{N}_+$ the term

$$s_n = \sum_{i=1}^{n} a_i \tag{5.511}$$

the n-th *partial sum* (of the series).

Notation 5.22. For any partially ordered set $(Y, \leq_Y)$ and any sequence $f = (a_n)_{n\in\mathbb{N}_+}$ in Y generating an increasingly or decreasingly convergent series $s = (s_n)_{n\in\mathbb{N}_+} = (\sum_{i=1}^{n} a_i)_{n\in\mathbb{N}_+}$, we write for the limit of s

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i. \tag{5.512}$$

Let us inspect a first example of a series and its tail characteristics.

Proposition 5.133. *For any set X and any addition $+_X$ on X such that X contains the zero element 0_X, as well as for any elements $\bar{a}, \bar{b} \in X$, it is true that*

a) the series $s = (s_n)_{n\in\mathbb{N}_+}$ generated by the sequence $f = (a_n)_{n\in\mathbb{N}_+}$ defined by (5.361) satisfies $s_1 = \bar{a}$ and

$$\forall n\,(n \in \mathbb{N}_+ \setminus \{1\} \Rightarrow s_n = \bar{a} +_X \bar{b}). \tag{5.513}$$

b) the 2nd tail of s is identical with the constant function on $\mathbb{N}_+$ with value $\bar{a} +_X \bar{b}$, i.e.

$$T_2^s = \mathbb{N}_+ \times \{\bar{a} +_X \bar{b}\} \quad [= g_{\bar{a}+_X\bar{b}}]. \tag{5.514}$$

Proof. We let X, $+_X$, $\bar{a}$ and $\bar{b}$ be arbitrary, assume that $+_X$ is an addition on X such that X contains the zero element 0_X, and assume moreover that $\bar{a}, \bar{b} \in X$ holds. Then, the sequence $f = (a_n)_{n\in\mathbb{N}_+}$ defined by (5.361) generates the series $s = (s_n)_{n\in\mathbb{N}_+}$ with $s_n = \sum_{i=1}^{n} a_i$ for any $n \in \mathbb{N}_+$. Concerning a), we first obtain with 5.411)

$$s_1 = \sum_{i=1}^{1} a_i = a_1 = \bar{a},$$

as claimed. Next, we observe in light of Exercise 4.10 that we may apply a proof by mathematical induction to establish (5.513). In the base case $(n = 2)$, we obtain with (5.413)

$$s_2 = \sum_{i=1}^{2} a_i = a_1 +_X a_2 = \bar{a} +_X \bar{b},$$

as desired. Letting now $n \in \mathbb{N}_+ \setminus \{1\}$ be arbitrary, we make the induction assumption $s_n = \bar{a} +_X \bar{b}$ and show that $s_{n^+} = \bar{a} +_X \bar{b}$ also holds. Since 2 is the beginning of the counting set $\mathbb{N}_+ \setminus \{1\}$, we evidently obtain the inequality $2 \leq n$ $[< n+1]$ with (4.184), and therefore $n+1 > 2$ with the Transitivity Formula for $\leq$ and $<$; thus, the definition of the sequence f yields $a_{n+1} = 0_X$. We therefore obtain the true equations

$$s_{n+1} = \sum_{i=1}^{n+1} a_i = \left(\sum_{i=1}^{n} a_i\right) +_X a_{n+1} = (\bar{a} +_X \bar{b}) +_X 0_X = \bar{a} +_X \bar{b}$$

using the definition of a series, (5.417), the induction assumption together with $a_{n+1} = 0_X$, and finally the definition of a neutral element. As n is arbitrary, we may infer from this the truth of the induction step, so that the proof by mathematical induction is complete.

Because X, $+_X$, $\bar{a}$ and $\bar{b}$ were arbitrary, we may further conclude that the proposed universal sentence a) is true. □

Exercise 5.55. Establish Part b) of Proposition 5.133.

(Hint: Proceed as in the proof of Proposition 5.98, using Corollary 4.49.)

Although the constant tail of the preceding series turns out to be an increasing/decreasing sequence, the underlying series is not necessarily monotone. In order for this to happen, we need make some further assumptions.

Proposition 5.134. *For any set X, any commutative addition $+_X$ on X and any reflexive partial ordering $\leq_X$ of X such that the zero element 0_X of X exists, such that $0_X \leq_X a$ holds for any $a \in X$ and such that the Monotony Law for $+$ and $\leq$ is satisfied, it is true that the series $(\sum_{i=1}^n a_i)_{n\in\mathbb{N}_+}$ generated by any sequence $(a_n)_{n\in\mathbb{N}_+}$ in X is increasing.*

Proof. We take an set X, an arbitrary commutative addition $+_X$ on X, an arbitrary reflexive partial ordering $\leq_X$ of X, and an arbitrary sequence $(a_n)_{n\in\mathbb{N}_+}$ in X, where we assume that the inequality $0_X \leq_X a$ concerning the zero element of X holds for any $a \in X$, and where we assume in addition the Monotony Law for $+$ and $\leq$ to apply. Thus, f defines the series $s = (s_n)_{n\in\mathbb{N}_+} = (\sum_{i=1}^n a_i)_{n\in\mathbb{N}_+}$. Next, we apply the Monotony Criterion for increasing sequences in connection with Exercise 4.29 and Proposition 5.48 to prove that the series s is increasing, that is, we verify

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow s_n \leq_X s_{n+1}). \tag{5.515}$$

For this purpose, we let $n \in \mathbb{N}_+$ be arbitrary and consider the restriction $(a_i \,|\, i \in \{1,\ldots,n+1\})$ of f. Since the terms a_{n+1} and $\sum_{i=1}^n a_i$ are both elements of X, they satisfy $0 \leq_X a_{n+1}$ and $0 \leq_X \sum_{i=1}^{n} a_i$ by assumption. Applying now the assumed monotony law, we obtain

$$0_X +_X \sum_{i=1}^{n} a_i \leq_X a_{n+1} +_X \sum_{i=1}^{n} a_i. \tag{5.516}$$

In view of the definition of a neutral element, the commutativity of $+_X$ and Corollary 5.109e), we may now write the preceding inequality as

$$\sum_{i=1}^{n} a_i \leq_X \sum_{i=1}^{n+1} a_i, \tag{5.517}$$

and then also as the desired $s_n \leq_X s_{n+1}$ by using the definition of a series. Since n was arbitrary, we may therefore conclude that (5.515) is true, which implies that the series $(s_n)_{n\in\mathbb{N}_+}$ is increasing. As X, $+_X$, $\leq_X$ and f were initially arbitrary, we may infer from this finding the truth of the proposition. □

Note 5.30. The preceding proposition holds in particular for every ordered elementary domain.

Proposition 5.135. *For any set X, any commutative addition $+_X$ on X and any reflexive partial ordering $\leq_X$ of X such that the zero element 0_X of X exists, such that $0_X \leq_X a$ holds for any $a \in X$ and such that the Monotony Law for $+$ and $\leq$ is satisfied, and for any elements $\bar{a}, \bar{b} \in X$, it is true that the series generated by the sequence $f = (a_n)_{n\in\mathbb{N}}$ in X defined by (5.361) converges increasingly to the sum of $\bar{a}$ and $\bar{b}$, i.e.*

$$\sum_{i=1}^{\infty} a_n = \bar{a} +_X \bar{b}. \tag{5.518}$$

Proof. Letting X, $\leq_X$, $+_X$, $\bar{a}$ and $\bar{b}$ be arbitrary, assuming $+_X$ to be a commutative addition on X where X contains the zero element 0_X, assuming $(X, \leq_X)$ to be a partially ordered set satisfying $0_X \leq_X a$ for any $a \in X$ as well as the Monotony Law for $+$ and $\leq$, and assuming X to contain $\bar{a}$ as well as $\bar{b}$, we may define the sequence $f = (a_n)_{n\in\mathbb{N}_+}$ in X according to (5.361). Then, the series $s = (s_n)_{n\in\mathbb{N}_+}$ with $s_n = \sum_{i=1}^{n}$ for any $n \in \mathbb{N}_+$ is also defined, and this series is increasing in view of Proposition 5.134. Furthermore, the 2nd tail T_2^s of this series is a constant sequence with value $\bar{a} +_X \bar{b}$, as shown in Proposition 5.133. Therefore, the tail T_2^s converges increasingly to the sum $L = \bar{a} +_X \bar{b}$ in view of Proposition 4.76. Because the underlying sequence s is increasing, we may then apply the Tail Criterion for convergence of an increasing sequence to infer from the previous findings that s converges increasingly to the same limit L, i.e. $L = \lim_{n\to\infty} s_n$, which we may then write also as (5.518) by using Notation 5.22. Since X, $\leq_X$, $+_X$, $\bar{a}$ and $\bar{b}$ were initially arbitrary, we may now finally conclude that the proposition holds. $\square$

In case a series does not have a constant tail, convergence of the series can be effected via the assumption of a complete lattice. Combining the findings of the preceding exercise and of Proposition 4.75 immediately gives us the following result.

Corollary 5.136. *For any sets X, $+_X$, $\leq_X$ and f such that*

1. *$+_X$ is a commutative addition on X,*
2. *X contains the zero element 0_X,*
3. *$(X, \leq_X)$ is a complete lattice satisfying $0_X \leq_X a$ for any $a \in X$ as well as the Monotony Law for $+$ and $\leq$,*
4. *f is a sequence $(a_n)_{n\in\mathbb{N}_+}$ in X,*

it is true that the generated series $s = (s_n)_{n\in\mathbb{N}_+} = (\sum_{i=1}^{n} a_i)_{n\in\mathbb{N}_+}$ is increasingly convergent.

5.6.4. Multiples and powers of natural numbers

Proposition 5.137. *Any natural number n is identical with the n-fold sum of the natural number 1, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow \sum_{i=1}^{n} 1 = n). \tag{5.519}$$

Proof. We carry out a proof by mathematical induction. In the base case ($n = 0$), we obtain

$$\sum_{i=1}^{n} 1 = \sum_{i=1}^{0} 1 = 0 = n$$

by applying substitution based on $n = 0$, then (5.409), and finally again substitution based on $n = 0$.

Regarding the induction step, we let $n \in \mathbb{N}$ be arbitrary, make the induction assumption $\sum_{i=1}^{n} 1 = n$, and show that this implies $\sum_{i=1}^{n+1} 1 = n + 1$. Indeed, we obtain the equations

$$\sum_{i=1}^{n+1} 1 = \left(\sum_{i=1}^{n} 1\right) + 1 = n + 1$$

with (5.417) and the induction assumption. Since n was arbitrary, we may therefore conclude that the proposed universal sentence (5.519) is true. □

We then obtain the following result immediately with Notation 5.20.

Corollary 5.138. *The following sentence holds for the semigroup* $(\mathbb{N}, +)$.

$$\forall n\,(n \in \mathbb{N} \Rightarrow n1 = n). \tag{5.520}$$

As a first application of the Generalized Distributive Law for semirings, we establish the fact that multiples of natural numbers may be view as products.

Proposition 5.139. *It is true that*

$$\forall k, n\,(k, n \in \mathbb{N} \Rightarrow nk = n \cdot_{\mathbb{N}} k). \tag{5.521}$$

Proof. Letting $k, n \in \mathbb{N}$ be arbitrary, we obtain the true equations

$$nk = \sum_{i=1}^{n} k = \sum_{i=1}^{n} (k \cdot_{\mathbb{N}} 1) = k \cdot_{\mathbb{N}} \sum_{i=1}^{n} 1 = k \cdot_{\mathbb{N}} n = n \cdot_{\mathbb{N}} k$$

by applying Notation 5.20, the fact that the sequences $(k \,|\, i \in \{1, \ldots, n\})$ and $(k \cdot_{\mathbb{N}} 1 \,|\, i \in \{1, \ldots, n\})$ are identical according to the Equality Criterion for functions because $k = k \cdot_{\mathbb{N}} 1$ holds (for any $i \in \{1, \ldots, n\}$) by virtue of the property of the neutral element 1 with respect to the multiplication on $\mathbb{N}$, the Generalized Distributive Law for semirings, (5.519), and the commutativity of $\cdot_{\mathbb{N}}$. Since k and n were initially arbitrary, we may therefore conclude that the proposition holds, as claimed. □

We see now that any natural number may be written as the sum of a multiple and a 'remainder' in a unique way.

Theorem 5.140 (Division of a natural number with remainder). *For any natural number $g \geq 2$ and for any natural number n, it is true that there exist unique natural numbers q and $r < g$ such that n is the sum of the multiple qg and r, i.e.*

$$\forall n\,(n \in \mathbb{N} \Rightarrow \exists! q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n = qg + r)). \tag{5.522}$$

Proof. We first let $g \in \mathbb{N}$ be arbitrary and assume $g \geq 2$ to hold. Evidently, we may write (5.522) also as

$$\begin{aligned} \forall n\,(n \in \mathbb{N} \Rightarrow [&\exists q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n = qg + r) \\ &\wedge\, \forall q, r, q', r'\,([q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n = qg + r \\ &\quad \wedge\, q', r' \in \mathbb{N} \wedge\, r' < g \,\wedge\, n = q'g + r'] \Rightarrow [q = q' \,\wedge\, r = r'])]). \end{aligned} \tag{5.523}$$

by using Notation 1.4. Next, we observe in light of the Distributive Law for sentences (1.46), the Distributive Law for quantification (1.74) and (5.521) that the preceding sentence is equivalent to the conjunction

$$\begin{aligned} &\forall n\,(n \in \mathbb{N} \Rightarrow \exists q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n = q \cdot g + r)) \\ &\quad \wedge\, \forall n\,(n \in \mathbb{N} \Rightarrow \forall q, r, q', r'\,([q, r \in \mathbb{N} \wedge\, r < g \,\wedge\, n = q \cdot g + r \\ &\qquad \wedge\, q', r' \in \mathbb{N} \wedge\, r' < g \,\wedge\, n = q' \cdot g + r'] \Rightarrow [q = q' \,\wedge\, r = r'])). \end{aligned} \tag{5.524}$$

To verify the first part

$$\forall n\,(n \in \mathbb{N} \Rightarrow \exists q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n = q \cdot g + r)), \tag{5.525}$$

we carry out a proof by mathematical induction. Considering the base case ($n = 0$), we observe the truth of $0 \in \mathbb{N}$, of $0 < (2 \leq) g$ (in view of (4.168) and the Transitivity Formula for $<$ and $\leq$) and of $0 \cdot g + 0 = 0$, so that the existential sentence

$$\exists q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, 0 = q \cdot g + r)$$

holds. Then, to prove the induction step

$$\begin{aligned} &\forall n\,(n \in \mathbb{N} \Rightarrow [\exists q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n = q \cdot g + r) \\ &\qquad \Rightarrow \exists q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n + 1 = q \cdot g + r)]), \end{aligned}$$

we let $n \in \mathbb{N}$ be arbitrary, we make the induction assumption

$$\exists q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n = q \cdot g + r),$$

and we show that this implies

$$\exists q, r\,(q, r \in \mathbb{N} \,\wedge\, r < g \,\wedge\, n + 1 = q \cdot g + r). \tag{5.526}$$

Because of the induction assumption, there are constants, say $\overline{q}$ and $\overline{r}$ which satisfy $\overline{q}, \overline{r} \in \mathbb{N}$, $\overline{r} < g$, and the equation

$$n = \overline{q} \cdot g + \overline{r}. \tag{5.527}$$

We then obtain the equations

$$\begin{aligned} n + 1 &= (\overline{q} \cdot g + \overline{r}) + 1 \\ &= \overline{q} \cdot g + (\overline{r} + 1) \end{aligned} \tag{5.528}$$

by applying substitution based on (5.527) and then the associativity of the addition on $\mathbb{N}$. Because of (4.157) and (5.217), the inequality $\overline{r} < g$ implies

$\overline{r}+1 \leq g$. Consequently, the disjunction $\overline{r}+1 < g \vee \overline{r}+1 = g$ is true by definition of an induced reflexive partial ordering, which we now use to prove the existential sentence (5.526) by cases.

In the first case of $\overline{r}+1 < g$, the constants $\overline{q}$ and $\overline{r}+1$ satisfy the conjunction of $\overline{q}, \overline{r}+1 \in \mathbb{N}$, $\overline{r}+1 < g$ and $n+1 = \overline{q}\cdot g + (\overline{r}+1)$, so that the existential sentence (5.526) is clearly true in the first case.

In the other case of $\overline{r}+1 = g$, we obtain the equations

$$\begin{aligned} n+1 &= \overline{q}\cdot g + (\overline{r}+1) \\ &= \overline{q}\cdot g + g \\ &= g\cdot\overline{q} + g\cdot 1 \\ &= g\cdot(\overline{q}+1) \\ &= (\overline{q}+1)\cdot g \\ &= (\overline{q}+1)\cdot g + 0. \end{aligned}$$

with (5.528), by applying substitution based on the current case assumption, then the commutativity of the multiplication on $\mathbb{N}$ together with the definition of a neutral element, subsequently the distributivity of the multiplication on $\mathbb{N}$ over the addition on $\mathbb{N}$, then again the commutativity of the multiplication on $\mathbb{N}$, and finally again the definition of a neutral element. Thus, the constants $\overline{q}+1$ and 0 evidently satisfy $(\overline{q}+1), 0 \in \mathbb{N}$, $0 < g$ and $n+1 = (\overline{q}+1)\cdot g + 0$, so that the existential sentence (5.526) is true also for the second case. Since n was arbitrary, we may therefore conclude that the induction step holds, besides the base case, so that the proof of (5.525) by mathematical induction is complete.

We now establish the second part

$$\begin{aligned} &\forall n\,(n \in \mathbb{N} \Rightarrow \forall q, r, q', r'\,([q, r \in \mathbb{N} \wedge r < g \,\wedge\, n = q\cdot g + r \\ &\quad \wedge\, q', r' \in \mathbb{N} \wedge r' < g \,\wedge\, n = q'\cdot g + r'] \Rightarrow [q = q' \,\wedge\, r = r'])). \qquad (5.529) \end{aligned}$$

of the conjunction (5.524). For this purpose, we let $n \in \mathbb{N}$ be arbitrary and let then q, r, q' and r' also be arbitrary such that the conjunction of $q, r \in \mathbb{N}$, $r < g$, $n = q\cdot g + r$, $q', r' \in \mathbb{N}$, $r' < g$ and $n = q'\cdot g + r'$ is true. Here, we may write the previous two equations for n also as

$$\begin{aligned} r + q\cdot g &= n, \\ r' + q'\cdot g &= n, \end{aligned}$$

applying the commutativity of the addition on $\mathbb{N}$. These equations in turn show in light of Note 5.28 that the differences

$$q\cdot g = n - r, \qquad (5.530)$$

$$q'\cdot g = n - r' \qquad (5.531)$$

are defined because the inequalities $r \leq n$ and $r' \leq n$ hold.

As the standard linear ordering of $\mathbb{N}$ is in particular connex, we have the true disjunction

$$q < q' \lor q' < q \lor q = q'. \tag{5.532}$$

We now carry out two proofs by contradiction to demonstrate that the negations of the first and second part of that disjunction are true.

Regarding the first part, we assume the negation of $\neg q < q'$ to hold, so that $q < q'$ follows to be true with the Double negation Law. Then, the disjunction $q < q' \lor q = q'$ also holds, which gives $q \leq q'$ by definition of a reflexive partial ordering. Thus, the difference $q' - q$ is defined, so that we may form the product of the natural number $q' - q$ and the natural number g. We then obtain the equations

$$\begin{aligned}(q' - q) \cdot g &= g \cdot (q' - q) \\ &= (g \cdot q') - (g \cdot q) \\ &= (q' \cdot g) - (q \cdot g) \\ &= (n - r') - (n - r)\end{aligned}$$

by applying the commutativity of the multiplication on $\mathbb{N}$, the Distributive Law for the difference of natural numbers, again the commutativity of the multiplication on $\mathbb{N}$, and then substitutions based on the equations (5.530) – (5.531). The last difference shows that $n - r \leq n - r'$ holds, and we may now prove by contradiction that the inequality $r' \leq r$ is also true. For this purpose, we assume $\neg r' \leq r$ to be true, which yields $r < r'$ with the Negation Formula for $\leq$. Together with the previously established inequality $r' \leq n$, this implies $n - r' < n - r$ with (5.354), and therefore $\neg n - r \leq n - r'$ with the Negation Formula for $\leq$. Evidently, we obtained a contradiction, so that $r' \leq r$ is indeed true. Thus, the conjunction $r \leq n \land r' \leq r$ holds, so that we may apply (5.348) to obtain the equation

$$[(q' - q) \cdot g =] \quad (n - r') - (n - r) = r - r'.$$

The previously established inequality $q < q'$ implies $0 < q' - q$ with (5.352) and then $(0^+ =)\ 1 \leq q' - q$ with (4.157). Because of $0 < g$, we may now use the Monotony Law for $\cdot$ and $\leq$ to obtain

$$1 \cdot g \leq (q' - q) \cdot g \quad [= r - r'],$$

that is, $g \leq r - r'$ (in view of the definition of a neutral element). Furthermore, the fact $0 \leq r'$ implies $0 + (r - r') \leq r' + (r - r')$ with the Monotony Law for $+$ and $\leq$, which we may write also as

$$r - r' \leq (r - r') + r'.$$

by applying the definition of a neutral element and the commutativity of the addition on $\mathbb{N}$. Here, $(r - r') + r' = r$ holds according to (5.343), where we initially assumed $r < g$ to be true. Consequently, we obtain the inequalities $r - r' \leq r < g$, which give $r - r' < g$ by means of the Transitivity Formula for $\leq$ and $<$, and then $\neg g \leq r - r'$ with the Negation Formula for $<$. This finding contradicts the previously established inequality $g \leq r - r'$, so that the proof of $\neg q < q'$ is complete.

Regarding the second part of the disjunction (5.532), we now assume the negation of $\neg q' < q$, so that the Double Negation Law yields now $q' < q$. We may actually apply exactly the same line of arguments to obtain a contradiction, where the roles of q and q' as well as the roles of r and r' are interchanged. To begin with, the preceding inequality evidently gives $q' \leq q$, so that the difference $q - q'$ and then the product $(q - q') \cdot g$ are defined. Clearly, we obtain then the true equations

$$\begin{aligned}(q - q') \cdot g &= g \cdot (q - q') \\ &= (g \cdot q) - (g \cdot q') \\ &= (q \cdot g) - (q' \cdot g) \\ &= (n - r) - (n - r').\end{aligned}$$

The difference $(n - r) - (n - r')$ reveals the inequality $n - r' \leq n - r$, and we may show via contradiction also that $r \leq r'$ holds. Indeed, the assumption $\neg r \leq r'$ gives $r' < r$, which implies together with $r \leq n$ the truth of $n - r < n - r'$, so that $\neg n - r' \leq n - r$ holds, in contradiction to the previously obtained $n - r' \leq n - r$. Thus, the proof of $r \leq r'$ is complete, so that the conjunction $r' \leq n \wedge r \leq r'$ is true. Then, we may infer from this finding the truth of the equations

$$[(q - q') \cdot g =] \quad (n - r) - (n - r') = r' - r.$$

Let us now observe that $q' < q$ gives $0 < q - q'$ and subsequently $(0^+ =) \; 1 \leq q - q'$, which further implies (with $0 < g$)

$$1 \cdot g \leq (q - q') \cdot g \quad [= r' - r],$$

that is, $g \leq r' - r$. Moreover, $0 \leq r$ implies $0 + (r' - r) \leq r + (r' - r)$, so that

$$r' - r \leq (r' - r) + r,$$

where $(r' - r) + r = r'$ and $r' < g$ hold. Thus, we have $r' - r \leq r' < g$, which inequalities result in $r' - r < g$, with the consequence that the negation $\neg g \leq r' - r$ is true. We therefore obtained a contradiction, which finding completes the proof of $\neg q' < q$.

We thus showed that the first and the second part of the disjunction (5.532) are false, so that its third part $q = q'$ is true, proving the first part of the desired conjunction $q = q' \wedge r = r'$ in (5.529). Let us now apply substitution based on the equation $q = q'$ to write the initially assumed equation $n = q' \cdot g + r'$ as $n = q \cdot g + r'$, and let us next combine the latter with the other assumed equation $n = q \cdot g + r$ to obtain

$$q \cdot g + r = q \cdot g + r'.$$

Then, an application of the Cancellation Law for $+_{\mathbb{N}}$ yields already the desired equation $r = r'$. Thus, the proof of the conjunction $q = q' \wedge r = r'$, and thus the proof of the second implication in (5.529), is complete. Because q, r, q' and r' are arbitrary, we may therefore conclude that the universal sentence with respect to these variables is true, which proves in turn the implication based in the antecedent $n \in \mathbb{N}$. Since n is arbitrary, we may now further conclude that (5.529) holds, and this completes the proof of the conjunction (5.524). As this sentence is equivalent to (5.522), we may infer from that conjunction the truth of the stated theorem. $\square$

Note 5.31. In the equation $n = qg + r$, we call the natural number

- n the *dividend*,
- g the *divisor*,
- q the *quotient*, and
- r the *remainder*.

We also say that n divides by g with remainder r.

The following result leads to the distinction of 'even' and 'odd' natural numbers.

Proposition 5.141. *It is true that any natural number n divides by 2 either with remainder 0 or with remainder 1, i.e.*

$$\begin{aligned}\forall n\,(n \in \mathbb{N} \Rightarrow ([\exists q\,(q \in \mathbb{N} \wedge n = q2) \vee \exists q\,(q \in \mathbb{N} \wedge n = q2 + 1)]\\ \wedge \neg[\exists q\,(q \in \mathbb{N} \wedge n = q2) \wedge \exists q\,(q \in \mathbb{N} \wedge n = q2 + 1)])).\end{aligned} \tag{5.533}$$

Proof. We let $n \in \mathbb{N}$ be arbitrary, so that there exist because of Theorem 5.140 unique natural numbers $\bar{q}$ and $\bar{r} < 2$ satisfying

$$n = \bar{q}2 + \bar{r}. \tag{5.534}$$

Here, $r \in \mathbb{N}$ implies $0 \leq \bar{r}$ with (4.187), so that the disjunction $0 < \bar{r} \vee 0 = \bar{r}$ follows to be true with the Characterization of an induced irreflexive partial ordering. We use this disjunction to prove the existential sentence in (5.533).

The first case $0 < \bar{r}$ implies $[1 =]\ 0^+ \leq \bar{r}$ with (2.291) and (4.157). Because of (2.292), we may write $\bar{r} < 2$ also as $\bar{r} < 1^+$. Then, the conjunction of $1 \leq \bar{r}$ and $\bar{r} < 1^+$ implies $\bar{r} = 1$ with (4.173). Due to this equation, (5.534) yields $n = \bar{q}2 + 1$ via substitution, which shows in light of $\bar{q} \in \mathbb{N}$ that the existential sentence $\exists q\,(q \in \mathbb{N} \wedge n = q2+1)$ holds. The disjunction

$$\exists q\,(q \in \mathbb{N} \wedge n = q2) \vee \exists q\,(q \in \mathbb{N} \wedge n = q2 + 1) \tag{5.535}$$

is then also true, proving the first part of the conjunction in (5.533) for the first case.

The second case $0 = \bar{r}$ gives us $m = \bar{q}2 + 0$ by means of substitution in (5.534), so that the existential sentence $\exists q\,(q \in \mathbb{N} \wedge n = q2)$ holds now. Consequently, the disjunction (5.535) and thus the first part of the conjunction in (5.533) hold also for the second case.

It now remains for us to establish the negation in (5.533), which we prove by contradiction. Assuming the negation of that negation to be true, it follows with the Double Negation Law that there is a natural number, say q', with $n = q'2$, and that there is a natural number, say q'', satisfying $n = q''2 + 1$. The first of these equations clearly shows that the remainder is $r' = 0$, whereas the remainder in the other equation is $r'' = 1$. Since the quotient for n is unique (according to Theorem 5.140), we obtain $q' = q''$, so that substitution gives $n = q'2 + 1$. Since $n = q'2 + 0$ also holds, and because the remainder for the division of n by 2 is unique, we also obtain $1 = 0$, in contradiction to the fact $0 \neq 1$ shown in (4.165). This finding completes the proof of the negation in (5.533), and as n was arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Note 5.32. By convention, we will write for instance $2q$ instead of $q2$, putting the explicit natural number in front.

The preceding proposition gives rise to the next definition.

Definition 5.27 (Even & odd natural number). We say that a natural number n

(1) is *even* iff n divides by 2 with remainder 0, i.e. iff

$$\exists q\,(q \in \mathbb{N} \wedge n = 2q). \tag{5.536}$$

(2) is *odd* iff n divides by 2 with remainder 1, i.e. iff

$$\exists q\,(q \in \mathbb{N} \wedge n = 2q + 1). \tag{5.537}$$

Proposition 5.142. *Any power of* 3 *is odd, that is,*

$$\forall n\,(n \in \mathbb{N} \Rightarrow \exists q\,(q \in \mathbb{N} \wedge 3^n = 2q+1)). \tag{5.538}$$

Proof. We apply a proof by mathematical induction. We obtain for the base case ($n = 0$) the equations

$$3^n = 3^0 = 1 = 0+1 = 2 \cdot 0 + 1 = 2n+1$$

by applying substitution, (5.476), the definition of a neutral element, the Cancellation Law for 0, and finally again substitution together with (5.521). Thus, the existential sentence in (5.538) is clearly true for $n = 0$.

To establish the induction step, we let $n \in \mathbb{N}$ be arbitrary, assume that there exists a natural number, say $\bar{q}$, such that $3^n = 2\bar{q}+1$ holds, and show that this implies the existential sentence

$$\exists q\,(q \in \mathbb{N} \wedge 3^{n+1} = 2q+1). \tag{5.539}$$

Let us now observe the truth of the equations

$$\begin{aligned}
3^{n+1} &= 3^n + 3 \\
&= (2\bar{q}+1) + 3 \\
&= 3 \cdot 2\bar{q} + 3 \cdot 1 \\
&= 3 \cdot (2 \cdot \bar{q}) + (2+1) \cdot 1 \\
&= 2 \cdot (3 \cdot \bar{q}) + (2 \cdot 1 + 1) \\
&= 2 \cdot (3 \cdot \bar{q} + 1) + 1 \\
&= 2(3 \cdot \bar{q} + 1) + 1
\end{aligned}$$

using (5.480), the induction assumption, the commutativity of $+_{\mathbb{N}}$ together with the distributivity of $\cdot_{\mathbb{N}}$ over $+_{\mathbb{N}}$, (5.521) alongside (2.293) in connection with (5.217), the associativity and commutativity of $\cdot_{\mathbb{N}}$ jointly with (5.475), the associativity of $+_{\mathbb{N}}$ together with the distributivity of $\cdot_{\mathbb{N}}$ over $+_{\mathbb{N}}$, and finally again (5.521). Because $3 \cdot \bar{q} + 1$ is a natural number, the desired existential sentence (5.539) holds, and since n was arbitrary, we may therefore conclude that the induction step is true, besides the base case. We thus completed the proof of the proposed universal sentence (5.538). □

Exercise 5.56. Show for any positive natural number n that the n-th power of 2 is even, that is,

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \exists q\,(q \in \mathbb{N} \wedge 2^n = 2q)). \tag{5.540}$$

(Hint: Use some of the arguments in the proof of Proposition 5.142.)

5.6.5. Laws for powers in ordered elementary domains

Proposition 5.143. *For any ordered elementary domain $(X,+,\cdot,<)$ it is true that any element in X strictly between the zero and the unity element is greater than its square, i.e.*

$$\forall a\,([a \in X \ \wedge\ 1_X < a] \Rightarrow a < a^2). \tag{5.541}$$

Proof. We let X, $+$, $\cdot$ and $<$ be arbitrary sets, assume that $(X,+,\cdot,<)$ is an ordered elementary domain, let a also be arbitrary, and assume moreover $a \in X$ and $1_X < a$ to be true. Recalling the truth of $1_X \neq 0_X$ in view of the Distinctness of the zero element and the unity element in ordered elementary domains, we now obtain with (5.275) the inequality $0_X < 1_X$. Then, we see in light of the Linear ordering of ordered elementary domains that $<$ is transitive, so that $0_X < 1_X$ and $1_X < a$ imply $0_X < a$. Based on the evident fact $0_X, 1_X, a \in X$ and the inequalities $0_X < a$ and $1_X < a$, the Monotony Law for $\cdot$ and $<$ yields $1_X \cdot a < a \cdot a$, which we may write as $a < a^2$ by applying the definition of a neutral element and (5.478). This inequality proves the implication in (5.541), and since a is arbitrary, we may therefore conclude that the universal sentence (5.541) holds. Then, as X, $+$, $\cdot$ and $<$ were initially arbitrary sets, we may further conclude that the proposition is true. □

Exercise 5.57. Prove the following sentences for any ordered elementary domain $(X,+,\cdot,<)$.

$$\forall a\,([a \in X \ \wedge\ 0_X < a < 1_X] \Rightarrow a^2 < a). \tag{5.542}$$

$$\forall a\,([a \in X \ \wedge\ 0_X < a \leq 1_X] \Rightarrow a^2 \leq a). \tag{5.543}$$

$$\forall a\,([a \in X \ \wedge\ 1_X \leq a] \Rightarrow a \leq a^2). \tag{5.544}$$

(Hint: Use similar arguments as in the proof of Proposition 5.143, using also (5.293) and (3.232).)

Lemma 5.144. *The following sentences are true for any ordered elementary domain $(X,+,\cdot,<)$.*

$$\forall a,b\,([a,b \in X \ \wedge\ a < b] \Rightarrow \forall n\,(n \in \mathbb{N}_+ \Rightarrow a^n < b^n)), \tag{5.545}$$

$$\forall a,m\,([a \in X \ \wedge\ 1_X < a \ \wedge\ m \in \mathbb{N}] \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow [m <_{\mathbb{N}} n \Rightarrow a^m < a^n])). \tag{5.546}$$

Proof. First we let X, $+$, $\cdot$ and $<$ be arbitrary and assume that $(X,+,\cdot,<)$ is an ordered elementary domain.

Concerning (5.545), we let also a and b be arbitrary and assume moreover that a and b are elements of X satisfying $a < b$. Let us observe that

(5.276) gives the disjunction $0_X < a \lor a = 0_X$ and therefore $0_X \leq a$ by definition of the total ordering $\leq$ induced by the linear ordering $<$ of X; the conjunction of the preceding inequality and the assumed inequality $a < b$ now implies $0_X < b$ with the Transitivity Formula for $\leq$ and $<$. Next, we prove the universal sentence with respect to n by carrying out mathematical induction. Regarding the base case ($n = 1$), we simply observe that we may write the assumed $a < b$ also as $a^1 < b^1$ because of (5.477), so that the desired inequality $a^n < b^n$ holds for $n = 1$.

Concerning the induction step, we let $n \in \mathbb{N}_+$ be arbitrary, make the induction assumption $a^n < b^n$, and show that this implies $a^{n+1} < b^{n+1}$. We prove this inequality by cases based on the disjunction $0_X < a^n \lor a^n = 0_X$, which is true because of (5.276) and the fact that a^n is an element of X by definition of an n-th power/n-fold binary operation (on X).

In the first case $0_X < a^n$, we have in light of the Monotony Law for $\cdot$ and $<$ that $a < b$ implies $a \cdot a^n < b \cdot a^n$, which yields with the commutativity of the multiplication $a^n \cdot a < a^n \cdot b$.

$$a^n \cdot a < a^n \cdot b. \tag{5.547}$$

Then, the previously established inequality $0_X < b$ implies together with the induction assumption $a^n < b^n$ the truth of the inequality

$$a^n \cdot b < b^n \cdot b \tag{5.548}$$

using again the Monotony Law for $\cdot$ and $<$. Since the linear ordering $<$ of X is transitive, the conjunction of the true inequalities (5.547) and (5.548) implies $a^n \cdot a < b^n \cdot b$, and therefore $a^{n+1} < b^{n+1}$ with (5.480), as desired.

In the other case $a^n = 0_X$, we obtain on the one hand the equations

$$\begin{aligned} a^{n+1} &= a^n \cdot a \\ &= 0_X \cdot a \\ &= 0_X \end{aligned} \tag{5.549}$$

by applying (5.480), substitution and the Cancellation Law for 0_X. On the other hand, the induction assumption $a^n < b^n$ gives via substitution $0_X < b^n$. This finding implies with the previously obtained $0_X < b$ and the Monotony Law for $\cdot$ and $<$ the true inequality $0_X \cdot b < b^n \cdot b$ and consequently $0_X < b^{n+1}$ with the Cancellation Law for 0_X and with (5.480). Now, substitution based on the equation (5.549) yields the desired $a^{n+1} < b^{n+1}$ again, so that the proof by cases is complete. As n was arbitrary, we may therefore conclude that the induction step is true (besides the base case), so that the proof by mathematical induction is also complete. Since a and

b were arbitrary, the universal sentence (5.545) follows then to be true.

Concerning (5.546), we take arbitrary a and m, assume $a \in X$, $1_X < a$ and $m \in \mathbb{N}$ to be true, and apply then a proof by mathematical induction. To establish the base case $n = 0$, we observe that $m <_{\mathbb{N}} 0$ is false according to (4.188), so that the implication $m <_{\mathbb{N}} 0 \Rightarrow a^m < a^0$ is true (having a false antecedent). To prove the induction step, we let $n \in \mathbb{N}$ be arbitrary, make the induction assumption

$$m <_{\mathbb{N}} n \Rightarrow a^m < a^n, \tag{5.550}$$

and we show that

$$m <_{\mathbb{N}} n + 1 \Rightarrow a^m < a^{n+1}, \tag{5.551}$$

follows to be true. To do this, we assume $m <_{\mathbb{N}} n + 1$ to be true, which implies $m+1 \leq_{\mathbb{N}} n+1$ with (4.157) and (5.217), so that we obtain $m \leq_{\mathbb{N}} n$ according to (4.159). By definition of an induced reflexive partial ordering, the preceding inequality yields now $m <_{\mathbb{N}} n \vee m = n$, which disjunction we use to prove the desired consequent $a^m < a^{n+1}$ by cases. Before considering the two cases, we observe that the fact $0_X < 1_X$ (see Corollary 5.69) and the assumed $1_X < a$ imply $0_X < a$ with the transitivity of the linear ordering $<$ of X. This inequality in turn implies $0_X \neq a$ with (5.275), which finding gives (with $a \in X$ and $n \in \mathbb{N}$) the inequality $a^n \neq 0_X$ because of (5.495), and therefore $0_X < a^n$ again with (5.275). Consequently, we may apply the Monotony Law for $\cdot$ and $<$ to infer from the assume $1_X < a$ the truth of $1_X \cdot a^n < a \cdot a^n$, which we may write also as $a^n < a^n \cdot a$ by using the definition of a neutral element as well as the commutativity of the multiplication on X. Then, this inequality yields $a^n < a^{n+1}$ with (5.480).

Now, the first case $m <_{\mathbb{N}} n$ implies $a^m < a^n$ with the induction assumption (5.550); since $a^n < a^{n+1}$ also holds, the transitivity of $<$ gives $a^m < a^{n+1}$. In the other case of $m = n$, we obtain via substitution $a^m = a^n$ ($< a^{n+1}$), so that $a^m < a^{n+1}$ is again true. Thus, the proof by cases is already complete, and the truth of $a^m < a^{n+1}$ establishes the truth of the implication (5.551). As n is arbitrary, we may therefore conclude that the induction step holds, completing the proof of the universal sentence with respect to n in (5.546). Since m and a are also arbitrary, we may further conclude that thhe universal sentence (5.546) is true.

Because X, $+$, $\cdot$ and $<$ were initially arbitrary sets, we may now finally conclude that the lemma holds. □

Corollary 5.145. *The following sentence is true for any ordered elementary domain* $(X, +, \cdot, <)$*.*

$$\forall a, b\,([a, b \in X \,\wedge\, a \leq b] \Rightarrow \forall n\,(n \in \mathbb{N}_+ \Rightarrow a^n \leq b^n)). \tag{5.552}$$

Proof. Letting X, $+$, $\cdot$, $<$, a and b be arbitrary such that $(X, +, \cdot, <)$ is an ordered elementary domain and such that $a, b \in X$ as well as $a \leq b$ holds, and letting n be arbitrary such that $n \in \mathbb{N}_+$ is true, we first observe that the assumption $a \leq b$ implies the truth of the disjunction $a < b \lor a = b$. On the one hand, if $a < b$ is true, then we obtain $a^n < b^n$ with Lemma 5.144; consequently, the disjunction $a^n < b^n \lor a^n = b^n$ is also true, which in turn gives the desired inequality $a^n \leq b^n$. On the other hand, if $a = b$ is true, then we obtain $a^n = b^n$ via substitution, so that the disjunction $a^n < b^n \lor a^n = b^n$ is again true. Thus, $a^n \leq b^n$ holds in both cases, and since n is arbitrary, we may therefore conclude that the universal sentence with respect to n holds. Then, as a and b are also arbitrary, we may further conclude that the universal sentence (5.552) is true. Because X, $+$, $\cdot$ and $<$ are arbitrary as well, we may finally conclude that the corollary holds. □

Exercise 5.58. Verify for any ordered elementary domain $(X, +, \cdot, <)$ the following sentence.

$$\forall a, m\,([a \in X \land 1_X \leq a \land m \in \mathbb{N}] \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow [m \leq_{\mathbb{N}} n \Rightarrow a^m \leq a^n])). \tag{5.553}$$

(Hint: Proceed in analogy to the proof of Corollary 5.145.)

Theorem 5.146 (Monotony Laws for the base & exponent and for $<$ & $\leq$ in ordered elementary domains). *The following laws hold for any ordered elementary domain $(X, +, \cdot, <)$.*

a) **Monotony Law for the base and $<$:**

$$\forall a, b, n\,([a, b \in X \land n \in \mathbb{N}_+] \Rightarrow [a < b \Leftrightarrow a^n < b^n]). \tag{5.554}$$

b) **Monotony Law for the exponent and $<$:**

$$\forall a, m, n\,([a \in X \land 1_X < a \land m, n \in \mathbb{N}] \Rightarrow [m <_{\mathbb{N}} n \Leftrightarrow a^m < a^n]). \tag{5.555}$$

c) **Monotony Law for the base and $\leq$:**

$$\forall a, b, n\,([a, b \in X \land n \in \mathbb{N}_+] \Rightarrow [a \leq b \Leftrightarrow a^n \leq b^n]). \tag{5.556}$$

d) **Monotony Law for the exponent and $\leq$:**

$$\forall a, m, n\,([a \in X \land 1_X < a \land m, n \in \mathbb{N}] \Rightarrow [m \leq_{\mathbb{N}} n \Leftrightarrow a^m \leq a^n]). \tag{5.557}$$

Exercise 5.59. Prove Theorem 5.146.

(Hint: Apply proofs by contradiction using Lemma 5.144, Corollary 5.145 and Exercise 5.58 in connection with Theorem 3.77.)

Proposition 5.147. *The following law holds for any ordered elementary domain* $(X, +, \cdot, <)$*.*

$$\forall a, m, n\,([a \in X \wedge 1_X < a \wedge m, n \in \mathbb{N}] \Rightarrow [a^m = a^n \Leftrightarrow m = n]). \quad (5.558)$$

Proof. We let $X, +, \cdot, <, a, m$ and n be arbitrary, we assume that $(X, +, \cdot, <)$ is an ordered elementary domain, and we also assume $a \in X$, $1_X < a$ and $m, n \in \mathbb{N}$ to be true. We now prove the first part ('$\Rightarrow$') of the stated equivalence by contraposition, assuming $m \neq n$ to hold. Since the standard linear ordering of $\mathbb{N}$ is in particular connex, the preceding inequality shows that $m <_{\mathbb{N}} n \vee n <_{\mathbb{N}} m$ is true, which disjunction we now use to prove $a^m \neq a^n$ by cases. On the one hand, the case $m <_{\mathbb{N}} n$ gives $a^m < a^n$ with (5.555) in view of the previously made assumptions $a \in X$, $1_X < a$ and $m, n \in \mathbb{N}$. Then, since the linear ordering $<$ of X is comparable and satisfies thus Theorem 3.49, we may infer from $a^m < a^n$ the truth of the desired inequality $a^m \neq a^n$. On the other hand, the case $n <_{\mathbb{N}} m$ evidently yields $a^n < a^m$ and therefore $a^n \neq a^m$ (applying the same arguments as for the first case), so that $a^m \neq a^n$ follows again to be true. Thus, the proof of the implication $a^m = a^n \Rightarrow m = n$ by contraposition is complete.

Regarding the second part ('$\Leftarrow$') of the equivalence in (5.558), we simply observe that assuming $m = n$ to be true yields already the desired $a^m = a^n$ via substitution.

Then, since X, $+$, $\cdot$, $<$, a, m and n were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Exercise 5.60. Verify the following sentence for any ordered elementary domain $(X, +, \cdot, <)$.

$$\forall a, b, n\,([a, b \in X \wedge n \in \mathbb{N}_+] \Rightarrow [a^n = b^n \Rightarrow a = b]). \quad (5.559)$$

(Hint: Use similar arguments as in the proof of Proposition 5.147.)

Exercise 5.61. Show for any natural number $p > 1$, any natural number m and any function $c : \{1, \ldots, m+1\} \to \mathbb{N}$ that there exists a unique function s with domain $\{1, \ldots, m+1\}$ and values satisfying

$$\forall i\,(i \in \{1, \ldots, m+1\} \Rightarrow s(i) = c_i \cdot p^{i-1}), \quad (5.560)$$

and show that s is a sequence in $\mathbb{N}$.

(Hint: Use Function definition by replacement in connection with (4.276), Corollary 5.88, and (1.109).)

Theorem 5.148 (p-adic expansion of positive natural numbers). *It is true for any for any natural number $p > 1$ that*

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \exists! m, c\,(m \in \mathbb{N} \,\wedge\, c : \{1, \ldots, m+1\} \to \mathbb{N} \tag{5.561}$$
$$\wedge\, \forall i\,(i \in \{1, \ldots, m+1\} \Rightarrow c_i < p) \,\wedge\, c_{m+1} \neq 0 \,\wedge\, n = \sum_{i=1}^{m+1} [c_i \cdot p^{i-1}])).$$

Proof. We take an arbitrary natural number p, assuming $1 < p$ to be true, and we prove

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow [n < p \Rightarrow \exists! m, c\,(m \in \mathbb{N} \,\wedge\, c : \{1, \ldots, m+1\} \to \mathbb{N} \tag{5.562}$$
$$\wedge\, \forall i\,(i \in \{1, \ldots, m+1\} \Rightarrow c_i < p) \,\wedge\, c_{m+1} \neq 0 \,\wedge\, n = \sum_{i=1}^{m+1} [c_i \cdot p^{i-1}])]),$$

taking an arbitrary positive natural number n and assuming then $n < p$ to be true. Let us consider the natural number

$$\bar{m} = 0 \tag{5.563}$$

and the singleton $\bar{c} = \{(1, n)\}$. The latter constitutes a function from

$$\{1\} = \{1, \ldots, 1\} = \{1, \ldots, 0+1\} = \{1, \ldots, \bar{m}+1\}$$

to $\{n\}$ according to Corollary 3.156, using the definition of an initial segment of $\mathbb{N}_+$ and the fact that 0 is the neutral element of $\mathbb{N}$ with respect to the addition on $\mathbb{N}$. Here, we notice that $(1, n) \in \{(1, n)\}$ $[= \bar{c}]$ holds, which we may write in function/sequence notation as $n = \bar{c}_1$. As $n \in \mathbb{N}_+$ implies $\{n\} \subseteq \mathbb{N}+$ with (2.184) and since the inclusion $\mathbb{N}_+ \subseteq \mathbb{N}$ is true according to (2.308), we therefore obtain the inclusion $\{n\} \subseteq \mathbb{N}$ with (2.13), which shows that $\mathbb{N}$ is also a codomain of $\bar{c}$. We thus have

$$\bar{c} : \{1, \ldots, \bar{m}+1\} \to \mathbb{N}. \tag{5.564}$$

Let us verify that $\bar{m}$ and $\bar{c}$ satisfy

$$\forall i\,(i \in \{1, \ldots, \bar{m}+1\} \Rightarrow \bar{c}_i < p). \tag{5.565}$$

We take an arbitrary i and assume $i \in \{1, \ldots, \bar{m}+1\}$ $[= \{1\}]$ to be true, so that we obtain $i = 1$ with (2.169). We therefore have

$$\bar{c}_i = \bar{c}_1 = n\ [< p],$$

and thus the desired consequent of the implication in (5.563). Since i was arbitrary, we may then infer from this indeed the truth of the universal sentence (5.565). Let us also observe the truth of

$$\bar{c}_{\bar{m}+1} = \bar{c}_{0+1} = \bar{c}_1 = n\ [\in \mathbb{N}_+],$$

which shows in light of (2.307) that $\bar{c}_{\bar{m}+1} \notin \{0\}$ holds, so that (2.169) yields

$$\bar{c}_{\bar{m}+1} \neq 0. \tag{5.566}$$

Furthermore, we obtain the equations

$$\begin{aligned}\sum_{i=1}^{\bar{m}+1}[\bar{c}_i \cdot p^{i-1}] = \sum_{i=1}^{0+1}[\bar{c}_i \cdot p^{i-1}] = \sum_{i=1}^{1}[\bar{c}_i \cdot p^{i-1}] = \bar{c}_1 \cdot p^{1-1} = n \cdot p^0 = n \cdot 1 \\ = n\end{aligned}$$

by using (5.563), the fact that 0 is the neutral element with respect to the addition on $\mathbb{N}$, (5.411), (5.345), (5.476), and the fact that 1 is the neutral element with respect to the multiplication on $\mathbb{N}$. Because of (5.563) – (5.566), we now see that the existential part of the uniquely existential sentence in (5.561) holds for the first case $n < p$.

To establish the uniqueness part, we take arbitrary m, c, m', c' satisfying $m, m' \in \mathbb{N}$, $c : \{1, \ldots, m+1\} \to \mathbb{N}$, $c' : \{1, \ldots, m'+1\} \to \mathbb{N}$, the universal sentences

$$\forall i\,(i \in \{1, \ldots, m+1\} \Rightarrow c_i < p), \tag{5.567}$$

$$\forall i\,(i \in \{1, \ldots, m'+1\} \Rightarrow c'_i < p), \tag{5.568}$$

the inequalities $c_{m+1} \neq 0$ and $c'_{m'+1} \neq 0$ and the equations

$$n = \sum_{i=1}^{m+1}[c_i \cdot p^{i-1}] \tag{5.569}$$

$$n = \sum_{i=1}^{m'+1}[c'_i \cdot p^{i-1}] \tag{5.570}$$

hold. According to (1.118), the task is now to show that $m = m'$ and $c = c'$ follow to be true. We first prove $m = 0$ by contradiction, assuming $m \neq 0$ to be true. We may write the sum (5.569) in its recursive form as

$$\begin{aligned}\sum_{i=1}^{m+1}[c_i \cdot p^{i-1}] &= \sum_{i=1}^{m}[c_i \cdot p^{i-1}] + [c_{m+1} \cdot p^{[m+1]-1}] \\ &= \sum_{i=1}^{m}[c_i \cdot p^{i-1}] + [c_{m+1} \cdot p^{m}] \end{aligned} \tag{5.571}$$

according to (5.417) and using also (5.344). Here, the previously assumed $c_{m+1} \neq 0$ implies $0 < c_{m+1}$ with (5.328), because the term c_{m+1} of the sequence $\bar{c}$ in $\mathbb{N}$ is evidently a natural number. We therefore obtain $[0^+ =]\ 1 \leq c_{m+1}$ with (2.291) and (4.157). Furthermore, because $0 < 1$ holds in view of (4.164) and because we initially assumed $1 < p$ to be true, we obtain $0 < p$ with the transitivity of the standard linear ordering of $\mathbb{N}$. Together with $1 \leq c_{m+1}$, this implies now $1 \cdot p \leq c_{m+1} \cdot p$ with the Monotony Law (5.293), and therefore evidently

$$p \leq c_{m+1} \cdot p. \tag{5.572}$$

Moreover, the previous assumption $m \neq 0$ implies $0 < m$ again with (5.328), consequently $[0^+ =]\ 1 \leq m$ with (2.291) and (4.157). Because if $1 < p$, the inequality $1 \leq m$ yields $p^1 \leq p^m$ with the Monotony Law for the exponent and $\leq$, consequently $p \leq p^m$ according to (5.477). Now, the previously established $0 < c_{m+1}$ and $p \leq p^m$ imply $p \cdot c_{m+1} \leq p^m \cdot c_{m+1}$ with the Monotony Law (5.293), and therefore

$$c_{m+1} \cdot p \leq c_{m+1} \cdot p^m \tag{5.573}$$

with the commutativity of the multiplication on $\mathbb{N}$. The conjunction of (5.572) and (5.573) further implies $p \leq c_{m+1} \cdot p^m$ with the transitivity of the standard total ordering of $\mathbb{N}$, and this finding implies in conjunction with the current case assumption $n < p$ that

$$n < c_{m+1} \cdot p^m \tag{5.574}$$

holds, using the Transitivity Formula for $<$ and $\leq$. Next, we observe that the sum $\sum_{i=1}^{m}[c_i \cdot p^{i-1}]$ is a natural number, so that $0 \leq \sum_{i=1}^{m}[c_i \cdot p^{i-1}]$ is true as shown by (4.187). We therefore obtain with the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$

$$c_{m+1} \cdot p^m \leq \sum_{i=1}^{m}[c_i \cdot p^{i-1}] + [c_{m+1} \cdot p^m] \tag{5.575}$$

Another application of the Transitivity Formula for $<$ and $\leq$ to (5.574) and (5.575) gives us now

$$n < \sum_{i=1}^{m}[c_i \cdot p^{i-1}] + [c_{m+1} \cdot p^m],$$

so that substitution based on (5.571) yields

$$n < \sum_{i=1}^{m+1}[c_i \cdot p^{i-1}].$$

Because the equation (5.569) is also true by assumption, we obtained a contradiction in view of the fact that the conjunction of these two sentences is false, because the standard linear ordering of $\mathbb{N}$ satisfies the Characterization of comparability. We thus completed the proof of $m = 0$ by contradiction. We may now prove also $m' = 0$ by contradiction by simply replacing in the preceding proof of $m = 0$ the constants m and c by m' and c', respectively. We therefore obtain via substitution $m = m'$, as desired. Consequently, the functions c and c' share the same domain

$$\{1, \ldots, m + 1\} = \{1, \ldots, m' + 1\} = \{1, \ldots, 0 + 1\} = \{1, \ldots, 1\} = \{1\},$$

and the assumed equations (5.569) – (5.570) evidently simplify to

$$n = \sum_{i=1}^{0+1}[c_i \cdot p^{i-1}] = \sum_{i=1}^{1}[c_i \cdot p^{i-1}] = c_1 \cdot p^{1-1} = c_1 \cdot p^0 = c_1 \cdot 1 = c_1$$

and

$$n = \sum_{i=1}^{0+1}[c'_i \cdot p^{i-1}] = \sum_{i=1}^{1}[c'_i \cdot p^{i-1}] = c'_1 \cdot p^{1-1} = c'_1 \cdot p^0 = c'_1 \cdot 1 = c'_1.$$

We are now in a position to conveniently prove $c = c'$ by means of the Equality Criterion for functions, letting i be an arbitrary element of their common domain $\{1\}$. We therefore obtain $i = 1$ with (2.169), and then the equations

$$c_i = c_1 = n = c'_1 = c'_i$$

by applying substitutions based on the previously found equations. Since i is arbitrary, we may therefore conclude that the functions c and c' are indeed identical. In conjunction with $m = m'$, this implies now the truth of the uniqueness part because m, c, m' and c' were arbitrary. Thus, the uniquely existential sentence in (5.562) is true, proving the implication based on the antecedent $n < p$. As n was initially arbitrary, we may now further conclude that the universal sentence (5.562) holds.

Next, we demonstrate the truth of the universal sentence

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow [p \leq n \Rightarrow \exists! m, c\,(m \in \mathbb{N} \,\wedge\, c : \{1, \ldots, m + 1\} \to \mathbb{N} \tag{5.576}$$
$$\wedge\, \forall i\,(i \in \{1, \ldots, m + 1\} \Rightarrow c_i < p) \,\wedge\, c_{m+1} \neq 0 \,\wedge\, n = \sum_{i=1}^{m+1}[c_i \cdot p^{i-1}])]),$$

by carrying out a proof by strong induction. For this purpose, we let $n \in \mathbb{N}_+$

be arbitrary, make the induction assumption

$$\forall k\,([k \in \mathbb{N}_+ \wedge k < n] \Rightarrow [p \leq k \Rightarrow \exists! m, c\,(m \in \mathbb{N} \wedge c : \{1, \ldots, m+1\} \to \mathbb{N}$$
$$\wedge\ \forall i\,(i \in \{1, \ldots, m+1\} \Rightarrow c_i < p)\ \wedge\ c_{m+1} \neq 0\ \wedge\ k = \sum_{i=1}^{m+1} [c_i \cdot p^{i-1}])]),$$

and show that this implies the truth of

$$p \leq n \Rightarrow \exists! m, c\,(m \in \mathbb{N}\ \wedge\ c : \{1, \ldots, m+1\} \to \mathbb{N} \qquad (5.577)$$
$$\wedge\ \forall i\,(i \in \{1, \ldots, m+1\} \Rightarrow c_i < p)\ \wedge\ c_{m+1} \neq 0\ \wedge\ n = \sum_{i=1}^{m+1} [c_i \cdot p^{i-1}]).$$

We prove this implication directly, assuming that $p \leq n$ is true. According to the Division of a natural number with remainder, it follows from $n \in \mathbb{N}_+$ (which evidently implies $n \in \mathbb{N}$) that there exist unique natural numbers $\bar{q}$ and $\bar{r}$ satisfying $\bar{r} < p$ and

$$n = \bar{q}p + \bar{r}. \qquad (5.578)$$

We prove now $\bar{q} \neq 0$ by contradiction, assuming the negation $\neg\bar{q} \neq 0$ to be true, so that the Double Negation Law yields the true sentence $\bar{q} = 0$. We then obtain $\bar{q}p = 0$ according to (5.471) and therefore via substitution $n = 0 + \bar{r}$, resulting evidently in $n = \bar{r}$. With this equation, the previously found $\bar{r} < p$ gives $n < p$, which in turn implies $\neg p \leq n$ with the Negation Formula for $\leq$. This finding clearly contradicts the previously made assumption $p \leq n$, so that the proof of $\bar{q} \neq 0$ is complete. Since $\bar{q}$ is a natural number, this inequality implies $0 < \bar{q}$ with (5.328). We may therefore apply the Monotony Law for $\cdot_{\mathbb{N}}$ and $<_{\mathbb{N}}$ to the initially assumed inequality $1 < p$ in order to obtain $1 \cdot \bar{q} < p \cdot \bar{q}$, which evidently yields

$$\bar{q} < \bar{q} \cdot p. \qquad (5.579)$$

Noting that $\bar{r} \in \mathbb{N}$ implies $0 \leq \bar{r}$ with (4.187), we obtain the inequality

$$\bar{q} \cdot p \leq (\bar{q} \cdot p) + \bar{r}$$

by means of the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$. In conjunction with (5.579), this gives us now $\bar{q} < (\bar{q} \cdot p) + \bar{r}$ with the Transitivity Formula for $<$ and $\leq$, which we may write also in the form

$$\bar{q} < \bar{q}p + \bar{r},$$

according to (5.521). Combining this inequality with the equation (5.578) via substitution results in $\bar{q} < n$. On the one hand, we observe now that the previous findings $\bar{q} \in \mathbb{N}$ and $\bar{q} \neq 0$ imply $\bar{q} \in \mathbb{N}_+$ with (2.310). On the other hand, we notice that the Law of the Excluded Middle yields the true disjunction $\bar{q} < p \lor \neg \bar{q} < p$, which we use now to establish the uniquely existential sentence

$$\exists! m, c\,(m \in \mathbb{N} \land c : \{1, \ldots, m+1\} \to \mathbb{N} \tag{5.580}$$
$$\land\ \forall i\,(i \in \{1, \ldots, m+1\} \Rightarrow c_i < p) \land c_{m+1} \neq 0 \land \bar{q} = \sum_{i=1}^{m+1} [c_i \cdot p^{i-1}])$$

by cases. Indeed, due to $\bar{q} \in \mathbb{N}_+$, the sentence (5.580) is implied in the first case $\bar{q} < p$ in view of the already proved universal sentence (5.562); in the second case $\neg \bar{q} < p$, which yields $p \leq \bar{q}$ with the Negation Formula for $<$, we obtain (5.580) in view of the induction assumption, recalling the truth of $\bar{q} < n$. Thus, there are unique constants m, c such that m is a natural number and c a function $c : \{1, \ldots, m+1\} \to \mathbb{N}$ satisfying $c_i < p$ for any $i \in \{1, \ldots, m+1\}$, the inequality $c_{m+1} \neq 0$, and moreover the equation

$$\bar{q} = \sum_{i=1}^{m+1} [c_i \cdot p^{i-1}]. \tag{5.581}$$

Here, the sum is evaluated for the function $s : \{1, \ldots, m+1\} \to \mathbb{N}$ with values

$$\forall i\,(i \in \{1, \ldots, m+1\} \Rightarrow s(i) = c_i \cdot p^{i-1}), \tag{5.582}$$

according to Exercise 5.61. We then obtain the true equations

$$n = \bar{q} \cdot p + \bar{r} = \left(\sum_{i=1}^{m+1} [c_i \cdot p^{i-1}] \right) \cdot p + \bar{r} = p \cdot \sum_{i=1}^{m+1} [c_i \cdot p^{i-1}] + \bar{r}$$
$$= \sum_{i=1}^{m+1} (p \cdot [c_i \cdot p^{i-1}]) + \bar{r} \tag{5.583}$$

using (5.578) in connection with (5.521), (5.581), the Commutative Law for the multiplication on $\mathbb{N}$ and the Generalized Distributive Law for semirings.

We use now Function definition by replacement to establish a sequence $d = (d_i \,|\, i \in \{1, \ldots, m+2\})$ whose terms satisfy

$$\forall i\,(i \in \{1, \ldots, m+2\} \tag{5.584}$$
$$\Rightarrow ([i = 1 \Rightarrow d_i = \bar{r}] \land [i \in \{2, \ldots, m+2\} \Rightarrow d_i = c_{i-1}])).$$

This task requires the verification of the universal sentence

$$\forall i\,(i \in \{1, \ldots, m+2\} \tag{5.585}$$
$$\Rightarrow \exists! y\,([i = 1 \Rightarrow y = \bar{r}] \land [i \in \{2, \ldots, m+2\} \Rightarrow y = c_{i-1}])).$$

We let i be arbitrary and assume $i \in \{1, \ldots, m+2\}$ to be true. Now, since $m \in \mathbb{N}$ gives the inequality $0 \leq m$ because of (4.187), we obtain $1 \leq m+1$ with the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$, where $m+1 < (m+1)+1$ is also true according to (5.217) and (4.153). Evidently, the latter inequality implies $m+1 \leq m+2$ and therefore – in conjunction with $1 \leq m+1$ – also $1 \leq m+2$. Consequently, we obtain

$$\{1\} \cup \{2, \ldots, m+2\} = \{1, \ldots, m+2\} \tag{5.586}$$

with (4.299) and the evident facts $\{1\} = \{1, \ldots, 1\}$ and $1^+ = 2$. Here, we note that

$$\{1\} \cap \{2, \ldots, m+2\} = \emptyset \tag{5.587}$$

holds because of (4.302). The preceding assumption $i \in \{1, \ldots, m+2\}$ implies now in view of the disjunction $i \in \{1\} \vee i \in \{2, \ldots, m+2\}$ by definition of the union of two sets, which allows us to prove the existential part of the uniquely existential sentence in (5.585) by cases. The first case $i \in \{1\}$ yields on the one hand evidently $i = 1$; on the other hand, (5.587) shows in light of the definition of the empty set that $\neg i \in \{1\} \cap \{2, \ldots, m+2\}$ holds. Thus,

$$\neg i \in \{1\} \vee \neg i \in \{2, \ldots, m+2\} \tag{5.588}$$

follows to be true with the definition of the intersection of two sets and De Morgan's Law for the conjunction. Due to the current case assumption, we have that $\neg i \in \{1\}$ is false, so that $\neg i \in \{2, \ldots, m+2\}$ is true. Thus, replacing the variable y by the given constant $\bar{r}$ gives us the true implications

$$[i = 1 \Rightarrow \bar{r} = \bar{r}] \wedge [i \in \{2, \ldots, m+2\} \Rightarrow \bar{r} = c_{i-1}],$$

as the antecedent and the consequent of the first implication are both true and as the antecedent of the second implication is false. In the second case $i \in \{2, \ldots, m+2\}$, we have that the second part of the disjunction (5.588) is now false, with the consequence that $\neg i \in \{1\}$ must be true, which we may write equivalently as $i \neq 1$ by means of (2.169) and the Law of Contraposition. Let us observe that the current case assumption implies $2 \leq i \leq m+2$ with (4.290); due to the evident fact $1 \leq 2$ and the evident consequence $1 \leq i$, we may apply the Monotony Law for $-_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$ to the inequalities $2 \leq i \leq m+2$ to obtain $2-1 \leq i-1 \leq (m+2)-1$. Clearly, we have therefore $1 \leq i-1 \leq m+1$, which shows in light of (4.290) that $i-1 \in \{1, \ldots, m+1\}$ is true, i.e. that $i-1$ is in the index set of the sequence c, so that c_{i-1} constitutes a term of that sequence. Thus, replacing the variable y by the given constant c_{i-1} yields the true implications

$$[i = 1 \Rightarrow c_{i-1} = \bar{r}] \wedge [i \in \{2, \ldots, m+2\} \Rightarrow c_{i-1} = c_{i-1}],$$

the antecedent of the implication being false and the antecedent and consequent of the second implication being true. This finding completes the proof of the existential part by cases, and we proceed in a similar way to establish also the uniqueness part. Letting y and y' be arbitrary and assuming the implications

$$[i = 1 \Rightarrow y = \bar{r}] \wedge [i \in \{2, \ldots, m+2\} \Rightarrow y = c_{i-1}]$$
$$[i = 1 \Rightarrow y' = \bar{r}] \wedge [i \in \{2, \ldots, m+2\} \Rightarrow y' = c_{i-1}]$$

to hold, we obtain in case of $i \in \{1\}$ the equation $i = 1$ and therefore $y = \bar{r} = y'$, and in case of $i \in \{2, \ldots, m+2\}$ the equations $y = c_{i-1} = y'$. Because y and y' are arbitrary, we may now infer from the truth of $y = y'$ (in both cases) the truth of the uniqueness part and thus the truth of the uniquely existential sentence in (5.585). Here, i was arbitrary, so that the universal sentence (5.585) follows now also to be true, and this sentence implies the unique existence of a function d with domain $\{1, \ldots, m+2\}$ such that (5.584). Let us verify that $\mathbb{N}$ is a codomain of d, i.e. that the range of d is included in $\mathbb{N}$. We let for this purpose y be arbitrary, and we assume $y \in \mathrm{ran}(d)$ to be true. Due to the definition of a range, there exists then a constant, say $\bar{k}$, such that $(\bar{k}, y) \in d$ holds, where $\bar{k} \in \{1, \ldots, m+2\}$ is true by definition of a domain. In view of (5.584), we obtain then $d(\bar{k}) = \bar{r}$ in case of $\bar{k} \in \{1\}$, and $d(\bar{k}) = c_{\bar{k}-1}$ in case of $\bar{k} \in \{2, \ldots, m+2\}$. Recalling that $\bar{r}$ is a natural number and noting that the function value $c_{\bar{k}-1}$ is an element of the codomain $\mathbb{N}$ of c, we find $\bar{k} \in \mathbb{N}$ to be true in any case. Writing now $(\bar{k}, y) \in d$ in function notation as $y = d_{\bar{k}}$, we arrive at $y \in \mathbb{N}$ through substitution. We thus showed that $y \in \mathrm{ran}(d)$ implies $y \in \mathbb{N}$, and since y was arbitrary, we may infer from the truth of that implication the truth of the inclusion $\mathrm{ran}(d) \subseteq \mathbb{N}$ (according to the definition of a subset). This shows that $\mathbb{N}$ is a codomain of d, that is, we have $d : \{1, \ldots, m+2\} \to \mathbb{N}$, which we can write also in the form $d : \{1, \ldots, (m+1)+1\} \to \mathbb{N}$. According to Exercise 5.61, this function in turn defines the function $S : \{1, \ldots, (m+1)+1\} \to \mathbb{N}$ whose values satisfy

$$\forall i\, (i \in \{1, \ldots, (m+1)+1\} \Rightarrow S(i) = d_i \cdot p^{i-1}). \tag{5.589}$$

We obtain then the equations

$$\begin{aligned}\sum_{i=1}^{m+2} S_i &= \sum_{i=1}^{1} S_i + \sum_{j=1}^{(m+2)-1} S_{1+j} = \sum_{j=1}^{m+1} S_{1+j} + S_1 \\ &= \sum_{j=1}^{m+1} [d_{1+j} \cdot p^{(1+j)-1}] + [d_1 \cdot p^{1-1}] = \sum_{j=1}^{m+1} [d_{j+1} \cdot p^j] + \bar{r}\end{aligned} \tag{5.590}$$

by using the Generalized Associative Law for semigroups, the commutativity of the addition on $\mathbb{N}$, (5.589), and (5.584) based on the evident facts $1 \in \{1, \ldots, m+2\}$ and $p^{1-1} = p^0 = 1$ in connection with (5.476). Concerning the $(m+1)$-fold sums in the preceding equation and in (5.583), we can now prove that the involved sequences

$$(p \cdot [c_i \cdot p^{i-1}] \,|\, i \in \{1, \ldots, m+1\}), \quad (d_{j+1} \cdot p^j \,|\, i \in \{1, \ldots, m+1\}) \tag{5.591}$$

are identical, so that the evaluation of their sums results in the same value. To do this, we apply the Equality Criterion for functions and let i be an arbitrary element of the joint domain $\{1, \ldots, m+1\}$. Evidently, this implies $1 \leq i \leq m+1$ and then $2 \leq i+1 \leq m+2$, so that $i+1 \in \{2, \ldots, m+2\}$ holds. By definition of the function d, we therefore have $d_{i+1} = c_{(i+1)-1} = c_i$. We then obtain

$$\begin{aligned} d_{i+1} \cdot p^i &= c_i \cdot p^i = c_i \cdot p^{(i-1)+1} = c_i \cdot (p^{i-1} \cdot p^1) = c_i \cdot (p^{i-1} \cdot p) \\ &= p \cdot (c_i \cdot p^{i-1}) \end{aligned}$$

by applying substitution, (5.343), the Addition Rule for powers, (5.477), and the associativity as well as commutativity of the multiplication on $\mathbb{N}$. Since i is arbitrary, we may therefore conclude that the two sequences in (5.591) are indeed identical, so that the associated sums are equal, i.e.

$$\sum_{i=1}^{m+1} (p \cdot [c_i \cdot p^{i-1}]) = \sum_{j=1}^{m+1} [d_{j+1} \cdot p^j],$$

recalling that $\sum_{i=1}^{m+1}$ is a function. Applying now a substitution based on this equation to (5.583) leads with (5.590) and (5.589) to

$$\begin{aligned} n &= \sum_{i=1}^{m+1} (p \cdot [c_i \cdot p^{i-1}]) + \overline{r} = \sum_{j=1}^{m+1} [d_{j+1} \cdot p^j] + \overline{r} = \sum_{i=1}^{m+2} S_i \\ &= \sum_{i=1}^{(m+1)+1} [d_i \cdot p^{i-1}]. \end{aligned} \tag{5.592}$$

Let us observe next that the evident fact $m+2 \in \{2, \ldots, m+2\}$ implies with (5.584) $d_{m+2} = c_{(m+2)-1}$ and therefore evidently $d_{(m+1)+1} = c_{m+1}$. Recalling now the truth of $c_{m+1} \neq \emptyset$, we thus find

$$d_{(m+1)+1} \neq \emptyset. \tag{5.593}$$

Furthermore, we may establish the truth of the universal sentence

$$\forall i \, (i \in \{1, \ldots, (m+1)+1\} \Rightarrow d_i < p), \tag{5.594}$$

letting $j \in \{1, \ldots, (m+1)+1\}$ be arbitrary. In case of $j \in \{1\}$, we obtain $d_j = \bar{r}$ in view of (5.584), where $\bar{r} < p$ is true as mentioned earlier, so that $d_j < p$ follows to be true via substitution. The other case $j \in \{2, \ldots, m+2\}$ yields on the one hand $d_j = c_{j-1}$ again with (5.584), and on the other hand $j-1 \in \{1, \ldots, m+1\}$ (as shown previously). We stated before that $c_i < p$ holds for any $i \in \{1, \ldots, m+1\}$, so that $c_{j-1} < p$ is true in particular. Thus, we find $d_j < p$ also for the second case. Here, j was arbitrary, and therefore the universal sentence (5.594) follows to be true.

We thus found a particular number $m+1 \in \mathbb{N}$ and a particular function $d : \{1, \ldots, (m+1)+1\} \to \mathbb{N}$ satisfying the universal sentence (5.594), the inequality (5.593) and the equation (5.592), so that the proof of the existential part of the uniquely existential sentence in (5.577) is now complete.

To prove the uniqueness part (according to Notation 1.4), we let m', c', m'' and c'' be arbitrary such that m' and m'' are natural numbers, such that c' is a function from $\{1, \ldots, m'+1\}$ to $\mathbb{N}$ and c'' a function from $\{1, \ldots, m''+1\}$ to $\mathbb{N}$, such that $c'_i < p$ holds for all $i \in \{1, \ldots, m'+1\}$ and $c''_i < p$ for all $i \in \{1, \ldots, m''+1\}$, such that c'_{m+1} and c''_{m+1} are different from zero, and such that the equations

$$n = \sum_{i=1}^{m+1} [c'_i \cdot p^{i-1}] \tag{5.595}$$

$$n = \sum_{i=1}^{m'+1} [c''_i \cdot p^{i-1}] \tag{5.596}$$

are satisfied. The task is now to establish $m' = m''$ and of $c' = c''$. To begin with, we observe that $m', m'' \in \mathbb{N}$ implies $0 \leq m'$ as well as $0 \leq m''$, so that the disjunctions $0 < m' \vee 0 = m'$ and $0 < m'' \vee 0 = m''$ are true according to the Characterization of an induced irreflexive partial ordering. Here, we can prove by contradiction that $\neg 0 = m' \wedge \neg 0 = m''$ holds. Assuming the negation of that conjunction to be true, we obtain the true disjunction $m' = 0 \vee m'' = 0$ with De Morgan's Law for the conjunction and the Double Negation Law. Let us use this disjunction to prove $n < p$ by cases. Clearly, (5.595) and (5.596) imply in the first and the second case, respectively,

$$n = \sum_{i=1}^{1} [c'_i \cdot p^{i-1}] = c'_1 \cdot p^{1-1} = c'_1 \cdot p^0 = c'_1$$

$$n = \sum_{i=1}^{1} [c''_i \cdot p^{i-1}] = c''_1 \cdot p^{1-1} = c''_1 \cdot p^0 = c''_1$$

and thus $n < p$ (recalling that $c'_i < p$ and $c''_i < p$ are true for any index).

Consequently, $\neg p \leq n$ follows to be true, in contradiction to the previous assumption $p \leq n$. Having established the truth of $\neg m' = 0$ and $\neg m'' = 0$, we can infer from these negations the truth of the first parts $0 < m'$ and $0 < m''$ of the preceding disjunctions. These inequalities further imply $1 \leq m'$ and $1 \leq m''$ with (4.157), which findings evidently allow us to rewrite (5.595) as

$$\begin{aligned}
n &= \sum_{i=1}^{1}[c'_i \cdot p^{i-1}] + \sum_{j=1}^{(m'+1)-1}[c'_{1+j} \cdot p^{(1+j)-1}] \\
&= \sum_{j=1}^{(m'-1)+1}[c'_{1+j} \cdot p^{(j-1)+1}] + c'_1 \cdot p^{1-1} \\
&= \sum_{j=1}^{(m'-1)+1}[c'_{1+j} \cdot p^{j-1} \cdot p^1] + c'_1 \cdot p^0 \\
&= \sum_{j=1}^{(m'-1)+1}[c'_{1+j} \cdot p^{j-1} \cdot p] + c'_1 \cdot 1 \\
&= p \cdot \sum_{j=1}^{(m'-1)+1}[c'_{1+j} \cdot p^{j-1}] + c'_1 \\
&= \left(\sum_{j=1}^{(m'-1)+1}[c'_{1+j} \cdot p^{j-1}]\right) \cdot p + c'_1,
\end{aligned}$$

and in the same way (5.596) as

$$n = \left(\sum_{j=1}^{(m''-1)+1}[c''_{1+j} \cdot p^{j-1}]\right) \cdot p + c''_1.$$

Recalling now from (5.578) that $\bar{q}$ and $\bar{r}$ are the only constants satisfying $n = \bar{q} \cdot p + \bar{r}$, we obtain according to Notation 1.4

$$\sum_{j=1}^{(m'-1)+1}[c'_{1+j} \cdot p^{j-1}] = \bar{q} = \sum_{j=1}^{(m''-1)+1}[c''_{1+j} \cdot p^{j-1}]$$

and $c'_1 = \bar{r} = c''_1$. Furthermore, we noted after proving (5.580) that there is a unique $m \in \mathbb{N}$ as well as a unique $c : \{1, \ldots, m+1\} \to \mathbb{N}$ satisfying

$\bar{q} = \sum_{i=1}^{m+1}[c_i \cdot p^{i-1}]$ in (5.581), so that we obtain $m' - 1 = m = m'' - 1$ and

$$\begin{aligned}(c'_{1+j} \,|\, j \in \{1, \ldots, (m'-1)+1\}) &= (c_i \,|\, i \in \{1, \ldots, m+1\}) \\ &= (c''_{1+j} \,|\, j \in \{1, \ldots, (m''-1)+1\}).\end{aligned}$$

These equations give us (using also the alternative notation according to the Generalized Associative Law for semigroups)

$$(c'_i \,|\, j \in \{2, \ldots, m'+1\}) = (c''_i \,|\, j \in \{2, \ldots, m'+1\}),$$

and the Equality Criterion for functions yields then

$$\forall i\, (i \in \{2, \ldots, m'+1\} \Rightarrow c'_i = c''_i), \tag{5.597}$$

which allows us to complete the proof of $c' = c''$. To do this, we apply again the Equality Criterion for functions and verify accordingly

$$\forall i\, (i \in \{1, \ldots, m'+1\} \Rightarrow c'_i = c''_i), \tag{5.598}$$

letting $i \in \{1, \ldots, m'+1\}$ be arbitrary. We noted previously that $0 \leq m'$ holds, so that $1 \leq m'+1$ follows to be true with the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$. Consequently, we have $1 \leq i$ according to (4.276), which gives us evidently the true disjunction $1 < i \lor 1 = i$. In case of $1 < i$, it follows that $2 \leq i$ is true, with the consequence that $i \in \{2, \ldots, m'+1\}$ holds (by definition of an intermediate segment). Thus, we obtain with (5.597) $c'_i = c''_i$, as desired. In the other case of $1 = i$, we recall the previously found $c'_1 = \bar{r} = c''_1$, so that substitution yields again the desired consequent $c'_i = c''_i$. Having completed the proof by cases, we may now infer from this the truth of the universal sentence (5.598), and therefore the truth of $c' = c''$. In conjunction with the previously established $m' = m''$, this allows us to conclude that the uniqueness part of the uniquely existential sentence in (5.577) holds, since m', c', m'' and c'' were arbitrary. We thus completed the proof of that uniquely existential sentence, which in turn completes the proof of the implication based on the antecedent $p \leq n$ in (5.577). As n was initially arbitrary, the proof of (5.576) via strong induction is now complete.

We are now in a position to prove (5.561). For this purpose, we let $n \in \mathbb{N}_+$ be arbitrary and consider the two cases $n < p$ and $\neg n < p$. In the first case $n < p$, the desired uniquely existential sentence in (5.561) follows to be true with (5.562). The second case $\neg n < p$ implies $p \leq n$ with the Negation Formula for $<$, and this inequality implies the desired uniquely existential sentence now with (5.576), completing the proof by cases. Because n was arbitrary, the universal sentence (5.576) follows then to be true, and as p was initially also arbitrary, we may finally conclude that the stated theorem holds. $\square$

5.6.6. Countability of Cartesian products

We begin with the construction of two useful injections.

Lemma 5.149. *It is true that there exists the unique function*

$$g : \mathbb{N}_+ \to \mathbb{N}_+ \times \mathbb{N}_+, \qquad n \mapsto (1, n), \tag{5.599}$$

and this function is an injection.

Proof. We apply Function definition by replacement to establish the unique function g with domain $\mathbb{N}_+$ and values satisfying

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow g(n) = (1, n)) \tag{5.600}$$

To do this, we prove the universal sentence

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow \exists! Y \, (Y = (1, n))), \tag{5.601}$$

letting $n \in \mathbb{N}_+$ be arbitrary. Then, $(1, n)$ constitutes the (uniquely specified) ordered pair formed by 1 and n, so that we may infer the truth of the uniquely existential sentence $\exists! Y \, (Y = (1, n))$ by means of (1.109). As n was arbitrary, we may therefore conclude that (5.606) holds, and the truth of this universal sentence implies now the unique existence of a function g with domain $\mathbb{N}_+$ such that (5.605).

Let us now verify that $\mathbb{N}_+ \times \mathbb{N}_+$ is a codomain of g, i.e. that the range of g is included in $\mathbb{N}_+ \times \mathbb{N}_+$. We apply for this purpose the definition of a subset and prove the equivalent universal sentence

$$\forall z \, (z \in \mathrm{ran}(g) \Rightarrow z \in \mathbb{N}_+ \times \mathbb{N}_+), \tag{5.602}$$

letting z be arbitrary and assuming $z \in \mathrm{ran}(g)$ to be true. The definition of a range gives us then a particular constant $\bar{n}$ satisfying $(\bar{n}, z) \in g$, where the definition of a domain shows that $\bar{n} \in \mathbb{N}_+$ [$= \mathrm{dom}(g)$] is true. This finding in turn implies $g(\bar{n}) = (1, \bar{n})$ with (5.605). Here, the evident truth of $1 \in \mathbb{N}_+$ and the truth of $\bar{n} \in \mathbb{N}_+$ yields $[g(\bar{n}) =]$ $(1, \bar{n}) \in \mathbb{N}_+ \times \mathbb{N}_+$ with the definition of the Cartesian product of two sets. Since we may write the previously found $(\bar{n}, z) \in g$ in function notation as $z = g(\bar{n})$, we obtain via substitution $z \in \mathbb{N}_+ \times \mathbb{N}_+$, as desired. Because z is arbitrary, we may now infer from this the truth of the universal sentence (5.608), and consequently the truth of the inclusion $\mathrm{ran}(g) \subseteq \mathbb{N}_+ \times \mathbb{N}_+$, which demonstrates that the latter Cartesian product is indeed a codomain of g.

It now remains for us to prove that g constitutes an injection, i.e. that g has the definite property

$$\forall n, n' \, ([n, n' \in \mathbb{N}_+ \wedge g(n) = g(n')] \Rightarrow n = n'). \tag{5.603}$$

We take arbitrary positive natural numbers n and n' such that the corresponding function values $g(n)$ and $g(n')$ are identical. Since $n, n' \in \mathbb{N}_+$ implies $g(n) = (1, n)$ as well as $g(n') = (1, n')$ by definition of the function g, the preceding assumption of $g(n) = g(n')$ yields $(1, n) = (1, n')$ by means of substitution. We then obtain with the Equality Criterion for ordered pairs especially the desired equation $n = n'$, which proves the implication in (5.609). Here, n and n' were arbitrary, so that the universal sentence (5.609) follows to be true, and this means that g is an injection by definition, i.e. $g : \mathbb{N}_+ \hookrightarrow \mathbb{N}_+ \times \mathbb{N}_+$. We thus completed the proof of the lemma. $\square$

Lemma 5.150. *It is true that there exists the unique function*

$$f : \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{N}_+, \qquad (m, n) \mapsto 2^m \cdot 3^n, \tag{5.604}$$

and this function is an injection.

Proof. We define the function by replacement, verifying first

$$\forall x \, (x \in \mathbb{N}_+ \times \mathbb{N}_+ \Rightarrow \exists! y \, (\exists m, n \, (x = (m, n) \, \wedge \, y = 2^m \cdot 3^n))). \tag{5.605}$$

For this purpose, we let $x \in \mathbb{N}_+ \times \mathbb{N}_+$ be arbitrary, so that there exist by definition of the Cartesian product of two sets elements of $\mathbb{N}_+$, say $\bar{m}$ and $\bar{n}$, such that $x = (\bar{m}, \bar{n})$ holds. Since $\bar{m}$ and $\bar{n}$ are evidently natural numbers, the powers $2^{\bar{m}}$ and $3^{\bar{n}}$ are clearly specified natural numbers, and their product $\bar{y} = 2^{\bar{m}} \cdot 3^{\bar{n}}$ is also a natural number. Thus, the existential part is true. Regarding the uniqueness part, we let y and y' be arbitrary such that the existential sentences

$$\exists m, n \, (x = (m, n) \, \wedge \, y = 2^m \cdot 3^n)$$
$$\exists m, n \, (x = (m, n) \, \wedge \, y' = 2^m \cdot 3^n)$$

are both true. This means that there are constants, say $\bar{m}$ and $\bar{n}$, with $x = (\bar{m}, \bar{n})$ and $y = 2^{\bar{m}} \cdot 3^{\bar{n}}$, and there are constants, say $\bar{M}$ and $\bar{N}$, with $x = (\bar{M}, \bar{N})$ and $y' = 2^{\bar{M}} \cdot 3^{\bar{N}}$. Combining the two equations for $\bar{x}$ yields $(\bar{m}, \bar{n}) = (\bar{M}, \bar{N})$ and then $\bar{m} = \bar{M}$ as well as $\bar{n} = \bar{N}$ with the Equality Criterion for ordered pairs. Then, we obtain

$$y' = 2^{\bar{M}} \cdot 3^{\bar{N}} = 2^{\bar{m}} \cdot 3^{\bar{n}} = y$$

via substitution, so that $y = y'$ holds. Since y and y' are arbitrary, we may therefore conclude that the uniqueness part of the uniquely existential sentence to be proven is also true. As x was also arbitrary, we may further conclude that (5.605) holds, which implies the existence of a function f with domain $\mathbb{N}_+ \times \mathbb{N}_+$ such that

$$\forall x\,(x \in \mathbb{N}_+ \times \mathbb{N}_+ \Rightarrow \exists m,n\,(x = (m,n) \,\wedge\, f(x) = 2^m \cdot 3^n). \tag{5.606}$$

It is now a simple task to establish

$$\forall m,n\,((m,n) \in \mathbb{N}_+ \times \mathbb{N}_+ \Rightarrow f((m,n)) = 2^m \cdot 3^n). \tag{5.607}$$

Letting m and n be arbitrary such that $(m,n) \in \mathbb{N}_+ \times \mathbb{N}_+$ holds, it follows with (5.606) that there are constants, say $\bar{m}$ and $\bar{n}$, satisfying $(m,n) = (\bar{m},\bar{n})$ and $f((m,n)) = 2^{\bar{m}} \cdot 3^{\bar{n}}$. As the former equation yields $m = \bar{m}$ and $n = \bar{n}$ with the Equality Criterion for ordered pairs, we obtain via substitutions $f((m,n)) = 2^m \cdot 3^n$, as desired. Since m and n are arbitrary, we may therefore conclude that the universal sentence (5.607) is true. Thus, any ordered pair $(m,n) \in \mathbb{N}_+ \times \mathbb{N}_+$ is mapped to $2^m \cdot 3^n$, as indicated in (5.604).

Next, we prove that $\mathbb{N}_+$ is a codomain of the function f. For this purpose, we verify

$$\forall y\,(y \in \operatorname{ran}(f) \Rightarrow y \in \mathbb{N}_+). \tag{5.608}$$

We let y be arbitrary and assume $y \in \operatorname{ran}(f)$ to be true, so that there exists by definition of a range a constant, say $\bar{x}$, such that $(\bar{x}, y) \in f$, i.e. such that $y = f(\bar{x})$ holds. We also see in light of the definition of a domain that $\bar{x} \in \mathbb{N}_+ \times \mathbb{N}_+$ is true, so that there are because of (5.606) constants, say $\bar{m}$ and $\bar{n}$, satisfying $\bar{x} = (\bar{m},\bar{n})$ and

$$[y =]\; f(\bar{x}) = 2^{\bar{m}} \cdot 3^{\bar{n}}.$$

Furthermore, we obtain via substitution $(\bar{m},\bar{n}) \in \mathbb{N}_+ \times \mathbb{N}_+$, which in turn yields $\bar{m} \in \mathbb{N}_+$ and $\bar{n} \in \mathbb{N}_+$ with the definition of the Cartesian product of two sets; thus, $\bar{m}$ and $\bar{n}$ are clearly natural numbers, so that

$$[y =]\; 2^{\bar{m}} \cdot 3^{\bar{n}} \in \mathbb{N} \tag{5.609}$$

holds (using the fact that the multiplication on $\mathbb{N}$ is a binary operation). Since $2 \neq 0$ and $3 \neq 0$ are evidently also true, we may therefore apply Corollary 5.128 to obtain $2^{\bar{m}} \neq 0$ as well as $3^{\bar{n}} \neq 0$. Consequently, the Criterion for zero-divisor freeness yields

$$[y =]\; 2^{\bar{m}} \cdot 3^{\bar{n}} \neq 0$$

and then $y \notin \{0\}$ with (2.169). In view of (5.609), we thus showed that $y \in \mathbb{N}$ and $y \notin \{0\}$ are both true, so that the desired $y \in \mathbb{N}_+$ follows to be true by definition of the set of positive natural numbers. Because y is arbitrary, we may therefore conclude that (5.608) holds, which universal sentence then implies $\mathrm{ran}(f) \subseteq \mathbb{N}_+$ by definition of a subset. Thus, $\mathbb{N}_+$ is indeed a codomain of f, so that f is a function from $\mathbb{N}_+ \times \mathbb{N}_+$ to $\mathbb{N}_+$, as claimed in (5.604).

It finally remains for us to demonstrate that f is an injection. To do this, we verify

$$\forall x, x' \, ([x, x' \in \mathbb{N}_+ \times \mathbb{N}_+ \,\wedge\, f(x) = f(x')] \Rightarrow x = x'), \tag{5.610}$$

letting x and x' be arbitrary and assuming $x, x' \in \mathbb{N}_+ \times \mathbb{N}_+$ as well as $f(x) = f(x')$ to hold. The former assumption implies (by definition of the Cartesian product of two sets) that there are elements of $\mathbb{N}_+$, say $\bar{m}$ and $\bar{n}$, with $(\bar{m}, \bar{n}) = x$, and that there are elements of $\mathbb{N}_+$, say $\bar{m}'$ and $\bar{n}'$, with $(\bar{m}', \bar{n}') = x'$. Thus, we may write for the function values

$$f(x) = f((\bar{m}, \bar{n})) = 2^{\bar{m}} \cdot 3^{\bar{n}},$$
$$f(x') = f((\bar{m}', \bar{n}')) = 2^{\bar{m}'} \cdot 3^{\bar{n}'},$$

which equations allow us to write the assumed $f(x) = f(x')$ as

$$2^{\bar{m}} \cdot 3^{\bar{n}} = 2^{\bar{m}'} \cdot 3^{\bar{n}'}. \tag{5.611}$$

Since $\bar{m}$, $\bar{n}$, $\bar{m}'$ and $\bar{n}'$ are then evidently also elements of $\mathbb{N}$, we obtain with the connexity of the standard linear ordering $<$ of $\mathbb{N}$ the true multiple disjunction

$$\bar{m} < \bar{m}' \,\vee\, \bar{m}' < \bar{m} \,\vee\, \bar{m} = \bar{m}', \tag{5.612}$$

for which it is now possible to prove via contradictions that the negations of the first two parts are true.

Regarding the first part, we assume the negation of $\neg \bar{m} < \bar{m}'$ to be true, so that the Double negation Law yields the true inequality $\bar{m} < \bar{m}'$. According to Property 7 of an ordered elementary domain and the definition of a difference, we then have $\bar{m}' - \bar{m} \neq 0$, and therefore evidently $\bar{m}' - \bar{m} \in \mathbb{N}_+$. Because of (5.540), there exists then a natural number, say $\bar{q}$, such that

$$2^{\bar{m}' - \bar{m}} = 2\bar{q}. \tag{5.613}$$

Furthermore, we obtain the equations

$$2^{\bar{m}'} = 2^{(\bar{m}' - \bar{m}) + \bar{m}} = 2^{\bar{m}' - \bar{m}} \cdot 2^{\bar{m}} = 2\bar{q} \cdot 2^{\bar{m}}$$

with (5.343), the Addition Rule for powers, and with (5.613). Therefore, the equation (5.611) yields

$$2^{\bar{m}} \cdot 3^{\bar{n}} = (2\bar{q} \cdot 2^{\bar{m}}) \cdot 3^{\bar{n}'} = 2^{\bar{m}} \cdot (2\bar{q} \cdot 3^{\bar{n}'}),$$

where we applied also the commutativity and the associativity of $\cdot_{\mathbb{N}}$. Then, we obtain with the Cancellation Rule for $\cdot_{\mathbb{N}}$ the equation $3^{\bar{n}} = 2\bar{q} \cdot 3^{\bar{n}'}$, which we may evidently write as

$$3^{\bar{n}} = 2(\bar{q} \cdot 3^{\bar{n}'}).$$

Let us observe here that the natural number $3^{\bar{n}}$ is odd according to (5.538), so that substitution based on the preceding equation shows that $2(\bar{q} \cdot 3^{\bar{n}'})$ is odd. Since $\bar{q} \cdot 3^{\bar{n}'}$ is a natural number, we also have that $2(\bar{q} \cdot 3^{\bar{n}'})$ is even, by definition. However, Proposition 5.141 shows that it is false that this number is both odd and even, so that we arrived at a contradiction, completing the proof of $\neg \bar{m} < \bar{m}'$.

Regarding the negation of the second part of the multiple disjunction (5.612), we may evidently apply exactly the same line of arguments as for the proof of the negation of the first part, where we simply exchange $\bar{n}$ and $\bar{n}'$ as well as $\bar{m}$ and $\bar{m}'$. We thus obtain the true negation $\neg \bar{m}' < \bar{m}$, so that the third part $\bar{m} = \bar{m}'$ of the multiple disjunction must be true. Consequently, we may apply substitution to (5.611) and write equivalently $2^{\bar{m}} \cdot 3^{\bar{n}} = 2^{\bar{m}} \cdot 3^{\bar{n}'}$, which in turn implies $3^{\bar{n}} = 3^{\bar{n}'}$ with the Cancellation Law for $\cdot_{\mathbb{N}}$. Now, because $3 \in \mathbb{N}$, $1 <_{\mathbb{N}} 3$ and $\bar{n}, \bar{n}'$ are true, we may apply Proposition 5.147 to the ordered elementary domain $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$ to infer from the preceding equation the truth of $\bar{n} = \bar{n}'$.

We thus showed that $\bar{m} = \bar{m}'$ and $\bar{n} = \bar{n}'$ are both true, so that the Equality Criterion for ordered pairs yields

$$[x =] \quad (\bar{m}, \bar{n}) = (\bar{m}', \bar{n}') \quad [= x'],$$

so that we obtain the desired equation $x = x'$, which proves the implication in (5.610). As x and x' are arbitrary, we may therefore conclude that the universal sentence (5.610) is true, so that f follows to be an injection by definition. □

The existence of injections $g : \mathbb{N}_+ \hookrightarrow \mathbb{N}_+ \times \mathbb{N}_+$ and $f : \mathbb{N}_+ \times \mathbb{N}_+ \hookrightarrow \mathbb{N}_+$ (according to Lemma 5.149 and Lemma 5.150) immediately yields the following results by means of the Cantor-Schröder-Bernstein theorem and the symmetry property (4.665) of equinumerosity.

Corollary 5.151. *It is true that the set $\mathbb{N}_+$ and the Cartesian product $\mathbb{N}_+ \times \mathbb{N}_+$ are equinumerous and vice versa, that is,*

$$\mathbb{N}_+ \sim \mathbb{N}_+ \times \mathbb{N}_+, \tag{5.614}$$

$$\mathbb{N}_+ \times \mathbb{N}_+ \sim \mathbb{N}_+. \tag{5.615}$$

Recalling now the truth of $\mathbb{N} \sim \mathbb{N}_+$ and $\mathbb{N}_+ \sim \mathbb{N}$ in view of (4.661) and (4.667), the transitivity property (4.666) equinumerosity gives us:

Corollary 5.152. *It is true that the set* $\mathbb{N}$ *and the Cartesian product* $\mathbb{N}_+ \times \mathbb{N}_+$ *are equinumerous and vice versa, that is,*

$$\mathbb{N} \sim \mathbb{N}_+ \times \mathbb{N}_+, \tag{5.616}$$

$$\mathbb{N}_+ \times \mathbb{N}_+ \sim \mathbb{N}. \tag{5.617}$$

Note 5.33. Since (5.616) means by definition of equinumerous sets that there exists a bijection from $\mathbb{N}$ to $\mathbb{N}_+ \times \mathbb{N}_+$, it means also that $\mathbb{N}_+ \times \mathbb{N}_+$ is a countably infinite set. Thus, $\mathbb{N}_+ \times \mathbb{N}_+$ is more generally a countable set.

Recalling next that $\mathbb{N}_+ \times \mathbb{N}_+ \sim \mathbb{N} \times \mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}_+ \times \mathbb{N}_+$ are true as shown by (4.679) – (4.680), further applications of the transitivity of '$\sim$' result in:

Corollary 5.153. *It is true that the set* $\mathbb{N}$ *and the Cartesian product* $\mathbb{N} \times \mathbb{N}$ *are equinumerous and vice versa, that is,*

$$\mathbb{N} \sim \mathbb{N} \times \mathbb{N}, \tag{5.618}$$

$$\mathbb{N} \times \mathbb{N} \sim \mathbb{N}. \tag{5.619}$$

Note 5.34. (5.618) shows in light of the definition of equinumerous sets that there is a bijection $d : \mathbb{N} \rightleftarrows \mathbb{N} \times \mathbb{N}$, which means that $\mathbb{N} \times \mathbb{N}$ constitutes a countably infinite set, and thus a countable set.

Corollary 5.154. *It is true that every countably infinite set is equivalent to the Cartesian product of that set with itself, that is,*

$$\forall A\,(A \text{ is countably infinite } \Rightarrow\ A \sim A \times A). \tag{5.620}$$

Proof. Letting A be an arbitrary countably infinite set, we thus have $\mathbb{N} \sim A$ according to Note 4.25. Due to the Idempotent Law for the conjunction, $\mathbb{N} \sim A \wedge \mathbb{N} \sim A$ is then also true, and this conjunction implies $\mathbb{N}\times\mathbb{N} \sim A\times A$ with (4.675). Because $\mathbb{N} \sim A$ yields $A \sim \mathbb{N}$ with the symmetry property (4.665) and since $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$ is true according to (5.618), we have the cascade of equivalences

$$A\ \sim\ \mathbb{N}\ \sim\ \mathbb{N} \times \mathbb{N}\ \sim\ A \times A.$$

The property of transitivity (4.666) allows us then to infer from these equivalences first

$$\mathbb{N} \sim A \times A \tag{5.621}$$

and then (using $A \sim \mathbb{N}$) the desired consequent $A \sim A\times A$. As the set A was initially arbitrary, we may therefore conclude that the universal sentence (5.620) holds, as claimed. □

Note 5.35. The equivalence (5.621) shows that the Cartesian product of any countably infinite set with itself is again countably infinite.

Corollary 5.155. *It is true that the Cartesian product of any countable set with itself is countable, that is,*

$$\forall A\,(A \text{ is countable } \Rightarrow\; A \times A \text{ is countable}). \tag{5.622}$$

Proof. We take an arbitrary set A and assume A to be countable, so that A is by definition finite or countably infinite. In case A is finite, the Cartesian product $A \times A$ is evidently finite by virtue of the Finiteness of the Cartesian product of two finite sets; thus, $A \times A$ is countable in the first case. In the other case that A is countably infinite, the Cartesian product $A \times A$ is countably infinite, as mentioned in Note 5.35. Thus, $A \times A$ is countable in any case, and since A was initially arbitrary, we may infer from this the truth of the corollary. □

Exercise 5.62. Show that the Cartesian product of two countably infinite sets A and B is itself countably infinite and moreover equivalent to A, i.e.

$$\forall A, B\,([A \text{ is countably infinite } \wedge\; B \text{ is countably infinite}] \Rightarrow\; A \sim A \times B). \tag{5.623}$$

(Hint: Proceed similarly as in the proof of Corollary 5.154, showing now that

$$\mathbb{N} \sim A \times B \tag{5.624}$$

holds.)

Theorem 5.156 (Countability of the Cartesian product of two countable sets). *It is true that the Cartesian product of two countable sets is itself countable, that is,*

$$\forall A, B\,([A \text{ is countable } \wedge\; B \text{ is countable}] \Rightarrow\; A \times B \text{ is countable}). \tag{5.625}$$

Proof. We take arbitrary sets A and B, and we assume both A and B to be countable. To establish the countability of the Cartesian product of these two sets, we consider the two cases $A = \emptyset \vee B = \emptyset$ and $\neg(A = \emptyset \vee B = \emptyset)$. The first case $A = \emptyset \vee B = \emptyset$ implies $A \times B = \emptyset$ with (3.27), where $\emptyset$ is the natural number 0 in view of (2.49) and therefore a finite set due to (4.464). Consequently, we find by means of substitution that $A \times B$ is finite (in the first case).

The second case $\neg(A = \emptyset \vee B = \emptyset)$ implies on the one hand $A \times B \neq \emptyset$ with (3.27), on the other hand $A \neq \emptyset$ and $B \neq \emptyset$ with De Morgan's Law for the disjunction. Since A and B are countable by assumption, we can apply the Countability Criterion (4.653) to infer from the previous findings that there exist particular surjections $f_A : \mathbb{N} \twoheadrightarrow A$ and $f_B : \mathbb{N} \twoheadrightarrow B$. We now use these functions to define a new function g with domain $\mathbb{N} \times \mathbb{N}$ such that

$$\begin{aligned} &\forall x\,(x \in \mathbb{N} \times \mathbb{N} \\ &\quad \Rightarrow [g(x) \in A \times B \,\wedge\, \exists m,n\,(x = (m,n) \,\wedge\, g(x) = (f_A(m), f_B(n)))]). \end{aligned} \tag{5.626}$$

We accomplish this task by means of Function definition by replacement, by verifying the universal sentence

$$\begin{aligned} &\forall x\,(x \in \mathbb{N} \times \mathbb{N} \\ &\quad \Rightarrow \exists! y\,(y \in A \times B \,\wedge\, \exists m,n\,(x = (m,n) \,\wedge\, y = (f_A(m), f_B(n))))). \end{aligned} \tag{5.627}$$

Letting $x \in \mathbb{N} \times \mathbb{N}$ be arbitrary, we see in light of the definition of the Cartesian product of two sets that there exist particular constants $\bar{m} \in \mathbb{N}$ and $\bar{n} \in \mathbb{N}$ with

$$x = (\bar{m}, \bar{n}). \tag{5.628}$$

These constants are thus elements of the domain of the functions f_A and f_B, so that we obtain the uniquely determined function values

$$f_A(\bar{m}) \in A \,\wedge\, f_B(\bar{n}) \in B \tag{5.629}$$

with the Function Criterion. We may then define the ordered pair

$$\bar{y} = (f_A(\bar{m}), f_B(\bar{n})), \tag{5.630}$$

which equation shows in conjunction with (5.628) that the existential sentence

$$\exists m,n\,(x = (m,n) \,\wedge\, y = (f_A(m), f_B(n))) \tag{5.631}$$

is true. Evidently, (5.629) implies that the ordered pair (5.630) is element of the Cartesian product $A \times B$, that is, $\bar{y} \in A \times B$. The conjunction of this finding and the existential sentence (5.631) implies now that the existential part of the uniquely existential sentence in (5.627) holds.

Concerning the uniqueness part, we let y' and y'' be arbitrary such that $y', y'' \in A \times B$ and the existential sentences

$$\begin{aligned} &\exists m,n\,(x = (m,n) \,\wedge\, y' = (f_A(m), f_B(n))) \\ &\exists m,n\,(x = (m,n) \,\wedge\, y'' = (f_A(m), f_B(n))) \end{aligned}$$

are satisfied. Thus, there are particular constants $\bar{m}'$, $\bar{n}'$, $\bar{m}''$ and $\bar{n}''$ such that the equations

$$\begin{aligned} x &= (\bar{m}', \bar{n}') \\ &= (\bar{m}'', \bar{n}'') \end{aligned}$$

and

$$y' = (f_A(\bar{m}'), f_B(\bar{n}'))$$
$$y'' = (f_A(\bar{m}''), f_B(\bar{n}''))$$

are true. On the one hand, the two equations for x imply $(\bar{m}', \bar{n}') = (\bar{m}'', \bar{n}'')$, which equations yields with the Equality Criterion for ordered pairs $\bar{m}' = \bar{m}''$ and $\bar{n}' = \bar{n}''$. On the other hand, the two equations for y' and y'' gives us then through substitutions

$$y' = (f_A(\bar{m}'), f_B(\bar{n}')) = (f_A(\bar{m}''), f_B(\bar{n}'')) = y''.$$

Since y' and y'' are arbitrary, we can infer from the resulting equation $y' = y''$ the truth also of the uniqueness part of the uniquely existential sentence in (5.627). Because x was also arbitrary, the universal sentence (5.627) follows then to be true as well. Consequently, there exists indeed a unique function g with domain $\mathbb{N} \times \mathbb{N}$ whose values are defined by (5.626).

Next, let us apply the Equality Criterion for sets to prove that $A \times B$ is the range of the function g, that is,

$$\forall y\,(y \in \mathrm{ran}(g) \Leftrightarrow y \in A \times B). \tag{5.632}$$

Letting y be arbitrary and assuming first $y \in \mathrm{ran}(g)$ to be true, we have then by definition of a range $(\bar{x}, y) \in g$ for some particular constant $\bar{x}$, which is in $\mathbb{N} \times \mathbb{N}$ by definition of a domain. Therefore, the definition of the function g in (5.626) yields especially $g(\bar{x}) \in A \times B$. Observing that $(\bar{x}, y) \in g$ can be written in function function as $y = g(\bar{x})$, we thus find $y \in A \times B$ to be true via substitution, so that the first part ('$\Rightarrow$') of the equivalence in (5.632) holds.

Concerning the second part ('$\Leftarrow$'), we assume now $y \in A \times B$ to be true. This assumption implies with the definition of the Cartesian product of two sets that there exist particular constants $\bar{a} \in A$ and $\bar{b} \in B$ such that $y = (\bar{a}, \bar{b})$. Recalling the surjectivity of f_A and f_B, we obtain then with the Surjection Criterion also particular constants $\bar{x}_1$ and $\bar{x}_2$ satisfying $f_A(\bar{x}_1) = \bar{a}$ and $f_B(\bar{x}_2) = \bar{b}$. These equations show clearly that $\bar{x}_1$ and $\bar{x}_2$ are in the domain $\mathbb{N}$ of the functions f_A and f_B, with the consequence that the ordered pair $(\bar{x}_1, \bar{x}_2)$ is in the domain $\mathbb{N} \times \mathbb{N}$ of the function g. Therefore, the definition of g gives rise to the existence of particular constants $\bar{m}, \bar{n}$ satisfying the conjunction of $(\bar{x}_1, \bar{x}_2) = (\bar{m}, \bar{n})$ and and $g((\bar{x}_1, \bar{x}_2)) = (f_A(\bar{m}), f_B(\bar{n}))$. The former equation yields with the Equality Criterion for ordered pairs $\bar{x}_1 = \bar{m}$ and $\bar{x}_2 = \bar{n}$, so that the latter equation becomes after substitutions

$$g((\bar{x}_1, \bar{x}_2)) = (f_A(\bar{x}_1), f_B(\bar{x}_2)) = (\bar{a}, \bar{b}) = y.$$

Writing the resulting equation $y = g((\bar{x}_1, \bar{x}_2))$ in the form $((\bar{x}_1, \bar{x}_2), y) \in g$, we now see clearly that y is in the range of g, completing the proof of the implication '$\Leftarrow$' in (5.632).

Since y is arbitrary, we may therefore conclude that the universal sentence (5.632) is true, which then further implies the truth of the equality $\mathrm{ran}(g) = A \times B$. Consequently, the function $g : \mathbb{N} \times \mathbb{N} \to A \times B$ is a surjection. Due to the equivalence $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$ in (5.618), there exists a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, say $\bar{h}$, which thus constitutes in particular a surjection. Then, $g \circ \bar{h}$ is a surjection from $\mathbb{N}$ to $A \times B$ according to the Surjectivity of the composition of two surjections. Recalling here the truth of $A \times B \neq \emptyset$, we see in light of the existence of a surjection from $\mathbb{N}$ to $A \times B$ and in light of the Countability Criterion (4.653) that the Cartesian product $A \times B$ is countable.

We thus completed the proof by cases, and as the sets A and B were initially arbitrary, we may finally conclude that the stated theorem is indeed true. □

Theorem 5.157 (Countability of the union of a countable system of countable sets). *It is true that the union of any countable system of countable sets constitutes a countable set, that is,*

$$\forall \mathcal{S}\,([\mathcal{S} \text{ is countable } \land\ \forall A\,(A \in \mathcal{S} \Rightarrow A \text{ is countable})] \Rightarrow \bigcup \mathcal{S} \text{ is countable}). \tag{5.633}$$

Proof. We establish first the universal sentence

$$\begin{aligned}&\forall \mathcal{S}\,([\mathcal{S} \text{ is countable } \land\ \emptyset \notin \mathcal{S}\ \land\ \forall A\,(A \in \mathcal{S} \Rightarrow A \text{ is countable})]\\ &\quad \Rightarrow \bigcup \mathcal{S} \text{ is countable}),\end{aligned} \tag{5.634}$$

letting $\mathcal{S}$ be an arbitrary set (system). Noting that the Law of the Excluded Middle gives rise to the true disjunction $\mathcal{S} = \emptyset \lor \mathcal{S} \neq \emptyset$, we prove now the implication by cases. In the first case, we assume both the antecedent and $\mathcal{S} = \emptyset$ to be true, which equation yields

$$\bigcup \mathcal{S} = \bigcup \emptyset = \emptyset = 0$$

by means of substitution, (2.205) and (2.49). Recalling that every natural number is a finite set (see Corollary 4.98), we have that $\bigcup \mathcal{S}$ is finite set in the first case.

In the second case, we assume that $\mathcal{S}$ is a nonempty countable set that does not contain the empty set, whose elements are all countable sets. Due to the Countability Criterion (4.653), there exists then a surjection from

$\mathbb{N}$ to $\mathcal{S}$, say $\bar{f} : \mathbb{N} \twoheadrightarrow \mathcal{S}$. Thus, the system $\mathcal{S}$ constitutes the range of $\bar{f}$, that is, $\mathcal{S} = \mathrm{ran}(\bar{f})$, where $\bar{f}$ can in view of its domain be written as the sequence $\bar{f} = (\bar{A}_n)_{n\in\mathbb{N}}$. According to the Function Criterion, every element n in the domain $\mathbb{N}$ of $\bar{f}$ is then associated with the unique value/term A_n in the codomain/range $\mathcal{S}$ of $\bar{f}$, i.e. we have

$$\forall n\,(n \in \mathbb{N} \Rightarrow A_n \in \mathcal{S}). \tag{5.635}$$

Consequently, we have

$$\forall n\,(n \in \mathbb{N} \Rightarrow A_n \text{ is countable}), \tag{5.636}$$

noting that any $n \in \mathbb{N}$ gives with (5.635) the element A_n in $\mathcal{S}$, which is countable by assumption. The preceding assumptions allow us also to choose for each of the countable sets in $\mathcal{S}$ a surjection from $\mathbb{N}$ to that set. To begin with, let us observe that the assumed $\emptyset \notin \mathcal{S}$ implies the universal sentence

$$\forall A\,(A \in \mathcal{S} \Rightarrow A \neq \emptyset) \tag{5.637}$$

with (2.5). This allows us to establish

$$\forall n\,(n \in \mathbb{N} \Rightarrow A_n \neq \emptyset), \tag{5.638}$$

because every $n \in \mathbb{N}$ gives us $A_n \in \mathcal{S}$ with (5.635) and then $A_n \neq \emptyset$ with (5.636). Moreover, we obtain now

$$\forall n\,(n \in \mathbb{N} \Rightarrow \exists g\,(g : \mathbb{N} \twoheadrightarrow A_n)), \tag{5.639}$$

since the assumption $n \in \mathbb{N}$ (for any arbitrary n) implies on the one hand with (5.636) that A_n is countable, and on the other hand with (5.638) that $A \neq \emptyset$ holds, so that the existential sentence in (5.639) follows to be true with the Countability Criterion (4.653). Based on these representation, we collect now for any index $n \in \mathbb{N}$ the surjections from $\mathbb{N}$ to A_n. For this purpose, we prove

$$\forall n\,(n \in \mathbb{N} \Rightarrow \exists! \mathcal{Y}\,\forall g\,(g \in \mathcal{Y} \Leftrightarrow [g \in A_n^{\mathbb{N}} \wedge g : \mathbb{N} \twoheadrightarrow A_n])), \tag{5.640}$$

letting $n \in \mathbb{N}$ be arbitrary and observing the truth of the uniquely existential sentence in light of the Axiom of Specification and the Equality Criterion for sets. According to Function definition by replacement, there exists then a unique function G with domain $\mathbb{N}$, in other words a unique sequence $(G_n)_{n\in\mathbb{N}}$, such that

$$\forall n\,(n \in \mathbb{N} \Rightarrow \forall g\,(g \in G(n) \Leftrightarrow [g \in A_n^{\mathbb{N}} \wedge g : \mathbb{N} \twoheadrightarrow A_n])). \tag{5.641}$$

Here, we can show that the empty set is not an element of the range of G, that is, $\emptyset \notin \mathrm{ran}(G)$. To do this, we apply once again (2.5) and demonstrate the truth of the equivalent universal sentence

$$\forall \mathcal{Y}\,(\mathcal{Y} \in \mathrm{ran}(G) \Rightarrow \mathcal{Y} \neq \emptyset). \tag{5.642}$$

We let $\mathcal{Y}$ be arbitrary and assume $\mathcal{Y} \in \mathrm{ran}(G)$ to be true, so that there exists by definition of a range a constant, say $\bar{n}$, satisfying $(\bar{n}, \mathcal{Y}) \in G$. Then, the definition of a domain yields $\bar{n} \in \mathbb{N}$ $[= \mathrm{dom}(G)]$, which in turn implies with (5.639) that there exists a constant, say $\bar{g}$, such that $\bar{g} : \mathbb{N} \twoheadrightarrow A_{\bar{n}}$. As a function from $\mathbb{N}$ to $A_{\bar{n}}$, $\bar{g}$ is then an element of the set of functions $A_{\bar{n}}^{\mathbb{N}}$, so that the surjection $\bar{g} : \mathbb{N} \twoheadrightarrow A_{\bar{n}}$ turns out to be an element of $G(\bar{n})$ because of (5.641). Noting that we can write the previous finding $(\bar{n}, \mathcal{Y}) \in G$ in function/sequence notation as $\mathcal{Y} = G_{\bar{n}}$, we obtain from $\bar{g} \in G(\bar{n})$ via substitution $\bar{g} \in \mathcal{Y}$, which shows clearly that $\mathcal{Y}$ is nonempty. Thus, the implication in (5.642) is true, in which $\mathcal{Y}$ is arbitrary, so that the universal sentence (5.642) follows to be true. Then, the equivalent negation $\emptyset \notin \mathrm{ran}(G)$ is also true, which allows us to apply the Axiom of Choice to establish the existence of a function, say $\bar{F} : \mathrm{ran}(G) \to \bigcup \mathrm{ran}(G)$, such that

$$\forall \mathcal{Y}\,(\mathcal{Y} \in \mathrm{ran}(G) \Rightarrow \bar{F}(\mathcal{Y}) \in \mathcal{Y})) \tag{5.643}$$

Since the sequence $(G_n)_{n \in \mathbb{N}}$ is evidently a function $G : \mathbb{N} \to \mathrm{ran}(G)$, we obtain the composition $\bar{F} \circ G : \mathbb{N} \to \bigcup \mathrm{ran}(G)$ with Proposition 3.178. Thus, $\bar{F} \circ G$ can be written as the sequence $(\bar{g}_n)_{n \in \mathbb{N}}$. To bring out more clearly the structure of this sequence, we demonstrate that each term $\bar{g}_n$ is a (chosen) surjection from $\mathbb{N}$ to A_n, that is,

$$\forall n\,(n \in \mathbb{N} \Rightarrow \bar{g}_n : \mathbb{N} \twoheadrightarrow A_n). \tag{5.644}$$

Letting $n \in \mathbb{N}$ be arbitrary, we have for the corresponding term $\bar{g}_n = (\bar{F} \circ G)(n) = \bar{F}(G(n))$ (using the notation for compositions), where $G(n)$ is evidently in the range of G, so that (5.643) gives $[\bar{g}_n =]$ $\bar{F}(G(n)) \in G(n)$. The resulting, $\bar{g}_n \in G(n)$ implies now especially $\bar{g}_n : \mathbb{N} \twoheadrightarrow A_n$ with (5.641), which is the desired consequent of the implication to be proven. Since n was arbitrary, we may therefore conclude that (5.644) is indeed true.

Our next task is to establish another function h with domain $\mathbb{N} \times \mathbb{N}$ whose values are defined by

$$\forall z\,(z \in \mathbb{N} \times \mathbb{N} \Rightarrow \exists m, n\,(h(z) \in \bigcup \mathcal{S} \,\wedge\, z = (m, n) \,\wedge\, h(z) = \bar{g}_m(n))). \tag{5.645}$$

We can accomplish this task through Function definition by replacement, by proving

$$\forall z\,(z \in \mathbb{N} \times \mathbb{N} \Rightarrow \exists! y\,(\exists m, n\,(y \in \bigcup \mathcal{S} \wedge z = (m, n) \wedge y = \bar{g}_m(n)))). \tag{5.646}$$

We take an arbitrary element $z \in \mathbb{N} \times \mathbb{N}$, so that there exist by definition of the Cartesian product of two sets particular elements $\bar{m} \in \mathbb{N}$ and $\bar{n} \in \mathbb{N}$ for which $(\bar{m}, \bar{n}) = z$ holds. Thus, the natural number $\bar{m}$ is associated with the term $\bar{g}_{\bar{m}}$ of the sequence $(\bar{g}_n)_{n \in \mathbb{N}}$, which term constitutes a surjection $\bar{g}_{\bar{m}} : \mathbb{N} \twoheadrightarrow A_{\bar{m}}$, as shown by (5.644). Then, the natural number $\bar{n}$ is associated with the value $\bar{y} = \bar{g}_{\bar{m}}(\bar{n})$ in the codomain/range $A_{\bar{m}}$ of that surjection. Because of $\bar{m} \in \mathbb{N}$, we obtain furthermore $A_{\bar{m}} \in \mathcal{S}$ with (5.635), which implies in conjunction with the preceding finding $\bar{y} \in A_{\bar{m}}$ that $\bar{y}$ is an element of the union of the set system $\mathcal{S}$. Having thus established the truth of $\bar{y} \in \bigcup \mathcal{S}$, of $z = (\bar{m}, \bar{n})$ and of $\bar{y} = \bar{g}_{\bar{m}}(\bar{n})$, we now see that the existential part of the uniquely existential sentence in (5.646) holds.

Having already found a particular constant $\bar{y}$ for which the existential sentence with respect to m and n holds, we may prove the uniqueness part by means of Method 1.18. For this purpose, we let y' be arbitrary, assume the existence of particular constants m', n' satisfying $y' \in \bigcup \mathcal{S}$, $z = (m', n')$ and $y' = \bar{g}_{m'}(n')$, and show that $\bar{y} = y'$ is implied. Combining the equations $z = (m', n')$ and $z = (\bar{m}, \bar{n})$ yields $(m', n') = (\bar{m}, \bar{n})$, and the Equality Criterion for ordered pairs gives us therefore $m' = \bar{m}$ as well as $n' = \bar{n}$. Applying now substitutions based on these equations, we obtain

$$y' = \bar{g}_{m'}(n') = \bar{g}_{\bar{m}}(\bar{n}) = \bar{y},$$

resulting in the desired consequent $\bar{y} = y'$. As y' was arbitrary, we may infer from this equation the truth of the uniqueness part and thus the truth of the uniquely existential sentence in (5.646). Because z was arbitrary, we may further conclude that the universal sentence (5.646) holds, so that there exists indeed a unique function h with domain $\mathbb{N} \times \mathbb{N}$ such that (5.645).

Let us check that $\mathcal{S}$ is a codomain of that function, i.e. that the range of h is included in $\bigcup \mathcal{S}$. To do this, we apply the definition of a subset and take accordingly an arbitrary constant y, assuming $y \in \operatorname{ran}(h)$ to be true. Then, the definition of a range gives us a particular constant $\bar{z}$ such that $(\bar{z}, y) \in h$ is true, which we may write in function notation as $y = h(\bar{z})$. Recalling that $\mathbb{N} \times \mathbb{N}$ is the domain of h, we also see that $\bar{z} \in \mathbb{N} \times \mathbb{N}$ holds. Consequently, (5.645) gives us $[y =]\ h(\bar{z}) \in \bigcup \mathcal{S}$. We thus proved that $y \in \operatorname{ran}(h)$ implies $y \in \bigcup \mathcal{S}$, and since y is arbitrary, we may infer from the truth of this implication the truth of the suggested inclusion $\operatorname{ran}(h) \subseteq \bigcup \mathcal{S}$. This demonstrates that h is a function from $\mathbb{N} \times \mathbb{N}$ to the union $\bigcup \mathcal{S}$.

In the next step, we prove conversely that the union $\bigcup \mathcal{S}$ is included in the range of h, letting y be arbitrary and assuming $y \in \bigcup \mathcal{S}$ to hold. By definition of the union of a set system, there exists then a set, say $\bar{X}$, for which $\bar{X} \in \mathcal{S}$ and $y \in \bar{X}$ both hold. Recalling the truth of the equation $\mathcal{S} = \operatorname{ran}(\bar{f})$, it follows with the definition of a range that there exists a constant, say $\bar{m}$, such that $(\bar{m}, \bar{X}) \in \bar{f}$ is true. Since $\bar{f}$ is the sequence $(\bar{A}_n)_{n\in\mathbb{N}}$, we obtain with the definition of a domain $\bar{m} \in \mathbb{N}$, and we can write for the corresponding term $\bar{X} = \bar{f}_{\bar{m}} = \bar{A}_{\bar{m}}$. Thus, $y \in \bar{X}$ implies $y \in \bar{A}_{\bar{m}}$ by means of substitution. In addition, $\bar{m} \in \mathbb{N}$ gives rise to the surjection $\bar{g}_{\bar{m}} : \mathbb{N} \twoheadrightarrow A_{\bar{m}}$ due to (5.644), so that y is in the range of $\bar{g}_{\bar{m}}$. Consequently, there exists a particular constant, say $\bar{n}$, satisfying $(\bar{n}, y) \in \bar{g}_{\bar{m}}$, which finding we can write in the form $y = \bar{g}_{\bar{m}}(\bar{n})$. Defining now the ordered pair $\bar{z} = (\bar{m}, \bar{n})$, we see in light of $y \in \bigcup \mathcal{S}$ and the preceding equations for z and y that the existential sentence

$$\exists m, n\,(y \in \bigcup \mathcal{S} \wedge \bar{z} = (m,n) \wedge y = \bar{g}_m(n)) \tag{5.647}$$

is true. Noting that $(\bar{n}, y) \in \bar{g}_{\bar{m}}$ yields $\bar{n} \in \mathbb{N}$ [$= \mathrm{dom}(\bar{g}_{\bar{m}})$] by definition of a domain, so that both $\bar{m}$ and $\bar{n}$ are natural numbers, we have that the ordered pair $\bar{z}$ is element of the Cartesian product $\mathbb{N} \times \mathbb{N}$. This fact implies with (5.645) that the corresponding function value $h(\bar{z})$ satisfies

$$\exists m, n\,(h(\bar{z}) \in \bigcup \mathcal{S} \wedge \bar{z} = (m,n) \wedge h(\bar{z}) = \bar{g}_m(n)). \tag{5.648}$$

Then, in view of the uniquely existential sentence in (5.646), we obtain the equation $y = h(\bar{z})$, which we can write also in the form $(\bar{z}, y) \in h$. Thus, y is an element of the range of h, so that the proof of the implication $y \in \bigcup \mathcal{S} \Rightarrow y \in \mathrm{ran}(h)$ is complete. Because y was arbitrary, we may therefore conclude (by definition of a subset) that the inclusion $\bigcup \mathcal{S} \subseteq \mathrm{ran}(h)$ holds.

Having thus established the truth of the two inclusions $\mathrm{ran}(h) \subseteq \bigcup \mathcal{S}$ and $\bigcup \mathcal{S} \subseteq \mathrm{ran}(h)$, we can use the Axiom of Extension to infer the truth of the equality $\mathrm{ran}(h) = \bigcup \mathcal{S}$. This shows that h is a surjection from $\mathbb{N} \times \mathbb{N}$ to $\bigcup \mathcal{S}$. As $\mathbb{N}$ and the preceding Cartesian product $\mathbb{N} \times \mathbb{N}$ are equinumerous, as shown by (5.618), there exists a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, say $\bar{b}$. This bijection is by definition a surjection, that is, $\bar{b} : \mathbb{N} \twoheadrightarrow \mathbb{N} \times \mathbb{N}$, so that the composition $h \circ \bar{b}$ constitutes a surjection from $\mathbb{N}$ to $\bigcup \mathcal{S}$, according to (3.649). We thus proved the existential sentence $\exists f\,(f : \mathbb{N} \twoheadrightarrow \bigcup \mathcal{S})$, which evidently implies with the Countability Criterion (4.653) that $\bigcup \mathcal{S}$ is a countable set.

This finding completes the proof of the implication in (5.634) by cases. Since $\mathcal{S}$ was initially arbitrary, we may therefore conclude that the universal sentence (5.634) holds.

We are now in a position to prove the proposed universal sentence (5.633), letting $\mathcal{S}$ be an arbitrary set (system) and considering the two cases $\emptyset \in \mathcal{S}$ and $\emptyset \notin \mathcal{S}$. In the first case, we assume $\mathcal{S}$ to be a countable set such that

$$\forall A\,(A \in \mathcal{S} \Rightarrow A \text{ is countable}). \tag{5.649}$$

and $\emptyset \in \mathcal{S}$ hold. We form then the singleton $\{\emptyset\}$ and observe in light of (4.656) that the set difference $\mathcal{S} \setminus \{\emptyset\}$ (being a subset of $\mathcal{S}$) is then also a countable set, for which

$$\emptyset \notin \mathcal{S} \setminus \{\emptyset\} \tag{5.650}$$

is true according to (2.179). The preceding inclusion allows us to establish

$$\forall A\,(A \in \mathcal{S} \setminus \{\emptyset\} \Rightarrow A \text{ is countable}).$$

Indeed, letting A be arbitrary and assuming $A \in \mathcal{S} \setminus \{\emptyset\}$ to be true, the definition of a subset yields $A \in \mathcal{S}$, which in turn implies with the assumed (5.649) that A is countable. Since A is arbitrary, the preceding universal sentence follows therefore to be true. In conjunction with the countability of $\mathcal{S} \setminus \{\emptyset\}$ and with (5.650), this implies with (5.634) that $\bigcup(\mathcal{S} \setminus \{\emptyset\})$ is a countable set. Since this union is identical with the union $\bigcup \mathcal{S}$ according to (2.203), it follows by means of substitution that the latter constitutes a countable set.

In the second case, we assume that $\mathcal{S}$ is a countable set satisfying (5.649) and $\emptyset \notin \mathcal{S}$. Thus, the union $\bigcup \mathcal{S}$ follows immediately to be true with (5.634). Because $\mathcal{S}$ was arbitrary, we can now infer from these findings the truth of the stated theorem. □

Exercise 5.63. Show that the union of any family having a countable index set and countable terms is itself countable, that is,

$$\begin{aligned}&\forall I, f\,([I \text{ is countable } \wedge\ f = (A_i)_{i \in I}\ \wedge\ \forall i\,(i \in I \Rightarrow A_i \text{ is countable})]\\ &\quad \Rightarrow \bigcup_{i \in I} A_i \text{ is countable}).\end{aligned} \tag{5.651}$$

(Hint: Use Proposition 4.141 and Theorem 5.157.)

Corollary 5.158. *It is true that the union of any countably infinite system of countably infinite sets constitutes a countably infinite set, that is,*

$$\begin{aligned}&\forall \mathcal{S}\,([\mathcal{S} \text{ is countably infinite } \wedge\ \forall A\,(A \in \mathcal{S} \Rightarrow A \text{ is countably infinite})]\\ &\quad \Rightarrow \bigcup \mathcal{S} \text{ is countably infinite}).\end{aligned} \tag{5.652}$$

Proof. We let $\mathcal{S}$ be an arbitrary set (system), assuming that $\mathcal{S}$ is a countably infinite set whose elements are all countably infinite sets. Thus, $\mathcal{S}$ is a countable set whose elements are countable, so that the union $\bigcup \mathcal{S}$ follows to be countable with Theorem 5.157. According to the Countability Criterion (4.652), there exists then an injection $g : \bigcup \mathcal{S} \hookrightarrow \mathbb{N}$.

In addition, the assumption that $\mathcal{S}$ is a countably infinite implies the existence of a particular bijective sequence $A : \mathbb{N} \rightleftarrows \mathcal{S}$, for which $A_1 \in \mathcal{S}$

holds in particular according to the Function Criterion. Since we assumed all elements of $\mathcal{S}$ to be countably infinite, there is then also a particular bijection $\bar{f} : \mathbb{N} \rightleftarrows A_1$. Thus, $\bar{f}$ is an injection $\bar{f} : \mathbb{N} \hookrightarrow A_1$, where $A_1 \in \mathcal{S}$ implies the inclusion $A_1 \subseteq \bigcup \mathcal{S}$ because of (2.201), so that we obtain $\bar{f}$ constitutes the injection $\bar{f} : \mathbb{N} \hookrightarrow \bigcup \mathcal{S}$ due to (3.623). This demonstrates the existence of an injection $f : \mathbb{N} \hookrightarrow \bigcup \mathcal{S}$.

The existence of injections $f : \mathbb{N} \hookrightarrow \bigcup \mathcal{S}$ and $g : \bigcup \mathcal{S} \hookrightarrow \mathbb{N}$ implies then with the Cantor-Schröder-Bernstein theorem that $\mathbb{N}$ and $\bigcup \mathcal{S}$ are equinumerous, that is, $\mathbb{N} \sim \bigcup \mathcal{S}$. This means that there exists a bijection from $\mathbb{N}$ to the union $\bigcup \mathcal{S}$, so that $\bigcup \mathcal{S}$ is countably infinite. As the set $\mathcal{S}$ was initially arbitrary, we may therefore conclude that the stated universal sentence is true. □

Exercise 5.64. Show that the union of any family having a countable, nonempty index set and countably infinite terms is itself countably infinite, that is,

$$\forall I, f\,([I \text{ is countable } \wedge\ I \neq \emptyset\ \wedge\ f = (A_i)_{i \in I} \tag{5.653}$$
$$\wedge\ \forall i\,(i \in I \Rightarrow A_i \text{ is countably infinite})] \Rightarrow \bigcup_{i \in I} A_i \text{ is countably infinite}).$$

(Hint: Proceed similarly as in the proof of Corollary 5.158, using here (5.651) and (2.42).)

Theorem 5.159 (Countability of Cartesian powers $\mathbb{N}^n$). *It is true*

a) *for any $n \in \mathbb{N}$ that $\mathbb{N}^n$ is countable.*

b) *for any $n \in \mathbb{N}_+$ that $\mathbb{N}^n$ is countably infinite.*

Proof. Let us begin with the observation that $\mathbb{N}^0$ denotes the Cartesian product of a constant sequence on $\{1, \ldots, 0\} = \emptyset$, i.e. of a family of sets with empty index set. It therefore follows with (3.863) that the Cartesian product $\mathbb{N}^0$ constitutes the singleton $\{\emptyset\}$, which is a finite set according to (4.470), and thus a countable set by definition.

Next, we verify that $\mathbb{N}^n$ is countably infinite for any positive natural number n, proving the universal sentence

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \exists d\,(d : \mathbb{N} \rightleftarrows \mathbb{N}^n)) \tag{5.654}$$

by means of mathematical induction. In the base case ($n = 1$), we apply Function definition by replacement to establish a function $\bar{d}$ with domain $\mathbb{N}$ such that

$$\forall m\,(m \in \mathbb{N} \Rightarrow [\bar{d}(m) \in \mathbb{N}^1\ \wedge\ [\bar{d}(m)](1) = m]). \tag{5.655}$$

For this purpose, we prove the universal sentence

$$\forall m\,(m \in \mathbb{N} \Rightarrow \exists! f\,(f \in \mathbb{N}^1 \,\wedge\, f(1) = m)), \tag{5.656}$$

letting $m \in \mathbb{N}$ be arbitrary. Observing that the Cartesian power $\mathbb{N}^1$ denotes the Cartesian product $\bigtimes_{i=1}^{1} \mathbb{N}$, we demonstrate now that the singleton $\bar{f} = \{(1, m)\}$ is an element of that Cartesian power. On the one hand, since the singleton $\{(1, m)\}$ constitutes a function from $\{1\}$ to $\{m\}$ according to Corollary 3.156, we may view it as a family with index set $\{1\} = \{1, \ldots, 1\}$ (recalling the notation for initial segments of $\mathbb{N}_+$). On the other hand, we may establish the universal sentence

$$\forall i\,(i \in \{1\} \Rightarrow \bar{f}(i) \in \mathbb{N}) \tag{5.657}$$

letting $i \in \{1\}$ be arbitrary gives $i = 1$ with (2.169) and therefore the unique value $\bar{f}(i) = \bar{f}(1)$ in $\{m\}$, so that another application of (2.169) gives $\bar{f}(i) = \bar{f}(1) = m$. Furthermore, in view of the assumed $m \in \mathbb{N}$, the preceding equations give $f(i) \in \mathbb{N}$, proving the implication in (5.657). Since i was arbitrary, we may now infer from this the truth of the universal sentence (5.657), which implies – in conjunction with the fact that $\bar{f} = \{(1, m)\}$ is a family with index set $\{1, \ldots, 1\}$ – that $\bar{f} \in \bigtimes_{i=1}^{1} \mathbb{N}$ is indeed true (by definition of the Cartesian product of a family of sets). Thus, we found $\bar{f} \in \mathbb{N}^1$ and $\bar{f}(1) = m$, so that the existential part of the uniquely existential sentence in (5.656) holds.

Regarding the uniqueness part, we let f and f' be arbitrary, assume $f \in \mathbb{N}^1$, $f(1) = m$, $f' \in \mathbb{N}^1$ and $f'(1) = m$ to be true, and show that $f = f'$ is implied. We prove this equation by means of the Equality Criterion for functions, noting that f and f', as elements of the Cartesian product $\mathbb{N}^1 = \bigtimes_{i=1}^{1} \mathbb{N}$, are families with common index set $\{1, \ldots, 1\}$, and constitute thus functions with common domain $\{1\}$. We therefore demonstrate the truth of the universal sentence

$$\forall i\,(i \in \{1\} \Rightarrow f(i) = f'(i)). \tag{5.658}$$

Letting $i \in \{1\}$ be arbitrary, we obtain $i = 1$ with (2.169) and therefore via substitutions

$$f(i) = f(1) = n = f'(1) = f'(i),$$

resulting in the desired $f(i) = f'(i)$. Because i is arbitrary, we may therefore conclude that the universal sentence (5.658) holds, which implies then (with the Equality Criterion for functions) $f = f'$. Here, f and f' were also arbitrary, so that the uniqueness part of the uniquely existential sentence in (5.656) holds as well. Since m was arbitrary, we may now further conclude that the universal sentence (5.656) holds. Consequently, there exists a unique function/sequence $\bar{d}$ with domain $\mathbb{N}$ and terms satisfying (5.655).

Next, we show that $\bar{d} : \mathbb{N} \to \mathbb{N}^1$, i.e. that $\mathbb{N}^1$ is a codomain of $\bar{d}$, which means that the range of $\bar{d}$ is included in $\mathbb{N}^1$. For this purpose, we verify the universal sentence

$$\forall f\,(f \in \mathrm{ran}(\bar{d}) \Rightarrow f \in \mathbb{N}^1), \tag{5.659}$$

taking an arbitrary constant f and assuming $f \in \mathrm{ran}(\bar{d})$ to hold. By definition of a range, there exists then a constant, say $\bar{m}$, such that $(\bar{m}, f) \in \bar{d}$. This shows that $\bar{m}$ is in the domain $\mathbb{N}$ of $\bar{d}$, which fact implies with (5.655) that $\bar{d}(\bar{m}) \in \mathbb{N}^1$ and $[\bar{d}(\bar{m})](1) = \bar{m}$ are both true. As we may write $(\bar{m}, f) \in \bar{d}$ in function notation as $f = \bar{d}(\bar{m})$, we obtain the desired $f \in \mathbb{N}^1$ via substitution. Since f is arbitrary, we may infer from this finding the truth of the universal sentence (5.659), and therefore the truth of the inclusion $\mathrm{ran}(\bar{d}) \subseteq \mathbb{N}^1$ (using the definition of a subset); thus, $\mathbb{N}^1$ is indeed a codomain of $\bar{d}$. We may now move on and prove that the reverse inclusion $\mathbb{N}^1 \subseteq \mathrm{ran}(\bar{d})$ holds, too. To do this, we establish accordingly the universal sentence

$$\forall f\,(f \in \mathbb{N}^1 \Rightarrow f \in \mathrm{ran}(\bar{d})), \tag{5.660}$$

letting $f \in \mathbb{N}^1$ $[= \bigtimes_{i=1}^{1} \mathbb{N}]$ be arbitrary, so that f is evidently a family with index set $\{1, \ldots, 1\}$ and terms satisfying $f_i \in \mathbb{N}$ for any $i \in \{1, \ldots, 1\}$. Since $1 \in \{1\}$ $[= \{1, \ldots, 1\}]$ is clearly true, we therefore have in particular $f_1 \in \mathbb{N}$, so that $\bar{d}(f_1) \in \mathbb{N}^1$ and $[\bar{d}(f_1)](1) = f_1$ follow to be true with (5.655). Let us observe here that $\bar{d}(f_1)$ is thus a family with the same index set $\{1, \ldots, 1\} = \{1\}$ (i.e., a function with the same domain) as the family/function f. We now use this fact and prove $f = \bar{d}(f_1)$ by means of the Equality Criterion for functions, that is, by proving

$$\forall i\,(i \in \{1\} \Rightarrow f_i = [\bar{d}(f_1)](i)). \tag{5.661}$$

Letting $i \in \{1\}$ be arbitrary, we evidently find $i = 1$, giving rise to the truth of the equations

$$f_i = f_1 = [\bar{d}(f_1)](1) = [\bar{d}(f_1)](i).$$

We thus obtain $f_i = [\bar{d}(f_1)](i)$, and as i was arbitrary, we may therefore infer from this equation the truth of (5.661) and consequently the truth of $f = \bar{d}(f_1)$. Writing this equation in the form $(f_1, f) \in \bar{d}$, we now see in light of the definition of a range that $f \in \mathrm{ran}(\bar{d})$ is also true. Because f is arbitrary, the universal sentence (5.660) follows therefore to be true, so that the inclusion $\mathbb{N}^1 \subseteq \mathrm{ran}(\bar{d})$ holds indeed. In conjunction with the reverse inclusion $\mathrm{ran}(\bar{d}) \subseteq \mathbb{N}^1$, this implies with the Axiom of Extension the equation $\mathrm{ran}(\bar{d}) = \mathbb{N}^1$, so that the function $\bar{d} : \mathbb{N} \to \mathbb{N}^1$ is a surjection.

Let us now prove that this surjection is also an injection. We let accordingly $m, m' \in \mathbb{N}$ be arbitrary, and we assume $\bar{d}(m) = \bar{d}(m')$ to be true.

On the one hand, $m, m' \in \mathbb{N}$ implies with (5.655) the truth of $\bar{d}(m) \in \mathbb{N}^1$, $[\bar{d}(m)](1) = m$, $\bar{d}(m') \in \mathbb{N}^1$ and $[\bar{d}(m')](1) = m'$. Thus, $\bar{d}(m)$ and $\bar{d}(m')$ are both families with index set $\{1, \ldots, 1\}$, i.e. functions sharing the same domain $\{1\}$. Because of the Equality Criterion for functions, the assumption $\bar{d}(m) = \bar{d}(m')$ implies the truth of $[\bar{d}(m)](i) = [\bar{d}(m')](i)$ for any $i \in \{1\}$, so that the evident fact $1 \in \{1\}$ implies

$$[m =] \quad [\bar{d}(m)](1) = [\bar{d}(m')](1) \quad [= m'].$$

We therefore find $m = m'$ to be true, and since m and m' are arbitrary, we may now conclude that the surjection $\bar{d} : \mathbb{N} \twoheadrightarrow \mathbb{N}^1$ is an injection from $\mathbb{N}$ to $\mathbb{N}^1$. Thus, $\bar{d}$ constitutes a bijection from $\mathbb{N}$ to $\mathbb{N}^1$, so that the existential sentence

$$\exists d\, (d : \mathbb{N} \rightleftarrows \mathbb{N}^1) \tag{5.662}$$

is true, as required by the base case.

In the induction step, we prove

$$\forall n\, (n \in \mathbb{N}_+ \Rightarrow [\exists d\, (d : \mathbb{N} \rightleftarrows \mathbb{N}^n) \Rightarrow \exists d\, (d : \mathbb{N} \rightleftarrows \mathbb{N}^{n+1})]), \tag{5.663}$$

letting $n \in \mathbb{N}_+$ be arbitrary and making the induction assumption that there exists a particular bijection

$$\bar{d} : \mathbb{N} \rightleftarrows \mathbb{N}^n,$$

so that the Bijectivity of inverse functions yields

$$\bar{d}^{-1} : \mathbb{N}^n \rightleftarrows \mathbb{N}.$$

Recalling that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ are equinumerous, as shown in (5.619), there is also a particular bijection

$$\bar{D} : \mathbb{N} \rightleftarrows \mathbb{N} \times \mathbb{N},$$

and the Bijectivity of inverse functions gives us

$$\bar{D}^{-1} : \mathbb{N} \times \mathbb{N} \rightleftarrows \mathbb{N}.$$

To prove of the existence of a bijection $d : \mathbb{N} \rightleftarrows \mathbb{N}^{n+1}$, we use in the following Function definition by replacement to establish a function e with domain $\mathbb{N}^{n+1}$ such that

$$\forall g\, (g \in \mathbb{N}^{n+1} \Rightarrow \exists k, m\, (e(g) = \bar{D}^{-1}((k, m)) \wedge g = \bar{d}(k) \cup \{(n+1, m)\})), \tag{5.664}$$

and we will also show that this defines a bijection from $\mathbb{N}^{n+1}$ to $\mathbb{N}$, whose inverse will then be a bijection from $\mathbb{N}$ to $\mathbb{N}^{n+1}$, proving thereby the desired existential sentence. According to this method, we are required to prove the universal sentence

$$\forall g\,(g \in \mathbb{N}^{n+1} \Rightarrow \exists! y\,(\exists k, m\,(y = \bar{D}^{-1}((k,m)) \,\wedge\, g = \bar{d}(k) \cup \{(n+1, m)\}))). \tag{5.665}$$

We let $g \in \mathbb{N}^{n+1}$ be arbitrary, so that g constitutes a family with index set $\{1, \ldots, n+1\}$ whose terms satisfy

$$\forall i\,(i \in \{1, \ldots, n+1\} \Rightarrow g_i \in \mathbb{N}). \tag{5.666}$$

Evidently, the fact $n < n+1$ implies the inclusion $\{1, \ldots, n\} \subseteq \{1, \ldots, n+1\}$ with (4.260), giving rise to the restriction

$$g \restriction \{1, \ldots, n\} : \{1, \ldots, n\} \to \mathbb{N}$$

because of (3.566). This restriction is thus a family with index set $\{1, \ldots, n\}$, whose terms satisfy

$$\forall i\,(i \in \{1, \ldots, n\} \Rightarrow [g \restriction \{1, \ldots, n\}]_i \in \mathbb{N}), \tag{5.667}$$

since any $i \in \{1, \ldots, n\}$ is associated with the unique function value $[g \restriction \{1, \ldots, n\}]_i$ in the codomain $\mathbb{N}$, according to the Function Criterion. This finding demonstrates that $g \restriction \{1, \ldots, n\} \in \mathbb{N}^n$ holds, which Cartesian power is the domain of the inverse $\bar{d}^{-1}$, so that the function value

$$\bar{k} = \bar{d}^{-1}(g \restriction \{1, \ldots, n\}) \tag{5.668}$$

is a uniquely determined element of the codomain $\mathbb{N}$ of the inverse. This equation implies furthermore

$$\bar{d}(\bar{k}) = g \restriction \{1, \ldots, n\}. \tag{5.669}$$

with (3.678). As $n+1$ is clearly an element of the index set $\{1, \ldots, n+1\}$ of the family g, we can denote the corresponding term by

$$\bar{m} = g_{n+1}. \tag{5.670}$$

Having thus established the particular constants $\bar{k}$ and $\bar{m}$, we define now

$$\bar{y} = \bar{D}^{-1}((\bar{k}, \bar{m})) \tag{5.671}$$

and prove in addition that $\bar{k}$ and $\bar{m}$ satisfy the required equation

$$g = \bar{d}(\bar{k}) \cup \{(n+1, \bar{m})\}, \tag{5.672}$$

using the Equality Criterion for sets. For this purpose, we establish the universal sentence

$$\forall z \, (z \in g \Leftrightarrow z \in \bar{d}(\bar{k}) \cup \{(n+1, \bar{m})\}), \tag{5.673}$$

letting z be arbitrary. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $z \in g$ to be true. Being a family, g is a binary relation, so that the preceding assumption implies the existence of particular constants $\bar{a}, \bar{b}$ such that $(\bar{a}, \bar{b}) = z$ holds. Then, substitution yields $(\bar{a}, \bar{b}) \in g$, which shows that $\bar{a}$ is in the domain $\{1, \ldots, n+1\}$ of g. Here, we note that $\{1, \ldots, n+1\} = \{1, \ldots, n\} \cup \{n+1\}$ holds according to (4.241), so that the definition of the union of two sets gives us the true disjunction

$$\bar{a} \in \{1, \ldots, n\} \lor \bar{a} \in \{n+1\}.$$

We use this disjunction to prove the disjunction

$$z \in \bar{d}(\bar{k}) \lor z \in \{(n+1, \bar{m})\}, \tag{5.674}$$

by cases. The first case $\bar{a} \in \{1, \ldots, n\}$ implies in conjunction with $(\bar{a}, \bar{b}) \in g$ – by definition of a restriction – that $(\bar{a}, \bar{b}) \in g \restriction \{1, \ldots, n\}$ holds. Consequently, substitution based on (5.669) gives us $[z =]$ $(\bar{a}, \bar{b}) \in \bar{d}(\bar{k})$, and the disjunction (5.674) is then also true. The second case $\bar{a} \in \{n+1\}$ implies $\bar{a} = n+1$ with (2.169), which allows us to rewrite $(\bar{a}, \bar{b}) \in g$ as $\bar{b} = g(\bar{a}) = g_{n+1} = \bar{m}$. Thus, the two equations $\bar{a} = n+1$ and $\bar{b} = \bar{m}$ are true, so that $[z =]$ $(\bar{a}, \bar{b}) = (n+1, \bar{m})$ holds according to the Equality Criterion for ordered pairs. The resulting equation $z = (n+1, \bar{m})$ in turn implies $z \in \{(n+1, \bar{m})\}$ with (2.169), so that the disjunction (5.674) is again true. Having thus completed the proof by cases, we may infer from the truth of that disjunction the truth of $z \in \bar{d}(\bar{k}) \cup \{(n+1, \bar{m})\}$ by definition of the union of two sets, which finding proves the first part of the equivalence in (5.673).

Regarding the second part ('$\Leftarrow$'), we assume $z \in \bar{d}(\bar{k}) \cup \{(n+1, \bar{m})\}$ to be true, which evidently implies the truth of the disjunction (5.674). Based on this disjunction, we prove $z \in g$ by cases. In case of $z \in \bar{d}(\bar{k})$, we obtain $z \in g \restriction \{1, \ldots, n\}$ with (5.669) and therefore $z \in g$ by definition of a restriction, as desired. In the other case $z \in \{(n+1, \bar{m})\}$, which evidently implies $z = (n+1, \bar{m})$, we observe that (5.670) can be written as $(n+1, \bar{m}) \in g$, so that the desired consequent $z \in g$ follows to be true via substitution. Thus, the proof by cases is complete, and the finding $z \in g$ shows us then that the second part of the equivalence in (5.673) holds, too.

Since z was arbitrary, we may therefore conclude that the universal sentence (5.673) is true, which in turn implies the truth of the equation (5.672).

In conjunction with (5.671), this demonstrates the truth of the existential sentence

$$\exists k, m\,(\bar{y} = \bar{D}^{-1}((k,m)) \,\wedge\, g = \bar{d}(k) \cup \{(n+1,m)\}),$$

and this in turn shows that the existential part of the uniquely existential sentence in (5.665) holds.

To prove the uniqueness part, we let y and y' be arbitrary, assume the existential sentences

$$\exists k, m\,(y = \bar{D}^{-1}((k,m)) \,\wedge\, g = \bar{d}(k) \cup \{(n+1,m)\})$$
$$\exists k, m\,(y' = \bar{D}^{-1}((k,m)) \,\wedge\, g = \bar{d}(k) \cup \{(n+1,m)\})$$

to be true, and show that $y = y'$ is implied. Thus, the preceding assumptions give us particular constants $\bar{k}, \bar{m}$ and $\bar{k}', \bar{m}'$ satisfying

$$y = \bar{D}^{-1}((\bar{k}, \bar{m})) \,\wedge\, g = \bar{d}(\bar{k}) \cup \{(n+1, \bar{m})\}, \tag{5.675}$$
$$y' = \bar{D}^{-1}((\bar{k}', \bar{m}')) \,\wedge\, g = \bar{d}(\bar{k}') \cup \{(n+1, \bar{m}')\}. \tag{5.676}$$

The second parts of these conjunctions show in light of the definition of the union of a set system that $(n+1, \bar{m}) \in g$ and $(n+1, \bar{m}') \in g$ hold, which findings imply $\bar{m} = \bar{m}'$ due to the fact that g is a function. Furthermore, noting that the values $\bar{d}(\bar{k})$ and $\bar{d}(\bar{k}')$ are elements of the range/codomain $\mathbb{N}^n$ of the bijection $\bar{d}$ and constitute therefore families with common index set $\{1, \ldots, n\}$, we prove $\bar{d}(\bar{k}) = \bar{d}(\bar{k}')$ by means of the Equality Criterion for functions. To do this, we verify the universal sentence

$$\forall i\,(i \in \{1, \ldots, n\} \Rightarrow [\bar{d}(\bar{k})](i) = [\bar{d}(\bar{k}')](i)), \tag{5.677}$$

letting $i \in \{1, \ldots, n\}$ be arbitrary. Observing that $[\bar{d}(\bar{k})]_i = [\bar{d}(\bar{k})](i)$ and $[\bar{d}(\bar{k}')]_i = [\bar{d}(\bar{k}')](i)$ constitute terms/values of families/functions $\bar{d}(\bar{k})$ and $\bar{d}(\bar{k}')$, respectively, we can write the equations also as $(i, [\bar{d}(\bar{k})]_i) \in \bar{d}(\bar{k})$ and $(i, [\bar{d}(\bar{k}')]_i) \in \bar{d}(\bar{k}')$. Consequently, we obtain with the second parts of the conjunctions in (5.675) and (5.676), by definition of the union of a set system, $(i, [\bar{d}(\bar{k})]_i) \in g$ as well as $(i, [\bar{d}(\bar{k}')]_i) \in g$. These findings imply then $[\bar{d}(\bar{k})]_i = [\bar{d}(\bar{k}')]_i$ by definition of a function, so that the equation in (5.677) follows to be true via substitutions. Since i is arbitrary, we may now infer from this the truth of the universal sentence (5.677) and therefore the truth of the equality $\bar{d}(\bar{k}) = \bar{d}(\bar{k}')$. Then, being a bijection, $\bar{d}$ is an injection by definition, so that the preceding equation implies $\bar{k} = \bar{k}'$. In conjunction with the previously found equation $\bar{m} = \bar{m}'$, this gives us $(\bar{k}, \bar{m}) = (\bar{k}', \bar{m}')$ with the Equality Criterion for ordered pairs. Applying now substitution based on this equation, we obtain with the first parts of the conjunctions (5.675) and (5.676)

$$y = \bar{D}^{-1}((\bar{k}, \bar{m})) = \bar{D}^{-1}((\bar{k}', \bar{m}')) = y',$$

resulting in the desired equation $y = y'$. As y and y' were arbitrary, we may therefore conclude that the uniqueness part holds as well (besides the existential part), so that the proof of the uniquely existential sentence in (5.665) is complete. Since g was also arbitrary, the universal sentence (5.665) follows then to be true. We thus proved that there exists indeed a unique function e with domain $\mathbb{N}^{n+1}$ and values defined according to (5.664).

Our next task is to verify that $\mathbb{N}$ is a codomain of the function e, which means $\mathrm{ran}(e) \subseteq \mathbb{N}$. To prove this inclusion, we apply the definition of a subset and establish the equivalent universal sentence

$$\forall y\,(y \in \mathrm{ran}(e) \Rightarrow y \in \mathbb{N}). \tag{5.678}$$

We take an arbitrary y and assume that $y \in \mathrm{ran}(e)$ holds. By definition of a range, there exists then a constant, say $\bar{g}$, such that $(\bar{g}, y) \in e$ is true. This implies with the definition of a domain $\bar{g} \in \mathbb{N}^{n+1}$, so that the definition of e in (5.664) yields the true existential sentence

$$\exists k, m\,(e(\bar{g}) = \bar{D}^{-1}((k, m)) \wedge \bar{g} = \bar{d}(k) \cup \{(n+1, m)\}).$$

This means that there are constants, say $\bar{k}$ and $\bar{m}$, for which the equations $e(\bar{g}) = \bar{D}^{-1}((\bar{k}, \bar{m}))$ and $\bar{g} = \bar{d}(\bar{k}) \cup \{(n+1, \bar{m})\}$ are satisfied. Since we can use function notation to write $(\bar{g}, y) \in e$ in the form $y = e(\bar{g})$, we obtain $y = \bar{D}^{-1}((\bar{k}, \bar{m}))$ by means of substitution. Thus, y is a value in the codomain/range $\mathbb{N}$ of the bijection $\bar{D}^{-1}$, so that the implication in (5.678) is true. Here, y is arbitrary, so that the universal sentence (5.678) holds as well, and this means that the inclusion $\mathrm{ran}(e) \subseteq \mathbb{N}$ is true. We thus proved that $\mathbb{N}$ is indeed a codomain of the function e.

We can establish also the converse inclusion $\mathbb{N} \subseteq \mathrm{ran}(e)$, which will prove that e is a surjection. For this purpose, we verify

$$\forall y\,(y \in \mathbb{N} \Rightarrow y \in \mathrm{ran}(e)), \tag{5.679}$$

taking an arbitrary natural number y. Being thus an element of the domain of $\bar{D}$, the associated function value $\bar{D}(y)$ is then in the codomain $\mathbb{N} \times \mathbb{N}$ of $\bar{D}$. By definition of the Cartesian product of two sets, there are then particular elements $\bar{k} \in \mathbb{N}$ and $\bar{m} \in \mathbb{N}$ with $(\bar{k}, \bar{m}) = \bar{D}(y)$. Here, we note that the latter equation implies

$$y = \bar{D}^{-1}((\bar{k}, \bar{m})). \tag{5.680}$$

Moreover $\bar{k} \in \mathbb{N}$ shows that $\bar{k}$ is in the domain of $\bar{d}$, so that the corresponding function value $\bar{d}(\bar{k})$ is in the codomain $\mathbb{N}^n$ of $\bar{d}$. Consequently, $\bar{d}(\bar{k})$ constitutes a family with index set $\{1, \ldots, n\}$, i.e. a function with domain $\{1, \ldots, n\}$. Clearly, it is true that $n + 1 \notin \{1, \ldots, n\}$, which fact allows us to apply Concatenation of functions, where Proposition 3.177 shows then that

$$\bar{g} = \bar{d}(\bar{k}) \cup \{(n + 1, \bar{m})\} \tag{5.681}$$

constitutes a function with domain $\{1, \ldots, n\} \cup \{n + 1\}$. This domain is identical with $\{1, \ldots, n + 1\}$ according to (4.241), which means that $\bar{g}$ is a family with index set $\{1, \ldots, n + 1\}$. We can readily show that

$$\forall i \, (i \in \{1, \ldots, n + 1\} \Rightarrow \bar{g}_i \in \mathbb{N}) \tag{5.682}$$

also holds. We let

$$i \in \{1, \ldots, n + 1\} \quad [= \{1, \ldots, n\} \cup \{n + 1\}]$$

be arbitrary, so that $i \in \{1, \ldots, n\}$ or $i \in \{n + 1\}$ holds by definition of the union of two sets. In case of

$$i \in \{1, \ldots, n\} \quad [= \mathrm{dom}(\bar{d}(\bar{k}))],$$

we take the corresponding term $[\bar{d}(\bar{k})]_i = [\bar{d}(\bar{k})](i)$ and write for this equation $(i, [\bar{d}(\bar{k})]_i) \in \bar{d}(\bar{k})$. This implies $(i, [\bar{d}(\bar{k})]_i) \in \bar{g}$ with (5.681), which we can write in function notation as $[\bar{d}(\bar{k})]_i = \bar{g}(i)$. Using the notation for families, we can write this also in the form $g_i = [\bar{d}(\bar{k})]_i$. Recalling that $\bar{d}(\bar{k})$ is element the Cartesian product $\mathbb{N}^n$, we thus have in particular $[\bar{d}(\bar{k})]_i \in \mathbb{N}$, so that substitution based on the preceding equation yields $g_i \in \mathbb{N}$, as desired. In the other case of $i \in \{n + 1\}$, we evidently have $i = n + 1$, so that the ordered pair $(n + 1, \bar{m})$ is equal to the ordered pair $(i, \bar{m})$. Because $(n + 1, \bar{m})$ is an element of the union $\bar{g}$ in (5.681), we therefore find $(i, \bar{m}) \in \bar{g}$ to be true, which we can write as $\bar{m} = \bar{g}_i$. Recalling now the truth of $\bar{m} \in \mathbb{N}$, we obtain the desired $\bar{g}_i \in \mathbb{N}$ also in the second case. We thus proved the implication in (5.682), in which i is arbitrary, so that the universal sentence (5.682) follows to be true. In connection with the previously mentioned fact that $\bar{g}$ is a family with index set $\{1, \ldots, n + 1\}$, this means that $\bar{g}$ is element of the Cartesian power $\mathbb{N}^{n+1}$. Consequently,

$$\exists k, m \, (e(\bar{g}) = \bar{D}^{-1}((k, m)) \wedge \bar{g} = \bar{d}(k) \cup \{(n + 1, m)\})$$

follows to be true with (5.664). Because the equations (5.680) and (5.681) demonstrate the truth of the existential sentence

$$\exists k, m \, (y = \bar{D}^{-1}((k, m)) \wedge \bar{g} = \bar{d}(k) \cup \{(n + 1, m)\}),$$

which is satisfied by a unique constant y according to (5.665), it follows with the Criterion for unique existence that $y = e(\bar{g})$. Writing this in the form $(\bar{g}, y) \in e$, we now clearly see that y is in the range of e, which finding completes the proof of the implication in (5.679). Since y was arbitrary, the universal sentence (5.679) follows therefore to be true, and this implies the truth of the inclusion $\mathbb{N} \subseteq \mathrm{ran}(e)$ by definition of a subset. Together with the previously established converse inclusion $\mathrm{ran}(e) \subseteq \mathbb{N}$, this further implies $\mathrm{ran}(e) = \mathbb{N}$ with the Axiom of Extension, so that the function $e : \mathbb{N}^{n+1} \to \mathbb{N}$ is a surjection, by definition.

Next, we prove that the surjection $e : \mathbb{N}^{n+1} \twoheadrightarrow \mathbb{N}$ is also an injection. Letting g and g' be arbitrary and assuming $g, g' \in \mathbb{N}^{n+1}$ as well as $e(g) = e(g')$ to be true, we demonstrate in the following the truth of $g = g'$. By definition of the function e, the former assumption implies the truth of the existential sentences

$$\exists k, m\,(e(g) = \bar{D}^{-1}((k,m)) \,\wedge\, g = \bar{d}(k) \cup \{(n+1, m)\}),$$
$$\exists k, m\,(e(g') = \bar{D}^{-1}((k,m)) \,\wedge\, g' = \bar{d}(k) \cup \{(n+1, m)\}),$$

so that there are particular constants $\bar{k}, \bar{m}$ and $\bar{k}', \bar{m}'$ such that

$$e(g) = \bar{D}^{-1}((\bar{k}, \bar{m})) \,\wedge\, g = \bar{d}(\bar{k}) \cup \{(n+1, \bar{m})\},$$
$$e(g') = \bar{D}^{-1}((\bar{k}', \bar{m}')) \,\wedge\, g' = \bar{d}(\bar{k}') \cup \{(n+1, \bar{m}')\}.$$

In view of the first parts of these conjunctions and the assumed equation $e(g) = e(g')$, we obtain by means of substitutions

$$\bar{D}^{-1}((\bar{k}, \bar{m})) = \bar{D}^{-1}((\bar{k}', \bar{m}')),$$

and therefore $(\bar{k}, \bar{m}) = (\bar{k}', \bar{m}')$ with the injectivity of the bijection $\bar{D}^{-1}$. The Equality Criterion for ordered pairs gives us then the two equations $\bar{k} = \bar{k}'$ and $\bar{m} = \bar{m}'$, which allow us to perform further substitutions to the second parts of the preceding conjunctions, with the consequence that

$$g = \bar{d}(\bar{k}) \cup \{(n+1, \bar{m})\} = \bar{d}(\bar{k}') \cup \{(n+1, \bar{m}') = g'.$$

Consequently, we find $g = g'$ indeed to be true, and as g and g' are arbitrary, we may now conclude that the function $e : \mathbb{N}^{n+1} \twoheadrightarrow \mathbb{N}$ is an injection, and thus a bijection. Applying now once more the Bijectivity of inverse functions, we obtain the bijection

$$e^{-1} : \mathbb{N} \rightleftarrows \mathbb{N}^{n+1},$$

so that the existential sentence $\exists d\,(d : \mathbb{N} \rightleftarrows \mathbb{N}^{n+1})$ is true. Because n is arbitrary, we can therefore conclude that the induction step (5.663) holds, besides the base case, and this completes the proof of the universal sentence (5.654). Thus, $\mathbb{N}^n$ is by definition a countably infinite set for any positive natural number, as claimed in b).

Moreover, this implies that $\mathbb{N}^n$ is a countable set for any $n \in \mathbb{N}_+$. Recalling our previous finding that $\mathbb{N}^0$ is also countable and observing the truth of the disjunction $n = 0 \vee n \in \mathbb{N}_+]$ for an arbitrary $n \in \mathbb{N}$ in light of (2.310), we see that $\mathbb{N}^n$ is a countable set both in the case of $n = 0$ and in the case of $\mathbb{N}_+$. We may therefore conclude that Part a) of the theorem is also true. □

Exercise 5.65. Prove that every countably infinite set A and any of its Cartesian powers A^n for $n \in \mathbb{N}_+$ are equinumerous, that is,

$$\forall A, n\,([A \text{ is countably infinite } \wedge\ n \in \mathbb{N}_+] \Rightarrow A \sim A^n). \tag{5.683}$$

(Hint: Proceed similarly as in the proof of Theorem 5.159b), using now a particular bijection $\bar{D} : A \rightleftarrows A \times A$ based on (5.620).)

Corollary 5.160. *It is true for any countably infinite set A and for any positive natural number n that the n-th Cartesian power of A is itself countably infinite, that is,*

$$\forall A, n\,([A \text{ is countably infinite } \wedge\ n \in \mathbb{N}_+] \Rightarrow A^n \text{ is countably infinite}). \tag{5.684}$$

Proof. Letting A and n be arbitrary sets such that A is countably infinite and such that n is a positive natural number, we obtain on the one hand $\mathbb{N} \sim A$ according to Note 4.25, and on the other hand $A \sim A^n$ with (5.683). Then,

$$\mathbb{N} \sim A^n \tag{5.685}$$

follows to be true with the transitivity property (4.666), which finding shows that A^n is countably infinite (in light of Note 4.25). Here, A and n were both arbitrary, so that the stated universal sentence holds indeed. □

Exercise 5.66. Show for any countably infinite set A and for any natural number n that the n-th Cartesian power of A is countable, that is,

$$\forall A, n\,([A \text{ is countably infinite } \wedge\ n \in \mathbb{N}] \Rightarrow A^n \text{ is countable}). \tag{5.686}$$

(Hint: Use (2.310) and recall the beginning of the proof of Theorem 5.159)

The preceding results can be extended to more general forms of Cartesian products.

Theorem 5.161 (Countability of the Cartesian product of a finite sequence of countable sets). *It is true for any natural number n and for any sequence $A = (A_i \,|\, i \in \{1, \ldots, n\})$ of countable sets that the Cartesian product $\bigtimes_{i=1}^{n} A_i$ is countable, that is,*

$$\forall n \, (n \in \mathbb{N} \Rightarrow \forall A \, ([A = (A_i \,|\, i \in \{1, \ldots, n\}) \\ \wedge \, \forall i \, (i \in \{1, \ldots, n\} \Rightarrow A_i \text{ is countable})] \Rightarrow \bigtimes_{i=1}^{n} A_i \text{ is countable})). \tag{5.687}$$

Proof. We prove the theorem by means of mathematical induction. In the base case $n = 0$, the index set $\{1, \ldots, n\}$ is empty because of (4.239), so that so that we obtain for an arbitrary sequence $(A_i \,|\, i \in \{1, \ldots, n\})$ with (3.863) the true equation $\bigtimes_{i=1}^{n} A_i = \{\emptyset\}$. Being thus a singleton, the Cartesian product $\bigtimes_{i=1}^{n} A_i$ is finite (see Corollary 4.100) and therefore countable. In the induction step, we take an arbitrary natural number n and make the induction assumption that the Cartesian product $\bigtimes_{i=1}^{n} A_i$ is countable for any sequence $A = (A_i \,|\, i \in \{1, \ldots, n\})$ of countable sets. We then take an arbitrary sequence $A = (A_i \,|\, i \in \{1, \ldots, n+1\})$ such that A_i is countable for every index $i \in \{1, \ldots, n+1\}$. Consequently, we obtain the equivalence

$$\bigtimes_{i=1}^{n+1} A_i \sim \left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}$$

with (4.681), so that there exists a bijection from the Cartesian product $\bigtimes_{i=1}^{n+1} A_i$ to the Cartesian product $\left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}$, say $\bar{f}$. This bijection constitutes then in particular an injection, that is,

$$\bar{f} : \bigtimes_{i=1}^{n+1} A_i \hookrightarrow \left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}.$$

Since all terms of the restricted sequence

$$A \upharpoonright \{1, \ldots, n\} = (A_i \,|\, i \in \{1, \ldots, n\})$$

are evidently terms of the sequence A and thus countable, it follows with the induction assumption that the Cartesian product $\bigtimes_{i=1}^{n} A_i$ is countable. Since the term A_{n+1} of the sequence A is also countable, we have by virtue of the Countability of the Cartesian product of two countable sets that $\left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1}$ is countable. According the Countability Criterion (4.652), there is then an injection from that Cartesian product to $\mathbb{N}$, say

$$\bar{g} : \left[\bigtimes_{i=1}^{n} A_i\right] \times A_{n+1} \hookrightarrow \mathbb{N}.$$

Due to the Injectivity of the composition of two injections, it follows that

$$\bar{g} \circ \bar{f} : \bigtimes_{i=1}^{n+1} A_i \hookrightarrow \mathbb{N},$$

demonstrating the existence of an injection from $\bigtimes_{i=1}^{n+1} A_i$ to $\mathbb{N}$. We therefore see in light of the Countability Criterion (4.652) that the Cartesian product $\bigtimes_{i=1}^{n+1} A_i$ is countable. Since the sequence A and the natural number n are arbitrary, we can now infer from the preceding finding that the induction step holds (besides the base case), so that the proof of the theorem via mathematical induction is complete. □

Corollary 5.162. *It is true for any countable set A and for any natural number n that the n-th Cartesian power of A is also countable, that is,*

$$\forall A, n\,([A \text{ is countable } \wedge\ n \in \mathbb{N}] \Rightarrow A^n \text{ is countable}). \tag{5.688}$$

Proof. Letting A and n be arbitrary sets such that A is countable and such that n is a natural number, we have that A defines the constant sequence

$$\{1, \ldots, n\} \times \{A\} = (A_i \,|\, i \in \{1, \ldots, n\}),$$

for which $A_i = A$ holds for every $i \in \{1, \ldots, n\}$ according to Corollary 3.154. Evidently, the countability of A implies then the countability of A_i for all $i \in \{1, \ldots, n\}$ via substitution, so that we can apply the preceding Theorem 5.161 to infer from this the countability of the Cartesian product $\bigtimes_{i=1}^{n} A_i$. Thus, the Cartesian power A^n is countable, where A and n are arbitrary, so that the universal sentence (5.688) follows now to be true. □

Before continuing our study concerning the countability of Cartesian products, we use the preceding corollary to establish the following corollary, which in turn serves as a basis for the subsequent theorem.

Corollary 5.163. *It is true for any countable set Y that the set $Y^{<\mathbb{N}_+}$ of all sequences in Y with domain $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$ constitutes a countable set.*

Proof. Letting Y be an arbitrary countable set, we recall from Exercise 4.32 that the set $Y^{<\mathbb{N}_+}$ is identical with the union $\bigcup_{n=0}^{\infty} Y^n$ of the sequence $(Y^n)_{n \in \mathbb{N}}$ of Cartesian powers of Y. Here, we observe that the index set $\mathbb{N}$ is countably infinite (see Corollary 4.137) and thus countable. Furthermore, (5.688) shows for the given countable set Y that $n \in \mathbb{N}$ implies the countability of Y^n for any $n \in \mathbb{N}$. We may therefore use (5.651) to infer from these findings that the union $Y^{<\mathbb{N}_+} = \bigcup_{n=0}^{\infty} Y^n$ is countable. Since Y was initially arbitrary, we may then conclude that the corollary holds. □

Theorem 5.164 (Countability of the set of all finite subsets of a countable set). *It is true for any countable set Y that there exists a unique set (system)*

$$\mathcal{X} = \{X : X \subseteq Y \,\wedge\, X \text{ is finite}\} \tag{5.689}$$

consisting of all finite subsets of Y, and this set is countable.

Proof. We take an arbitrary set Y and assume it to be countable. We then see in light of the Axiom of Specification and the Equality Criterion for sets that there exists a unique set $\mathcal{X}$ such that

$$\forall X\,(X \in \mathcal{X} \Leftrightarrow [X \in \mathcal{P}(Y) \,\wedge\, X \text{ is finite}]). \tag{5.690}$$

By definition of a power set, $X \in \mathcal{P}(Y)$ is equivalent to $X \subseteq Y$ for any X, so that $\mathcal{X}$ represents the set (5.689) consisting of all finite subsets of Y. Next, we apply Function definition by replacement to establish a unique function f with domain $Y^{<\mathbb{N}_+}$ such that

$$\forall s\,(s \in Y^{<\mathbb{N}_+} \Rightarrow f(s) = \mathrm{ran}(s)). \tag{5.691}$$

For this purpose, we verify the universal sentence

$$\forall s\,(s \in Y^{<\mathbb{N}_+} \Rightarrow \exists! y\,(y = \mathrm{ran}(s))). \tag{5.692}$$

Letting s be arbitrary in $Y^{<\mathbb{N}_+}$, the range of s is a uniquely specified set, so that the uniquely existential sentence is true according to (1.109). Since s is arbitrary, (5.692) follows to be true, so that there exists indeed a unique function function with domain $Y^{<\mathbb{N}_+}$ satisfying (5.691).

We now prove that the set (5.689) constitutes the range of the function f. To do this, we apply the Equality Criterion for sets and establish the equivalent universal sentence

$$\forall y\,(y \in \mathrm{ran}(f) \Leftrightarrow y \in \mathcal{X}). \tag{5.693}$$

We let y be arbitrary, and we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $y \in \mathrm{ran}(f)$ to be true. By definition of a range, there is then a constant, say $\bar{s}$, such that $(\bar{s}, y) \in f$. This yields with the definition of a domain $\bar{s} \in Y^{<\mathbb{N}_+}$, so that $\bar{s}$ is a sequence in Y. Thus, Y is a codomain of $\bar{s}$, which means that the inclusion $\mathrm{ran}(\bar{s}) \subseteq Y$ holds. Because $\bar{s} \in Y^{<\mathbb{N}_+}$ implies $f(\bar{s}) = \mathrm{ran}(\bar{s})$ with (5.691), we obtain from the preceding inclusion by means of substitution $f(\bar{s}) \subseteq Y$. Writing the previous finding $(\bar{s}, y) \in f$ now in function notation, we obtain $y = f(\bar{s})$ and therefore $y \subseteq Y$, so that $y \in \mathcal{P}(Y)$ follows to be true by definition of a power set. Moreover, $\bar{s} \in Y^{<\mathbb{N}_+}$ implies that $\mathrm{ran}(\bar{s})$ is finite due to (4.605). Applying now substitutions based on the equations $y = f(\bar{s}) = \mathrm{ran}(\bar{s})$, we thus have that y is finite, which implies in conjunction with $y \in \mathcal{P}(Y)$ that y is an element of $\mathcal{X}$, according to (5.690). This finding completes the proof of the first part of the equivalence in (5.693).

Concerning the second part ('$\Leftarrow$'), we conversely assume $y \in \mathcal{X}$ to hold, so that y is evidently a finite subset of Y. By definition of a finite set, there exist then a natural number, say $\bar{n}$, and a bijection from $\{1, \ldots, \bar{n}\}$ to y, say $\bar{c}$. Thus, $\bar{c}$ is a surjection, so that the codomain y equals the range of $\bar{c}$. The truth of the preceding inclusion $y \subseteq Y$ implies now via substitution the inclusion $\mathrm{ran}(\bar{c}) \subseteq Y$, which demonstrates that Y is also a codomain of $\bar{c}$. Consequently, the function $\bar{c}: \{1, \ldots, \bar{n}\} \to Y$ is an element of $Y^{\{1,\ldots,\bar{n}\}}$, which fact evidently implies $\bar{c} \in Y^{<\mathbb{N}_+}$ with the specification of that set in (4.395). This sequence $\bar{c}$ is associated with the function value $f(\bar{c}) = \mathrm{ran}(\bar{c})$ according to (5.691), so that the previously found equality $y = \mathrm{ran}(\bar{c})$ yields $y = f(\bar{c})$. We can write this also in the form $(\bar{c}, y) \in f$, which gives us with the definition of a range the desired consequent $y \in \mathrm{ran}(f)$ of the implication '$\Leftarrow$' in (5.693).

We thus completed the proof of the equivalence in (5.693), in which y is arbitrary, so that the universal sentence (5.693) follows now to be true. This in turn completes the proof of the equality $\mathrm{ran}(f) = \mathcal{X}$, which shows that the function f is a surjection from $Y^{<\mathbb{N}_+}$ to $\mathcal{X}$, that is,

$$f : Y^{<\mathbb{N}_+} \twoheadrightarrow \{X : X \subseteq Y \wedge X \text{ is finite}\}.$$

Let us observe now that the assumed countability of Y implies the countability of the set $Y^{<\mathbb{N}_+}$, according to Corollary 5.163. Since Exercise 4.32d) shows that $Y^{<\mathbb{N}_+}$ is nonempty even for $Y = \emptyset$, it follows with the Countability Criterion (4.653) that there exists a surjection from $\mathbb{N}$ to $Y^{<\mathbb{N}_+}$, say

$$g : \mathbb{N} \twoheadrightarrow Y^{<\mathbb{N}_+}.$$

Due to the Surjectivity of the composition of two surjections, we obtain therefore from f and g the surjection

$$f \circ g : \mathbb{N} \twoheadrightarrow \{X : X \subseteq Y \wedge X \text{ is finite}\}.$$

In view of the Countability Criterion (4.653), the existence of such a surjection implies that the set (5.689) of all finite subsets of the countable set Y is countable. Since Y was arbitrary, we may therefore conclude that the stated theorem is true. □

The following theorem generalizes the Countability of the Cartesian product of a finite sequence of countable sets, by allowing the index set to be any finite set rather than a specific initial segment of $\mathbb{N}_+$.

Theorem 5.165 (Countability of the Cartesian product of a family of countable sets with finite index set). *It is true that*

a) the Cartesian product of any family of sets $A = (A_j)_{j\in J}$ having a finite index set J is equivalent to the Cartesian product of the sequence $([A \circ c]_i \,|\, i \in \{1, \dots, n\})$ for some natural number n and for some bijection c from $\{1, \dots, n\}$ to J, that is,

$$\forall J, A\,([J \text{ is finite } \wedge\ A = (A_j)_{j\in J}] \tag{5.694}$$
$$\Rightarrow \exists n, c\,(n \in \mathbb{N} \wedge c : \{1, \dots, n\} \rightleftarrows J \wedge \bigtimes_{j\in J} A_j \sim \bigtimes_{i=1}^{n} [A \circ c]_i)).$$

b) if all terms of a family of sets $(A_j)_{j\in J}$ having a finite index set are countable, then the Cartesian product of this family is countable, that is,

$$\forall J, A\,([J \text{ is finite } \wedge\ A = (A_j)_{j\in J} \tag{5.695}$$
$$\wedge\ \forall j\,(j \in J \Rightarrow A_j \text{ is countable})] \Rightarrow \bigtimes_{j\in J} A_j \text{ is countable}).$$

Proof. We let J and A be arbitrary sets, assume J to be finite, and assume moreover that A is a family of sets $(A_j)_{j\in J}$ with index set J. By definition of a finite sets, there are then constants, say $\bar{n}$ and $\bar{c}$, such that $\bar{n}$ is a natural number and $\bar{c}$ a bijection from $\{1, \dots, \bar{n}\}$ to J. Since we may view the family A as being a function $A : J \to \text{ran}(A)$, we have that the composition of A and $\bar{c}$ constitutes the function $A \circ \bar{c} : \{1, \dots, \bar{n}\} \to \text{ran}(A)$, according to Proposition 3.178. Thus, we can write this composition as the sequence $([A \circ \bar{c}]_i \,|\, i \in \{1, \dots, \bar{n}\})$. It remains for us to establish the equivalence $\bigtimes_{j\in J} A_j \sim \bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i$, that is, to demonstrate the existence of a bijection from the former to the latter Cartesian product.

For this purpose, we apply first Function definition by replacement to establish a unique function F with domain $\bigtimes_{j\in J} A_j$ such that

$$\forall f\,(f \in \bigtimes_{j\in J} A_j \Rightarrow F(f) = f \circ \bar{c}), \tag{5.696}$$

which task requires the verification of the universal sentence

$$\forall f\,(f \in \bigtimes_{j\in J} A_j \Rightarrow \exists! y\,(y = f \circ \bar{c})). \tag{5.697}$$

Letting f be arbitrary and assuming $f \in \bigtimes_{j\in J} A_j$, we have by definition of the Cartesian product of a family of sets that f is a family with index set J, which is a function with domain J (and codomain, say, $\mathrm{ran}(f)$). Consequently, the composition of f and the bijection $\bar{c} : \{1, \ldots, \bar{n}\} \to J$ constitutes the function $f \circ \bar{c} : \{1, \ldots, \bar{n}\} \to \mathrm{ran}(f)$ (using again Proposition 3.178). This is a uniquely specified set, so that the uniquely existential sentence in (5.697) follows to be true with (1.109). Since f is arbitrary, we may therefore conclude that the universal sentence (5.697) also holds, which in turn implies the existence of a unique function F with domain $\bigtimes_{j \in J} A_j$ and values defined by (5.696).

Next, we establish the Cartesian product $\bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i$ as the range of that function. To do this, we apply the Equality Criterion for sets and prove

$$\forall y \, (y \in \mathrm{ran}(F) \Leftrightarrow y \in \bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i), \tag{5.698}$$

taking an arbitrary set y. We prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $y \in \mathrm{ran}(F)$ to be true. By definition of a range, there exists then a particular constant $\bar{f}$ such that $(\bar{f}, y) \in F$. By definition of a domain, we thus have $\bar{f} \in \bigtimes_{j \in J} A_j$ [$= \mathrm{dom}(F)$], so that the definition of the function F in (5.696) yields $F(\bar{f}) = \bar{f} \circ \bar{c}$. Furthermore, $\bar{f} \in \bigtimes_{j \in J} A_j$ evidently means that $\bar{f}$ is a family with index set J and terms satisfying

$$\forall j \, (j \in J \Rightarrow \bar{f}_j \in A_j). \tag{5.699}$$

Thus, $\bar{f}$ is a function from J to $\mathrm{ran}(\bar{f})$, so that the composition $F(\bar{f})$ is a function from $\{1, \ldots, \bar{n}\}$ to $\mathrm{ran}(\bar{f})$, and this is a family/sequence with index set $\{1, \ldots, \bar{n}\}$. Noting that we can write the previous finding $(\bar{f}, y) \in F$ in function notation as $y = F(\bar{f})$, we thus have that y is a family with index set $\{1, \ldots, \bar{n}\}$. We can also show that the terms of y satisfy

$$\forall i \, (i \in \{1, \ldots, \bar{n}\} \Rightarrow y_i \in [A \circ \bar{c}]_i). \tag{5.700}$$

Letting i be arbitrary and assuming $i \in \{1, \ldots, \bar{n}\}$, it is true that i is in the domain of $\bar{c} : \{1, \ldots, \bar{n}\} \to J$, so that the associated function value $\bar{c}(i)$ is in the codomain J (according to the Function Criterion). Thus, $\bar{c}(i)$ is an element of the domain J of the function $\bar{f}$, which fact gives us the value $\bar{f}(\bar{c}(i)) \in A_{\bar{c}(i)}$ with (5.699). In view of the previously established equations, we can write on the one hand

$$\bar{f}(\bar{c}(i)) = [\bar{f} \circ \bar{c}](i) = [F(\bar{f})]_i = y_i,$$

and on the other hand

$$A_{\bar{c}(i)} = A(\bar{c}(i)) = [A \circ \bar{c}]_i,$$

allowing us to infer from $\bar{f}(\bar{c}(i)) \in A_{\bar{c}(i)}$ the truth of the desired consequent $y_i \in [A \circ \bar{c}]_i$ of the implication in (5.700). Here, i was arbitrary, so that the universal sentence (5.700) follows to be true. In conjunction with the fact that y is a family with index set $\{1, \ldots, \bar{n}\}$, this implies now that y is in the Cartesian product $\bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i$, proving the implication '$\Rightarrow$' in (5.698).

Concerning the implication '$\Leftarrow$', assuming $y \in \bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i$ to be true, it is thus true that y is a family with index set $\{1, \ldots, \bar{n}\}$ satisfying (5.700). We may view y also a function $y : \{1, \ldots, \bar{n}\} \to \operatorname{ran}(y)$. Due to the Bijectivity of inverse functions, the bijection $\bar{c} : \{1, \ldots, \bar{n}\} \rightleftarrows J$ gives rise to the bijection $\bar{c}^{-1} : J \rightleftarrows \{1, \ldots, \bar{n}\}$, so that we evidently obtain the composition $y \circ \bar{c}^{-1} : J \to \operatorname{ran}(y)$. This finding demonstrates that $y \circ \bar{c}^{-1}$ is a family with index set J, for which we prove now

$$\forall j \, (j \in J \Rightarrow [y \circ \bar{c}^{-1}]_j \in A_j). \tag{5.701}$$

Taking an arbitrary constant j and assuming $j \in J$ to be true, we obtain the value $\bar{c}^{-1}(j) \in \{1, \ldots, \bar{n}\}$, and this further implies

$$y_{\bar{c}^{-1}(j)} \in [A \circ \bar{c}]_{\bar{c}^{-1}(j)} \tag{5.702}$$

with (5.700). Here, we can write on the one hand

$$y_{\bar{c}^{-1}(j)} = y(\bar{c}^{-1}(j)) = [y \circ \bar{c}^{-1}]_j,$$

using the notation for functions and the notation for compositions, and on the other hand

$$\begin{aligned} [A \circ \bar{c}]_{\bar{c}^{-1}(j)} &= [A \circ \bar{c}](\bar{c}^{-1}(j)) \quad = ([A \circ \bar{c}] \circ \bar{c}^{-1})(j) \\ &= (A \circ [\bar{c} \circ \bar{c}^{-1}])(j) = (A \circ \mathrm{id}_J)(j) \\ &= A_j \end{aligned}$$

applying the notation for functions, the notation for compositions, the Associative Law for function composition, (3.680), and the Neutrality of identity functions under compositions. Because of the preceding equations, (5.702) implies $[y \circ \bar{c}^{-1}]_j \in A_j$ via substitutions, and since j is arbitrary, we may therefore conclude that the universal sentence (5.702) is true. Consequently, the family $y \circ \bar{c}^{-1}$ is an element of the Cartesian product $\bigtimes_{j \in J} A_j$. As this Cartesian product is the domain of the function F, the value corresponding to the preceding family is given by

$$\begin{aligned} F(y \circ \bar{c}^{-1}) &= (y \circ \bar{c}^{-1}) \circ \bar{c} \\ &= y \circ (\bar{c}^{-1} \circ \bar{c}) \\ &= y \circ \mathrm{id}_{\{1, \ldots, \bar{n}\}} \\ &= y \end{aligned}$$

due to (5.696), the Associative Law for function composition, (3.679) and the Neutrality of identity functions under compositions. We can write the resulting equation $y = F(y \circ \bar{c}^{-1})$ also in the form $(y \circ \bar{c}^{-1}, y) \in F$, which shows clearly that y is in the range of F, so that the proof of the second part of the equivalence in (5.698) is now complete. Here, y was arbitrary, which allows us to infer from the truth of that equivalence the truth of the universal sentence (5.698), and therefore the truth of the equality $\mathrm{ran}(F) = \bigtimes_{i=1}^{\bar{n}}[A \circ \bar{c}]_i$. Thus, F constitutes a surjection from the Cartesian product $\bigtimes_{j\in J} A_j$ to the Cartesian product $\bigtimes_{i=1}^{\bar{n}}[A \circ \bar{c}]_i$.

We now verify that the surjection F is injective, by proving the universal sentence

$$\forall f, f' \, ([f, f' \in \bigtimes_{j\in J} A_j \, \wedge \, F(f) = F(f')] \Rightarrow f = f'). \tag{5.703}$$

We let f and f' be arbitrary sets such that $f, f' \in \bigtimes_{j\in J} A_j$ and $F(f) = F(f')$ are true. Evaluating F at f and f' yields then $F(f) = f \circ \bar{c}$ and $F(f') = f' \circ \bar{c}$, so that the assumed equation becomes $f \circ \bar{c} = f' \circ \bar{c}$ after carrying out substitutions. We obtain therefore evidently

$$\begin{aligned} f &= f \circ \mathrm{id}_J &&= f \circ (\bar{c} \circ \bar{c}^{-1}) \\ &= (f \circ \bar{c}) \circ \bar{c}^{-1} &&= (f' \circ \bar{c}) \circ \bar{c}^{-1} \\ &= f' \circ (\bar{c} \circ \bar{c}^{-1}) &&= f' \circ \mathrm{id}_J \\ &= f', \end{aligned}$$

and consequently $f = f'$ as required. As f and f' were arbitrary, we may infer from the preceding equation the truth of the universal sentence (5.703), which means that the surjection $F : \bigtimes_{j\in J} A_j \twoheadrightarrow \bigtimes_{i=1}^{\bar{n}}[A \circ \bar{c}]_i$ is indeed an injection and thus a bijection.

Having demonstrated the existence of a one-to-one correspondence from $\bigtimes_{j\in J} A_j$ to $\bigtimes_{i=1}^{\bar{n}}[A \circ \bar{c}]_i$, it follows that these two Cartesian products are equinumerous by definition, that is, $\bigtimes_{j\in J} A_j \sim \bigtimes_{i=1}^{\bar{n}}[A \circ \bar{c}]_i$. Recalling that $\bar{n}$ is a natural number and $\bar{c}$ a bijection from $\{1, \ldots, \bar{n}\}$ to J, we thus showed that the existential sentence in (5.694) holds. Initially, the sets J and A were arbitrary, so that the universal sentence (5.694) is also true.

Concerning b), we let again J and A be arbitrary such that J is a finite set and such that $A = (A_j)_{j\in J}$ constitutes a family of sets. We assume now in addition that A_j is countable for every $j \in J$. Because of a), we obtain the equivalence $\bigtimes_{j\in J} A_j \sim \bigtimes_{i=1}^{\bar{n}}[A \circ \bar{c}]_i$ for some particular natural number $\bar{n}$ and some particular bijection $\bar{c} : \{1, \ldots, \bar{n}\} \rightleftarrows J$. We can prove now that

$[A \circ \bar{c}]_i$ is countable for every index i, that is,

$$\forall i\,(i \in \{1, \ldots, \bar{n}\} \Rightarrow [A \circ \bar{c}]_i \text{ is countable}). \tag{5.704}$$

Letting i be arbitrary and assuming that $i \in \{1, \ldots, \bar{n}\}$ holds, we obtain the index value $\bar{c}(i) \in J$ and therefore the term $A_{\bar{c}(i)}$, which is by assumption countable. Since this term can be written as $A_{\bar{c}(i)} = A(\bar{c}(i)) = [A \circ \bar{c}]_i$ (using the notations for functions and compositions), we thus have that $[A \circ \bar{c}]_i$ is countable. Here, i is arbitrary, so that (5.704) turns out to be true indeed. Because $\bar{n}$ is a natural number and $([A \circ \bar{c}]_i \,|\, i \in \{1, \ldots, \bar{n}\})$ a sequence of countable sets, we may apply Theorem 5.161 to infer from these two findings that the Cartesian product $\bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i$ is countable. According to the Countability Criterion (4.652), there exists then a particular injection

$$\bar{f} : \bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i \hookrightarrow \mathbb{N}.$$

Furthermore, the equinumerosity of the Cartesian products $\bigtimes_{j \in J} A_j$ and $\bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i$ implies the existence of a particular bijection $\bar{F}$ from the former to the latter, which is especially an injection, that is,

$$\bar{F} : \bigtimes_{j \in J} A_j \hookrightarrow \bigtimes_{i=1}^{\bar{n}} [A \circ \bar{c}]_i.$$

Then, the Injectivity of the composition of two injections yields

$$\bar{f} \circ \bar{F} : \bigtimes_{j \in J} A_j \hookrightarrow \mathbb{N},$$

which proves the existence of an injection from $\bigtimes_{j \in J} A_j$ to $\mathbb{N}$, so that the Cartesian product $\bigtimes_{j \in J} A_j$ constitutes a countable set according to the previously mentioned Countability Criterion. Because J and A were arbitrary, we therefore conclude that Part b) of the theorem holds, too. □

Theorem 5.166 (Countability of the Cartesian product of a family of singletons except for finitely many countable terms). *It is true that*

a) *the Cartesian product of any family of sets $A = (A_i)_{i \in I}$ whose terms are singletons except for a finite number of terms is equivalent to the Cartesian product of the family $([A \restriction J]_j)_{j \in J}$, that is,*

$$\forall I, J, A, a\,([A = (A_i)_{i \in I} \,\wedge\, a = (a_i)_{i \in I \setminus J} \,\wedge\, J \text{ is finite} \,\wedge\, J \subseteq I$$
$$\wedge\, \forall i\,(i \in I \setminus J \Rightarrow A_i = \{a_i\})] \Rightarrow \bigtimes_{i \in I} A_i \sim \bigtimes_{j \in J} [A \restriction J]_j)). \tag{5.705}$$

b) the Cartesian product of any family of sets $A = (A_i)_{i\in I}$ whose terms are countable for indexes in a finite subset J of I and whose terms are singletons for the remaining indexes (in $I \setminus J$) is countable, i.e.

$$\forall I, J, A, a\,([A = (A_i)_{i\in I} \,\wedge\, a = (a_i)_{i\in I\setminus J} \,\wedge\, J \text{ is finite} \,\wedge\, J \subseteq I \\ \wedge\, \forall i\,([i \in J \Rightarrow A_i \text{ is countable}] \,\wedge\, [i \in I \setminus J \Rightarrow A_i = \{a_i\}])] \\ \Rightarrow \bigtimes_{i\in I} A_i \text{ is countable}). \tag{5.706}$$

Proof. We take arbitrary sets I, J, A and a, we assume A and a to be families of sets with index sets I and $I \setminus J$, respectively, we assume that J is a finite subset of I, and we assume the universal sentence

$$\forall i\,(i \in I \setminus J \Rightarrow A_i = \{a_i\}) \tag{5.707}$$

to be true. To prove the equivalence $\bigtimes_{i\in I} A_i \sim \bigtimes_{j\in J}[A \upharpoonright J]_j$, we first establish a function F with domain $\bigtimes_{i\in I} A_i$ whose values satisfy

$$\forall f\,(f \in \bigtimes_{i\in I} A_i \Rightarrow F(f) = f \upharpoonright J). \tag{5.708}$$

According to Function definition by replacement, we thus need to verify the universal sentence

$$\forall f\,(f \in \bigtimes_{i\in I} A_i \Rightarrow \exists! y\,(y = f \upharpoonright J)). \tag{5.709}$$

To do this, we take an arbitrary set f and assume $f \in \bigtimes_{i\in I} A_i$ to be true. This means in view of the definition of the Cartesian product of a family of sets that f is a family with index set I such that f_i is in A_i for every index $i \in I$. Thus, f is a binary relation, so that the restriction $f \upharpoonright J$ is a defined set. Therefore, the uniquely existential sentence $\exists! y\,(y = f \upharpoonright J)$ is true because of (1.109). As the set f was arbitrary, we may infer from this the truth of (5.709), so that there exists indeed a unique function F with domain $\bigtimes_{i\in I} A_i$ such that (5.708) is satisfied.

Let us verify now that $\bigtimes_{j\in J}[A \upharpoonright J]_j$ is a codomain of the function F, i.e. that $\mathrm{ran}(F)$ is included in the preceding Cartesian product. We apply for this purpose the definition of a subset and prove the equivalent universal sentence

$$\forall y\,(y \in \mathrm{ran}(F) \Rightarrow y \in \bigtimes_{j\in J}[A \upharpoonright J]_j). \tag{5.710}$$

We let y be arbitrary, and we assume $y \in \mathrm{ran}(F)$, so that the definition of range yields a particular constant $\bar{f}$ for which $(\bar{f}, y) \in F$ holds. This finding shows in light of the definition of a domain that $\bar{f} \in \bigtimes_{i\in I} A_i$ [$= \mathrm{dom}(F)$] is true. Thus, $\bar{f}$ is a family with index set I, that is, a function from I to its range, whose values/terms satisfy

$$\forall i\,(i \in I \Rightarrow \bar{f}_i \in A_i). \tag{5.711}$$

Moreover, $\bar{f} \in \bigtimes_{i\in I} A_i$ implies $F(\bar{f}) = \bar{f} \upharpoonright J$ with (5.708), so that the previous finding $\bar{f} : I \to \mathrm{ran}(\bar{f})$ and the initial assumption $J \subseteq i$ imply that the restriction $F(\bar{f})$ is a function with domain J (see Proposition 3.164). Noting that the previously established $(\bar{f}, y) \in F$ can be written in function notation as $y = F(\bar{f})$, we thus have that y constitutes a restricted family with index set J, for which we establish now

$$\forall j\,(j \in J \Rightarrow y_j \in [A \upharpoonright J]_j). \tag{5.712}$$

Letting j be arbitrary and assuming $j \in J$, the assumption $J \subseteq I$ gives

$$[y_j = F(\bar{f})_j =] \quad \bar{f} \upharpoonright J(j) = \bar{f}(j)$$

with (3.567) and also $j \in I$ with the definition of a subset. Consequently, $[y_j =]\ \bar{f}(j) \in A_j$ turns out to be true because of (5.711), where we can write $A_j = [A \upharpoonright J]_j$ due to $J \subseteq I$ and (3.567), so that substitution yields the desired consequent $y_j \in [A \upharpoonright J]_j$. As j was arbitrary, we may therefore conclude that (5.712) holds, which shows us now that the restricted family y (with index set J) is element of the Cartesian product $\bigtimes_{j\in J}[A \upharpoonright J]_j$. This proves the implication in (5.710), in which y is arbitrary, so that the universal sentence (5.710) follows to be true. We may then further conclude that the inclusion $\mathrm{ran}(F) \subseteq \bigtimes_{j\in J}[A \upharpoonright J]_j$, and this finding demonstrates that F is a function from $\bigtimes_{i\in I} A_i$ to $\bigtimes_{j\in J}[A \upharpoonright J]_j$.

Next, we use the Surjection Criterion to prove that F is surjection, establishing the truth of

$$\forall y\,(y \in \bigtimes_{j\in J}[A \upharpoonright J]_j \Rightarrow \exists f\,(F(f) = y)). \tag{5.713}$$

We take an arbitrary y and assume the antecedent to be true, which means that y is a family with index set J whose terms satisfy (5.712). Recalling the truth of (5.707), let us now consider the Cartesian product of the restriction of the family $A = (A_i)_{i\in I}$ to $I \setminus J$, that is,

$$\bigtimes_{i\in I\setminus J} A_i = \bigtimes_{i\in I\setminus J} \{a_i\}.$$

Here, it is clearly true for any $i \in I \setminus J$ that $a_i \in \{a_i\}$ holds, so that the singleton $\{a_i\}$ is nonempty for every $i \in I \setminus J$. Since the latter implies the truth of $\neg\exists i\,(i \in I \setminus J \,\wedge\, \{a_i\} = \emptyset)$ with the Negation Law for existential conjunction, it follows with the Emptiness Criterion for Cartesian products of families of sets that $\bigtimes_{i\in I\setminus J}\{a_i\}$ is nonempty. Thus, there exists a particular element in that Cartesian product, say z, which constitutes a family/function with index set/domain $I\setminus J$ and whose terms/values satisfy

$$\forall i\,(i \in I \setminus J \Rightarrow z_i \in \{a_i\}). \tag{5.714}$$

Observing the truth of $J\cap[I\setminus J] = \emptyset$ in light of the Generation of two disjoint sets, we thus have that the domain of the families y and z are disjoint. Consequently, y and z are compatible functions in view of Exercise 3.73, which fact allows us then to apply Concatenation of functions according to Proposition 3.176. This gives us the new function $y \cup z$ with domain $J \cup (I \setminus J)$, for which the assumed inclusion $J \subseteq I$ implies $J \cup (I \setminus J) = I$ with (2.263) and the Commutative Law for the union of two sets. We may therefore view $y \cup z$ as a family with index set I, and we prove in the following that its terms satisfy

$$\forall i\,(i \in I \Rightarrow [y \cup z]_i \in A_i). \tag{5.715}$$

We take an arbitrary constant i and assume $i \in I$ $[= J \cup (I \setminus J)]$ to hold. This assumption implies with the definition of the union of two sets that the disjunction $i \in J \,\vee\, i \in I \setminus J$ is true, and we use this disjunction to prove the desired consequent $[y \cup z]_i \in A_i$ by cases. In the first case $i \in J$, we see that i is in the domain J of the family y, so that i is associated with the term/value $y_i = y(i)$. We can write this also in the form $(i, y_i) \in y$, so that the definition of the union of a set system yields $(i, y_i) \in y \cup z$, and we can write this in function/family notation as $y_i = [y \cup z]_i$. The current case assumption $i \in J$ implies also with the true sentence (5.712) $y_i \in [A \upharpoonright J]_i$ and moreover $[A \upharpoonright J]_i = A_i$ according to (3.567), so that we obtain by means of substitutions $[y \cup z]_i \in A_i$, as desired. In the second case $i \in I \setminus J$, we notice that i is contained in the domain $I \setminus J$ of the family z, giving rise to the term/value $z_i = z(i)$. Writing this in the form $(i, z_i) \in z$, it evidently follows that (i, z_i) is also element of the union/family $y \cup z$, where we can then write $z_i = [y \cup z]_i$. The current case assumption $i \in I \setminus J$ gives us also $z_i \in \{a_i\}$ with (5.714), and this finding implies $z_i \in A_i$ via substitution based on (5.707). Thus, the desired consequent $[y \cup z]_i \in A_i$ turns out to be true also for the second case, and as i was arbitrary, we may now infer from this the truth of the universal sentence (5.715). Therefore, the family $y \cup z$ (having the index set I) is an element of the Cartesian product $\bigtimes_{i\in I} A_i$, and thus an element of the domain of F. By definition of the function F in (5.708), $y \cup z$ is then associated with the value

$$F(y \cup z) = (y \cup z) \upharpoonright J,$$

and we can prove now that this restriction is identical with y (noting that the domains of the restriction $(y \cup z) \upharpoonright J$ and of y are identical to J). For this purpose, we apply the Equality Criterion for functions and establish accordingly the truth of

$$\forall j\,(j \in J \Rightarrow [(y \cup z) \upharpoonright J]_j = y_j). \tag{5.716}$$

We let j be arbitrary and assume that $j \in J$ is true. Since $y \cup z$ is a function with domain I and since the inclusion $J \subseteq I$ holds, we obtain $[(y \cup z) \upharpoonright J]_j = (y \cup z)_j$ with (3.567). In addition, $j \in J$ gives in analogy to the first case of the preceding proof by cases the value $y_j = y(j)$ of the function y, so that the ordered pair (j, y_j) is an element of y, and then also an element of the union $y \cup z$. Recalling that this union is a function/family, we can write $y_j = (y \cup z)_j$, and this gives us $y_j = [(y \cup z) \upharpoonright J]_j$ through substitution. Because j was arbitrary, we may infer from the preceding equation the truth of (5.716) and therefore the truth of the equality

$$[F(y \cup z) =] \quad (y \cup z) \upharpoonright J = y.$$

The resulting equation $F(y \cup z) = y$ clearly demonstrates the truth of the existential sentence $\exists f\,(F(f) = y)$, thereby proving the implication in (5.713). As y was initially arbitrary, we can therefore conclude that the universal sentence (5.713) is true, and this finding completes the proof that is a surjection, in the sense that

$$F : \bigtimes_{i \in I} A_i \twoheadrightarrow \bigtimes_{j \in J} [A \upharpoonright J]_j. \tag{5.717}$$

Next, we establish the injectivity of this function, by proving the universal sentence

$$\forall f, f'\,([f, f' \in \bigtimes_{i \in I} A_i \wedge F(f) = F(f')] \Rightarrow f = f'). \tag{5.718}$$

Letting f, f' be arbitrary and assuming $f, f' \in \bigtimes_{i \in I} A_i$ and $F(f) = F(f')$ to be true, we demonstrate that $f = f'$ is implied. By definition of the function F, the values associated with f and f' are given by $F(f) = f \upharpoonright J$ and $F(f') = f' \upharpoonright J$, so that the assumed equality yields $f \upharpoonright J = f' \upharpoonright J$. Due to the Equality Criterion for functions and the fact that the two restrictions have the same domain J, the preceding equation implies the truth of the universal sentence

$$\forall j\,(j \in J \Rightarrow [f \upharpoonright J]_j = [f' \upharpoonright J]_j). \tag{5.719}$$

We are now in a position to prove $f = f'$ by means of the Equality Criterion for functions, by proving

$$\forall i \, (i \in I \Rightarrow f_i = f'_i). \tag{5.720}$$

To do this, we let i be arbitrary, assume $i \in I$ and recall the equation $I = J \cup (I \setminus J)$, so that the assumption implies the disjunction $i \in J \vee i \in I \setminus J$ (using the definition of the union of two sets). On the one hand, we obtain in case of $i \in J$ the equations

$$f_i = [f \upharpoonright J]_i = [f' \upharpoonright J]_i = f'_i$$

by applying first (3.567), then (5.719), and subsequently again (3.567). Thus, the desired $f_i = f'_i$ follows to be true.

On the other hand, the remaining case $i \in I \setminus J$ implies $A_i = \{a_i\}$ with (5.707). Furthermore, the inclusion $I \setminus J \subseteq I$ holds according to (2.125), so that the case assumption implies $i \in I$. Let us observe now that the previous assumption $f, f' \in \bigtimes_{i \in I} A_i$ demonstrates that f and f' are families with index sets I and terms satisfying

$$\forall i \, (i \in I \Rightarrow f_i \in A_i),$$
$$\forall i \, (i \in I \Rightarrow f'_i \in A_i).$$

Consequently, the finding $i \in I$ implies $f_i \in A_i \; [= \{a_i\}]$ as well as $f'_i \in A_i \; [= \{a_i\}]$. The resulting sentences $f_i \in \{a_i\}$ and $f'_i \in \{a_i\}$ in turn imply $f_i = a_i$ and $f'_i = a_i$ with (2.169), so that we obtain via substitution $f_i = f'_i$, as desired.

Having completed the proof of $f_i = f'_i$ by cases, the universal sentence (5.720) follows now to be true, because i was initially arbitrary. We thus completed the proof of the equality $f = f'$ and therefore also the proof of the implication in (5.718). Since f and f' were arbitrary, we may then further conclude that the universal sentence (5.718) holds, which means that the surjection (5.717) is an injection, that is,

$$F : \bigtimes_{i \in I} A_i \hookrightarrow \bigtimes_{j \in J} [A \upharpoonright J]_j. \tag{5.721}$$

This also means that the function F is a bijection, which fact proves the existence of a bijection from the Cartesian product $\bigtimes_{i \in I} A_i$ to the Cartesian product $\bigtimes_{j \in J} [A \upharpoonright J]_j$. Thus, the two Cartesian products are equinumerous by definition, and this find proves in turn the implication in (5.705). Then, as the sets I, J, A and a were arbitrary, we can finally conclude that Part a) of the theorem holds.

Concerning b), we let again I, J, A and a be arbitrary such that A and a are families of sets with index sets I and $I \setminus J$, respectively, and such that J is a finite subset of I. Moreover, we assume now

$$\forall i\,([i \in J \Rightarrow A_i \text{ is countable}] \wedge [i \in I \setminus J \Rightarrow A_i = \{a_i\}]), \tag{5.722}$$

which implies with the Distributive Law for quantification (1.74) the truth of the universal sentences

$$\forall i\,(i \in J \Rightarrow A_i \text{ is countable}), \tag{5.723}$$

$$\forall i\,(i \in I \setminus J \Rightarrow A_i = \{a_i\}). \tag{5.724}$$

Due to the initial assumptions and (5.724), we obtain the equivalence $\bigtimes_{i\in I} A_i \sim \bigtimes_{j\in J}[A \upharpoonright J]_j$ with Part a) of the theorem. Thus, there exists a particular bijection $\bar{F}$ from the former Cartesian product to the latter, and this bijection is in particular an injection, that is,

$$\bar{F} : \bigtimes_{i\in I} A_i \hookrightarrow \bigtimes_{j\in J}[A \upharpoonright J]_j. \tag{5.725}$$

Let us verify now that (5.723) implies

$$\forall j\,(j \in J \Rightarrow [A \upharpoonright J]_j \text{ is countable}). \tag{5.726}$$

Letting j be arbitrary and assuming $j \in J$ to be true, we obtain $[A \upharpoonright J]_j = A_j$ in view of the assumed inclusion $J \subseteq I$ and (3.567). Because A_j follows to be countable with (5.723), $[A \upharpoonright J]_j$ turns out to be countable as well via substitution. As j was arbitrary, we may therefore conclude that the universal sentence (5.726) is indeed true. In conjunction with the assumptions that J is finite and that $A = (A_j)_{j\in J}$ is a family with index set J, this implies now with Theorem 5.165b) that the Cartesian product $\bigtimes_{j\in J}[A \upharpoonright J]_j$ is countable. The Countability Criterion (4.652) gives us then a particular injection

$$\bar{f} : \bigtimes_{j\in J}[A \upharpoonright J]_j \hookrightarrow \mathbb{N},$$

whose composition with the injection (5.725) yields (according to the Injectivity of the composition of two injections)

$$\bar{f} \circ \bar{F} : \bigtimes_{i\in I} A_i \hookrightarrow \mathbb{N}.$$

This demonstrates the existence of an injection from $\bigtimes_{i\in I} A_i$ to $\mathbb{N}$, so that the Cartesian product $\bigtimes_{i\in I} A_i$ follows to be a countable set in view of the previously mentioned Countability Criterion. As the sets I, J, A and a were arbitrary, we can conclude now that Part b) of the theorem is also true. □

Part III.

Numerical Observation Data

Chapter 6.

The Ordered Integral Domain of Integers

6.1. Groups $(X, *)$

Definition 6.1 (Inverse (element)). For any set X and any binary operation $*$ on X such that the identity element e of X with respect to $*$ exists, and for any elements $a, a^- \in X$, we say that a^- is an *inverse* (*element*) of a with respect to $*$ iff

$$a * a^- = e \ \wedge \ a^- * a = e. \tag{6.1}$$

Inverse elements exist uniquely in the context of a semigroup.

Proposition 6.1. *For any semigroup $(X, *)$ such that the identity element e of X with respect to $*$ exists and for any element $a \in X$ such that an inverse element a^- of a with respect to $*$ exists, it is true that a^- is the only inverse element of a with respect to $*$, i.e.*

$$\forall a\,([a \in X \wedge \exists a^-\,(a{*}a^- = e \wedge a^-{*}a = e)] \Rightarrow \exists! a^-\,(a{*}a^- = e \wedge a^-{*}a = e)). \tag{6.2}$$

Proof. We let X and $*$ be arbitrary sets, assuming that $(X, *)$ is a semigroup such that the identity element e of X with respect to $*$ exists. Next, we let a be arbitrary, assume a to be an element X, and assume furthermore that there exists an inverse element of a with respect to $*$, say $\bar{a}^-$, satisfying thus

$$a * \bar{a}^- = e \ \wedge \ \bar{a}^- * a = e. \tag{6.3}$$

We may now apply Method 1.18 to establish the desired uniquely existential sentence. For this purpose, we let a^- be arbitrary, assume the conjunction

$$a * a^- = e \ \wedge \ a^- * a = e \tag{6.4}$$

to be true, and we show that $\bar{a}^- = a^-$ is implied. Let us observe that, for instance $a{*}\bar{a}^- = e$ and $a{*}a^-$ can be written, respectively, as $((a, \bar{a}^-), e) \in *$

and $((a, a^-), e) \in *$. These findings clearly show in light of the definition of a domain that $(a, \bar{a}^-), (a, a^-) \in X \times X$ [$= \mathrm{dom}(*)$] holds. Thus, $\bar{a}^-$ and a^- are elements of X, according to the definition of the Cartesian product of two sets. Then, $\bar{a}^- \in X$ and $a^- \in X$ imply by definition of the identity element e the truth of the equations

$$a^- = a^- * e \wedge e * \bar{a}^- = \bar{a}^-. \tag{6.5}$$

We then obtain the true equations

$$a^- = a^- * (a * \bar{a}^-) = (a^- * a) * \bar{a}^- = e * \bar{a}^- = \bar{a}^-$$

by applying substitution based on the first equation in (6.3) to the first equation in (6.5), then the associativity of $*$, subsequently substitution based on the second equation in (6.4), and finally substitution based on the second equation in (6.5). Since a^- was arbitrary, we may now infer from the resulting equation $\bar{a}^- = a^-$ the truth of the uniquely existential sentence in (6.2). Because a, X and $*$ were initially also arbitrary, we can therefore conclude that the proposed universal sentence is true. $\square$

Notation 6.1. We will then write for the unique inverse element a^- of an element a with respect to a semigroup $(X, *)$, if it exists, also

$$a^{-1} \tag{6.6}$$

Exercise 6.1. Show for any semigroup $(X, *)$ such that the identity element e of X with respect to $*$ exists that the inverse element e^{-1} of e exists (uniquely), and that

$$e^{-1} = e \tag{6.7}$$

holds.

(Hint: Use Definition 6.1, Definition 5.5, and Proposition 6.1.)

Exercise 6.2. Show for any semigroup $(X, *)$ with identity element e and for any element $a \in X$ such that an inverse element of a with respect to $*$ exists that the implication

$$a \neq e \Rightarrow a^{-1} \neq e \tag{6.8}$$

is true.

(Hint: Apply Method 1.10 in connection with Theorem 1.10.)

Whenever we encounter a semigroup of the following type, the existence of the identity element and of the unique inverse element for any of its elements is guaranteed.

Definition 6.2 (Group, commutative/Abelian group). We say that a semigroup $(X, *)$ is a *group* iff

1. the identity element e of X with respect to $*$ exists, i.e.

$$\exists e\,(e \in X \;\wedge\; \forall a\,(a \in X \Rightarrow [a * e = a \;\wedge\; e * a = a])), \tag{6.9}$$

   and

2. the inverse element of any element of X with respect to $*$ exists, i.e.

$$\forall a\,(a \in X \Rightarrow \exists a^{-1}\,(a^{-1} \in X \;\wedge\; a * a^{-1} = e \;\wedge\; a^{-1} * a = e)). \tag{6.10}$$

Furthermore, we say that a group $(X, *)$ is *commutative* or *Abelian* iff $*$ is commutative.

6.2. Transformation Groups $(T(X), \circ)$

Proposition 6.2. *For any function $f : X \to Y$ and any set $T_{fi}(X)$ of invertible transformations under which f is invariant such that $T_{fi}(X)$ contains id_X, it holds that the ordered pair $(T_{fi}(X), \circ)$ constitutes a group.*

Proof. Letting $f : X \to Y$ be an arbitrary function and $T_{fi}(X)$ an arbitrary set of invertible transformations under which f is invariant, and assuming id_X to be an element of $T_{fi}(X)$, we recall from Note 5.13 that the ordered pair $(T_{fi}(X), \circ_{T_{fi}(X)})$ is a semigroup. Furthermore, we observe in light of Note 5.9 that id_X is the identity element of $T_{fi}(X)$ with respect to $\circ_{T_{fi}(X)}$, so that $\circ_{T_{fi}(X)}$ possesses already Property 1 of a group. Thus, it remains for us to prove that the inverse element of any element of $T_{fi}(X)$ with respect to $\circ_{T_{fi}(X)}$ exists. To do this, we let $t \in T_{fi}(X)$ be arbitrary, so that $t \in T(X)$ follows to be true with Note 3.28. This means that t is a bijection $t : X \rightleftarrows X$ by definition of the set $T(X)$. Thus, the inverse of that function is a bijection $t^{-1} : X \rightleftarrows X$ due to Theorem 3.212 and therefore an element of $T(X)$. Furthermore, $t \in T_{fi}(X)$ implies $f \circ t = f$, so that we obtain the equations

$$f \circ t^{-1} = (f \circ t) \circ t^{-1} = f \circ (t \circ t^{-1}) = f \circ \mathrm{id}_X = f,$$

using substitution, the Associative Law for function composition, Lemma 3.211, and the Neutrality of identity functions under composition. The resulting equation $f \circ t^{-1} = f$ and the previous finding $t^{-1} \in T(X)$ imply now $t^{-1} \in T_{fi}(X)$. This inverse function satisfies $t^{-1} \circ t = \mathrm{id}_X$ and $t \circ t^{-1} = \mathrm{id}_X$ in view of (3.679) and (3.680). We also note that $t^{-1} \circ t = t^{-1} \circ_{T_{fi}(X)} t$

and $t \circ t^{-1} = t \circ_{T_{fi}(X)} t^{-1}$ are true according to (5.77). The previous findings demonstrate the truth of the existential sentence

$$\exists t^{-1}\,(t^{-1} \in T_{fi}(X) \,\wedge\, t \circ_{T_{fi}(X)} t^{-1} = \mathrm{id}_X \,\wedge\, t^{-1} \circ_{T_{fi}(X)} t = \mathrm{id}_X).$$

Since t is an arbitrary element of $T_{fi}(X)$, we may therefore conclude that $\circ_{T_{fi}(X)}$ possesses also Property 2 of a group. Thus, $(T_{fi}(X), \circ_{T_{fi}(X)})$ constitutes a group indeed, which is then evidently true for any set such function $f : X \to Y$ and set $T_{fi}(X)$. □

Exercise 6.3. Show for any set X and any set $T(X)$ of invertible transformations on X containing id_X that the ordered pair $(T(X), \circ_{T(X)})$ constitutes a group.

(Hint: The proof is a simplified version of the proof of Proposition 6.2.)

Note 6.1. In particular for any finite set X, the ordered pair $(\Pi(X), \circ_{\Pi(X)})$ formed by the set of permutations of X and the binary operation of composition on that set is a group.

Definition 6.3 (Transformation group, permutation group). For any set X, we call

$$(\mathcal{T}(X), \circ) = (\mathcal{T}(X), \circ_{\mathcal{T}(X)}) \tag{6.11}$$

the *transformation group on* X. For any subset $T(X) \subseteq \mathcal{T}(X)$ containing id_X, we say that

$$(T(X), \circ) = (T(X), \circ_{T(X)}) \tag{6.12}$$

is *the transformation group formed by* $T(X)$, or simply *a transformation group.* In particular for any finite set X, we say that

$$(\Pi(X), \circ) = (\Pi(X), \circ_{\Pi(X)}) \tag{6.13}$$

is the *permutation group* of X.

Definition 6.4 (Transitive transformation group). We say that a transformation group $(T(X), \circ)$ is *transitive* (on X) iff for any elements x_1 and x_2 of X, the element x_2 is the value of some transformation on X evaluated at x_1, that is

$$\forall x_1, x_2\,(x_1, x_2 \in X \Rightarrow \exists t\,(t \in T(X) \,\wedge\, x_1 = t(x_2))). \tag{6.14}$$

Definition 6.5 (Maximal invariant). We say for any set X, any transformation group $(T(X), \circ)$ on X and any invariant $f : X \to Y$ under $T(X)$ that f is a *maximal invariant* (*under* $T(X)$) iff f has the definite property

$$\forall x_1, x_2\,(x_1, x_2 \in X \Rightarrow [f(x_1) = f(x_2) \Rightarrow \exists t\,(t \in T(X) \,\wedge\, x_2 = t(x_1))]). \tag{6.15}$$

Corollary 6.3. *It is true for any transitive transformation group* $(T(X), \circ)$ *that every invariant* $f : X \to Y$ *under* $T(X)$ *is a maximal invariant.*

Proof. Letting $(T(X), \circ)$ be an arbitrary transitive transformation group and $f : X \to Y$ an arbitrary invariant under $T(X)$. To prove that f is a maximal invariant under $T(X)$, we let x_1 and x_2 be arbitrary elements of X, and we assume $f(x_1) = f(x_2)$ to be true. By definition of a transitive transformation group, the assumption $x_2, x_1 \in X$ implies already the truth of the desired existential sentence in (6.14). As x_1 and x_2 are arbitrary elements of X, we may therefore conclude that f is indeed a maximal invariant under $T(X)$. Since $(T(X), \circ)$ and f were also arbitrary, we may then further conclude that the stated universal sentence holds. □

Note 6.2. Recalling from Corollary 3.179 that the empty function $\emptyset : \emptyset \to Y$ is an invariant under any set of transformations, we will consider in the case of a transitive transformation group $(T(X), \circ)$ that empty invariant $\emptyset : \emptyset \to Y$ as the standard maximal invariant under $T(X)$.

Proposition 6.4. *It is true for any nonempty sets* X *and* Y*, any transitive transformation group* $(T(X), \circ)$ *and any function* $f : X \to Y$ *that* f *is an invariant under* $T(X)$ *iff* f *is some constant function on* X*, that is,*

$$\forall t\,(t \in T(X) \Rightarrow f \circ t = f) \;\Leftrightarrow\; \exists c\,(f = X \times \{c\}). \tag{6.16}$$

Proof. We let X and Y be arbitrary sets, $(T(X), \circ)$ an arbitrary transitive transformation group, and $f : X \to Y$ an arbitrary function. To prove the first part of the equivalence ('$\Rightarrow$'), we assume that f is an invariant under $T(X)$. Since the assumption that $X \neq \emptyset$ implies $f \neq \emptyset$ with (3.120), f evidently has some element, say $\bar{z}$. Due to the initial assumption that $f : X \to Y$, we have that f is a subset of $X \times Y$, according to (3.514). Thus, the previous finding $\bar{z} \in f$ implies $\bar{z} \in X \times Y$ by definition of a subset. Consequently, there exist particular elements $\bar{x} \in X$ and $\bar{y} \in Y$ satisfying $(\bar{x}, \bar{y}) = \bar{z}$, according to Exercise 3.4. Therefore, $\bar{z} \in f$ yields $(\bar{x}, \bar{y}) \in f$ via substitution, which we may also write as $\bar{y} = f(\bar{x})$ since f is a function. We may now prove that f is identical with the constant function $g_{\bar{y}} = X \times \{\bar{y}\}$. For this purpose, we apply the Equality Criterion for functions, letting $x \in X$ be arbitrary. Now, as $T(X)$ constitutes a transitive transformation group, the fact that $\bar{x}, x \in X$ implies the existence of a particular element $\bar{t} \in T(X)$ with $\bar{x} = \bar{t}(x)$. Furthermore, the assumption that f is invariant under $T(X)$ implies $f(\bar{t}(x)) = f(x)$ (see Note 3.21). Combining the previous two equations gives us therefore

$$f(x) = f(\bar{x}) \quad [= \bar{y} = g_{\bar{y}}(x)],$$

using also Corollary 3.154. Since x was arbitrary, we may infer from the truth of the resulting equation $f(x) = g_{\bar{y}}(x)$ the truth of the equality $f = g_{\bar{y}}$ $[= X \times \{\bar{y}\}]$, so that $f = X \times \{\bar{y}\}$. This equation demonstrates the truth of the existential sentence in (6.16), so that the first part of the equivalence holds. To establish the second part ('$\Leftarrow$'), we now assume f to be the constant function $g_{\bar{c}} = X \times \{\bar{c}\}$ for some particular constant $\bar{c}$ and observe in light of Exercise 3.75 that this function is an invariant under $\mathcal{T}$. Since X, Y, $T(X)$ and f were initially arbitrary, we may therefore conclude that the proposition is true. □

We now see that maximal invariants can be constructed also for transformation groups that are not necessarily transitive.

Theorem 6.5 (Existence of maximal invariants). *For any transformation group* $(T(X), \circ)$

a) there exists a unique set $\sim_O$ *consisting of all the ordered pairs* (x_1, x_2) *with* $x_1, x_2 \in X$ *such that* $x_1 = t(x_2)$ *holds for some transformation* t *in* $T(X)$*, and this set* $\sim_O$ *is an equivalence relation on* X*.*

b) there exists a unique function $f_{\sim_O}$ *on* X *mapping every element* x *of* X *to the the equivalence class of* x *with respect to* $\sim_O$*, i.e.*

$$\forall x\,(x \in X \Rightarrow f_{\sim_O}(x) = [x]_{\sim_O}), \tag{6.17}$$

and this function is a maximal invariant under $T(X)$*.*

Proof. We let $(T(X), \circ)$ be an arbitrary transformation group. We may evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of a set $\sim_O$ such that

$$\begin{aligned}\forall Z\,(Z \in \sim_O \Leftrightarrow [&Z \in X \times X \,\wedge\, \exists x_1, x_2, t\,(x_1, x_2 \in X \,\wedge\, t \in T(X)\\ &\wedge\, x_1 = t(x_2) \,\wedge\, (x_1, x_2) = Z)]).\end{aligned} \tag{6.18}$$

Since $Z \in \sim_O$ implies $Z \in X \times X$ for any Z, it follows by definition of a subset that $\sim_O$ is included in $X \times X$ and therefore constitutes a binary relation on X. We now verify that $\sim_O$ has all three properties of an equivalence relation. Regarding Property 1 (i.e., reflexivity), we show that

$$\forall x\,(x \in X \Rightarrow x \,\sim_O\, x)$$

holds. Letting $x \in X$ be arbitrary, we note that $(x, x) \in X \times X$ holds by definition of a Cartesian product. Furthermore, the set $T(X)$ contains the identity element id_X, which yields $x = \mathrm{id}_X(x)$. Clearly, $(x, x) = (x, x)$ is also true, so that the existential sentence in (6.18) is satisfied by the ordered pair $Z = (x, x)$. These findings imply due to (6.18) that $(x, x) \in \sim_O$ is true. As x was arbitrary, we therefore conclude that $\sim_O$ is reflexive.

Regarding Property 2 (i.e., symmetry), we prove

$$\forall x_1, x_2 \,(x_1, x_2 \in X \Rightarrow [x_1 \sim_O x_2 \Rightarrow x_2 \sim_O x_1]).$$

To do this, we let $x_1, x_2 \in X$ be arbitrary and assume $(x_1, x_2) \in \sim_O$. This implies with the specification of $\sim_O$ that there exists an element of $T(X)$, say $\bar{t}$, such that $x_1 = \bar{t}(x_2)$. Since $(T(X), \circ)$ is a group, the inverse $\bar{t}^{-1}$ is a specified element of $T(X)$. Then, we obtain

$$\bar{t}^{-1}(x_1) = \bar{t}^{-1}(\bar{t}(x_2)) = (\bar{t}^{-1} \circ \bar{t})(x_2) = \mathrm{id}_X(x_2) = x_2,$$

by applying substitution, the notation for the composition of functions, (3.679), and the definition of an identity function. Therefore, $(x_2, x_1) \in \sim_O$ evidently follows to be true by the specification of $\sim_O$, so that $x_2 \sim_O x_1$. Since x_1 and x_2 were arbitrary, we conclude that $\sim_O$ is symmetric, too.

Finally, regarding Property 3 (i.e., transitivity), we establish

$$\forall x_1, x_2, x_3 \,(x_1, x_2, x_3 \in X \Rightarrow [(x_1 \sim_O x_2 \wedge x_2 \sim_O x_3) \Rightarrow x_1 \sim_O x_3]).$$

For this purpose, we let $x_1, x_2, x_3 \in X$ be arbitrary and assume (x_1, x_2), $(x_2, x_3) \in \sim_O$. This implies with the specification of $\sim_O$ that there exist elements of $T(X)$, say $\bar{t}_1$ and $\bar{t}_2$, such that $x_1 = \bar{t}_1(x_2)$ and $x_2 = \bar{t}_2(x_3)$. Substitution of the latter into the former equation then yields

$$x_1 = \bar{t}_1(x_2) = \bar{t}_1(\bar{t}_2(x_3)) = (\bar{t}_1 \circ \bar{t}_2)(x_3).$$

Since $\circ$ is a binary operation on $T(X)$, we have that $\bar{t}_1 \circ \bar{t}_2$ is an element of $T(X)$. It follows with the specification of $\sim_O$ that $(x_1, x_3) \in \sim_O$, thus $x_1 \sim_O x_3$. As x_1, x_2 and x_3 were arbitrary, we conclude that $\sim_O$ is also transitive and thus constitutes an equivalence relation on X, by definition.

To establish the function in b), we may apply function definition by replacement. Letting for this purpose $x \in X$ be arbitrary, we see in light of Proposition 3.55 that there exists a unique set $[x]_{\sim_O}$ having the definite property of an equivalence class of x with respect to $\sim_O$. Since x is arbitrary, we may therefore conclude that there indeed exists a unique function $f_{\sim_O}$ on X such that (6.17). Next, we prove that this function is an invariant under $T(X)$, letting $t \in T(X)$ and $x \in X$ be arbitrary. Being thus an invertible transformation on X, t constitutes a function from X to X. By the Function Criterion, x is associated with the unique value $y = t(x)$ in X. Forming now the ordered pair (x, y), we thus see that there are constant x_1, x_2 and t such that $x_1, x_2 \in X$, $t \in T(X)$, $x_1 = t(x_2)$ and

$(x_1, x_2) = (x, y)$. Since $x, y \in X$ imply $(x, y) \in X \times X$ by definition of a Cartesian product, it follows with (6.18) that $(y, x) \in \sim_O$. Since $\sim_O$ is a binary relation, we may write the latter as $y \sim_O x$, and substitution based on the previous equation for y gives us then $t(x) \sim_O x$. This in turn implies $[t(x)]_{\sim_O} = [x]_{\sim_O}$ with the Equality Criterion for equivalence classes. Since the elements $t(x)$ and x of X are associated with the values $f_{\sim_O}(t(x)) = [t(x)]_{\sim_O}$ and $f_{\sim_O}(x) = [x]_{\sim_O}$, we find $f_{\sim_O}(t(x)) = f_{\sim_O}(x)$ through substitutions. Applying now the notation for compositions of functions, we may write $(f_{\sim_O} \circ t)(x) = f_{\sim_O}(x)$. Since x is an arbitrary element of X, we may infer from the truth of this equation the truth of the equality $f_{\sim_O} \circ t = f_{\sim_O}$ by means of the Equality Criterion for functions. As t is an arbitrary element of $T(X)$, we may further conclude that $f_{\sim_O}$ is invariant under $T(X)$, by definition.

It remains to show that $f_{\sim_O}$ is a maximal invariant under $T(X)$. We let $x_1, x_2 \in X$ be arbitrary and assume $f_{\sim_O}(x_1) = f_{\sim_O}(x_2)$. By definition of $f_{\sim_O}$, the elements x_1 and x_2 are thus mapped into the identical equivalence classes $[x_1]_{\sim_O} = [x_2]_{\sim_O}$. Since x_2 is an element of its own equivalence class $[x_2]_{\sim_O}$ due to Corollary 3.57, we obtain via substitution $x_2 \in [x_1]_{\sim_O}$, so that evidently $x_2 \sim_O x_1$. Writing this finding as $(x_2, x_1) \in \sim_O$, it follows from this that there exist constants $\bar{x}_1, \bar{x_2} \in X$ and $\bar{t} \in T(X)$ satisfying $\bar{x}_1 = \bar{t}(\bar{x}_2)$ and $(\bar{x}_1, \bar{x_2}) = (x_2, x_1)$. The latter equation implies $\bar{x}_1 = x_2$ and $\bar{x}_2 = x_1$ with the Equality Criterion for ordered pairs, so that the former equation becomes $x_2 = \bar{t}(x_1)$ after substitutions. This proves that $f_{\sim_O}(x_1) = f_{\sim_O}(x_2)$ implies $x_2 = t(x_1)$ for some $t \in T(X)$, and since x_1 and x_2 are arbitrary elements of X, we may therefore conclude that $f_{\sim_O}$ is a maximal invariant under $T(X)$ by definition. This completes the proof of b), and as X and $T(X)$ were initially arbitrary, it follows that the theorem is true indeed. □

Corollary 6.6. *It is true for any transitive transformation group* $(T(X), \circ)$ *that the equivalence relation* $\sim_O$ *(consisting of all the ordered pairs* (x_1, x_2) *with* $x_1, x_2 \in X$ *such that* $x_1 = t(x_2)$ *holds for some transformation* $t \in T(X)$*) is identical with the Cartesian product* $X \times X$*.*

Proof. We let $(T(X), \circ)$ be an arbitrary transitive transformation group, which gives rise to the equivalence relation $\sim_O$ specified by (6.18). We now apply the Equality Criterion for sets to prove $\sim_O = X \times X$, letting Z be arbitrary. Then, the assumption $Z \in \sim_O$ implies in particular $Z \in X \times X$. The converse assumption $Z \in X \times X$ implies the existence of particular elements $\bar{x}_1, \bar{x}_2 \in X$ with $(\bar{x}_1, \bar{x}_2) = Z$, according to Exercise 3.4. By definition of a transitive transformation group, there is then also a particular transformation $\bar{t} \in T(X)$ with $\bar{x}_1 = \bar{t}(\bar{x}_2)$. These findings demonstrate the truth of the existential sentence

$$\exists x_1, x_2, t\,(x_1, x_2 \in X \,\wedge\, t \in T(X) \,\wedge\, x_1 = t(x_2) \,\wedge\, (x_1, x_2) = Z),$$

which implies – in conjunction with $Z \in X \times X$ – the truth of $Z \in \sim_O$ by virtue of (6.18). Since Z is arbitrary, we may therefore conclude that $\sim_O = X \times X$ holds indeed. This is then evidently true for any transitive transformation group $(T(X), \circ)$. □

Theorem 6.7 (Characterization of maximal invariants). *For any sets X and Y, any transformation group $(T(X), \circ)$ and any maximal invariant $m : X \to Y$ under $T(X)$, it is true that a function $f : X \to Z$ is an invariant under $T(X)$ iff there exists a function h such that f is the composition of h and m, that is,*

$$\forall t\,(t \in T(X) \Rightarrow f \circ t = f) \;\Leftrightarrow\; \exists h\,(h : \mathrm{ran}(m) \to \mathrm{ran}(f) \,\wedge\, f = h \circ m). \tag{6.19}$$

Proof. We let X, Y, Z be arbitrary sets, $T(X)$ an arbitrary transformation group, $m : X \to Y$ an arbitrary maximal invariant under $T(X)$, and $f :\to Z$ an arbitrary function. To prove the first part ('⇒') of the equivalence, we assume that f is an invariant under $T(X)$. We may now evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of a set h consisting of all the ordered pairs $(m(x), f(x))$ with $x \in X$, that is,

$$\forall Z\,(Z \in h \Leftrightarrow [Z \in \mathrm{ran}(m) \times \mathrm{ran}(f) \,\wedge\, \exists x\,(x \in X \,\wedge\, (m(x), f(x)) = Z)]). \tag{6.20}$$

Since $Z \in h$ implies $Z \in \mathrm{ran}(m) \times \mathrm{ran}(f)$ for any z, it follows by definition of a subset that h is a binary relation with $h \subseteq \mathrm{ran}(m) \times \mathrm{ran}(f)$. We now apply the Function Criterion to establish $h : \mathrm{ran}(m) \to \mathrm{ran}(f)$. To do this, we let $y \in \mathrm{ran}(m)$ be arbitrary and demonstrate the truth of the uniquely existential sentence $\exists! z\,(z \in \mathrm{ran}(f) \,\wedge\, (y, z) \in h)$. To begin with, we note that $y \in \mathrm{ran}(m)$ implies by definition of a range that there is a particular constant $\bar{x}$ with $(\bar{x}, y) \in m$, which we may write also as $y = m(\bar{x})$. Moreover, we see in light of the definition of a domain that $\bar{x} \in X$ $[= \mathrm{dom}(m)]$. Since X is also the domain of the invariant f, there is another particular constant $\bar{y}$ with $(\bar{x}, \bar{y}) \in f$, which we may write as $\bar{y} = f(\bar{x})$. Furthermore, $\bar{y} \in \mathrm{ran}(f)$ holds by definition of a range. In conjunction with the previous assumption $y \in \mathrm{ran}(m)$, this implies $(y, \bar{y}) \in \mathrm{ran}(m) \times \mathrm{ran}(f)$. In addition, the equations for y and $\bar{y}$ yield $(y, \bar{y}) = (m(\bar{x}), f(\bar{x}))$, which shows in conjunction with the previous finding $\bar{x} \in X$ that there is some $x \in X$ satisfying $(m(x), f(x)) = (y, \bar{y})$. Thus, the ordered pair $(y, \bar{y})$ turns out to be an element of h, in view of (6.20). Since $\bar{y} \in \mathrm{ran}(f)$ is also true, we see that there is some $z \in \operatorname{ran}(f)$ with $(y, z) \in h$. We thus established the existential part of the uniquely existential sentence to be proven. Regarding the uniqueness part, we let $z, z' \in \operatorname{ran}(f)$ be arbitrary such that $(y, z_1) \in h$ and $(y, z_2) \in h$ are satisfied. These assumptions imply with (6.20) that there are elements $\bar{x}_1, \bar{x}_2 \in X$ satisfying $(m(\bar{x}_1), f(\bar{x}_1)) = (y, z_1)$ and $(m(\bar{x}_2), f(\bar{x}_2)) = (y, z_2)$. Two applications of the Equality Criterion for ordered pairs give us then the equations $m(\bar{x}_1) = y = m(\bar{x}_2)$, $f(\bar{x}_1) = z_1$ and $f(\bar{x}_2) = z_2$. The first two equations yield $m(\bar{x}_1) = m(\bar{x}_2)$ and therefore $\bar{x}_2 = \bar{t}(\bar{x}_1)$ for some particular element $\bar{t} \in T(X)$, since m is a maximal invariant under $T(X)$. We obtain then the equations

$$z_2 = f(\bar{x}_2) = f(\bar{t}(\bar{x}_1)) = f(\bar{x}_1) = z_1$$

using the current assumption that f is invariant under $T(X)$. Since z_1 and z_2 were arbitrary, we may infer from the truth of the resulting equation $z_1 = z_2$ the truth of the desired uniquely existential sentence. As x was also arbitrary, we may therefore conclude that h is indeed a function from $\operatorname{ran}(m)$ to $\operatorname{ran}(f)$.

Since $\operatorname{ran}(m)$ is evidently a codomain of the given maximal invariant m on X, we have $m : X \to \operatorname{ran}(m)$. The composition of $h : \operatorname{ran}(m) \to \operatorname{ran}(f)$ and m is therefore a function $h \circ m : X \to \operatorname{ran}(f)$. As the domain of the given invariant f is also given by the set X, we may apply the Equality Criterion for functions to check whether $f = h \circ m$ is found to be true. To do this, we let $\bar{x} \in X$ be arbitrary, so that $m(\bar{x})$ and $f(\bar{x})$ are uniquely specified values. Clearly, these values in the ranges of the corresponding functions, so that $(m(\bar{x}), f(\bar{x})) \in \operatorname{ran}(m) \times \operatorname{ran}(f)$ follows to be true by definition of the Cartesian product of two sets. We evidently found some element $x \in X$ satisfying $(m(x), f(x)) = (m(\bar{x}), f(\bar{x}))$, so that $(m(\bar{x}), f(\bar{x}))$ follows to be an element of h by virtue of (6.20). Using function notation, we thus have $f(\bar{x}) = h(m(\bar{x}))$, which we may write also as $f(\bar{x}) = (h \circ m)(\bar{x})$ by using the notation for function composition. As $\bar{x}$ was arbitrary, we may therefore conclude that the functions f and $h \circ m$ are indeed identical, which finding completes the proof of the existential sentence in (6.19).

To prove the second part ('$\Rightarrow$') of the equivalence, we now assume that there exists a function from $\operatorname{ran}(m)$ to $\operatorname{ran}(f)$, say $\bar{h}$, such that $f = \bar{h} \circ m$ holds. To verify that f is then an invariant under $T(X)$, we let $t \in T(X)$ be arbitrary and derive the equations

$$f \circ t = (\bar{h} \circ m) \circ t = \bar{h} \circ (m \circ t) = \bar{h} \circ m = f$$

by applying substitution, the Associative Law for function composition (to the given functions $t : X \to X$, $m : X \to \operatorname{ran}(m)$ and $\bar{h} : \operatorname{ran}(m) \to \operatorname{ran}(f)$), the fact that the maximal invariant m under $T(X)$ is an invariant under $T(X)$, and finally again substitution. Since t is an arbitrary element of $T(X)$, we may infer from the truth of the resulting equation $f \circ t = f$ that the universal sentence in (6.19), i.e., that f is invariant under $T(X)$.

We thus completed the proof of the equivalence, and since X, Y, $T(X)$ and f were initially arbitrary, we may now finally conclude that the theorem is true. □

Note 6.3. Theorem 6.7 shows for any transformation group $(T(X), \circ)$ that a maximal invariant under $T(X)$ is characterized by the fact that every invariant under $T(X)$ is a function of that maximal invariant.

Theorem 6.8 (Maximal invariance of compositions of bijections with maximal invariants). *For any transformation group* $(T(X), \circ)$*,*

a) for any maximal invariant $m : X \to Y$ *under* $T(X)$ *and for any function* $h : \mathrm{ran}(m) \to \mathrm{ran}(h)$*, it is true that* h *is a bijection iff the composition* $h \circ m$ *is a maximal invariant under* $T(X)$*.*

b) and for any maximal invariants $m_1 : X \to Y$ *and* $m_2 : X \to Z$ *under* $T(X)$*, it is true that* m_2 *is the composition of some bijection* h *and* m_1*.*

Proof. We let $(T(X), \circ)$ be an arbitrary transformation group. Concerning a), we also let $m : X \to Y$ be an arbitrary maximal invariant under $T(X)$, and $h : \mathrm{ran}(m) \to \mathrm{ran}(h)$ an arbitrary function. Treating m as the function $m : X \to \mathrm{ran}(m)$, the composition of h and m is a function $h \circ m : X \to \mathrm{ran}(h)$ by Proposition 3.178. To prove the first part ('⇒') of the equivalence, we assume that h is a bijection. By Proposition 3.183, the composition of the function h and the (maximal) invariant m is itself an invariant under $T(X)$. To prove that $h \circ m$ is even a maximal invariant, we let $x_1, x_2 \in X$ be arbitrary, assume that $(h \circ m)(x_1) = (h \circ m)(x_2)$, and we show that there exists an invertible transformation $t \in T(X)$ such that $x_2 = t(x_1)$. We first derive the equations

$$\begin{aligned} f(x_1) &= (\mathrm{id}_{\mathrm{ran}(m)} \circ m)(x_1) &&= ([h^{-1} \circ h] \circ m)(x_1) \\ &= (h^{-1} \circ [h \circ m])(x_1) &&= h^{-1}([h \circ m](x_1)) \\ &= h^{-1}([h \circ m](x_2)) &&= (h^{-1} \circ [h \circ m])(x_2) \\ &= ([h^{-1} \circ h] \circ m)(x_2) &&= (\mathrm{id}_{\mathrm{ran}(m)} \circ m)(x_2) \\ &= f(x_2) \end{aligned}$$

by applying the Neutrality of identity functions under composition, (3.679), the Associative Law for function composition, Notation 3.6, and substitution. As m is a maximal invariant under $T(X)$, it follows from $m(x_1) =$

$m(x_2)$ that there exists an invertible transformation $t \in T(X)$ such that $x_2 = t(x_1)$, which is precisely what we wanted to show. Then, since x_1 and x_2 are arbitrary elements of X, we may conclude that $h \circ m$ indeed constitutes a maximal invariant under $T(X)$.

To prove the second part ('$\Leftarrow$') of the proposed equivalence, we assume that $h \circ m$ is a maximal invariant under $T(X)$, and we show that h is a bijection. We first verify that h is an injection, i.e., that $h : \mathrm{ran}(m) \to \mathrm{ran}(h)$ satisfies

$$\forall y, y' \, ([y, y' \in \mathrm{ran}(m) \wedge h(y) = h(y')] \Rightarrow y = y'). \tag{6.21}$$

For this purpose, we let y and y' be arbitrary and assume $y, y' \in \mathrm{ran}(m)$ as well as $h(y) = h(y')$. The former assumption implies with the definition of a range that there exist two constants, say $\bar{x}$ and $\bar{x}'$, such that $(\bar{x}, y), (\bar{x}', y') \in m$. Since m is a function with domain X, we may write $y = m(\bar{m})$ and $y' = m(\bar{m}')$, where $\bar{x}, \bar{x}' \in X$. Now, it follows from the assumption $h(y) = h(y')$ that $h(m(\bar{x})) = h(m(\bar{x}'))$, which we may also write as $(h \circ m)(\bar{x}) = (h \circ m)(\bar{x}')$ since the maximal invariant $h \circ m$ is a composed function. By definition of a maximal invariant under $T(X)$, the latter equation implies that there exists an element of $T(X)$, say $\bar{t}$, such that $\bar{x}' = \bar{t}(\bar{x})$. Since m is then invariant under $\bar{t}$, we obtain the equations

$$[y' =] \, m(\bar{x}') = m(\bar{t}(\bar{x})) = (m \circ \bar{t})(\bar{x}) = m(\bar{x}) \, [= y],$$

giving the desired $y = y'$. Since y and y' were arbitrary, we may therefore conclude that the universal sentence (6.21) is true, so that h is indeed an injection from $\mathrm{ran}(m)$ to Z. As we specified the codomain of h to be the range of h, that function is by definition a surjection. Then, as an injection and a surjection, h is by definition a bijection. Since $(T(X), \circ)$, $m : X \to Y$ and h were arbitrary, we conclude that Part a) of the theorem holds.

To establish b), we let again $(T(X), \circ)$ be an arbitrary transformation group. Furthermore, we let $m_1 : X \to Y$ and $m_2 : X \to Z$ be arbitrary maximal invariants under $T(X)$. Since m_2 is then more generally an invariant under $T(X)$, we may apply the Characterization of maximal invariants to conclude that m_2 is the composition of some particular function $\bar{h} : \mathrm{ran}(m_1) \to \mathrm{ran}(m_2)$ and the maximal invariant m_1, symbolically $m_2 = \bar{h} \circ m_1$. Clearly, the range of $\bar{h}$ is also a codomain of that function, so that $\bar{h} : \mathrm{ran}(m_1) \to \mathrm{ran}(\bar{h})$. Since m_2 is a maximal invariant under $T(X)$, it follows with the preceding equation via substitution that the composition $\bar{h} \circ m_1$ is a maximal invariant under $T(X)$. These findings imply with a) that $\bar{h}$ is a bijection. $(T(X), \circ)$, $m_1 : X \to Y$ and $m_2 : X \to Z$ were arbitrary, we therefore conclude that Part b) of the theorem is also true. $\square$

6.3. Basic Laws for Groups

Theorem 6.9 (Inversion Laws for groups). *The following sentences are true for any group* $(X, *)$.

a) *Any element is equal to the inverse of its inverse, that is,*

$$\forall a\,(a \in X \Rightarrow [a^{-1}]^{-1} = a). \tag{6.22}$$

b) *Equality of the inverses of two elements implies equality of the elements, that is,*

$$\forall a, b\,(a, b \in X \Rightarrow [a^{-1} = b^{-1} \Rightarrow a = b]). \tag{6.23}$$

c) *Inversion is distributive over the binary group operation with reversal of order, in the sense that*

$$\forall a, b\,(a, b \in X \Rightarrow (a * b)^{-1} = b^{-1} * a^{-1}). \tag{6.24}$$

Proof. We let in the following X and $*$ be arbitrary sets such that $(X, *)$ constitutes a group.

Concerning a), we let a arbitrary and assume $a \in X$ to be true. Thus, the inverse element a^{-1} of a is defined, which satisfies by definition especially

$$a * a^{-1} = e. \tag{6.25}$$

The inverse element a^{-1} is itself evidently an element of X, so that the inverse element $[a^{-1}]^{-1}$ of a^{-1} exists also (in X), which thus satisfies in particular

$$a^{-1} * [a^{-1}]^{-1} = e. \tag{6.26}$$

We then obtain the true equations

$$\begin{aligned} a &= a * e \\ &= a * (a^{-1} * [a^{-1}]^{-1}) \\ &= (a * a^{-1}) * [a^{-1}]^{-1} \\ &= e * [a^{-1}]^{-1} \\ &= [a^{-1}]^{-1} \end{aligned}$$

by using the definition of the identity element e, (6.25), the associativity of $*$, (6.26), and finally again the definition of the identity element e. We thus find $[a^{-1}]^{-1} = a$ to be true, where a is arbitrary, so that (6.22) follows to be true.

Concerning b), we take arbitrary a and b, assuming that $a, b \in X$ holds, and assuming then also $a^{-1} = b^{-1}$ to be true. We now get

$$a = [a^{-1}]^{-1} = [b^{-1}]^{-1} = b.$$

by applying (6.22), substitution based on the assumed equation and again (6.22). Since a and b are arbitrary, we may now infer from the resulting equation $a = b$ the truth of the universal sentence (6.23).

Concerning c), we let again a and b be arbitrary such that $a, b \in X$ holds, and we demonstrate that $b^{-1} * a^{-1}$ is the inverse element of $a * b$, using the fact that $a * b$ is in X so that the inverse element $(a * b)^{-1}$ of $a * b$ with respect to $*$ is defined. Let us observe the truth of the equations

$$\begin{aligned}
(a * b) * (b^{-1} * a^{-1}) &= ((a * b) * b^{-1}) * a^{-1} \\
&= (a * (b * b^{-1})) * a^{-1} \\
&= (a * e) * a^{-1} \\
&= a * a^{-1} \\
&= e
\end{aligned}$$

in light of the associativity of $*$, the definition of an inverse element and the definition of the neutral element e. Using the same arguments, we also find

$$\begin{aligned}
(b^{-1} * a^{-1}) * (a * b) &= b^{-1} * (a^{-1} * (a * b)) \\
&= b^{-1} * ((a^{-1} * a) * b) \\
&= b^{-1} * (e * b) \\
&= b^{-1} * b \\
&= e.
\end{aligned}$$

Therefore, $b^{-1} * a^{-1}$ is by definition an inverse element of $a * b$. As the inverse element $(a * b)^{-1}$ of $a * b$ exists uniquely by virtue of Proposition 6.1, it follows that $b^{-1} * a^{-1} = (a * b)^{-1}$, so that the implication in (6.24) is true. Because a and b were arbitrary, we may now further conclude that c) holds.

As the sets X and $*$ were initially arbitrary, we may infer from the truth of a) – c) the truth of the stated theorem. □

Corollary 6.10. *It is true for any group $(X, *)$ that*

$$\forall a, b\, (a, b \in X \Rightarrow (a * b^{-1})^{-1} = b * a^{-1}), \tag{6.27}$$

$$\forall a, b\, (a, b \in X \Rightarrow (a^{-1} * b)^{-1} = b^{-1} * a). \tag{6.28}$$

Proof. Letting X, $*$, a and b be arbitrary such that $(X, *)$ is a group and such that $a, b \in X$ holds, we obtain

$$(a * b^{-1})^{-1} = [b^{-1}]^{-1} * a^{-1} = b * a^{-1}$$
$$(a^{-1} * b)^{-1} = b^{-1} * [a^{-1}]^{-1} = b^{-1} * a$$

with (6.24) and (6.22), so that the implications in (6.27) and (6.28) follow to be true. Since X, $*$, a and b are arbitrary, we may therefore conclude that the stated sentences hold. □

Note 6.4. We may for formulate the Inversion Laws for groups and (6.27) – (6.28) more generally as: For any semigroup $(X, *)$ such that the identity element e with respect to $*$ exists,

a) any element a whose inverse a^{-1} exists is equal to the inverse of that inverse, i.e.
$$(a^{-1})^{-1} = a. \tag{6.29}$$

b) and for any elements $a, b \in X$ whose inverses a^{-1} and b^{-1} exist, equality of these inverses implies equality of the elements, that is,
$$a^{-1} = b^{-1} \;\Rightarrow\; a = b. \tag{6.30}$$

c) and for any elements $a, b \in X$ whose inverses a^{-1} and b^{-1} exist, inversion is distributive over the binary group operation with reversal of order, in the sense that
$$(a * b)^{-1} = b^{-1} * a^{-1}. \tag{6.31}$$

d) and for any elements $a, b \in X$ whose inverses a^{-1} and b^{-1} exist, it is true that
$$(a * b^{-1})^{-1} = b * a^{-1}, \tag{6.32}$$
$$(a^{-1} * b)^{-1} = b^{-1} * a. \tag{6.33}$$

Proposition 6.11. *The following sentence is true for any group* $(X, *)$.

$$\forall a, b\, (a, b \in X \Rightarrow \exists d\, (a * d = b)) \tag{6.34}$$

Proof. We let X, $*$, a and b be arbitrary, assuming $(X, *)$ to be a group (with identity element e) and assuming $a, b \in X$ to be true. We can now show that replacing the variable d in the equation $a * d = b$ by the constant $a^{-1} * b$ gives a true sentence. Indeed, we obtain the equations

$$a * (a^{-1} * b) = (a * a^{-1}) * b = e * b = b$$

by means of the associativity of $*$, the definition of an inverse element and the definition of the identity element e. Thus, the resulting equation $a*(a^{-1}*b) = b$ demonstrates the truth of the existential sentence in (6.34). Since X, $*$, a and b are arbitrary, we may therefore conclude that the proposition is indeed true. □

Theorem 6.12 (Cancellation Law for groups). *The following sentence is true for any group* $(X, *)$.

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a * b = a * c \Leftrightarrow b = c]) \tag{6.35}$$

Proof. Letting X, $*$, a, b and c be arbitrary such that $(X, *)$ is a group (with identity element e) and such that $a, b, c \in X$ holds, and assuming moreover the equation $a * b = a * c$ to be true, we obtain the true equations

$$\begin{aligned} b &= e * b = (a^{-1} * a) * b = a^{-1} * (a * b) = a^{-1} * (a * c) = (a^{-1} * a) * c \\ &= e * c = c \end{aligned}$$

using the definition of the identity element e, the definition of an inverse element, the associativity of $*$, the assumed equation, again the associativity of $*$, again the definition of an inverse element, and finally again the definition of the identity element e. Thus, the resulting equation $b = c$ proves the first part ('$\Rightarrow$') of the equivalence in (6.35). To establish the second part ('$\Leftarrow$'), we assume now $b = c$ to be true, and we note that the equation $a * b = a * b$ holds to. Applying now substitution to the latter equation based on the assumed equation $b = c$, we obtain $a * b = a * c$, as desired. Since X, $*$, a, b and c were all arbitrary, the theorem follows then to true. □

We now specialize these concepts to the binary operation of addition.

Notation 6.2. For any semigroup $(X, +)$ such that the zero element 0_X of X with respect to the addition on X exists and for any element $a \in X$ such that the inverse a^{-1} of a with respect to the addition exists, we symbolize a^{-1} also by

$$-a \tag{6.36}$$

and speak of the *additive inverse* or the *negative* of a (with respect to the addition on X).

Note 6.5. For any semigroup $(X, +)$ with zero element 0_X, we see in light of the definition of an inverse element that the negative $-a$ of an $a \in X$, if it exists, satisfies the equations

$$a + (-a) = 0_X \;\wedge\; -a + a = 0_X. \tag{6.37}$$

In the former equation we use brackets by convention since in order to indicate more clearly that the negative of a is being added. Furthermore, we have according to Exercise 6.1 that the negative -0_X of 0_X exists uniquely and satisfies

$$-0_X = 0_X. \tag{6.38}$$

In addition, Exercise 6.2 yields for any $a \in X$ whose negative exists

$$a \neq 0_X \;\Rightarrow\; -a \neq 0_X. \tag{6.39}$$

In case of a group $(X, +)$, the negative exists for every element in X, which fact allows us to establish the following binary operation.

Exercise 6.4. Verify for any group $(X, +)$ that there exists the unique binary operation

$$-_X : X \times X \to X, \quad (a, b) \mapsto a -_X b = a + (-b). \tag{6.40}$$

(Hint: Proceed in analogy to the proof of Proposition 5.2.)

Definition 6.6 (Subtraction, difference). For any group $(X, +)$ we call the binary operation $-_X$ in (6.40) the *subtraction* on X. We then call for any $a, b \in X$ the value $a -_X b = a + (-b)$ the *difference* of a and b, which we symbolize also by

$$a - b \tag{6.41}$$

Proposition 6.13. *For any group $(X, +)$ with zero element 0, the ordered pair $(G_L(X), \circ_{G_L(X)})$ formed by the set of translations on X and the binary operation of composition on that set constitutes a group.*

Proof. Letting $(X, +)$ be an arbitrary semigroup with zero element 0, we recall from Note 5.13 that the ordered pair $(G_L(X), \circ_{G_L(X)})$ constitutes a semigroup. Furthermore, we recall from Proposition 5.32 that id_X is the identity element of $G_L(X)$ with respect to $\circ_{G_L(X)}$, so that $(G_L(X), \circ_{G_L(X)})$ possesses already Property 1 of a group. Thus, it remains for us to prove that the inverse element of any element of $G_L(X)$ with respect to $\circ_{G_L(X)}$ exists. To do this, we let $g \in G_L(X)$ be arbitrary, so that g is a transformation on X satisfying

$$\forall x\, (x \in X \Rightarrow g(x) = x + a)$$

for some particular $a \in X$. Thus, g is the translation g_{La} on X, by definition. As $(X, +)$ constitutes a group, the negative $-a$ of a exists (in X). It then follows with Exercise 5.9a) that $-a$ defines a translation on X by $-a$, that is, $g_{L-a} \in G_L(\mathbb{R})$. Recalling from Proposition 5.32 that id_X is the identity element of $G_L(X)$ with respect to $\circ_{G_L(X)}$, we may now prove that g_{L-a} is the negative of g_{La} with respect to $\circ_{G_L(X)}$, by establishing the truth of the equations

$$g_{La} \circ_{G_L(X)} g_{L-a} = \mathrm{id}_X \ \wedge \ g_{L-a} \circ_{G_L(X)} g_{La} = \mathrm{id}_X. \tag{6.42}$$

For this purpose, we apply the Equality Criterion for functions twice, letting $x \in X$ be arbitrary. We then obtain the equations

$$\begin{aligned}
(g_{La} \circ_{G_L(X)} g_{L-a})(x) &= (g_{La} \circ g_{L-a})(x) \\
&= g_{La}(g_{L-a}(x)) \\
&= g_{La}(x + (-a)) \\
&= (x + (-a)) + a \\
&= x + (-a + a) \\
&= x + 0 \\
&= x \\
&= \mathrm{id}_X(x)
\end{aligned}$$

as well as

$$\begin{aligned}
(g_{L-a} \circ_{G_L(X)} g_{La})(x) &= (g_{L-a} \circ g_{La})(x) \\
&= g_{L-a}(g_{La}(x)) \\
&= g_{L-a}(x + a) \\
&= (x + a) + (-a) \\
&= x + (a + (-a)) \\
&= x + 0 \\
&= x \\
&= \mathrm{id}_X(x)
\end{aligned}$$

by using (5.152), the notation for function compositions, the definition of a translation (twice), the associativity of $+$, the definition of an inverse element, the definition of an identity element, and the definition of an identity function. Because x is an arbitrary element of X, we may infer from the truth of these equations the truth of the equalities in (6.42). We thus proved that the inverse element of $g = g_{La}$ exists, and as g is an arbitrary element of $G_L(X)$, we may therefore conclude that $(G_L(X), \circ_{G_L(X)})$ possesses also Property 2 of a group. Since $(X, +)$ was initially arbitrary, we may now further conclude that the proposed universal sentence is true. $\square$

Definition 6.7 (Translation group). For any group $(X, +)$, we call

$$(G_L(X), \circ) = (G_L(X), \circ_{G_L(X)}) \tag{6.43}$$

the *translation group* on X.

Proposition 6.14. *It is true for any commutative group $(X, +)$ that the translation group $(G_L(X), \circ)$ is a transitive group of transformations.*

Proof. We let $(X, +)$ be an arbitrary commutative group, which gives rise to the translation group $(G_L(X), \circ)$, where $G_L(X)$ is a subset of X^X according to Note 5.5, and where $\circ$ is the restriction (5.146) of $\circ_X$ to $G_L(X) \times G_L(X)$. Next, we let x_1, x_2 arbitrary elements of X and observe the truth of the equations

$$\begin{aligned} x_2 &= x_2 + 0 \\ &= x_2 + (x_1 + (-x_1)) \\ &= (x_2 + x_1) + (-x_1) \\ &= (x_1 + x_2) + (-x_1) \\ &= x_1 + (x_2 + (-x_1)) \end{aligned}$$

in light of the definition of an identity element, the definition of an inverse element, the associativity of $+$ and the commutativity of $+$. Defining now $a = x_2 + (-x_1)$, we have $a \in X$ since $+$ is a binary operation X. Therefore, a defines the translation g_{La} on X, so that $g_{La} \in G_L(X)$ holds. In addition, the definition of a translation on X by a gives us then the equations

$$g_{La}(x_1) = x_1 + a = x_1 + (x_2 + (-x_1)) = x_2.$$

These findings demonstrate the truth of the desired existential sentence

$$\exists t\, (t \in G_L(X) \wedge x_2 = t(x_1)),$$

and as x_1, x_2 were arbitrary elements of X, we may therefore conclude that the translation group $(G_L(X), \circ)$ is a transitive group of transformations, by definition. □

The following exercise shows that, once we have a group, we may easily form also a corresponding 'group of functions' whose codomain is given by the original set.

Exercise 6.5. Establish the following sentences for any set X, any group $(Y, +_Y)$ and any function $f : X \to Y$.

a) There exists a unique function $-f : X \to Y$ satisfying

$$\forall x\,(x \in X \Rightarrow [-f](x) = -[f(x)]). \tag{6.44}$$

b) $-f$ is the negative of f with respect to the pointwise addition $+_{Y^X}$ of functions in Y^X.

Corollary 6.15. *It is true for any set X and any (commutative) group $(Y, +_Y)$ that the ordered pair $(Y^X, +_{Y^X})$ containing the pointwise addition of functions in Y^X is itself a (commutative) group, and the subtraction*

$$-_{Y^X} : Y^X \times Y^X \to Y^X, \quad (f, g) \mapsto f -_{Y^X} g = f +_{Y^X} (-g) \tag{6.45}$$

is defined.

Proof. Letting X be an arbitrary set and $(Y, +_Y)$ be an arbitrary (commutative) group, it is by definition also a (commutative) semigroup, which induces the (commutative) semigroup $(Y^X, +_{Y^X})$, as mentioned in Note 5.14. Furthermore, the zero element of Y^X with respect to the pointwise addition of functions in Y^X exists according to Proposition 5.19. Finally, Exercise 6.5 shows that the inverse of any element $f \in Y^X$ exists. Therefore, $(Y^X, +_{Y^X})$ is a (commutative) group by definition, and the subtraction is defined according to Exercise 6.4. Since $(Y, +_Y)$ is an arbitrary group, the preceding findings are then true for any set X and any group $(Y, +_Y)$. □

Recalling Note 5.14, we obtain the particular type of matrix group.

Corollary 6.16. *It is true for any $m, n \in \mathbb{N}_+$ and any (commutative) group $(Y, +_Y)$ that the (commutative) semigroup $(Y^{m\times n}, +_{Y^{m\times n}})$ with respect to the addition of m-by-n matrices with values in Y constitutes a (commutative) group, so that the subtraction*

$$-_{Y^{m\times n}} : Y^{m\times n} \times Y^{m\times n} \to Y^{m\times n}, \quad (A, B) \mapsto A -_{Y^{m\times n}} B = A +_{Y^{m\times n}} (-B) \tag{6.46}$$

is defined.

Corollary 6.17 (Characterization of the negative of a matrix). *Show for any positive natural numbers m, n and any group $(Y, +_Y)$ that the negative $-\boldsymbol{A}$ of any m-by-n matrix*

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \tag{6.47}$$

with values in Y satisfies

$$\forall i, j\,([i \in \{1, \ldots, m\} \wedge j \in \{1, \ldots, n\}] \Rightarrow [-\boldsymbol{A}]((i, j)) = -[a_{i,j}]). \tag{6.48}$$

Proof. Letting m and n be arbitrary positive natural numbers, $(Y, +_Y)$ an arbitrary group, $\boldsymbol{A}$ an arbitrary m-by-n matrix with values in Y given on the right-hand side of (6.47), i an arbitrary element of $\{1,\ldots,m\}$ and j an arbitrary element of $\{1,\ldots,n\}$, we may define the Cartesian product $(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}$. Since $\boldsymbol{A}$ is a function from $\{1,\ldots,m\} \times \{1,\ldots,n\}$ to Y, the negative $-\boldsymbol{A}$ of $\boldsymbol{A}$ satisfies the desired equations $[-\boldsymbol{A}]((i,j)) = -[\boldsymbol{A}((i,j))] = -[a_{i,j}]$ in view of Exercise 6.5 and (4.713). We may therefore conclude that the corollary holds as claimed. □

Notation 6.3. The preceding corollary shows that we may write the negative of a given m-by-n matrix (6.47) as

$$-\boldsymbol{A} = \begin{bmatrix} -a_{1,1} & \cdots & -a_{1,n} \\ \vdots & \ddots & \vdots \\ -a_{m,1} & \cdots & -a_{m,n} \end{bmatrix}. \tag{6.49}$$

We now rewrite the Inversion Laws for groups, Proposition 6.11 and the Cancellation Law for groups specifically in terms of addition/subtraction.

Corollary 6.18 (Sign Laws for $-$ and $+$). *The following sentences are true for any group $(X, +)$.*

a) Any element is equal to the negative of its negative, that is,

$$\forall a\,(a \in X \Rightarrow -(-a) = a). \tag{6.50}$$

b) Equality of the negatives of two elements implies equality of the elements, that is,

$$\forall a, b\,(a, b \in X \Rightarrow [-a = -b \Rightarrow a = b]). \tag{6.51}$$

c) The negative of a sum can be written as the difference of the negative of the second summand and the other summand, that is,

$$\forall a, b\,(a, b \in X \Rightarrow -(a + b) = -b - a). \tag{6.52}$$

d) The negative of a difference can be written as a difference by reversing the elements, in the sense that

$$\forall a, b\,(a, b \in X \Rightarrow -(a - b) = b - a). \tag{6.53}$$

Corollary 6.19. *The following sentence is true for any group $(X, +)$.*

$$\forall a, b\,(a, b \in X \Rightarrow \exists d\,(d \in X \wedge a + d = b)). \tag{6.54}$$

Corollary 6.20 (Cancellation Law for $+$). *The following sentence is true for any group $(X, +)$.*

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a + b = a + c \Rightarrow b = c]). \tag{6.55}$$

6.4. Rings $(X, +, \cdot, -)$

The following new structure combines a commutative addition group with an associative multiplication which is distributive over the addition.

Definition 6.8 (Ring, commutative/Abelian ring, Boolean ring). For any group $(X, +)$ with induced subtraction $-$ on X and for any multiplication $\cdot$ on X, we say that the ordered quadruple

$$(X, +, \cdot, -) \tag{6.56}$$

is a *ring* iff

(1) $(X, +)$ is a commutative group,

(2) $(X, \cdot)$ is a semigroup, and

(3) the multiplication $\cdot$ is distributive over the addition $+$.

Furthermore, we say that a ring $(X, +, \cdot, -)$ is *commutative* or *Abelian* iff the multiplication $\cdot$ is commutative. In addition, we call a ring

$$(X, +, \cdot, +) \tag{6.57}$$

(for which the subtraction is thus identical with the addition) a *Boolean ring* iff the multiplication is idempotent.

Note 6.6. Thus, the ordered triple $(X, +, \cdot)$ is a semiring where $(X, +)$ is now a commutative group instead of a commutative semigroup, so that the binary subtraction operation $-$ is defined (in terms of the addition and the negatives).

We combine now the observations made in Corollary 5.39 and Corollary 6.15 to obtain a first ring type.

Corollary 6.21. *For any set X and any (commutative) ring $(Y, +_Y, \cdot_Y, -_Y)$, it is true that the ordered quadruple $(Y^X, +_{Y^X}, \cdot_{Y^X}, -_{Y^X})$ containing the pointwise addition and multiplication of functions in Y^X also constitutes a (commutative) ring.*

Definition 6.9 (Ring of functions from X to Y). For any set X and any ring $(Y, +_Y, \cdot_Y, -_Y)$, we call the ordered quadruple

$$(Y^X, +_{Y^X}, \cdot_{Y^X}, -_{Y^X}) \tag{6.58}$$

(containing the pointwise addition and multiplication of functions in Y^X) the *ring of functions from X to Y*.

Definition 6.10 (Trivial ring, nontrivial ring). We say that a ring $(X, +, \cdot, -)$ is *trivial* iff $X = \{0_X\}$, and we say that a ring $(X, +, \cdot, -)$ is *nontrivial* iff $X \neq \{0_X\}$.

Exercise 6.6. Show for any Boolean ring $(X, +, \cdot, +)$ with unity element 1_X that there exist the binary operation

$$\vee_X : X \times X \to X, \quad (a, b) \mapsto a \vee_X b = (a + b) + (a \cdot b) \tag{6.59}$$

and the function

$$'_X : X \to X, \quad a \mapsto a' = a + 1_X. \tag{6.60}$$

Definition 6.11 (Boolean algebra). For any Boolean ring $(X, +_X, \cdot_X, +_X)$ with unity element 1_X we call the ordered quadruple

$$(X, \vee_X, \cdot_X, '_X) \tag{6.61}$$

a *Boolean algebra*, which we also symbolize by $(X, \vee, \wedge, ')$.

The ring properties alone implies the following cancellation law for 0_X in analogy to the one for ordered elementary domains, not requiring any linear ordering of the set X.

Theorem 6.22 (Cancellation Law for 0_X in rings). *The following sentence is true for any ring $(X, +, \cdot, -)$.*

$$\forall a\, (a \in X \Rightarrow [a \cdot 0_X = 0_X \,\wedge\, 0_X \cdot a = 0_X]). \tag{6.62}$$

Exercise 6.7. Prove the Cancellation Law for 0_X in rings in analogy to the Cancellation Law for 0_X in ordered elementary domains.

The following theorem states two frequently arising interactions between the formation of negatives and the multiplication.

Theorem 6.23 (Sign Laws for $-$ and $\cdot$). *The following equations are true for any ring $(X, +, \cdot, -)$ and any $a, b \in X$.*

$$a \cdot (-b) = -(a \cdot b) \tag{6.63}$$

$$(-a) \cdot b = -(a \cdot b) \tag{6.64}$$

$$(-a) \cdot (-b) = a \cdot b \tag{6.65}$$

Proof. We let X, $+$, $\cdot$, $-$, a and b be arbitrary, assuming $(X, +, \cdot, -)$ to be a ring and assuming $a, b \in X$ to hold. Concerning the equation in (6.63), we show that $a \cdot (-b)$ is the additive inverse of $a \cdot b$. We obtain

$$(a \cdot (-b)) + (a \cdot b) = (a \cdot b) + (a \cdot (-b)) = a \cdot (b + (-b)) = a \cdot 0_X = 0_X$$

by applying the commutativity of the addition, the (left-)distributivity of the multiplication over the addition and the Cancellation Law for 0_X in rings, so that the equations

$$(a \cdot b) + (a \cdot (-b)) = 0_X \land (a \cdot (-b)) + (a \cdot b) = 0_X$$

hold. Thus, $a \cdot (-b)$ is by definition an inverse element of $a \cdot b$. Because $-(a \cdot b)$ is the unique inverse element of $a \cdot b$, the equation (6.63) follows to be true, as desired.

The second law (6.64) can be proved similarly by exploiting the right-distributivity of the multiplication over the addition.

Concerning (6.65), we observe the truth of

$$(-a) \cdot (-b) = -((-a) \cdot b) = -(-(a \cdot b)) = a \cdot b.$$

in light of (6.63), (6.64) and (6.50).

Since X, $+$, $\cdot$, $-$, a and b were arbitrary, we may now infer from these findings the truth of the stated laws. □

Exercise 6.8. Prove the Sign Law (6.64).

(Hint: Proceed in analogy to the proof of (6.63), using now (5.121).)

Proposition 6.24. *The following sentence is true for any ring* $(X, +, \cdot, -)$.

$$\forall a, b\, (a, b \in X \Rightarrow \exists! d\, (a + d = b)). \tag{6.66}$$

Proof. Letting X, $+$, $\cdot$, $-$, a and b be arbitrary such that $(X, +, \cdot, -)$ is a ring and such that a and b are elements of X, we see in light of Corollary 6.34 that the existential part $\exists d\, (a + d = b)$ of the uniquely existential sentence to be proven holds, since $(X, +)$ is a group (by definition of a ring). To prove the uniqueness part, we let d and d' be arbitrary such that $a + d = b$ and $a + d' = b$ are true, and we verify that $d = d'$ is implied. Let us observe here that $a + d = b$ can be written in the form $((a, d), b) \in +$, so that $(a, d) \in X \times X$ [$= \mathrm{dom}(+)$] holds by definition of a domain. Thus, d turns out to be an element of X by definition of the Cartesian product of two sets. Combining now the equations $a + d = b$ and $a + d' = b$, we obtain $a + d = a + d'$, which in turn implies the desired $d = d'$ with the Cancellation Law for $+$ in rings. As d and d' are arbitrary, we may infer from the truth of the preceding equation the truth of the uniqueness part and therefore the truth of the uniquely existential sentence in (6.66). Because X, $+$, $\cdot$, $-$, a and b were initially all arbitrary, we may now conclude that the proposition holds, as claimed. □

Let us recall that the set $(X, +)$ underlying an ordered elementary domain $(X, +, \cdot, <)$ is a semigroup with zero element 0_X, so that the negative of 0_X is given by $-0_X = 0_X$, as shown by (6.38). Other elements of X do not possess a negative.

Proposition 6.25. *For any ordered elementary domain* $(X, +, \cdot, <)$ *and any nonzero element of* X*, it is true that the negative of that element does not exist, i.e.*

$$\forall a\,([a \in X \,\wedge\, a \neq 0_X] \Rightarrow \neg\exists a^-\,(a + a^- = 0_X \,\wedge\, a^- + a = 0_X)). \qquad (6.67)$$

Proof. We let X, $+$, $\cdot$, $<$ and a be arbitrary, we assume $(X, +, \cdot, <)$ to be an ordered elementary domain, and we prove the implication by contradiction, assuming a to be an element of X such that $a \neq 0_X$ holds and assuming the negation of the negated existential sentence in (6.67) to be true. The latter assumption implies with the Double Negation Law that there exists a constant, say $\bar{a}^-$, satisfying the equations $a + \bar{a}^- = 0_X$ and $\bar{a}^- + a = 0_X$. We note that $(a, \bar{a}^-)$ is in the domain $X \times X$ of the addition $+$ on X, so that $\bar{a}^-$ is an element of X by definition of the Cartesian product of two sets. Then, the truth of $a, \bar{a}^- \in X$ and of $a \neq 0_X$ implies $a + \bar{a}^- \neq 0_X$ because of (5.278), which is in contradiction to the previously established equation $a + \bar{a}^- = 0_X$. We thus completed the proof of the implication in (6.67), and since X, $+$, $\cdot$, $<$ and a were initially arbitrary, we may therefore conclude that the proposed universal sentence is true. $\square$

As a second example, a commutative group involving the addition of matrices can be expanded to a ring. We prepare this by defining a suitable multiplication operation for matrices.

Exercise 6.9 (Product of matrices). Establish the following sentences for any positive natural numbers m, n, p, any set Y, any addition $+_Y$ on Y such that the zero element 0_Y exists, any multiplication $\cdot_Y$ on Y, any m-by-n matrix $\boldsymbol{A}$ with values in Y, and any n-by-p matrix $\boldsymbol{B}$ also with values in Y.

a) For any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, p\}$, the n-tuple

$$(\boldsymbol{A}_i(i,k) \cdot_Y \boldsymbol{B}^{(j)}(k,j) \,|\, k \in \{1, \ldots, n\}) \qquad (6.68)$$

in Y exists uniquely.

(Hint: Proceed similarly as in Exercise 5.5.)

b) There exists a unique m-by-p matrix $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$ with values in Y satisfying

$$c_{i,j} = \sum_{k=1}^{n} \left(\boldsymbol{A}_i(i,k) \cdot_Y \boldsymbol{B}^{(j)}(k,j)\right) \qquad (6.69)$$

for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, p\}$.

(Hint: Apply first Function definition by replacement to establish

$$\forall z \, (z \in \{1, \ldots, m\} \times \{1, \ldots, p\} \Rightarrow$$
$$\exists i, j \, (z = (i,j) \wedge \boldsymbol{C}(z) = \sum_{k=1}^{n} \Big(\boldsymbol{A}_i(i,k) \cdot_Y \boldsymbol{B}^{(j)}(k,j) \Big)))$$

and then the universal sentence b) by using (3.3) and (4.713).)

Note 6.7. Recalling that $\boldsymbol{A}_i : \{i\} \times \{1, \ldots, n\} \to Y$ is the ith row of $\boldsymbol{A}$ and that $\boldsymbol{B}^{(j)} : \{1, \ldots, n\} \times \{j\} \to Y$ is the jth column of $\boldsymbol{B}$, we observe in light of Proposition 4.106 that the domain $\{1, \ldots, n\} \times \{j\}$ of $\boldsymbol{B}^{(j)}$ has the cardinality n. Furthermore, since the sets $\{i\} \times \{1, \ldots, n\}$ and $\{1, \ldots, n\} \times \{i\}$ are equinumerous in view of Exercise 4.37d), we see that the domain of $\boldsymbol{A}_i$ has the same cardinality n as the domain of $\boldsymbol{B}^{(j)}$. Thus, the ith row of $\boldsymbol{A}$ and the jth column of $\boldsymbol{B}$ have the same number of elements, so that it makes sense to multiply both sets ‘pointwise’, similarly to the pointwise multiplication of two functions sharing the same domain. This finding is a direct consequence of the fact that the number of columns in $\boldsymbol{A}$ coincides with the number of rows in $\boldsymbol{B}$.

Note 6.8. For $m, n \in \mathbb{N}_+$, any set Y and any addition $+_Y$ such that the zero element 0_Y exists, for any multiplication $\cdot_Y$ and any m-by-n matrix $\boldsymbol{A}$, the transpose $\boldsymbol{A}^T$ of that matrix is an n-by-m matrix, so that the product $\boldsymbol{A}^T \boldsymbol{A}$ constitutes a square matrix of order n.

Definition 6.12 (Gram matrix). For any $m, n \in \mathbb{N}_+$, any set Y and any addition $+_Y$ such that the zero element 0_Y exists, for any multiplication $\cdot_Y$ and any m-by-n matrix $\boldsymbol{A}$, we call the product

$$\boldsymbol{A}^T \boldsymbol{A} \tag{6.70}$$

the *Gram matrix* formed by $\boldsymbol{A}$.

Theorem 6.26 (Transposition rule for matrix products). *It is true for any positive natural numbers m, n, p, any set Y and any addition $+_Y$ on Y such that the zero element 0_Y exists, any multiplication $\cdot_Y$ on Y, any m-by-n matrix $\boldsymbol{A}$ with values in Y and for any n-by-p matrix $\boldsymbol{B}$ also with values in Y that the transpose of the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is given by the product of the transpose of $\boldsymbol{B}$ and the transpose of $\boldsymbol{A}$, i.e.*

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T. \tag{6.71}$$

Proof. We take arbitrary $m, n, p \in \mathbb{N}_+$, an arbitrary set Y and an arbitrary addition $+_Y$ on Y with zero element 0_Y, an arbitrary multiplication $\cdot_Y$ on

Y, an arbitrary m-by-n matrix $\boldsymbol{A}$ with values in Y, and an arbitrary n-by-p matrix $\boldsymbol{B}$ also with values in Y. Therefore, $\boldsymbol{A}^T$ is an n-by-m matrix and $\boldsymbol{B}^T$ is a p-by-n matrix, by definition of the transpose of a matrix. then, the product $\boldsymbol{C} = \boldsymbol{AB}$ follows to be an m-by-p matrix and $\boldsymbol{C}' = \boldsymbol{B}^T\boldsymbol{A}^T$ a p-by-m matrix. Since the transpose $\boldsymbol{C}^T = (\boldsymbol{AB})^T$ constitutes also a p-by-m matrix, the domains $\{1, \ldots, p\} \times \{1, \ldots, m\}$ of the last two matrices coincide. To establish their equality (6.71), we apply the Equality Criterion for functions, letting z be an arbitrary element of that domain. According to Exercise 3.4, there are then particular elements $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, m\}$ with $(i, j) = z$. The latter equation allows us to write

$$(\boldsymbol{AB})^T(z) = c^T_{i,j} = c_{j,i} = \sum_{k=1}^{n} \Big(\boldsymbol{A}_j(j,k) \cdot_Y \boldsymbol{B}^{(i)}(k,i) \Big) \tag{6.72}$$

by applying substitution, subsequently (4.725), and then (6.69). Similarly, we obtain

$$(\boldsymbol{B}^T\boldsymbol{A}^T)(z) = c'_{i,j} = \sum_{k=1}^{n} \Big(\boldsymbol{B}^T_i(i,k) \cdot_Y \boldsymbol{A}^{T(j)}(k,j) \Big). \tag{6.73}$$

Noting that the n-fold sums in (6.72) and (6.73) involve the two n-tuples

$$(\boldsymbol{A}_j(j,k) \cdot_Y \boldsymbol{B}^{(i)}(k,i) \,|\, k \in \{1, \ldots, n\}), \tag{6.74}$$

$$(\boldsymbol{B}^T_i(i,k) \cdot_Y \boldsymbol{A}^{T(j)}(k,j) \,|\, k \in \{1, \ldots, n\}) \tag{6.75}$$

sharing the same domain $\{1, \ldots, n\}$, we now prove their equality, letting k be arbitrary and assuming $k \in \{1, \ldots, n\}$ to be true. We then obtain

$$\begin{aligned}
\boldsymbol{B}^T_i(i,k) \cdot_Y \boldsymbol{A}^{T(j)}(k,j) &= \boldsymbol{B}^T((i,k)) \cdot_Y \boldsymbol{A}^T((k,j)) \\
&= b^T_{i,k} \cdot_Y a^T_{k,j} \\
&= a_{j,k} \cdot_Y b_{k,i} \\
&= \boldsymbol{A}((j,k)) \cdot_Y \boldsymbol{B}((k,i)) \\
&= \boldsymbol{A}_j(j,k) \cdot_Y \boldsymbol{B}^{(i)}(k,i)
\end{aligned}$$

by applying (3.567), the definition (4.713) of an entry, and (4.725). Since k is arbitrary, we may therefore conclude by virtue of the Equality Criterion for functions that the n-tuples (6.74) and (6.75) are identical indeed. The corresponding n-fold sums (6.72) and (6.73) are then evidently identical as well. Since z is arbitrary, we may now further conclude that the matrices $(\boldsymbol{AB})^T$ and $(\boldsymbol{B}^T\boldsymbol{A}^T)$ are identical, too. Having thus proven (6.71), we may finally conclude that the theorem is true since m, n, p, Y, $+_Y$, $\cdot_Y$, $\boldsymbol{A}$ and $\boldsymbol{B}$ were initially all arbitrary. $\square$

Corollary 6.27. *It is true for any $m, n \in \mathbb{N}_+$, any set Y and any addition $+_Y$ such that the zero element 0_Y exists, for any multiplication $\cdot_Y$ and for any m-by-n matrix $\boldsymbol{A}$ with values in Y that the Gram matrix formed by $\boldsymbol{A}$ is symmetric.*

Proof. Letting m and n be arbitrary positive natural numbers, letting Y be an arbitrary set and $+_Y$ an arbitrary addition on Y such that the zero element 0_Y exists, letting $\cdot_Y$ be an arbitrary multiplication on Y, and letting $\boldsymbol{A}$ be an arbitrary m-by-n matrix $\boldsymbol{A}$ with values in Y, we obtain the true equations

$$(\boldsymbol{A}^T\boldsymbol{A})^T = \boldsymbol{A}^T(\boldsymbol{A}^T)^T = \boldsymbol{A}^T\boldsymbol{A}$$

by applying the Transposition rule for matrix products and the Double Transposition Law. The resulting equation $(\boldsymbol{A}^T\boldsymbol{A})^T = \boldsymbol{A}^T\boldsymbol{A}$ shows then that $\boldsymbol{A}^T\boldsymbol{A}$ is symmetric, by definition. Since all of the occurring constants were arbitrary, we may therefore conclude that the stated universal sentence is true. □

To obtain a proper binary multiplication operation (with coinciding domain and codomain) from this specification of product, all involved matrices are required be square.

Proposition 6.28 (Multiplication of matrices). *It is true for any positive natural number n, any set Y, any addition $+_Y$ on Y such that the zero element exists, and any multiplication $\cdot_Y$ on Y that there exists a unique binary multiplication operation $\cdot_{Y^{n\times n}}$ in the sense of (6.69), which satisfies*

$$\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\,(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in Y^{n\times n} \Rightarrow [\boldsymbol{C} = \boldsymbol{A} \cdot_{Y^{n\times n}} \boldsymbol{B} \tag{6.76}$$
$$\Leftrightarrow \forall i, j\,(i, j \in \{1, \ldots, n\} \Rightarrow c_{i,j} = \sum_{k=1}^{n}(a_{i,k} \cdot_Y b_{k,j}))]).$$

Proof. We let n be an arbitrary positive natural number, Y an arbitrary set, $+_Y$ an arbitrary addition on Y such that the zero element exists, and $\cdot_Y$ an arbitrary multiplication on Y. First, we apply Function definition by replacement to prove the universal sentence

$$\forall x\,(x \in Y^{n\times n} \times Y^{n\times n} \Rightarrow \exists! y\,(\exists \boldsymbol{A}, \boldsymbol{B}\,(x = (\boldsymbol{A}, \boldsymbol{B}) \wedge \forall i, j\,(i, j \in \{1, \ldots, n\}$$
$$\Rightarrow y(i,j) = \sum_{k=1}^{n}\Big(\boldsymbol{A}_i(i,k) \cdot_Y \boldsymbol{B}^{(j)}(k,j)\Big)))))). \tag{6.77}$$

For this purpose, we let x be arbitrary and assume $x \in Y^{n\times n} \times Y^{n\times n}$ to be true. In view of (3.38), there exist then particular matrices $\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}} \in Y^{n\times n}$

with $x = (\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}})$. We know from Exercise 6.9b) that there exists then a unique n-by-n matrix $\boldsymbol{C}$ with values in Y such that

$$\forall i, j\,(i, j \in \{1, \ldots, n\} \Rightarrow c_{i,j} = \sum_{k=1}^{n} (\bar{a}_{i,k} \cdot_Y \bar{b}_{k,j})).$$

Since $c_{i,j}$ can be written as $\boldsymbol{C}((i,j))$ or $\boldsymbol{C}(i,j)$, the existential sentence in (6.77) with respect to y is true. Regarding the uniqueness part, we take arbitrary y and y' such that the existential sentences

$$\exists \boldsymbol{A}, \boldsymbol{B}\,(x = (\boldsymbol{A}, \boldsymbol{B}) \wedge \forall i, j\,(i, j \in \{1, \ldots, n\}$$
$$\Rightarrow y(i,j) = \sum_{k=1}^{n} \Big(\boldsymbol{A}_i(i,k) \cdot_Y \boldsymbol{B}^{(j)}(k,j)\Big))),$$
$$\exists \boldsymbol{A}, \boldsymbol{B}\,(x = (\boldsymbol{A}, \boldsymbol{B}) \wedge \forall i, j\,(i, j \in \{1, \ldots, n\}$$
$$\Rightarrow y'(i,j) = \sum_{k=1}^{n} \Big(\boldsymbol{A}_i(i,k) \cdot_Y \boldsymbol{B}^{(j)}(k,j)\Big)))$$

hold. Thus, there are particular constants $\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{A}}', \bar{\boldsymbol{B}}'$ satisfying $x = (\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}}) = (\bar{\boldsymbol{A}}', \bar{\boldsymbol{B}}')$, so that the Equality Criterion for ordered pairs yields $\bar{\boldsymbol{A}} = \bar{\boldsymbol{A}}'$ and $\bar{\boldsymbol{B}} = \bar{\boldsymbol{B}}'$. These constants also satisfy the universal sentences

$$\forall i, j\,(i, j \in \{1, \ldots, n\} \Rightarrow y(i,j) = \sum_{k=1}^{n} \Big(\bar{\boldsymbol{A}}_i(i,k) \cdot_Y \bar{\boldsymbol{B}}^{(j)}(k,j)\Big)), \tag{6.78}$$

$$\forall i, j\,(i, j \in \{1, \ldots, n\} \Rightarrow y'(i,j) = \sum_{k=1}^{n} \Big(\bar{\boldsymbol{A}}_i(i,k) \cdot_Y \bar{\boldsymbol{B}}^{(j)}(k,j)\Big)). \tag{6.79}$$

Clearly, y and y' are n-by-n matrices with values in Y, thus functions from $\{1, \ldots, n\} \times \{1, \ldots, n\}$ to Y. To prove that they are identical, we apply the Equality Criterion for functions and establish accordingly

$$\forall z\,(z \in \{1, \ldots, n\} \times \{1, \ldots, n\} \Rightarrow y(z) = y'(z)). \tag{6.80}$$

Letting $z \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ be arbitrary, there are then evidently particular indexes $\bar{i}, \bar{j} \in \{1, \ldots, n\}$ with $(\bar{i}, \bar{j}) = z$. Consequently, (6.78) and (6.79) yield

$$y(i,j) = \sum_{k=1}^{n} \Big(\bar{\boldsymbol{A}}_i(i,k) \cdot_Y \bar{\boldsymbol{B}}^{(j)}(k,j)\Big) = y'(i,j),$$

and therefore

$$y(z) = y((i,j)) = y(i,j) = y'(i,j) = y'((i,j)) = y'(z).$$

Since z is arbitrary, we may infer from the truth of the resulting equation $y(z) = y'(z)$ the truth of the universal sentence (6.80), which in turn implies $y = y'$. As y and y' are also arbitrary, we may conclude that the uniquely existential sentence in (6.77) is true. As a consequence, the universal sentence (6.77) also holds because x was arbitrary. Therefore, there exists a unique function $\cdot_{Y^{n\times n}}$ with domain $Y^{n\times n} \times Y^{n\times n}$ satisfying

$$\forall x\,(x \in Y^{n\times n} \times Y^{n\times n} \Rightarrow \exists \boldsymbol{A}, \boldsymbol{B}\,(x = (\boldsymbol{A}, \boldsymbol{B}) \wedge \forall i, j\,(i, j \in \{1, \ldots, n\}$$
$$\Rightarrow [\cdot_{Y^{n\times n}}(\boldsymbol{A}, \boldsymbol{B})](i, j) = \sum_{k=1}^{n} \Big(\boldsymbol{A}_i(i, k) \cdot_Y \boldsymbol{B}^{(j)}(k, j)\Big)))). \tag{6.81}$$

Recalling that the unique function value y is an n-by-n matrices with values in Y for each $x \in Y^{n\times n} \times Y^{n\times n}$, we see that $Y^{n\times n}$ is a codomain of $\cdot_{Y^{n\times n}}$. Thus, $\cdot_{Y^{n\times n}}$ constitutes a binary operation on $Y^{n\times n}$, which can be shown to satisfy also (6.76) - see the following exercise. Initially, the sets n, Y and $\cdot_Y$ were all arbitrary, so that the proposition holds as claimed. □

Exercise 6.10. Complete the proof of Proposition 6.28 by establishing (6.76).

Notation 6.4. We also write $\boldsymbol{AB}$ instead of $\boldsymbol{A} \cdot_{Y^{n\times n}} \boldsymbol{B}$.

Definition 6.13 (Idempotent matrix). For any positive natural number n, any set Y, any addition $+_Y$ with zero element 0_Y, any multiplication $\cdot_Y$ with identity element 1_Y, and any n-by-n matrix $\boldsymbol{A}$ with values in Y, we say that $\boldsymbol{A}$ is *idempotent* iff the multiplication of $\boldsymbol{A}$ with itself does not change $\boldsymbol{A}$, i.e., iff

$$\boldsymbol{AA} = \boldsymbol{A}. \tag{6.82}$$

Matrix multiplication provides a convenient mechanism for adding up all entries of a matrix.

Theorem 6.29 (Characterization of double n-fold sums). *The following equations hold for any positive natural number n, any set Y, any addition $+_Y$ with zero element 0_Y, any multiplication $\cdot_Y$ with identity element 1_Y, and any n-by-n matrix $\boldsymbol{A}$ with values in Y.*

$$\begin{bmatrix} 1_Y & \cdots & 1_Y \end{bmatrix} \left(\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} 1_Y \\ \vdots \\ 1_Y \end{bmatrix} \right) = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \right], \tag{6.83}$$

$$\left(\begin{bmatrix} 1_Y & \cdots & 1_Y \end{bmatrix} \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \right) \begin{bmatrix} 1_Y \\ \vdots \\ 1_Y \end{bmatrix} = \left[\sum_{j=1}^{n} \sum_{i=1}^{n} a_{i,j} \right] \tag{6.84}$$

Proof. Letting n, Y, $+_Y$, $\cdot_Y$ and $\boldsymbol{A}$ be arbitrary, we assume n to be a positive natural number, $+_Y$ to be an addition on Y with zero element 0_Y, $\cdot_Y$ to be a multiplication on Y, and $\boldsymbol{A}$ to be an n-by-n matrix with values in Y. The specification of the product of matrices gives us then the equations

$$\begin{aligned}
&\begin{bmatrix} 1_Y & \cdots & 1_Y \end{bmatrix} \left(\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} 1_Y \\ \vdots \\ 1_Y \end{bmatrix} \right) \\
&= \begin{bmatrix} 1_Y & \cdots & 1_Y \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} (a_{1,j} \cdot_Y 1_Y) \\ \vdots \\ \sum_{j=1}^{n} (a_{n,j} \cdot_Y 1_Y) \end{bmatrix} \\
&= \left[\sum_{i=1}^{n} \left(1_Y \cdot_Y \sum_{j=1}^{n} (a_{i,j} \cdot_Y 1_Y) \right) \right],
\end{aligned}$$

which further imply (6.83) with the definition of an identity element. Equation (6.84) is established similarly. Thus, the double sums $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}$ and $\sum_{j=1}^{n} \sum_{i=1}^{n} a_{i,j}$ are well-defined entries of corresponding 1-by-1 matrices. Since n, Y, $+_Y$, $\cdot_Y$ and $\boldsymbol{A}$ were initially arbitrary, we may therefore conclude that the proposition holds. □

Exercise 6.11. Complete the proof of Theorem 6.29 by establishing (6.84).

Theorem 6.30 (Interchange of two nested n-fold sums). *The following sentence holds for any set Y, any associative addition $+_Y$ with zero element 0_Y and any multiplication $\cdot_Y$ with identity element 1_Y. For any positive natural number n and any n-by-n matrix $\boldsymbol{A}$ with values in Y, two nested n-fold sums can be interchanged in the sense of*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{i,j}. \tag{6.85}$$

Proof. Letting Y be an arbitrary set, $+_Y$ an arbitrary associative addition on Y with zero element 0_Y and $\cdot_Y$ an arbitrary multiplication on Y with identity element 1_Y, we prove the proposed universal sentence with respect to n by means of mathematical induction. Concerning the base case ($n = 1$), we let $\boldsymbol{A} = \begin{bmatrix} a_{1,1} \end{bmatrix}$ be an arbitrary 1-by-1 matrix with value in Y and observe the truth of the equations

$$\sum_{i=1}^{1} \sum_{j=1}^{1} a_{i,j} = \sum_{i=1}^{1} a_{i,1} = a_{1,1} = \sum_{j=1}^{1} a_{1,j} = \sum_{j=1}^{1} \sum_{i=1}^{1} a_{i,j}$$

in light of (5.411). Since $\boldsymbol{A}$ was arbitrary, we may infer from the truth of these equations the truth of the base case. Regarding the induction step, we let n be an arbitrary positive natural number and make the induction assumption (6.85). Then, we let $\boldsymbol{A}$ be an arbitrary $n+1$-by-$n+1$ matrix with values in Y and observe the truth of the equations

$$\begin{aligned}
\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_{i,j} &= \sum_{i=1}^{n+1}\left(\sum_{j=1}^{n} a_{i,j} +_Y a_{i,n+1}\right) \\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{i,j} +_Y a_{i,n+1}\right) +_Y \sum_{j=1}^{n} a_{n+1,j} +_Y a_{n+1,n+1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} +_Y \sum_{i=1}^{n} a_{i,n+1} +_Y \sum_{j=1}^{n} a_{n+1,j} +_Y a_{n+1,n+1} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} a_{i,j} +_Y \sum_{j=1}^{n} a_{n+1,j} +_Y \sum_{i=1}^{n} a_{i,n+1} +_Y a_{n+1,n+1} \\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{i,j} +_Y a_{n+1,j}\right) +_Y \left(\sum_{i=1}^{n} a_{i,n+1} +_Y a_{n+1,n+1}\right) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n+1} a_{i,j} +_Y \sum_{i=1}^{n+1} a_{i,n+1} \\
&= \sum_{j=1}^{n+1}\sum_{i=1}^{n+1} a_{i,j}
\end{aligned}$$

in light of (5.417), Theorem 5.113a), the induction assumption, the commutativity of $+_Y$, and the associativity of $+_Y$. As $\boldsymbol{A}$ and also n are arbitrary, we may conclude that the induction step holds as well, so that the proof by mathematical induction is now complete. Initially, the sets Y, $+_Y$ and $\cdot_Y$ were all arbitrary, so that the theorem follows to be true. □

Theorem 6.31 (Associative Law for the multiplication on $Y_{n\times n}$). *It is true for any positive natural number n and any commutative semiring $(Y,+,\cdot)$ with zero element 0_Y and unity element 1_Y that the multiplication $\cdot_{Y^{n\times n}}$ on $Y^{n\times n}$ is associative.*

Proof. Letting n be an arbitrary positive natural number and $(Y,+,\cdot)$ an arbitrary commutative semiring with zero element 0_Y and unity element 1_Y, we establish the associativity of $\cdot_{Y^{n\times n}}$ by proving the universal sentence

$$\forall \boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\,(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\in Y^{n\times n} \Rightarrow (\boldsymbol{A}\boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{C})), \tag{6.86}$$

letting $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C} \in Y^{n\times n}$ be arbitrary. As these matrices as well as their products are functions with common domain $\{1,\ldots,n\}\times\{1,\ldots,n\}$, we may apply the Equality Criterion for functions to establish the desired equation in (6.86), by proving

$$\forall z\,(z \in \{1,\ldots,n\} \times \{1,\ldots,n\} \Rightarrow [(\boldsymbol{AB})\boldsymbol{C}](z) = [\boldsymbol{A}(\boldsymbol{BC})](z)). \qquad (6.87)$$

We take an arbitrary $z \in \{1,\ldots,n\}\times\{1,\ldots,n\}$, so that there are particular indexes $i,j \in \{1,\ldots,n\}$ satisfying $z=(i,j)$, according to Exercise 3.4. Let us observe now the truth of the equations

$$\begin{aligned}
(\boldsymbol{AB})\boldsymbol{C} &= \left(\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{bmatrix}\right) \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{k=1}^{n}(a_{1,k}\cdot b_{k,1}) & \cdots & \sum\limits_{k=1}^{n}(a_{1,k}\cdot b_{k,n}) \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^{n}(a_{n,k}\cdot b_{k,1}) & \cdots & \sum\limits_{k=1}^{n}(a_{n,k}\cdot b_{k,n}) \end{bmatrix} \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{\ell=1}^{n}\left(\sum\limits_{k=1}^{n}(a_{1,k}\cdot b_{k,\ell})\cdot c_{\ell,1}\right) & \cdots & \sum\limits_{\ell=1}^{n}\left(\sum\limits_{k=1}^{n}(a_{1,k}\cdot b_{k,\ell})\cdot c_{\ell,n}\right) \\ \vdots & \ddots & \vdots \\ \sum\limits_{\ell=1}^{n}\left(\sum\limits_{k=1}^{n}(a_{n,k}\cdot b_{k,\ell})\cdot c_{\ell,1}\right) & \cdots & \sum\limits_{\ell=1}^{n}\left(\sum\limits_{k=1}^{n}(a_{n,k}\cdot b_{k,\ell})\cdot c_{\ell,n}\right) \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{BC}) &= \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \left(\begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{bmatrix}\right) \\
&= \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} \sum\limits_{\ell=1}^{n}(b_{1,\ell}\cdot c_{\ell,1}) & \cdots & \sum\limits_{\ell=1}^{n}(b_{1,\ell}\cdot c_{\ell,n}) \\ \vdots & \ddots & \vdots \\ \sum\limits_{\ell=1}^{n}(b_{n,\ell}\cdot c_{\ell,1}) & \cdots & \sum\limits_{\ell=1}^{n}(b_{n,\ell}\cdot c_{\ell,n}) \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{k=1}^{n}\left(a_{1,k}\cdot\sum\limits_{\ell=1}^{n}(b_{k,\ell}\cdot c_{\ell,1})\right) & \cdots & \sum\limits_{k=1}^{n}\left(a_{1,k}\cdot\sum\limits_{\ell=1}^{n}(b_{k,\ell}\cdot c_{\ell,n})\right) \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^{n}\left(a_{n,k}\cdot\sum\limits_{\ell=1}^{n}(b_{k,\ell}\cdot c_{\ell,1})\right) & \cdots & \sum\limits_{k=1}^{n}\left(a_{n,k}\cdot\sum\limits_{\ell=1}^{n}(b_{k,\ell}\cdot c_{\ell,n})\right) \end{bmatrix}
\end{aligned}$$

in light of Proposition 6.28. We obtain then the true equations

$$\begin{aligned}
[(\boldsymbol{AB})\boldsymbol{C}](z) &= [(\boldsymbol{AB})\boldsymbol{C}]((i,j)) \\
&= \sum_{\ell=1}^{n}\left(\sum_{k=1}^{n}(a_{i,k}\cdot b_{k,\ell})\cdot c_{\ell,j}\right) \\
&= \sum_{\ell=1}^{n}\left(c_{\ell,j}\cdot\sum_{k=1}^{n}(a_{i,k}\cdot b_{k,\ell})\right) \\
&= \sum_{\ell=1}^{n}\sum_{k=1}^{n}(c_{\ell,j}\cdot(a_{i,k}\cdot b_{k,\ell})) \\
&= \sum_{\ell=1}^{n}\sum_{k=1}^{n}((a_{i,k}\cdot b_{k,\ell})\cdot c_{\ell,j}) \\
&= \sum_{\ell=1}^{n}\sum_{k=1}^{n}(a_{i,k}\cdot(b_{k,\ell}\cdot c_{\ell,j})) \\
&= \sum_{k=1}^{n}\sum_{\ell=1}^{n}(a_{i,k}\cdot(b_{k,\ell}\cdot c_{\ell,j})) \\
&= \sum_{k=1}^{n}\left(a_{i,k}\cdot\sum_{\ell=1}^{n}(b_{k,\ell}\cdot c_{\ell,j})\right) \\
&= [\boldsymbol{A}(\boldsymbol{BC})]((i,j)) \\
&= [\boldsymbol{A}(\boldsymbol{BC})](z)
\end{aligned}$$

by applying substitution, the previous matrix representation for $(\boldsymbol{AB})\boldsymbol{C}$, the commutativity of the multiplication on Y, the Generalized Distributive Law for semirings, again the commutativity of the multiplication on Y, the associativity of the multiplication on Y, Interchange of two nested n-fold sums, again the Generalized Distributive Law for semirings, the previous matrix representation for $\boldsymbol{A}(\boldsymbol{BC})$, and finally again substitution. Since z was arbitrary, we may therefore conclude that the products $(\boldsymbol{AB})\boldsymbol{C}$ and $\boldsymbol{A}(\boldsymbol{BC})$ are indeed identical. As $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ were also arbitrary, we may further conclude that the multiplication $\cdot_{Y^{n\times n}}$ is associative. Finally, we may infer from this finding the truth of the stated theorem because n and $(Y,+,\cdot)$ were initially arbitrary, too. □

Note 6.9. The previous finding that double summations may be viewed as being applied to the entries of a given matrix can be easily verified here for the double sum $\sum_{k=1}^{n}\sum_{\ell=1}^{n}(a_{i,k}\cdot(b_{k,\ell}\cdot c_{\ell,j}))$, by specifying for arbitrary

$i, j \in \{1, \ldots, n\}$ the auxiliary matrix

$$\boldsymbol{H} : \{1, \ldots, n\} \times \{1, \ldots, n\} \to Y, \qquad (k, \ell) \mapsto \boldsymbol{H}_{k,\ell} = a_{i,k} \cdot (b_{k,\ell} \cdot c_{\ell,j}).$$

Corollary 6.32. *It is true for any positive natural number n and any commutative semiring $(Y, +, \cdot)$ with zero element 0_Y and unity element 1_Y that the ordered pair*

$$(Y^{n \times n}, \cdot_{Y^{n \times n}}) \tag{6.88}$$

constitutes a semigroup.

Theorem 6.33 (Distributive Law for $Y_{n \times n}$). *It is true for any positive natural number n and any commutative semiring $(Y, +, \cdot)$ with zero element 0_Y and unity element 1_Y that the multiplication $\cdot_{Y^{n \times n}}$ is distributive over the addition $+_{Y^{n \times n}}$.*

Exercise 6.12. Establish the Distributive Law for $Y_{n \times n}$.

(Hint: Show that $\cdot_{Y^{n \times n}}$ is both left- and right-distributive over $+_{Y^{n \times n}}$, using Theorem 5.113.)

Corollary 6.34. *It is true for any positive natural number n and any commutative ring $(Y, +, \cdot)$ with zero element 0_Y and unity element 1_Y that the ordered quadruple $(Y^{n \times n}, +_{Y^{n \times n}}, \cdot_{Y^{n \times n}}, -_{Y^{n \times n}})$ constitutes a ring.*

Definition 6.14 (Ring of n-by-n matrices). We call for any positive natural number n

$$(Y^{n \times n}, +_{Y^{n \times n}}, \cdot_{Y^{n \times n}}, -_{Y^{n \times n}}) \tag{6.89}$$

the *ring of n-by-n matrices*.

Definition 6.15 (Identity matrix). We say for any positive natural number n, any set Y, any addition $+_Y$ such that the zero element 0_Y exists, and any multiplication $\cdot_Y$ such that the unity element 1_Y exists that a matrix

$$\boldsymbol{I}_n \tag{6.90}$$

of order n is the *identity matrix* (*of order n*) iff all main diagonal entries are equal to 1_Y and all other entries equal to 0_Y, i.e., iff

$$\forall i\, (i \in \{1, \ldots, n\} \Rightarrow I_n((i, i)) = 1_Y), \tag{6.91}$$

$$\forall i, j\, ([i, j \in \{1, \ldots, n\} \wedge i \neq j] \Rightarrow I_n((i, j)) = 0_Y). \tag{6.92}$$

Notation 6.5. The identity matrix of order n can be written as

$$\boldsymbol{I}_n = \begin{bmatrix} 1_Y & \cdots & 0_Y \\ \vdots & \ddots & \vdots \\ 0_Y & \cdots & 1_Y \end{bmatrix}. \tag{6.93}$$

Whenever the order is immediately in a given situation, we will usually omit the subscript n in $\boldsymbol{I}_n$ and write shorter $\boldsymbol{I}$.

Note 6.10. By virtue of the definite property (6.92), every identity matrix is a diagonal matrix.

Proposition 6.35. *It is true for any positive natural number n, any ring $(Y, +_Y, \cdot_Y, -_Y)$ with zero element 0_Y and unity element 1_Y that the identity matrix of order n is the identity/unity element of $Y^{n\times n}$ with respect to the multiplication $\cdot_{Y^{n\times n}}$ of matrices.*

Proof. We take an arbitrary positive natural number n and an arbitrary ring $(Y, +_Y, \cdot_Y, -_Y)$ with zero element 0_Y and unity element 1_Y. To prove that $\boldsymbol{I} = \boldsymbol{I}_n$ is the unity element of $Y^{n\times n}$ with respect to $\cdot_{Y^{n\times n}}$, we demonstrate the truth of

$$\forall \boldsymbol{A}\,(\boldsymbol{A} \in Y^{n\times n} \Rightarrow [\boldsymbol{I}\boldsymbol{A} = \boldsymbol{A} \wedge \boldsymbol{A}\boldsymbol{I} = \boldsymbol{A}]) \tag{6.94}$$

(omitting the multiplication symbol $\cdot_{Y^{n\times n}}$ for the sake of brevity of expressions). For this purpose, we let $\boldsymbol{A} \in Y^{n\times n}$ be arbitrary, define the matrix product

$$\boldsymbol{A}' = \boldsymbol{I}\boldsymbol{A} \tag{6.95}$$

and apply the Equality Criterion for functions to prove $\boldsymbol{A}' = \boldsymbol{A}$. To do this, we establish the truth of

$$\forall z\,(z \in \{1,\ldots,n\} \times \{1,\ldots,n\} \Rightarrow \boldsymbol{A}'(z) = \boldsymbol{A}(z)), \tag{6.96}$$

letting $z \in \{1,\ldots,n\} \times \{1,\ldots,n\}$ be arbitrary. This assumption implies the existence of particular elements $i, j \in \{1,\ldots,n\}$ satisfying $(i,j) = z$ (according to Exercise 3.4). According to the multiplication of matrices, (6.95) implies

$$a'_{i,j} = \sum_{k=1}^{n} (e_{i,k} \cdot_Y a_{k,j}), \tag{6.97}$$

where we write $e_{i,k} = \boldsymbol{I}((i,k))$ and where $(e_{i,k} \cdot_Y a_{k,j} \,|\, k \in \{1,\ldots,n\})$ constitutes an n-tuple in Y. The ith term of that n-tuple can be re-expressed by

$$e_{i,i} \cdot_Y a_{i,j} = \boldsymbol{I}((i,i)) \cdot_Y a_{i,j} = 1_Y \cdot_Y a_{i,j} = a_{i,j}$$

by using (6.91) and the property of a unity element, resulting in the equation

$$e_{i,i} \cdot_Y a_{i,j} = a_{i,j} \tag{6.98}$$

We may now establish the universal sentence

$$\forall k\,(k \in \{1,\ldots,n\} \setminus \{i\} \Rightarrow e_{i,k} \cdot_Y a_{k,j} = 0_Y). \tag{6.99}$$

Letting k be arbitrary and assuming $k \in \{1, \ldots, n\} \setminus \{i\}$ to be true, we obtain $k \in \{1, \ldots, n\}$ and $k \neq i$ using the definition of a set difference and (2.169). Since $i \in \{1, \ldots, n\}$ also holds, these findings give us $[e_{i,k} =]\ I_n((i,j)) = 0_Y$ with (6.92), so that

$$e_{i,k} \cdot_Y a_{k,j} = 0_Y \cdot_Y a_{k,j} = 0_Y$$

follows to be true with the Cancellation Law for 0_X in rings. This proves the implication in (6.99), in which k is arbitrary, so that the universal sentence (6.99) follows to be true, too. In conjunction with (6.98), this demonstrates the truth of the existential sentence

$$\begin{aligned}&\exists I\, (I \in \{1, \ldots, n\} \wedge e_{i,I} \cdot_Y a_{I,j} = a_{i,j} \\ &\quad \wedge \forall k\, (k \in \{1, \ldots, n\} \setminus \{I\} \Rightarrow e_{I,k} \cdot_Y a_{k,j} = 0_Y)),\end{aligned}$$

which in turn implies that the equation

$$\sum_{i=1}^{n} (e_{i,k} \cdot_Y a_{k,j}) = a_{i,j}$$

is true, according to Exercise 5.48. Combining this equation with (6.97) yields then

$$[\boldsymbol{A}'(z) = \boldsymbol{A}'((i,j)) =] \quad a'_{i,j} = a_{i,j} \quad [= \boldsymbol{A}((i,j)) = \boldsymbol{A}(z)],$$

which proves the implication in (6.96). As z is arbitrary, we may therefore conclude that the universal sentence (6.96) holds as well, so that $\boldsymbol{A}' = \boldsymbol{A}$ is true indeed. In view of (6.95), we thus proved the first part of the conjunction in (6.94). The second part can be proven similarly. As $\boldsymbol{A}$ was arbitrary, we may now further conclude that the universal sentence (6.94) also holds, which means that $\boldsymbol{I}$ is the neutral element of $Y^{n \times n}$ with respect to the multiplication $\cdot_{Y^{n\times n}}$. Initially, n and $(Y, +_Y, \cdot_Y, -_Y)$ were arbitrary, so that the proposition finally follows to be true. □

Exercise 6.13. Complete the proof of Proposition 6.35.

Exercise 6.14. Prove that every identity matrix is symmetric.

(Hint: Establish $\boldsymbol{I}^T = \boldsymbol{I}$ by means of the Equality Criterion for functions and a proof by cases (as occurring in the definition of an identity matrix).)

Note 6.11 (Inversion Laws for matrices). Since every ring of n-by-n matrices with respect to a commutative ring $(Y, +, \cdot)$ with zero element 0_Y and unity element 1_Y has the identity element $\boldsymbol{I}$, we see in light of Exercise 6.1 that this identity matrix is identical with its inverse, that is,

$$\boldsymbol{I}^{-1} = \boldsymbol{I}. \tag{6.100}$$

Furthermore, the inverse (element) $\boldsymbol{A}^{-1}$ of a matrix $\boldsymbol{A} \in Y^{n\times n}$ by definition satisfies

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \;\wedge\; \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \tag{6.101}$$

if it exists. In this case, we obtain from the general inversion laws listed in Note 6.4 for any such matrix

$$(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}, \tag{6.102}$$

and for any matrices $\boldsymbol{A}, \boldsymbol{B} \in Y^{n\times n}$ for which the inverses $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$ exist also

$$\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1} \Rightarrow \boldsymbol{A} = \boldsymbol{B}, \tag{6.103}$$

$$(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}, \tag{6.104}$$

$$(\boldsymbol{A}\boldsymbol{B}^{-1})^{-1} = \boldsymbol{B}\boldsymbol{A}^{-1}, \tag{6.105}$$

$$(\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}. \tag{6.106}$$

Definition 6.16 (Orthogonal matrix). We say for any positive natural number n and any commutative ring $(Y, +, \cdot)$ with zero element 0_Y and unity element 1_Y that an n-by-n matrix $\boldsymbol{A}$ is *orthogonal* iff the transpose of $\boldsymbol{A}$ is identical with the inverse of $\boldsymbol{A}$, that is, iff

$$\boldsymbol{A}^T = \boldsymbol{A}^{-1}. \tag{6.107}$$

Note 6.12. As every identity matrix satisfies $\boldsymbol{I}^T = \boldsymbol{I} = \boldsymbol{I}^{-1}$ according to Exercise 6.14 and (6.100), we see that $\boldsymbol{I}$ is an orthogonal matrix.

Note 6.13. In view of (6.107) and (6.101), every orthogonal matrix $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I} \;\wedge\; \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}. \tag{6.108}$$

We will establish the geometric meaning of orthogonality within the context of vector spaces in Chapter 10.

Corollary 6.36. *For any positive natural number n and any commutative ring $(Y, +, \cdot)$ with zero element 0_Y and unity element 1_Y, it is true that the inverse of any orthogonal n-by-n matrix $\boldsymbol{A}$ is itself orthogonal.*

Proof. Letting n be an arbitrary element of $\mathbb{N}_+$, $(Y, +, \cdot)$ an arbitrary commutative ring with zero element 0_Y and unity element 1_Y, and $\boldsymbol{A}$ an arbitrary orthogonal n-by-n matrix, we obtain the true equations

$$(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^T)^T = \boldsymbol{A} = (\boldsymbol{A}^{-1})^{-1}$$

using the definition of an orthogonal matrix, the Double Transposition Law and (6.102). The resulting equation $(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^{-1})^{-1}$ shows that $\boldsymbol{A}^{-1}$ is an orthogonal matrix, by definition. Since n, $(Y, +, \cdot)$ and $\boldsymbol{A}$ were initially arbitrary, we conclude that the stated universal sentence holds. □

6.5. The Ring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}})$

Proposition 6.37. *The following sentences are true.*

a) There exists a unique set $\sim_d$ *such that*

$$\begin{aligned}\forall Q\,(Q \in {\sim_d} \Leftrightarrow [Q \in (\mathbb{N}\times\mathbb{N}) \times (\mathbb{N}\times\mathbb{N}) \\ \wedge\ \exists m, n, M, N\,(m +_{\mathbb{N}} N = M +_{\mathbb{N}} n\ \wedge\ ((m, n), (M, N)) = Q)]).\end{aligned} \tag{6.109}$$

b) This set $\sim_d$ *satisfies*

$$\begin{aligned}\forall Q\,(Q \in {\sim_d} \Leftrightarrow \\ \exists m, n, M, N\,(m +_{\mathbb{N}} N = M +_{\mathbb{N}} n\ \wedge\ ((m, n), (M, N)) = Q))\end{aligned} \tag{6.110}$$

and constitutes an equivalence relation on $\mathbb{N}\times\mathbb{N}$.

Proof. We note that the uniquely existential sentence a) holds due to the Axiom of Specification and the Equality Criterion for sets. Letting now Q be arbitrary, the assumption $Q \in {\sim_d}$ implies with (6.109) especially the existential sentence in (6.110), so that the implication '$\Rightarrow$' in (6.110) holds. Assuming conversely that existential sentence to be true, there are particular natural numbers, say $\bar{m}$, $\bar{n}$, $\bar{M}$ and $\bar{N}$, satisfying the equations $\bar{m} +_{\mathbb{N}} \bar{N} = \bar{M} +_{\mathbb{N}} \bar{n}$ and $((\bar{m}, \bar{n}), (\bar{M}, \bar{N})) = Q$. Thus, the ordered pairs $(\bar{m}, \bar{n})$ and $(\bar{M}, \bar{N})$ are evidently elements of the domain $\mathbb{N}\times\mathbb{N}$ of the addition $+_{\mathbb{N}}$, which fact implies with the definition of the Cartesian product of two sets that

$$[Q =]\quad (\bar{m}, \bar{n}) \times (\bar{M}, \bar{N}) \in (\mathbb{N}\times\mathbb{N}) \times (\mathbb{N}\times\mathbb{N})$$

is true. In conjunction with the assumed existential sentence, this finding further implies $Q \in {\sim_d}$ with (6.109), proving the implication '$\Leftarrow$'. Thus, the equivalence in (6.110) is true, and as Q was arbitrary, we may therefore conclude that $\sim_d$ satisfies indeed the universal sentence (6.110). Moreover, because $Q \in {\sim_d}$ implies $Q \in (\mathbb{N}\times\mathbb{N}) \times (\mathbb{N}\times\mathbb{N})$ for any Q, in view of (6.109), it follows with the definition of a subset that the inclusion

$${\sim_d}\ \subseteq\ (\mathbb{N}\times\mathbb{N}) \times (\mathbb{N}\times\mathbb{N}) \tag{6.111}$$

holds. According to Note 3.3, the set $\sim_d$ constitutes therefore a binary relation on $\mathbb{N}\times\mathbb{N}$.

We now verify that this binary relation is reflexive, i.e. that $\sim_d$ satisfies

$$\forall z\,(z \in \mathbb{N}\times\mathbb{N} \Rightarrow z \sim_d z). \tag{6.112}$$

To do this, we let $z \in \mathbb{N}\times\mathbb{N}$ be arbitrary, which implies (by definition of the Cartesian product of two sets) that there are particular constants $\bar{m} \in \mathbb{N}$ and $\bar{n} \in \mathbb{N}$ with $(\bar{m}, \bar{n}) = z$. Let us observe now the truth of the equations

$$\bar{m} +_{\mathbb{N}} \bar{n} = \bar{m} +_{\mathbb{N}} \bar{n} \,\wedge\, ((\bar{m}, \bar{n}), (\bar{m}, \bar{n})) = ((\bar{m}, \bar{n}), (\bar{m}, \bar{n})),$$

and then also the truth of the existential sentence

$$\exists m, n, M, N\, (m +_{\mathbb{N}} N = M +_{\mathbb{N}} n \,\wedge\, ((m, n), (M, N)) = ((\bar{m}, \bar{n}), (\bar{m}, \bar{n}))).$$

In view of (6.110), the consequence of the existential sentence is $((\bar{m}, \bar{n}), (\bar{m}, \bar{n})) \in {\sim_d}$, which we may write also in the form $(\bar{m}, \bar{n}) \sim_d (\bar{m}, \bar{n})$, because $\sim_d$ is a binary relation. Consequently, substitution based on the previously established equation $(\bar{m}, \bar{n}) = z$ yields $z \sim_d z$, as desired. As z was arbitrary, we may therefore conclude that (6.112) holds, which means that $\sim_d$ is indeed a reflexive binary relation.

Next, we demonstrate the symmetry of $\sim_d$, by verifying

$$\forall x, y\, (x, y \in \mathbb{N} \times \mathbb{N} \Rightarrow [x \sim_d y \Rightarrow y \sim_d x]). \tag{6.113}$$

Letting x and y be arbitrary in $\mathbb{N} \times \mathbb{N}$ and assuming $x \sim_d y$ to be true, we thus have $(x, y) \in {\sim_d}$, which implies with (6.110) that there are particular constants $\bar{m}$, $\bar{n}$, $\bar{M}$ and $\bar{N}$ such that $\bar{m} +_{\mathbb{N}} \bar{N} = \bar{M} +_{\mathbb{N}} \bar{n}$ and $((\bar{m}, \bar{n}), (\bar{M}, \bar{N})) = (x, y)$ hold. Here, we may write the former equation also as

$$\bar{M} +_{\mathbb{N}} \bar{n} = \bar{m} +_{\mathbb{N}} \bar{N}, \tag{6.114}$$

whereas the latter implies with the Equality Criterion for ordered pairs $(\bar{m}, \bar{n}) = x$ as well as $(\bar{M}, \bar{N}) = y$. Consequently, we may form the ordered pair

$$((\bar{M}, \bar{N}), (\bar{m}, \bar{n})) = (y, x),$$

which equation shows in connection with the equation (6.114) that there exist constants m, n, M and N for which $m +_{\mathbb{N}} N = M +_{\mathbb{N}} n$ and $((m, n), (M, N)) = (y, x)$ are satisfied. Consequently, the ordered pair (y, x) turns out to be an element of $\sim_d$ because of (6.110), and we may write this finding also as $y \sim_d x$, which proves the implication $x \sim_d y \Rightarrow y \sim_d x$. Since x and y were initially arbitrary, we may now conclude that (6.113) is true, so that the binary relation $\sim_d$ is symmetric, by definition.

It remains for us to establish the transitivity of $\sim_d$, i.e. the universal sentence

$$\forall x, y, z\, (x, y, z \in \mathbb{N} \times \mathbb{N} \Rightarrow [(x \sim_d y \,\wedge\, y \sim_d z) \Rightarrow x \sim_d z]). \tag{6.115}$$

We take arbitrary $x, y, z \in \mathbb{N} \times \mathbb{N}$, and we assume both $x \sim_d y$ and $y \sim_d z$ to be true, which we may write equivalently as $(x, y) \in {\sim_d}$ and $(y, z) \in {\sim_d}$.

Therefore, there are on the one hand particular constants $\bar{m}$, $\bar{n}$, $\bar{M}$ and $\bar{N}$ satisfying

$$\bar{m} +_{\mathbb{N}} \bar{N} = \bar{M} +_{\mathbb{N}} \bar{n} \wedge ((\bar{m}, \bar{n}), (\bar{M}, \bar{N})) = (x, y). \tag{6.116}$$

On the other hand, there are particular constants $\bar{a}$, $\bar{b}$, $\bar{A}$ and $\bar{B}$ satisfying

$$\bar{a} +_{\mathbb{N}} \bar{B} = \bar{A} +_{\mathbb{N}} \bar{b} \wedge ((\bar{a}, \bar{b}), (\bar{A}, \bar{B})) = (y, z). \tag{6.117}$$

The second equation in (6.116) and the second equation in (6.117) imply with the Equality Criterion for ordered pairs $(\bar{m}, \bar{n}) = x$, $(\bar{M}, \bar{N}) = y$, $(\bar{a}, \bar{b}) = y$ and $(\bar{A}, \bar{B}) = z$. Combining now the two equations for y yields $(\bar{M}, \bar{N}) = (\bar{a}, \bar{b})$, and this equation evidently further implies the truth of the equations $\bar{M} = \bar{a}$ and $\bar{N} = \bar{b}$. Therefore, we may apply substitutions to the first equation in (6.117) in order to get the new equation

$$\bar{M} +_{\mathbb{N}} \bar{B} = \bar{A} +_{\mathbb{N}} \bar{N}. \tag{6.118}$$

We obtain then also the true equations

$$\begin{aligned}
(\bar{m} +_{\mathbb{N}} \bar{N}) +_{\mathbb{N}} (\bar{A} +_{\mathbb{N}} \bar{B}) &= (\bar{M} +_{\mathbb{N}} \bar{n}) +_{\mathbb{N}} (\bar{A} +_{\mathbb{N}} \bar{B}) \\
&= (\bar{M} +_{\mathbb{N}} \bar{B}) +_{\mathbb{N}} (\bar{A} +_{\mathbb{N}} \bar{n}) \\
&= (\bar{A} +_{\mathbb{N}} \bar{N}) +_{\mathbb{N}} (\bar{A} +_{\mathbb{N}} \bar{n})
\end{aligned} \tag{6.119}$$

by applying substitution based on the first equation in (6.116), the associativity and commutativity of $+_{\mathbb{N}}$ (multiple times), and another substitution based on (6.118). We may apply the associativity and commutativity of $+_{\mathbb{N}}$ also directly to the left-hand side of the equation (6.119) and write

$$(\bar{m} +_{\mathbb{N}} \bar{N}) +_{\mathbb{N}} (\bar{A} +_{\mathbb{N}} \bar{B}) = (\bar{A} +_{\mathbb{N}} \bar{N}) +_{\mathbb{N}} (\bar{m} +_{\mathbb{N}} \bar{B}).$$

Combining this with the equation (6.119) gives us

$$(\bar{A} +_{\mathbb{N}} \bar{N}) +_{\mathbb{N}} (\bar{m} +_{\mathbb{N}} \bar{B}) = (\bar{A} +_{\mathbb{N}} \bar{N}) +_{\mathbb{N}} (\bar{A} +_{\mathbb{N}} \bar{n}),$$

which equation can be simplified by means of the Cancellation Law for $+_{\mathbb{N}}$, with the consequence that

$$\bar{m} +_{\mathbb{N}} \bar{B} = \bar{A} +_{\mathbb{N}} \bar{n}. \tag{6.120}$$

Clearly, the equation $((\bar{m}, \bar{n}), (\bar{A}, \bar{B})) = ((\bar{m}, \bar{n}), (\bar{A}, \bar{B}))$ is also true, so that there exist constants m, n, M and N which satisfy both $m +_{\mathbb{N}} N = M +_{\mathbb{N}} n$ and $((m, n), (M, N)) = ((\bar{m}, \bar{n}), (\bar{A}, \bar{B}))$. Consequently, $((\bar{m}, \bar{n}), (\bar{A}, \bar{B}))$ is evidently an element of $\sim_d$, so that substitutions give us the desired consequent $x \sim_d z$ of the second implication in (6.115). Since x, y and z were arbitrary, we may therefore conclude that the universal sentence (6.115) holds, which shows that $\sim_d$ is transitive.

We thus proved that $\sim_d$ is a reflexive, symmetric and transitive binary relation on $\mathbb{N} \times \mathbb{N}$, so that $\sim_d$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. □

Exercise 6.15. Show that the equivalence relation $\sim_d$ satisfies also

$$\forall m, n, M, N\,(\,(m,n) \sim_d (M,N) \;\Leftrightarrow\; m +_{\mathbb{N}} N = M +_{\mathbb{N}} n\,). \tag{6.121}$$

(Hint: Use (6.110) and the Equality Criterion for ordered pairs.)

Proposition 6.38. *It is not true that the ordered pairs* $(1,0)$ *and* $(0,0)$ *are equivalent with respect to* $\sim_d$*, that is,*

$$\neg(1,0) \sim_d (0,0). \tag{6.122}$$

Proof. We note the truth of $1 \neq 0$ in view of (4.165) and the truth of the equations $1 = 1 +_{\mathbb{N}} 0$ and $0 = 0 +_{\mathbb{N}} 0$ in light of the definition of the zero element, so that substitutions give us the negation

$$\neg 1 +_{\mathbb{N}} 0 = 0 +_{\mathbb{N}} 0.$$

This negation in turn implies the negation (6.122) with (6.121) and the Law of Contraposition. □

Proposition 6.39. *It is true that the ordered pairs* $(m +_{\mathbb{N}} n, m)$ *and* $(n,0)$ *are equivalent with respect to* $\sim_d$ *for any natural numbers* m *and* n*, that is,*

$$\forall m, n\,(\,m, n \in \mathbb{N} \Rightarrow (m +_{\mathbb{N}} n, m) \sim_d (n, 0)\,). \tag{6.123}$$

Proof. Letting $m, n \in \mathbb{N}$ be arbitrary, we observe the truth of the equation

$$(m +_{\mathbb{N}} n) +_{\mathbb{N}} 0 = n +_{\mathbb{N}} m$$

in light of the definition of the zero element and the Commutative Law for the addition on $\mathbb{N}$, so that (6.121) gives $(m +_{\mathbb{N}} n, m) \sim_d (n, 0)$, as desired. Here, m and n are arbitrary, so that the stated universal sentence follows to be true. □

Exercise 6.16. Show that the ordered pairs $(m, m +_{\mathbb{N}} n)$ and $(0, n)$ are equivalent with respect to $\sim_d$ for any natural numbers m and n, that is,

$$\forall m, n\,(\,m, n \in \mathbb{N} \Rightarrow (m, m +_{\mathbb{N}} n) \sim_d (0, n)\,). \tag{6.124}$$

(Hint: The proof is a slightly simpler version of the proof of Proposition 6.39.)

Definition 6.17 (Set of integers, integer). We call the quotient set

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim_d \tag{6.125}$$

the *set of integers* and every element of $\mathbb{Z}$ an *integer*.

Corollary 6.40. *The equivalence classes* $[(1,0)]_{\sim_d}$ *and* $[(0,0)]_{\sim_d}$ *constitute distinct integers, that is,*

$$[(1,0)]_{\sim_d}, [(0,0)]_{\sim_d} \in \mathbb{Z} \tag{6.126}$$

and the negation

$$[(1,0)]_{\sim_d} \neq [(0,0)]_{\sim_d} \tag{6.127}$$

are both true.

Proof. We begin with the observation that 1 and 0 are natural numbers, so that the ordered pairs $(1,0)$ and $(0,0)$ are elements of the Cartesian product $\mathbb{N} \times \mathbb{N}$. Since $\sim_d$ is an equivalence relation on that Cartesian product, the equivalence classes $[(1,0)]_{\sim_d}$ and $[(0,0)]_{\sim_d}$ are defined. Furthermore, $(1,0) \in \mathbb{N}\times\mathbb{N}$ and $(0,0) \in \mathbb{N}\times\mathbb{N}$ show that there exists an element x of $\mathbb{N}\times\mathbb{N}$ such that $[x]_{\sim_d} = [(1,0)]_{\sim_d}$, and that there is also an element y of $\mathbb{N}\times\mathbb{N}$ with $[y]_{\sim_d} = [(0,0)]_{\sim_d}$. Consequently, both $[(1,0)]_{\sim_d}$ and $[(0,0)]_{\sim_d}$ are elements of $\mathbb{N} \times \mathbb{N} / \sim_d$ by definition of a quotient set. Thus, (6.126) holds, and the negation (6.122) implies then the negation (6.126) with the Equality Criterion for equivalence classes and the Law of Contraposition. □

Corollary 6.41. *It is true that the equivalence classes* $[(m, m +_{\mathbb{N}} n)]_{\sim_d}$ *and* $[(0,n)]_{\sim_d}$ *constitute identical integers for any natural numbers* m *and* n*, that is,*

$$\forall m, n\, (\, m, n \in \mathbb{N} \Rightarrow [(m, m +_{\mathbb{N}} n)]_{\sim_d} = [(0,n)]_{\sim_d}\,). \tag{6.128}$$

Proof. We let $m, n \in \mathbb{N}$ be arbitrary and note that $m +_{\mathbb{N}} n$ is then also in $\mathbb{N}$, so that the ordered pairs $(m, m +_{\mathbb{N}} n)$ and $(0,n)$ are both elements of the Cartesian product $\mathbb{N}\times\mathbb{N}$. Thus, the equivalence classes $[(m, m +_{\mathbb{N}} n)]_{\sim_d}$ and $[(0,n)]_{\sim_d}$ with respect to the equivalence relation $\sim_d$ are defined, and they evidently constitute elements of $\mathbb{N}\times\mathbb{N}/\sim_d$ $[= \mathbb{Z}]$, by definition of a quotient set. As $(m, m +_{\mathbb{N}} n)$ and $(0,n)$ are equivalent due to (6.124), it follows with the Equality Criterion for equivalence classes that $[(m, m +_{\mathbb{N}} n)]_{\sim_d}$ and $[(0,n)]_{\sim_d}$ are identical. This finding proves the implication in (6.128), and because m and n were initially arbitrary, we may therefore conclude that the universal sentence (6.128) is true. □

Exercise 6.17. Show that $[(m +_{\mathbb{N}} n, m)]_{\sim_d}$ and $[(n, 0)]_{\sim_d}$ constitute identical integers for any natural numbers m and n, that is,

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [(m +_{\mathbb{N}} n, m)]_{\sim_d} = [(n, 0)]_{\sim_d}\,). \tag{6.129}$$

Corollary 6.42. *It is true that the equivalence classes* $[(m, m)]_{\sim_d}$ *and* $[(0, 0)]_{\sim_d}$ *are identical integers for any natural number* m*, that is,*

$$\forall m\,(m \in \mathbb{N} \Rightarrow [(m, m)]_{\sim_d} = [(0, 0)]_{\sim_d}\,). \tag{6.130}$$

Proof. We take an arbitrary natural number m and consider the natural number $n = 0$, for which Exercise 6.17 gives

$$[(m +_{\mathbb{N}} 0, m)]_{\sim_d} = [(0, 0)]_{\sim_d},$$

$[(m +_{\mathbb{N}} 0, m)]_{\sim_d} \in \mathbb{Z}$ and $[(0, 0)]_{\sim_d} \in \mathbb{Z}$. Since $m +_{\mathbb{N}} 0 = m$ is true by definition of the zero element, these equations imply $[(m, m)]_{\sim_d} = [(0, 0)]_{\sim_d}$, as desired. Because m was initially arbitrary, the stated universal sentence follows therefore to be true. □

Proposition 6.43. *Any integer different from* $[(0, 0)]_{\sim_d}$ *can be written as the equivalence class of an ordered pair which has one zero and some nonzero component, that is,*

$$\begin{aligned} &\forall m\,([m \in \mathbb{Z} \wedge m \neq [(0, 0)]_{\sim_d}] \\ &\quad \Rightarrow \exists n\,(n \in \mathbb{N} \wedge n \neq 0 \wedge [m = [(0, n)]_{\sim_d} \vee m = [(n, 0)]_{\sim_d}])). \end{aligned} \tag{6.131}$$

Proof. We let $m \in \mathbb{Z}$ be arbitrary such that $m \neq [(0, 0)]_{\sim_d}$ is true. Thus, m is in the quotient set $\mathbb{N} \times \mathbb{N} / \sim_d$, which means that there exists an element of $\mathbb{N} \times \mathbb{N}$, say $\bar{z}$, with $[\bar{z}]_{\sim_d} = m$. By definition of the Cartesian product of two sets, there are then also elements of $\mathbb{N}$, say $\bar{M}$ and $\bar{N}$, such that $(\bar{M}, \bar{N}) = \bar{z}$. Combining the previous two equations via substitution yields then $m = [(\bar{M}, \bar{N})]_{\sim_d}$. With this equation, the initial assumption $m \neq [(0, 0)]_{\sim_d}$ can be written also as $[(\bar{M}, \bar{N})]_{\sim_d} \neq [(0, 0)]_{\sim_d}$, which implies the negation $\neg(\bar{M}, \bar{N}) \sim_d (0, 0)$ with the Equality Criterion for equivalence classes and the Law of Contraposition. This negation in turn implies $\bar{M} +_{\mathbb{N}} 0 \neq 0 +_{\mathbb{N}} \bar{N}$ with (6.121) and the Law of Contraposition. Since the equations $\bar{M} +_{\mathbb{N}} 0 = \bar{M}$ and $0 +_{\mathbb{N}} \bar{N} = \bar{N}$ are true by definition of the zero element, the preceding inequality gives us now $\bar{M} \neq \bar{N}$ by means of substitutions. Because the standard linear ordering $<_{\mathbb{N}}$ of $\mathbb{N}$ is connex, the inequality implies the truth of the disjunction $\bar{M} <_{\mathbb{N}} \bar{N} \vee \bar{N} <_{\mathbb{N}} \bar{M}$, which we use in the following to prove the desired existential sentence in (6.131) by cases.

The first case $\bar{M} <_{\mathbb{N}} \bar{N}$ implies with (5.325) that there exists a constant, say $\bar{d}$, satisfying

$$\bar{d} \in \mathbb{N} \wedge \bar{d} \neq 0 \tag{6.132}$$

and the equation $\bar{M} +_{\mathbb{N}} \bar{d} = \bar{N}$. Consequently, substitutions and (6.128) give

$$m = [(\bar{M}, \bar{N})]_{\sim_d} = [(\bar{M}, \bar{M} +_{\mathbb{N}} \bar{d})]_{\sim_d} = [(0, \bar{d})]_{\sim_d},$$

and the disjunction

$$m = [(0, \bar{d})]_{\sim_d} \lor m = [(\bar{d}, 0)]_{\sim_d} \tag{6.133}$$

is then also true. The conjunction of (6.132) – (6.133) shows now that the existential sentence in (6.131) is indeed true.

Similarly, the second case $\bar{N} <_{\mathbb{N}} \bar{M}$ implies with (5.325) the existence of a particular $\bar{d}$ such that (6.132) and the equation $\bar{N} +_{\mathbb{N}} \bar{d} = \bar{M}$ holds. We therefore obtain via substitutions and by using now (6.129)

$$m = [(\bar{M}, \bar{N})]_{\sim_d} = [(\bar{N} +_{\mathbb{N}} \bar{d}, \bar{N})]_{\sim_d} = [(\bar{d}, 0)]_{\sim_d},$$

so that the disjunction (6.133) is again true. Thus, the existential sentence in (6.131) holds also in the second case, which finding completes the proof of the consequent in (6.131) by cases. Since m was initially arbitrary, we may now infer from this the truth of the proposition. □

Theorem 6.44 (Identification of $\mathbb{N}$ in $\mathbb{Z}$). *It is true that there exists a unique function $f_{\mathbb{N}}^{\mathbb{Z}}$ with domain $\mathbb{N}$ such that*

$$\forall n\, (n \in \mathbb{N} \Rightarrow f_{\mathbb{N}}^{\mathbb{Z}}(n) = [(n, 0)]_{\sim_d}), \tag{6.134}$$

and $f_{\mathbb{N}}^{\mathbb{Z}}$ is an injection from $\mathbb{N}$ to $\mathbb{Z}$.

Proof. We first apply Function definition by replacement and prove the universal sentence

$$\forall n\, (n \in \mathbb{N} \Rightarrow \exists! y\, (y = [(n, 0)]_{\sim_d})). \tag{6.135}$$

Letting n be an arbitrary natural number, the ordered pair formed by n and the natural number 0 constitutes then an element of the Cartesian product $\mathbb{N} \times \mathbb{N}$. Therefore, the equivalence class $[(n, 0)]_{\sim_d}$ of that ordered pair with respect to the equivalence relation $\sim_d$ on $\mathbb{N} \times \mathbb{N}$ is then a uniquely specified set. It then follows with (1.109) that the uniquely existential sentence $\exists! y\, (y = [(n, 0)]_{\sim_d})$ is true. Since n was arbitrary, we may now infer from this finding the truth of the universal sentence (6.135) and therefore the unique existence of a function $f_{\mathbb{N}}^{\mathbb{Z}}$ such that (6.134) holds.

Next, we prove that $\mathbb{Z}$ is a codomain of $f_{\mathbb{N}}^{\mathbb{Z}}$, i.e. that the range of $f_{\mathbb{N}}^{\mathbb{Z}}$ is included in the set of integers. For this purpose, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall y\, (y \in \operatorname{ran}(f_{\mathbb{N}}^{\mathbb{Z}}) \Rightarrow y \in \mathbb{Z}). \tag{6.136}$$

We take an arbitrary set y and assume $y \in \operatorname{ran}(f_{\mathbb{N}}^{\mathbb{Z}})$ to be true. Consequently, there exists a constant, say $\bar{n}$, such that $(\bar{n}, y) \in f_{\mathbb{N}}^{\mathbb{Z}}$ is true, according to the definition of a range. This finding shows on the one hand in light of the definition of a domain that $\bar{n} \in \mathbb{N}$ [$= \operatorname{dom}(f_{\mathbb{N}}^{\mathbb{Z}})$] holds, which implies with (6.134) $f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n}) = [(\bar{n}, 0)]_{\sim_d}$. On the other hand, we may write $(\bar{n}, y) \in f_{\mathbb{N}}^{\mathbb{Z}}$ in function notation as the equation $y = f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})$. Combining the previous two equations yields now $y = [(\bar{n}, 0)]_{\sim_d}$, which is evidently an equivalence class with respect to $\sim_d$. Therefore, y is an element of $\mathbb{N} \times \mathbb{N} / \sim_d$ by definition of a quotient set, so that $y \in \mathbb{Z}$ follows to be true by definition of the set of integers. Because y was arbitrary, we may therefore infer the truth of the universal sentence (6.136) and thus the truth of the inclusion $\operatorname{ran}(f_{\mathbb{N}}^{\mathbb{Z}}) \subseteq \mathbb{Z}$. Consequently, $\mathbb{Z}$ is a codomain of $f_{\mathbb{N}}^{\mathbb{Z}}$, by definition.

To establish the injectivity of $f_{\mathbb{N}}^{\mathbb{Z}} : \mathbb{N} \to \mathbb{Z}$, we verify

$$\forall n, n' \, ([n, n' \in \mathbb{N} \wedge f_{\mathbb{N}}^{\mathbb{Z}}(n) = f_{\mathbb{N}}^{\mathbb{Z}}(n')] \Rightarrow n = n'), \tag{6.137}$$

letting $n, n' \in \mathbb{N}$ be arbitrary and assuming $f_{\mathbb{N}}^{\mathbb{Z}}(n) = f_{\mathbb{N}}^{\mathbb{Z}}(n')$ to be true. In view of (6.134), we find $f_{\mathbb{N}}^{\mathbb{Z}}(n) = [(n, 0)]_{\sim_d}$ and $f_{\mathbb{N}}^{\mathbb{Z}}(n') = [(n', 0)]_{\sim_d}$, and we therefore obtain $[(n, 0)]_{\sim_d} = [(n', 0)]_{\sim_d}$ by means of substitution. This equation in turn implies $(n, 0) \sim_d (n', 0)$ with the Equality Criterion for equivalence classes, and this equivalence further implies the truth of the equation $n +_{\mathbb{N}} 0 = n' +_{\mathbb{N}} 0$ with the characterization of the equivalence relation $\sim_d$ in (6.121). Here, the definition of the zero element gives us the equations $n +_{\mathbb{N}} 0 = n$ and $n' +_{\mathbb{N}} 0 = n'$, so that $n = n'$ follows to be true via substitutions, as desired. Because n and n' were initially arbitrary, we may now conclude that $f_{\mathbb{N}}^{\mathbb{Z}}$ has the property (6.137) and constitutes thus an injection from $\mathbb{N}$ to $\mathbb{Z}$. □

Note 6.14. The injection $f_{\mathbb{N}}^{\mathbb{Z}} : \mathbb{N} \hookrightarrow \mathbb{Z}$ constitutes the bijection

$$f_{\mathbb{N}}^{\mathbb{Z}} : \mathbb{N} \rightleftarrows \operatorname{ran}(f_{\mathbb{N}}^{\mathbb{Z}})$$

in view of Corollary 3.204. Here the range is identical with the image of $\mathbb{N}$ under $f_{\mathbb{N}}^{\mathbb{Z}}$ according to Corollary 3.216, i.e.

$$\operatorname{ran}(f_{\mathbb{N}}^{\mathbb{Z}}) = f_{\mathbb{N}}^{\mathbb{Z}}[\mathbb{N}], \tag{6.138}$$

so that we can write the preceding bijection in the form

$$f_{\mathbb{N}}^{\mathbb{Z}} : \mathbb{N} \rightleftarrows f_{\mathbb{N}}^{\mathbb{Z}}[\mathbb{N}]. \tag{6.139}$$

Due to the Bijectivity of inverse functions, we have then also the (bijective) function

$$[f_{\mathbb{N}}^{\mathbb{Z}}]^{-1} : f_{\mathbb{N}}^{\mathbb{Z}}[\mathbb{N}] \rightleftarrows \mathbb{N} \tag{6.140}$$

at our disposal, allowing us to switch 'back and forth' between $\mathbb{N}$ and its image, in the sense that any natural number n can on the one hand be transformed into its representation $[(n,0)]_{\sim_d}$ as an integer and on the other hand be unambiguously recovered from its integer form.

Notation 6.6. We write for the image of $\mathbb{N}$ under $f_{\mathbb{N}}^{\mathbb{Z}}$

$$\mathbb{N}_{\mathbb{Z}} = f_{\mathbb{N}}^{\mathbb{Z}}[\mathbb{N}], \tag{6.141}$$

whose elements we call the *natural numbers in* $\mathbb{Z}$. With this, the bijection (6.139) reads

$$f_{\mathbb{N}}^{\mathbb{Z}} : \mathbb{N} \rightleftarrows \mathbb{N}_{\mathbb{Z}}. \tag{6.142}$$

We will especially in the following chapters follow the usual convention and drop the explicit reference to $\mathbb{Z}$ by writing

$$\mathbb{N} = \mathbb{N}_{\mathbb{Z}}, \tag{6.143}$$

overloading deliberately the symbol '$\mathbb{N}$'.

Exercise 6.18. Establish the inclusion

$$\mathbb{N}_{\mathbb{Z}} \subseteq \mathbb{Z}. \tag{6.144}$$

(Hint: Use Definition 2.2, (6.141), (6.138), Theorem 6.44 and (3.507).)

Our next task is to define the multiplication and addition for integers.

Proposition 6.45. *It is true that there exists a unique function* $\cdot_{\mathbb{Z}}$ *with domain* $\mathbb{Z} \times \mathbb{Z}$ *such that*

$$\begin{aligned}\forall x\,(x \in \mathbb{Z} \times \mathbb{Z} \Rightarrow [\cdot_{\mathbb{Z}}(x) \in \mathbb{Z} \,\wedge\, \exists E, F, m, n, M, N\,(x = (E,F) \,\wedge\, (m,n) \in E \\ \wedge\, (M,N) \in F \,\wedge\, (m \cdot M + n \cdot N, m \cdot N + n \cdot M) \in \cdot_{\mathbb{Z}}(x))]).\end{aligned} \tag{6.145}$$

This function $\cdot_{\mathbb{Z}}$ *is a binary operation on* $\mathbb{Z}$ *satisfying*

$$\begin{aligned}&\forall m, n, M, N\,(m, n, M, N \in \mathbb{N} \\ &\quad \Rightarrow [(m,n)]_{\sim_d} \cdot_{\mathbb{Z}} [(M,N)]_{\sim_d} = [(m \cdot M + n \cdot N, m \cdot N + n \cdot M)]_{\sim_d}).\end{aligned} \tag{6.146}$$

Proof. We apply Function definition by replacement and prove accordingly the universal sentence

$$\begin{aligned}\forall x\,(x \in \mathbb{Z} \times \mathbb{Z} \Rightarrow \exists! y\,(y \in \mathbb{Z} \,\wedge\, \exists E, F, m, n, M, N\,(x = (E,F) \,\wedge\, (m,n) \in E \\ \wedge\, (M,N) \in F \,\wedge\, (m \cdot M + n \cdot N, m \cdot N + n \cdot M) \in y))),\end{aligned} \tag{6.147}$$

letting $x \in \mathbb{Z} \times \mathbb{Z}$ be arbitrary. It then follows with the definition of the Cartesian product of two sets that there are particular elements $\bar{E}, \bar{F} \in \mathbb{Z}$ with

$$x = (\bar{E}, \bar{F}). \tag{6.148}$$

According to (6.125), we thus have $\bar{E}, \bar{F} \in \mathbb{N} \times \mathbb{N}/ \sim_d$. By definition of a quotient set, these findings imply now the existence of particular elements $\bar{e}, \bar{f} \in \mathbb{N} \times \mathbb{N}$ satisfying, respectively, $[\bar{e}]_{\sim_d} = \bar{E}$ and $[\bar{f}]_{\sim_d} = \bar{F}$. Here, we have $\bar{e} \in [\bar{e}]_{\sim_d}$ and $\bar{f} \in [\bar{f}]_{\sim_d}$ according to (3.183), i.e. that $\bar{e}$ and $\bar{f}$ are representatives of the equivalence classes $[\bar{e}]_{\sim_d}$ and $[\bar{f}]_{\sim_d}$, respectively. Since $\sim_d$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$, the representatives of the equivalence classes are elements of $\mathbb{N}\times\mathbb{N}$, so that $\bar{e} \in \mathbb{N}\times\mathbb{N}$ and $\bar{f} \in \mathbb{N}\times\mathbb{N}$ are true. Consequently, the definition of the Cartesian product of two sets gives us particular constants $\bar{m}, \bar{n} \in \mathbb{N}$ with $(\bar{m}, \bar{n}) = \bar{e}$ and particular constants $\bar{M}, \bar{N} \in \mathbb{N}$ with $(\bar{M}, \bar{N}) = \bar{f}$. These equations imply via substitutions first $(\bar{m}, \bar{n}) \in [\bar{e}]_{\sim_d}$ and $(\bar{M}, \bar{N}) \in [\bar{f}]_{\sim_d}$, then also

$$(\bar{m}, \bar{n}) \in \bar{E} \wedge (\bar{M}, \bar{N}) \in \bar{F}. \tag{6.149}$$

Furthermore, since $\bar{m}$, $\bar{n}$, $\bar{M}$ and $\bar{N}$ are natural numbers, the products $\bar{m} \cdot \bar{M}$, $\bar{n} \cdot \bar{N}$, $\bar{m} \cdot \bar{N}$ and $\bar{n} \cdot \bar{M}$, as well as the sums $\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}$ and $\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}$ are uniquely determined natural numbers, so that we may form the ordered pair $(\bar{m}\cdot\bar{M}+\bar{n}\cdot\bar{N}, \bar{m}\cdot\bar{N}+\bar{n}\cdot\bar{M})$, which is then evidently an element of the Cartesian product $\mathbb{N}\times\mathbb{N}$. Recalling that $\sim_d$ is an equivalence relation on that Cartesian product, the equivalence class

$$\bar{y} = [(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M})]_{\sim_d} \tag{6.150}$$

is therefore defined as well, and this equivalence class contains the ordered pair $(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M})$ because of (3.183), that is,

$$(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}) \in \bar{y}. \tag{6.151}$$

Since (6.150) shows that there exists an element z of $\mathbb{N} \times \mathbb{N}$ for which $[z]_{\sim_d} = \bar{y}$ holds, the equivalence class $\bar{y}$ turns out to be an element of $\mathbb{N} \times \mathbb{N} \sim_d$ $[= \mathbb{Z}]$ by definition of a quotient set, i.e.

$$\bar{y} \in \mathbb{Z}. \tag{6.152}$$

Because the findings (6.148) – (6.151) demonstrate the existence of constants E, F, m, n, M and N satisfying $x = (E, F)$, $(m, n) \in E$, $(M, N) \in F$ and $(m\cdot M+n\cdot N, m\cdot N+n\cdot M) \in \bar{y}$, it follows from the conjunction of this existential sentence and (6.152) that the existential part of the uniquely existential sentence in (6.147) holds.

To establish the uniqueness part, we take arbitrary y and y', assuming that $y, y' \in \mathbb{Z}$ holds, assuming that there exist particular constants $\bar{E}$, $\bar{F}$, $\bar{m}$, $\bar{n}$, $\bar{M}$ and $\bar{N}$ satisfying $x = (\bar{E}, \bar{F})$,

$$(\bar{m}, \bar{n}) \in \bar{E} \wedge (\bar{M}, \bar{N}) \in \bar{F} \tag{6.153}$$

as well as

$$(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}) \in y, \tag{6.154}$$

and assuming moreover that there are particular constants $\bar{E}'$, $\bar{F}'$, $\bar{m}'$, $\bar{n}'$, $\bar{M}'$ and $\bar{N}'$ satisfying $x = (\bar{E}', \bar{F}')$,

$$(\bar{m}', \bar{n}') \in \bar{E}' \wedge (\bar{M}', \bar{N}') \in \bar{F}' \tag{6.155}$$

as well as

$$(\bar{m}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N}', \bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}') \in y'. \tag{6.156}$$

To show that this implies $y = y'$, we observe first that we obtain from the two equations for x via substitution $(\bar{E}, \bar{F}) = (\bar{E}', \bar{F}')$, which equation in turn implies $\bar{E} = \bar{E}'$ and $\bar{F} = \bar{F}'$ with the Equality Criterion for ordered pairs. Further substitutions in (6.155) based on these two equations then yield

$$(\bar{m}', \bar{n}') \in \bar{E} \wedge (\bar{M}', \bar{N}') \in \bar{F}. \tag{6.157}$$

Recalling the truth of $[(\bar{E}, \bar{F}) =]\ x \in \mathbb{Z} \times \mathbb{Z}$, we evidently have $\bar{E}, \bar{F} \in \mathbb{Z}$ and thus $\bar{E}, \bar{F} \in \mathbb{N} \times \mathbb{N}/\sim_d$. According to the definition of a quotient set, there exist then elements, say $\bar{e}, \bar{f} \in \mathbb{N} \times \mathbb{N}$, with $[\bar{e}]_{\sim_d} = \bar{E}$ and $[\bar{f}]_{\sim_d} = \bar{F}$. We therefore obtain from (6.153) and (6.157) via substitutions

$$(\bar{m}, \bar{n}), (\bar{m}', \bar{n}') \in [\bar{e}]_{\sim_d},$$
$$(\bar{M}, \bar{N}), (\bar{M}', \bar{N}') \in [\bar{f}]_{\sim_d},$$

and consequently (by definition of an equivalence class) the equivalences

$$(\bar{m}, \bar{n}) \sim_d \bar{e} \wedge (\bar{m}', \bar{n}') \sim_d \bar{e},$$
$$(\bar{M}, \bar{N}) \sim_d \bar{f} \wedge (\bar{M}', \bar{N}') \sim_d \bar{f}.$$

Here, $(\bar{m}', \bar{n}') \sim_d \bar{e}$ and $(\bar{M}', \bar{N}') \sim_d \bar{f}$ imply, respectively, $\bar{e} \sim_d (\bar{m}', \bar{n}')$ and $\bar{f} \sim_d (\bar{M}', \bar{N}')$ with the symmetry of the equivalence relation $\sim_d$, so that we have the true conjunctions

$$(\bar{m}, \bar{n}) \sim_d \bar{e} \wedge \bar{e} \sim_d (\bar{m}', \bar{n}'),$$
$$(\bar{M}, \bar{N}) \sim_d \bar{f} \wedge \bar{f} \sim_d (\bar{M}', \bar{N}').$$

These imply then with the transitivity of $\sim_d$

$$(\bar{m}, \bar{n}) \sim_d (\bar{m}', \bar{n}'),$$
$$(\bar{M}, \bar{N}) \sim_d (\bar{M}', \bar{N}'),$$

which equivalences allow us to infer the truth of the equations

$$\bar{m} + \bar{n}' = \bar{m}' + \bar{n}, \tag{6.158}$$
$$\bar{M} + \bar{N}' = \bar{M}' + \bar{N} \tag{6.159}$$

by virtue of (6.121).

We note that the initial assumptions $y, y' \in \mathbb{Z}$ $[= \mathbb{N} \times \mathbb{N}/ \sim_d]$ imply by definition of a quotient set that y and y' are equivalence classes with respect to $\sim_d$, i.e. that there exist particular elements $\bar{u}, \bar{u}' \in \mathbb{N} \times \mathbb{N}$ with $[\bar{u}]_{\sim_d} = y$ and $[\bar{u}']_{\sim_d} = y'$. Thus, we may write (6.154) and (6.156) equivalently as

$$(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}) \in [\bar{u}]_{\sim_d},$$
$$(\bar{m}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N}', \bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}') \in [\bar{u}']_{\sim_d}.$$

By definition of an equivalence class, we thus have the equivalences

$$(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}) \sim_d \bar{u},$$
$$(\bar{m}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N}', \bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}') \sim_d \bar{u}',$$

so that the Equality Criterion for equivalence classes yields the equations

$$[(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M})]_{\sim_d} = [\bar{u}]_{\sim_d} \quad [= y], \tag{6.160}$$
$$[(\bar{m}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N}', \bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}')]_{\sim_d} = [\bar{u}']_{\sim_d} \quad [= y']. \tag{6.161}$$

We now establish

$$(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}) \sim_d (\bar{m}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N}', \bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}'), \tag{6.162}$$

(which will imply the desired $y = y'$ by means of the Equality Criterion for equivalence classes and substitution). To begin with, we obtain the equations

$$\begin{aligned} &(\bar{m} + \bar{n}') \cdot \bar{M} + (\bar{m}' + \bar{n}) \cdot \bar{N} + \bar{m}' \cdot (\bar{M} + \bar{N}') + \bar{n}' \cdot (\bar{M}' + \bar{N}) \\ &= (\bar{m}' + \bar{n}) \cdot \bar{M} + (\bar{m} + \bar{n}') \cdot \bar{N} + \bar{m}' \cdot (\bar{M}' + \bar{N}) + \bar{n}' \cdot (\bar{M} + \bar{N}') \end{aligned}$$

by applying substitutions based on (6.158) – (6.159) and the associativity of $+_{\mathbb{N}}$ (which allows us to omit brackets). Using now the distributivity of $\cdot_{\mathbb{N}}$ over $+_{\mathbb{N}}$, we can evidently rewrite the preceding equation in the form

$$\begin{aligned} &\bar{m} \cdot \bar{M} + \bar{n}' \cdot \bar{M} + \bar{m}' \cdot \bar{N} + \bar{n} \cdot \bar{N} + \bar{m}' \cdot \bar{M} + \bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N} \\ &= \bar{m}' \cdot \bar{M} + \bar{n} \cdot \bar{M} + \bar{m} \cdot \bar{N} + \bar{n}' \cdot \bar{N} + \bar{m}' \cdot \bar{M}' + \bar{m}' \cdot \bar{N} + \bar{n}' \cdot \bar{M} + \bar{n}' \cdot \bar{N}'. \end{aligned}$$

We then rearrange the summands by exploiting again the associativity and also the commutativity of $+_{\mathbb{N}}$, writing the preceding equation as

$$\begin{aligned}
&(\bar{n}' \cdot \bar{M} + \bar{m}' \cdot \bar{N} + \bar{m}' \cdot \bar{M} + \bar{n}' \cdot \bar{N}) \\
&\qquad + (\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}) + (\bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}') \\
= &(\bar{n}' \cdot \bar{M} + \bar{m}' \cdot \bar{N} + \bar{m}' \cdot \bar{M} + \bar{n}' \cdot \bar{N}) \\
&\qquad + (\bar{m}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N}') + (\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}),
\end{aligned}$$

and we simplify this equation by means of the Cancellation Law for $+_{\mathbb{N}}$ to

$$(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}) + (\bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}') = (\bar{m}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N}') + (\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}).$$

This equation implies now the desired equivalence (6.162) with (6.121), which in turn implies with the Equality Criterion for equivalence classes

$$[(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M})]_{\sim_d} = [(\bar{m}' \cdot \bar{M}' + \bar{n}' \cdot \bar{N}', \bar{m}' \cdot \bar{N}' + \bar{n}' \cdot \bar{M}')]_{\sim_d},$$

and then $y = y'$ through substitutions based on the equations in (6.160) – (6.161). Since y and y' were arbitrary, we may therefore conclude that the uniqueness part also holds, so that the proof of the uniquely existential sentence in (6.147) is complete. Here, x was also arbitrary, so that the universal sentence (6.147) follows to be also true. Consequently, there exists a unique function $\cdot_{\mathbb{Z}}$ with domain $\mathbb{Z} \times \mathbb{Z}$ such that (6.145).

Our next task within this proof is to demonstrate that $\mathbb{Z}$ is a codomain of $\cdot_{\mathbb{Z}}$, i.e. that the inclusion $\mathrm{ran}(\cdot_{\mathbb{Z}}) \subseteq \mathbb{Z}$ holds. For this purpose, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall y\, (y \in \mathrm{ran}(\cdot_{\mathbb{Z}}) \Rightarrow y \in \mathbb{Z}), \tag{6.163}$$

letting y be arbitrary and assuming that $y \in \mathrm{ran}(\cdot_{\mathbb{Z}})$ holds. According to the definition of a range, there exists then a constant, say $\bar{x}$, such that $(\bar{x}, y) \in \cdot_{\mathbb{Z}}$ is true. This finding shows in light of the definition of a domain that $\bar{x} \in \mathbb{Z} \times \mathbb{Z}$ $[= \mathrm{dom}(\cdot_{\mathbb{Z}})]$ holds, which implies with (6.145) especially $\cdot_{\mathbb{Z}}(\bar{x}) \in \mathbb{Z}$. Noting that we can write $(\bar{x}, y) \in \cdot_{\mathbb{Z}}$ in function notation as $y = \cdot_{\mathbb{Z}}(\bar{x})$, we therefore obtain $y \in \mathbb{Z}$ by means of substitution. We thus completed already the proof of the implication in (6.163), and since y is arbitrary, we may infer from the truth of that implication the truth of the universal sentence (6.163), and therefore the truth of the inclusion $\mathrm{ran}(\cdot_{\mathbb{Z}}) \subseteq \mathbb{Z}$. This shows that $\mathbb{Z}$ is a codomain of the function $\cdot_{\mathbb{Z}}$, which constitutes then a binary operation on $\mathbb{Z}$, recalling that $\mathbb{Z} \times \mathbb{Z}$ is the domain of $\cdot_{\mathbb{Z}}$.

We now prove that the binary operation $\cdot_{\mathbb{Z}}$ on $\mathbb{Z}$ satisfies (6.146). We let m, n, M and N be arbitrary natural numbers and note that $(m, n), (M, N) \in$

$\mathbb{N} \times \mathbb{N}$ is then evidently true, so that the equivalence classes $[(m,n)]_{\sim_d}$ and $[(M,N)]_{\sim_d}$ are clearly defined. Consequently, these equivalence classes are evidently elements of the quotient set $\mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim_d$, i.e. we have $[(m,n)]_{\sim_d} \in \mathbb{Z}$ and $[(M,N)]_{\sim_d} \in \mathbb{Z}$. Then $([(m,n)]_{\sim_d}, [(M,N)]_{\sim_d})$ is an element of the Cartesian product $\mathbb{Z} \times \mathbb{Z}$, so that we obtain, according to (6.145),

$$\cdot_{\mathbb{Z}}(([(m,n)]_{\sim_d}, [(M,N)]_{\sim_d})) \in \mathbb{Z} \quad [= \mathbb{N} \times \mathbb{N}/\sim_d] \tag{6.164}$$

and particular constants $\bar{E}$, $\bar{F}$, $\bar{m}$, $\bar{n}$, $\bar{M}$, $\bar{N}$ such that

$$([(m,n)]_{\sim_d}, [(M,N)]_{\sim_d}) = (\bar{E}, \bar{F}) \wedge (\bar{m}, \bar{n}) \in \bar{E} \wedge (\bar{M}, \bar{N}) \in \bar{F}$$

and

$$(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}) \in \cdot_{\mathbb{Z}}(([(m,n)]_{\sim_d}, [(M,N)]_{\sim_d})) \tag{6.165}$$

hold. The preceding equation implies with the Equality Criterion for ordered pairs $[(m,n)]_{\sim_d} = \bar{E}$ and $[(M,N)]_{\sim_d} = \bar{F}$. We therefore obtain by means of substitutions $(\bar{m}, \bar{n}) \in [(m,n)]_{\sim_d}$, $(\bar{M}, \bar{N}) \in [(M,N)]_{\sim_d}$. Consequently, we have $(\bar{m}, \bar{n}) \sim_d (m,n)$ and $(\bar{M}, \bar{N}) \sim_d (M,N)$, which equivalences evidently mean that the equations $\bar{m}+n = m+\bar{n}$ and $\bar{M}+N = M+\bar{N}$ are true. We obtain then also the equations

$$\begin{aligned} &(\bar{m}+n) \cdot \bar{M} + (m+\bar{n}) \cdot \bar{N} + m \cdot (\bar{M}+N) + n \cdot (M+\bar{N}) \\ &= (m+\bar{n}) \cdot \bar{M} + (\bar{m}+n) \cdot \bar{N} + m \cdot (M+\bar{N}) + n \cdot (\bar{M}+N), \end{aligned}$$

which we may simplify to

$$\begin{aligned} &\bar{m} \cdot \bar{M} + n \cdot \bar{M} + m \cdot \bar{N} + \bar{n} \cdot \bar{N} + m \cdot \bar{M} + m \cdot N + n \cdot M + n \cdot \bar{N} \\ &= m \cdot \bar{M} + \bar{n} \cdot \bar{M} + \bar{m} \cdot \bar{N} + n \cdot \bar{N} + m \cdot M + m \cdot \bar{N} + n \cdot \bar{M} + n \cdot N. \end{aligned}$$

Rearranging terms, we obtain

$$\begin{aligned} &(n \cdot \bar{M} + m \cdot \bar{N} + m \cdot \bar{M} + n \cdot \bar{N}) \\ &\quad + (\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}) + (m \cdot N + n \cdot M) \\ &= (n \cdot \bar{M} + m \cdot \bar{N} + m \cdot \bar{M} + n \cdot \bar{N}) \\ &\quad + (m \cdot M + n \cdot N) + (\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}), \end{aligned}$$

and we can simplify this equation to

$$(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}) + (m \cdot N + n \cdot M) = (m \cdot M + n \cdot N) + (\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}),$$

which shows that $(\bar{m}\cdot\bar{M}+\bar{n}\cdot\bar{N}, \bar{m}\cdot\bar{N}+\bar{n}\cdot\bar{M})$ and $(m\cdot M+n\cdot N, m\cdot N+n\cdot M)$ are equivalent. Therefore, the equivalence classes these elements represent are identical, i.e.

$$[(\bar{m}\cdot\bar{M}+\bar{n}\cdot\bar{N}, \bar{m}\cdot\bar{N}+\bar{n}\cdot\bar{M})]_{\sim_d} = [(m\cdot M+n\cdot N, m\cdot N+n\cdot M)]_{\sim_d}. \quad (6.166)$$

Next, we note that (6.164) implies by definition of a quotient set that $\cdot_{\mathbb{Z}}(([(m,n)]_{\sim_d}, [(M,N)]_{\sim_d}))$ is an equivalence class with respect to $\sim_d$, i.e. that there exists a particular element $\bar{u} \in \mathbb{N}\times\mathbb{N}$ with

$$[\bar{u}]_{\sim_q} = \cdot_{\mathbb{Z}}(([(m,n)]_{\sim_d}, [(M,N)]_{\sim_d})). \quad (6.167)$$

Thus, we may write (6.165) equivalently as

$$(\bar{m}\cdot\bar{M}+\bar{n}\cdot\bar{N}, \bar{m}\cdot\bar{N}+\bar{n}\cdot\bar{M}) \in [\bar{u}]_{\sim_d},$$

so that we have the equivalence

$$(\bar{m}\cdot\bar{M}+\bar{n}\cdot\bar{N}, \bar{m}\cdot\bar{N}+\bar{n}\cdot\bar{M}) \sim_d \bar{u},$$

and this implies that the corresponding equivalence classes are equal, i.e.

$$[(\bar{m}\cdot\bar{M}+\bar{n}\cdot\bar{N}, \bar{m}\cdot\bar{N}+\bar{n}\cdot\bar{M})]_{\sim_d} = [\bar{u}]_{\sim_d}. \quad (6.168)$$

Combining now (6.166) – (6.168) gives

$$\begin{aligned}
[(m\cdot M+n\cdot N, m\cdot N+n\cdot M)]_{\sim_d} &= [(\bar{m}\cdot\bar{M}+\bar{n}\cdot\bar{N}, \bar{m}\cdot\bar{N}+\bar{n}\cdot\bar{M})]_{\sim_d} \\
&= [\bar{u}]_{\sim_d} \\
&= \cdot_{\mathbb{Z}}(([(m,n)]_{\sim_d}, [(M,N)]_{\sim_d})) \\
&= [(m,n)]_{\sim_d} \cdot_{\mathbb{Z}} [(M,N)]_{\sim_d},
\end{aligned}$$

as desired. Since m, n, M and N are arbitrary, we may therefore conclude that (6.146) is satisfied. □

Definition 6.18 (Multiplication on the set of integers). We call

$$\cdot_{\mathbb{Z}} : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}, \quad (m,n) \mapsto m \cdot_{\mathbb{Z}} n \quad (6.169)$$

the *multiplication on the set of integers*.

Theorem 6.46 (Commutative Law for the multiplication on $\mathbb{Z}$). *It is true that the multiplication $\cdot_{\mathbb{Z}}$ on $\mathbb{Z}$ is commutative.*

Proof. To prove that $\cdot_{\mathbb{Z}}$ is commutative, we verify

$$\forall p, q\,(p, q \in \mathbb{Z} \Rightarrow p \cdot_{\mathbb{Z}} q = q \cdot_{\mathbb{Z}} p). \tag{6.170}$$

We take arbitrary sets p and q and assume $p, q \in \mathbb{Z}$ to be true. This means that p and q are equivalence classes $p = [\bar{e}]_{\sim_d}$ and $q = [\bar{f}]_{\sim_d}$ in the quotient set $\mathbb{N} \times \mathbb{N}/\sim_d]$ for some particular elements $\bar{e}, \bar{f} \in \mathbb{N} \times \mathbb{N}$. According to the definition of the Cartesian product of two sets, we therefore have $(\bar{m}, \bar{n}) = \bar{e}$ and $(\bar{M}, \bar{N}) = \bar{f}$ for some particular natural numbers $\bar{m}$, $\bar{n}$, $\bar{M}$ and $\bar{N}$. The preceding equations give us $p = [(\bar{m}, \bar{n})]_{\sim_d}$ and $q = [(\bar{M}, \bar{N})]_{\sim_d}$ by means of substitutions, and $\bar{m}, \bar{n}, \bar{M}, \bar{N} \in \mathbb{N}$ implies then because of (6.146)

$$\begin{aligned} p \cdot_{\mathbb{Z}} q &= [(\bar{m}, \bar{n})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{M}, \bar{N})]_{\sim_d} \\ &= [(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M})]_{\sim_d}. \end{aligned}$$

Here, we have $\bar{m} \cdot \bar{M} = \bar{M} \cdot \bar{m}$, $\bar{n} \cdot \bar{N}) = \bar{N} \cdot \bar{n}$, $\bar{m} \cdot \bar{N} = \bar{N} \cdot \bar{m}$ and $\bar{n} \cdot \bar{M} = \bar{M} \cdot \bar{n}$ due to the commutativity of $\cdot_{\mathbb{N}}$, so that we obtain by means of substitutions

$$p \cdot_{\mathbb{Z}} q = [(\bar{M} \cdot \bar{m} + \bar{N} \cdot \bar{n}, \bar{M} \cdot \bar{n} + \bar{N} \cdot \bar{m})]_{\sim_d}.$$

Let us observe now the truth of $\bar{M}, \bar{N}, \bar{m}, \bar{n} \in \mathbb{N}$, which gives with (6.146)

$$\begin{aligned} q \cdot_{\mathbb{Z}} p &= [(\bar{M}, \bar{N})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{m}, \bar{n})]_{\sim_d} \\ &= [(\bar{M} \cdot \bar{m} + \bar{N} \cdot \bar{n}, \bar{M} \cdot \bar{n} + \bar{N} \cdot \bar{m})]_{\sim_d}. \end{aligned}$$

Combining the previous equations yields then $p \cdot_{\mathbb{Z}} q = q \cdot_{\mathbb{Z}} p$, as desired. Since p and q are arbitrary, we may therefore conclude that the universal sentence (6.170) is true, which means that $\cdot_{\mathbb{Z}}$ is a commutative binary operation. □

Theorem 6.47 (Associative Law for the multiplication on $\mathbb{Z}$). *It is true that the multiplication $\cdot_{\mathbb{Z}}$ on $\mathbb{Z}$ is associative.*

Proof. We establish the associativity of $\cdot_{\mathbb{Z}}$ by demonstrating the truth of

$$\forall a, b, c\,(a, b, c \in \mathbb{Z} \Rightarrow (a \cdot_{\mathbb{Z}} b) \cdot_{\mathbb{Z}} c = a \cdot_{\mathbb{Z}} (b \cdot_{\mathbb{Z}} c)). \tag{6.171}$$

Letting a, b and c be arbitrary and assuming that $a, b, c \in \mathbb{Z}$ holds, so that we have $a = [\bar{e}]_{\sim_d}$, $b = [\bar{f}]_{\sim_d}$ and $c = [\bar{g}]_{\sim_d}$ for particular elements $\bar{e}, \bar{f}, \bar{g} \in \mathbb{N} \times \mathbb{N}$. These elements of the Cartesian product $\mathbb{N} \times \mathbb{N}$ satisfy $(\bar{m}, \bar{n}) = \bar{e}$, $(\bar{M}, \bar{N}) = \bar{f}$ and $(\bar{p}, \bar{q}) = \bar{g}$ for particular natural numbers $\bar{m}$, $\bar{n}$, $\bar{M}$, $\bar{N}$, $\bar{p}$ and $\bar{q}$. Then, substitutions give then rise to the equations $a = [(\bar{m}, \bar{n})]_{\sim_d}$, $b = [(\bar{M}, \bar{N})]_{\sim_d}$ and $c = [(\bar{p}, \bar{q})]_{\sim_d}$. On the one hand, the fact $\bar{m}, \bar{n}, \bar{M}, \bar{N} \in \mathbb{N}$ implies with (6.146)

$$\begin{aligned} a \cdot_{\mathbb{Z}} b &= [(\bar{m}, \bar{n})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{M}, \bar{N})]_{\sim_d} \\ &= [(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M})]_{\sim_d}. \end{aligned}$$

Since $\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}$, $\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}$, $\bar{p}$ and $\bar{q}$ are evidently all natural numbers, another application of (6.146) gives us with the distributivity of $\cdot_{\mathbb{N}}$ over $+_{\mathbb{N}}$ and with the associativity of $+_{\mathbb{N}}$ (which allows us to omit some of the brackets)

$$\begin{aligned}
(a \cdot_{\mathbb{Z}} b) \cdot_{\mathbb{Z}} c &= [(\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}, \bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{p}, \bar{q})]_{\sim_d} \\
&= [([\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}] \cdot \bar{p} + [\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}] \cdot \bar{q}, \\
&\qquad [\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}] \cdot \bar{q} + [\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}] \cdot \bar{p})]_{\sim_d} \\
&= [(\bar{m} \cdot \bar{M} \cdot \bar{p} + \bar{n} \cdot \bar{N} \cdot \bar{p} + \bar{m} \cdot \bar{N} \cdot \bar{q} + \bar{n} \cdot \bar{M} \cdot \bar{q}, \\
&\qquad \bar{m} \cdot \bar{M} \cdot \bar{q} + \bar{n} \cdot \bar{N} \cdot \bar{q} + \bar{m} \cdot \bar{N} \cdot \bar{p} + \bar{n} \cdot \bar{M} \cdot \bar{p})]_{\sim_d}
\end{aligned}$$

On the other hand, the truth of $\bar{M}, \bar{N}, \bar{p}, \bar{q} \in \mathbb{N}$ implies

$$\begin{aligned}
b \cdot_{\mathbb{Z}} c &= [(\bar{M}, \bar{N})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{p}, \bar{q})]_{\sim_d} \\
&= [(\bar{M} \cdot \bar{p} + \bar{N} \cdot \bar{q}, \bar{M} \cdot \bar{q} + \bar{N} \cdot \bar{p})]_{\sim_d},
\end{aligned}$$

and since $\bar{m}$, $\bar{n}$, $\bar{M} \cdot \bar{p} + \bar{N} \cdot \bar{q}$ and $\bar{M} \cdot \bar{q} + \bar{N} \cdot \bar{p}$ are elements of $\mathbb{N}$, we obtain then with (6.146), with the distributivity of $\cdot_{\mathbb{N}}$ over $+_{\mathbb{N}}$, with the associativity of $+_{\mathbb{N}}$ and with the commutativity of $+_{\mathbb{N}}$ the equations

$$\begin{aligned}
a \cdot_{\mathbb{Z}} (b \cdot_{\mathbb{Z}} c) &= [(\bar{m}, \bar{n})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{M} \cdot \bar{p} + \bar{N} \cdot \bar{q}, \bar{M} \cdot \bar{q} + \bar{N} \cdot \bar{p})]_{\sim_d} \\
&= [(\bar{m} \cdot [\bar{M} \cdot \bar{p} + \bar{N} \cdot \bar{q}] + \bar{n} \cdot [\bar{M} \cdot \bar{q} + \bar{N} \cdot \bar{p}], \\
&\qquad \bar{m} \cdot [\bar{M} \cdot \bar{q} + \bar{N} \cdot \bar{p}] + \bar{n} \cdot [\bar{M} \cdot \bar{p} + \bar{N} \cdot \bar{q}])]_{\sim_d} \\
&= [(\bar{m} \cdot \bar{M} \cdot \bar{p} + \bar{m} \cdot \bar{N} \cdot \bar{q} + \bar{n} \cdot \bar{M} \cdot \bar{q} + \bar{n} \cdot \bar{N} \cdot \bar{p}, \\
&\qquad \bar{m} \cdot \bar{M} \cdot \bar{q} + \bar{m} \cdot \bar{N} \cdot \bar{p} + \bar{n} \cdot \bar{M} \cdot \bar{p} + \bar{n} \cdot \bar{N} \cdot \bar{q})]_{\sim_d} \\
&= [(\bar{m} \cdot \bar{M} \cdot \bar{p} + \bar{n} \cdot \bar{N} \cdot \bar{p} + \bar{m} \cdot \bar{N} \cdot \bar{q} + \bar{n} \cdot \bar{M} \cdot \bar{q}, \\
&\qquad \bar{m} \cdot \bar{M} \cdot \bar{q} + \bar{n} \cdot \bar{N} \cdot \bar{q} + \bar{m} \cdot \bar{N} \cdot \bar{p} + \bar{n} \cdot \bar{M} \cdot \bar{p})]_{\sim_d} \\
&= (a \cdot_{\mathbb{Z}} b) \cdot_{\mathbb{Z}} c
\end{aligned}$$

We thus proved the implication in (6.171), where a, b and c were arbitrary, so that the universal sentence (6.171) follows to be true. This finding shows that $\cdot_{\mathbb{Z}}$ is an associative binary operation. □

Note 6.15. The ordered pair $(\mathbb{Z}, \cdot_{\mathbb{Z}})$ constitutes a commutative semigroup due to the Commutative and the Associative Law for $\cdot_{\mathbb{Z}}$.

Proposition 6.48. *It is true that* $[(1,0)]_{\sim_d}$ *is the neutral element of* $\mathbb{Z}$ *with respect to the multiplication* $\cdot_{\mathbb{Z}}$ *on* $\mathbb{Z}$.

Proof. Recalling from (6.126) that $[(1,0)]_{\sim_d}$ is an element of $\mathbb{Z}$, we verify

$$\forall p\, (p \in \mathbb{Z} \Rightarrow [[(1,0)]_{\sim_d} \cdot_{\mathbb{Z}} p = p \,\wedge\, p \cdot_{\mathbb{Z}} [(1,0)]_{\sim_d} = p]), \tag{6.172}$$

letting $p \in \mathbb{Z}$ $[= \mathbb{N} \times \mathbb{N}/ \sim_d]$ be arbitrary. Thus, p is an equivalence class in that quotient set, that is, there exists an element of $\mathbb{N} \times \mathbb{N}$, say $\bar{z}$, with $[\bar{z}]_{\sim_d} = p$. By definition of the Cartesian product of two sets, there are then also natural numbers, say $\bar{m}$ and $\bar{n}$, with $(\bar{m}, \bar{n}) = \bar{z}$, so that substitution yields $p = [(\bar{m}, \bar{n})]_{\sim_d}$. We have $1, 0, \bar{m}, \bar{n} \in \mathbb{N}$, which implies

$$\begin{aligned}
[(1,0)]_{\sim_d} \cdot_{\mathbb{Z}} p &= [(1,0)]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{m}, \bar{n})]_{\sim_d} \\
&= [(1 \cdot \bar{m} + 0 \cdot \bar{n}, 1 \cdot \bar{n} + 0 \cdot \bar{m})]_{\sim_d} \\
&= [(\bar{m}, \bar{n})]_{\sim_d} \\
&= p
\end{aligned}$$

with (6.146), the Cancellation Law for 0 and the definitions of the unity and of the zero element (with respect to $\mathbb{N}$). Due to the commutativity of $\cdot_{\mathbb{Z}}$, we obtain also

$$\begin{aligned}
p \cdot_{\mathbb{Z}} [(1,0)]_{\sim_d} &= [(1,0)]_{\sim_d} \cdot_{\mathbb{Z}} p \\
&= p.
\end{aligned}$$

The previously established equations show that the implication in (6.172) is true, and since p was arbitrary, we may therefore conclude that the universal sentence (6.172) holds, so that $[(1,0)]_{\sim_d}$ is by definition the unity element of $\mathbb{Z}$. □

Lemma 6.49. *It is true that the restriction of the multiplication on $\mathbb{Z}$ to $\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}$ constitutes a binary operation on $\mathbb{N}_{\mathbb{Z}}$, that is,*

$$\cdot_{\mathbb{Z}} \restriction (\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}) : \mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}} \to \mathbb{N}_{\mathbb{Z}}. \tag{6.173}$$

Proof. Since the inclusion $\mathbb{N}_{\mathbb{Z}} \subseteq \mathbb{Z}$ is true in view of (6.144), we obtain the inclusion

$$\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}} \subseteq \mathbb{Z} \times \mathbb{Z} \tag{6.174}$$

by means of the Idempotent Law for the conjunction and (3.40). Because the multiplication on $\mathbb{Z}$ is a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, we obtain for its restriction to $\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}$

$$\cdot_{\mathbb{Z}} \restriction (\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}) : \mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}} \to \mathbb{Z}. \tag{6.175}$$

because of (3.566). It remains for us to demonstrate that $\mathbb{N}_{\mathbb{Z}}$ is also a codomain of the restriction, i.e. that the range of the restriction is included in $\mathbb{N}_{\mathbb{Z}}$. For this purpose, we apply the definition of a subset and verify accordingly

$$\forall m \, (m \in \mathrm{ran}(\cdot_{\mathbb{Z}} \restriction (\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}})) \Rightarrow m \in \mathbb{N}_{\mathbb{Z}}). \tag{6.176}$$

Letting m be arbitrary and assuming the antecedent to be true, we obtain by definition of a range a particular constant $\bar{z}$ such that

$$(\bar{z}, m) \in \cdot_{\mathbb{Z}} \upharpoonright (\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}). \tag{6.177}$$

As the restricted multiplication is a function with domain $\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}$, we have (by definition) that $\bar{z}$ is an element of that Cartesian product, so that there exist (by definition) particular elements $a \in \mathbb{N}_{\mathbb{Z}}$ and $b \in \mathbb{N}_{\mathbb{Z}}$ with $(a, b) = \bar{z}$. We may therefore apply substitution to rewrite (6.177) as

$$((a, b), m) \in \cdot_{\mathbb{Z}} \upharpoonright (\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}),$$

which implies with the definition of a restriction $((a, b), m) \in \cdot_{\mathbb{Z}}$. Using now the notation for binary operations, we may write this also as

$$m = a \cdot_{\mathbb{Z}} b. \tag{6.178}$$

Let us observe next that $a \in \mathbb{N}_{\mathbb{Z}}$ and $b \in \mathbb{N}_{\mathbb{Z}}$ give the values $(f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a) \in \mathbb{N}$ and $(f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b) \in \mathbb{N}$ under the inverse of the bijection $f_{\mathbb{N}}^{\mathbb{Z}} : \mathbb{N} \rightleftarrows \mathbb{N}_{\mathbb{Z}}$ in (6.142). Evaluating now these natural numbers with that bijection, we obtain

$$f_{\mathbb{N}}^{\mathbb{Z}}((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a)) = [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a), 0)]_{\sim_d},$$
$$f_{\mathbb{N}}^{\mathbb{Z}}((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b)) = [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b), 0)]_{\sim_d},$$

where the left-hand sides can be simplified to

$$f_{\mathbb{N}}^{\mathbb{Z}}((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a)) = (f_{\mathbb{N}}^{\mathbb{Z}} \circ (f_{\mathbb{N}}^{\mathbb{Z}})^{-1})(a) = \mathrm{id}_{\mathbb{N}_{\mathbb{Z}}}(a) = a,$$
$$f_{\mathbb{N}}^{\mathbb{Z}}((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b)) = (f_{\mathbb{N}}^{\mathbb{Z}} \circ (f_{\mathbb{N}}^{\mathbb{Z}})^{-1})(b) = \mathrm{id}_{\mathbb{N}_{\mathbb{Z}}}(a) = b$$

by applying the notation for function compositions, (3.680) and the definition of the identity function. Combining the previous equations yields $a = [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a), 0)]_{\sim_d}$ as well as $b = [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b), 0)]_{\sim_d}$, so that the product (6.178) becomes

$$\begin{aligned} m &= [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a), 0)]_{\sim_d} \cdot_{\mathbb{Z}} [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b), 0)]_{\sim_d} \\ &= [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a) \cdot (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b) + 0 \cdot 0, (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a) \cdot 0 + 0 \cdot (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b))]_{\sim_d} \\ &= [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a) \cdot (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b) + 0, 0 + 0)]_{\sim_d} \\ &= [((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a) \cdot (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b), 0)]_{\sim_d} \\ &= f_{\mathbb{N}}^{\mathbb{Z}}((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a) \cdot (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b)) \end{aligned}$$

by means of substitutions, (6.146), the Cancellation Law for 0, the definition of the zero element and (6.134). We may write the last equation also in the form

$$((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(a) \cdot (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(b), m) \in f_{\mathbb{N}}^{\mathbb{Z}},$$

which shows in light of the definition of a range that $m \in \mathbb{N}_{\mathbb{Z}}$ [$= \mathrm{ran}(f_{\mathbb{N}}^{\mathbb{Z}})$] is true, which is the desired consequent of the implication in (6.176). As m was arbitrary, we may infer from the truth of that implication the truth of the universal sentence (6.176) and therefore the truth of the inclusion the restricted multiplication's range in $\mathbb{N}_{\mathbb{Z}}$. We thus completed the proof of (6.173). □

Notation 6.7. We denote the restricted binary operation (6.173) also by

$$\cdot_{\mathbb{N}_{\mathbb{Z}}} : \mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}} \to \mathbb{N}_{\mathbb{Z}}. \tag{6.179}$$

Corollary 6.50. *It is true that the product of two natural numbers in $\mathbb{Z}$ with respect to the multiplication on $\mathbb{N}_{\mathbb{Z}}$ is the same as the product of these numbers with respect to the multiplication on $\mathbb{Z}$, that is,*

$$\forall m, n\,(m, n \in \mathbb{N}_{\mathbb{Z}} \Rightarrow m \cdot_{\mathbb{N}_{\mathbb{Z}}} n = m \cdot_{\mathbb{Z}} n). \tag{6.180}$$

Proof. Letting m and n be arbitrary and assuming $m, n \in \mathbb{N}_{\mathbb{Z}}$ to be true, we note that $(m, n) \in \mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}$ holds by definition of the Cartesian product of two sets. We then obtain the true equations

$$\begin{aligned} m \cdot_{\mathbb{N}_{\mathbb{Z}}} n &= \cdot_{\mathbb{N}_{\mathbb{Z}}}((m, n)) \\ &= [\cdot_{\mathbb{Z}} \restriction (\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}})]((m, n)) \\ &= \cdot_{\mathbb{Z}}((m, n)) \\ &= m \cdot_{\mathbb{Z}} n \end{aligned}$$

by applying the notation for binary operations, Notation 6.7, then (3.567) in connection with the inclusion (6.174) and $(m, n) \in \mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}$, and finally again the notation for binary operations. Here, m and n are arbitrary, so that the corollary follows to be true. □

Theorem 6.51 (Isomorphism from $(\mathbb{N}, \cdot_{\mathbb{N}})$ to $(\mathbb{N}_{\mathbb{Z}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$). *It is true that $f_{\mathbb{N}}^{\mathbb{Z}}$ constitutes an isomorphism from $(\mathbb{N}, \cdot_{\mathbb{N}})$ to $(\mathbb{N}_{\mathbb{Z}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$, that is,*

$$f_{\mathbb{N}}^{\mathbb{Z}} : (\mathbb{N}, \cdot_{\mathbb{N}}) \rightleftarrows (\mathbb{N}_{\mathbb{Z}}, \cdot_{\mathbb{N}_{\mathbb{Z}}}). \tag{6.181}$$

Proof. To prove the assertion, we establish

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow f_{\mathbb{N}}^{\mathbb{Z}}(m \cdot_{\mathbb{N}} n) = f_{\mathbb{N}}^{\mathbb{Z}}(m) \cdot_{\mathbb{N}_{\mathbb{Z}}} f_{\mathbb{N}}^{\mathbb{Z}}(n)), \tag{6.182}$$

letting m and n be arbitrary natural numbers. We note that the corresponding values $f_{\mathbb{N}}^{\mathbb{Z}}(m)$ and $f_{\mathbb{N}}^{\mathbb{Z}}(n)$ are elements of the range $\mathbb{N}_{\mathbb{Z}}$ of the bijection

$f_{\mathbb{N}}^{\mathbb{Z}} : \mathbb{N} \rightleftarrows \mathbb{N}_{\mathbb{Z}}$ established in (6.134) and (6.142). We obtain then

$$\begin{aligned} f_{\mathbb{N}}^{\mathbb{Z}}(m) \cdot_{\mathbb{N}_{\mathbb{Z}}} f_{\mathbb{N}}^{\mathbb{Z}}(n) &= f_{\mathbb{N}}^{\mathbb{Z}}(m) \cdot_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(n) \\ &= [(m,0)]_{\sim_d} \cdot_{\mathbb{Z}} [(n,0)]_{\sim_d} \\ &= [(m \cdot n + 0 \cdot 0, m \cdot 0 + 0 \cdot n)]_{\sim_d} \\ &= [(m \cdot n, 0)]_{\sim_d} \\ &= f_{\mathbb{N}}^{\mathbb{Z}}(m \cdot_{\mathbb{N}} n) \end{aligned}$$

by applying (6.180), then (6.134), (6.146), the Cancellation Law for 0 together with the definition of the zero element, and finally again (6.134). Since m and n are arbitrary, we may therefore conclude that (6.182) holds. Since $f_{\mathbb{N}}^{\mathbb{Z}} : \mathbb{N} \rightleftarrows \mathbb{N}_{\mathbb{Z}}$ is a bijection, it is an isomorphism from $(\mathbb{N}, \cdot_{\mathbb{N}})$ to $(\mathbb{Z}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$, by definition. □

Corollary 6.52. *It is true that*

$$\forall m, n \, (m, n \in \mathbb{N} \Rightarrow f_{\mathbb{N}}^{\mathbb{Z}}(m \cdot_{\mathbb{N}} n) = f_{\mathbb{N}}^{\mathbb{Z}}(m) \cdot_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(n)). \tag{6.183}$$

Proof. Letting $m, n \in \mathbb{N}$ be arbitrary, we find $f_{\mathbb{N}}^{\mathbb{Z}}(m), f_{\mathbb{N}}^{\mathbb{Z}}(n) \in \mathbb{Z}$ since $f_{\mathbb{N}}^{\mathbb{Z}}$ is a function from $\mathbb{N}$ to $\mathbb{Z}$ (according to the Identification of $\mathbb{N}$ in $\mathbb{Z}$). We therefore obtain the equations

$$\begin{aligned} f_{\mathbb{N}}^{\mathbb{Z}}(m \cdot_{\mathbb{N}} n) &= f_{\mathbb{N}}^{\mathbb{Z}}(m) \cdot_{\mathbb{N}_{\mathbb{Z}}} f_{\mathbb{N}}^{\mathbb{Z}}(n) \\ &= f_{\mathbb{N}}^{\mathbb{Z}}(m) \cdot_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(n) \end{aligned}$$

with (6.182) and (6.180). As m and n were arbitrary, we may therefore conclude that the universal sentence (6.183) holds. □

Exercise 6.19. Show that there exists a unique function $+_{\mathbb{Z}}$ with domain $\mathbb{Z} \times \mathbb{Z}$ such that

$$\begin{aligned} &\forall x \, (x \in \mathbb{Z} \times \mathbb{Z} \Rightarrow [+_{\mathbb{Z}}(x) \in \mathbb{Z} \wedge \exists E, F, m, n, M, N \, (x = (E, F) \\ &\quad \wedge \, (m, n) \in E \wedge (M, N) \in F \wedge (m +_{\mathbb{N}} M, n +_{\mathbb{N}} N) \in +_{\mathbb{Z}}(x))]), \end{aligned} \tag{6.184}$$

and demonstrate that this function $+_{\mathbb{Z}}$ is a binary operation on $\mathbb{Z}$ satisfying

$$\begin{aligned} &\forall m, n, M, N \, (m, n, M, N \in \mathbb{N} \\ &\quad \Rightarrow [(m,n)]_{\sim_d} +_{\mathbb{Z}} [(M,N)]_{\sim_d} = [(m +_{\mathbb{N}} M, n +_{\mathbb{N}} N)]_{\sim_d}). \end{aligned} \tag{6.185}$$

(Hint: Proceed similarly as in the proof of Proposition 6.45, establishing now instead of (6.162)

$$(\bar{m} + \bar{M}, \bar{n} + \bar{N}) \sim_d (\bar{m}' + \bar{M}', \bar{n}' + \bar{N}'),$$

directly by means of (6.121).)

Definition 6.19 (Addition on the set of integers). We call

$$+_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m +_{\mathbb{Z}} n \tag{6.186}$$

the *addition on the set of integers.*

Theorem 6.53 (Commutative Law for the addition on $\mathbb{Z}$). *It is true that the addition $+_{\mathbb{Z}}$ on $\mathbb{Z}$ is commutative.*

Exercise 6.20. Prove the Commutative Law for the addition on $\mathbb{Z}$.

(Hint: Proceed similarly as in the proof of the Commutative Law for the multiplication on $\mathbb{Z}$.)

Theorem 6.54 (Associative Law for the addition on $\mathbb{Z}$). *It is true that the addition $+_{\mathbb{Z}}$ on $\mathbb{Z}$ is associative.*

Exercise 6.21. Prove the Associative Law for the addition on $\mathbb{Z}$.

(Hint: The proof is a simpler version of the verification of the Associative Law for the multiplication on $\mathbb{Z}$.)

Note 6.16. The Commutative and the Associative Law for $+_{\mathbb{Z}}$ show that the ordered pair $(\mathbb{Z}, +_{\mathbb{Z}})$ constitutes a commutative semigroup.

Exercise 6.22. Show that $[(0,0)]_{\sim_d}$ is the neutral element of $\mathbb{Z}$ with respect to the addition $+_{\mathbb{Z}}$ on $\mathbb{Z}$, i.e. that $[(0,0)]_{\sim_d}$ is an element of $\mathbb{Z}$ which satisfies

$$\forall p\, (p \in \mathbb{Z} \Rightarrow [[(0,0)]_{\sim_d} +_{\mathbb{Z}} p = p \wedge p +_{\mathbb{Z}} [(0,0)]_{\sim_d} = p]). \tag{6.187}$$

Lemma 6.55. *It is true that the restriction of the addition on $\mathbb{Z}$ to $\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}$ constitutes a binary operation on $\mathbb{N}_{\mathbb{Z}}$, that is,*

$$+_{\mathbb{Z}} \upharpoonright (\mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}}) : \mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}} \to \mathbb{N}_{\mathbb{Z}}. \tag{6.188}$$

Exercise 6.23. Prove Lemma 6.55.

(Hint: Proceed similarly as in the proof of Lemma 6.49.)

Notation 6.8. We denote the restricted binary operation (6.188) also by

$$+_{\mathbb{N}_{\mathbb{Z}}} : \mathbb{N}_{\mathbb{Z}} \times \mathbb{N}_{\mathbb{Z}} \to \mathbb{N}_{\mathbb{Z}}. \tag{6.189}$$

Exercise 6.24. Verify that the sum of two natural numbers in $\mathbb{Z}$ with respect to the addition on $\mathbb{N}_{\mathbb{Z}}$ is the same as the sum of these numbers with respect to the addition on $\mathbb{Z}$, that is,

$$\forall m, n\, (m, n \in \mathbb{N}_{\mathbb{Z}} \Rightarrow m +_{\mathbb{N}_{\mathbb{Z}}} n = m +_{\mathbb{Z}} n). \tag{6.190}$$

(Hint: Recall Corollary 6.50.)

Theorem 6.56 (Isomorphism from $(\mathbb{N}, +_\mathbb{N})$ to $(\mathbb{N}_\mathbb{Z}, +_{\mathbb{N}_\mathbb{Z}})$). *It is true that $f_\mathbb{N}^\mathbb{Z}$ constitutes an isomorphism from $(\mathbb{N}, +_\mathbb{N})$ to $(\mathbb{N}_\mathbb{Z}, +_{\mathbb{N}_\mathbb{Z}})$, that is,*

$$f_\mathbb{N}^\mathbb{Z} : (\mathbb{N}, +_\mathbb{N}) \rightleftarrows (\mathbb{N}_\mathbb{Z}, +_{\mathbb{N}_\mathbb{Z}}). \tag{6.191}$$

Exercise 6.25. Establish

a) the Isomorphism from $(\mathbb{N}, +_\mathbb{N})$ to $(\mathbb{N}_\mathbb{Z}, +_{\mathbb{N}_\mathbb{Z}})$.
(Hint: Demonstrate the truth of

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow f_\mathbb{N}^\mathbb{Z}(m +_\mathbb{N} n) = f_\mathbb{N}^\mathbb{Z}(m) +_{\mathbb{N}_\mathbb{Z}} f_\mathbb{N}^\mathbb{Z}(n)), \tag{6.192}$$

using similar arguments as in the proof of Theorem 6.51.)

b) the universal sentence

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow f_\mathbb{N}^\mathbb{Z}(m +_\mathbb{N} n) = f_\mathbb{N}^\mathbb{Z}(m) +_\mathbb{Z} f_\mathbb{N}^\mathbb{Z}(n)). \tag{6.193}$$

(Hint: Recall the proof of Corollary 6.52.)

Proposition 6.57. *The semigroup $(\mathbb{Z}, +_\mathbb{Z})$ constitutes a group.*

Proof. We already proved that the zero element of $\mathbb{Z}$ exists, which is given by $[(0,0)]_{\sim_d}$, as shown in Exercise 6.22. It therefore remains for us to verify that the additive inverse exists for every integer, i.e.

$$\forall p\, (p \in \mathbb{Z} \Rightarrow \exists -p\, (p +_\mathbb{Z} -p = [(0,0)]_{\sim_d} \wedge -p +_\mathbb{Z} p = [(0,0)]_{\sim_d})). \tag{6.194}$$

To do this, we take an arbitrary integer p and observe that p is thus in the quotient set $\mathbb{N} \times \mathbb{N} / \sim_d$, so that there is a particular element $\bar{z} \in \mathbb{N} \times \mathbb{N}$ with $[\bar{z}]_{\sim_d} = p$. As an element of a Cartesian product of two sets, $\bar{z}$ in turn can be written as the ordered pair $(\bar{m}, \bar{n})$ where $\bar{m}$ and $\bar{n}$ are particular natural numbers. Thus, we can write the preceding equation as $[(\bar{m}, \bar{n})]_{\sim_d} = p$ by applying substitution. Let us observe now that the ordered pair $(\bar{n}, \bar{m})$ is also an element of the Cartesian product $\mathbb{N} \times \mathbb{N}$, so that the equivalence class $[(\bar{n}, \bar{m})]_{\sim_d}$ is defined as well. We then obtain the true equations

$$\begin{aligned} p +_\mathbb{Z} [(\bar{n}, \bar{m})]_{\sim_d} &= [(\bar{m}, \bar{n})]_{\sim_d} +_\mathbb{Z} [(\bar{n}, \bar{m})]_{\sim_d} \\ &= [(\bar{m} +_\mathbb{N} \bar{n}, \bar{n} +_\mathbb{N} \bar{m})]_{\sim_d} \\ &= [(\bar{m} +_\mathbb{N} \bar{n}, \bar{m} +_\mathbb{N} \bar{n})]_{\sim_d} \\ &= [(0,0)]_{\sim_d} \end{aligned} \tag{6.195}$$

by applying substitution, (6.185), the Commutative Law for the addition on $\mathbb{N}$ and (6.130). Because the equation

$$p +_\mathbb{Z} [(\bar{n}, \bar{m})]_{\sim_d} = [(\bar{n}, \bar{m})]_{\sim_d} +_\mathbb{Z} p$$

is also true due to the Commutative Law for the addition on $\mathbb{Z}$, we can carry out another substitution to get the equation

$$[(\bar{n}, \bar{m})]_{\sim_d} +_{\mathbb{Z}} p = [(0,0)]_{\sim_d}. \tag{6.196}$$

Having found the particular integer $[(\bar{n}, \bar{m})]_{\sim_d}$ satisfying the two equations (6.195) and (6.196), we now see that the existential sentence in (6.194) is true. Since p was initially arbitrary, we may therefore conclude that $(\mathbb{Z}, +_{\mathbb{Z}})$ satisfies also Property 2 of a group. □

Exercise 6.26. Show that the negative of an integer is obtained by interchanging the two components within the equivalence class of the ordered pair formed by these components, that is,

$$\forall m, n\, (m, n \in \mathbb{N} \Rightarrow [(n, m)]_{\sim_d} = -[(m, n)]_{\sim_d}). \tag{6.197}$$

Proposition 6.58. *Any integer is a natural number in $\mathbb{Z}$ or its negative is a natural number in $\mathbb{Z}$, that is,*

$$\forall m\, (m \in \mathbb{Z} \Rightarrow [m \in \mathbb{N}_{\mathbb{Z}} \lor -m \in \mathbb{N}_{\mathbb{Z}}]). \tag{6.198}$$

Proof. We let m be an arbitrary integer and observe the truth of the disjunction

$$m = [(0,0)]_{\sim_d} \lor m \neq [(0,0)]_{\sim_d}$$

in light of the Law of the Excluded Middle, which we use now to prove the disjunction in (6.198) by cases.

In the first case $m = [(0,0)]_{\sim_d}$, we find because of $0 \in \mathbb{N}$ the equation $f_{\mathbb{N}}^{\mathbb{Z}}(0) = [(0,0)]_{\sim_d}$ by definition of the function $f_{\mathbb{N}}^{\mathbb{Z}}$ in (6.134). Combining the previous two equations yields $m = f_{\mathbb{N}}^{\mathbb{Z}}(0)$, which we may write also in the form $(0, m) \in f_{\mathbb{N}}$, and this yields $m \in \mathbb{N}_{\mathbb{Z}}$ with the definition of a range and the fact that $f_{\mathbb{N}}^{\mathbb{Z}}$ constitutes a bijection and thus a surjection from $\mathbb{N}$ to $\mathbb{N}_{\mathbb{Z}}$, as shown by (6.142). Then, the desired disjunction in (6.198) is also true.

In the second case $m \neq [(0,0)]_{\sim_d}$, there exists according to (6.131) a particular natural number $\bar{n} \neq 0$ which satisfies the disjunction

$$m = [(0, \bar{n})]_{\sim_d} \lor m = [(\bar{n}, 0)]_{\sim_d}.$$

We prove the disjunction in (6.198) by considering two corresponding subcases. In the first sub-case $m = [(0, \bar{n})]_{\sim_d}$, we obtain the equations

$$-m = -[(0, \bar{n})]_{\sim_d} = [(\bar{n}, 0)]_{\sim_d} = f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})$$

by applying substitution, (6.197) and the definition of the function $f_{\mathbb{N}}^{\mathbb{Z}}$ in (6.134). Then, we may write the resulting equation $-m = f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})$ also as

$(\bar{n}, -m) \in f_{\mathbb{N}}^{\mathbb{Z}}$, which gives us $-m \in \mathbb{N}_{\mathbb{Z}}$ with the previously mentioned fact that $\mathbb{N}_{\mathbb{Z}}$ is the range of $f_{\mathbb{N}}^{\mathbb{Z}}$. Consequently, the disjunction in (6.198) holds as well. In the second sub-case $m = [(\bar{n}, 0)]_{\sim_d}$, we recall from the proof of the first sub-case the equation $[(\bar{n}, 0)]_{\sim_d} = f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})$, so that we obtain now $m = f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})$ by combining these equations. Writing this equation in the form $(\bar{n}, m) \in f_{\mathbb{N}}^{\mathbb{Z}}$, we see that $m \in \mathbb{N}_{\mathbb{Z}}$ $[= \mathrm{ran}(f_{\mathbb{N}}^{\mathbb{Z}})]$ holds, with the consequence that the disjunction in (6.198) is true again.

We thus completed the two nested proofs by cases, and as m was initially arbitrary, we may therefore conclude that the proposition holds, as claimed. □

The group property of $(\mathbb{Z}, +_{\mathbb{Z}})$ allows us to define the subtraction on the set of integers.

Definition 6.20 (Subtraction on the set of integers). We call

$$-_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m -_{\mathbb{Z}} n = m +_{\mathbb{Z}} (-n) \tag{6.199}$$

the *subtraction on the set of integers*.

Theorem 6.59 (Distributive Law for $\mathbb{Z}$). *The multiplication $\cdot_{\mathbb{Z}}$ on the set of integers is distributive over the addition $+_{\mathbb{Z}}$ on $\mathbb{Z}$.*

Proof. Since the multiplication $\cdot_{\mathbb{Z}}$ is commutative, it will be sufficient to prove that $\cdot_{\mathbb{Z}}$ is left-distributive over the addition $+_{\mathbb{Z}}$ in view of Proposition 5.28. For this purpose, we prove the universal sentence

$$\forall a, b, c\, (a, b, c \in \mathbb{Z} \Rightarrow a \cdot_{\mathbb{Z}} (b +_{\mathbb{Z}} c) = (a \cdot_{\mathbb{Z}} b) +_{\mathbb{Z}} (a \cdot_{\mathbb{Z}} c)), \tag{6.200}$$

taking arbitrary elements $a, b, c \in \mathbb{Z}$. Recalling that $\mathbb{Z}$ is the quotient set $\mathbb{N} \times \mathbb{N} / \sim_d$, there exist constants, say $\bar{e}, \bar{f}, \bar{g} \in \mathbb{N} \times \mathbb{N}$ for which $a = [\bar{e}]_{\sim_d}$, $b = [\bar{f}]_{\sim_d}$ and $c = [\bar{g}]_{\sim_d}$ hold. By definition of the Cartesian product of two sets, there exist then also natural numbers, say $\bar{m}$, $\bar{n}$, $\bar{M}$, $\bar{N}$, $\bar{p}$ and $\bar{q}$, such that $(\bar{m}, \bar{n}) = \bar{e}$, $(\bar{M}, \bar{N}) = \bar{f}$ and $(\bar{p}, \bar{q}) = \bar{g}$ are satisfied. Combining these findings through substitutions results in the equations $a = [(\bar{m}, \bar{n})]_{\sim_d}$, $b = [(\bar{M}, \bar{N})]_{\sim_d}$ and $c = [(\bar{p}, \bar{q})]_{\sim_d}$.

We now firstly observe that $\bar{M}, \bar{N}, \bar{p}, \bar{q} \in \mathbb{N}$ implies with (6.185)

$$\begin{aligned} b +_{\mathbb{Z}} c &= [(\bar{M}, \bar{N})]_{\sim_d} +_{\mathbb{Z}} [(\bar{p}, \bar{q})]_{\sim_d} \\ &= [(\bar{M} + \bar{p}, \bar{N} + \bar{q})]_{\sim_d}. \end{aligned}$$

As $\bar{m}$, $\bar{n}$, $\bar{M} + \bar{p}$ and $\bar{N} + \bar{q}$ are clearly elements of $\mathbb{N}$, we may now apply (6.146) to form the product

$$\begin{aligned} a \cdot_{\mathbb{Z}} (b +_{\mathbb{Z}} c) &= [(\bar{m}, \bar{n})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{M} + \bar{p}, \bar{N} + \bar{q})]_{\sim_d} \\ &= [(\bar{m} \cdot [\bar{M} + \bar{p}] + \bar{n} \cdot [\bar{N} + \bar{q}], \bar{m} \cdot [\bar{N} + \bar{q}] + \bar{n} \cdot [\bar{M} + \bar{p}]])]_{\sim_d} \\ &= [(\bar{m}\bar{M} + \bar{m}\bar{p} + \bar{n}\bar{N} + \bar{n}\bar{q}, \bar{m}\bar{N} + \bar{m}\bar{q} + \bar{n}\bar{M} + \bar{n}\bar{p})]_{\sim_d}, \end{aligned}$$

where we used also the Distributive Law for $\mathbb{N}$ in connection with the Associative Law for the addition on $\mathbb{N}$ to get rid of some of the brackets, as well as (5.521) to circumvent the notationally inconvenient multiplication dots.

Secondly, we note that $\bar{m}, \bar{n}, \bar{M}, \bar{N} \in \mathbb{N}$ and $\bar{m}, \bar{n}, \bar{p}, \bar{q} \in \mathbb{N}$ imply, respectively,

$$\begin{aligned} a \cdot_{\mathbb{Z}} b &= [(\bar{m}, \bar{n})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{M}, \bar{N})]_{\sim_d} \\ &= [(\bar{m}\bar{M} + \bar{n}\bar{N}, \bar{m}\bar{N} + \bar{n}\bar{M})]_{\sim_d}, \\ a \cdot_{\mathbb{Z}} c &= [(\bar{m}, \bar{n})]_{\sim_d} \cdot_{\mathbb{Z}} [(\bar{p}, \bar{q})]_{\sim_d} \\ &= [(\bar{m}\bar{p} + \bar{n}\bar{q}, \bar{m}\bar{q} + \bar{n}\bar{p})]_{\sim_d}. \end{aligned}$$

with (6.146). Here, $\bar{m} \cdot \bar{M} + \bar{n} \cdot \bar{N}$, $\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}$, $\bar{m} \cdot \bar{p} + \bar{n} \cdot \bar{q}$ and $\bar{m} \cdot \bar{q} + \bar{n} \cdot \bar{p}$ evidently constitute natural numbers, so that we may form the sum (whose terms we rearrange by using the Commutative Law for the addition on $\mathbb{N}$)

$$\begin{aligned} a \cdot_{\mathbb{Z}} b +_{\mathbb{Z}} a \cdot_{\mathbb{Z}} c &= [(\bar{m}\bar{M} + \bar{n}\bar{N}, \bar{m}\bar{N} + \bar{n}\bar{M})]_{\sim_d} +_{\mathbb{Z}} [(\bar{m}\bar{p} + \bar{n}\bar{q}, \bar{m}\bar{q} + \bar{n}\bar{p})]_{\sim_d} \\ &= [(\bar{m}\bar{M} + \bar{n}\bar{N} + \bar{m}\bar{p} + \bar{n}\bar{q}, \bar{m}\bar{N} + \bar{n}\bar{M} + \bar{m}\bar{q} + \bar{n}\bar{p})]_{\sim_d} \\ &= [(\bar{m}\bar{M} + \bar{m}\bar{p} + \bar{n}\bar{N} + \bar{n}\bar{q}, \bar{m}\bar{N} + \bar{m}\bar{q} + \bar{n}\bar{M} + \bar{n}\bar{p})]_{\sim_d} \\ &= a \cdot_{\mathbb{Z}} (b +_{\mathbb{Z}} c). \end{aligned}$$

This proves the implication in (6.200), in which a, b and c are arbitrary, so that the universal sentence (6.200) follows to be true. Thus, $\cdot_{\mathbb{Z}}$ is left-distributive over $+_{\mathbb{Z}}$, which implies that $\cdot_{\mathbb{Z}}$ is distributive over $+_{\mathbb{Z}}$, as mentioned at the beginning. □

Note 6.17. The ordered quadruple $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}})$ is a commutative ring as

1. $(\mathbb{Z}, +_{\mathbb{Z}})$ is a commutative group (see Note 6.16 and Proposition 6.57),
2. $(\mathbb{Z}, \cdot_{\mathbb{Z}})$ is a commutative semigroup (see Note 6.15),
3. $\cdot_{\mathbb{Z}}$ is distributive over $+_{\mathbb{Z}}$ (see Theorem 6.59).

Since $\mathbb{N}_{\mathbb{Z}}$ is included in the integers and since both the addition and the multiplication on $\mathbb{N}_{\mathbb{Z}}$ constitute corresponding restrictions of the addition and the multiplication on the integers, we can derive the following properties of $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$ from the commutative ring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}})$.

Lemma 6.60. *It is true that the ordered triple*

$$(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}}) \tag{6.201}$$

constitutes a commutative semiring with zero element

$$f_{\mathbb{N}}^{\mathbb{Z}}(0) = [(0,0)]_{\sim_d} \tag{6.202}$$

and unity element

$$f_{\mathbb{N}}^{\mathbb{Z}}(1) = [(1,0)]_{\sim_d}. \tag{6.203}$$

Proof. We verify first that the addition on $\mathbb{N}_{\mathbb{Z}}$ is commutative, i.e.

$$\forall m, n\,(m, n \in \mathbb{N}_{\mathbb{Z}} \Rightarrow m +_{\mathbb{N}_{\mathbb{Z}}} n = n +_{\mathbb{N}_{\mathbb{Z}}} m). \tag{6.204}$$

We let m and n be arbitrary, assume $m, n \in \mathbb{N}_{\mathbb{Z}}$ to be true, and observe the truth of the equations

$$\begin{aligned} m +_{\mathbb{N}_{\mathbb{Z}}} n &= m +_{\mathbb{Z}} n \\ &= n +_{\mathbb{Z}} m \\ &= n +_{\mathbb{N}_{\mathbb{Z}}} m \end{aligned}$$

in light of (6.190) and the Commutative Law for the addition on $\mathbb{Z}$. Since m and n are arbitrary, we may infer from the truth of the resulting equation $m +_{\mathbb{N}_{\mathbb{Z}}} n = n +_{\mathbb{N}_{\mathbb{Z}}} m$ the truth of the universal sentence (6.204) and thus the commutativity of $+_{\mathbb{N}_{\mathbb{Z}}}$.

To establish the associativity of the addition on $\mathbb{N}_{\mathbb{Z}}$, we prove

$$\forall m, n, p\,(m, n, p \in \mathbb{N}_{\mathbb{Z}} \Rightarrow (m +_{\mathbb{N}_{\mathbb{Z}}} n) +_{\mathbb{N}_{\mathbb{Z}}} p = m +_{\mathbb{N}_{\mathbb{Z}}} (n +_{\mathbb{N}_{\mathbb{Z}}} p)). \tag{6.205}$$

Letting m, n and p be arbitrary and assuming that $m, n, p \in \mathbb{N}_{\mathbb{Z}}$ is true, we obtain

$$\begin{aligned} (m +_{\mathbb{N}_{\mathbb{Z}}} n) +_{\mathbb{N}_{\mathbb{Z}}} p &= (m +_{\mathbb{Z}} n) +_{\mathbb{Z}} p \\ &= m +_{\mathbb{Z}} (n +_{\mathbb{Z}} p) \\ &= m +_{\mathbb{N}_{\mathbb{Z}}} (n +_{\mathbb{N}_{\mathbb{Z}}} p) \end{aligned}$$

with (6.190) and the Associative Law for the addition on $\mathbb{Z}$. These equations evidently prove (6.205) since m, n and p were arbitrary, so that $+_{\mathbb{N}_{\mathbb{Z}}}$ is indeed associative. Thus, $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}})$ is a commutative semigroup. Let us verify now that (6.202) is the neutral element in $\mathbb{N}_{\mathbb{Z}}$ with respect to the addition on $\mathbb{N}_{\mathbb{Z}}$, i.e. that $f_{\mathbb{N}}^{\mathbb{Z}}(0)$ satisfies

$$\forall p\,(p \in \mathbb{N}_{\mathbb{Z}} \Rightarrow [f_{\mathbb{N}}^{\mathbb{Z}}(0) +_{\mathbb{N}_{\mathbb{Z}}} p = p \wedge p +_{\mathbb{N}_{\mathbb{Z}}} f_{\mathbb{N}}^{\mathbb{Z}}(0) = p]). \tag{6.206}$$

We note that $0 \in \mathbb{N}$ implies indeed $f_{\mathbb{N}}^{\mathbb{Z}}(0) = [(0,0)]_{\sim_d}$ according to the definition of the function $f_{\mathbb{N}}^{\mathbb{Z}}$ in (6.134), which value is an element of the range $\mathbb{N}_{\mathbb{Z}}$ of the bijection $f_{\mathbb{N}}^{\mathbb{Z}}$. Letting now p be arbitrary such that $p \in \mathbb{N}_{\mathbb{Z}}$

holds, it follows with the inclusion (6.144) and the definition of a subset that $p \in \mathbb{Z}$ is true. This in turn implies

$$[(0,0)]_{\sim_d} +_{\mathbb{Z}} p = p \wedge p +_{\mathbb{Z}} [(0,0)]_{\sim_d} = p$$

with (6.187), which yields

$$[(0,0)]_{\sim_d} +_{\mathbb{N}_{\mathbb{Z}}} p = p \wedge p +_{\mathbb{N}_{\mathbb{Z}}} [(0,0)]_{\sim_d} = p$$

by using (6.190), and substitutions give us then also the equation in (6.206). Because p is arbitrary, we may therefore conclude that $f_{\mathbb{N}}^{\mathbb{Z}}(0) = [(0,0)]_{\sim_d}$ is indeed the zero element with respect to $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}})$.

The proof that $(\mathbb{N}_{\mathbb{Z}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$ constitutes a commutative semigroup with unity element (6.203) is similar.

Next, we prove that $\cdot_{\mathbb{N}_{\mathbb{Z}}}$ is left-distributive over $+_{\mathbb{N}_{\mathbb{Z}}}$, i.e.

$$\forall m, n, p\, (m, n, p \in \mathbb{N}_{\mathbb{Z}} \Rightarrow m \cdot_{\mathbb{N}_{\mathbb{Z}}} (n +_{\mathbb{N}_{\mathbb{Z}}} p) = m \cdot_{\mathbb{N}_{\mathbb{Z}}} n +_{\mathbb{N}_{\mathbb{Z}}} m \cdot_{\mathbb{N}_{\mathbb{Z}}} p). \quad (6.207)$$

Letting m, n and p be arbitrary and assuming $m, n, p \in \mathbb{N}_{\mathbb{Z}}$, we get

$$\begin{aligned} m \cdot_{\mathbb{N}_{\mathbb{Z}}} (n +_{\mathbb{N}_{\mathbb{Z}}} p) &= m \cdot_{\mathbb{Z}} (n +_{\mathbb{Z}} p) \\ &= m \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} m \cdot_{\mathbb{Z}} p \\ &= m \cdot_{\mathbb{N}_{\mathbb{Z}}} n +_{\mathbb{N}_{\mathbb{Z}}} m \cdot_{\mathbb{N}_{\mathbb{Z}}} p \end{aligned}$$

by using (6.180) together with (6.190) and the Distributive Law for $\mathbb{Z}$. Here, m, n and p were arbitrary, so that (6.207) follows to be true. This property of left-distributivity implies then with Proposition 5.28 that $\cdot_{\mathbb{N}_{\mathbb{Z}}}$ is distributive over $+_{\mathbb{N}_{\mathbb{Z}}}$.

The preceding findings demonstrate that $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$ constitutes indeed a commutative semiring, for which the zero and the unity element exist, as claimed. □

Exercise 6.27. Prove in detail that $(\mathbb{N}_{\mathbb{Z}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$ is a commutative semigroup with unity element (6.203).

Note 6.18. Following the idea of an isomorphism, we may more generally view the semirings $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$ and $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$, including the corresponding zero and unity elements, as 'isomorphic' under the mapping $f_{\mathbb{N}}^{\mathbb{Z}}$.

6.6. Ordered Integral Domains $(X, +, \cdot, -, <)$

Definition 6.21 (Ordered integral domain). For any ring $(X, +, \cdot, -)$ and any linear ordering $<$ of X, we say that the ordered quadruple

$$(X, +, \cdot, -, <) \tag{6.208}$$

is an *ordered integral domain* iff

1. $(X, +, \cdot, -)$ is a commutative ring,
2. $(X, +, \cdot)$ is zero-divisor free,
3. $(X, +, \cdot)$ is non-trivial, and
4. the unity element 1_X exists, and
5. the monotony laws

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a < b \Rightarrow a + c < b + c]) \tag{6.209}$$

$$\forall a, b, c\,([a, b, c \in X \wedge 0_X < c] \Rightarrow [a < b \Rightarrow a \cdot c < b \cdot c]) \tag{6.210}$$

   are satisfied.

Note 6.19. In any ordered integral domain $(X, +, \cdot, -, <)$,

- the Sign Laws for $-$ and $+$,
- the Cancellation Law for $+$,
- the Cancellation Law for 0_X,
- the Sign Laws for $-$ and $\cdot$

hold naturally as a consequence of the assumed ring structure of $(X, +, \cdot, -)$. Furthermore, since $(X, +)$ and $(X, \cdot)$ constitute semigroups with corresponding neutral elements, n-fold additions and multiplications are defined. Thus, multiples na and powers a^n of all elements of X exist in particular (with $n \in \mathbb{N}$), and these satisfy Corollary 5.122, for instance. Moreover,

- the Generalized Associative Law for semirings and
- the Generalized Distributive Law for semirings

apply directly to ordered integral domains.

Proposition 6.61. *It is true for any ordered integral domain* $(X, +, \cdot, -, <)$ *that two elements of* X *are nonzero if their product is nonzero, i.e.*

$$\forall a, b\,([a, b \in X \wedge a \cdot b \neq 0_X] \Rightarrow [a \neq 0_X \wedge b \neq 0_X]). \tag{6.211}$$

Proof. We take arbitrary X, $+$, $\cdot$, $-$, $<$, a and b, assuming $(X, +, \cdot, -, <)$ to be an ordered integral domain, assuming a and b to be elements of X, and assuming moreover $a \cdot b \neq 0_X$ to hold. We can then prove the conjunction $a \neq 0_X \wedge b \neq 0_X$ by contradiction, assuming its negation to be true. We therefore obtain $a = 0_X \vee b = 0_X$ by means of De Morgan's Law for the conjunction and the Double Negation Law. We now use this disjunction to prove $a \cdot b = 0_X$ by cases. Indeed, the first case $a = 0_X$ yields $a \cdot b = 0_X \cdot b = 0_X$, and the second case $b = 0_X$ gives $a \cdot b = a \cdot 0_X = 0_X$, applying substitutions and the Cancellation Law for 0_X. Since $a \cdot b \neq 0_X$ and $a \cdot b = 0_X$ are both true, we arrived at a contradiction, so that the desired conjunction holds indeed. This in turn proves the implication in (6.211), in which X, $+$, $\cdot$, $-$, $<$, a and b are all arbitrary, so that the proposition follows to be true. □

Definition 6.22 (Negative constant, nonnegative constant, positive constant). We say for any ordered integral domain $(X, +, \cdot, -, <)$ that an element a of X is

(1) *negative* iff
$$a < 0_X. \tag{6.212}$$

(2) *nonnegative* iff
$$0_X \leq a. \tag{6.213}$$

(3) *positive* iff
$$0_X < a. \tag{6.214}$$

Note 6.20. We see in light of the Axiom of Specification and the Equality Criterion for sets that there exist unique sets X_-, X_+^0 and X_+ consisting, respectively, of all constants that are, respectively, negative, nonnegative and positive, that is,

$$\forall a\,(a \in X_- \Leftrightarrow [a \in X \wedge a < 0_X]), \tag{6.215}$$
$$\forall a\,(a \in X_+^0 \Leftrightarrow [a \in X \wedge 0_X \leq a]), \tag{6.216}$$
$$\forall a\,(a \in X_+ \Leftrightarrow [a \in X \wedge 0_X < a]). \tag{6.217}$$

Since $a \in X_-$, $a \in X_+^0$ and $a \in X_+$ all imply $a \in X$ for any a, the inclusions

$$X_- \subseteq X, \tag{6.218}$$
$$X_+^0 \subseteq X, \tag{6.219}$$
$$X_+ \subseteq X \tag{6.220}$$

are true by definition of a subset.

Exercise 6.28. Show for any ordered integral domain $(X, +, \cdot, -, <)$ that two nonnegative constants a, b in X are both zero if their sum is zero, i.e.

$$\forall a, b\,([0_X \leq a \,\wedge\, 0_X \leq b] \Rightarrow [a + b = 0_X \Rightarrow (a = 0_X \,\wedge\, b = 0_X)]). \quad (6.221)$$

(Hint: Apply a Proof by contraposition, using some of the ideas within the proof of Proposition 6.61 in connection with (3.239), (6.209), (3.233) and (3.156).)

The following theorem makes use in particular of Property 2 of an ordered integral domain.

Theorem 6.62 (Cancellation Law for $\cdot$). *The following law holds for every ordered integral domain* $(X, +, \cdot, -, <)$*.*

$$\forall a, b, c\,([a, b, c \in X \,\wedge\, a \neq 0_X] \Rightarrow [a \cdot b = a \cdot c \Leftrightarrow b = c]). \qquad (6.222)$$

Proof. Letting X, $+$, $\cdot$, $-$, $<$, a, b and c be arbitrary sets such that $(X, +, \cdot, -, <)$ is an ordered integral domain, we assume $a, b, c \in X$ and $a \neq 0_X$ to be true. We prove the first part ('$\Rightarrow$') of the equivalence in (6.222) directly, assuming also that $a \cdot b = a \cdot c$ holds. We may therefore apply substitution to obtain the new equation

$$a \cdot b + [-(a \cdot c)] = a \cdot c + [-(a \cdot c)],$$

which we may write equivalently as

$$a \cdot b + [a \cdot (-c)] = 0_X,$$

using the Sign Law (6.63) and the definition of the negative. Because the multiplication is distributive over the addition, we obtain now

$$a \cdot [b + (-c)] = 0_X.$$

Due to Property 2 of an ordered integral domain, the preceding equation evidently implies with the Criterion for zero-divisor freeness (5.175) the disjunction

$$a = 0_X \,\vee\, b + (-c) = 0_X,$$

whose first part is false in view of the initial assumption $a \neq 0_X$, so that its second part $b + (-c) = 0_X$ must be true. Therefore, substitution yields the true equation

$$[b + (-c)] + c = 0_X + c,$$

which we can write equivalently as

$$b + [(-c) + c] = c$$

by applying the associativity of the addition and the definition of the zero element. Using now again the definition of the negative, we get $b + 0_X = c$ and consequently $b = c$ (applying once again the definition of the zero element). We thus completed the proof of the first part of the equivalence in (6.222). Regarding the second part ('$\Leftarrow$'), we simply observe that assuming $b = c$ gives us the desired consequent $a \cdot b = a \cdot c$ immediately by means of substitution (into the evidently true equation $a \cdot b = a \cdot b$). Having completed the proof of the equivalence, we may now infer from this the truth of the theorem, as the sets X, $+$, $\cdot$, $-$, $<$, a, b and c were initially arbitrary. □

The next theorem is founded essentially on Property 1, Property 3 and Property 4 of an ordered integral domain.

Theorem 6.63 (Distinctness of 1_X and 0_X). *It is true for every ordered integral domain $(X, +, \cdot, -, <)$ that the unity and the zero element are distinct, i.e.*

$$1_X \neq 0_X. \tag{6.223}$$

Proof. Letting X, $+$, $\cdot$, $-$ and $<$ be arbitrary sets such that $(X, +, \cdot, -, <)$ is an ordered integral domain, we prove the stated inequality by contradiction, assuming its negation $\neg 1_X \neq 0_X$ to be true (and noting that 1_X exists by virtue of Property 4 of an ordered integral domain). Because of the Double Negation Law, we thus have the true equation $1_X = 0_X$, which we now use to prove the equation $X = \{0_X\}$. For this purpose, we apply the Equality Criterion for sets and establish the equivalent universal sentence

$$\forall a\, (a \in X \Leftrightarrow a \in \{0_X\}). \tag{6.224}$$

To do this, we take an arbitrary a and assume first $a \in X$ to be true. Let us observe now the truth of the equation $a = a \cdot 1_X$ in light of the definition of the unity element. As the equation $a \cdot 1_X = a \cdot 1_X$ is clearly true, we may apply substitution to the latter based on the former equation, with the consequence that $a \cdot 1_X = a \cdot 0_X$ also holds. Thus, we find $a = a \cdot 0_X$, where the right-hand side satisfies $a \cdot 0_X = 0_X$ because of the Cancellation Law for 0_X, so that $a = 0_X$ follows to be true. This finding in turn implies $a \in \{0_X\}$ with (2.169), which proves the first part ('$\Rightarrow$') of the equivalence in (6.224).

Concerning the second part ('$\Leftarrow$'), we note that $a \in \{0_X\}$ implies $a = 0_X$ with (2.169) and that $0_X \in X$ holds according to the group property of $(X, +)$. We therefore obtain the desired consequent $a \in X$ through substitution, so that the proof of the equivalence in (6.224) is now complete. Since a is arbitrary, we may infer from the truth of this equivalence the truth of the universal sentence (6.224), and consequently the truth of the equation $X = \{0_X\}$. Because $(X, +, \cdot)$ is nontrivial by Property 3 of an ordered integral domain, the inequality $X \neq \{0_X\}$ is also true, so that we arrived at a contradiction. Thus, the proof of $1_X \neq 0_X$ is complete, and as the sets X, $+$, $\cdot$, $-$ and $<$ were initially arbitrary, we may now conclude that the theorem is indeed true. □

Property 5 of an ordered integral domain allow us to establish also the following laws.

Lemma 6.64. *The following sentences hold for any ordered integral domain* $(X, +, \cdot, -, <)$.

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a \leq b \Rightarrow a + c \leq b + c]), \tag{6.225}$$

$$\forall a, b, c\,([a, b, c \in X \,\wedge\, 0_X < c] \Rightarrow [a \leq b \Rightarrow a \cdot c \leq b \cdot c]) \tag{6.226}$$

Exercise 6.29. Prove Lemma 6.64.

(Hint: Recall the proof of Lemma 5.63.)

Theorem 6.65 (Monotony Laws for ordered integral domains). *The following laws hold for any ordered integral domain* $(X, +, \cdot, -, <)$.

a) **Monotony Law for $+$ and $<$:**

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a < b \Leftrightarrow a + c < b + c]) \tag{6.227}$$

b) **Monotony Law for $+$ and $\leq$:**

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a \leq b \Leftrightarrow a + c \leq b + c]) \tag{6.228}$$

c) **Monotony Law for $\cdot$ and $<$:**

$$\forall a, b, c\,([a, b, c \in X \,\wedge\, 0_X < c] \Rightarrow [a < b \Leftrightarrow a \cdot c < b \cdot c]) \tag{6.229}$$

d) **Monotony Law for $\cdot$ and $\leq$:**

$$\forall a, b, c\,([a, b, c \in X \,\wedge\, 0_X < c] \Rightarrow [a \leq b \Leftrightarrow a \cdot c \leq b \cdot c]) \tag{6.230}$$

Exercise 6.30. Prove the Monotony Laws for ordered integral domains.

(Hint: Proceed similarly as in the proofs of the Monotony Laws for ordered elementary domains.)

In the remainder of this section, we establish some useful consequences of the Monotony Laws.

Theorem 6.66 (Additivity of $<$- & $\leq$-inequalities for ordered integral domains). *The following laws hold for any ordered integral domain* $(X, +, \cdot, -, <)$.

a) **Additivity of $<$-inequalities***:*

$$\forall a, b, c, d\,(a, b, c, d \in X \Rightarrow [(a < b \wedge c < d) \Rightarrow a+c < b+d]). \quad (6.231)$$

b) **Additivity of $\leq$-inequalities***:*

$$\forall a, b, c, d\,(a, b, c, d \in X \Rightarrow [(a \leq b \wedge b \leq d) \Rightarrow a+c \leq b+d]). \quad (6.232)$$

Exercise 6.31. Establish the Additivity of inequalities for any ordered integral domain $(X, +, \cdot, -, <)$.

(Hint: Recall the proof of the Additivity of inequalities for ordered elementary domains $(X, +, \cdot, <)$.)

Proposition 6.67. *The following sentence is true for any ordered integral domain* $(X, +, \cdot, -, <)$.

$$\forall a, b\,(a, b \in X \Rightarrow [a < b \Leftrightarrow -b < -a]) \qquad (6.233)$$

Proof. We take arbitrary sets X, $+$, $\cdot$, $-$, $<$, a and b, we assume that $(X, +, \cdot, -, <)$ constitutes an ordered integral domain, and we assume that both a and b are elements of X. We then obtain the true equivalences

$$\begin{aligned}
a \leq b &\Leftrightarrow a + (-a) \leq b + (-a) \\
&\Leftrightarrow 0_X \leq -a + b \\
&\Leftrightarrow 0_X + (-b) \leq (-a + b) + (-b) \\
&\Leftrightarrow -b \leq -a + (b + (-b)) \\
&\Leftrightarrow -b \leq -a + 0_X \\
&\Leftrightarrow -b \leq -a
\end{aligned}$$

by applying the Monotony Law for $+$ and $\leq$ (noting that the negative of a exists by virtue of the group property of X), the definition of the negative together with the commutativity of the addition on X, again the Monotony Law for $+$ and $\leq$, the definition of a neutral element alongside the associativity of the addition on X, again the definition of the negative, and finally again the definition of a neutral element. Since X, $+$, $\cdot$, $-$, $<$, a and b are arbitrary, we may now infer from the truth of the resulting equivalence $a < b \Leftrightarrow -b < -a$ the truth of the proposed universal sentence. $\square$

Exercise 6.32. Establish the following sentence for any ordered integral domain $(X, +, \cdot, -, <)$.

$$\forall a, b\,(a, b \in X \Rightarrow [a \leq b \Leftrightarrow -b \leq -a]) \tag{6.234}$$

(Hint: Proceed similarly as in the proof of Proposition 6.67.)

Proposition 6.68. *It is true for any ordered integral domain* $(X, +, \cdot, -, <)$ *that the product of two elements of* X *is positive iff the two elements are both positive or both negative, i.e.*

$$\forall a, b\,(a, b \in X \Rightarrow [a \cdot b > 0_X \Leftrightarrow ([a > 0_X \wedge b > 0_X] \vee [a < 0_X \wedge b < 0_X])]). \tag{6.235}$$

Proof. We take arbitrary sets X, $+$, $\cdot$, $-$, $<$, a and b, assuming $(X, +, \cdot, -, <$ $)$ to be an ordered integral domain and assuming a and b to be elements of X. We prove now the first part ('$\Rightarrow$') of the equivalence directly, assuming also $a \cdot b > 0_X$ to be true. Then, since $<$ is a linear ordering, satisfying thus the Characterization of comparability, we have that $a \cdot b = 0_X$ is false, in other words that $a \cdot b \neq 0_X$ is true. Consequently, $a \neq 0_X$ and $b \neq 0_X$ follow to be true as well with (6.211). Because of the connexity of the linear ordering $<$, we thus have the true disjunctions $a > 0_X \vee a < 0_X$ and $b > 0_X \vee b < 0_X$. We use the former disjunction to prove the desired consequent

$$[a > 0_X \wedge b > 0_X] \vee [a < 0_X \wedge b < 0_X] \tag{6.236}$$

by cases.

In the first case $a > 0_X$, we can prove $\neg b < 0_X$ by contradiction, assuming the negation of that negation to be true, so that the Double Negation Law gives us the true sentence $b < 0_X$. This in turn implies $-0_X < -b$ with (6.233) and furthermore $0_X < -b$ with (6.38). In conjunction with the case assumption $0_X < a$, this further implies $0_X \cdot (-b) < a \cdot (-b)$ with the Monotony Law for $\cdot$ and $<$. Then, an application of the Cancellation Law for 0_X and of the Sign Law (6.63) yields $0_X < -(a \cdot b)$, and therefore $a \cdot b < 0_X$ by virtue of (6.233) in connection with (6.38). In view of the comparability of the linear ordering of X, this means that $a \cdot b > 0_X$ is false, in contradiction to the initial assumption that $a \cdot b > 0_X$ is true. We thus completed the proof of $\neg b < 0_X$ by contradiction, which shows that $b < 0_X$ is false, so that the first part of the true disjunction $b > 0_X \vee b < 0_X$ must be true. The truth of $a > 0_X$ and of $b > 0_X$ demonstrates the truth of the first part of the disjunction (6.236), which itself is then also true.

In the second case $a < 0_X$, we can now prove $\neg b > 0_X$ by contradiction. To do this, we assume $\neg\neg b > 0_X$, so that we evidently find $b > 0_X$ to be true, that is, $0_X < b$. The Monotony Law for $\cdot$ and $<$ yields then $a \cdot b < 0_X \cdot b$, which we can simplify to $a \cdot b < 0_X$ by means of the Cancellation Law for 0_X. As in the first case, this finding contradicts the initial assumption $a \cdot b > 0_X$, so that $\neg b > 0_X$ holds indeed. Thus, the second part of the true disjunction $b > 0_X \lor b < 0_X$ must be true, so that we found the conjunction $a < 0_X \land b < 0_X$ to be true. Then, the disjunction (6.236) also holds, and the proof of the consequent of the first part of the equivalence in (6.235) by cases is now complete.

To prove the second part ('$\Leftarrow$'), we assume conversely (6.236) to be true, and we use this disjunction to prove $a \cdot b > 0_X$ by cases. In the first case, we have $0_X < a$ and $0_X < b$, so that the Monotony Law for $\cdot$ and $<$ gives us $0_X \cdot b < a \cdot b$, which gives the desired $a \cdot b > 0_X$ by means of the Cancellation Law for 0_X. The second case $a < 0_X \land b < 0_X$ implies $0_X < -a$ and $0_X < -b$ with (6.233) and (6.38), therefore $0_X \cdot (-b) < (-a) \cdot (-b)$ again with the Monotony Law for $\cdot$ and $<$. Consequently, we obtain $a \cdot b > 0_X$ by means of the Sign Law (6.65) and the Cancellation Law for 0_X. Having thus completed the proof by cases, the second part of the equivalence is also true.

The truth of this equivalence establishes also the truth of the implication in (6.235), in which a and b are arbitrary, so that the proposed universal sentence follows now to be true. □

Exercise 6.33. Show for any ordered integral domain $(X, +, \cdot, -, <)$ that the product of two elements of X is negative iff one of the elements is positive and the other one is negative, i.e.

$$\forall a, b\,(a, b \in X \Rightarrow [a \cdot b < 0_X \Leftrightarrow ([a > 0_X \land b < 0_X] \lor [a < 0_X \land b > 0_X])]). \tag{6.237}$$

(Hint: Use similar arguments as in the proof of Proposition 6.68.)

Proposition 6.69. *The following universal sentence holds for any ordered integral domain $(X, +, \cdot, -, <)$.*

$$\forall a\,([a \in X \land a \neq 0_X] \Rightarrow a^2 > 0_X) \tag{6.238}$$

Proof. Letting X, $+$, $\cdot$, $-$, $<$ and a be arbitrary such that $(X, +, \cdot, -, <)$ is an ordered integral domain and such that a is a nonzero element of X. Due to the connexity of the linear ordering $<$ and due to the assumption $a \neq 0_X$, the disjunction $a < 0_X \lor a > 0_X$ is true, which we use now to prove $0_X < a \cdot a$ by cases.

The first case $a < 0_X$ implies $-0_X < -a$ with (6.233) and therefore $0_X < -a$ via substitution based on (6.38). Next, we apply the Monotony Law for $\cdot$ and $<$ to infer from this the truth of the inequality $0_X \cdot (-a) < (-a) \cdot (-a)$. Here, the equations $0_X \cdot (-a) = 0_X$ and $(-a) \cdot (-a) = a \cdot a$ hold according to the Cancellation Law for 0_X and the Sign Law (6.65), so that we obtain $0_X < a \cdot a$ via substitutions.

The second case $0_X < a$ gives us with the Monotony Law for $\cdot$ and $<$ the inequality $0_X \cdot a < a \cdot a$ and then $0_X < a \cdot a$ with the Cancellation Law for 0_X. Thus, $0_X < a \cdot a$ holds in any case, and we can write this inequality also in the desired form $a^2 > 0_X$, according to (5.478). Since X, $+$, $\cdot$, $-$, $<$ and a are arbitrary, we may therefore conclude that the proposition holds. □

Corollary 6.70. *The zero element* 0_X *is less than the unity element* 1_X *for any ordered integral domain* $(X, +, \cdot, -, <)$*, i.e.*

$$0_X < 1_X. \tag{6.239}$$

Proof. Letting X, $+$, $\cdot$, $-$ and $<$ be arbitrary and assuming $(X, +, \cdot, -, <)$ to be an ordered integral domain, Theorem 6.63 gives us $1_X \neq 0_X$, which implies $1_X^2 > 0_X$ with (6.238). We therefore have $1_X \cdot 1_X > 0_X$ according to (5.478), where $1_X \cdot 1_X = 1_X$ holds by definition of the unity element. Thus, substitution yields $1_X > 0_X$, which we may write also in the form (6.239). Since X, $+$, $\cdot$, $-$ and $<$ were arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Proposition 6.71. *It is true for any ordered integral domain* $(X, +, \cdot, -, <)$ *that* $(X, +, \cdot, <)$ *is not an ordered elementary domain.*

Proof. We let X, $+$, $\cdot$, $-$ and $<$ be arbitrary and assume that $(X, +, \cdot, -, <)$ is an ordered integral domain. We now prove by contradiction that $(X, +, \cdot, <)$ is not an ordered elementary domain, assuming the negation of that sentence to be true. It then follows with the Double Negation Law that $(X, +, \cdot, <)$ is an ordered elementary domain. Due to Property 7 of an ordered elementary domain, the elements $0_X, -1_X \in X$ satisfy then

$$0_X < -1_X \;\Leftrightarrow\; \exists d\,(d \in X \,\wedge\, d \neq 0_X \,\wedge\, 0_X + d = -1_X). \tag{6.240}$$

Since $0_X < 1_X$ is true, as shown by (6.239), the inequality $-1_X < -0_X$ follows to be also true with (6.233). Consequently, substitution based on (6.38) yields

$$-1_X < 0_X. \tag{6.241}$$

The truth of this inequality shows in light of the Characterization of comparability with respect to the linear ordering $<$ that $-1_X = 0_X$ and $-1_X > 0_X$ are both false. The former finding means that $-1_X \neq 0_X$ is true, whereas the falsity of the latter demonstrates that the left-hand side $0_X < -1_X$ of the equivalence (6.240) is false; thus, the negation $\neg 0_X < -1_X$ is true. Let us observe now the truth of the equation

$$0_X + (-1_X) = -1_X$$

in view of the definition of the zero element. In connection with the evident $-1_X \in X$ and the previously established $-1_X \neq 0_X$, this equation shows that the existential sentence on the right-hand side of the equivalence (6.240) is true, so that the implication '$\Leftarrow$' gives the true consequent $0_X < -1_X$. We thus found $0_X < -1_X$ and $\neg 0_X < -1_X$ to be both true, which means that we obtained a contradiction. This finding completes the proof of the assertion that $(X, +, \cdot, <)$ is not an ordered elementary domain. Since X, $+$, $\cdot$, $-$ and $<$ were initially arbitrary, we may therefore conclude that the proposition holds, as claimed. $\square$

Proposition 6.72. *The following universal sentence holds for any ordered integral domain* $(X, +, \cdot, -, <)$.

$$\forall a\, (a \in X \Rightarrow [a = 0_X \Leftrightarrow a^2 = 0_X]) \tag{6.242}$$

Proof. We take arbitrary sets X, $+$, $\cdot$, $-$, $<$, a such that $(X, +, \cdot, -, <)$ is an ordered integral domain. On the one hand, assuming $a = 0_X$ implies

$$a^2 = a \cdot a = 0_X \cdot 0_X = 0_X$$

by means of (5.478), substitution and the Cancellation Law for 0_X. On the other hand, assuming $a^2 = 0_X$ and $a \neq 0_X$ to be true yields with (6.238) $a^2 > 0_X$ and thus a contradiction since the conjunction $a^2 = 0_X \wedge a^2 > 0_X$ is false according to the Characterization of linearly ordered sets and the definition of a comparable binary relation. Therefore, the equivalence in (6.242) is true, where X, $+$, $\cdot$, $-$, $<$ and a are arbitrary, so that the proposed universal sentence follows to be true. $\square$

Exercise 6.34. Prove the following universal sentence for any ordered integral domain $(X, +, \cdot, -, <)$.

$$\forall a\, (a \in X \Rightarrow a^2 \geq 0_X) \tag{6.243}$$

(Hint: Apply Method 1.8, using Theorem 3.81 with (6.238) and (6.242).)

Exercise 6.35. Prove for any ordered integral domain $(X, +, \cdot, -, <)$ that any sum of squares is nonnegative, in the sense that

$$\forall n\, (n \in \mathbb{N} \Rightarrow \forall s\, (s \in X^{\{1,\ldots,n\}} \Rightarrow \sum_{i=1}^{n} a_i^2 \geq 0_X)). \tag{6.244}$$

(Hint: Apply a proof by mathematical induction, using (4.239), (3.120), (5.409), (3.145), (5.417), (6.243), and (5.295).)

Proposition 6.73. *It is true for any ordered integral domain* $(X, +, \cdot, -, <)$ *that any sum of squares is nonnegative, in the sense that*

$$\forall n\, (n \in \mathbb{N} \Rightarrow \forall s\, (s \in X^{\{1,\ldots,n\}} \Rightarrow \left[\sum_{i=1}^{n} a_i^2 = 0_X \Leftrightarrow \forall i\, (i \in \{1, \ldots, n\} \Rightarrow a_i = 0_X)\right])). \tag{6.245}$$

Proof. Letting $(X, +, \cdot, -, <)$ be an arbitrary ordered integral domain, we apply a proof by mathematical induction. Regarding the base case ($n = 0$), we let $s \in X^{\{1,\ldots,n\}}$ be arbitrary, so that $s = \emptyset$ due to (4.239) and (3.120). Let us observe on the one hand that $\sum_{i=1}^{0} a_i^2 = 0_X$ holds by definition of a square and (5.409). On the other hand, letting i be arbitrary, we see that $i \in \{1, \ldots, 0\}$ $[= \emptyset]$ is false, so that the implication $i \in \{1, \ldots, n\} \Rightarrow a_i = 0_X$ is true. As i is arbitrary, we conclude that the universal sentence $\forall i\, (i \in \{1, \ldots, n\} \Rightarrow a_i = 0_X)$ holds, too. Thus, the equivalence in (6.245) is true for $n = 0$. Then, because s is arbitrary, we may infer from this finding the truth of the base case. Regarding the induction step, we let $n \in \mathbb{N}$ be arbitrary, make the induction assumption

$$\forall s\, (s \in X^{\{1,\ldots,n\}} \Rightarrow \left[\sum_{i=1}^{n} a_i^2 = 0_X \Leftrightarrow \forall i\, (i \in \{1, \ldots, n\} \Rightarrow a_i = 0_X)\right]),$$

and show that

$$\forall s\, (s \in X^{\{1,\ldots,n+1\}} \Rightarrow \left[\sum_{i=1}^{n+1} a_i^2 = 0_X \Leftrightarrow \forall i\, (i \in \{1, \ldots, n+1\} \Rightarrow a_i = 0_X)\right]) \tag{6.246}$$

follows to be true. Letting $s \in X^{\{1,\ldots,n+1\}}$ be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\sum_{i=1}^{n+1} a_i^2 = 0_X$ to be true and demonstrating the truth of

$$\forall i\, (i \in \{1, \ldots, n+1\} \Rightarrow a_i = 0_X). \tag{6.247}$$

Due to the assumed equation, we thus have

$$\sum_{i=1}^{n} a_i^2 + a_{n+1}^2 = 0_X. \tag{6.248}$$

in view of (5.417), where $\sum_{i=1}^{n} a_i^2 \geq 0_X$ and $a_{n+1}^2 \geq 0_X$ hold according to (6.244) and (6.243). Consequently, the equation (6.248) implies $\sum_{i=1}^{n} a_i^2 = 0_X$ and $a_{n+1}^2 = 0_X$ with (6.221). Here, the former gives us with the induction assumption the true universal sentence

$$\forall i\, (i \in \{1, \ldots, n\} \Rightarrow a_i = 0_X). \tag{6.249}$$

We are now in a position to prove the universal sentence (6.247). To do this, we let $i \in \{1, \ldots, n+1\}$ be arbitrary, so that

$$i \in \{1, \ldots, n\} \cup \{n+1\} \tag{6.250}$$

holds by virtue of (4.241). By definition of the union of two sets, this means that the disjunction

$$i \in \{1, \ldots, n\} \vee i = n+1 \tag{6.251}$$

is true (using also (2.169)). We now use this true disjunction to prove $a_i = 0_X$ by cases. The first case $i \in \{1, \ldots, n\}$ immediately gives the desired equation with (6.249). The second case $i = n+1$ yields first $a_i^2 = 0_X$ via substitution into the previously established equation $a_{n+1}^2 = 0_X$, and then also $a_i = 0_X$ with (6.242), which thus holds in any case. Since i is arbitrary, we therefore conclude that (6.247) is true, so that the proof of the first part of the equivalence in (6.246) is complete.

To establish the second part ('$\Leftarrow$'), we assume now (6.247) to be true, and we show that $\sum_{i=1}^{n+1} a_i^2 = 0_X$ holds, too. For this purpose, we prove (6.249), letting i be arbitrary and assuming $i \in \{1, \ldots, n\}$ to hold. Then, the disjunction (6.251) is also true, which evidently implies (6.250) and then $i \in \{1, \ldots, n+1\}$. This in turn implies $a_i = 0_X$ with the assumption (6.247). As i was arbitrary, we therefore conclude that (6.249) is true, and this universal sentence gives us now the true equation $\sum_{i=1}^{n} a_i^2 = 0_X$ with the induction assumption. Since the evident fact $n+1 \in \{1, \ldots, n+1\}$ implies also $a_{n+1} = 0_X$ with the assumption (6.247), we evidently obtain the true equation (6.248), which finding completes the proof of the equivalence.

Since s was arbitrary, we may therefore conclude that the universal sentence (6.246) holds. Then, as n was also arbitrary, we may further conclude that the induction step is true. Thus, the proof of (6.73) by mathematical induction is complete. Here, $(X, +, \cdot, -, <)$ was initially also arbitrary, so that the proposition follows to be true. □

Proposition 6.74. *It is true for any ordered integral domain* $(X, +, \cdot, -, <)$ *and any elements* a, b, c *of* X *that*

a) the product of b *and* c *is positive if the products of* a *and* b *and of* a *and* c *are both positive, i.e.*

$$\forall a, b, c\,([a, b, c \in X \wedge a \cdot b > 0_X \wedge a \cdot c > 0_X] \Rightarrow b \cdot c > 0_X). \tag{6.252}$$

b) the product of b *and* c *is positive if the products of* a *and* b *and of* a *and* c *are both negative, i.e.*

$$\forall a, b, c\,([a, b, c \in X \wedge a \cdot b < 0_X \wedge a \cdot c < 0_X] \Rightarrow b \cdot c > 0_X). \tag{6.253}$$

Proof. We take an arbitrary ordered integral domain $(X, +, \cdot, -, <)$ and arbitrary elements a, b, c in X. Concerning a), we assume that $a \cdot b > 0_X$ and $a \cdot c > 0_X$ are satisfied. Since $<$ is a linear ordering, it satisfies the Characterization of comparability, so that the assumed inequalities imply the falseness of $a \cdot b = 0_X$ and of $a \cdot c = 0_X$. This means that the negations $a \cdot b \neq 0_X$ and $a \cdot c \neq 0_X$ are true, which imply now the truth of the conjunctions $a \neq 0_X \wedge b \neq 0_X$ and $a \neq 0_X \wedge c \neq 0_X$ with (6.211). Let us observe here that $a \neq 0_X$ implies $0_X < a \cdot a$ with (6.238) and (5.478). Now, the assumed inequalities $0_X < a \cdot b$ and $0_X < a \cdot c$ imply also

$$0_X \cdot (a \cdot c) < (a \cdot b) \cdot (a \cdot c)$$

with the Monotony Law for $\cdot$ and $<$, which yields with the Cancellation Law for 0_X

$$0_X < (a \cdot b) \cdot (a \cdot c),$$

and then also

$$0_X \cdot (a \cdot a) < (b \cdot c) \cdot (a \cdot a)$$

with the Associative Law and the Commutative Law for the multiplication on X. Because of $0_X < a \cdot a$, we can apply once again the Monotony Law for $\cdot$ and $<$, with the consequence that $0_X < b \cdot c$ is true, and this finding proves the implication in (6.252). Since a, b and c are arbitrary, we therefore conclude that the universal sentence (6.252) holds.

Concerning b), we let now $a, b, c \in X$ be arbitrary such that $a \cdot b < 0_X$ and $a \cdot c < 0_X$. We obtain then $-0_X < -(a \cdot b)$ and $-0_X < -(a \cdot c)$ with (6.233), so that $0_X < -(a \cdot b)$ and $0_X < -(a \cdot c)$ follow to be true with (6.38). Consequently, the Sign Law (6.63) yields $0_X < a \cdot (-b)$ and $0_X < a \cdot (-c)$, and the conjunction $a \cdot (-b) > 0_X \wedge a \cdot (-c) > 0_X$ is then also true. In view of a), we obtain therefore $(-b) \cdot (-c) > 0_X$, which implies then the desired consequent $b \cdot c > 0_X$ of the implication in (6.253) by means of the Sign Law (6.65). Because a, b and c are arbitrary, we may now infer from the truth of that implication the truth of the universal sentence (6.253).

Initially, $(X, +, \cdot, -, <)$ and a, b, c was also arbitrary, so that the proposition follows to be true. □

Exercise 6.36. Verify for any ordered integral domain $(X, +, \cdot, -, <)$ and any elements a, b, c of X that

a) the product of b and c is negative if the product of a and b is negative and the product of a and c positive, i.e.

$$\forall a, b, c\,([a, b, c \in X \;\wedge\; a \cdot b < 0_X \;\wedge\; a \cdot c > 0_X] \Rightarrow b \cdot c < 0_X). \quad (6.254)$$

(Hint: Proceed as in the proof of Proposition 6.74.)

b) the product of b and c is negative if the product of a and b is positive and the product of a and c negative, i.e.

$$\forall a, b, c\,([a, b, c \in X \,\wedge\, a \cdot b > 0_X \,\wedge\, a \cdot c < 0_X] \Rightarrow b \cdot c < 0_X). \quad (6.255)$$

(Hint: Use a) and the commutativity of the multiplication on X.)

Theorem 6.75 (Induced ordered elementary domains). *It is true for any ordered integral domain $(X, +, \cdot, -, <)$ that*

$$(X^0_+, +_{X^0_+}, \cdot_{X^0_+}, <_{X^0_+}) \quad (6.256)$$

constitutes an ordered elementary domain.

Proof. We let $(X, +, \cdot, -, <)$ be an arbitrary ordered integral domain, we define the set X^0_+, which is a subset of X in view of (6.219). Next, we define $+_{X^0_+}$ as well as $\cdot_{X^0_+}$ to be the restrictions of, respectively, $+$ and $\cdot$ to $X^0_+ \times X^0_+$. These restrictions are functions from $X^0_+ \times X^0_+$ to X (see Note 5.1). We now show that X^0_+ is also a codomain of these restrictions, i.e., that their ranges are included in X^0_+.

Letting first y be an arbitrary element of $\mathrm{ran}(\cdot_{X^0_+})$, we find by definition of range $(\bar{z}, y) \in \cdot_{X^0_+}$ for a particular constant $\bar{z}$, so that $y = \cdot_{X^0_+}(\bar{z})$. Here, we have $\bar{z} \in X^0_+ \times X^0_+$ $[= \mathrm{dom}(\cdot_{X^0_+})]$ by definition of a domain. Therefore, there are particular elements $\bar{x}, \bar{x}^* \in X^0_+$ with $(\bar{x}, \bar{x}^*) = \bar{z}$, in view of (3.38). The preceding equation for y can then be written as $y = \cdot_{X^0_+}(\bar{x}, \bar{x}^*) = \bar{x} \cdot_{X^0_+} \bar{x}^*$. Let us observe here that $\bar{x}, \bar{x}^* \in X^0_+$ implies $0_X \leq \bar{x}$ and $0_X \leq \bar{x}^*$ by definition of the set X^0_+. The latter inequality implies with the Characterization of induced reflexive partial orderings that $0_X = \bar{x}^*$ or $0_X < \bar{x}^*$ holds. We may use this true disjunction to prove $0_X \leq \bar{x} \cdot \bar{x}^*$ by cases. On the one hand, $0_X = \bar{x}^*$ implies $\bar{x} \cdot \bar{x}^* = \bar{x} \cdot 0_X = 0_X$; since the induced total ordering $\leq$ is reflexive, $0_X \leq 0_X$ is true, so that substitution based on the preceding equations gives $0_X \leq \bar{x} \cdot \bar{x}^*$, as desired. On the other hand, $0_X < \bar{x}^*$ implies with $0_X \leq \bar{x}$ the inequality $[0_X =]\ 0_X \cdot \bar{x}^* \leq \bar{x} \cdot \bar{x}^*$ with the Monotony Law for $\cdot$ and $\leq$, and therefore again the desired $0_X \leq \bar{x} \cdot \bar{x}^*$. Since $\bar{x}$ and $\bar{x}^*$ are both in X^0_+, we can write $[0_X \leq]\ \bar{x} \cdot \bar{x}^* = \bar{x} \cdot_{X^0_+} \bar{x}^*$ $[= y]$. We thus find $0_X \leq y$, and this gives us now evidently $y \in X^0_+$. As y was arbitrary, we may therefore conclude that $y \in \mathrm{ran}(\cdot_{X^0_+})$ implies $y \in X^0_+$ for any y, so that the range of $\cdot_{X^0_+}$ is a subset of X^0_+, by definition. This means that X^0_+ is a codomain of $\cdot_{X^0_+}$. The proof that X^0_+ is also a codomain of $+_{X^0_+}$ is a slightly simpler version of the proof given in the preceding paragraph with respect to $\cdot_{X^0_+}$. Thus, $+_{X^0_+}$ and $\cdot_{X^0_+}$ are both binary operations on X^0_+.

Recall now that $(X, +)$ and $(X, \cdot)$ constitute semigroups with corresponding neutral elements 0_X and 1_X (see Note 6.19). Since the linear ordering $<$ of X induces the total ordering of X, which is reflexive, we see that $0_X \leq 0_X$ holds besides $0_X \in X$; consequently, $0_X \in X_+^0$ follows to be true with the specification (6.216) of the set X_+^0. Because of Proposition 5.13, 0_X is then also the identity element of X_+^0 with respect to $+_{X_+^0}$. Thus, the zero element of X_+^0 exists in that set.

Moreover, the fact $0_X < 1_X$ in (6.239) implies $0_X \leq 1_X$ with the Characterization of induced reflexive partial orderings. In conjunction with the aforementioned fact $1_X \in X$, this implies $1_X \in X_+^0$ by definition of the set X_+^0. In other words, the unity element of X_+^0 also exists.

Let us observe now that $(X, +, \cdot, -)$ is a commutative ring, according to Property 1 of an ordered integral domain. This implies that $(X, +, \cdot)$ is a commutative semiring (see Note 6.6). Next, we recall the fact $1_X \neq 0_X$ from (6.223), so that $1_X \notin \{0_X\}$ follows to be true with (2.169). This allows to prove $X_+^0 \neq \{0_X\}$ by contradiction. Assuming for this purpose the negation of that negation to be true, we obtain the equation $X_+^0 = \{0_X\}$ with the Double Negation Law. Consequently, $1_X \notin \{0_X\}$ yields $1_X \notin X_+^0$ via substitution, in contradiction with the previously established fact $1_X \in X_+^0$. We thus completed the proof of $X_+^0 \neq \{0_X\}$, so that the semiring $(X_+^0, +_{X_+^0}, \cdot_{X_+^0})$ is non-trivial, by definition. As the semiring $(X, +, \cdot)$ is zero-divisor free, the semiring $(X_+^0, +_{X_+^0}, \cdot_{X_+^0})$ is also zero-divisor free (see Proposition 5.43).

Furthermore, since $(X, <)$ is linearly ordered and since $X_+^0 \subseteq X$ holds, the set $(X_+^0, <_{X_+^0})$ is also linearly ordered, in view of the Linear ordering of subsets. It follows with the Characterization of linearly ordered sets that $<_{X_+^0}$ is comparable.

It remains for us to prove that $<_{X_+^0}$ satisfies

$$\forall a, b\, (a, b \in X_+^0 \Rightarrow [a <_{X_+^0} b \Leftrightarrow \exists d\, (d \in X_+^0 \, \wedge \, d \neq 0_X \, \wedge \, a +_{X_+^0} d = b)]). \tag{6.257}$$

To do this, we let $a, b \in X_+^0$ be arbitrary. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $a <_{X_+^0} b$ to be true, which gives us $a < b$, according to the Irreflexive partial ordering of subsets. Then, the Monotony Law for $+$ and $<$ yields $a + (-a) < b + (-a)$ and therefore evidently $0_X < b + (-a)$. This inequality in turn implies $0_X \leq b + (-a)$ with the Characterization of induced reflexive partial orderings; thus, $b + (-a) \in X_+^0$ holds by definition of X_+^0. Let us observe also that $0_X < b + (-a)$ implies $b + (-a) \neq 0_X$ with the Characterization of comparability with respect to the linear ordering $<$. Moreover, we obtain $a + [b + (-a)] = b$ with the commutativity & associativity of $+$. Here, a and $b + (-a)$ are both elements of X_+^0, so that $a + [b + (-a)] = a +_{X_+^0} [b + (-a)]$ holds according to (5.4). Combining the previous two equations gives us now $a +_{X_+^0} [b + (-a)] = b$, which demonstrates in light of the previous findings that there exists a an element $d \in X_+^0$ satisfying $d \neq 0_X$ and $a +_{X_+^0} d = b$. Thus, the first part of the equivalence in (6.257) holds.

Regarding the second part ('$\Leftarrow$'), we conversely assume that there is a nonzero element of X_+^0, say $\bar{d}$, such that $a +_{X_+^0} \bar{d} = b$. Here, $\bar{d} \in X_+^0$ evidently implies $0_X \leq \bar{d}$; in conjunction with $\bar{d} \neq 0_X$, this gives us then $0_X < \bar{d}$ (applying the Characterization of induced reflexive partial orderings). Consequently, we obtain $[0_X \leq]\ a < a + \bar{d}$ using the Monotony Law for $+$ and $<$ (and the fact that the initial assumption $a \in X_+^0$ implies $0_X \leq a$). These inequalities further imply $0_X < a + \bar{d}$ with the Transitivity Formula for $\leq$ and $<$, so that $0_X \leq a + \bar{d}$ is evidently also true; we thus have $a + \bar{d} \in X_+^0$. Recalling the truth of $0_X \in X_+^0$, we can therefore write the previously established inequality $a < a + \bar{d}$ as $a <_{X_+^0} a + \bar{d}$. Here, we can also write $a + \bar{d} = a +_{X_+^0} \bar{d}$ due to the fact that $a, \bar{d} \in X_+^0$, where $a +_{X_+^0} \bar{d} = b$ holds by assumption. We thus find the desired inequality $a <_{X_+^0} b$, which completes the proof of the equivalence. Since a and b were arbitrary, we may therefore conclude that $<_{X_+^0}$ satisfies indeed (6.257).

In summary, we demonstrated that (6.256) has all of the properties of an ordered elementary domain. As $(X, +, \cdot, -, <)$ was initially arbitrary, we may therefore conclude that the stated theorem holds indeed. □

Exercise 6.37. Complete the missing part in the proof of Theorem 6.75 that $+_{X_+^0}$ is a binary operation on X_+^0.

(Hint: Use the Additivity of $\leq$-inequalities.)

Note 6.21. According to the irreflexive orderings of subsets, the linear ordering $<_{X_+^0}$ satisfies

$$\forall a, b\, (a, b \in X_+^0 \Rightarrow [a <_{X_+^0} b \Leftrightarrow a <_X b]) \tag{6.258}$$

and induces the total ordering $\leq_{X_+^0}$.

Corollary 6.76. *For any ordered integral domain $(X, +, \cdot, -, <)$, it is true that the total ordering $\leq_{X_+^0}$ induced by the linear ordering $<_{X_+^0}$ of the induced ordered elementary domain $(X_+^0, +_{X_+^0}, \cdot_{X_+^0}, <_{X_+^0})$ satisfies*

$$\forall a, b\, (a, b \in X_+^0 \Rightarrow [a \leq_{X_+^0} b \Leftrightarrow a \leq_X b]). \tag{6.259}$$

Proof. For an arbitrary ordered integral domain $(X, +, \cdot, -, <)$ and arbitrary $a, b \in X_+^0$, we find the equivalences

$$\begin{aligned} a \leq_{X_+^0} b &\Leftrightarrow [a <_{X_+^0} b \vee a = b] \\ &\Leftrightarrow [a <_X b \vee a = b] \\ &\Leftrightarrow a \leq_X b \end{aligned}$$

by means of the Characterization of induced reflexive partial orderings and the equivalence in (6.258). □

6.7. The Ordered Integral Domain $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}}, <_{\mathbb{Z}})$

Exercise 6.38. Show that $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}})$ is a nontrivial ring.

(Hint: Proceed in analogy to the proof of Corollary 5.56, using the findings of Corollary 6.40.)

Theorem 6.77 (Zero-divisor freeness of $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}})$). *The semiring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}})$ is zero-divisor free.*

Proof. Recalling from Exercise 6.22 that $[(0,0)]_{\sim_d}$ is the zero element in $\mathbb{Z}$, we apply the Criterion for zero-divisor freeness (5.175) and prove accordingly

$$\begin{aligned}&\forall m,n\,(m,n \in \mathbb{Z} \Rightarrow [(m \cdot_{\mathbb{Z}} n = [(0,0)]_{\sim_d} \lor n \cdot_{\mathbb{Z}} m = [(0,0)]_{\sim_d})\\ &\quad \Rightarrow (m = [(0,0)]_{\sim_d} \lor n = [(0,0)]_{\sim_d})]).\end{aligned} \tag{6.260}$$

We let $m,n \in \mathbb{Z}$ be arbitrary and observe that $n \in \mathbb{N} \times \mathbb{N}/\sim_d$ is then true in particular. This means by definition of a quotient set that there exists a particular element $\bar{z} \in \mathbb{N} \times \mathbb{N}$ for which $[\bar{z}]_{\sim_d} = n$ holds. Thus, $\bar{z}$ can be written as the ordered pair $(\bar{p}, \bar{q})$ where $\bar{p}$ and $\bar{q}$ are particular natural numbers, and this allows us to express n in the form $n = [(\bar{p}, \bar{q})]_{\sim_d}$. Next, we assume the disjunction

$$m \cdot_{\mathbb{Z}} n = [(0,0)]_{\sim_d} \lor n \cdot_{\mathbb{Z}} m = [(0,0)]_{\sim_d} \tag{6.261}$$

to be true, and we use this disjunction to prove in the following the desired consequent

$$m = [(0,0)]_{\sim_d} \lor n = [(0,0)]_{\sim_d} \tag{6.262}$$

by cases. In the first case $m \cdot_{\mathbb{Z}} n = [(0,0)]_{\sim_d}$, we observe in light of the Law of the Excluded Middle that the disjunction

$$m = [(0,0)]_{\sim_d} \lor m \neq [(0,0)]_{\sim_d}$$

holds and consider accordingly the two further subcases for the proof of (6.262). In the first subcase $m = [(0,0)]_{\sim_d}$, the desired disjunction (6.262) is also true. The second subcase $m \neq [(0,0)]_{\sim_d}$ implies because of (6.131) that there exists a particular natural number $\bar{d} \neq 0$, such that the disjunction

$$m = [(0,\bar{d})]_{\sim_d} \lor m = [(\bar{d},0)]_{\sim_d}$$

holds. We use this disjunction within the current subcase to prove the equation $\bar{p} = \bar{q}$ by cases.

On the one hand, we obtain in case of $m = [(0, \bar{d})]_{\sim_d}$

$$\begin{aligned}
[(0,0)]_{\sim_d} &= m \cdot_\mathbb{Z} n \\
&= [(0, \bar{d})]_{\sim_d} \cdot_\mathbb{Z} [(\bar{p}, \bar{q})]_{\sim_d} \\
&= [(0 \cdot \bar{p} + \bar{d} \cdot \bar{q}, 0 \cdot \bar{q} + \bar{d} \cdot \bar{p})]_{\sim_d} \\
&= [(\bar{d} \cdot \bar{q}, \bar{d} \cdot \bar{p})]_{\sim_d}
\end{aligned}$$

using the current case assumptions, the previously obtained expression for n, (6.146) and the Cancellation Law for 0. Therefore, the Equality Criterion for equivalence classes yields $(0,0) \sim_d (\bar{d} \cdot \bar{q}, \bar{d} \cdot \bar{p})$, so that the equation

$$0 +_\mathbb{N} (\bar{d} \cdot \bar{p}) = (\bar{d} \cdot \bar{q}) +_\mathbb{N} 0$$

holds according to (6.121). By definition of the zero element, this equation simplifies to

$$\bar{d} \cdot \bar{p} = \bar{d} \cdot \bar{q}.$$

Recalling now the truth of $\bar{d} \neq 0$, the Cancellation Law for $\cdot_\mathbb{N}$ gives us now the desired equation $\bar{p} = \bar{p}$.

On the other hand, if $m = [(\bar{d}, 0)]_{\sim_d}$ is true, then we may use the same arguments as before and write

$$\begin{aligned}
[(0,0)]_{\sim_d} &= m \cdot_\mathbb{Z} n \\
&= [(\bar{d}, 0)]_{\sim_d} \cdot_\mathbb{Z} [(\bar{p}, \bar{q})]_{\sim_d} \\
&= [(\bar{d} \cdot \bar{p} + 0 \cdot \bar{q}, \bar{d} \cdot \bar{q} + 0 \cdot \bar{p})]_{\sim_d} \\
&= [(\bar{d} \cdot \bar{p}, \bar{d} \cdot \bar{q})]_{\sim_d},
\end{aligned}$$

so that we obtain this time the equivalence $(0,0) \sim_d (\bar{d} \cdot \bar{p}, \bar{d} \cdot \bar{q})$. Consequently, the equation

$$0 +_\mathbb{N} (\bar{d} \cdot \bar{q}) = (\bar{d} \cdot \bar{p}) +_\mathbb{N} 0$$

is true, which simplifies first to

$$\bar{d} \cdot \bar{q} = \bar{d} \cdot \bar{p}$$

and subsequently to $\bar{q} = \bar{p}$. We therefore find $\bar{p} = \bar{q}$ to be true for both cases, and substitution based on this equation gives us now with (6.130)

$$n = [(\bar{p}, \bar{q})]_{\sim_d} = [(\bar{q}, \bar{q})]_{\sim_d} = [(0,0)]_{\sim_d}.$$

Then, the resulting equation $n = [(0,0)]_{\sim_d}$ implies the truth of the desired disjunction (6.262), so that the proof with respect to the second subcase, and thus the proof of the first case $m \cdot_\mathbb{Z} n = [(0,0)]_{\sim_d}$ is now complete.

In the second case $n \cdot_{\mathbb{Z}} m = [(0,0)]_{\sim_d}$, we observe that the Commutative Law for the multiplication on $\mathbb{Z}$ yields the equation $m \cdot_{\mathbb{Z}} n = n \cdot_{\mathbb{Z}} m$. Then, substitution leads to $m \cdot_{\mathbb{Z}} n = [(0,0)]_{\sim_d}$, which equation implies (6.262) as shown in the proof of the first case. We thus completed the proof by cases, and since m and n were initially arbitrary, we may therefore finally conclude that the universal sentence (6.260) holds. This sentence implies then that the semiring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}})$ is zero-divisor free. □

Proposition 6.78. *The following sentences are true.*

a) There exists a unique set $\leq_{\mathbb{Z}}$ *such that*

$$\begin{aligned}&\forall X\,(X \in \,\leq_{\mathbb{Z}} \Leftrightarrow [X \in \mathbb{Z} \times \mathbb{Z} \\ &\quad \wedge\ \exists m,n,p,q\,(X = ([(m,n)]_{\sim_d}, [(p,q)]_{\sim_d}) \ \wedge\ m +_{\mathbb{N}} q \leq_{\mathbb{N}} n +_{\mathbb{N}} p)]).\end{aligned} \tag{6.263}$$

This set $\leq_{\mathbb{Z}}$ *is a binary relation on* $\mathbb{Z}$ *satisfying*

$$\begin{aligned}&\forall X\,(X \in \,\leq_{\mathbb{Z}} \\ &\quad \Leftrightarrow \exists m,n,p,q\,(X = ([(m,n)]_{\sim_d}, [(p,q)]_{\sim_d}) \ \wedge\ m +_{\mathbb{N}} q \leq_{\mathbb{N}} n +_{\mathbb{N}} p))\end{aligned} \tag{6.264}$$

and

$$\forall m,n,p,q\,([(m,n)]_{\sim_d} \leq_{\mathbb{Z}} [(p,q)]_{\sim_d} \Leftrightarrow m +_{\mathbb{N}} q \leq_{\mathbb{N}} n +_{\mathbb{N}} p). \tag{6.265}$$

b) Furthermore, the binary relation $\leq_{\mathbb{Z}}$ *is a total ordering of* $\mathbb{Z}$.

Proof. Concerning a), we observe in light of the Axiom of Specification and the Equality Criterion for sets that the stated uniquely existential sentence holds. In addition, we note that $X \in \,\leq_{\mathbb{Z}}$ implies with (6.263) especially $X \in \mathbb{Z} \times \mathbb{Z}$ for any X, so that $\leq_{\mathbb{Z}}$ is by definition a subset of $\mathbb{Z} \times \mathbb{Z}$ and thus by definition a binary relation on $\mathbb{Z}$. To establish (6.264), we take an arbitrary set X. On the one hand, assuming $X \in \,\leq_{\mathbb{Z}}$ to be true implies with (6.263) especially the existential sentence in (6.264), so that the first part ('$\Rightarrow$') of the equivalence in (6.264) holds. On the other hand, assuming that existential sentence to be true yields particular constants $\bar{m}$, $\bar{n}$, $\bar{p}$, $\bar{q}$ such that $X = ([(\bar{m},\bar{n})]_{\sim_d}, [(\bar{p},\bar{q})]_{\sim_d})$ and $\bar{m} +_{\mathbb{N}} \bar{q} \leq_{\mathbb{N}} \bar{n} +_{\mathbb{N}} \bar{p}$ are satisfied. Recalling that $\sim_d$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$, we thus see that the ordered pairs $(\bar{m},\bar{n})$ and $(\bar{p},\bar{q})$ are elements of the Cartesian product $\mathbb{N} \times \mathbb{N}$ and that the equivalence classes $[(\bar{m},\bar{n})]_{\sim_d}$ and $[(\bar{p},\bar{q})]_{\sim_d}$ are in the quotient set $\mathbb{N} \times \mathbb{N}/\sim_d$ $[= \mathbb{Z}]$, i.e. elements of $\mathbb{Z}$. Consequently, the ordered pair X formed by these equivalence classes is contained in the Cartesian product $\mathbb{Z} \times \mathbb{Z}$. This finding $X \in \mathbb{Z} \times \mathbb{Z}$ implies now in conjunction with the assumed existential sentence that $X \in \,\leq_{\mathbb{Z}}$ is true, according to (6.263). This proves the second part ('$\Leftarrow$') of the equivalence in (6.264), in which X is arbitrary, so that the universal sentence (6.264) follows to be true.

We now prove (6.265), letting $\bar{m}$, $\bar{n}$, $\bar{p}$ and $\bar{q}$ be arbitrary and assuming first that the inequality $[(\bar{m},\bar{n})]_{\sim_d} \leq_{\mathbb{Z}} [(\bar{p},\bar{q})]_{\sim_d}$ is true, which we may write also in the form $([(\bar{m},\bar{n})]_{\sim_d}, [(\bar{p},\bar{q})]_{\sim_d}) \in \leq_{\mathbb{Z}}$. The latter implies because of (6.264) that there exist constants, say $\bar{M}$, $\bar{N}$, $\bar{P}$ and $\bar{Q}$, such that equation

$$([(\bar{m},\bar{n})]_{\sim_d}, [(\bar{p},\bar{q})]_{\sim_d}) = ([(\bar{M},\bar{N})]_{\sim_d}, [(\bar{P},\bar{Q})]_{\sim_d})$$

and the inequality

$$\bar{M} +_{\mathbb{N}} \bar{Q} \ \leq_{\mathbb{N}} \ \bar{N} +_{\mathbb{N}} \bar{P} \tag{6.266}$$

are satisfied. The former equation implies with the Equality Criterion for ordered pairs the truth of the two equations

$$[(\bar{m},\bar{n})]_{\sim_d} = [(\bar{M},\bar{N})]_{\sim_d},$$
$$[(\bar{p},\bar{q})]_{\sim_d} = [(\bar{P},\bar{Q})]_{\sim_d},$$

which in turn imply with the Equality Criterion for equivalence classes

$$(\bar{m},\bar{n}) \sim_d (\bar{M},\bar{N}),$$
$$(\bar{p},\bar{q}) \sim_d (\bar{P},\bar{Q}).$$

According to the characterization of the equivalence relation $\sim_d$ in (6.121), the equations

$$\bar{m} +_{\mathbb{N}} \bar{N} = \bar{M} +_{\mathbb{N}} \bar{n} \tag{6.267}$$
$$\bar{p} +_{\mathbb{N}} \bar{Q} = \bar{P} +_{\mathbb{N}} \bar{q} \tag{6.268}$$

are therefore true. Let us observe now that (6.266) implies with the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$

$$(\bar{M} +_{\mathbb{N}} \bar{Q}) +_{\mathbb{N}} (\bar{n} +_{\mathbb{N}} \bar{p}) \ \leq_{\mathbb{N}} \ (\bar{N} +_{\mathbb{N}} \bar{P}) +_{\mathbb{N}} (\bar{n} +_{\mathbb{N}} \bar{p}),$$

in which we rearrange the terms on the left-hand side by means of the Associative and the Commutative Law for the addition on $\mathbb{N}$ and write

$$(\bar{M} +_{\mathbb{N}} \bar{n}) +_{\mathbb{N}} (\bar{p} +_{\mathbb{N}} \bar{Q}) \ \leq_{\mathbb{N}} \ (\bar{N} +_{\mathbb{N}} \bar{P}) +_{\mathbb{N}} (\bar{n} +_{\mathbb{N}} \bar{p}).$$

Applying now substitutions based on (6.267) – (6.268), we obtain

$$(\bar{m} +_{\mathbb{N}} \bar{N}) +_{\mathbb{N}} (\bar{P} +_{\mathbb{N}} \bar{q}) \ \leq_{\mathbb{N}} \ (\bar{N} +_{\mathbb{N}} \bar{P}) +_{\mathbb{N}} (\bar{n} +_{\mathbb{N}} \bar{p}).$$

Further applications of the Associative and the Commutative Law for the addition on $\mathbb{N}$ give us then

$$(\bar{m} +_{\mathbb{N}} \bar{q}) +_{\mathbb{N}} (\bar{P} +_{\mathbb{N}} \bar{N}) \ \leq_{\mathbb{N}} \ (\bar{n} +_{\mathbb{N}} \bar{p}) +_{\mathbb{N}} (\bar{P} +_{\mathbb{N}} \bar{N}).$$

Using now the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$, we arrive at the inequality

$$\bar{m} +_{\mathbb{N}} \bar{q} \ \leq_{\mathbb{N}} \ \bar{n} +_{\mathbb{N}} \bar{p}, \tag{6.269}$$

which constitutes the desired consequent of the implication '$\Leftarrow$' in (6.265). Having thus established the first part of the equivalence to be proven, we now conversely assume the preceding inequality to be true. Thus, $\bar{m}$, $\bar{n}$, $\bar{p}$ and $\bar{q}$ are evidently natural numbers, so that the the ordered pairs $(\bar{m}, \bar{n})$ and $(\bar{p}, \bar{q})$ are elements of the Cartesian product $\mathbb{N} \times \mathbb{N}$. Then, the equivalence classes $[(\bar{m}, \bar{n})]_{\sim_d}$ and $[(\bar{p}, \bar{q})]_{\sim_d}$ with respect to the equivalence relation $\sim_d$ on $\mathbb{N} \times \mathbb{N}$ are also defined. Forming the ordered pair

$$\bar{X} = ([(\bar{m}, \bar{n})]_{\sim_d}, [(\bar{p}, \bar{q})]_{\sim_d}), \tag{6.270}$$

we see in light of the assumed inequality (6.269) that the existential sentence

$$\exists m, n, p, q \, (\bar{X} = ([(m, n)]_{\sim_d}, [(p, q)]_{\sim_d}) \ \wedge \ m +_{\mathbb{N}} q \leq_{\mathbb{N}} n +_{\mathbb{N}} p)$$

holds, so that X turns out to be an element of $\leq_{\mathbb{Z}}$ in view of (6.264). Since we established $\leq_{\mathbb{Z}}$ already as a binary relation, we may write $\bar{X} \in \leq_{\mathbb{Z}}$ because of (6.270) also as

$$[(\bar{m}, \bar{n})]_{\sim_d} \ \leq_{\mathbb{Z}} \ [(\bar{p}, \bar{q})]_{\sim_d}.$$

This finding proves the implication '$\Leftarrow$' in (6.265), so that the equivalence in (6.265) holds. As $\bar{m}$, $\bar{n}$, $\bar{p}$ and $\bar{q}$ were arbitrary, we may therefore conclude that $\leq_{\mathbb{Z}}$ satisfies (6.265).

Concerning b), we prove first that the binary relation $\leq_{\mathbb{Z}}$ is reflexive, i.e.

$$\forall m \, (m \in \mathbb{Z} \Rightarrow m \leq_{\mathbb{Z}} m). \tag{6.271}$$

Letting m be an arbitrary integer, m is thus in the quotient set $\mathbb{N} \times \mathbb{N} / \sim_d$, so that there is a particular element $\bar{z}$ of the Cartesian product $\mathbb{N} \times \mathbb{N}$ whose equivalence class $[\bar{z}]_{\sim_d}$ equals m. Here, $\bar{z}$ can be written as the ordered pair $(\bar{p}, \bar{q})$ for some particular natural numbers $\bar{p}$ and $\bar{q}$, so that the integer/equivalence class reads

$$m = [(\bar{p}, \bar{q})]_{\sim_d}. \tag{6.272}$$

Now, the Commutative Law for the addition on $\mathbb{N}$ yields the true equation $\bar{p} +_{\mathbb{N}} \bar{q} = \bar{q} +_{\mathbb{N}} \bar{p}$, and the reflexivity of the standard total ordering $\leq_{\mathbb{N}}$ gives the inequality $\bar{p} +_{\mathbb{N}} \bar{q} \leq_{\mathbb{N}} \bar{p} +_{\mathbb{N}} \bar{q}$. Consequently, substitution yields

$$\bar{p} +_{\mathbb{N}} \bar{q} \ \leq_{\mathbb{N}} \ \bar{q} +_{\mathbb{N}} \bar{p},$$

which inequality implies then

$$[(\bar{p}, \bar{q})]_{\sim_d} \ \leq_{\mathbb{Z}} \ [(\bar{p}, \bar{q})]_{\sim_d}$$

with (6.265). This means in view of (6.272) that the desired consequent $m \leq_{\mathbb{Z}} m$ of the implication in (6.271) holds. Since m was arbitrary, we may therefore conclude that the universal sentence (6.271) is true, so that $\leq_{\mathbb{Z}}$ is a reflexive binary relation on $\mathbb{Z}$ by definition.

Next, we prove that $\leq_{\mathbb{Z}}$ is also antisymmetric, that is,

$$\forall m, n\, (m, n \in \mathbb{Z} \Rightarrow [(m \leq_{\mathbb{Z}} n \wedge n \leq_{\mathbb{Z}} m) \Rightarrow m = n]). \tag{6.273}$$

We let m and n be arbitrary in $\mathbb{Z}$ $[= \mathbb{N} \times \mathbb{N}/ \sim_d]$, which integers thus constitute the equivalence classes $[\bar{e}]_{\sim_d} = m$ and $[\bar{f}]_{\sim_d} = n$ for some particular elements $\bar{e}, \bar{f} \in \mathbb{N} \times \mathbb{N}$. These elements in turn constitute the ordered pairs $(\bar{p}, \bar{q}) = \bar{e}$ and $(\bar{r}, \bar{s}) = \bar{f}$ for some particular elements $\bar{p}, \bar{q}, \bar{r}, \bar{s} \in \mathbb{N}$. We may therefore write the integers in the form $m = [(\bar{p}, \bar{q})]_{\sim_d}$ and $n = [(\bar{r}, \bar{s})]_{\sim_d}$, by carrying out substitutions. Next, we assume the conjunction in (6.273) to be true, and we show that $m = n$ is implied. Due to the preceding equations for m and n, we may write the desired consequent $m = n$ equivalently as

$$[(\bar{p}, \bar{q})]_{\sim_d} = [(\bar{r}, \bar{s})]_{\sim_d}, \tag{6.274}$$

and the assumed conjunction $m \leq_{\mathbb{Z}} n \wedge n \leq_{\mathbb{Z}} m$ implies

$$[(\bar{p}, \bar{q})]_{\sim_d} \leq_{\mathbb{Z}} [(\bar{r}, \bar{s})]_{\sim_d} \ \wedge \ [(\bar{r}, \bar{s})]_{\sim_d} \leq_{\mathbb{Z}} [(\bar{p}, \bar{q})]_{\sim_d}.$$

In view of the equivalence in (6.265), this conjunction further implies

$$\bar{p} +_{\mathbb{N}} \bar{s} \leq_{\mathbb{N}} \bar{q} +_{\mathbb{N}} \bar{r} \ \wedge \ \bar{r} +_{\mathbb{N}} \bar{q} \leq_{\mathbb{N}} \bar{s} +_{\mathbb{N}} \bar{p}.$$

Observing now the truth of the equations

$$\bar{s} +_{\mathbb{N}} \bar{p} = \bar{p} +_{\mathbb{N}} \bar{s}$$
$$\bar{q} +_{\mathbb{N}} \bar{r} = \bar{r} +_{\mathbb{N}} \bar{q}$$

in light of the Commutative Law for the addition on $\mathbb{N}$, we obtain by means of substitutions

$$\bar{p} +_{\mathbb{N}} \bar{s} \leq_{\mathbb{N}} \bar{r} +_{\mathbb{N}} \bar{q} \ \wedge \ \bar{r} +_{\mathbb{N}} \bar{q} \leq_{\mathbb{N}} \bar{p} +_{\mathbb{N}} \bar{s}.$$

Because the standard total ordering $\leq_{\mathbb{N}}$ is antisymmetric, this conjunction implies

$$\bar{p} +_{\mathbb{N}} \bar{s} = \bar{r} +_{\mathbb{N}} \bar{q},$$

which shows that the equivalence $(\bar{p},\bar{p}) \sim_d (\bar{r},\bar{s})$ holds, according to (6.121). Consequently, the Equality Criterion for equivalence classes yields (6.274) and thus the desired equation $m = n$. This finding completes the proof of the implications in (6.273), where m and n are arbitrary, so that the universal sentence (6.273) follows now to be true. This means by definition that $\leq_{\mathbb{Z}}$ is an antisymmetric binary relation. To demonstrate the transitivity of $\leq_{\mathbb{Z}}$, we establish

$$\forall m,n,p\,(m,n,p \in \mathbb{Z} \Rightarrow [(m \leq_{\mathbb{Z}} n \,\wedge\, n \leq_{\mathbb{Z}} p) \Rightarrow m \leq_{\mathbb{Z}} p]), \tag{6.275}$$

taking arbitrary constants m, n and p such that $m,n,p \in \mathbb{Z}$ and the conjunction of $m \leq_{\mathbb{Z}} n$ and $n \leq_{\mathbb{Z}} p$ are true. The three integers, as elements of the quotient set $\mathbb{N} \times \mathbb{N}/ \sim_d$, constitute equivalence classes $m = [\bar{e}]_{\sim_d}$, $n = [\bar{f}]_{\sim_d}$ and $p = [\bar{g}]_{\sim_d}$, where $\bar{e}$, $\bar{f}$ and $\bar{g}$ are particular elements of the Cartesian product $\mathbb{N} \times \mathbb{N}$. Therefore, we may write for these elements $\bar{e} = (a,b)$, $\bar{f} = (c,d)$ and $\bar{g} = (e,f)$, which ordered pairs are formed by particular natural numbers. Substitutions allow us now to write for the selected integers $m = [(a,b)]_{\sim_d}$, $n = [(c,d)]_{\sim_d}$ and $p = [(e,f)]_{\sim_d}$, then for the assumed antecedent

$$[(a,b)]_{\sim_d} \leq_{\mathbb{Z}} [(c,d)]_{\sim_d} \,\wedge\, [(c,d)]_{\sim_d} \leq_{\mathbb{Z}} [(e,f)]_{\sim_d},$$

and furthermore for the desired consequent

$$[(a,b)]_{\sim_d} \leq_{\mathbb{Z}} [(e,f)]_{\sim_d}.$$

Let us use here the equivalence in (6.265) to rewrite the consequent equivalently as

$$a +_{\mathbb{N}} f \;\leq_{\mathbb{N}}\; b +_{\mathbb{N}} e \tag{6.276}$$

and the assumed antecedent in the form

$$\begin{aligned} a +_{\mathbb{N}} d \;&\leq_{\mathbb{N}}\; b +_{\mathbb{N}} c, \\ c +_{\mathbb{N}} f \;&\leq_{\mathbb{N}}\; d +_{\mathbb{N}} e. \end{aligned}$$

The latter two inequalities imply with the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$

$$\begin{aligned} (a +_{\mathbb{N}} d) +_{\mathbb{N}} f \;&\leq_{\mathbb{N}}\; (b +_{\mathbb{N}} c) +_{\mathbb{N}} f, \\ (c +_{\mathbb{N}} f) +_{\mathbb{N}} b \;&\leq_{\mathbb{N}}\; (d +_{\mathbb{N}} e) +_{\mathbb{N}} b. \end{aligned}$$

Due to the Associative and the Commutative Law for the addition on $\mathbb{N}$, we can rewrite these two inequalities as

$$\begin{aligned} (a +_{\mathbb{N}} f) +_{\mathbb{N}} d \;&\leq_{\mathbb{N}}\; (b +_{\mathbb{N}} c) +_{\mathbb{N}} f, \\ (b +_{\mathbb{N}} c) +_{\mathbb{N}} f \;&\leq_{\mathbb{N}}\; (b +_{\mathbb{N}} e) +_{\mathbb{N}} d, \end{aligned}$$

whose conjunction implies now with the transitivity of the standard total ordering $\leq_{\mathbb{N}}$

$$(a +_{\mathbb{N}} f) +_{\mathbb{N}} d \ \leq_{\mathbb{N}} \ (b +_{\mathbb{N}} e) +_{\mathbb{N}} d.$$

This inequality implies now (6.276) with the Monotony Law for $+_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$, so that the consequent in the equivalent forms $[(a, b)]_{\sim_d} \leq_{\mathbb{Z}} [(e, f)]_{\sim_d}$ and $m \leq_{\mathbb{Z}} p$ are true. We may therefore conclude that $\leq_{\mathbb{Z}}$ satisfies the universal sentence (6.275) and constitutes thus a transitive binary relation on $\mathbb{Z}$. In summary, $\leq_{\mathbb{Z}}$ is a reflexive partial ordering of $\mathbb{Z}$.

Finally, we prove that the reflexive partial ordering $\leq_{\mathbb{Z}}$ is total, that is,

$$\forall m, n \, (m, n \in \mathbb{Z} \Rightarrow [m \leq_{\mathbb{Z}} n \ \vee \ n \leq_{\mathbb{Z}} m]). \tag{6.277}$$

We take arbitrary integers m and n, so that there evidently exist natural numbers, say $\bar{p}, \bar{q}$ and $\bar{r}, \bar{s}$, for which these integers can be written as the equivalence classes $m = [(\bar{p}, \bar{q})]_{\sim_d}$ and $n = [(\bar{r}, \bar{s})]_{\sim_d}$. We may therefore apply substitutions and write the desired disjunction in (6.273) equivalently as

$$[(\bar{p}, \bar{q})]_{\sim_d} \leq_{\mathbb{Z}} [(\bar{r}, \bar{s})]_{\sim_d} \ \vee \ [(\bar{r}, \bar{s})]_{\sim_d} \leq_{\mathbb{Z}} [(\bar{p}, \bar{q})]_{\sim_d}.$$

Due to the equivalence in (6.265), this disjunction is also equivalent to

$$\bar{p} +_{\mathbb{N}} \bar{s} \leq_{\mathbb{N}} \bar{q} +_{\mathbb{N}} \bar{r} \ \vee \ \bar{r} +_{\mathbb{N}} \bar{q} \leq_{\mathbb{N}} \bar{s} +_{\mathbb{N}} \bar{p}.$$

Here, we note in view of the Commutative Law for the addition on $\mathbb{N}$ that the preceding disjunction is equivalent to

$$\bar{s} +_{\mathbb{N}} \bar{p} \leq_{\mathbb{N}} \bar{r} +_{\mathbb{N}} \bar{q} \ \vee \ \bar{r} +_{\mathbb{N}} \bar{q} \leq_{\mathbb{N}} \bar{s} +_{\mathbb{N}} \bar{p}.$$

Since $\leq_{\mathbb{N}}$ is total, this disjunction is actually true, so that the equivalent disjunction $m \leq_{\mathbb{Z}} n \vee n \leq_{\mathbb{Z}} m$ holds as well. As m and n were arbitrary, we may now infer from this finding the truth of the universal sentence (6.277). Thus, $\leq_{\mathbb{Z}}$ constitutes a total ordering of $\mathbb{Z}$. □

Note 6.22. The total ordering $\leq_{\mathbb{Z}}$ of $\mathbb{Z}$ induces then the linear ordering $<_{\mathbb{Z}}$ of $\mathbb{Z}$.

Definition 6.23 (Standard total & linear ordering of $\mathbb{Z}$). We call $\leq_{\mathbb{Z}}$ the *standard total ordering of* $\mathbb{Z}$ and $<_{\mathbb{Z}}$ the *standard linear ordering of* $\mathbb{Z}$.

Note 6.23. Due to the inclusion $\mathbb{N}_{\mathbb{Z}} \subseteq \mathbb{Z}$ established in Exercise 6.18, the total ordering $\leq_{\mathbb{Z}}$ of the set of integers gives rise to the total ordering $\leq_{\mathbb{N}_{\mathbb{Z}}}$ of the set of natural numbers in $\mathbb{Z}$ (according to the Total ordering of subsets), which satisfies (according to the Reflexive partial ordering of subsets)

$$\forall m, n \, (m, n \in \mathbb{N}_{\mathbb{Z}} \Rightarrow [m \leq_{\mathbb{N}_{\mathbb{Z}}} n \Leftrightarrow m \leq_{\mathbb{Z}} n]). \tag{6.278}$$

Then, $\leq_{\mathbb{N}_{\mathbb{Z}}}$ in turn induces the linear ordering $<_{\mathbb{N}_{\mathbb{Z}}}$.

Corollary 6.79. *It is true that the induced linear ordering* $<_{\mathbb{N}_\mathbb{Z}}$ *satisfies*

$$\forall m, n\,(m, n \in \mathbb{N}_\mathbb{Z} \Rightarrow [m <_{\mathbb{N}_\mathbb{Z}} n \Leftrightarrow m <_\mathbb{Z} n]). \tag{6.279}$$

Proof. Letting m and n be arbitrary and assuming $m, n \in \mathbb{N}_\mathbb{Z}$ to be true, we obtain the true equivalences

$$\begin{aligned} m <_{\mathbb{N}_\mathbb{Z}} n \;&\Leftrightarrow\; m \leq_{\mathbb{N}_\mathbb{Z}} n \wedge m \neq n \\ &\Leftrightarrow\; m \leq_\mathbb{Z} n \wedge m \neq n \\ &\Leftrightarrow\; m <_\mathbb{Z} n \end{aligned}$$

by applying the Characterization of induced irreflexive partial orderings (with respect to $<_{\mathbb{N}_\mathbb{Z}}$), the equivalence in (6.278), and then again the Characterization of induced irreflexive partial orderings (now with respect to $<_\mathbb{Z}$). Since m and n were initially arbitrary, we may infer from these findings the truth of the stated corollary. □

Theorem 6.80 (Order-embedding from $(\mathbb{N}, \leq_\mathbb{N})$ **to** $(\mathbb{Z}, \leq_\mathbb{Z})$ **and order-isomorphism from** $(\mathbb{N}, \leq_\mathbb{N})$ **to** $(\mathbb{N}_\mathbb{Z}, \leq_{\mathbb{N}_\mathbb{Z}})$**).** *It is true that the function* $f_\mathbb{N}^\mathbb{Z}$ *defining the Identification of* $\mathbb{N}$ *in* $\mathbb{Z}$ *constitutes*

a) an order-embedding from $(\mathbb{N}, \leq_\mathbb{N})$ *to* $(\mathbb{Z}, \leq_\mathbb{Z})$*, that is,*

$$f_\mathbb{N}^\mathbb{Z} : (\mathbb{N}, \leq_\mathbb{N}) \hookrightarrow (\mathbb{Z}, \leq_\mathbb{Z}), \qquad n \mapsto [(n, 0)]_{\sim_d}. \tag{6.280}$$

b) an order-isomorphism from $(\mathbb{N}, \leq_\mathbb{N})$ *to* $(\mathbb{N}_\mathbb{Z}, \leq_{\mathbb{N}_\mathbb{Z}})$*, that is,*

$$f_\mathbb{N}^\mathbb{Z} : (\mathbb{N}, \leq_\mathbb{N}) \rightleftarrows (\mathbb{N}_\mathbb{Z}, \leq_{\mathbb{N}_\mathbb{Z}}), \qquad n \mapsto [(n, 0)]_{\sim_d}. \tag{6.281}$$

Proof. We first prove the universal sentence

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \leq_\mathbb{N} n \Leftrightarrow f_\mathbb{N}^\mathbb{Z}(m) \leq_\mathbb{Z} f_\mathbb{N}^\mathbb{Z}(n)]), \tag{6.282}$$

letting m and n be arbitrary natural numbers, so that the definition of $f_\mathbb{N}^\mathbb{Z}$ yields the corresponding values $f_\mathbb{N}^\mathbb{Z}(m) = [(m, 0)]_{\sim_d}$ and $f_\mathbb{N}^\mathbb{Z}(n) = [(n, 0)]_{\sim_d}$. We obtain then the true equivalences

$$\begin{aligned} f_\mathbb{N}^\mathbb{Z}(m) \leq_\mathbb{Z} f_\mathbb{N}^\mathbb{Z}(n) \;&\Leftrightarrow\; [(m, 0)]_{\sim_d} \leq_\mathbb{Z} [(n, 0)]_{\sim_d} \\ &\Leftrightarrow\; m +_\mathbb{N} 0 \leq_\mathbb{N} 0 +_\mathbb{N} n \\ &\Leftrightarrow\; m \leq_\mathbb{N} n \end{aligned}$$

by applying substitution, then the equivalence in (6.265), and finally the definition of the zero element. These equivalences prove the implication in (6.282), with the consequence that the universal sentence (6.282) is true, since m and n were arbitrary. Thus, $f_\mathbb{N}^\mathbb{Z}$ is indeed an order-embedding from $(\mathbb{N}, \leq_\mathbb{N})$ to $(\mathbb{Z}, \leq_\mathbb{Z})$.

Since $f_\mathbb{N}^\mathbb{Z}$ is a bijection from $\mathbb{N}$ to $\mathbb{N}_\mathbb{Z}$ (see Note 6.14), it is by definition a surjection. Next, we prove that $f_\mathbb{N}^\mathbb{Z}$ is an order-embedding from $(\mathbb{N}, \leq_\mathbb{N})$ to $(\mathbb{N}_\mathbb{Z}, \leq_{\mathbb{N}_\mathbb{Z}})$, by verifying

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m \leq_\mathbb{N} n \Leftrightarrow f_\mathbb{N}^\mathbb{Z}(m) \leq_{\mathbb{N}_\mathbb{Z}} f_\mathbb{N}^\mathbb{Z}(n)]). \tag{6.283}$$

We let m and n be arbitrary natural numbers and observe that the corresponding values $f_\mathbb{N}^\mathbb{Z}(m)$ and $f_\mathbb{N}^\mathbb{Z}(n)$ are elements of the range $\mathbb{N}_\mathbb{Z}$ of the bijection/surjection $f_\mathbb{N}^\mathbb{Z} : \mathbb{N} \rightleftarrows \mathbb{N}_\mathbb{Z}$. We obtain the true equivalences

$$\begin{aligned} m \leq_\mathbb{N} n \;&\Leftrightarrow\; f_\mathbb{N}^\mathbb{Z}(m) \leq_\mathbb{Z} f_\mathbb{N}^\mathbb{Z}(n) \\ &\Leftrightarrow\; f_\mathbb{N}^\mathbb{Z}(m) \leq_{\mathbb{N}_\mathbb{Z}} f_\mathbb{N}^\mathbb{Z}(n) \end{aligned}$$

with (6.282) and (3.204) – according to the Reflexive partial ordering of subsets. Therefore, the equivalence in (6.283) follows to be true, and because m and n were initially arbitrary, we may infer from this the truth of the universal sentence (6.283). We thus proved that $f_\mathbb{N}^\mathbb{Z}$ is a surjective order-embedding from $(\mathbb{N}, \leq_\mathbb{N})$ to $(\mathbb{N}_\mathbb{Z}, \leq_{\mathbb{N}_\mathbb{Z}})$, and this is an order-isomorphism by definition. □

Exercise 6.39. Verify

$$\forall m, n\,(m, n \in \mathbb{N} \Rightarrow [m <_\mathbb{N} n \Leftrightarrow f_\mathbb{N}^\mathbb{Z}(m) <_\mathbb{Z} f_\mathbb{N}^\mathbb{Z}(n)]). \tag{6.284}$$

(Hint: Use the Characterization of induced irreflexive partial orderings, (6.282), the Injection Criterion, and Proposition 3.149.)

Note 6.24. The fact $0 <_\mathbb{N} 1$ from (4.164) implies

$$f_\mathbb{N}^\mathbb{Z}(0) <_\mathbb{Z} f_\mathbb{N}^\mathbb{Z}(1), \tag{6.285}$$

and therefore

$$[(0,0)]_{\sim_d} <_\mathbb{Z} [(1,0)]_{\sim_d} \tag{6.286}$$

with the Identification of $\mathbb{N}$ in $\mathbb{Z}$.

Proposition 6.81. *Any integer greater than* $[(0,0)]_{\sim_d}$ *can be written as the equivalence class of an ordered pair whose second component is* 0*, i.e.*

$$\forall m\,([m \in \mathbb{Z} \wedge [(0,0)]_{\sim_d} <_\mathbb{Z} m] \Rightarrow \exists n\,(0 <_\mathbb{N} n \wedge m = [(n,0)]_{\sim_d})). \tag{6.287}$$

Proof. We take an arbitrary integer and assume $[(0,0)]_{\sim_d} <_\mathbb{Z} m$ to be true. Let us observe now that the linear ordering $<_\mathbb{Z}$ satisfies the Characterization of comparability, so that $m = [(0,0)]_{\sim_d}$ must be false. Then, $m \neq [(0,0)]_{\sim_d}$ implies with 6.131 that there exists a particular natural number $\bar{n} \neq 0$ such that the disjunction

$$m = [(0,\bar{n})]_{\sim_d} \vee m = [(\bar{n},0)]_{\sim_d} \tag{6.288}$$

is satisfied. We prove now the negation $m \neq [(0,\bar{n})]_{\sim_d}$ by contradiction, assuming the negation of that negation to be true, so that the equation $m = [(0,\bar{n})]_{\sim_d}$ follows to be true with the Double Negation Law. We therefore obtain from the initial assumption via substitution

$$[(0,0)]_{\sim_d} <_{\mathbb{Z}} [(0,\bar{n})]_{\sim_d},$$

which implies with the definition of an induced irreflexive partial ordering

$$[(0,0)]_{\sim_d} \leq_{\mathbb{Z}} [(0,\bar{n})]_{\sim_d}.$$

According to (6.265), the equation

$$0 +_{\mathbb{N}} \bar{n} \leq_{\mathbb{N}} 0 +_{\mathbb{N}} 0$$

is then also true, which we may simplify to $\bar{n} \leq_{\mathbb{N}} 0$ by using the definition of the zero element. We now have also that the previous findings $\bar{n} \in \mathbb{N}$ and $\bar{n} \neq 0$ imply $0 <_{\mathbb{N}} \bar{n}$ with (5.328) and consequently $\neg \bar{n} \leq_{\mathbb{N}} 0$ with the Negation Formula for $<$. We thus obtained a contradiction, so that the proof of $m \neq [(0,\bar{n})]_{\sim_d}$ is complete. This means that the first part of the true disjunction (6.288) is false, and therefore its second part $m = [(\bar{n},0)]_{\sim_d}$ must be true. In conjunction with $0 <_{\mathbb{N}} \bar{n}$, this demonstrates the truth of the existential sentence in (6.287), and since m was initially arbitrary, we may now infer from this the truth of the proposition. □

Corollary 6.82. *It is true that any integer greater than or equal to* $[(0,0)]_{\sim_d}$ *is a natural number in* $\mathbb{Z}$*, that is,*

$$\forall m\,([m \in \mathbb{Z} \wedge [(0,0)]_{\sim_d} \leq_{\mathbb{Z}} m] \Rightarrow m \in \mathbb{N}_{\mathbb{Z}}). \tag{6.289}$$

Proof. We take an arbitrary integer m and assume $[(0,0)]_{\sim_d} \leq_{\mathbb{Z}} m$, which implies with the Characterization of an induced irreflexive partial ordering the disjunction

$$[(0,0)]_{\sim_d} <_{\mathbb{Z}} m \vee [(0,0)]_{\sim_d} = m.$$

We now prove $m \in \mathbb{N}_{\mathbb{Z}}$ by cases, based on the preceding disjunction. The first case $[(0,0)]_{\sim_d} <_{\mathbb{Z}} m$ implies that m can be written as $m = [(\bar{n},0)]_{\sim_d}$ for some particular natural number $0 <_{\mathbb{N}} \bar{n}$. As $\bar{n} \in \mathbb{N}$ implies $f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n}) =$

$[(\bar{n}, 0)]_{\sim_d}$ according to the Identification of $\mathbb{N}$ in $\mathbb{Z}$, we obtain via substitution $m = f_\mathbb{N}^\mathbb{Z}(\bar{n})$. Recalling that $f_\mathbb{N}^\mathbb{Z}$ is a bijection with range $\mathbb{N}_\mathbb{Z}$ (see Note 6.14), the value m is evidently an element of that range. In the second case $[(0,0)]_{\sim_d} = m$, we have $f_\mathbb{N}^\mathbb{Z}(0) = [(0,0)]_{\sim_d}$, with the consequence that $m = f_\mathbb{N}^\mathbb{Z}(0)$ is again in the range $\mathbb{N}_\mathbb{Z}$. Thus, the implication in (6.289) holds, where m was arbitrary, so that the corollary follows then to be true. □

Exercise 6.40. Show that every natural number in $\mathbb{Z}$ is greater than or equal to $[(0,0)]_{\sim_d}$ with respect to the total ordering of $\mathbb{N}_\mathbb{Z}$, that is,

$$\forall m\, (m \in \mathbb{N}_\mathbb{Z} \Rightarrow [(0,0)]_{\sim_d} \leq_{\mathbb{N}_\mathbb{Z}} m). \tag{6.290}$$

(Hint: Use the properties of $f_\mathbb{N}^\mathbb{Z}$, (6.265), (3.680) and (6.278).)

Theorem 6.83 (Ordered integral domain of integers). *It is true that the set*

$$(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z}, -_\mathbb{Z}, <_\mathbb{Z}) \tag{6.291}$$

constitutes an ordered integral domain.

Proof. We already found out that $(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z}, -_\mathbb{Z})$ is a commutative ring (see Note 6.17), for which $(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z})$ is zero-divisor free and nontrivial (according to Theorem 6.77 and Exercise 6.38). Furthermore, Proposition 6.48 shows that the unity element of $\mathbb{Z}$ exists. Recalling that $<_\mathbb{Z}$ is a linear ordering of $\mathbb{Z}$ (see Note 6.22), we thus have that $(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z}, -_\mathbb{Z}, <_\mathbb{Z})$ satisfies Property 1 – 4 of an ordered integral domain. We establish now also the required monotony laws

$$\forall m, n, p\, (m, n, p \in \mathbb{Z} \Rightarrow [m <_\mathbb{Z} n \Rightarrow m +_\mathbb{Z} p <_\mathbb{Z} n +_\mathbb{Z} p]) \tag{6.292}$$

$$\forall m, n, p\, ([m, n, p \in \mathbb{Z} \wedge [(0,0)]_{\sim_d} <_\mathbb{Z} p] \Rightarrow [m <_\mathbb{Z} n \Rightarrow m \cdot_\mathbb{Z} p <_\mathbb{Z} n \cdot_\mathbb{Z} p]) \tag{6.293}$$

letting m, n and p in the following be arbitrary. Concerning (6.292), we assume $m, n, p \in \mathbb{Z}$ and prove the implication

$$m <_\mathbb{Z} n \Rightarrow m +_\mathbb{Z} p <_\mathbb{Z} n +_\mathbb{Z} p \tag{6.294}$$

by contraposition, by verifying instead

$$\neg m +_\mathbb{Z} p <_\mathbb{Z} n +_\mathbb{Z} p \Rightarrow \neg m <_\mathbb{Z} n. \tag{6.295}$$

To do this, we assume that the antecedent of the preceding implication to be true, so that the Negation Formula for $<$ yields

$$n +_\mathbb{Z} p \ \leq_\mathbb{Z} \ m +_\mathbb{Z} p. \tag{6.296}$$

As integers, m, n and p are by definition elements of the quotient set $\mathbb{N}\times\mathbb{N}/\sim_d$, so that there exist particular elements x, y and z in the Cartesian product $\mathbb{N}\times\mathbb{N}$ such that $[x]_{\sim_d} = m$, $[y]_{\sim_d} = n$ and $[z]_{\sim_d} = p$. Then, there exist also particular natural numbers for which $x = (a,b)$, $y = (c,d)$ and $z = (e,f)$, so that the chosen integers can be written as $m = [(a,b)]_{\sim_d}$, $n = [(c,d)]_{\sim_d}$ and $p = [(e,f)]_{\sim_d}$. With these equations, (6.296) becomes

$$[(c,d)]_{\sim_d} +_{\mathbb{Z}} [(e,f)]_{\sim_d} \ \leq_{\mathbb{Z}} \ [(a,b)]_{\sim_d} +_{\mathbb{Z}} [(e,f)]_{\sim_d}.$$

Using now the characterization of the addition on $\mathbb{Z}$ in (6.185), we obtain

$$[(c +_{\mathbb{N}} e, d +_{\mathbb{N}} f)]_{\sim_d} \ \leq_{\mathbb{Z}} \ [(a +_{\mathbb{N}} e, b +_{\mathbb{N}} f)]_{\sim_d}.$$

Applying then the characterization of $\leq_{\mathbb{Z}}$ in (6.265) gives

$$(c +_{\mathbb{N}} e) +_{\mathbb{N}} (b +_{\mathbb{N}} f) \ \leq_{\mathbb{N}} \ (d +_{\mathbb{N}} f) +_{\mathbb{N}} (a +_{\mathbb{N}} e).$$

The Associative and the Commutative Law for the addition on $\mathbb{N}$ allows us to rearrange the terms and write instead

$$(e +_{\mathbb{N}} f) +_{\mathbb{N}} (c +_{\mathbb{N}} b) \ \leq_{\mathbb{N}} \ (e +_{\mathbb{N}} f) +_{\mathbb{N}} (d +_{\mathbb{N}} a),$$

which can now be simplified by means of the Cancellation Law for $+_{\mathbb{N}}$ to

$$c +_{\mathbb{N}} b \ \leq_{\mathbb{N}} \ d +_{\mathbb{N}} a.$$

According to (6.265), this inequality means

$$[(c,d)]_{\sim_d} \ \leq_{\mathbb{Z}} \ [(a,b)]_{\sim_d},$$

that is, $n \leq_{\mathbb{Z}} m$. This inequality implies now $\neg m <_{\mathbb{Z}} n$ with the Negation Formula for $<$, which finding proves the implication (6.295) and thus the desired implication (6.294).

Concerning (6.293), we assume again $m,n,p \in \mathbb{Z}$ and now also $[(0,0)]_{\sim_d} <_{\mathbb{Z}} p$, which assumption allows us to write p as the equivalence class $[(e,0)]_{\sim_d}$ for some particular natural number $0 <_{\mathbb{N}} e$ because of (6.287). As in the proof of (6.292), we write the integers m and n as the equivalence classes $m = [(a,b)]_{\sim_d}$ and $n = [(c,d)]_{\sim_d}$, where a, b, c and d are particular natural numbers. We prove the implication

$$m <_{\mathbb{Z}} n \Rightarrow m \cdot_{\mathbb{Z}} p <_{\mathbb{Z}} n \cdot_{\mathbb{Z}} p \tag{6.297}$$

by contraposition and prove for this purpose the implication

$$\neg m \cdot_{\mathbb{Z}} p <_{\mathbb{Z}} n \cdot_{\mathbb{Z}} p \Rightarrow \neg m <_{\mathbb{Z}} n. \tag{6.298}$$

We note that the desired consequent $\neg m <_\mathbb{Z} n$ is equivalent to

$$[(c,d)]_{\sim_d} \leq_\mathbb{Z} [(a,b)]_{\sim_d}$$

by virtue of the Negation Formula for $<$ and the equations for m and n, and due to (6.265) also equivalent to

$$c +_\mathbb{N} b \ \leq_\mathbb{N} \ d +_\mathbb{N} a. \tag{6.299}$$

To establish this consequent, we assume the antecedent of the implication (6.298) to be true, so that the Negation Formula for $<$ and the equations for m, n and p give

$$[(c,d)]_{\sim_d} \cdot_\mathbb{Z} [(e,0)]_{\sim_d} \ \leq_\mathbb{Z} \ [(a,b)]_{\sim_d} \cdot_\mathbb{Z} [(e,0)]_{\sim_d}.$$

Applying then (6.146) yields

$$[(c \cdot_\mathbb{N} e +_\mathbb{N} d \cdot_\mathbb{N} 0, c \cdot_\mathbb{N} 0 +_\mathbb{N} d \cdot_\mathbb{N} e)]_{\sim_d} \ \leq_\mathbb{Z} \ [(a \cdot_\mathbb{N} e +_\mathbb{N} b \cdot_\mathbb{N} 0, a \cdot_\mathbb{N} 0 +_\mathbb{N} b \cdot_\mathbb{N} e)]_{\sim_d},$$

which inequality we simplify to

$$[(c \cdot_\mathbb{N} e, d \cdot_\mathbb{N} e)]_{\sim_d} \ \leq_\mathbb{Z} \ [(a \cdot_\mathbb{N} e, b \cdot_\mathbb{N} e)]_{\sim_d}$$

by applying the Cancellation Law for 0 and the definition of the zero element. In view of (6.265), we obtain now

$$c \cdot_\mathbb{N} e +_\mathbb{N} b \cdot_\mathbb{N} e \ \leq_\mathbb{N} \ d \cdot_\mathbb{N} e +_\mathbb{N} a \cdot_\mathbb{N} e.$$

Let us use here the Distributive Law for $\mathbb{N}$ and write the preceding inequality in the form

$$(c +_\mathbb{N} b) \cdot_\mathbb{N} e \ \leq_\mathbb{N} \ (d +_\mathbb{N} a) \cdot_\mathbb{N} e.$$

Recalling now the truth of $0 <_\mathbb{N} e$, we may apply now the Monotony Law for $\cdot_\mathbb{N}$ and $\leq_\mathbb{N}$ to obtain the desired consequent (6.299), which proves the implication (6.298) and thus the original implication (6.297).

Since m, n and p were initially arbitrary, we may therefore conclude that both of the required monotony laws are satisfied, so that the set $(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z}, -_\mathbb{Z}, <_\mathbb{Z})$ is an ordered integral domain by definition. □

Note 6.25. The ordered integral domain (6.291) has all of the properties we derived for a generic ordered integral domain in Section 6.6. Instead of rewriting these laws specifically for the integers, we refer to them in their general form.

Theorem 6.84 (Ordered elementary domain of $\mathbb{N}$ in $\mathbb{Z}$). *It is true that the ordered quadruple*

$$(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}}, <_{\mathbb{N}_{\mathbb{Z}}}) \tag{6.300}$$

constitutes an ordered elementary domain.

Proof. To show that $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$ is a nontrivial semiring, we may proceed similarly as in Exercise 6.38. We recall from Lemma 6.60 that $[(1,0)]_{\sim_d}$ and $[(0,0)]_{\sim_d}$, as the unity and the zero element of $\mathbb{N}_{\mathbb{Z}}$, are elements of the latter set. We now prove $\mathbb{N}_{\mathbb{Z}} \neq \{[(0,0)]_{\sim_d}\}$ by contradiction, assuming the negation of that inequality to be true, so that $\mathbb{N}_{\mathbb{Z}} = \{[(0,0)]_{\sim_d}\}$ follows to be true with the Double Negation Law. As $\mathbb{N}_{\mathbb{Z}}$ is a singleton, we obtain with (2.180) the uniquely existential sentence $\exists! m\,(m \in \mathbb{N}_{\mathbb{Z}})$, whose uniqueness part is given by

$$\forall m, m'([m \in \mathbb{N}_{\mathbb{Z}} \wedge m' \in \mathbb{N}_{\mathbb{Z}}] \Rightarrow m = m').$$

Therefore, $[(1,0)]_{\sim_d} \in \mathbb{N}_{\mathbb{Z}}$ and $[(0,0)]_{\sim_d} \in \mathbb{N}_{\mathbb{Z}}$ imply $[(1,0)]_{\sim_d} = [(0,0)]_{\sim_d}$, which contradicts the fact that $[(1,0)]_{\sim_d} \neq [(0,0)]_{\sim_d}$ holds according to (6.127). This completes the proof of $\mathbb{N}_{\mathbb{Z}} \neq \{[(0,0)]_{\sim_d}\}$, so that the semiring $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$ is nontrivial by definition. Thus, $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}}, <_{\mathbb{N}_{\mathbb{Z}}})$ satisfies Property 1 – Property 4 of an ordered elementary domain.

Concerning Property 5, we recall that $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}})$ is zero-divisor free and satisfies thus (6.260). Let us now apply the Criterion for zero-divisor freeness and establish the truth of

$$\begin{aligned}\forall m, n\,(m, n \in \mathbb{N}_{\mathbb{Z}} \Rightarrow [(m \cdot_{\mathbb{N}_{\mathbb{Z}}} n = [(0,0)]_{\sim_d} \vee n \cdot_{\mathbb{N}_{\mathbb{Z}}} m = [(0,0)]_{\sim_d})\\ \Rightarrow (m = [(0,0)]_{\sim_d} \vee n = [(0,0)]_{\sim_d})]),\end{aligned} \tag{6.301}$$

letting $m, n \in \mathbb{N}_{\mathbb{Z}}$ be arbitrary, assuming the disjunction

$$m \cdot_{\mathbb{N}_{\mathbb{Z}}} n = [(0,0)]_{\sim_d} \vee n \cdot_{\mathbb{N}_{\mathbb{Z}}} m = [(0,0)]_{\sim_d}$$

to be true. Here, we observe in light of (6.180) that the equations $m \cdot_{\mathbb{N}_{\mathbb{Z}}} n = m \cdot_{\mathbb{Z}} n$ and $n \cdot_{\mathbb{N}_{\mathbb{Z}}} m = n \cdot_{\mathbb{Z}}$ are true, so that we may apply substitutions to the preceding disjunction and write it equivalently as

$$m \cdot_{\mathbb{Z}} n = [(0,0)]_{\sim_d} \vee n \cdot_{\mathbb{Z}} m = [(0,0)]_{\sim_d}.$$

This disjunction implies now already the desired disjunction $m = [(0,0)]_{\sim_d} \vee n = [(0,0)]_{\sim_d}$ with (6.260). Since m and n were initially arbitrary, we may therefore conclude that (6.301) holds, so that $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$ is indeed zero-divisor free.

Next, we recall from Note 6.23 that $<_{\mathbb{N}_\mathbb{Z}}$ is a linear ordering of $\mathbb{N}_\mathbb{Z}$, which is comparable in view of the Characterization of linearly ordered sets, as required by Property 6 of an ordered elementary domain.

Finally, we establish Property 7 by proving

$$\begin{aligned}&\forall m, n\,(m, n \in \mathbb{N}_\mathbb{Z}\\&\quad\Rightarrow [m <_{\mathbb{N}_\mathbb{Z}} n \Leftrightarrow \exists d\,(d \in \mathbb{N}_\mathbb{Z} \wedge d \neq [(0,0)]_{\sim_d} \wedge m +_{\mathbb{N}_\mathbb{Z}} d = n)]).\end{aligned} \tag{6.302}$$

For this purpose, we let $m, n \in \mathbb{N}_\mathbb{Z}$ be arbitrary and assume first $m <_{\mathbb{N}_\mathbb{Z}} n$ to be true. These assumptions imply now $m <_{\mathbb{Z}} n$ with (6.279). Let us now form the difference $\bar{d} = n -_{\mathbb{Z}} m$, which we may write also as the sum $\bar{d} = n +_{\mathbb{Z}} (-m)$. We obtain then the equations

$$\begin{aligned}m +_{\mathbb{Z}} \bar{d} &= m +_{\mathbb{Z}} (n +_{\mathbb{Z}} (-m))\\&= (m +_{\mathbb{Z}} n) +_{\mathbb{Z}} (-m)\\&= (n +_{\mathbb{Z}} m) +_{\mathbb{Z}} (-m)\\&= n +_{\mathbb{Z}} (m +_{\mathbb{Z}} (-m))\\&= n +_{\mathbb{Z}} [(0,0)]_{\sim_d}\\&= n\end{aligned} \tag{6.303}$$

by applying substitution, the Associative Law for the addition on $\mathbb{Z}$, the Commutative Law for the addition on $\mathbb{Z}$, again the preceding Associative Law, the definition of an additive inverse in connection with fact that $[(0,0)]_{\sim_d}$ constitutes the zero element of $\mathbb{Z}$ (see Exercise 6.22), and finally the property of a zero element. Let us observe now that the Monotony Law for $+$ and $<$ applies to the ordered integral domain $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}}, <_{\mathbb{Z}})$, so that the previously established $m <_{\mathbb{Z}} n$ implies $m +_{\mathbb{Z}} (-m) <_{\mathbb{Z}} n +_{\mathbb{Z}} (-m)$, and therefore evidently $[(0,0)]_{\sim_d} <_{\mathbb{Z}} \bar{d}$. On the one hand, this implies

$$\bar{d} \in \mathbb{N}_\mathbb{Z} \tag{6.304}$$

with (6.289), which shows that m, $\bar{d}$ and $\bar{n}$ are all elements of $\mathbb{N}_\mathbb{Z}$, so that the sum (6.303) can be written equivalently as

$$m +_{\mathbb{N}_\mathbb{Z}} \bar{d} = n \tag{6.305}$$

by using (6.190). On the other hand, since $<_{\mathbb{Z}}$ is a linear ordering, it satisfies the Characterization of comparability, so that $[(0,0)]_{\sim_d} <_{\mathbb{Z}} \bar{d}$ implies the falsity of the equality $[(0,0)]_{\sim_d} = \bar{d}$, which means that

$$\bar{d} \neq [(0,0)]_{\sim_d} \tag{6.306}$$

is true. In view of the findings (6.304) – (6.306), we now see that the existential sentence in (6.302) is true, which proves the first part ('$\Rightarrow$') of the equivalence in (6.302).

Regarding the second part ('$\Leftarrow$'), we assume now conversely that there is a constant, say $\bar{d}$, satisfying (6.304) – (6.306). Here, $\bar{d} \in \mathbb{N}_{\mathbb{Z}}$ implies $[(0,0)]_{\sim_d} \leq_{\mathbb{N}_{\mathbb{Z}}} \bar{d}$ with (6.290). In conjunction with $[(0,0)]_{\sim_d} \neq \bar{d}$, this gives us $[(0,0)]_{\sim_d} <_{\mathbb{N}_{\mathbb{Z}}} \bar{d}$ with the Characterization of an induced irreflexive partial ordering, and we can evidently write this inequality equivalently as $[(0,0)]_{\sim_d} <_{\mathbb{Z}} \bar{d}$. This implies with the Monotony Law for $+_{\mathbb{Z}}$ and $<_{\mathbb{Z}}$

$$[(0,0)]_{\sim_d} +_{\mathbb{Z}} m \ <_{\mathbb{Z}} \ \bar{d} +_{\mathbb{Z}} m,$$

which yields $m <_{\mathbb{Z}} m +_{\mathbb{Z}} \bar{d}$ with the definition of a zero element and the Commutative Law for $+_{\mathbb{Z}}$. Now, substitution based on (6.305) gives us $m <_{\mathbb{Z}} n$, which we can write equivalently as $m <_{\mathbb{N}_{\mathbb{Z}}} n$, recalling that m and n are elements of $\mathbb{N}_{\mathbb{Z}}$ by assumption. Thus, the proof of the equivalence in (6.302) is complete, and as m was arbitrary, we may therefore conclude that the universal sentence (6.302) holds. This means that $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}}, <_{\mathbb{N}_{\mathbb{Z}}})$ satisfies also Property 7 of an ordered elementary domain, so that the proof of the theorem is now complete. □

Note 6.26. We could now extend the idea of isomorphic semirings (see Note 6.18) and view the ordered elementary domain $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}}, <_{\mathbb{N}_{\mathbb{Z}}})$ as an 'isomorphic copy' of the ordered elementary domain $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, <_{\mathbb{N}})$ carrying all essential properties of the latter. Instead of operating on the original number numbers, we will in the following usually, out of convenience, handle their mapped versions in $\mathbb{Z}$ by means of the binary operations on $\mathbb{Z}$ and by means of the linear/total ordering of $\mathbb{Z}$. If desired, the isomorphism $f_{\mathbb{N}}^{\mathbb{Z}}$ allows us always to go back into the original domain after carrying out integer-based actions.

Notation 6.9. As the zero element $[(0,0)]_{\sim_d}$ and the unity element $[(1,0)]_{\sim_d}$ of $\mathbb{Z}$ are natural numbers in $\mathbb{Z}$ and moreover the 'isomorphic copies' of the zero element 0 and the unity element 1 of $\mathbb{N}$ (under the mapping $f_{\mathbb{N}}^{\mathbb{Z}}$), we abbreviate

$$0 = [(0,0)]_{\sim_d}, \tag{6.307}$$

$$1 = [(1,0)]_{\sim_d}, \tag{6.308}$$

thereby overloading the symbols '0' and '1'. Then, we may write for (6.286), in analogy to $0 <_{\mathbb{N}} 1$ in (4.164), also

$$0 <_{\mathbb{Z}} 1. \tag{6.309}$$

Proposition 6.85. *If an integer m is smaller than an integer n, then m is less than or equal to $n-1$, that is,*

$$\forall m, n\,(m, n \in \mathbb{Z} \Rightarrow [m <_{\mathbb{Z}} n \Rightarrow m \leq_{\mathbb{Z}} n -_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1)]). \tag{6.310}$$

Proof. We take tow arbitrary integers m and n, assuming $m <_{\mathbb{Z}} n$ to hold. An application of the Monotony Law for $+$ and $-$ gives us then $m +_{\mathbb{Z}} (-m) <_{\mathbb{Z}} n +_{\mathbb{Z}} (-m)$, and consequently $[(0,0)]_{\sim_d} <_{\mathbb{Z}} n -_{\mathbb{Z}} m$ with the properties of a negative and of a subtraction. Here, we have $f_{\mathbb{N}}^{\mathbb{Z}}(0) = [(0,0)]_{\sim_d}$ according to the Identification of $\mathbb{N}$ in $\mathbb{Z}$; thus,

$$f_{\mathbb{N}}^{\mathbb{Z}}(0) <_{\mathbb{Z}} n -_{\mathbb{Z}} m. \tag{6.311}$$

This inequality implies then $f_{\mathbb{N}}^{\mathbb{Z}}(0) \leq_{\mathbb{Z}} n -_{\mathbb{Z}} m$ in view of the Characterization of induced irreflexive partial orderings, and this yields $n -_{\mathbb{Z}} m \in \mathbb{N}_{\mathbb{Z}}$ by virtue of (6.289). Let us observe now the truth of the equations

$$\begin{aligned} f_{\mathbb{N}}^{\mathbb{Z}}((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(n -_{\mathbb{Z}} m)) &= (f_{\mathbb{N}}^{\mathbb{Z}} \circ (f_{\mathbb{N}}^{\mathbb{Z}})^{-1})(n -_{\mathbb{Z}} m) = \mathrm{id}_{\mathbb{N}_{\mathbb{Z}}}(n -_{\mathbb{Z}} m) \\ &= n -_{\mathbb{Z}} m \end{aligned} \tag{6.312}$$

in light of the notation for function compositions, (3.680) and the definition of an identity function. We may therefore write the inequality (6.311) as

$$f_{\mathbb{N}}^{\mathbb{Z}}(0) <_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(n -_{\mathbb{Z}} m)).$$

Noting that 0 and $(f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(n -_{\mathbb{Z}} m)$ are elements of the domain $\mathbb{N}$ of the function $f_{\mathbb{N}}^{\mathbb{Z}}$, the inequality $0 <_{\mathbb{N}} (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(n -_{\mathbb{Z}} m)$ follows to be true because of (6.284). This gives us $1 \leq_{\mathbb{N}} (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(n -_{\mathbb{Z}} m)$ with (4.157). Another application of (6.284) results in $f_{\mathbb{N}}^{\mathbb{Z}}(1) \leq_{\mathbb{N}} n -_{\mathbb{Z}} m$, using (6.312). Then, $m \leq_{\mathbb{N}} n -_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1)$ evidently follows to be true by virtue of the Monotony Law for $+$ and $<$. As m and n were initially arbitrary, we may therefore conclude that the proposed universal sentence (6.310) is true. □

Notation 6.10. We may write (6.310) shorter as

$$\forall m, n\,(m, n \in \mathbb{Z} \Rightarrow [m <_{\mathbb{Z}} n \Rightarrow m \leq_{\mathbb{Z}} n -_{\mathbb{Z}} 1]) \tag{6.313}$$

by using the Identification of $\mathbb{N}$ in $\mathbb{Z}$ and the notation (6.308).

Corollary 6.86. *If an integer m is smaller than an integer n, then $m+1$ is less than or equal to n, that is,*

$$\forall m, n\,(m, n \in \mathbb{Z} \Rightarrow [m <_{\mathbb{Z}} n \Rightarrow m +_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1) \leq_{\mathbb{Z}} n]). \tag{6.314}$$

Proof. Letting $m, n \in \mathbb{Z}$ be arbitrary such that $m <_{\mathbb{Z}} n$, we find $m \leq_{\mathbb{Z}} n -_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1)$ with (6.310). The Monotony Law for $+$ and $\leq$ gives us therefore

$$m +_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1) \leq_{\mathbb{Z}} [n -_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1)] +_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1).$$

Exploiting here the associativity of the addition on $\mathbb{Z}$ and the property of a negative/inverse, we find the desired inequality to be true. As m and n were arbitrary, we may therefore conclude that (6.314) holds. $\square$

This useful result allows us to conveniently establish to extend the fact that every nonempty and bounded from below/above subset of $\mathbb{N}$ has a least/greatest element (see Proposition 4.42 and Corollary 4.41) to the set of integers.

Proposition 6.87. *It is true that*

a) every nonempty and bounded-from-below subset of $\mathbb{Z}$ has a least element, that is,

$$\forall A\,([A \subseteq \mathbb{Z} \wedge A \neq \emptyset \wedge \exists a\,(a \in \mathbb{Z} \wedge \forall m\,(m \in A \Rightarrow a \leq_{\mathbb{Z}} m)] \Rightarrow \exists i\,(i = \overset{\leq_{\mathbb{Z}}}{\min} A))). \tag{6.315}$$

b) every nonempty and bounded-from-above subset of $\mathbb{Z}$ has a greatest element, that is,

$$\forall A\,([A \subseteq \mathbb{Z} \wedge A \neq \emptyset \wedge \exists u\,(u \in \mathbb{Z} \wedge \forall m\,(m \in A \Rightarrow m \leq_{\mathbb{Z}} u)] \Rightarrow \exists s\,(s = \overset{\leq_{\mathbb{Z}}}{\max} A))). \tag{6.316}$$

Proof. Concerning a), we let A be an arbitrary nonempty and bounded-from-below subset of $\mathbb{Z}$. Here, the assumption $A \neq \emptyset$ implies that there is an element in A, say $\bar{m}$. Thus, $\bar{m}$ turns out to be an integer in view of the assumed inclusion $A \subseteq \mathbb{Z}$ and the definition of a subset. To show that the minimum of A with respect to $\leq_{\mathbb{Z}}$ exists, we prove by mathematical induction that the implication

$$\exists a\,(\forall m\,(m \in A \Rightarrow a \leq_{\mathbb{Z}} m) \wedge f_{\mathbb{N}}^{\mathbb{Z}}(n) = \bar{m} -_{\mathbb{Z}} a) \;\Rightarrow\; \exists i\,(i = \overset{\leq_{\mathbb{Z}}}{\min} A) \tag{6.317}$$

is true for any $n \in \mathbb{N}$, where the antecedent means that an arbitrary natural number n (in $\mathbb{Z}$) can be written as the difference of the given element $\bar{m}$ of A and some lower bound for A. In the base case, we set $n = 0$ and assume there is some lower bound for A, say $\bar{a}$, such that $f_\mathbb{N}^\mathbb{Z}(0) = \bar{m} -_\mathbb{Z} \bar{a}$. We obtain then the equations

$$\begin{aligned}
\bar{m} &= \bar{m} +_\mathbb{Z} [(0,0)]_{\sim_d} \\
&= \bar{m} +_\mathbb{Z} (\bar{a} +_\mathbb{Z} [-\bar{a}]) \\
&= \bar{m} +_\mathbb{Z} (-\bar{a} +_\mathbb{Z} \bar{a}) \\
&= (\bar{m} -_\mathbb{Z} \bar{a}) +_\mathbb{Z} \bar{a} \\
&= f_\mathbb{N}^\mathbb{Z}(n) +_\mathbb{Z} \bar{a} \\
&= f_\mathbb{N}^\mathbb{Z}(0) +_\mathbb{Z} \bar{a} \\
&= [(0,0)]_{\sim_d} +_\mathbb{Z} \bar{a} \\
&= \bar{a}
\end{aligned}$$

by using the property of a zero/neutral element, the property of a negative/inverse element, the commutativity of the addition on $\mathbb{Z}$, the associativity of the addition on $\mathbb{Z}$ alongside the definition of a subtraction, and the previous two equations involving n. Applying now a substitution based on the resulting equation $\bar{m} = \bar{a}$ to the previously found $\bar{m} \in A$ yields $\bar{a} \in A$, so that the lower bound $\bar{a}$ for A is the minimum of A (with respect to $\leq_\mathbb{Z}$), by definition. This demonstrates the truth of the existential sentence $\exists i\, (i = \min^{\leq_\mathbb{Z}} A)$, and thus the truth of the implication (6.317), in the base case.

In the base case, we let n be an arbitrary natural number, and we assume that n satisfies the implication (6.317), as the induction assumption. To prove the induction step, we need to demonstrate the truth of the implication

$$\exists a\, (\forall m\, (m \in A \Rightarrow a \leq_\mathbb{Z} m) \wedge f_\mathbb{N}^\mathbb{Z}(n+1) = \bar{m} -_\mathbb{Z} a) \;\Rightarrow\; \exists i\, (i = \overset{\leq_\mathbb{Z}}{\min} A). \tag{6.318}$$

For this purpose, we assume there to be some particular constant $\bar{a}$ that has the lower bound property

$$\forall m\, (m \in A \Rightarrow \bar{a} \leq_\mathbb{Z} m) \tag{6.319}$$

and that satisfies $f_\mathbb{N}^\mathbb{Z}(n+1) = \bar{m} -_\mathbb{Z} \bar{a}$. We prove the desired existential sentence $\exists i\, (i = \min^{\leq_\mathbb{Z}} A)$ by cases, based on the fact that the disjunction $\bar{a} \in X \vee \bar{a} \notin X$ is true by the Law of the Excluded Middle. The first case $\bar{a} \in X$ immediately implies $\bar{a} = \min^{\leq_\mathbb{Z}} A$ as $\bar{a}$ is a lower bound for A (with respect to $\leq_\mathbb{Z}$); thus, the desired existential sentence holds in the first case. In the second case $\bar{a} \notin X$, we consider the integer $a^* = \bar{a} +_\mathbb{Z} f_\mathbb{N}^\mathbb{Z}(1)$. We may show that a^* constitutes a lower bound for A, that is,

$$\forall m\, (m \in A \Rightarrow a^* \leq_\mathbb{Z} m). \tag{6.320}$$

Letting $m \in A$ be arbitrary, we find $\bar{a} \leq_{\mathbb{Z}} m$ with (6.319), and this implies the disjunction $\bar{a} <_{\mathbb{Z}} m \vee \bar{a} = m$ due to the Characterization of induced irreflexive partial orderings. Since $m \in X$ and the current case assumption $\bar{a} \notin X$ give $m \neq \bar{a}$ by means of (2.4), we see that the second part of the disjunction is false. Therefore, its first part $\bar{a} <_{\mathbb{Z}} m$ is true. This inequality implies now $\bar{a} +_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1) \leq_{\mathbb{Z}} m$ with (6.314), so that a substitution based on the previous definition of a^* yields $a^* \leq_{\mathbb{Z}} m$. This proves the implication in (6.320), in which m is arbitrary, so that we may infer from this the truth of the universal sentence (6.320). Furthermore, we evidently obtain the true equations

$$\begin{aligned}
\bar{m} -_{\mathbb{Z}} a^* &= \bar{m} -_{\mathbb{Z}} (\bar{a} +_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1)) \\
&= (\bar{m} -_{\mathbb{Z}} \bar{a}) -_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1) \\
&= f_{\mathbb{N}}^{\mathbb{Z}}(n+1) -_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1) \\
&= f_{\mathbb{N}}^{\mathbb{Z}}(n) +_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1) -_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1) \\
&= f_{\mathbb{N}}^{\mathbb{Z}}(n),
\end{aligned}$$

using in particular (6.193).

Thus, a^* satisfies both (6.320) and $f_{\mathbb{N}}^{\mathbb{Z}}(n) = \bar{m} -_{\mathbb{Z}} a^*$, so that the existential sentence in (6.317) is true. Being thus the antecedent of the induction assumption (6.317), the desired existential sentence $\exists i\, (i = \min^{\leq_{\mathbb{Z}}} A)$ follows no be true. This proves the implication in (6.318), and since n was arbitrary, we may therefore conclude that the induction step holds, besides the base case.

Now, as the set A was initially assumed to be bounded from below, there is a constant, say $\bar{a}^*$, such that

$$\forall m\, (m \in A \Rightarrow \bar{a}^* \leq_{\mathbb{Z}} m). \tag{6.321}$$

The initially found element $\bar{m} \in A$ satisfies therefore the inequality $\bar{a}^* \leq_{\mathbb{Z}} \bar{m}$, which evidently yields $0 \leq_{\mathbb{Z}} \bar{m} -_{\mathbb{Z}} \bar{a}^*$. Consequently, we find $\bar{m} -_{\mathbb{Z}} \bar{a}^* \in \mathbb{N}_{\mathbb{Z}}$ by means of (6.289). As $f_{\mathbb{N}}^{\mathbb{Z}}$ constitutes a bijection from $\mathbb{N}$ to $\mathbb{N}_{\mathbb{Z}}$, we find the number $\bar{n} = (f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(\bar{m} -_{\mathbb{Z}} \bar{a}^*)$ to be an element of $\mathbb{N}$, according to Proposition 3.673. The preceding equation yields then

$$\begin{aligned}
f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n}) &= f_{\mathbb{N}}^{\mathbb{Z}}((f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(\bar{m} -_{\mathbb{Z}} \bar{a}^*)) \\
&= (f_{\mathbb{N}}^{\mathbb{Z}} \circ (f_{\mathbb{N}}^{\mathbb{Z}})^{-1})(\bar{m} -_{\mathbb{Z}} \bar{a}^*) \\
&= \mathrm{id}_{\mathbb{N}_{\mathbb{Z}}}(\bar{m} -_{\mathbb{Z}} \bar{a}^*) \\
&= \bar{m} -_{\mathbb{Z}} \bar{a}^*
\end{aligned}$$

by applying substitution, the notation for function compositions, (3.680) and the definition of an identity function. Now, the resulting equation

$f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n}) = \bar{m} -_{\mathbb{Z}} \bar{a}^*$ and (6.321) show that the natural number $\bar{n}$ satisfies the existential sentence

$$\exists a\,(\forall m\,(m \in A \Rightarrow a \leq_{\mathbb{Z}} m) \wedge f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n}) = \bar{m} -_{\mathbb{Z}} a).$$

It therefore follows by virtue of the previously proven implication (6.317), which holds for any $n \in \mathbb{N}$, that the existential sentence in (6.315) with respect to s is true. Since A was arbitrary, we may therefore conclude that Part a) of the proposition holds. Part b) can be proved similarly. □

Exercise 6.41. Establish Part b) of Proposition 6.87.

Theorem 6.88 (Division of an integer with remainder). *It is true*

a) for any integer n greater than 0 and for any integer m that there exist a unique integer q and a unique natural number $r <_{\mathbb{Z}} n$ such that m is the sum of the product $q \cdot_{\mathbb{Z}} n$ and r, i.e.

$$\begin{aligned}&\forall n\,(0 <_{\mathbb{Z}} n \Rightarrow \forall m\,(m \in \mathbb{Z} \\ &\quad \Rightarrow \exists! q, r\,(q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_{\mathbb{Z}} n \wedge m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r)))\end{aligned} \tag{6.322}$$

b) for any integer n less than 0 and for any integer m that there are a unique integer q and a unique natural number $r <_{\mathbb{Z}} -n$ such that m is the sum of the product $q \cdot_{\mathbb{Z}} n$ and r, i.e.

$$\begin{aligned}&\forall n\,(n <_{\mathbb{Z}} 0 \Rightarrow \forall m\,(m \in \mathbb{Z} \\ &\quad \Rightarrow \exists! q, r\,(q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_{\mathbb{Z}} -n \wedge m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r)))\end{aligned} \tag{6.323}$$

Proof. Let us begin by establishing (6.322) taking an arbitrary constant n such that $0 <_{\mathbb{Z}} n$ is true. Instead of proving

$$\forall m\,(m \in \mathbb{Z} \Rightarrow \exists! q, r\,(q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_{\mathbb{Z}} n \wedge m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r)) \tag{6.324}$$

immediately, we prove first

$$\forall m\,(m \in \mathbb{N} \Rightarrow \exists q, r\,(q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_{\mathbb{Z}} n \wedge m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r)) \tag{6.325}$$

by mathematical induction. In the base case $(m = 0)$, treating m as the integer $[(0,0)]_{\sim_d} = 0$, we note that the integer $\bar{q} = [(0,0)]_{\sim_d} = 0$ and the natural number $\bar{r} = [(0,0)]_{\sim_d} = 0$ (in $\mathbb{Z}$) satisfy

$$\begin{aligned}m &= 0 \\ &= 0 \cdot_{\mathbb{Z}} n \\ &= 0 \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} 0 \\ &= \bar{q} \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} \bar{r}\end{aligned}$$

because of the Cancellation Law for 0_X with respect to the ordered integral domain $(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z}, -_\mathbb{Z}, <_\mathbb{Z})$ and due to the definition of a zero element. Thus, the existential part of the existential sentence in (6.325) holds. Concerning the induction step, we let $m \in \mathbb{N}$ be arbitrary and make the induction assumption that there exist particular numbers $\bar{q} \in \mathbb{Z}$ and $\bar{r} \in \mathbb{N}$ satisfying $\bar{r} <_\mathbb{Z} n$ and the equation $m = \bar{q} \cdot_\mathbb{Z} n +_\mathbb{Z} \bar{r}$. We obtain then

$$\begin{aligned} m +_\mathbb{N} 1 &= (\bar{q} \cdot_\mathbb{Z} n +_\mathbb{Z} \bar{r}) +_\mathbb{N} 1 \\ &= (\bar{q} \cdot_\mathbb{Z} n +_\mathbb{Z} \bar{r}) +_\mathbb{Z} 1 \\ &= \bar{q} \cdot_\mathbb{Z} n +_\mathbb{Z} (\bar{r} +_\mathbb{Z} 1) \\ &= \bar{q} \cdot_\mathbb{Z} n +_\mathbb{Z} (\bar{r} +_\mathbb{N} 1) \end{aligned} \tag{6.326}$$

by applying substitution based on the induction assumption, by treating 1 as the integer $[(1, 0)]_{\sim_d}$, the Associative Law for the addition on $\mathbb{Z}$, and the fact that $\bar{r}$ and 1 are both natural numbers. Since the case assumption $0 <_\mathbb{Z} n$ implies $0 <_\mathbb{Z} n \vee 0 = n$ and therefore $0 \leq_\mathbb{Z} n$ with the Characterization of an induced irreflexive partial ordering, it follows with Corollary 6.82 that n is a natural number in $\mathbb{Z}$. Treating n as an element of $\mathbb{N}$ and recalling that $\bar{r}$ is also a natural number, we may then write $\bar{r} <_\mathbb{Z} n$ equivalently as $\bar{r} <_\mathbb{N} n$, which implies $\bar{r} +_\mathbb{N} 1 \leq_\mathbb{N} n$ with (4.157). According to the Characterization of an induced irreflexive partial ordering, this inequality implies the disjunction $\bar{r} +_\mathbb{N} 1 <_\mathbb{N} n \vee \bar{r} +_\mathbb{N} 1 = n$, which we use now to prove the existential sentence (i.e., the consequent of the induction step)

$$\exists q, r\, (q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_\mathbb{Z} n \wedge m +_\mathbb{N} 1 = q \cdot_\mathbb{Z} n +_\mathbb{Z} r) \tag{6.327}$$

by cases. In case $\bar{r} +_\mathbb{N} 1 <_\mathbb{N} n$ holds, this existential sentence is true in view of $\bar{q} \in \mathbb{Z}$, $\bar{r} +_\mathbb{N} 1 \in \mathbb{N}$, the preceding inequality and (6.326). In the other case $\bar{r} +_\mathbb{N} 1 = n$, we obtain

$$\begin{aligned} m +_\mathbb{N} 1 &= \bar{q} \cdot_\mathbb{Z} n +_\mathbb{Z} n \\ &= \bar{q} \cdot_\mathbb{Z} n +_\mathbb{Z} 1 \cdot_\mathbb{Z} n \\ &= (\bar{q} +_\mathbb{Z} 1) \cdot_\mathbb{Z} n \\ &= (\bar{q} +_\mathbb{Z} 1) \cdot_\mathbb{Z} n +_\mathbb{Z} 0 \end{aligned} \tag{6.328}$$

by applying substitution to the equation (6.326) based on that case assumption, the definition of the unity element, the Distributive Law for $\mathbb{Z}$, and the definition of the zero element. Having thus found the numbers $\bar{q} +_\mathbb{Z} 1 \in \mathbb{Z}$ and $0 \in \mathbb{N}$ satisfying $0 <_\mathbb{Z} n$ and the equation (6.328), we now see that the existential sentence (6.327) is true also in the second case. As m was arbitrary, we may therefore conclude that the induction step holds (besides the base case), so that the proof of (6.325) is complete.

Next, we prove the universal sentence

$$\forall m\,(m \in \mathbb{Z} \Rightarrow \exists q, r\,(q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_{\mathbb{Z}} n \wedge m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r)), \quad (6.329)$$

letting $m \in \mathbb{Z}$ be arbitrary. In view of the totality of $\leq_{\mathbb{Z}}$, we have then the true disjunction $0 \leq_{\mathbb{Z}} m \vee m \leq_{\mathbb{Z}} 0$, which we use in the following to prove the desired consequent by cases. In the first case $0 \leq_{\mathbb{Z}} m$, it follows with Corollary 6.82 that m is a natural number in $\mathbb{Z}$, and we denote its isomorphic counterpart in $\mathbb{N}$ also by m. Then, $m \in \mathbb{N}$ implies the desired existential sentence in (6.329) with (6.325).

The second case $m \leq_{\mathbb{Z}} 0$ implies with the Monotony Law for $+$ and $\leq$ the inequality $m +_{\mathbb{Z}} (-m) \leq_{\mathbb{Z}} 0 +_{\mathbb{Z}} (-m)$ and therefore $0 \leq_{\mathbb{Z}} -m$ with the definitions of an additive inverse and of a zero element. In analogy to the first case, we may treat now $-m$ as an element of $\mathbb{N}$, so that there exist – according to (6.325) – particular numbers $\bar{q} \in \mathbb{Z}$ and $\bar{r} \in \mathbb{N}$ satisfying the inequality $\bar{r} <_{\mathbb{Z}} n$ and the equation

$$-m = \bar{q} \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} \bar{r}. \quad (6.330)$$

Here, $\bar{r} \in \mathbb{N}$ implies $0 \leq_{\mathbb{N}} \bar{r}$ with (4.187), which inequality we write in the form $0 \leq_{\mathbb{Z}} \bar{r}$ (treating $\bar{r}$ as a natural number in $\mathbb{Z}$). This inequality implies now the disjunction $0 <_{\mathbb{Z}} \bar{r} \vee 0 = \bar{r}$, which allows us to consider two sub-cases within the current second case.

The first sub-case $0 <_{\mathbb{Z}} \bar{r}$ $[<_{\mathbb{Z}} n]$ evidently implies with the Monotony Law for $+$ and $<$ the inequalities

$$-\bar{r} \;<_{\mathbb{Z}}\; 0 \;<_{\mathbb{Z}}\; n +_{\mathbb{Z}} (-\bar{r}).$$

Here, the first inequality implies $n +_{\mathbb{Z}} (-\bar{r}) <_{\mathbb{Z}} n$ (again with the Monotony Law for $+$ and $<$), and the second one implies $0 \leq_{\mathbb{Z}} n +_{\mathbb{Z}} (-\bar{r})$ (with the Characterization of an induced irreflexive partial ordering), so that $n -_{\mathbb{Z}} \bar{r}$ can be treated as a natural number (according to Corollary 6.82) satisfying $n -_{\mathbb{Z}} \bar{r} <_{\mathbb{Z}} n$. Thus, the difference $n -_{\mathbb{Z}} \bar{r}$ qualifies as a remainder.

Furthermore, we obtain the equations

$$\begin{aligned}
m &= -(-m) \\
&= -(\bar{q} \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} \bar{r}) \\
&= -(\bar{q} \cdot_{\mathbb{Z}} n) -_{\mathbb{Z}} \bar{r} \\
&= (-\bar{q}) \cdot_{\mathbb{Z}} n -_{\mathbb{Z}} \bar{r} \\
&= (-\bar{q}) \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} 0 -_{\mathbb{Z}} \bar{r} \\
&= (-\bar{q}) \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} (-n) +_{\mathbb{Z}} n -_{\mathbb{Z}} \bar{r} \\
&= (-\bar{q}) \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} 1 \cdot_{\mathbb{Z}} (-n) +_{\mathbb{Z}} n -_{\mathbb{Z}} \bar{r} \\
&= (-\bar{q}) \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} (-[1 \cdot_{\mathbb{Z}} n]) +_{\mathbb{Z}} n -_{\mathbb{Z}} \bar{r} \\
&= (-\bar{q}) \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} (-1) \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} n -_{\mathbb{Z}} \bar{r} \\
&= [(-\bar{q}) +_{\mathbb{Z}} (-1)] \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} [n -_{\mathbb{Z}} \bar{r}]
\end{aligned}$$

using the Sign Law (6.50) with respect to the group $(\mathbb{Z}, +_{\mathbb{Z}})$, substitution based on (6.330), the Sign Law (6.52), the Sign Law (6.64) with respect to the ring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}})$, the definition of the zero element in connection with the Associative Law for the addition on $\mathbb{Z}$ (which allows us to omit brackets), the definition of an additive inverse, the definition of the unity element, the Sign Law (6.63), again the Sign Law (6.64), and the Distributive Law for $\mathbb{Z}$. We thus found the integer $-\bar{q} -_{\mathbb{Z}} 1$ and the natural number $n -_{\mathbb{Z}} \bar{r}$ satisfying both $n -_{\mathbb{Z}} \bar{r} <_{\mathbb{Z}} n$ and the preceding equation, so that the existential sentence in (6.329) is true in the first sub-case.

In the second sub-case $0 = \bar{r}$, the inequality $\bar{r} <_{\mathbb{Z}} n$ gives us $0 <_{\mathbb{Z}} n$, so that the natural number 0 is a possible remainder. Moreover, the equation (6.330) yields evidently $-m = \bar{q} \cdot_{\mathbb{Z}} n$, and therefore

$$m = -(-m) = -(\bar{q} \cdot_{\mathbb{Z}} n) = (-\bar{q}) \cdot_{\mathbb{Z}} n = (-\bar{q}) \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} 0,$$

where $-\bar{q}$ is clearly an integer. These findings demonstrate that the existential sentence in (6.329) holds also in the second sub-case, so that the proof proof by (sub-)cases within the current second case is complete. This in turn completes the proof of the second case and thus the proof of the existential sentence in (6.329) by cases. Since m was initially arbitrary, we may now infer from this the truth of the universal sentence (6.329).

We prove now also the universal sentence

$$\begin{aligned}
&\forall m\,(m \in \mathbb{Z} \Rightarrow \forall q, r, q', r'\,([(q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_{\mathbb{Z}} n \wedge m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r) \\
&\quad \wedge (q' \in \mathbb{Z} \wedge r' \in \mathbb{N} \wedge r' <_{\mathbb{Z}} n \wedge m = q' \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r')] \Rightarrow [q = q' \wedge r = r']))
\end{aligned} \tag{6.331}$$

letting $m \in \mathbb{Z}$, $q, q' \in \mathbb{Z}$ and $r, r' \in \mathbb{N}$ be arbitrary, and assuming the inequalities $r <_{\mathbb{Z}} n$ and $r' <_{\mathbb{Z}} n$ as well as the equations $m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r$ and $m = q' \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r'$ to be true. Let us observe here that $r, r' \in \mathbb{N}$ imply $0 \leq_{\mathbb{N}} r$ and $0 \leq_{\mathbb{N}} r'$ with (4.187), which inequalities we may write as $0 \leq_{\mathbb{Z}} r$ and $0 \leq_{\mathbb{Z}} r'$ (treating 0, r and r' as natural numbers in $\mathbb{Z}$). Now, as $\leq_{\mathbb{Z}}$ is a total ordering, the disjunction $r \leq_{\mathbb{Z}} r' \vee r' \leq_{\mathbb{Z}} r$ is true, which will allows us to prove the equation $q = q'$ by cases.

The first case $[0 \leq_{\mathbb{Z}}]\ r \leq_{\mathbb{Z}} r'$ implies

$$0 +_{\mathbb{Z}} (-r) \ \leq_{\mathbb{Z}} \ r +_{\mathbb{Z}} (-r) \ \leq_{\mathbb{Z}} \ r' +_{\mathbb{Z}} (-r)$$

with the Monotony Law for + and $\leq$, with the consequence that

$$-r \ \leq_{\mathbb{Z}} \ 0 \ \leq_{\mathbb{Z}} \ r' -_{\mathbb{Z}} r, \tag{6.332}$$

using the definitions of the zero element, of the additive inverse and of the subtraction. Here, $-r \leq_{\mathbb{Z}} 0$ implies $-r +_{\mathbb{Z}} n \leq_{\mathbb{Z}} 0 +_{\mathbb{Z}} n$ (again with the Monotony Law for + and $\leq$) and therefore $n -_{\mathbb{Z}} r \leq_{\mathbb{Z}} n$ (using the Commutative Law for the addition on $\mathbb{Z}$ and the definition of the zero element). Furthermore, the assumed $r' <_{\mathbb{Z}} n$ implies with the Monotony Law for + and $<$ the inequality $r' +_{\mathbb{Z}} (-r) <_{\mathbb{Z}} n +_{\mathbb{Z}} (-r)$, which yields then (by definition of the subtraction)

$$r' -_{\mathbb{Z}} r \ <_{\mathbb{Z}} \ n -_{\mathbb{Z}} r \quad [\leq_{\mathbb{Z}} n]. \tag{6.333}$$

The Transitivity Formula for $<$ and $\leq$ gives us now $r' -_{\mathbb{Z}} r <_{\mathbb{Z}} n$, which means in connection with the second inequality in (6.332) that

$$0 \ \leq_{\mathbb{Z}} \ r' -_{\mathbb{Z}} r <_{\mathbb{Z}} n \tag{6.334}$$

Combining the assumed equations $m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r$ and $m = q' \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r'$ to

$$q' \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r' = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r, \tag{6.335}$$

we obtain then also

$$\begin{aligned} r' -_{\mathbb{Z}} r &= 0 +_{\mathbb{Z}} (r' +_{\mathbb{Z}} [-r]) \\ &= (q' \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} -[q' \cdot_{\mathbb{Z}} n]) +_{\mathbb{Z}} (r' +_{\mathbb{Z}} [-r]) \\ &= (q' \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r') +_{\mathbb{Z}} (-[q' \cdot_{\mathbb{Z}} n] +_{\mathbb{Z}} [-r]) \\ &= (q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r) +_{\mathbb{Z}} (-[q' \cdot_{\mathbb{Z}} n] +_{\mathbb{Z}} [-r]) \\ &= (q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} -[q' \cdot_{\mathbb{Z}} n]) +_{\mathbb{Z}} (r +_{\mathbb{Z}} [-r]) \\ &= (q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} [-q'] \cdot_{\mathbb{Z}} n) +_{\mathbb{Z}} 0 \\ &= (q +_{\mathbb{Z}} [-q']) \cdot_{\mathbb{Z}} n \end{aligned} \tag{6.336}$$

applying the definition of the zero element with the definition of a subtraction, the definition of the additive inverse, the Associative together with the Commutative Law for the addition on $\mathbb{Z}$, substitution based on (6.335), again the preceding Associative and Commutative Law, the definition of the additive inverse alongside the Sign Law (6.64), and finally the Distributive Law for $\mathbb{Z}$ together with the definition of the zero element. We may therefore carry out a substitution to (6.334) and write instead

$$0 \cdot_{\mathbb{Z}} n \ \leq_{\mathbb{Z}} \ (q +_{\mathbb{Z}} [-q']) \cdot_{\mathbb{Z}} n \ <_{\mathbb{Z}} \ 1 \cdot_{\mathbb{Z}} n,$$

using also the Cancellation Law for 0 and the definition of a unity element. The previous finding $0 <_{\mathbb{Z}} n$ allows us now to apply the Monotony Law for $\cdot$ and $\leq$ to the left inequality and to apply the Monotony Law for $\cdot$ and $<$ to the right inequality, resulting in

$$0 \ \leq_{\mathbb{Z}} \ q +_{\mathbb{Z}} [-q'] \ <_{\mathbb{Z}} \ 1.$$

Here, the first inequality implies with (6.289) that $q -_{\mathbb{Z}} q'$ is a natural number in $\mathbb{Z}$ (besides 0 and 1), so that we may rewrite the inequalities in the form

$$0 \ \leq_{\mathbb{N}} \ q +_{\mathbb{Z}} [-q'] \ <_{\mathbb{N}} \ 1.$$

This finding implies then $q +_{\mathbb{Z}} [-q'] = 0$ because of (4.173) and the fact $0^+ = 1$. Evidently, this gives us

$$\begin{aligned} q &= q +_{\mathbb{Z}} 0 = q +_{\mathbb{Z}} (-q' +_{\mathbb{Z}} q') = (q +_{\mathbb{Z}} [-q']) +_{\mathbb{Z}} q' = 0 +_{\mathbb{Z}} q' \\ &= q'. \end{aligned}$$

The resulting equation $q = q'$ evidently yields then in view of (6.336)

$$\begin{aligned} r' +_{\mathbb{Z}} [-r] &= (q +_{\mathbb{Z}} [-q']) \cdot_{\mathbb{Z}} n = (q' +_{\mathbb{Z}} [-q']) \cdot_{\mathbb{Z}} n = 0 \cdot_{\mathbb{Z}} n \\ &= 0, \end{aligned}$$

with the consequence that

$$\begin{aligned} r' &= r' +_{\mathbb{Z}} 0 = r' +_{\mathbb{Z}} (-r +_{\mathbb{Z}} r) = (r' +_{\mathbb{Z}} [-r]) +_{\mathbb{Z}} r = 0 +_{\mathbb{Z}} r \\ &= r. \end{aligned}$$

We thus find the desired consequent $q = q' \wedge r = r'$ to be true in the first case. The proof of the second case can be carried out by interchanging r and r' as well as q and q' in the preceding proof of the first case. Then, since q, r, q', r' and m were initially arbitrary, we may therefore conclude that the universal sentence (6.331) is true.

We are now in a position to prove the universal sentence (6.324). Letting

$m \in \mathbb{Z}$ be arbitrary, we see in light of (6.329) and (6.331) that the uniquely existential sentence

$$\exists! q, r\, (q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_{\mathbb{Z}} n \wedge m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r)$$

holds, according to Notation 1.4. As m was arbitrary, we may therefore conclude that the universal sentence (6.324) holds. Furthermore, as n was initially also arbitrary, (6.322) follows to be true as well.

Let us establish now the truth of (6.323), letting $n <_{\mathbb{Z}} 0$ and $m \in \mathbb{Z}$ be arbitrary. The preceding inequality evidently implies $0 <_{\mathbb{Z}} -n$, so that $-n$ satisfies the universal sentence (6.329), as shown in the first case. Therefore, $m \in \mathbb{Z}$ implies the existence of particular elements $\bar{q} \in \mathbb{Z}$ and $\bar{r} \in \mathbb{N}$ such that $\bar{r} <_{\mathbb{Z}} (-n)$ and $m = \bar{q} \cdot_{\mathbb{Z}} (-n) +_{\mathbb{Z}} \bar{r}$. Here, we may apply the two Sign Laws (6.63) and (6.64) to write the equation in the required form $m = (-q) \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} \bar{r}$. Thus, the existential part of the uniquely existential sentence in (6.323) holds. Regarding the uniqueness part

$$\begin{aligned} &\forall q, r, q', r'\, ([(q \in \mathbb{Z} \wedge r \in \mathbb{N} \wedge r <_{\mathbb{Z}} -n \wedge m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r) \\ &\quad \wedge (q' \in \mathbb{Z} \wedge r' \in \mathbb{N} \wedge r' <_{\mathbb{Z}} -n \wedge m = q' \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r')] \\ &\quad \Rightarrow [q = q' \wedge r = r']) \end{aligned} \tag{6.337}$$

we let q, r, q' and r' be arbitrary such that $q, q' \in \mathbb{Z}$, $r, r' \in \mathbb{N}$, $r <_{\mathbb{Z}} -n$, $r' <_{\mathbb{Z}} -n$, $m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r$ and $m = q' \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r'$ are all satisfied. The Sign Law (6.65) allows us to write the previous two equations as $m = (-q) \cdot_{\mathbb{Z}} (-n) +_{\mathbb{Z}} r$ and $m = (-q') \cdot_{\mathbb{Z}} (-n) +_{\mathbb{Z}} r'$. In view of $-q, -q' \in \mathbb{Z}$, $r, r' \in \mathbb{N}$, the inequalities $r <_{\mathbb{Z}} -n$ and $r' <_{\mathbb{Z}} -n$, and the preceding equations, we obtain with (6.331) the identities $-q = -q'$ and $r = r'$. The former equation implies $q = q'$ with the Sign Law (6.51), so that the implications in (6.337) are clearly true. As q, r, q' and r' were arbitrary, we may then infer from the truth of the implications the truth of the universal sentence (6.337) and thus the truth of the uniqueness part. Thus, the uniquely existential sentence in (6.323) holds, and since m and n were initially also arbitrary, the universal sentence (6.323) is also true. □

Note 6.27. In the equation $m = q \cdot_{\mathbb{Z}} n +_{\mathbb{Z}} r$, we call

- m the *dividend*,
- n the *divisor*,
- q the *quotient*, and
- r the *remainder*.

We also say that m divides by n with remainder r.

Similarly to the natural numbers, the particular case of $n = 2$ allows for the following distinction of 'even' and 'odd' integers.

Exercise 6.42. Prove that every integer m divides by 2 either with remainder 0 or with remainder 1, i.e.

$$\forall m \, (m \in \mathbb{Z} \Rightarrow ([\exists q \, (q \in \mathbb{Z} \wedge m = q \cdot_{\mathbb{Z}} 2) \vee \exists q \, (q \in \mathbb{Z} \wedge m = q \cdot_{\mathbb{Z}} 2 +_{\mathbb{Z}} 1)] \wedge \neg[\exists q \, (q \in \mathbb{Z} \wedge m = q \cdot_{\mathbb{Z}} 2) \wedge \exists q \, (q \in \mathbb{Z} \wedge m = q \cdot_{\mathbb{Z}} 2 +_{\mathbb{Z}} 1)])). \tag{6.338}$$

(Hint: Proceed in analogy to the proof of Proposition 5.141.)

Definition 6.24 (Even & odd integer). We say that an integer m

(1) is *even* iff
$$\exists q \, (q \in \mathbb{Z} \wedge m = 2 \cdot_{\mathbb{Z}} q). \tag{6.339}$$

(2) is *odd* iff
$$\exists q \, (q \in \mathbb{Z} \wedge m = 2 \cdot_{\mathbb{Z}} q +_{\mathbb{Z}} 1). \tag{6.340}$$

Note 6.28. In light of the previous Exercise 6.42, every integer is either even or odd.

The next definition, exercise and lemma prepare us for the concept of a 'greatest common divisor'. For greater clarity, we will often write $+$ instead of $+_{\mathbb{Z}}$ and $\cdot$ instead of $\cdot_{\mathbb{Z}}$.

Definition 6.25 (Common factor). For any integers $m_1 \neq 0$ and $m_2 \neq 0$ we say that an integer n is a *common factor of* m_1 and m_2 iff

$$\exists q_1 \, (q_1 \in \mathbb{Z} \wedge m_1 = q_1 \cdot n) \wedge \exists q_2 \, (q_2 \in \mathbb{Z} \wedge m_2 = q_2 \cdot n). \tag{6.341}$$

Proposition 6.89. *It is true for any integers $m_1 \neq 0$ and $m_2 \neq 0$ that the integer 0 is not a common factor of m_1 and m_2, that is,*

$$\forall m_1, m_2 \, ([m_1, m_2 \in \mathbb{Z} \wedge m_1 \neq 0 \wedge m_2 \neq 0] \Rightarrow \neg[\exists q_1 \, (q_1 \in \mathbb{Z} \wedge m_1 = q_1 \cdot 0) \wedge \exists q_2 \, (q_2 \in \mathbb{Z} \wedge m_2 = q_2 \cdot 0)]), \tag{6.342}$$

Proof. Letting $m_1, m_2 \in \mathbb{Z}$ be arbitrary and assuming $m_1 \neq 0$ and $m_2 \neq 0$, we may write the desired consequent equivalently as

$$\neg\exists q_1 \, (q_1 \in \mathbb{Z} \wedge m_1 = q_1 \cdot 0) \vee \neg\exists q_2 \, (q_2 \in \mathbb{Z} \wedge m_2 = q_2 \cdot 0)$$

by using De Morgan's Law for the conjunction. This disjunction is also equivalent to

$$\forall q_1 \, (q_1 \in \mathbb{Z} \Rightarrow m_1 \neq q_1 \cdot 0) \vee \forall q_2 \, (q_2 \in \mathbb{Z} \Rightarrow m_2 \neq q_2 \cdot 0)$$

because of the Negation Law for existential conjunctions. Letting now $q_1 \in \mathbb{Z}$ be arbitrary, we observe the truth of $q_1 \cdot 0 = 0$ in light of the Cancellation Law for 0. Then, the initial assumption $m_1 \neq 0$ gives us $m_1 \neq q_1 \cdot 0$ by means of substitution, as desired. Here, q_1 is arbitrary, so that the first part of the preceding disjunction follows to be true. Then, that disjunction is itself true, and consequently the equivalent negation in (6.342) holds. Since m_1 and m_2 were initially arbitrary, we may therefore conclude that the proposed universal holds, as claimed. □

Exercise 6.43. Verify that

a) 1 is a common factor of any two nonzero integers, that is,

$$\begin{aligned} &\forall m_1, m_2\, ([m_1, m_2 \in \mathbb{Z} \wedge m_1 \neq 0 \wedge m_2 \neq 0] \\ &\Rightarrow [\exists q_1\, (q_1 \in \mathbb{Z} \wedge m_1 = q_1 \cdot 1) \wedge \exists q_2\, (q_2 \in \mathbb{Z} \wedge m_2 = q_2 \cdot 1)]), \end{aligned} \tag{6.343}$$

b) -1 is a common factor of any two nonzero integers, that is,

$$\begin{aligned} &\forall m_1, m_2\, ([m_1, m_2 \in \mathbb{Z} \wedge m_1 \neq 0 \wedge m_2 \neq 0] \\ &\Rightarrow [\exists q_1\, (q_1 \in \mathbb{Z} \wedge m_1 = q_1 \cdot -1) \wedge \exists q_2\, (q_2 \in \mathbb{Z} \wedge m_2 = q_2 \cdot -1)]). \end{aligned} \tag{6.344}$$

(Hint: Use (5.90) and (6.51).)

Lemma 6.90. *The following sentences are true for any integers $m_1 \neq 0$ and $m_2 \neq 0$.*

a) There exists a unique set $\mathcal{X}$ consisting of all the sums

$$x \cdot_{\mathbb{Z}} m_1 + y \cdot m_2 \tag{6.345}$$

with $x, y \in \mathbb{Z}$, and there exists a unique set $\mathcal{Y}$ consisting of all the positive elements of $\mathcal{X}$.

b) The set $\mathcal{X}$ is not empty, containing in particular m_1, m_2, and 0.

c) The negative of every element of $\mathcal{X}$ is also in $\mathcal{X}$, that is,

$$\forall M\, (M \in \mathcal{X} \Rightarrow -M \in \mathcal{X}). \tag{6.346}$$

d) The set $\mathcal{Y}$ is not empty and has a least element.

e) Every element M of $\mathcal{X}$ can be written as the product $q \cdot n$ for any common factor n of m_1 and m_2 and for some integer q, that is,

$$\begin{aligned} &\forall M, n\, ([M \in \mathcal{X} \wedge n \text{ is a common factor of } m_1 \text{ and } m_2] \\ &\Rightarrow \exists q\, (q \in \mathbb{Z} \wedge M = q \cdot n)). \end{aligned} \tag{6.347}$$

Proof. We let m_1 and m_2 be arbitrary integers and assume $m_1 \neq 0$ as well as $m_2 \neq 0$. Concerning a), we first observe that $x \cdot m_1$ as well as $y \cdot m_2$ are integers for arbitrary $x, y \in \mathbb{Z}$, consequently also every sum $x \cdot m_1 + y \cdot m_2$. We also see in light of the Axiom of Specification and the Equality Criterion for sets on the one hand that there exists a unique set $\mathcal{X}$ such that

$$\forall M\,(M \in \mathcal{X} \Leftrightarrow [M \in \mathbb{Z} \wedge \exists x, y\,(x, y \in \mathbb{Z} \wedge M = x \cdot m_1 + y \cdot m_2)]). \quad (6.348)$$

We now verify that the set $\mathcal{X}$ satisfies also

$$\forall M\,(M \in \mathcal{X} \Leftrightarrow \exists x, y\,(x, y \in \mathbb{Z} \wedge M = x \cdot m_1 + y \cdot m_2)). \quad (6.349)$$

Letting M be arbitrary, the assumption $M \in \mathcal{X}$ implies the existential sentence in (6.349) due to (6.348). Conversely, the assumption that there exist constants, say $\bar{x}$ and $\bar{y}$, satisfying $\bar{x}, \bar{y} \in \mathbb{Z}$ and the equation $M = \bar{x} \cdot m_1 + \bar{y} \cdot m_2$ clearly shows that M is an integer, being the sum of two integers. This finding $M \in \mathbb{Z}$ and the assumed existential sentence imply now $M \in \mathcal{X}$ with (6.349), so that the equivalence in (6.349) holds. As M was arbitrary, we may therefore conclude that $\mathcal{X}$ satisfies indeed the universal sentence (6.349). Then, the Axiom of Specification and the Equality Criterion for sets show that there exists also a unique set $\mathcal{Y}$ with

$$\forall N\,(N \in \mathcal{Y} \Leftrightarrow [N \in \mathcal{X} \wedge N >_{\mathbb{Z}} 0). \quad (6.350)$$

Concerning b), we observe that the equations

$$\begin{aligned} m_1 &= 1 \cdot m_1 + 0 \cdot m_2 \\ m_2 &= 0 \cdot m_1 + 1 \cdot m_2 \\ 0 &= 0 \cdot m_1 + 0 \cdot m_2 \end{aligned}$$

are true because of the Cancellation Law for 0, the definition of the zero element and the definition of the unity element. Thus, the existential sentences

$$\begin{aligned} \exists x, y\,(x, y \in \mathbb{Z} \wedge m_1 &= x \cdot m_1 + y \cdot m_2) \\ \exists x, y\,(x, y \in \mathbb{Z} \wedge m_2 &= x \cdot m_1 + y \cdot m_2) \\ \exists x, y\,(x, y \in \mathbb{Z} \wedge 0 &= x \cdot m_1 + y \cdot m_2) \end{aligned}$$

hold, which imply then $m_1 \in \mathcal{X}$, $m_2 \in \mathcal{X}$ and $0 \in \mathcal{X}$ with (6.349).

Concerning c), we let M be an arbitrary element of $\mathcal{X}$, so that there exist according to (6.349) two integers, say $\bar{x}$ and $\bar{y}$, such that

$$M = \bar{x} \cdot m_1 + \bar{y} \cdot m_2.$$

Then, the negatives $-\bar{x}$ and $-\bar{y}$ are also integers, so that the integer

$$N = (-\bar{x}) \cdot m_1 + (-\bar{y}) \cdot m_2$$

is evidently also in $\mathcal{X}$. We now observe that

$$\begin{aligned} N &= (-\bar{x}) \cdot m_1 + (-\bar{y}) \cdot m_2 \\ &= -(\bar{x} \cdot m_1) - (\bar{y} \cdot m_2) \\ &= -(\bar{x} \cdot m_1 + \bar{y} \cdot m_2) \\ &= -M \end{aligned}$$

holds with the Sign Laws (6.63) and (6.52), so that the previous finding $N \in \mathcal{X}$ yields $-M \in \mathcal{X}$ via substitution. Since M was arbitrary, we may therefore conclude that (6.346) is indeed true.

Concerning d), since $\mathcal{X}$ contains for instance the integer m_1, which was assumed to be nonempty, so that the disjunction $m_1 <_{\mathbb{Z}} 0 \vee m_1 >_{\mathbb{Z}} 0$ follows to be true because of the connexity of the standard linear $<_{\mathbb{Z}}$ ordering. Let us use this disjunction to prove

$$\exists N \, (N \in \mathcal{Y}) \tag{6.351}$$

by cases. The first case $m_1 <_{\mathbb{Z}} 0$ implies with the Monotony Law for $+$ and $<$ the inequality $m_1 + (-m_1) <_{\mathbb{Z}} 0 + (-m_1)$ and therefore $0 <_{\mathbb{Z}} -m_1$ with the definition of an additive inverse and with the definition of the zero element. Furthermore, recalling from c) the truth of $m_1 \in \mathcal{X}$, we also find $-m_1 \in \mathcal{X}$ to be true. The conjunction of this finding and the previously found $-m_1 >_{\mathbb{Z}} 0$ implies then $-m_1 \in \mathcal{Y}$ with (6.350). Thus, there exists an element in $\mathcal{Y}$ in the first case.

The second case $m_1 >_{\mathbb{Z}} 0$ immediately implies – in conjunction with $m_1 \in \mathcal{X}$ – the truth of $m_1 \in \mathcal{Y}$ in view of with (6.350), so that the existential (6.351) is true again and holds thus for both cases.

Then, the truth of (6.351) implies the truth of $\mathcal{Y} \neq \emptyset$ with (2.42). Let us now verify also that $\mathcal{Y}$ is a subset of $\mathbb{N}$, i.e. that the universal sentence

$$\forall N \, (N \in \mathcal{Y} \Rightarrow N \in \mathbb{N}) \tag{6.352}$$

holds. Letting $N \in \mathcal{Y}$ be arbitrary, we obtain especially $N >_{\mathbb{Z}} 0$ with (6.350), so that the disjunction $0 <_{\mathbb{Z}} N \vee 0 = N$ is true. Consequently, $0 \leq_{\mathbb{Z}} N$ holds according to the Characterization of an induced irreflexive partial ordering, so that $N \in \mathbb{N}_{\mathbb{Z}}$ follows evidently to be true with (6.289). Thus, N is a natural number in $\mathbb{Z}$, whose isomorphic counterpart N is in $\mathbb{N}$, so that we may consider the implication in (6.351) as true. As N was arbitrary, we may therefore conclude that the universal sentence (6.352) is also true. We thus demonstrated that $\mathcal{Y}$ is a nonempty subset of $\mathbb{N}$, so that the least element of $\mathcal{Y}$ exists due to the well-ordering of $\mathbb{N}$.

Concerning e), we let M be an arbitrary element of $\mathcal{X}$ (so that $M = \bar{x} \cdot m_1 + \bar{y} \cdot m_2$ holds for particular elements $\bar{x}, \bar{y} \in \mathbb{Z}$) and we let n an arbitrary common factor of m_1 and m_2 (so that $m_1 = \bar{q}_1 \cdot n$ and $m_2 = \bar{q}_2 \cdot n$ are true for particular elements $\bar{q}_1, \bar{q}_2 \in \mathbb{Z}$). Consequently, we obtain

$$\begin{aligned} M &= \bar{x} \cdot (\bar{q}_1 \cdot n) + \bar{y} \cdot (\bar{q}_2 \cdot n) \\ &= (\bar{x} \cdot \bar{q}_1) \cdot n + (\bar{y} \cdot \bar{q}_2) \cdot n \\ &= (\bar{x} \cdot \bar{q}_1 + \bar{y} \cdot \bar{q}_2) \cdot n \end{aligned}$$

by applying substitutions, the Associative Law for the multiplication on $\mathbb{Z}$, and the Distributive Law for $\mathbb{Z}$. Clearly, the sum $\bar{q} = \bar{x} \cdot \bar{q}_1 + \bar{y} \cdot \bar{q}_2$ is an integer, so that we have $M = \bar{q} \cdot n$ with $\bar{q} \in \mathbb{Z}$. Thus, the existential sentence in (6.347) is true, and since M and n are arbitrary, we can infer from this the truth of the universal sentence (6.347). □

Theorem 6.91 (Unique existence of the greatest common factor). *It is true for any integers $m_1 \neq 0$ and $m_2 \neq 0$ that there exists a positive natural number N such that N is a common factor of m_1, m_2 which can be written as the product $N = q \cdot n$ for any common factor n of m_1, m_2 and for some integer q, that is,*

$$\begin{aligned} &\forall m_1, m_2\, ([m_1, m_2 \in \mathbb{Z} \wedge m_1 \neq 0 \wedge m_2 \neq 0] \\ &\quad \Rightarrow \exists! N\, (N \in \mathbb{N}_+ \wedge N \text{ is a common factor of } m_1 \text{ and } m_2 \\ &\qquad \wedge \forall n\, (n \text{ is a common factor of } m_1 \text{ and } m_2 \Rightarrow \exists q\, (q \in \mathbb{Z} \wedge N = q \cdot n)))). \end{aligned} \tag{6.353}$$

Then, this common factor N of m_1 and m_2 is greater than or equal to all common factors of m_1 and m_2, that is,

$$\forall n\, (n \text{ is a common factor of } m_1 \text{ and } m_2 \Rightarrow N \geq_{\mathbb{Z}} n)). \tag{6.354}$$

Proof. We let m_1 and m_2 be arbitrary nonzero integers and show that the least element N of the set $\mathcal{Y}$ consisting of all the positive integers $x \cdot m_1 + y \cdot m_2$ with $x, y \in \mathbb{Z}$, as specified in (6.350), is the desired integer. For this purpose, we verify first that N is a common factor of m_1 and m_2. Since the least element N of $\mathcal{Y}$ is by definition an element of $\mathcal{Y}$, it follows with (6.350) that N is also an element of the set $\mathcal{X}$ specified by (6.348). Thus, $N \in \mathcal{X}$ implies the existence of particular integers $\bar{x}$ and $\bar{y}$ satisfying

$$N = \bar{x} \cdot m_1 + \bar{y} \cdot m_2. \tag{6.355}$$

Furthermore, $N \in \mathcal{Y}$ implies with (6.350) that $N >_{\mathbb{Z}} 0$ is true, so that the division of the integers m_1 and m_2 by N yields, according to (6.322), unique integers q_1, q_2 and unique natural numbers r_1, r_2 satisfying the inequalities $r_2 <_{\mathbb{Z}} N$ and $r_2 <_{\mathbb{Z}} N$ as well as the equations

$$m_1 = q_1 \cdot N + r_1, \tag{6.356}$$

$$m_2 = q_2 \cdot N + r_2. \tag{6.357}$$

We obtain then the equations

$$\begin{aligned} r_1 &= m_1 + (-m_1) + r_1 \\ &= m_1 + (-[q_1 \cdot N + r_1]) + r_1 \\ &= m_1 + (-[q_1 \cdot N] - r_1) + r_1 \\ &= m_1 + (-q_1) \cdot N + (-r_1) + r_1 \\ &= m_1 + (-q_1) \cdot (\bar{x} \cdot m_1 + \bar{y} \cdot m_2) \\ &= m_1 + (-q_1) \cdot (\bar{x} \cdot m_1) + (-q_1) \cdot (\bar{y} \cdot m_2) \\ &= 1 \cdot m_1 + [(-q_1) \cdot \bar{x}] \cdot m_1 + [(-q_1) \cdot \bar{y}] \cdot m_2 \\ &= [1 + (-q_1) \cdot \bar{x}] \cdot m_1 + [(-q_1) \cdot \bar{y}] \cdot m_2 \end{aligned}$$

by applying the definition of the zero element together with the definition of an additive inverse and the Associative Law for the addition on $\mathbb{Z}$ (allowing us in the following to omit brackets with respect to the multiple sums), substitution based on (6.356), the Sign Law (6.52), the Sign Law (6.64) together with the definition of the subtraction, substitution based on (6.355) together with the definition of an additive inverse and with the definition of the zero element, the Distributive Law for $\mathbb{Z}$, the definition of the unity element alongside the Associative Law for the multiplication on $\mathbb{Z}$, and finally again the Distributive Law for $\mathbb{Z}$. Using essentially the same arguments, applying now substitution based on (6.357) and additionally the Commutative Law for the addition on $\mathbb{Z}$, we get also

$$\begin{aligned} r_2 &= m_2 + (-m_2) + r_2 \\ &= m_2 + (-[q_2 \cdot N + r_2]) + r_2 \\ &= m_2 + (-[q_2 \cdot N] - r_2) + r_2 \\ &= m_2 + (-q_2) \cdot N + (-r_2) + r_2 \\ &= m_2 + (-q_2) \cdot (\bar{x} \cdot m_1 + \bar{y} \cdot m_2) \\ &= m_2 + (-q_2) \cdot (\bar{x} \cdot m_1) + (-q_2) \cdot (\bar{y} \cdot m_2) \\ &= [(-q_2) \cdot \bar{x}] \cdot m_1 + 1 \cdot m_2 + [(-q_2) \cdot \bar{y}] \cdot m_2 \\ &= [(-q_2) \cdot \bar{x}] \cdot m_1 + [1 + (-q_2) \cdot \bar{y}] \cdot m_2. \end{aligned}$$

As sums/products with respect the addition/multiplication on $\mathbb{Z}$, the numbers $[1 + (-q_1) \cdot \bar{x}]$, $[(-q_1) \cdot \bar{y}]$, $[(-q_2) \cdot \bar{x}]$ and $[1 + (-q_2) \cdot \bar{y}]$ are integers, so that r_1 and r_2 turn out to be elements of $\mathcal{X}$, according to (6.349). Let us recall the fact $r_1, r_2 \in \mathbb{N}$, so that $0 \leq_{\mathbb{N}} r_1$ and $0 \leq_{\mathbb{N}} r_2$ hold due to (4.187). According to the Characterization of an induced irreflexive partial ordering, these inequalities imply the disjunctions

$$0 <_{\mathbb{N}} r_1 \lor 0 = r_1 \tag{6.358}$$

$$0 <_{\mathbb{N}} r_2 \lor 0 = r_2, \tag{6.359}$$

whose first parts we now prove to be false by establishing contradictions. To prove the negations $\neg 0 <_{\mathbb{N}} r_1$ and $\neg 0 <_{\mathbb{N}} r_2$, we assume then the negations of these negations to be true, so that the Double Negation Law gives us the true sentences $0 <_{\mathbb{N}} r_1$ and $0 <_{\mathbb{N}} r_2$. Viewing 0, r_1 and r_2 as integers, we may write these inequalities equivalently as $0 <_{\mathbb{Z}} r_1$ and $0 <_{\mathbb{Z}} r_2$. Thus, the conjunctions

$$r_1 \in \mathcal{X} \land r_1 >_{\mathbb{Z}} 0,$$
$$r_2 \in \mathcal{X} \land r_2 >_{\mathbb{Z}} 0$$

are true, which in turn imply $r_1, r_2 \in \mathcal{Y}$ with (6.350). Since the least element N of $\mathcal{Y}$ is by definition a lower bound for that set, we obtain then $N \leq_{\mathbb{Z}} r_1$ and $N \leq_{\mathbb{Z}} r_2$. Consequently, the Negation Formula for $<$ yields $\neg r_1 <_{\mathbb{Z}} N$ as well as $\neg r_2 <_{\mathbb{Z}} N$, in contradiction to the previous findings $r_2 <_{\mathbb{Z}} N$ and $r_2 <_{\mathbb{Z}} N$. We thus completed the proof of the negations $\neg 0 <_{\mathbb{N}} r_1$ and $\neg 0 <_{\mathbb{N}} r_2$, so that the first parts of the true disjunctions (6.358) – (6.359) are indeed false. This means that the second parts $r_1 = 0$ and $r_2 = 0$ must be true. Applying now substitutions to (6.356) and (6.357) based on the preceding equations results in $m_1 = q_1 \cdot N$ and $m_2 = q_2 \cdot N$ (applying the definition of the zero element), so that N is by definition a common factor of m_1 and m_2.

Let us recall the previous finding $N >_{\mathbb{Z}} 0$, which implies on the one hand $0 \leq_{\mathbb{Z}} N$ with the Characterization of an induced irreflexive partial ordering, and on the other hand $N \neq 0$ with the Characterization of comparability regarding the linear ordering $<_{\mathbb{Z}}$. We may therefore view N as a nonzero natural number, and thus as an element of $\mathbb{N}_+$ according to (2.310).

Next, we establish the third part of the conjunction in (6.353), that is, the universal sentence

$$\forall n\,(n \text{ is a common factor of } m_1 \text{ and } m_2 \Rightarrow \exists q\,(q \in \mathbb{Z} \land N = q \cdot n)). \tag{6.360}$$

Letting n be an arbitrary common factor of m_1 and m_2 and noting that the previously established $N \in \mathcal{Y}$ implies $N \in \mathcal{X}$ with (6.350), the existential sentence in (6.360) immediately follows to be true with Lemma 6.90e). Since n is arbitrary, we may therefore conclude that the universal sentence (6.360) holds, so that the proof of the existential part of the uniquely existential sentence in (6.353) is now complete.

Next, we take an arbitrary set N^* such N^* is both a positive natural number and a common factor of m_1 and m_2 satisfying

$$\forall n\,(n \text{ is a common factor of } m_1 \text{ and } m_2 \Rightarrow \exists q\,(q \in \mathbb{Z} \wedge N^* = q \cdot n)), \tag{6.361}$$

and we demonstrate that $N^* = N$ follows to be true. Let us observe first that $N, N^* \in \mathbb{N}_+$ implies $0 <_{\mathbb{N}} 1 \leq_{\mathbb{N}} N$ and $0 <_{\mathbb{N}} 1 \leq_{\mathbb{N}} N^*$ with (4.164) and (4.278). Consequently, we obtain $0 <_{\mathbb{N}} N$ and $0 <_{\mathbb{N}} N^*$ with the Transitivity Formula for $<$ and $\leq$, which inequalities show in light of the Characterization of comparability that $N \neq 0$ and $N^* \neq 0$ are true. As N satisfies (6.360) and as N^* is a common factor of m_1 and m_2, there exists an integer, say $\bar{q}$, for which $N = \bar{q} \cdot N^*$ holds. Furthermore, since N^* satisfies (6.361) and since N is a common factor of m_1 and m_2, there is also an integer, say $\bar{q}^*$, such that $N^* = \bar{q}^* \cdot N$.

We now carry out a proof by contradiction to establish $0 <_{\mathbb{N}} \bar{q}^*$, assuming the negation $\neg 0 <_{\mathbb{N}} \bar{q}^*$ to be true, so that we have $\bar{q}^* \leq_{\mathbb{N}} 0$ according to the Negation Formula for $<$. Because of the previously found $0 <_{\mathbb{N}} N$, the Monotony Law for $\cdot_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$ yields now $\bar{q}^* \cdot_{\mathbb{N}} N \leq_{\mathbb{N}} 0 \cdot_{\mathbb{N}} N$, and therefore $N^* \leq_{\mathbb{N}} 0$ after applying substitution and the Cancellation Law for 0. Because this inequality implies $\neg 0 <_{\mathbb{N}} N^*$ with the Negation Formula for $<$ and as we found $0 <_{\mathbb{N}} N^*$ to be also true, we obtained a contradiction, so that $0 <_{\mathbb{N}} \bar{q}^*$ holds indeed.

We can use similar arguments to prove also $0 <_{\mathbb{N}} \bar{q}$ by contradiction. Indeed, the assumption $\neg 0 <_{\mathbb{N}} \bar{q}$ implies $\bar{q} \leq_{\mathbb{N}} 0$, and this gives us in view of the previously found $0 <_{\mathbb{N}} N^*$ the new inequality $\bar{q} \cdot_{\mathbb{N}} N^* \leq_{\mathbb{N}} 0 \cdot_{\mathbb{N}} N^*$, so that evidently $N \leq_{\mathbb{N}} 0$. This means that the negation $\neg 0 <_{\mathbb{N}} N$ holds, which is in contradiction to $0 <_{\mathbb{N}} N$.

Moreover, we obtain the equations

$$N \cdot 1 = N = \bar{q} \cdot N^* = \bar{q} \cdot (\bar{q}^* \cdot N) = N \cdot (\bar{q} \cdot \bar{q}^*)$$

by applying the definition of the unity element, substitution based on $N = \bar{q} \cdot N^*$, then substitution based on $N^* = \bar{q}^* \cdot N$, and finally the Associative in connection with the Commutative Law for the multiplication on $\mathbb{Z}$. Because of $N \neq 0$, we can apply the Cancellation Law for $\cdot$ and simplify the resulting equation $N \cdot 1 = N \cdot (\bar{q} \cdot \bar{q}^*)$ to $1 = \bar{q} \cdot \bar{q}^*$.

Let us observe now that the previously established inequalities $0 <_{\mathbb{N}} \bar{q}$ and $0 <_{\mathbb{N}} \bar{q}^*$ imply $1 \leq_{\mathbb{N}} \bar{q}$ and $1 \leq_{\mathbb{N}} \bar{q}^*$ with (2.291) and (4.157). The former inequality gives us now (due to the Characterization of an induced irreflexive partial ordering) the true disjunction $1 <_{\mathbb{N}} \bar{q}^* \vee 1 = \bar{q}^*$, whose first part we demonstrate now to be false by contradiction. To establish the negation $\neg 1 <_{\mathbb{N}} \bar{q}^*$, we assume its negation to be true, so that we obtain the true inequality $1 <_{\mathbb{N}} \bar{q}^*$ with the Double Negation Law. This yields because of $0 <_{\mathbb{N}} \bar{q}$, the inequality $1 \cdot \bar{q} <_{\mathbb{N}} \bar{q}^* \cdot \bar{q}$ with the Monotony Law for $\cdot$ and $<$, and then $\bar{q} <_{\mathbb{N}} \bar{q} \cdot \bar{q}^*$ $[= 1]$ with the definition of the unity element and the Commutative Law for the multiplication on $\mathbb{Z}$, that is, $\bar{q} <_{\mathbb{N}} 1$. Consequently, the negation $\neg 1 \leq_{\mathbb{N}} \bar{q}$ holds, in contradiction to the previous finding $1 \leq_{\mathbb{N}} \bar{q}$. This completes the proof of the negation $\neg 1 <_{\mathbb{N}} \bar{q}^*$, which means that the first part of the disjunction $1 <_{\mathbb{N}} \bar{q}^* \vee 1 = \bar{q}^*$ is false. Thus, its second part $\bar{q}^* = 1$ is true, and this equation allows us to rewrite $N^* = \bar{q}^* \cdot N$ as $N^* = 1 \cdot N$, which clearly demonstrates the truth of the desired equation $N^* = N$. Therefore, the uniquely existential sentence in (6.353) follows evidently to be true, according to the proof technique (1.111).

Regarding the proof of (6.354), we let n be an arbitrary common factor of m_1 and m_2, and we prove the desired consequent $N \geq_{\mathbb{Z}} n$ by contradiction, assuming its negation $\neg N \geq_{\mathbb{Z}} n$ to be true. This assumption gives us $N <_{\mathbb{Z}} n$ with the Negation Formula for $\leq$, which implies then in conjunction with the previously established fact $0 <_{\mathbb{Z}} N$ the truth of the inequality $0 <_{\mathbb{Z}} n$ by means of the transitivity of the linear ordering $<_{\mathbb{Z}}$. Now, because N satisfies (6.360), we can write this number as the product $N = \bar{q} \cdot n$ for a particular integer $\bar{q}$. Let us prove here $0 <_{\mathbb{Z}} \bar{q}$ by contradiction, assuming $\neg 0 <_{\mathbb{Z}} \bar{q}$ to be true, so that the Negation Formula for $<$ gives us $\bar{q} \leq_{\mathbb{Z}} 0$. This implies then in view of $0 <_{\mathbb{Z}} n$ also the inequality $\bar{q} \cdot n \leq_{\mathbb{Z}} 0 \cdot n$ with the Monotony Law for $\cdot$ and $\leq$, which in turn yields $N \leq_{\mathbb{Z}} 0$ by means of substitution and the Cancellation Law for 0. Consequently, the negation $\neg N >_{\mathbb{Z}} 0$ follows to be true with the Negation Formula for $<$, which finding contradicts the fact $N >_{\mathbb{Z}} 0$, completing the proof of $0 <_{\mathbb{Z}} \bar{q}$. This inequality clearly shows that we may treat $\bar{q}$ as a natural number, so that we may write the inequality also as $0 <_{\mathbb{N}} \bar{q}$, which further implies $1 \leq_{\mathbb{N}} \bar{q}$ with (2.291) and (4.157). Taking $\bar{q}$ for an integer again, we thus have $1 \leq_{\mathbb{Z}} \bar{q}$, which implies because of $0 <_{\mathbb{Z}} n$ the truth of $1 \cdot n \leq_{\mathbb{Z}} \bar{q} \cdot n$ with the Monotony Law for $\cdot$ and $\leq$. We therefore obtain $n \leq_{\mathbb{Z}} N$ with the definition of the unity element and with the equation $N = \bar{q} \cdot n$, which inequality clearly contradicts the assumed negation $\neg N \geq_{\mathbb{Z}} n$. Having completed the proof of $N \geq_{\mathbb{Z}} n$ by contradiction, we may now infer from this the truth of (6.354), since n was arbitrary.

As m_1 and m_2 were initially also arbitrary, we may finally conclude that the stated theorem holds. □

Notation 6.11. We symbolize the greatest common factor N of any integers $m_1 \neq 0$ and $m_2 \neq 0$ (established in preceding Theorem 6.91) also by

$$\operatorname{gcf}(m_1, m_2). \tag{6.362}$$

We see from the proof of the previous Theorem 6.91 that the greatest common factor exists uniquely for any two nonzero integers, that it is greater than or equal to 1, and that it satisfies the following equation. Equation (6.355) in the proof of the Unique existence of the greatest common factor N of any two nonzero integers m_1 and m_2 immediately gives us the following statement.

Corollary 6.92 (Bézout's Identity). *It is true that the greatest common factor of any integers $m_1 \neq 0$ and $m_2 \neq 0$ satisfies the equation*

$$\operatorname{gcf}(m_1, m_2) = x \cdot m_1 + y \cdot m_2. \tag{6.363}$$

for some integers x and y.

Definition 6.26 (Coprime/relatively prime integers). We say that any two integers $m_1 \neq 0$ and $m_2 \neq 0$ are *coprime* or *relatively prime* iff the greatest common factor of m_1 and m_2 equals 1, that is, iff

$$\operatorname{gcf}(m_1, m_2) = 1. \tag{6.364}$$

Theorem 6.93 (Coprimality of the quotients with respect to the greatest common factor). *It is true that dividing two nonzero integers by their greatest common factor yields unique coprime quotients, that is,*

$$\begin{aligned}
&\forall m_1, m_2 \, ([m_1, m_2 \in \mathbb{Z} \wedge m_1 \neq 0 \wedge m_2 \neq 0] \\
&\quad \Rightarrow \exists! q_1, q_2 \, (q_1, q_2 \in \mathbb{Z} \wedge q_1 \neq 0 \wedge q_2 \neq 0 \wedge m_1 = q_1 \cdot \operatorname{gcf}(m_1, m_2) \\
&\qquad\qquad \wedge m_2 = q_2 \cdot \operatorname{gcf}(m_1, m_2) \wedge q_1 \text{ and } q_2 \text{ are coprime})).
\end{aligned} \tag{6.365}$$

Proof. We let m_1 and m_2 be arbitrary nonzero integers, so that the greatest common factor $\operatorname{gcf}(m_1, m_2)$ of m_1 and m_2 is a uniquely determined. Therefore, there exists by definition of a common factor on the one hand a particular integer $\bar{q}_1$ satisfying

$$m_1 = \bar{q}_1 \cdot \operatorname{gcf}(m_1, m_2), \tag{6.366}$$

and there exists on the other hand a particular integer $\bar{q}_2$ satisfying

$$m_2 = \bar{q}_2 \cdot \operatorname{gcf}(m_1, m_2). \tag{6.367}$$

Here, we can prove by contradiction that $\bar{q}_1$ and $\bar{q}_2$ are nonzero. Indeed, assuming first $\neg \bar{q}_1 \neq 0$ to be true, which yields $\bar{q}_1 = 0$ with the Double

Negation Law, we obtain $m_1 = 0 \cdot \mathrm{gcf}(m_1, m_2) = 0$ by means of substitution in (6.366) and the Cancellation Law for 0, so that the resulting equation $m_1 = 0$ contradicts the assumption $m_1 \neq 0$. Similarly, assuming $\neg \bar{q}_2 \neq 0$ to be true gives us evidently $\bar{q}_2 = 0$ and therefore $m_2 = 0 \cdot \mathrm{gcf}(m_1, m_2) = 0$ in view of (6.367), so that the resultant $m_2 = 0$ contradicts the other assumption $m_2 \neq 0$. We thus found

$$\bar{q}_1 \neq 0 \wedge \bar{q}_2 \neq 0. \tag{6.368}$$

Next, we prove now by contradiction that $\bar{q}_1$ and $\bar{q}_2$ are coprime, assuming that they are not. Since $\bar{q}_1$ and $\bar{q}_2$ are nonzero, their greatest common factor $\mathrm{gcf}(\bar{q}_1, \bar{q}_2)$ is indeed defined, and the preceding assumption implies then $\mathrm{gcf}(\bar{q}_1, \bar{q}_2) \neq 1$ by definition of coprime integers (in connection with the Law of Contraposition). According to the Unique existence of the greatest common factor, we have $\mathrm{gcf}(\bar{q}_1, \bar{q}_2) \in \mathbb{N}_+$ and therefore $1 \leq_{\mathbb{N}} \mathrm{gcf}(\bar{q}_1, \bar{q}_2)$ according to (4.278). Due to the preceding inequality, we thus have in view of the Characterization of an induced irreflexive partial ordering

$$1 <_{\mathbb{N}} \mathrm{gcf}(\bar{q}_1, \bar{q}_2). \tag{6.369}$$

Furthermore, the greatest common factor $\mathrm{gcf}(\bar{q}_1, \bar{q}_2)$, as a common factor of $\bar{q}_1$ and $\bar{q}_2$, satisfies the equations

$$\begin{aligned} \bar{q}_1 &= \bar{Q}_1 \cdot \mathrm{gcf}(\bar{q}_1, \bar{q}_2) \\ \bar{q}_2 &= \bar{Q}_2 \cdot \mathrm{gcf}(\bar{q}_1, \bar{q}_2) \end{aligned}$$

for some particular integers $\bar{Q}_1$ and $\bar{Q}_2$. Applying substitutions to (6.366) and (6.367) based on the preceding equations and using the Commutative Law for the multiplication on $\mathbb{Z}$ gives now

$$\begin{aligned} m_1 &= [\bar{Q}_1 \cdot \mathrm{gcf}(\bar{q}_1, \bar{q}_2)] \cdot \mathrm{gcf}(m_1, m_2) = \bar{Q}_1 \cdot [\mathrm{gcf}(\bar{q}_1, \bar{q}_2) \cdot \mathrm{gcf}(m_1, m_2)], \\ m_2 &= [\bar{Q}_2 \cdot \mathrm{gcf}(\bar{q}_1, \bar{q}_2)] \cdot \mathrm{gcf}(m_1, m_2) = \bar{Q}_2 \cdot [\mathrm{gcf}(\bar{q}_1, \bar{q}_2) \cdot \mathrm{gcf}(m_1, m_2)]. \end{aligned}$$

Consequently, the conjunction of the existential sentences

$$\begin{aligned} &\exists q_1 \, (q_1 \in \mathbb{Z} \wedge m_1 = q_1 \cdot [\mathrm{gcf}(\bar{q}_1, \bar{q}_2) \cdot \mathrm{gcf}(m_1, m_2)]) \\ &\exists q_2 \, (q_2 \in \mathbb{Z} \wedge m_2 = q_2 \cdot [\mathrm{gcf}(\bar{q}_1, \bar{q}_2) \cdot \mathrm{gcf}(m_1, m_2)]) \end{aligned}$$

is clearly true, which means that the product $\mathrm{gcf}(\bar{q}_1, \bar{q}_2) \cdot \mathrm{gcf}(m_1, m_2)$ is a common factor of m_1 and m_2. According to Theorem 6.91, this implies

$$\mathrm{gcf}(m_1, m_2) \;\geq_{\mathbb{Z}}\; \mathrm{gcf}(\bar{q}_1, \bar{q}_2) \cdot \mathrm{gcf}(m_1, m_2). \tag{6.370}$$

Let us observe now that the greatest common factor $\mathrm{gcf}(m_1, m_2)$ is an element of $\mathbb{N}_+$, which thus satisfies $0 <_{\mathbb{N}} 1 \leq_{\mathbb{N}} \mathrm{gcf}(m_1, m_2)$ due to (4.164) and

(4.278), so that $0 <_{\mathbb{N}} \operatorname{gcf}(m_1, m_2)$ follows to be true with the Transitivity Law for $<$ and $\leq$. With this finding, the inequality (6.369) implies with the definition of the unity element and the Monotony Law for $\cdot$ and $<$

$$[\operatorname{gcf}(m_1, m_2) =] \quad 1 \cdot \operatorname{gcf}(m_1, m_2) \; <_{\mathbb{N}} \; \operatorname{gcf}(\bar{q}_1, \bar{q}_2) \cdot \operatorname{gcf}(m_1, m_2).$$

Viewing all of the numbers as integers and applying the Negation Formula for $\leq$, we obtain the true negation

$$\neg \operatorname{gcf}(m_1, m_2) \; \geq_{\mathbb{Z}} \; \operatorname{gcf}(\bar{q}_1, \bar{q}_2) \cdot \operatorname{gcf}(m_1, m_2),$$

which means in view of the simultaneous truth of (6.370) that we arrived at a contradiction. This completes the proof that $\bar{q}_1$ and $\bar{q}_2$ are coprime, which implies in conjunction with (6.366) – (6.368) the truth of the existential part of the uniquely existential sentence (6.365).

To establish the uniqueness part, we now take arbitrary sets q_1, q_2, q_1', q_2', assuming $q_1, q_2, q_1', q_2' \in \mathbb{Z} \setminus \{0\}$, assuming the equations

$$m_1 = q_1 \cdot \operatorname{gcf}(m_1, m_2) \wedge m_2 = q_2 \cdot \operatorname{gcf}(m_1, m_2),$$
$$m_1 = q_1' \cdot \operatorname{gcf}(m_1, m_2) \wedge m_2 = q_2' \cdot \operatorname{gcf}(m_1, m_2),$$

and assuming that q_1 and q_2 as well as q_1' and q_2' are coprime. Applying the Commutative Law for the multiplication on $\mathbb{Z}$ and combining the two equations for m_1 as well as the two equations for m_2 gives us the new equations

$$\operatorname{gcf}(m_1, m_2) \cdot q_1 = \operatorname{gcf}(m_1, m_2) \cdot q_1'$$
$$\operatorname{gcf}(m_1, m_2) \cdot q_2 = \operatorname{gcf}(m_1, m_2) \cdot q_2'$$

Recalling the truth of $0 <_{\mathbb{N}} \operatorname{gcf}(m_1, m_2)$, we thus have $\operatorname{gcf}(m_1, m_2) \neq 0$ according to the Characterization of comparability with respect to the standard linear ordering of $\mathbb{N}$. We may therefore apply the Cancellation Law for $\cdot$ to the preceding equations and infer the truth of $q_1 = q_1'$ and $q_2 = q_2'$. Since q_1, q_2, q_1' and q_2' were arbitrary, we may now conclude in view of Notation 1.4 that the uniquely existential sentence in (6.365) holds. Here, m_1 and m_2 were initially arbitrary, so that the theorem follows to be true. □

6.8. Multiples and Powers of Group Elements

Exercise 6.44. Show for any group $(X, +_X)$ with zero element 0_X, for any integer $m <_{\mathbb{Z}} 0$ and for any element $a \in X$ that the multiple $-ma$ in the sense of (5.468) is defined.

Notation 6.12. For any group $(X, +_X)$ with zero element 0_X, for any integer $m <_{\mathbb{Z}} 0$ and for any element $a \in X$, we write

$$ma = (-m)(-a) \tag{6.371}$$

and speak of a *multiple* of a; here, we call m the *multiplier*.

Exercise 6.45. Show for any group $(X, \cdot_X)$ with unity element 1_X, for any integer $m <_{\mathbb{Z}} 0$ and for any element $a \in X$ that the power a^{-m} in the sense of (5.470) is defined.

Notation 6.13. For any group $(X, \cdot_X)$ with unity element 1_X, for any integer $m <_{\mathbb{Z}} 0$ and for any element $a \in X$, we write

$$a^m = (a^{-1})^{-m} \tag{6.372}$$

and speak of the *m-th power* of a; here, we call a the *base* and m the *exponent* (group).

Proposition 6.94. *It is true for any group $(X, \cdot_X)$ with unity element 1_X and for any element $a \in X$ that*

$$\forall m \, (m \in \mathbb{Z} \Rightarrow a^{-m} = (a^{-1})^m). \tag{6.373}$$

Proof. We let X, $\cdot_X$, a and m be arbitrary such that $(X, \cdot_X)$ is a group with unity element 1_X, such that a is an element of X and such that m is an integer. Thus, the inverse element of a exists (in X). Noting that the connexity of the linear ordering $<_{\mathbb{Z}}$ gives the true disjunction $m <_{\mathbb{Z}} 0 \vee m = 0 \vee m >_{\mathbb{Z}} 0$, we now prove $a^{-m} = (a^{-1})^m$ by cases. In the first case $m <_{\mathbb{Z}} 0$, we obtain the true equations

$$(a^{-1})^m = ([a^{-1}]^{-1})^{-m} = a^{-m}$$

with (6.372) and (6.22). In the second case $m = 0$, we get

$$(a^{-1})^m = (a^{-1})^0 = 1_X = a^0 = a^{-0} = a^{-m}$$

by applying substitutions, (5.476) and (6.38) in connection with (semi)group $(\mathbb{Z}, +_{\mathbb{Z}})$. Finally, the third case $m >_{\mathbb{Z}} 0$ yields $-m <_{\mathbb{Z}} 0$ by applying the

Monotony Law for + and < in connection with the definition of an additive inverse and the definition of the zero element of $\mathbb{Z}$, and therefore

$$a^{-m} = (a^{-1})^{-(-m)} = (a^{-1})^m$$

with (6.372) and the Sign Law (6.50). Having found the desired equation in all cases, we may now infer from this the truth of the proposition, since X, $\cdot_X$, a and m were initially all arbitrary. $\square$

Exercise 6.46. Prove for any group $(X, +_X)$ with zero element 0_X and for any element $a \in X$ that

$$\forall m\,(m \in \mathbb{Z} \Rightarrow [-m]a = m[-a]). \tag{6.374}$$

(Hint: Apply a similar proof as for Proposition 6.373.)

Proposition 6.95. *It is true for any group $(X, \cdot)$ with unity element 1_X that multiplying two powers with commuting bases and identical exponents corresponds to multiplying the bases, that is,*

$$\forall a, b, m\,([a, b \in X \,\wedge\, a \cdot b = b \cdot a \,\wedge\, m \in \mathbb{Z}] \Rightarrow (a \cdot b)^m = a^m \cdot b^m). \tag{6.375}$$

Proof. Letting X, $\cdot$, a, b and m be arbitrary, assuming $a, b \in X$, assuming $a \cdot b = b \cdot a$ and assuming $m \in \mathbb{Z}$ to be true, we consider the two cases $m <_{\mathbb{Z}} 0$ and $\neg m <_{\mathbb{Z}} 0$. The first case $m <_{\mathbb{Z}} 0$ implies $0 <_{\mathbb{Z}} -m$ with the Monotony Law for + and <, and therefore $-m \in \mathbb{N}$ (as shown in Exercise 6.45). We obtain then the equations

$$\begin{aligned}(a \cdot b)^m &= ([a \cdot b]^{-1})^{-m} \\ &= ([b \cdot a]^{-1})^{-m} \\ &= (a^{-1} \cdot b^{-1})^{-m} \\ &= (a^{-1})^{-m} \cdot (b^{-1})^{-m} \\ &= a^m \cdot b^m\end{aligned}$$

by using (6.372) with the current case assumption $m <_{\mathbb{Z}} 0$, the assumption $a \cdot b = b \cdot a$, the Inversion Law (6.24), (5.484) in connection with the previously established fact $-m \in \mathbb{N}$, and finally again (6.372).

The second case $\neg m <_{\mathbb{Z}} 0$ implies $0 \leq_{\mathbb{Z}} m$ with the Negation Formula for <, so that we may treat m as a natural number. Consequently, the desired equation $(a \cdot b)^m = a^m \cdot b^m$ is true because of (5.484).

Having thus completed the proof by cases, we may now conclude that the stated universal sentence is true, because X, $\cdot$, a, b and m were initially arbitrary. $\square$

Exercise 6.47. Prove for any group $(X,+)$ with zero element 0_X that

$$\forall a,b,m\,([a,b \in X \wedge a+b = b+a \wedge m \in \mathbb{Z}] \Rightarrow m(a+b) = ma+mb). \quad (6.376)$$

(Hint: Recall (5.483).)

Proposition 6.96. *It is true for any group $(X,\cdot)$ with unity element 1_X that any power of the unity element equals that element, i.e.*

$$\forall m\,(m \in \mathbb{Z} \Rightarrow [1_X]^m = 1_X). \quad (6.377)$$

Proof. We let X and $\cdot$ be arbitrary sets, assume $(X,\cdot)$ to be a group such that the unity element of X exists, and we consider the two cases $m <_{\mathbb{Z}} 0$ and $\neg m <_{\mathbb{Z}} 0$. The first case $m <_{\mathbb{Z}} 0$ implies $0 <_{\mathbb{Z}} -m$ with the Monotony Law for $+$ and $<$, and therefore $-m \in \mathbb{N}$ (as shown in Exercise 6.45). We obtain then the equations

$$[1_X]^m = [(1_X)^{-1}]^{-m} = [1_X]^{-m} = 1_X$$

by using (6.372) with the case assumption $m <_{\mathbb{Z}} 0$, (6.7) and (5.486).

The second case $\neg m <_{\mathbb{Z}} 0$ implies $0 \leq_{\mathbb{Z}} m$ with the Negation Formula for $<$, so that we may treat m as a natural number. Consequently, the desired equation $[1_X]^m = 1_X$ is true because of (5.486).

Thus, the proof by cases is complete, and as X, $\cdot$ and m were arbitrary, we may now conclude that the stated universal sentence holds. □

Exercise 6.48. Verify for any group $(X,+)$ with zero element 0_X that any multiple of the zero element equals that element, i.e.

$$\forall m\,(m \in \mathbb{Z} \Rightarrow m0_X = 0_X). \quad (6.378)$$

(Hint: Use (5.485).)

Proposition 6.97. *It is true for any group $(X,\cdot)$ with unity element 1_X, for any element a of X and for any integer m that a^{-m} is the inverse element of a^m, i.e.*

$$\forall a,m\,([a \in X \;\wedge\; m \in \mathbb{Z}] \Rightarrow a^{-m} = (a^m)^{-1}). \quad (6.379)$$

Proof. We take arbitrary X, $\cdot$, a and m such that $(X,\cdot)$ is a group having the unity element 1_X, such that a is in X, and such that m is in $\mathbb{Z}$. We apply now the definition of an inverse element and prove the equations $a^m \cdot a^{-m} = 1_X$ and $a^{-m} \cdot a^m = 1_X$. For this purpose, we observe first (in light of the definition of an inverse element) the truth of the equations

$$a \cdot a^{-1} = 1_X,$$
$$a^{-1} \cdot a = 1_X,$$

which give us via substitution

$$a \cdot a^{-1} = a^{-1} \cdot a,$$
$$a^{-1} \cdot a = a \cdot a^{-1}.$$

These equations in turn imply with (6.375)

$$(a \cdot a^{-1})^m = a^m \cdot (a^{-1})^m,$$
$$(a^{-1} \cdot a)^m = (a^{-1})^m \cdot a^m,$$

where we obtain for the left-hand sides

$$(a \cdot a^{-1})^m = (1_X)^m = 1_X,$$
$$(a^{-1} \cdot a)^m = (1_X)^m = 1_X$$

by using (6.377) and for the right-hand sides

$$a^m \cdot (a^{-1})^m = a^m \cdot a^{-m},$$
$$(a^{-1})^m \cdot a^m = a^{-m} \cdot a^m$$

by means of (6.373). Combining the new left-hand sides with the new right-hand sides yields

$$1_X = a^m \cdot a^{-m},$$
$$1_X = a^{-m} \cdot a^m,$$

which equations clearly show that a^{-m} is indeed the inverse element of a^m, that is, $a^{-m} = (a^m)^{-1}$. Thus, the proof of the implication in (6.379) is complete, and since X, $\cdot$, a and m were arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Exercise 6.49. Prove for any group $(X, +)$ with zero element 0_X, for any element $a \in X$ and for any $m \in \mathbb{Z}$ that $(-m)a$ is the negative of ma, i.e.

$$\forall a, m\,([a \in X \,\wedge\, m \in \mathbb{Z}] \Rightarrow (-m)a = -(ma)). \tag{6.380}$$

(Hint: Proceed in analogy to the proof of Proposition 6.97.)

Combining the findings (6.373) and (6.379) as well as (6.374) and (6.380) immediately yields the following relationships.

Corollary 6.98. *It is true*

a) *for any group $(X, +_X)$ with zero element 0_X that*

$$\forall a, m\,([a \in X \,\wedge\, m \in \mathbb{Z}] \Rightarrow m[-a] = -[ma]). \tag{6.381}$$

b) for any group $(X, \cdot_X)$ with identity element 1_X that

$$\forall a, m\,([a \in X \;\wedge\; m \in \mathbb{Z}] \Rightarrow (a^{-1})^m = (a^m)^{-1}). \tag{6.382}$$

Theorem 6.99 (Addition & Multiplication Rules for multiples & powers in a group). *The following laws hold for any group $(X, +_X)$ with zero element 0_X resp. any group $(X, \cdot_X)$ with identity element 1_X.*

a) **Addition Rule for multiples in a group***:*

$$\forall a, m, n\,([a \in X \;\wedge\; m, n \in \mathbb{Z}] \Rightarrow (m+n)a = ma +_X na). \tag{6.383}$$

b) **Multiplication Rule for multiples in a group***:*

$$\forall a, m, n\,([a \in X \;\wedge\; m, n \in \mathbb{Z}] \Rightarrow (m \cdot n)a = n[ma]). \tag{6.384}$$

c) **Addition Rule for powers in a group***:*

$$\forall a, m, n\,([a \in X \;\wedge\; m, n \in \mathbb{Z}] \Rightarrow a^{m+n} = a^m \cdot_X a^n). \tag{6.385}$$

d) **Multiplication Rule for powers in a group***:*

$$\forall a, m, n\,([a \in X \;\wedge\; m, n \in \mathbb{Z}] \Rightarrow a^{m \cdot n} = [a^m]^n). \tag{6.386}$$

Proof. We let X and $\cdot$ be arbitrary sets and assume that $(X, \cdot_X)$ is a group for which the identity element 1_X exists.

Concerning c), we let also a, m and n be arbitrary, assuming a to be contained in X and assuming m as well as n to be contained in $\mathbb{Z}$. We consider now the two cases $m <_{\mathbb{Z}} 0$ and $\neg m <_{\mathbb{Z}} 0$, and within theses cases also the two sub-cases $n <_{\mathbb{Z}} 0$ and $\neg n <_{\mathbb{Z}} 0$.

The first case $m <_{\mathbb{Z}} 0$ in connection with the first sub-case $n <_{\mathbb{Z}} 0$ yields $m + n <_{\mathbb{Z}} 0$ with the Additivity of inequalities for ordered integral domains and the definition of the zero element. Then, the inequalities $0 <_{\mathbb{Z}} -m$, $0 <_{\mathbb{Z}} -n$ and $0 <_{\mathbb{Z}} -(m+n)$ follow to be true with the Monotony Law for $+$ and $<$, so that we can evidently treat $-m$, $-n$ and $-(m+n)$ as natural numbers (see Exercise 6.45). We obtain then the equations

$$\begin{aligned}
a^{m+n} &= (a^{-1})^{-(m+n)} \\
&= (a^{-1})^{-(n+m)} \\
&= (a^{-1})^{-m+(-n)} \\
&= (a^{-1})^{-m} \cdot_X (a^{-1})^{-n} \\
&= a^m \cdot_X a^n
\end{aligned}$$

by means of (6.372), the Commutative Law for the addition on $\mathbb{Z}$, the Sign Law (6.52) in connection with the definition of a subtraction, the Addition Rule for powers in a semigroup and (6.372).

In the second sub-case $\neg n <_{\mathbb{Z}} 0$, the Negation Formula for $<$ gives $0 \leq_{\mathbb{Z}} n$. We consider now the two further cases $m + n <_{\mathbb{Z}} 0$ and $\neg m + n <_{\mathbb{Z}} 0$. If $m + n <_{\mathbb{Z}} 0$ is true, we have $0 <_{\mathbb{Z}} -(m + n)$ as for the first sub-case. Thus, we can now treat $-m$, n and $-(m + n)$ as natural numbers, and we obtain the equations

$$\begin{aligned}
a^{m+n} &= (a^{-1})^{-(m+n)} \cdot_X 1_X \\
&= (a^{-1})^{-(m+n)} \cdot_X [a^{-n} \cdot_X a^n] \\
&= (a^{-1})^{-(m+n)} \cdot_X [(a^{-1})^n \cdot_X a^n] \\
&= [(a^{-1})^{-(m+n)} \cdot_X (a^{-1})^n] \cdot_X a^n \\
&= (a^{-1})^{-(m+n)+n} \cdot_X a^n \\
&= (a^{-1})^{-m} \cdot_X a^n \\
&= a^m \cdot_X a^n
\end{aligned}$$

by applying (6.372) alongside the definition of the unity element, Proposition 6.97 with the definition of an inverse element, (6.373), the associativity of $\cdot_X$, the Addition Rule for powers in a semigroup, the Sign Law (6.52) together with the commutativity & associativity of $+_{\mathbb{Z}}$ and alongside the definitions of an additive inverse & of the zero element, and finally (6.372). If by contrast $\neg m + n <_{\mathbb{Z}} 0$ is true, so that $0 \leq_{\mathbb{Z}} m + n$ holds evidently, we have that $-m$, n and $m + n$ are natural numbers. Using some of the preceding arguments, we get

$$\begin{aligned}
a^{m+n} &= 1_X \cdot_X a^{m+n} \\
&= [a^m \cdot_X a^{-m}] \cdot_X a^{m+n} \\
&= a^m \cdot_X [a^{-m} \cdot_X a^{m+n}] \\
&= a^m \cdot_X a^{(-m+m)+n} \\
&= a^m \cdot_X a^n.
\end{aligned}$$

The second case $\neg m <_{\mathbb{Z}} 0$ in connection with the first sub-case $n <_{\mathbb{Z}} 0$ gives us $0 \leq_{\mathbb{Z}} m$ and $0 <_{\mathbb{Z}} -n$. Then, if $m + n <_{\mathbb{Z}} 0$ is true, we also have $0 <_{\mathbb{Z}} -(m + n)$, so that m, $-n$ and $-(m + n)$ can be viewed as natural numbers. In analogy to the case of $m + n <_{\mathbb{Z}} 0$ within the second sub-case of the first case, the equations

$$\begin{aligned} a^{m+n} &= 1_X \cdot_X (a^{-1})^{-(m+n)} \\ &= [a^m \cdot_X a^{-m}] \cdot_X (a^{-1})^{-(m+n)} \\ &= [a^m \cdot_X (a^{-1})^m] \cdot_X (a^{-1})^{-(m+n)} \\ &= a^m \cdot_X [(a^{-1})^m \cdot_X (a^{-1})^{-(m+n)}] \\ &= a^m \cdot_X (a^{-1})^{m+[-(m+n)]} \\ &= a^m \cdot_X (a^{-1})^{-n} \\ &= a^m \cdot_X a^n \end{aligned}$$

follow now to be true. On the other hand, if $\neg m + n <_{\mathbb{Z}} 0$ is true, so that $0 \leq_{\mathbb{Z}} m + n$ holds and so that m, $-n$ and $m + n$ are natural numbers, we get (again in analogy to the second sub-case of the first case) the true equations

$$\begin{aligned} a^{m+n} &= a^{m+n} \cdot_X 1_X \\ &= a^{m+n} \cdot_X [a^{-n} \cdot_X a^n] \\ &= [a^{m+n} \cdot_X a^{-n}] \cdot_X a^n \\ &= a^{m+[n+(-n)]} \cdot_X a^n \\ &= a^m \cdot_X a^n. \end{aligned}$$

In the second sub-case $\neg n <_{\mathbb{Z}} 0$, we have $0 \leq_{\mathbb{Z}} n$ as before, so that we can treat m and n directly as natural numbers. This allows us to immediately apply the Addition Rule for powers in a semigroup to infer the truth of the desired equation $a^{m+n} = a^m \cdot_X a^n$, which thus holds in all of the cases. As a, m and n were arbitrary, we may then conclude that the Addition Rule for powers in a group is a true sentence.

Concerning d), we let again a, m and n be arbitrary such that $a \in X$ and $m, n \in \mathbb{Z}$ are true, and we consider the same two cases and sub-cases as in the proof of c).

In the first case $m <_{\mathbb{Z}} 0$ and the first sub-case $n <_{\mathbb{Z}} 0$, we recall that $-m, -n \in \mathbb{N}$ holds, and we derive the equations

$$\begin{aligned} [a^m]^n &= [(a^{-1})^{-m}]^n = ([(a^{-1})^{-m}]^{-1})^{-n} = ([(a^{-1})^{-1}]^{-m})^{-n} = a^{(-m)\cdot(-n)} \\ &= a^{m \cdot n} \end{aligned}$$

by using (6.372) twice in a row, (6.382), the Inversion Law (6.22) jointly with the Multiplication Rule for powers in a semigroup, and the Sign Law

(6.65). In the second subcase $\neg n <_{\mathbb{Z}} 0$, we now have $-m, n \in \mathbb{N}$ and obtain therefore

$$[a^m]^n = [(a^{-1})^{-m}]^n = (a^{-1})^{(-m)\cdot n} = (a^{-1})^{-(m\cdot n)} = a^{m\cdot n}$$

with (6.372), the Multiplication Rule for powers in a semigroup, the Sign Law (6.64) and (6.372).

In the second case $\neg m <_{\mathbb{Z}} 0$ in connection with the first sub-case $n <_{\mathbb{Z}} 0$, which assumptions give $m, -n \in \mathbb{N}$, we can establish the equations

$$[a^m]^n = [(a^m)^{-1}]^{-n} = [(a^{-1})^m]^{-n} = (a^{-1})^{m\cdot(-n)} = (a^{-1})^{-(m\cdot n)} = a^{m\cdot n}$$

by applying (6.372), (6.382), the Multiplication Rule for powers in a semigroup, the Sign Law (6.63) and again (6.372). Finally, in the second subcase $\neg n <_{\mathbb{Z}} 0$, in which m and n are both natural numbers, the desired equation $a^{m\cdot n} = [a^m]^n$ holds directly by virtue of the Multiplication Rule for powers in a semigroup.

We thus completed the proof by cases, and since a, m and n are arbitrary, we may therefore conclude that Part d) of the theorem holds as well.

As X and $\cdot$ were initially arbitrary, we may now infer from these findings the truth of the theorem. □

Exercise 6.50. Establish the Addition Rule and the Multiplication Rule for multiples.

(Hint: Proceed in analogy to the proofs of the Addition Rule and the Multiplication Rule for powers.)

Theorem 6.100 (First, Second & Third Binomial Formula). *The following formulae hold for any commutative ring $(X, +, \cdot, -)$ and any elements a, b of X.*

a) **First Binomial Formula***:*

$$(a+b)^2 = a^2 + 2(a\cdot b) + b^2. \tag{6.387}$$

b) **Second Binomial Formula***:*

$$(a-b)^2 = a^2 - 2(a\cdot b) + b^2. \tag{6.388}$$

c) **Third Binomial Formula***:*

$$(a+b)\cdot(a-b) = a^2 - b^2. \tag{6.389}$$

Proof. We let X, $+$, $\cdot$, $-$, a and b be arbitrary such that $(X, +, \cdot, -)$ is a commutative ring and such that $a, b \in X$ holds. Concerning (6.387), we establish the equations

$$\begin{aligned}(a+b)^2 &= (a+b)\cdot(a+b)\\ &= a\cdot(a+b)+b\cdot(a+b)\\ &= (a\cdot a+a\cdot b)+(b\cdot a+b\cdot b)\\ &= a\cdot a+(a\cdot b+a\cdot b)+b\cdot b\\ &= a^2+2(a\cdot b)+b^2\end{aligned}$$

by applying (5.478), the distributivity of $\cdot$ over $+$, again the distributivity of $\cdot$ over $+$, the commutativity of $\cdot$ together with the associativity of $+$, and finally (5.478) alongside (5.473). Since a and b are arbitrary, we may infer from these equations the truth of the First Binomial Formula.

Concerning (6.388), we derive the equations

$$\begin{aligned}(a-b)^2 &= (a+[-b])^2\\ &= a^2+2(a\cdot[-b])+[-b]^2\\ &= a^2+2(-[a\cdot b])+(-b)\cdot(-b)\\ &= a^2+[(-[a\cdot b])+(-[a\cdot b])]+b\cdot b\\ &= a^2+[-(a\cdot b)-(a\cdot b)]+b^2\\ &= a^2-[a\cdot b+a\cdot b]+b^2\\ &= a^2-2(a\cdot b)+b^2\end{aligned}$$

by using the definition of a subtraction, the First Binomial Formula, the Sign Law (6.63) together with (5.478), (5.473) alongside the Sign Law (6.65), the definition of a subtraction jointly with (5.478), the Sign Law (6.52), and finally (5.473). Here, a and b are arbitrary, so that the Second Binomial Formula follows to be true as well.

The Third Binomial Formula can be proved by using some of the previous arguments in connection with the definitions of an additive inverse and of a zero element.

Because X, $+$, $\cdot$ and $-$ were initially arbitrary, we may infer from these findings the truth of the theorem. □

Exercise 6.51. Establish the Third Binomial Formula.

Proposition 6.101. *It is true that*

$$\forall k, m\,(k, m \in \mathbb{Z} \Rightarrow mk = m \cdot_{\mathbb{Z}} k). \tag{6.390}$$

Proof. We take arbitrary integers k and m and consider the two cases $k <_{\mathbb{Z}} 0$ and $\neg k <_{\mathbb{Z}} 0$, and within theses cases also the two sub-cases $m <_{\mathbb{Z}} 0$ and $\neg m <_{\mathbb{Z}} 0$.

The first case $k <_{\mathbb{Z}} 0$ in connection with the first sub-case $m <_{\mathbb{Z}} 0$ gives us $0 <_{\mathbb{Z}} -k$ and $0 <_{\mathbb{Z}} -m$ with the Monotony Law for $+$ and $<$, and therefore evidently $-k, -m \in \mathbb{N}$. We obtain now the true equations

$$mk = (-m)(-k) = (-m) \cdot_{\mathbb{N}} (-k) = (-m) \cdot_{\mathbb{Z}} (-k) = m \cdot_{\mathbb{Z}} k$$

by applying (6.371) for the group $(\mathbb{Z}, +_{\mathbb{Z}})$, (5.521), the fact that we can treat $-k$ and $-m$ both as natural numbers or as integers, and finally the Sign Law (6.65) for the group $(\mathbb{Z}, +_{\mathbb{Z}})$. In the second sub-case $\neg m <_{\mathbb{Z}} 0$, we obtain with the Negation Formula for $<$ the inequality $0 \leq_{\mathbb{Z}} m$, so that $-k$ and m are now natural numbers. We then get the equations

$$\begin{aligned} mk = m[-(-k)] &= -[m(-k)] \\ = -[m \cdot_{\mathbb{N}} (-k)] &= -[m \cdot_{\mathbb{Z}} (-k)] \\ = -[-(m \cdot_{\mathbb{Z}} k)] &= m \cdot_{\mathbb{Z}} k \end{aligned}$$

using the Sign Law (6.50) with the group $(\mathbb{Z}, +)$, (6.381) with the group $(\mathbb{Z}, +)$, (5.521), the Sign Law (6.63) with the group $(\mathbb{Z}, +)$, and finally again the Sign Law (6.50).

The second case $\neg k <_{\mathbb{Z}} 0$ in connection with the first sub-case $m <_{\mathbb{Z}} 0$ evidently yields $0 \leq_{\mathbb{Z}} k$ and $0 <_{\mathbb{Z}} -m$, so that $k, -m \in \mathbb{N}$. We observe then the truth of the equations

$$\begin{aligned} mk = (-m)(-k) &= -[(-m)k] \\ = -[(-m) \cdot_{\mathbb{N}} k] &= -[(-m) \cdot_{\mathbb{Z}} k] \\ = -[-(m \cdot_{\mathbb{Z}} k)] &= m \cdot_{\mathbb{Z}} k \end{aligned}$$

in light of (6.371), (6.381) for the group $(\mathbb{Z}, +)$, (5.521), the Sign Law (6.64) for the group $(\mathbb{Z}, +)$ and the Sign Law (6.50). Because the second sub-case $\neg m <_{\mathbb{Z}} 0$ implies $0 \leq_{\mathbb{Z}} m$, we have that k and m are both natural numbers, which fact allows us to use directly (5.521) to infer the truth of the equation $mk = m \cdot_{\mathbb{N}} k$ and thus the truth of $mk = m \cdot_{\mathbb{N}} k$.

Since k and m were initially arbitrary, we may therefore conclude that the proposition is true. □

The property of the unity element 1 of $\mathbb{Z}$ leads then directly to the following special case of (6.390).

Corollary 6.102. *It is true that*

$$\forall m \, (m \in \mathbb{Z} \Rightarrow m1 = m). \tag{6.391}$$

Proposition 6.103. *If an integer is odd, then its square is also odd, that is,*

$$\forall m\,(m \text{ is an odd integer } \Rightarrow\; m^2 \text{ is an odd integer}). \tag{6.392}$$

Proof. We take an arbitrary integer m and assume m to be odd, which means by definition that there exists an integer, say $\bar{q}$, such that $m = 2 \cdot_{\mathbb{Z}} \bar{q} +_{\mathbb{Z}} 1$. We obtain now the true equations

$$\begin{aligned} m^2 &= (2 \cdot \bar{q} + 1)^2 \\ &= (2 \cdot \bar{q})^2 + 2([2 \cdot \bar{q}] \cdot 1) + 1^2 \\ &= (2 \cdot \bar{q}) \cdot (2 \cdot \bar{q}) + 2(2 \cdot \bar{q}) + 1 \\ &= 2 \cdot (2\bar{q}^2) + 2 \cdot (2\bar{q}) + 1 \\ &= 2 \cdot (2\bar{q}^2 + 2\bar{q}) + 1 \end{aligned}$$

by applying substitution, the First Binomial Formula in connection with the commutative ring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}})$, then (5.478), together with the definition of the unity element, subsequently the Associative & Commutative Law for the multiplication on $\mathbb{Z}$ alongside (6.390) and (5.478), and finally the Distributive Law for $\mathbb{Z}$. These equations clearly show that m^2 can be written as $m^2 = 2 \cdot q + 1$ for some integer q, so that m^2 is by definition an odd integer (noting that m^2 and $2\bar{q}^2 + 2\bar{q}$ are integers as values of the addition/multiplication on $\mathbb{Z}$). As m was arbitrary, we may then infer from this finding the truth of the proposed universal sentence. □

Exercise 6.52. Prove that if an integer is even, then its square is also even, that is,

$$\forall m\,(m \text{ is an even integer } \Rightarrow\; m^2 \text{ is an even integer}). \tag{6.393}$$

(Hint: The proof is a simpler version of the proof of Proposition 6.103.)

Corollary 6.104. *It is true that an integer is even iff its square is even, that is,*

$$\forall m\,(m \in \mathbb{Z} \Rightarrow [m^2 \text{ is even } \Leftrightarrow\; m \text{ is even}). \tag{6.394}$$

Proof. Letting m be an arbitrary integer, we apply a proof by contraposition to establish the first part ('⇒') of the equivalence, assuming that m is not even. As every integer is either even or odd (see Note 6.28), it follows that the integer m is odd, so that m^2 is also an odd integer because of (6.392). In view of the preceding note, m^2 is therefore not even, so that the proof by contraposition is complete. The second part ('⇐') of the equivalence is true according to (6.393). Since m was arbitrary, we therefore conclude that the corollary holds indeed. □

6.9. Countability of $\mathbb{Z}$

Theorem 6.105 (Countable Infinity of $\mathbb{Z}$). *The set $\mathbb{Z}$ is countably infinite.*

Proof. Let us apply Function definition by replacement to establish a unique function f with domain $\mathbb{Z}$ such that

$$\forall m\,(m \in \mathbb{Z} \Rightarrow ([m < 0 \Rightarrow f(m) = -2m - 1] \wedge [m \geq 0 \Rightarrow f(m) = 2m])). \tag{6.395}$$

For this purpose, we prove the universal sentence

$$\forall m\,(m \in \mathbb{Z} \Rightarrow \exists! y([m < 0 \Rightarrow y = -2m-1] \wedge [m \geq 0 \Rightarrow y = 2m])), \tag{6.396}$$

letting $m \in \mathbb{Z}$ be arbitrary, and considering then concerning the existential part the two cases $m < 0$ and $\neg m < 0$. In the first case $m < 0$, which implies $\neg m \geq 0$ with the Negation Formula for $\leq$, we obtain the true implications

$$[m < 0 \Rightarrow -2m - 1 = -2m - 1] \wedge [m \geq 0 \Rightarrow -2m - 1 = 2m],$$

noting that the antecedent and the consequent of the first implication are both true, whereas the antecedent of the second implication is false. In the second case $\neg m < 0$, which evidently implies $m \geq 0$, we have then the true implications

$$[m < 0 \Rightarrow 2m = -2m - 1] \wedge [m \geq 0 \Rightarrow 2m = 2m],$$

the antecedent of the first one being now false, and the antecedent and consequent of the second one being both true. Having found the particular constant $\bar{y} = -2m-1$ in the first case and the particular constant $\bar{y} = 2m$ in the second case, the proof of the existential part by cases is thus complete. To establish the uniqueness part, we take arbitrary sets y and y' such that

$$[m < 0 \Rightarrow y = -2m - 1] \wedge [m \geq 0 \Rightarrow y = 2m]$$
$$[m < 0 \Rightarrow y' = -2m - 1] \wedge [m \geq 0 \Rightarrow y' = 2m]$$

are satisfied, and we consider the same two cases as in the proof of the existential part. Due to the preceding implications, the first case $m < 0$ implies $y = -2m - 1 = y'$, and the second case $\neg m < 0$ yields $m \geq 0$ and implies therefore $y = 2m = y'$. We thus find $y = y'$ to be true in both cases, and since y, y' are arbitrary, we may therefore conclude that the uniqueness part holds, too. Having completed the proof of the uniquely existential sentence in (6.396), we can infer from this also the truth of the universal sentence (6.396), because m was arbitrary. Consequently, there exists indeed a unique function f with domain $\mathbb{Z}$ and values satisfying (6.395).

Next, we verify that $\mathbb{N}$ is a codomain of f, i.e. that the range of f is included in $\mathbb{N}$. By definition of a subset, this means

$$\forall n\,(n \in \mathrm{ran}(f) \Rightarrow n \in \mathbb{N}), \tag{6.397}$$

which universal sentence we prove by letting n be arbitrary. Assuming now $n \in \mathrm{ran}(f)$ to be true, it follows with the definition of a range that there exists a set, say $\bar{m}$, such that $(\bar{m}, n) \in f$ holds. This shows in light of the definition of a domain that $\bar{m} \in \mathbb{Z}$ is true, so that the definition of f in (6.395) yields the value $f(\bar{m}) = -2\bar{m} - 1$ in case of $\bar{m} < 0$ and the value $f(\bar{m}) = 2\bar{m}$ in case of $\bar{m} \geq 0$. We now prove $f(\bar{m}) \in \mathbb{N}$ by cases. The first case $\bar{m} < 0$ implies $0 < -\bar{m}$ with the Monotony Law for $+$ and $<$, so that $-\bar{m}$ is a positive natural number. This finding implies $1 \leq_{\mathbb{N}} -\bar{m}$, and because $0 <_{\mathbb{N}} 2$ holds as well, we obtain with the Monotony Law for $\cdot_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$ the inequality $1 \cdot_{\mathbb{N}} 2 \leq_{\mathbb{N}} -\bar{m} \cdot_{\mathbb{N}} 2$, which we may write equivalently as $2 \leq -2\bar{m}$ by means of (6.390). The Monotony Law for $+$ and $\leq$ gives us then $2 - 1 \leq -2\bar{m} - 1$, which simplifies to $1 \leq -2\bar{m} - 1$. Thus, $f(\bar{m}) = -2\bar{m} - 1$ is clearly a (positive) natural number in the first case. The second case $\neg\bar{m} < 0$ implies $0 \leq \bar{m}$ and therefore $0 \cdot 2 \leq \bar{m} \cdot 2$ with the Monotony Law for $\cdot$ and $\leq$, which inequality we can write also as $0 \leq 2\bar{m}$. Thus, $f(\bar{m}) = 2\bar{m}$ is again a natural number. Noting that the previous finding $(\bar{m}, n) \in f$ reads in function notation $n = f(\bar{m})$, we now see that $n \in \mathbb{N}$ holds, as desired. As n was arbitrary, we may therefore conclude that (6.397) is true, so that the inclusion $\mathrm{ran}(f) \subseteq \mathbb{N}$ holds. Thus, f is a function from $\mathbb{Z}$ to $\mathbb{N}$.

We now prove that $f : \mathbb{Z} \to \mathbb{N}$ is an injection, letting $m, m' \in \mathbb{Z}$ be arbitrary, assuming $f(m) = f(m')$ and demonstrating the truth of $m = m'$ by consider the two cases $m < 0$ and $\neg m < 0$. and within each of these cases the subcases $m' < 0$ and $\neg m' < 0$.

The first case $m < 0$ gives us the corresponding function value $f(m) = -2m - 1$, and we can prove by contradiction that $m' < 0$ also holds. Assuming the negation $\neg m' < 0$ to be true, so that $m' \geq 0$ holds, we obtain the function value $f(m') = 2m'$. Therefore, the assumed equation $f(m) = f(m')$ and (6.390) give $-2 \cdot m - 1 = 2 \cdot m'$, which implies $2 \cdot (-m - m') = 2 \cdot 0 + 1$. We thus have an integer which is both even and odd, in contradiction to the fact that any integer is either even or odd (see Note 6.28). Having completed the proof of $m' < 0$, we obtain for the corresponding function value $f(m') = -2m' - 1$. Recalling the truth of $f(m) = -2m - 1$, the equation $-2 \cdot m - 1 = -2 \cdot m' - 1$ follows then to be true, which we can simplify to $-2 \cdot m = -2 \cdot m'$. Applying now the Cancellation Law for $\cdot$ with respect to the ordered integral domain $(\mathbb{Z}, +, \cdot, -, <)$ and in connection with the fact $-2 \neq 0$ yields $m = m'$, as desired.

The second case $\neg m < 0$ implies $m \geq 0$ and therefore $f(m) = 2m$. In analogy to the first case, we can prove now by contradiction that $m' \geq 0$ is true as well. For this purpose, we observe that the assumption $\neg m' \geq 0$ implies $m' < 0$ and therefore $f(m') = -2m' - 1$. Due to $f(m) = f(m')$, we have $2m = -2m' - 1$ and consequently $2 \cdot 0 + 1 = 2 \cdot (-m' - m)$. As in the first case, we thus found an integer that is both odd and even, resulting in a contradiction. Therefore, $m' \geq 0$ is indeed true, and the corresponding value is then $f(m') = 2m'$. Because of $f(m) = 2m$ and the assumed $f(m) = f(m')$, we obtain via substitution $2m = 2m'$, which we can write in the form $2 \cdot m = 2 \cdot m'$. Noting that $2 \neq 0$, the preceding equation gives us again the desired consequent $m = m'$ by means of the Cancellation Law for $\cdot$.

We thus showed that $m, m' \in \mathbb{Z} \wedge f(m) = f(m')$ implies $m = m'$ for any m, m' (in any case), so that $f : \mathbb{Z} \to \mathbb{N}$ is an injection by definition. We now prove that this injection is also a surjection, by demonstrating the truth of

$$\forall n\, (n \in \mathbb{N} \Rightarrow n \in \mathrm{ran}(f)), \tag{6.398}$$

We let $n \in \mathbb{N}$ be arbitrary and consider the two cases that n is an even or an odd natural number.

In case n is even, there exists a particular natural number $\bar{q}$ satisfying $n = 2 \cdot_{\mathbb{N}} \bar{q}$. Here, we may treat $\bar{q}$ as an integer with $\bar{q} \geq 0$, so that $2\bar{q} = f(\bar{q})$ follows to be true by definition of f. The preceding equations imply $n = f(\bar{q})$, which we may write also as $(\bar{q}, n) \in f$. Thus, n turns out to be an element of the range of f.

In the other case that n is odd, there is a particular natural number $\bar{q}$ such that $n = 2\bar{q} + 1$. We now rewrite n as

$$n = 2 \cdot \bar{q} + 2 \cdot 1 - 1 = 2 \cdot (\bar{q} + 1) - 1 = -2(-[\bar{q} + 1]) - 1,$$

and we observe that the inequalities $0 \leq_{\mathbb{N}} \bar{q} <_{\mathbb{N}} \bar{q} +_{\mathbb{N}} 1$ imply $0 <_{\mathbb{N}} \bar{q} +_{\mathbb{N}} 1$ with the Transitivity Formula for $\leq$ and $<$, so that $-[\bar{q} + 1] < 0$. These findings imply with the definition of f the truth of $f(-[\bar{q} + 1]) = n$, which demonstrates that n is in the range of f again.

As n was arbitrary, we may therefore conclude that (6.398) holds, and this universal sentence implies the inclusion $\mathbb{N} \subseteq \mathrm{ran}(f)$ with the definition of a subset. In conjunction with the previously established inclusion $\mathrm{ran}(f) \subseteq \mathbb{N}$, this further implies $\mathrm{ran}(f) = \mathbb{N}$ with the Axiom of Extension, so that the injection $f : \mathbb{Z} \to \mathbb{N}$ constitutes a surjection. Thus, f is a bijection from $\mathbb{Z}$ to $\mathbb{N}$, which finding shows that there exists a bijection $f : \mathbb{Z} \rightleftarrows \mathbb{N}$, and this existential implies that the set $\mathbb{Z}$ is countably infinite. $\square$

Chapter 7.

The Ordered Field of Rational Numbers

7.1. The Ring $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, -_{\mathbb{Q}})$

Exercise 7.1. Establish the truth of the following sentences.

a) There exists a unique set $\sim_q$ such that

$$\begin{aligned}\forall Q\,(Q \in \sim_q \Leftrightarrow [Q \in (\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]) \times (\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]) \\ \wedge\ \exists m, n, M, N\,(m \cdot_{\mathbb{Z}} N = M \cdot_{\mathbb{Z}} n \wedge ((m,n),(M,N)) = Q)]).\end{aligned} \tag{7.1}$$

(Hint: Proceed similarly as in the proof of Proposition 6.37.)

b) This set $\sim_q$ satisfies also

$$\begin{aligned}\forall Q\,(Q \in \sim_q \Leftrightarrow \exists m, n, M, N\,(n \neq 0 \wedge N \neq 0 \\ \wedge\ m \cdot_{\mathbb{Z}} N = M \cdot_{\mathbb{Z}} n \wedge ((m,n),(M,N)) = Q))\end{aligned} \tag{7.2}$$

and constitutes an equivalence relation on $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$.

(Hint: Concerning the transitivity of $\sim_q$, apply the Cancellation Law for $\cdot$ by finding a single nonzero multiplier.)

c) Furthermore, the equivalence relation $\sim_q$ is characterized by

$$\begin{aligned}\forall m, n, M, N\,(\,(m,n) \sim_q (M,N) \\ \Leftrightarrow [n \neq 0 \wedge N \neq 0 \wedge m \cdot_{\mathbb{Z}} N = M \cdot_{\mathbb{Z}} n]).\end{aligned} \tag{7.3}$$

(Hint: Recall the proof in Exercise 6.15.)

d) It is not true that the ordered pairs $(1,1)$ and $(0,1)$ are equivalent with respect to $\sim_q$, that is,

$$\neg(1,1) \sim_q (0,1). \tag{7.4}$$

(Hint: The proof can be carried out in analogy to the proof of Proposition 6.38.)

The next two results constitute the multiplicative counterpart of Proposition 6.39 and Exercise 6.16 in the context of integers.

Proposition 7.1. *It is true that the ordered pairs* $(m, m \cdot_{\mathbb{Z}} n)$ *and* $(1, n)$ *are equivalent with respect to* $\sim_q$ *for any nonzero integers* m *and* n*, that is,*

$$\forall m, n\,(\,m, n \in \mathbb{Z} \setminus \{0\} \Rightarrow (m, m \cdot_{\mathbb{Z}} n) \sim_q (1, n)\,). \tag{7.5}$$

Proof. Letting $m, n \in \mathbb{Z} \setminus \{0\}$ be arbitrary, we have by definition of a subset $m, n \in \mathbb{Z}$ and $m, n \notin \{0\}$, where the latter implies $m \neq 0$ and $n \neq 0$ with (2.169). Next, we now observe the truth of the equation

$$m \cdot_{\mathbb{Z}} n = 1 \cdot_{\mathbb{Z}} (m \cdot_{\mathbb{Z}} n)$$

in light of the definition of the unity element, where the previous findings $m \neq 0$ and $n \neq 0$ imply $m \cdot_{\mathbb{Z}} n \neq 0$ with the Criterion for zero-divisor freeness – applied to the semiring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}})$. In conjunction with $n \neq 0$ and the preceding equation, this implies $(m, m \cdot_{\mathbb{Z}} n) \sim_q (1, n)$ with (7.3), as desired. Because m and n are arbitrary, the proposed universal sentence follows then to be true. □

Exercise 7.2. Demonstrate that $(m \cdot_{\mathbb{Z}} n, m)$ and $(n, 1)$ are equivalent with respect to $\sim_q$ for any nonzero integer m and any integer n, that is,

$$\forall m, n\,(\,[m, n \in \mathbb{Z} \wedge m \neq 0] \Rightarrow (m \cdot_{\mathbb{Z}} n, m) \sim_q (n, 1)\,). \tag{7.6}$$

Proposition 7.2. *It is true that*

$$\forall c, m, n\,([c, m, n \in \mathbb{Z} \wedge c \neq 0 \wedge n \neq 0] \Rightarrow [(c \cdot_{\mathbb{Z}} m, c \cdot_{\mathbb{Z}} n)]_{\sim_q} = [(m, n)]_{\sim_q}). \tag{7.7}$$

Proof. Letting c be an arbitrary nonzero integer, m be an arbitrary integer, and n be an arbitrary nonzero integer, we observe the truth of the equations

$$(c \cdot_{\mathbb{Z}} m) \cdot_{\mathbb{Z}} n = (m \cdot_{\mathbb{Z}} c) \cdot_{\mathbb{Z}} n = m \cdot_{\mathbb{Z}} (c \cdot_{\mathbb{Z}} n)$$

in light of the Commutative and the Associative Law for the multiplication on $\mathbb{Z}$. Here, the assumptions $c \neq 0$ and $n \neq 0$ imply with the Criterion for zero-divisor freeness $c \cdot_{\mathbb{Z}} n \neq 0$. Then, the conjunction

$$n \neq 0 \wedge c \cdot_{\mathbb{Z}} n \neq 0 \wedge (c \cdot_{\mathbb{Z}} m) \cdot_{\mathbb{Z}} n = m \cdot_{\mathbb{Z}} (c \cdot_{\mathbb{Z}} n)$$

implies

$$(c \cdot_{\mathbb{Z}} m, c \cdot_{\mathbb{Z}} n) \sim_q (m, n)$$

with (7.3). Consequently, we obtain the desired equation

$$[(c \cdot_{\mathbb{Z}} m, c \cdot_{\mathbb{Z}} n)]_{\sim_q} = [(m,n)]_{\sim_q}$$

with the Equality Criterion for equivalence classes, so that the proof of the implication in (7.7) is complete. Since c, m and n were arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Definition 7.1 (Set of rational numbers, rational number). We call the quotient set

$$\mathbb{Q} = \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]/\sim_q \tag{7.8}$$

the *set of rational numbers* and every element of $\mathbb{Q}$ a *rational number*.

Notation 7.1. We symbolize a given rational number $p = [(m,n)]_{\sim_q}$ also by

$$\frac{m}{n}. \tag{7.9}$$

Theorem 7.3 (Generation of rational numbers). *Any integers m and n with $n \neq 0$ define the equivalence class $[(m,n)]_{\sim_q}$ and the rational number $\frac{m}{n}$.*

Proof. We let m and n be arbitrary such that $m, n \in \mathbb{Z}$ and $n \neq 0$ hold. The latter yields $n \notin \{0\}$ with (2.169), so that $n \in \mathbb{Z} \setminus \{0\}$ holds by definition of a set difference. Consequently, $(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ is true by definition of a set difference. We then obtain $(m,n) \in [(m,n)]_{\sim_q}$ with Corollary 3.57. Denoting that equivalence class by $p = [(m,n)]_{\sim_q}$, we now see that q satisfies the existential sentence

$$\exists x\, (x \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\} \,\wedge\, [x]_{\sim_q} = p),$$

so that $p \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]/\sim_q$ follows to be true according to Proposition 3.60. By definition of the set $\mathbb{Q}$ in (7.8), we thus find $[(m,n)]_{\sim_q} \in \mathbb{Q}$ via substitution. In view of Notation 7.1, we see that $\frac{m}{n}$ is indeed a rational number. Since m and n were initially arbitrary, we may therefore conclude that the stated theorem is true. □

Theorem 7.4 (Characterization of rational numbers). *Every rational number p can be written as the equivalence class $[(m,n)]_{\sim_q}$ for some integers m and n with $n \neq 0$, that is,*

$$\forall p\, (p \in \mathbb{Q} \Rightarrow \exists m, n\, (m, n \in \mathbb{Z} \,\wedge\, n \neq 0 \,\wedge\, p = [(m,n)]_{\sim_q})). \tag{7.10}$$

Note 7.1. In the following, we will frequently use given integers m and n with $n \neq 0$ to define the equivalence class $[(m,n)]_{\sim_q}$ (i.e., the rational number $\frac{m}{n}$) without explicit reference to Theorem 7.3, and we will oftentimes express a given rational number p as such an equivalence class or fraction without mentioning Theorem 7.4.

Exercise 7.3. Prove Theorem 7.4.

(Hint: Use some of the arguments from the proof of Theorem 7.3 in connection with Exercise 3.4.)

Exercise 7.4. Prove that the equivalence classes $[(1,1)]_{\sim_q}$ and $[(0,1)]_{\sim_q}$ constitute distinct rational numbers, that is, prove

$$[(1,1)]_{\sim_q}, [(0,1)]_{\sim_q} \in \mathbb{Q}, \tag{7.11}$$

and

$$[(1,1)]_{\sim_q} \neq [(0,1)]_{\sim_1}. \tag{7.12}$$

(Hint: Proceed in analogy to the proof of Corollary 6.40.)

Exercise 7.5. Verify that the equivalence classes

a) $[(m \cdot_{\mathbb{Z}} n, m)]_{\sim_q}$ and $[(n,1)]_{\sim_q}$ constitute identical rational numbers for any nonzero integer m and any integer n, that is,

$$\forall m,n\,(\,[m,n \in \mathbb{Z} \,\wedge\, m \neq 0] \Rightarrow [(m \cdot_{\mathbb{Z}} n, m)]_{\sim_q} = [(n,1)]_{\sim_q}\,). \tag{7.13}$$

b) $[(m, m \cdot_{\mathbb{Z}} n)]_{\sim_q}$ and $[(1,n)]_{\sim_q}$ are identical rational numbers for any nonzero integers m and n, that is,

$$\forall m,n\,(\,m,n \in \mathbb{Z} \setminus \{0\} \Rightarrow [(m, m \cdot_{\mathbb{Z}} n)]_{\sim_q} = [(1,n)]_{\sim_q}\,). \tag{7.14}$$

c) $[(m,m)]_{\sim_q}$ and $[(1,1)]_{\sim_q}$ are identical rational numbers for any nonzero integer m, that is,

$$\forall m\,(\,m \in \mathbb{Z} \setminus \{0\} \Rightarrow [(m,m)]_{\sim_q} = [(1,1)]_{\sim_q}\,). \tag{7.15}$$

(Hint: Use similar arguments as in the proofs of Corollary 6.41, Exercise 6.17, Proposition 7.1, and Corollary 6.42.)

Corollary 7.5. *It is true that*

$$\forall m,n\,([m,n \in \mathbb{Z} \,\wedge\, n \neq 0] \Rightarrow [\,[(m,n)]_{\sim_q} \neq [(0,1)]_{\sim_q} \Rightarrow m \neq 0\,]). \tag{7.16}$$

Proof. Letting m be an arbitrary integer and n an arbitrary nonzero integer, we prove the second implication by contraposition, assuming $\neg m \neq 0$ to be true. We therefore obtain with the Double Negation Law $m = 0$, so that we obtain

$$[(m,n)]_{\sim_q} = [(0,n)]_{\sim_q} = [(n \cdot_{\mathbb{Z}} 0, n)]_{\sim_q} = [(0,1)]_{\sim_q}$$

by means of substitution, the Cancellation Law for 0 and (7.13). This yields with the Double Negation Law the true negation $\neg[(m,n)]_{\sim_q} \neq [(0,1)]_{\sim_q}$, completing the proof by contraposition. Since m and n were initially arbitrary, the corollary follows therefore to be true. □

Theorem 7.6 (Identification of $\mathbb{Z}$ in $\mathbb{Q}$). *It is true that there exists a unique function $f_{\mathbb{Z}}^{\mathbb{Q}}$ with domain $\mathbb{Z}$ such that*

$$\forall m\,(m \in \mathbb{Z} \Rightarrow f_{\mathbb{Z}}^{\mathbb{Q}}(m) = [(m,1)]_{\sim_q}), \tag{7.17}$$

and $f_{\mathbb{Z}}^{\mathbb{Q}}$ is an injection from $\mathbb{Z}$ to $\mathbb{Q}$.

Exercise 7.6. Establish the Identification of $\mathbb{Z}$ in $\mathbb{Q}$ in analogy to the Identification of $\mathbb{N}$ in $\mathbb{Z}$.

Note 7.2. The injection $f_{\mathbb{Z}}^{\mathbb{Q}} : \mathbb{Z} \hookrightarrow \mathbb{Q}$ turns into the bijection

$$f_{\mathbb{Z}}^{\mathbb{Q}} : \mathbb{Z} \rightleftarrows \operatorname{ran}(f_{\mathbb{Z}}^{\mathbb{Q}})$$

according to Corollary 3.204. Because the range of $f_{\mathbb{Z}}^{\mathbb{Q}}$ is identical with the image of $\mathbb{Z}$ under that function in view of Corollary 3.216, that is,

$$\operatorname{ran}(f_{\mathbb{Z}}^{\mathbb{Q}}) = f_{\mathbb{Z}}^{\mathbb{Q}}[\mathbb{Z}], \tag{7.18}$$

we may alternatively write for the preceding bijection

$$f_{\mathbb{Z}}^{\mathbb{Q}} : \mathbb{Z} \rightleftarrows f_{\mathbb{Z}}^{\mathbb{Q}}[\mathbb{Z}]. \tag{7.19}$$

The Bijectivity of inverse functions gives us then in addition the bijection

$$[f_{\mathbb{Z}}^{\mathbb{Q}}]^{-1} : f_{\mathbb{Z}}^{\mathbb{Q}}[\mathbb{Z}] \rightleftarrows \mathbb{Z}. \tag{7.20}$$

Thus, every integer m can be mapped into its representation $[(m,1)]_{\sim_q}$ by a rational number and conversely be retrieved from the image without ambiguity.

Notation 7.2. We symbolize the image of $\mathbb{Z}$ under $f_{\mathbb{Z}}^{\mathbb{Q}}$ also by $\mathbb{Z}_{\mathbb{Q}}$, that is,

$$\mathbb{Z}_{\mathbb{Q}} = f_{\mathbb{Z}}^{\mathbb{Q}}[\mathbb{Z}], \tag{7.21}$$

and we will call its elements *integers in* $\mathbb{Q}$. The bijection (7.19) can then be written in the form

$$f_{\mathbb{Z}}^{\mathbb{Q}} : \mathbb{Z} \rightleftarrows \mathbb{Z}_{\mathbb{Q}}. \tag{7.22}$$

For simplicity, we will usually omit the explicit reference to $\mathbb{Q}$ and set

$$\mathbb{Z} = \mathbb{Z}_{\mathbb{Q}}, \tag{7.23}$$

in analogy to the overloading of the symbol '$\mathbb{N}$' in (6.143).

Exercise 7.7. Establish the inclusion

$$\mathbb{Z}_{\mathbb{Q}} \subseteq \mathbb{Q}. \tag{7.24}$$

(Hint: Proceed as in Exercise 6.18.)

We define now the addition on the set of rational numbers. For greater clarity, we will denote the addition and the multiplication on $\mathbb{Z}$ simply by $+$ and $\cdot$, respectively (instead of using the explicit symbols $+_{\mathbb{Z}}$ and $\cdot_{\mathbb{Z}}$).

Exercise 7.8. Show that there exists a unique function $+_{\mathbb{Q}}$ with domain $\mathbb{Q} \times \mathbb{Q}$ such that

$$\begin{aligned} &\forall x\, (x \in \mathbb{Q} \times \mathbb{Q} \Rightarrow [+_{\mathbb{Q}}(x) \in \mathbb{Q} \wedge \exists E, F, m, n, M, N\, (x = (E, F) \\ &\quad \wedge (m, n) \in E \wedge (M, N) \in F \wedge (m \cdot N + n \cdot M, n \cdot N) \in +_{\mathbb{Q}}(x))]), \end{aligned} \tag{7.25}$$

and show that this function $+_{\mathbb{Q}}$ is a binary operation on $\mathbb{Q}$ satisfying

$$\begin{aligned} &\forall m, n, M, N\, ([m, n, M, N \in \mathbb{Z} \wedge n \neq 0 \wedge N \neq 0] \\ &\quad \Rightarrow [(m, n)]_{\sim_q} +_{\mathbb{Q}} [(M, N)]_{\sim_q} = [(m \cdot N + n \cdot M, n \cdot N)]_{\sim_q}). \end{aligned} \tag{7.26}$$

(Hint: The proof can be carried out similarly as for Proposition 6.45; here, the uniqueness part requires then the distributivity, associativity and commutativity with respect to $\cdot_{\mathbb{Z}}$, but not the Cancellation Law!)

Definition 7.2 (Addition on the set of rational numbers). We call

$$+_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (p, q) \mapsto p +_{\mathbb{Q}} q \tag{7.27}$$

the *addition on the set of rational numbers*.

Theorem 7.7 (Commutative Law for the addition on $\mathbb{Q}$). *It is true that the addition $+_{\mathbb{Q}}$ on $\mathbb{Q}$ is commutative.*

Proof. Let us verify the universal sentence

$$\forall p, q\,(p, q \in \mathbb{Q} \Rightarrow p +_{\mathbb{Q}} q = q +_{\mathbb{Q}} p), \tag{7.28}$$

taking arbitrary elements $p, q \in \mathbb{Q}$. Thus, p and q constitute equivalence classes $p = [(\bar{m}, \bar{n})]_{\sim_q}$ and $q = [(\bar{M}, \bar{N})]_{\sim_q}$ for some particular $\bar{m}, \bar{n}, \bar{M}, \bar{N} \in \mathbb{Z}$ with $\bar{n} \neq 0$ and $\bar{N} \neq 0$ (according to the Characterization of rational numbers). These equations give according to (7.26)

$$\begin{aligned} p +_{\mathbb{Q}} q &= [(\bar{m}, \bar{n})]_{\sim_q} +_{\mathbb{Q}} [(\bar{M}, \bar{N})]_{\sim_q} \\ &= [(\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}, \bar{n} \cdot \bar{N})]_{\sim_q}. \end{aligned} \tag{7.29}$$

In view of the Commutative Laws for the Multiplication and for the Addition on $\mathbb{Z}$, the equations

$$\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M} = \bar{M} \cdot \bar{n} + \bar{N} \cdot \bar{m}$$

and $\bar{n} \cdot \bar{N} = \bar{N} \cdot \bar{n}$ are both true, allowing us to carry out substitutions to (7.29), that is,

$$p \cdot_{\mathbb{Q}} q = [(\bar{M} \cdot \bar{n} + \bar{N} \cdot \bar{m}, \bar{N} \cdot \bar{n})]_{\sim_q}. \tag{7.30}$$

Observing the truth of $\bar{M}, \bar{N}, \bar{m}, \bar{n} \in \mathbb{Z}$, $\bar{N} \neq 0$ and $\bar{n} \neq 0$, another application of (7.26) yields also

$$\begin{aligned} q \cdot_{\mathbb{Q}} p &= [(\bar{M}, \bar{N})]_{\sim_q} +_{\mathbb{Q}} [(\bar{m}, \bar{n})]_{\sim_q} \\ &= [(\bar{M} \cdot \bar{n} + \bar{N} \cdot \bar{m}, \bar{N} \cdot \bar{n})]_{\sim_q}. \end{aligned}$$

Substitution based on (7.30) gives us now the desired result $p +_{\mathbb{Q}} q = q +_{\mathbb{Q}} p$, where p and q were arbitrary, so that (7.28) follows to be true. Thus, the binary operation $+_{\mathbb{Q}}$ is commutative by definition. □

Theorem 7.8 (Associative Law for the addition on $\mathbb{Q}$). *It is true that the addition $+_{\mathbb{Q}}$ on $\mathbb{Q}$ is associative.*

Proof. According to the definition of an associative binary operation, we must show that $+_{\mathbb{Q}}$ satisfies

$$\forall a, b, c\,(a, b, c \in \mathbb{Q} \Rightarrow (a +_{\mathbb{Q}} b) +_{\mathbb{Q}} c = a +_{\mathbb{Q}} (b +_{\mathbb{Q}} c)).$$

We let a, b and c be arbitrary rational numbers, so that we find $a = [(\bar{m}, \bar{n})]_{\sim_q}$, $b = [(\bar{M}, \bar{N})]_{\sim_q}$ and $c = [(\bar{p}, \bar{q})]_{\sim_q}$ for some particular integers $\bar{m}$, $\bar{n}$, $\bar{M}$, $\bar{N}$, $\bar{p}$ and $\bar{q}$ where $\bar{n}$, $\bar{N}$ and $\bar{q}$ are nonzero. Clearly, these findings imply because of (7.26)

$$\begin{aligned} a +_{\mathbb{Q}} b &= [(\bar{m}, \bar{n})]_{\sim_q} +_{\mathbb{Q}} [(\bar{M}, \bar{N})]_{\sim_q} \\ &= [(\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}, \bar{n} \cdot \bar{N})]_{\sim_q} \end{aligned}$$

as well as

$$\begin{aligned} b +_{\mathbb{Q}} c &= [(\bar{M}, \bar{N})]_{\sim_q} +_{\mathbb{Q}} [(\bar{p}, \bar{q})]_{\sim_q} \\ &= [(\bar{M} \cdot \bar{q} + \bar{N} \cdot \bar{p}, \bar{N} \cdot \bar{q})]_{\sim_q}, \end{aligned}$$

Further applications of (7.26) to these equations lead to

$$\begin{aligned} (a +_{\mathbb{Q}} b) +_{\mathbb{Q}} c &= [(\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}, \bar{n} \cdot \bar{N})]_{\sim_q} +_{\mathbb{Q}} [(\bar{p}, \bar{q})]_{\sim_q} \\ &= [([\bar{m} \cdot \bar{N} + \bar{n} \cdot \bar{M}] \cdot \bar{q} + [\bar{n} \cdot \bar{N}] \cdot \bar{p}, [\bar{n} \cdot \bar{N}] \cdot \bar{q}]_{\sim_q} \\ &= [(\bar{m} \cdot \bar{N} \cdot \bar{q} + \bar{n} \cdot \bar{M} \cdot \bar{q} + \bar{n} \cdot \bar{N} \cdot \bar{p}, \bar{n} \cdot \bar{N} \cdot \bar{q})]_{\sim_q} \end{aligned}$$

as well as

$$\begin{aligned} a +_{\mathbb{Q}} (b +_{\mathbb{Q}} c) &= [(\bar{m}, \bar{n})]_{\sim_q} +_{\mathbb{Q}} [(\bar{M} \cdot \bar{q} + \bar{N} \cdot \bar{p}, \bar{N} \cdot \bar{q})]_{\sim_q} \\ &= [(\bar{m} \cdot [\bar{N} \cdot \bar{q}] + \bar{n} \cdot [\bar{M} \cdot \bar{q} + \bar{N} \cdot \bar{p}], \bar{n} \cdot [\bar{N} \cdot \bar{q}])]_{\sim_q} \\ &= [(\bar{m} \cdot \bar{N} \cdot \bar{q} + \bar{n} \cdot \bar{M} \cdot \bar{q} + \bar{n} \cdot \bar{N} \cdot \bar{p}, \bar{n} \cdot \bar{N} \cdot \bar{q})]_{\sim_q} \end{aligned}$$

using the Distributive Law for $\mathbb{Z}$ and the Associative Law for the multiplication on $\mathbb{Z}$. Thus, the sums $(a +_{\mathbb{Q}} b) +_{\mathbb{Q}} c$ and $a +_{\mathbb{Q}} (b +_{\mathbb{Q}} c)$ are identical, and since a, b and c were initially arbitrary, we may therefore conclude that $+_{\mathbb{Q}}$ is indeed an associative binary operation. □

Note 7.3. The ordered pair $(\mathbb{Q}, +_{\mathbb{Q}})$ constitutes a commutative semigroup in view of the Commutative and Associative Law for the addition on $\mathbb{Q}$.

Proposition 7.9. *It is true that $[(0,1)]_{\sim_q}$ constitutes the neutral element of $\mathbb{Q}$ with respect to the addition $+_{\mathbb{Q}}$ on $\mathbb{Q}$.*

Proof. We already established in (7.11) that $[(0,1)]_{\sim_q}$ is an element of $\mathbb{Q}$, for which we now demonstrate the truth of

$$\forall p \, (p \in \mathbb{Q} \Rightarrow [[(0,1)]_{\sim_q} +_{\mathbb{Q}} p = p \,\wedge\, p +_{\mathbb{Q}} [(0,1)]_{\sim_q} = p]). \tag{7.31}$$

We take an arbitrary rational number p, in other words, some equivalence class $[(\bar{m}, \bar{n})]_{\sim_q}$ that equals p. In connection with (7.26), this gives

$$\begin{aligned} [(0,1)]_{\sim_q} +_{\mathbb{Q}} p &= [(0,1)]_{\sim_q} +_{\mathbb{Q}} [(\bar{m}, \bar{n})]_{\sim_q} \\ &= [(0 \cdot \bar{n} + 1 \cdot \bar{m}, 1 \cdot \bar{n})]_{\sim_q} \\ &= [(\bar{m}, \bar{n})]_{\sim_q} \\ &= p, \end{aligned}$$

where we used also the Cancellation Law for 0 and the definitions of the unity and of the zero element (with respect to $\mathbb{Z}$). Because to the commutativity of $+_{\mathbb{Q}}$, we obtain also

$$\begin{aligned} p +_{\mathbb{Q}} [(0,1)]_{\sim_q} &= [(0,1)]_{\sim_q} +_{\mathbb{Q}} p \\ &= p. \end{aligned}$$

These equations prove the implication in (7.31), in which p is arbitrary, so that the universal sentence (7.31) follows to be true. This means by definition that $[(0,1)]_{\sim_q}$ constitutes the zero element of $\mathbb{Q}$. □

Lemma 7.10. *It is true that the restriction of the addition on $\mathbb{Q}$ to $\mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}}$ is a binary operation on $\mathbb{Z}_{\mathbb{Q}}$, that is,*

$$+_{\mathbb{Q}} \upharpoonright (\mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}}) : \mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}} \to \mathbb{Z}_{\mathbb{Q}}. \tag{7.32}$$

Exercise 7.9. Prove Lemma 7.10.

(Hint: Proceed as in the proof of Lemma 6.49, showing first that the inclusion

$$\mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}} \subseteq \mathbb{Q} \times \mathbb{Q} \tag{7.33}$$

holds.)

Notation 7.3. We symbolize the restricted binary operation (7.32) also by

$$+_{\mathbb{Z}_{\mathbb{Q}}} : \mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}} \to \mathbb{Z}_{\mathbb{Q}}. \tag{7.34}$$

Corollary 7.11. *It is true that the sum of two integers in $\mathbb{Q}$ with respect to the addition on $\mathbb{Z}_{\mathbb{Q}}$ is identical with the sum of these numbers with respect to the addition on $\mathbb{Q}$, that is,*

$$\forall m, n\, (m, n \in \mathbb{Z}_{\mathbb{Q}} \Rightarrow m +_{\mathbb{Z}_{\mathbb{Q}}} n = m +_{\mathbb{Q}} n). \tag{7.35}$$

Proof. We take arbitrary integers m and n in $\mathbb{Z}_{\mathbb{Q}}$ and observe that $(m,n) \in \mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}}$ follows to be true by definition of the Cartesian product of two sets. We obtain then the equations

$$\begin{aligned} m +_{\mathbb{Z}_{\mathbb{Q}}} n &= +_{\mathbb{Z}_{\mathbb{Q}}}((m,n)) = [+_{\mathbb{Q}} \upharpoonright (\mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}})]((m,n)) = +_{\mathbb{Q}}((m,n)) \\ &= m +_{\mathbb{Q}} n \end{aligned}$$

using the notation for binary operations, Notation 7.3, (3.567) in connection with the inclusion (7.33) and $(m,n) \in \mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}}$, and finally again the notation for binary operations. Since m and n were arbitrary, we may therefore conclude that the corollary holds. □

Theorem 7.12 (Isomorphism from $(\mathbb{Z}, +_\mathbb{Z})$ to $(\mathbb{Z}_\mathbb{Q}, +_{\mathbb{Z}_\mathbb{Q}})$). *It is true that $f_\mathbb{Z}^\mathbb{Q}$ constitutes an isomorphism from $(\mathbb{Z}, +_\mathbb{Z})$ to $(\mathbb{Z}_\mathbb{Q}, +_{\mathbb{Z}_\mathbb{Q}})$, that is,*

$$f_\mathbb{Z}^\mathbb{Q} : (\mathbb{Z}, +_\mathbb{Z}) \rightleftarrows (\mathbb{Z}_\mathbb{Q}, +_{\mathbb{Z}_\mathbb{Q}}). \tag{7.36}$$

Proof. We establish first the universal sentence

$$\forall m, n\, (m, n \in \mathbb{Z} \Rightarrow f_\mathbb{Z}^\mathbb{Q}(m +_\mathbb{Z} n) = f_\mathbb{Z}^\mathbb{Q}(m) +_{\mathbb{Z}_\mathbb{Q}} f_\mathbb{Z}^\mathbb{Q}(n)), \tag{7.37}$$

letting m and n be arbitrary integers. The associated values $f_\mathbb{Z}^\mathbb{Q}(m)$ and $f_\mathbb{Z}^\mathbb{Q}(n)$ are then elements of the codomain/range $\mathbb{Z}_\mathbb{Q}$ of the bijection $f_\mathbb{Z}^\mathbb{Q} : \mathbb{Z} \rightleftarrows \mathbb{Z}_\mathbb{Q}$ given in (7.17) and (7.22). We obtain now the equations

$$\begin{aligned} f_\mathbb{Z}^\mathbb{Q}(m) +_{\mathbb{Z}_\mathbb{Q}} f_\mathbb{Z}^\mathbb{Q}(n) &= f_\mathbb{Z}^\mathbb{Q}(m) +_\mathbb{Q} f_\mathbb{Z}^\mathbb{Q}(n) \\ &= [(m,1)]_{\sim_q} +_\mathbb{Q} [(n,1)]_{\sim_q} \\ &= [(m \cdot 1 + 1 \cdot n, 1 \cdot 1)]_{\sim_q} \\ &= [(m+n, 1)]_{\sim_q} \\ &= f_\mathbb{Z}^\mathbb{Q}(m +_\mathbb{Z} n) \end{aligned}$$

by applying (7.35), then (7.17), (7.26), the definition of the unity element, and finally again (7.17). As m and n were arbitrary, we may then infer from the preceding equations the truth of the universal sentence (7.37). Because of the bijectivity of $f_\mathbb{Z}^\mathbb{Q} : \mathbb{Z} \rightleftarrows \mathbb{Z}_\mathbb{Q}$, that function is by definition an isomorphism in the sense of (7.36). □

Corollary 7.13. *The following universal sentence holds:*

$$\forall m, n\, (m, n \in \mathbb{Z} \Rightarrow f_\mathbb{Z}^\mathbb{Q}(m +_\mathbb{Z} n) = f_\mathbb{Z}^\mathbb{Q}(m) +_\mathbb{Q} f_\mathbb{Z}^\mathbb{Q}(n)), \tag{7.38}$$

Proof. Letting $m, n \in \mathbb{Z}$ be arbitrary, we find

$$\begin{aligned} f_\mathbb{Z}^\mathbb{Q}(m +_\mathbb{Z} n) &= f_\mathbb{Z}^\mathbb{Q}(m) +_{\mathbb{Z}_\mathbb{Q}} f_\mathbb{Z}^\mathbb{Q}(n) \\ &= f_\mathbb{Z}^\mathbb{Q}(m) +_\mathbb{Q} f_\mathbb{Z}^\mathbb{Q}(n) \end{aligned}$$

by virtue of (7.37) and (7.35), noting that the values $f_\mathbb{Z}^\mathbb{Q}(m)$ and $f_\mathbb{Z}^\mathbb{Q}(n)$ are elements of the range $\mathbb{Z}_\mathbb{Q}$ of the bijection $f_\mathbb{Z}^\mathbb{Q} : \mathbb{Z} \rightleftarrows \mathbb{Z}_\mathbb{Q}$. As m and n are arbitrary, we therefore conclude that the corollary holds. □

Proposition 7.14. *It is true that the semigroup $(\mathbb{Q}, +_\mathbb{Q})$ is a group.*

Proof. Let us recall from Proposition 7.9 that the zero element of $\mathbb{Q}$ is given by $[(0,1)]_{\sim_q}$, and let us now establish the truth of

$$\forall p\, (p \in \mathbb{Q} \Rightarrow \exists -p\, (p +_\mathbb{Q} -p = [(0,1)]_{\sim_q} \wedge -p +_\mathbb{Q} p = [(0,1)]_{\sim_q})). \tag{7.39}$$

For this purpose, we let $p \in \mathbb{Q}$ be arbitrary, which rational we may express as $p = [(\bar{m}, \bar{n})]_{\sim_q}$ for particular $\bar{m}, \bar{n} \in \mathbb{Z}$ with $\bar{n} \neq 0$. Due to the group property of $(\mathbb{Z}, +_{\mathbb{Z}})$ (see Proposition 6.57), the negative $-\bar{m}$ is also an element of $\mathbb{Z}$, so that $\bar{m}$ and $\bar{n}$ evidently define the equivalence class $[(-\bar{m}, \bar{n})]_{\sim_q}$. We obtain then

$$\begin{aligned} p +_{\mathbb{Q}} [(-\bar{m}, \bar{n})]_{\sim_q} &= [(\bar{m}, \bar{n})]_{\sim_q} +_{\mathbb{Q}} [(-\bar{m}, \bar{n})]_{\sim_q} \\ &= [(\bar{m} \cdot_{\mathbb{Z}} \bar{n} +_{\mathbb{Z}} \bar{n} \cdot_{\mathbb{Z}} [-\bar{m}], \bar{n} \cdot_{\mathbb{Z}} \bar{n})]_{\sim_q} \\ &= [(\bar{m} \cdot_{\mathbb{Z}} \bar{n} +_{\mathbb{Z}} -[\bar{n} \cdot_{\mathbb{Z}} \bar{m}], \bar{n} \cdot_{\mathbb{Z}} \bar{n})]_{\sim_q} \\ &= [(\bar{m} \cdot_{\mathbb{Z}} \bar{n} +_{\mathbb{Z}} -[\bar{m} \cdot_{\mathbb{Z}} \bar{n}], \bar{n} \cdot_{\mathbb{Z}} \bar{n})]_{\sim_q} \\ &= [(0, \bar{n} \cdot_{\mathbb{Z}} \bar{n})]_{\sim_q} \\ &= [([\bar{n} \cdot_{\mathbb{Z}} \bar{n}] \cdot_{\mathbb{Z}} 0, \bar{n} \cdot_{\mathbb{N}} \bar{n})]_{\sim_q} \\ &= [(0, 1)]_{\sim_q} \end{aligned} \tag{7.40}$$

by applying substitution, (7.26), the Sign Law (6.63) with respect to the ring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}})$, the Commutative Law for the multiplication on $\mathbb{Z}$, the definition of an additive inverse, the Cancellation Law for 0, and finally (7.13). Since

$$p +_{\mathbb{Q}} [(-\bar{m}, \bar{n})]_{\sim_q} = [(-\bar{m}, \bar{n})]_{\sim_q} +_{\mathbb{Q}} p$$

also holds because of the Commutative Law for the addition on $\mathbb{Q}$, we may carry out a vsubstitution to obtain

$$[(-\bar{m}, \bar{n})]_{\sim_q} +_{\mathbb{Q}} p = [(0, 1)]_{\sim_q}. \tag{7.41}$$

We thus found a particular integer $[(-\bar{m}, \bar{n})]_{\sim_q}$ for which the two equations (7.40) and (7.41) are true, so that the existential sentence in (7.39) also holds. Because p is arbitrary, we may infer from this finding that Property 2 of a group is satisfied by $(\mathbb{Q}, +_{\mathbb{Q}})$, too. □

Exercise 7.10. Establish the universal sentence

$$\forall m, n\, ([m, n \in \mathbb{Z} \wedge n \neq 0] \Rightarrow [(-m, n)]_{\sim_q} = -[(m, n)]_{\sim_q}). \tag{7.42}$$

(Hint: Consider the equations (7.40) and (7.41).)

Corollary 7.15. *The following universal sentence is true:*

$$\forall m\, (m \in \mathbb{Z} \Rightarrow f_{\mathbb{Z}}^{\mathbb{Q}}(-m) = -f_{\mathbb{Z}}^{\mathbb{Q}}(m)). \tag{7.43}$$

Proof. Letting $m \in \mathbb{Z}$ be arbitrary, we find

$$f_{\mathbb{Z}}^{\mathbb{Q}}(-m) = [(-m, 1)]_{\sim_q} = -[(m, 1)]_{\sim_q} = -f_{\mathbb{Z}}^{\mathbb{Q}}(m)$$

by means of (7.17) and (7.42). As m was arbitrary, we therefore conclude that (7.43) holds. □

Proposition 7.16. *It is true that the negative of every integer in $\mathbb{Q}$ is again an integer in $\mathbb{Q}$.*

Proof. Let us establish the truth of

$$\forall p\,(p \in \mathbb{Z}_{\mathbb{Q}} \Rightarrow -p \in \mathbb{Z}_{\mathbb{Q}}), \tag{7.44}$$

by taking an arbitrary element $p \in \mathbb{Z}_{\mathbb{Q}}$. The inclusion (7.24) and the definition of a subset give us then $p \in \mathbb{Q}$, which implies with (7.39) that the negative $-p$ of p exists and satisfies

$$p +_{\mathbb{Q}} -p = [(0,1)]_{\sim_q} \wedge -p +_{\mathbb{Q}} p = [(0,1)]_{\sim_q}).$$

To show that $-p$ is an element of $\mathbb{Z}_{\mathbb{Q}}$, we denote by $m_p = (f_{\mathbb{Z}}^{\mathbb{Q}})^{-1}(p)$ the value of p under the inverse of the bijection $f_{\mathbb{Z}}^{\mathbb{Q}} : \mathbb{Z} \rightleftarrows \mathbb{Z}_{\mathbb{Q}}$ in (7.22). Then, we obtain

$$p = \mathrm{id}_{\mathbb{Z}_{\mathbb{Q}}}(p) = (f_{\mathbb{Z}}^{\mathbb{Q}} \circ (f_{\mathbb{Z}}^{\mathbb{Q}})^{-1})(p) = f_{\mathbb{Z}}^{\mathbb{Q}}((f_{\mathbb{Z}}^{\mathbb{Q}})^{-1}(p)) = f_{\mathbb{Z}}^{\mathbb{Q}}(m_p) = [(m_p,1)]_{\sim_q}$$

by applying the definition of the identity function, (3.680), the notation for function compositions, substitution, and (7.17). Now, the resulting equation $p = [(m_p,1)]_{\sim_q}$ yields

$$-p = -[(m_p,1)]_{\sim_q} = [(-m_p,1)]_{\sim_q} = f_{\mathbb{Z}}^{\mathbb{Q}}(-m_p)$$

with (7.42) and (7.17), which value is in the codomain/range $\mathbb{Z}_{\mathbb{Q}}$ of the function $f_{\mathbb{Z}}^{\mathbb{Q}} : \mathbb{Z} \rightleftarrows \mathbb{Z}_{\mathbb{Q}}$. This finding proves the implication in (7.44), in which p is arbitrary, so that the proposition follows to be true. □

Having established the group property for $(\mathbb{Q}, +_{\mathbb{Q}})$, the difference of any two elements of $\mathbb{Q}$ is thus defined.

Definition 7.3 (Subtraction on the set of rational numbers). We say that

$$-_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (p,q) \mapsto p -_{\mathbb{Q}} q = p +_{\mathbb{Q}} (-q) \tag{7.45}$$

is the *subtraction on the set of rational numbers*.

Let us now switch over to the multiplication of rational numbers.

Exercise 7.11. Show that there exists a unique function $\cdot_{\mathbb{Q}}$ with domain $\mathbb{Q} \times \mathbb{Q}$ such that

$$\begin{aligned} &\forall x\,(x \in \mathbb{Q} \times \mathbb{Q} \Rightarrow [\cdot_{\mathbb{Q}}(x) \in \mathbb{Q} \wedge \exists E, F, m, n, M, N\,(x = (E,F) \\ &\quad \wedge (m,n) \in E \wedge (M,N) \in F \wedge (m \cdot M, n \cdot N) \in \cdot_{\mathbb{Q}}(x))]), \end{aligned} \tag{7.46}$$

and show that this function $\cdot_{\mathbb{Q}}$ is a binary operation on $\mathbb{Q}$ satisfying

$$\begin{aligned}&\forall m, n, M, N\,([m, n, M, N \in \mathbb{Z} \wedge n \neq 0 \wedge N \neq 0] \\ &\quad \Rightarrow [(m, n)]_{\sim_q} \cdot_{\mathbb{Q}} [(M, N)]_{\sim_q} = [(m \cdot M, n \cdot N)]_{\sim_q}).\end{aligned} \tag{7.47}$$

(Hint: Replace in Exercise 6.19 additions by multiplications.)

Definition 7.4 (Multiplication on the set of rational numbers). We call

$$\cdot_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, \quad (p, q) \mapsto p \cdot_{\mathbb{Q}} q \tag{7.48}$$

the *multiplication on the set of rational numbers*.

Theorem 7.17 (Commutative Law for the multiplication on $\mathbb{Q}$). *It is true that the multiplication $\cdot_{\mathbb{Q}}$ on $\mathbb{Q}$ is commutative.*

Exercise 7.12. Prove the Commutative Law for the multiplication on $\mathbb{Q}$.

(Hint: Proceed in analogy to the proof of the Commutative Law for the addition on $\mathbb{Z}$.)

Theorem 7.18 (Associative Law for the multiplication on $\mathbb{Q}$). *It is true that the multiplication $\cdot_{\mathbb{Q}}$ on $\mathbb{Q}$ is associative.*

Exercise 7.13. Prove the Associative Law for the multiplication on $\mathbb{Z}$.

(Hint: Recall the verification of the Associative Law for the addition on $\mathbb{Z}$.)

Note 7.4. The Commutative and the Associative Law for $\cdot_{\mathbb{Q}}$ demonstrate that $(\mathbb{Q}, \cdot_{\mathbb{Q}})$ forms a commutative semigroup.

Exercise 7.14. Verify that $[(1,1)]_{\sim_q}$ is the neutral element of $\mathbb{Q}$ with respect to the multiplication $\cdot_{\mathbb{Q}}$ on $\mathbb{Q}$, i.e. that $[(1,1)]_{\sim_q}$ is an element of $\mathbb{Q}$ with the property

$$\forall p\,(p \in \mathbb{Q} \Rightarrow [[(1,1)]_{\sim_q} \cdot_{\mathbb{Q}} p = p \wedge p \cdot_{\mathbb{Q}} [(1,1)]_{\sim_q} = p]). \tag{7.49}$$

Lemma 7.19. *It is true that the restriction of the multiplication on $\mathbb{Q}$ to $\mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}}$ is a binary operation on $\mathbb{Z}_{\mathbb{Q}}$, i.e.*

$$\cdot_{\mathbb{Q}} \upharpoonright (\mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}}) : \mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}} \to \mathbb{Z}_{\mathbb{Q}}. \tag{7.50}$$

Exercise 7.15. Verify Lemma 7.19.

Notation 7.4. We symbolize the restricted binary operation (7.50) also by

$$\cdot_{\mathbb{Z}_{\mathbb{Q}}} : \mathbb{Z}_{\mathbb{Q}} \times \mathbb{Z}_{\mathbb{Q}} \to \mathbb{Z}_{\mathbb{Q}}. \tag{7.51}$$

Exercise 7.16. Verify that the product of two integers in $\mathbb{Q}$ with respect to the multiplication on $\mathbb{Z}_{\mathbb{Q}}$ is the same as their product with respect to the multiplication on $\mathbb{Q}$, that is,

$$\forall m, n \, (m, n \in \mathbb{Z}_{\mathbb{Q}} \Rightarrow m \cdot_{\mathbb{Z}_{\mathbb{Q}}} n = m \cdot_{\mathbb{Q}} n). \tag{7.52}$$

(Hint: Replace in the proof of Corollary 7.11 addition by multiplication.)

Theorem 7.20 (Isomorphism from $(\mathbb{Z}, \cdot_{\mathbb{Z}})$ to $(\mathbb{Z}_{\mathbb{Q}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}})$). *It is true that $f_{\mathbb{Z}}^{\mathbb{Q}}$ constitutes an isomorphism from $(\mathbb{Z}, \cdot_{\mathbb{Z}})$ to $(\mathbb{Z}_{\mathbb{Q}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}})$, that is,*

$$f_{\mathbb{Z}}^{\mathbb{Q}} : (\mathbb{Z}, \cdot_{\mathbb{Z}}) \rightleftarrows (\mathbb{Z}_{\mathbb{Q}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}}). \tag{7.53}$$

Exercise 7.17. Prove Theorem 7.20 in analogy to Theorem 7.12, and then

$$\forall m, n \, (m, n \in \mathbb{Z} \Rightarrow f_{\mathbb{Z}}^{\mathbb{Q}}(m \cdot_{\mathbb{Z}} n) = f_{\mathbb{Z}}^{\mathbb{Q}}(m) \cdot_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(n)), \tag{7.54}$$

Theorem 7.21 (Distributive Law for $\mathbb{Q}$). *The multiplication $\cdot_{\mathbb{Q}}$ on the set of rational numbers is distributive over the addition $+_{\mathbb{Q}}$ on $\mathbb{Q}$.*

Exercise 7.18. Prove the Distributive Law for $\mathbb{Q}$. (Hint: Apply (7.7).)

Note 7.5. The set $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, -_{\mathbb{Q}})$ constitutes a commutative ring because

1. $(\mathbb{Q}, +_{\mathbb{Q}})$ is a commutative group (see Note 7.3 and Proposition 7.14),
2. $(\mathbb{Q}, \cdot_{\mathbb{Q}})$ is a commutative semigroup (as mentioned in Note 7.4), and
3. $\cdot_{\mathbb{Q}}$ is distributive over $+_{\mathbb{Q}}$ (according to Theorem 7.21).

Lemma 7.22. *It is true that the ordered quadruple*

$$(\mathbb{Z}_{\mathbb{Q}}, +_{\mathbb{Z}_{\mathbb{Q}}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}}, -_{\mathbb{Z}_{\mathbb{Q}}}) \tag{7.55}$$

constitutes a commutative ring with zero element

$$f_{\mathbb{Z}}^{\mathbb{Q}}(0) = [(0,1)]_{\sim_q} \tag{7.56}$$

and unity element

$$f_{\mathbb{Z}}^{\mathbb{Q}}(1) = [(1,1)]_{\sim_q}. \tag{7.57}$$

Exercise 7.19. Prove Lemma 7.22.

(Hint: Proceed as in the proof of Lemma 7.22, and verify here in addition that $(\mathbb{Z}_{\mathbb{Q}}, +_{\mathbb{Z}_{\mathbb{Q}}})$ satisfies Property 2 of a group by using (7.44).)

Note 7.6. The preceding lemma extends the idea of isomorphic semirings $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}})$ and $(\mathbb{N}_{\mathbb{Z}}, +_{\mathbb{N}_{\mathbb{Z}}}, \cdot_{\mathbb{N}_{\mathbb{Z}}})$ (established for the set of natural numbers under the mapping $f_{\mathbb{N}}^{\mathbb{Z}}$) to the notion of 'isomorphic rings' $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}})$ and $(\mathbb{Z}_{\mathbb{Q}}, +_{\mathbb{Z}_{\mathbb{Q}}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}}, -_{\mathbb{Z}_{\mathbb{Q}}})$ (now under the mapping $f_{\mathbb{Z}}^{\mathbb{Q}}$).

7.2. Ordered Fields $(X, +, \cdot, -, /, <)$

We considered so far only inverse elements with respect to an addition. Similarly to Notation 6.2 concerning such *negatives*, the following notation fixes the conventional symbol for multiplicative inverses.

Notation 7.5. For any semigroup $(X, \cdot)$ with unity element 1_X and for any element $a \in X$ such that the inverse a^{-1} of a with respect to the multiplication exists, we write

$$\frac{1}{a} = \frac{1_X}{a} = 1_X/a = a^{-1} \tag{7.58}$$

the *multiplicative inverse* or the *reciprocal* of a (with respect to the multiplication on X).

We can now translate Note 6.5 by replacing the addition and negatives by the multiplication and reciprocals.

Note 7.7. If the reciprocal $\frac{1}{a}$ of an $a \in X$ with respect to a semigroup $(X, \cdot)$ with unity element 1_X exists, then it satisfies the equations

$$a \cdot \frac{1}{a} = 1_X \ \wedge \ \frac{1}{a} \cdot a = 1_X \tag{7.59}$$

by definition of an inverse element. Moreover, the reciprocal $\frac{1}{1_X}$ of 1_X exists uniquely and satisfies

$$\frac{1}{1_X} = 1_X, \tag{7.60}$$

according to Exercise 6.1.

It is a mere exercise to define (in analogy to Exercise 6.4) the binary operation whose values are the products of group elements and reciprocals of group elements.

Exercise 7.20. Verify for any group $(X, \cdot)$ that there exists the unique binary operation

$$/_X : X \times X \to X, \quad (a, b) \mapsto a/_X b = a \cdot \frac{1}{b}. \tag{7.61}$$

Definition 7.5 (Division, quotient, fraction, numerator, denominator). For any group $(X, \cdot)$ we call the binary operation $/_X$ in (7.61) the *division* on X. We then call for any $a, b \in X$ the value $a/_X b = a \cdot \frac{1}{b}$ the *quotient* of (alternatively, the *fraction* formed by) a and b, which we symbolize also by

$$\frac{a}{b}. \tag{7.62}$$

Here, we say that a is the *numerator* and a the *denominator*.

Note 7.8. In view of Definition 7.5, (7.59) and the definition of the unity element, we have for any group $(X, \cdot)$ and any $a \in X$

$$\frac{a}{a} = a \cdot \frac{1}{a} = 1_X = \frac{1}{a} \cdot a \tag{7.63}$$

and furthermore

$$\frac{1_X}{a} = 1_X \cdot \frac{1}{a} = \frac{1}{a}. \tag{7.64}$$

Exercise 7.21. Show that for any group $(X, \cdot)$ that the every element of X is identical with the quotient of that element and the unity element of X, that is,

$$\forall a \, (a \in X \Rightarrow \frac{a}{1_X} = a). \tag{7.65}$$

(Hint: Use (7.60).)

Exercise 7.22. Demonstrate for any group $(X, \cdot)$ that the ordered pair $(G_S(X), \circ_{G_S(X)})$ formed by the set of scalings on X and the binary operation of composition on that set constitutes a group.

(Hint: Proceed as in the proof of Proposition 6.13, establishing now

$$g_{Sa} \circ_{G_L(X)} g_{S\frac{1}{a}} = \mathrm{id}_X \;\wedge\; g_{S\frac{1}{a}} \circ_{G_S(X)} g_{Sa} = \mathrm{id}_X \tag{7.66}$$

for some particular $a \in X$.)

Definition 7.6 (Scaling group). For any group $(X, \cdot)$, we call

$$(G_S(X), \circ) = (G_S(X), \circ_{G_S(X)}) \tag{7.67}$$

the *scaling group* on X.

We obtain the multiplicative counterparts of the Sign Laws for $-$ and $+$ by rewriting the Inversion Laws for groups, Proposition 6.11 and the Cancellation Law for groups accordingly.

Corollary 7.23 (Laws for reciprocals). *The following sentences are true for any group* $(X, \cdot)$.

a) *Any element is equal to the reciprocal of its reciprocal, that is,*

$$\forall a \, (a \in X \Rightarrow \frac{1}{\frac{1}{a}} = a). \tag{7.68}$$

b) *Equality of the reciprocals of two elements implies equality of the elements, that is,*

$$\forall a, b \, (a, b \in X \Rightarrow \left[\frac{1}{a} = \frac{1}{b} \Rightarrow a = b\right]). \tag{7.69}$$

c) *The reciprocal of a product can be written as the fraction formed by the reciprocal of the second factor and the other factor, that is,*

$$\forall a, b\,(a, b \in X \Rightarrow \frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}). \tag{7.70}$$

d) *The reciprocal of a quotient can be written as a quotient by reversing the elements, in the sense that*

$$\forall a, b\,(a, b \in X \Rightarrow \frac{1}{\frac{a}{b}} = \frac{b}{a}). \tag{7.71}$$

Note 7.9. The Law (7.70) can also be written in the form

$$\forall a, b\,(a, b \in X \Rightarrow \frac{1}{a \cdot b} = \frac{\frac{1}{a}}{b}) \tag{7.72}$$

or – applying the commutativity of the multiplication to the equation (7.70) – in the form

$$\forall a, b\,(a, b \in X \Rightarrow \frac{1}{a \cdot b} = \frac{\frac{1}{b}}{a}). \tag{7.73}$$

Proposition 6.11 and the Cancellation Law for groups can be written in the following forms.

Corollary 7.24. *It is true for any group* $(X, \cdot)$ *that*

$$\forall a, b\,(a, b \in X \Rightarrow \exists q\,(q \in X \;\wedge\; a \cdot q = b)). \tag{7.74}$$

Corollary 7.25 (Cancellation Law for $\cdot$). *It is true for any group* $(X, \cdot)$ *that*

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a \cdot b = a \cdot c \Rightarrow b = c]). \tag{7.75}$$

We now intend to incorporate into a commutative, nontrivial ring the binary operation of division on the underlying set without its zero element.

Theorem 7.26. *For any nontrivial, commutative ring* $(X, +, \cdot, -)$ *for which the unity element* 1_X *and the reciprocal of every element in* $X \setminus \{0_X\}$ *exist (with respect to* $\cdot$*), it is true that*

a) *the product of any two elements in* $X \setminus \{0_X\}$ *is itself an element of* $X \setminus \{0_X\}$*, that is,*

$$\forall a, b\,(a, b \in X \setminus \{0_X\} \Rightarrow a \cdot b \in X \setminus \{0_X\}). \tag{7.76}$$

b) the restriction of the multiplication to $X \setminus \{0_X\} \times X \setminus \{0_X\}$ *constitutes a binary (multiplication) operation on* $X \setminus \{0_X\}$*, i.e.*

$$\cdot \restriction [X \setminus \{0_X\} \times X \setminus \{0_X\}] : X \setminus \{0_X\} \times X \setminus \{0_X\} \to X \setminus \{0_X\}. \quad (7.77)$$

c) the product of any two elements in $X \setminus \{0_X\}$ *with respect to the multiplication on* $X \setminus \{0_X\}$ *is the same as the product of these elements with respect to the multiplication on* X*, that is,*

$$\forall a, b\,(a, b \in X \setminus \{0_X\} \Rightarrow a \cdot_{X \setminus \{0_X\}} b = a \cdot b). \quad (7.78)$$

d) the unity element is an element of $X \setminus \{0_X\}$*, i.e.*

$$1_X \in X \setminus \{0_X\}. \quad (7.79)$$

e) the reciprocal of any element of $X \setminus \{0_X\}$ *is also in* $X \setminus \{0_X\}$*, that is,*

$$\forall a\,(a \in X \setminus \{0_X\} \Rightarrow \frac{1}{a} \in X \setminus \{0_X\}). \quad (7.80)$$

f) the ordered pair

$$(X \setminus \{0_X\}, \cdot \restriction [X \setminus \{0_X\} \times X \setminus \{0_X\}]) \quad (7.81)$$

constitutes a commutative group with unity element 1_X*.*

Proof. We let X, $+$, $\cdot$ and $-$ be arbitrary sets, and we assume $(X, +, \cdot, -)$ to be a commutative ring such that the unity element 1_X exists and such that

$$\forall a\,(a \in X \setminus \{0_X\} \Rightarrow \exists \frac{1}{a}\,(a \cdot \frac{1}{a} = 1_X \,\wedge\, \frac{1}{a} \cdot a = 1_X)) \quad (7.82)$$

holds. Concerning a), also let a and b be arbitrary elements of $X \setminus \{0_X\}$, so that the reciprocals $\frac{1}{a}$ and $\frac{1}{b}$ are defined. Furthermore, we obtain $a, b \in X$, $a \neq 0_X$ and $b \neq 0_X$ with the definition of a set difference and with (2.169). We now prove the negation $\neg a \cdot b = 0_X$ by contradiction, assuming the negation of that negation to be true, so that the Double Negation Law yields the true equation $a \cdot b = 0_X$. Then, we obtain the equations

$$a = a \cdot 1_X = a \cdot (b \cdot \frac{1}{b}) = (a \cdot b) \cdot \frac{1}{b} = 0_X \cdot \frac{1}{b} = 0_X$$

by applying the definition of the neutral element with respect to the multiplication, (7.59), the associativity of the multiplication, substitution based on the true equation $a \cdot b = 0_X$, and finally the Cancellation Law for 0_X in rings. Evidently, the resulting equation $a = 0_X$ contradicts the previously obtained $a \neq 0_X$, so that the negation $\neg a \cdot b = 0_X$ is indeed true. This finding in turn implies $a \cdot b \notin \{0_X\}$ with (2.169), where $a \cdot b$ is a value of the binary operation $\cdot$ on X, and constitutes thus an element of X. The definition of a set difference gives us then the desired $a \cdot b \in X \setminus \{0_X\}$, where a and b are arbitrary, so that (7.76) follows to be true.

Part b) – Part d) can now be established in analogy to Lemma 6.49, Corollary 6.50 and Theorem 5.68.

Concerning e), we let $a \in X \setminus \{0_X\}$ be arbitrary, so that the reciprocal $\frac{1}{a}$ with respect to the multiplication on X exists (in X). We now prove the negation $\frac{1}{a} \neq 0_X$ by contradiction, assuming $\neg \frac{1}{a} \neq 0_X$, which yields $\frac{1}{a} = 0_X$ with the Double Negation Law. Let us observe now the truth of the equations

$$1_X = a \cdot \frac{1}{a} = a \cdot 0_X = 0_X$$

in light of (7.59), the equation $\frac{1}{a} = 0$ and the Cancellation Law for 0_X in rings. The resulting equation $1_X = 0_X$ implies then $1_X \in \{0_X\}$ with (2.169), in contradiction to the fact that $1_X \notin \{0_X\}$ holds according to d) (in view of the definition of a set difference). Having thus completed the proof by contradiction, we may evidently infer from the truth of $\frac{1}{a} \neq 0$ the truth of $\frac{1}{a} \notin \{0\}$ and then also the truth of $\frac{1}{a} \in X \setminus \{0_X\}$ (recalling the truth of $\frac{1}{a} \in X$). Because a was arbitrary, we therefore conclude that e) also holds.

Part f) can be proved by taking a similar approach as in Lemma 6.60 and Proposition 6.57, by using c) – e).

Since X, $+$, $\cdot$ and $-$ were initially arbitrary sets, we can infer from the findings a) – f) the truth of the stated theorem. □

Exercise 7.23. Prove the parts b), c), d) and f) of Theorem 7.26.

Note 7.10. The group (7.81) allows then for the definition of the division on $X \setminus \{0_X\}$, that is,

$$/_{X \setminus \{0_X\}} : X \setminus \{0_X\} \times X \setminus \{0_X\} \to X \setminus \{0_X\}, \quad (a, b) \mapsto \frac{a}{b}. \tag{7.83}$$

Moreover, since the reciprocal $\frac{1}{b}$ of any element $b \in X \setminus \{0_X\}$ is in X, we have for any $a \in X$ that the product $a \cdot \frac{1}{b}$ is in X, being a value of the binary operation $\cdot$ on X.

Notation 7.6. For any nontrivial, commutative ring $(X, +, \cdot, -)$ with unity element 1_X such that the reciprocal of every element in $X \setminus \{0_X\}$ exists, we will use the convenient abbreviations

$$(X \setminus \{0_X\}, \cdot) = (X \setminus \{0_X\}, \cdot \restriction [X \setminus \{0_X\} \times X \setminus \{0_X\}]) \tag{7.84}$$

and

$$/ = /_X = /_{X \setminus \{0_X\}}. \tag{7.85}$$

Furthermore, we write for any $a, b \in X$ with $b \neq 0_X$

$$\frac{a}{b} = a \cdot \frac{1}{b}, \tag{7.86}$$

and we call this a *quotient* or *fraction* as if it was the result of the division (7.83).

Definition 7.7 (Field). For any nontrivial, commutative ring $(X, +, \cdot, -)$ for which the unity element 1_X and the reciprocal of every element in $X \setminus \{0_X\}$ exists (with respect to $\cdot$), we call

$$(X, +, \cdot, -, /) \tag{7.87}$$

a *field*.

Note 7.11. In view of Theorem 7.26f), it is true for every field $(X, +, \cdot, -, /)$ that $(X \setminus \{0_X\}, \cdot)$ constitutes a commutative group.

The next result follows immediately from Theorem 7.26d) with the definition of a set difference and (2.169).

Corollary 7.27. *The unity and zero element with any field* $(X, +, \cdot, -, /)$ *are distinct, i.e.*

$$1_X \neq 0_X. \tag{7.88}$$

Theorem 7.28 (Cancellation Law for $\cdot$ in fields). *The following sentence is true for any field* $(X, +, \cdot, -, /)$.

$$\forall a, b, c\, ([a, b, c \in X \ \wedge\ a \neq 0] \Rightarrow [a \cdot b = a \cdot c \Rightarrow b = c]). \tag{7.89}$$

Proof. Letting X, $+$, $\cdot$, $-$, $/$, a, b, c be arbitrary such that $(X, +, \cdot, -, /)$ is a field, such that a, b and c are elements of X and such that a is nonzero, we see that the reciprocal $\frac{1}{a}$ is defined (by definition of a field). We obtain then the true equations

$$b = 1_X \cdot b = \left(\frac{1}{a} \cdot a\right) \cdot b = \frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot (a \cdot c) = \left(\frac{1}{a} \cdot a\right) \cdot c = 1_X \cdot c = c$$

by applying the definition of the unity element 1_X, (7.59), the associativity of the multiplication on X, substitution based on the assumed equation, again the associativity of the multiplication on X, again (7.59), and finally again the definition of the unity element 1_X. The resulting equation $b = c$ proves the implication $a \cdot b = a \cdot c \Rightarrow b = c$, whose truth in turn establishes the truth of the first implication in (7.89). Since X, $+$, $\cdot$, $-$, $/$, a, b and c were initially all arbitrary, we may therefore conclude that the stated Cancellation Law holds. $\square$

Theorem 7.29 (Zero-divisor freeness of fields). *It is true that the semiring* $(X, +, \cdot)$ *underlying any field* $(X, +, \cdot, -, /)$ *is zero-divisor free.*

Proof. We let X, $+$, $\cdot$, $-$, $/$ be arbitrary and assume that $(X, +, \cdot, -, /)$ constitutes a field. We now apply the Criterion for zero-divisor freeness, letting a and b be arbitrary and assuming $a, b \in X$ to be true. Next, we assume also the disjunction $a \cdot b = 0_X \lor b \cdot a = 0_X$ to hold, and we show that the disjunction $a = 0_X \lor b = 0_X$ is implied. For this purpose, we carry out a proof by cases. In case $a \cdot b = 0_X$ holds, we consider also the true disjunction $a = 0_X \lor a \neq 0_X$, arising by virtue of the Law of the Excluded Middle, and we prove $a = 0_X \lor b = 0_X$ by (sub-)cases. Then, the first sub-case $a = 0_X$ implies that disjunction immediately. In the other sub-case $a \neq 0_X$, we apply the Cancellation Law for 0_X in rings and write the case assumption $a \cdot b = 0_X$ equivalently as

$$a \cdot b = a \cdot 0_X.$$

The truth of $a, b, 0_X \in X$, of $a \neq 0_X$ and of the preceding equation implies now $b = 0_X$ with the Cancellation Law for $\cdot$ in fields. The desired disjunction $a = 0_X \lor b = 0_X$ holds therefore also for the second sub-case, completing the proof of the first case.

In the second case that $b \cdot a = 0_X$ holds, the equation $a \cdot b = 0_X$ is also true because of the commutativity of the multiplication on X, which equation we already demonstrated to imply $a = 0_X \lor b = 0_X$. As the latter disjunction thus holds in any case, and since a and b were arbitrary, we can infer from the truth of that disjunction that the semiring $(X, +, \cdot)$ underlying the given field $(X, +, \cdot, -, /)$ has no zero divisors. Here, X, $+$, $\cdot$, $-$ and $/$ were arbitrary, so that the stated theorem follows to be true. $\square$

In analogy to the fact that an equation $a + d = b$ is uniquely 'solved' by the difference $b - a$ in any ring, we now see that an equation $a \cdot q = b$ is now solved uniquely by the quotient b/a in the context of a field.

Proposition 7.30. *It is true for any field $(X, +, \cdot, -, /)$ and any elements a and b in X where a is nonzero that there exists a unique element q in X that satisfies the equation $a \cdot q = b$, i.e.*

$$\forall a, b\,([a, b \in X \ \wedge \ a \neq 0_X] \Rightarrow \exists! q\,(q \in X \ \wedge \ a \cdot q = b)). \tag{7.90}$$

Proof. We let X, $+$, $\cdot$, $-$, $/$, a, and b be arbitrary, assume $(X, +, \cdot, -, /)$ to be a field, assume a and b to be elements of X, and assume furthermore that $a \neq 0_X$ holds. According to Notation 7.86, we may therefore form the quotient $\bar{q} = \frac{b}{a} = b \cdot \frac{1}{a}$. We obtain then the true equations

$$a \cdot \bar{q} = a \cdot \frac{b}{a} = a \cdot \left(b \cdot \frac{1}{a}\right) = a \cdot \left(\frac{1}{a} \cdot b\right) = \left(a \cdot \frac{1}{a}\right) \cdot b = 1_X \cdot b = b$$

by applying substitutions, the commutativity of the multiplication on X, the associativity of the multiplication on X, (7.59), and the definition of the unity element. The resulting equation $a \cdot \bar{q} = b$ demonstrates in connection with the previous finding $b \cdot \frac{1}{a} \in X$ that there exists an element q in X satisfying $a \cdot q = b$, so that the existential part of the uniquely existential sentence in (7.90) holds.

To prove the uniqueness part, we take an arbitrary q' such that $q' \in X$ and $a \cdot q' = b$ are satisfied. Combining the two equations for b yields then

$$a \cdot \bar{q} = a \cdot q',$$

and this equation implies because of $a, \bar{q}, q' \in X$ and because of $a \neq 0_X$ the equation $\bar{q} = q'$ with the Cancellation Law for $\cdot$ in fields. Because q' was arbitrary, we may therefore conclude that the uniqueness part also holds, according to Method 1.18. Having thus completed the proof of the uniquely existential sentence, we see that the implication in (7.90) is true, and as X, $+$, $\cdot$, $-$, $/$, a, and b were initially arbitrary, the proposed universal sentence follows now to be true. □

The following theorem constitutes a compilation of rules that we will apply frequently when dealing with fractions.

Theorem 7.31 (Calculus of fractions). *The following sentences are true* *for any field* $(X, +, \cdot, -, /)$ *and any* $a, b, c, d, e \in X$ *with* $b, d, e \neq 0_X$.

$$\frac{a}{b} = \frac{c}{d} \;\Leftrightarrow\; a \cdot d = c \cdot b. \tag{7.91}$$

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d}. \tag{7.92}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}. \tag{7.93}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}. \tag{7.94}$$

$$\frac{\frac{a}{b}}{\frac{d}{e}} = \frac{a \cdot e}{b \cdot d}. \tag{7.95}$$

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}. \tag{7.96}$$

Proof. We let $(X, +, \cdot, -, /)$ be an arbitrary field and a, b, c, d, e arbitrary elements of X where $b \neq 0$, $d \neq 0$ and $e \neq 0$.

Concerning the equivalence (7.91), we assume first $\frac{a}{b} = \frac{c}{d}$ to be true and show that $a \cdot d = c \cdot b$ is implied. Indeed, we obtain the equations

$$\begin{aligned}
a \cdot d &= (a \cdot d) \cdot 1_X = (a \cdot d) \cdot \left(\frac{1}{b} \cdot b\right) = \left((a \cdot d) \cdot \frac{1}{b}\right) \cdot b = \left(a \cdot \left(d \cdot \frac{1}{b}\right)\right) \cdot b \\
&= \left(a \cdot \left(\frac{1}{b} \cdot d\right)\right) \cdot b = \left(\left(a \cdot \frac{1}{b}\right) \cdot d\right) \cdot b = \left(\frac{a}{b} \cdot d\right) \cdot b = \left(\frac{c}{d} \cdot d\right) \cdot b \\
&= \left(\left(c \cdot \frac{1}{d}\right) \cdot d\right) \cdot b = \left(c \cdot \left(\frac{1}{d} \cdot d\right)\right) \cdot b = (c \cdot 1_X) \cdot b = c \cdot b
\end{aligned}$$

by applying the definition of the unity element, (7.59), the associativity of the multiplication on X (twice), the commutativity of the multiplication on X, again the associativity of the multiplication on X, the notation (7.86), substitution based on the assumed equation, again the notation (7.86), again the associativity of the multiplication on X, again (7.59), and finally again the definition of the unity element. Thus, the resulting equation $a \cdot d = c \cdot b$ proves the first part ('$\Rightarrow$') of the equivalence (7.91).

To prove the second part ('$\Leftarrow$'), we now assume $a \cdot d = c \cdot b$ and verify that $\frac{a}{b} = \frac{c}{d}$ follows to be true. Indeed, we evidently obtain

$$\begin{aligned}\frac{a}{b} &= a \cdot \frac{1}{b} = \left(a \cdot \frac{1}{b}\right) \cdot 1_X = \left(a \cdot \frac{1}{b}\right) \cdot \left(d \cdot \frac{1}{d}\right) = \left(\left(a \cdot \frac{1}{b}\right) \cdot d\right) \cdot \frac{1}{d} \\ &= \left(a \cdot \left(\frac{1}{b} \cdot d\right)\right) \cdot \frac{1}{d} = \left(a \cdot \left(d \cdot \frac{1}{b}\right)\right) \cdot \frac{1}{d} = \left((a \cdot d) \cdot \frac{1}{b}\right) \cdot \frac{1}{d} \\ &= \left((c \cdot b) \cdot \frac{1}{b}\right) \cdot \frac{1}{d} = \left(c \cdot \left(b \cdot \frac{1}{b}\right)\right) \cdot \frac{1}{d} = (c \cdot 1_X) \cdot \frac{1}{d} = c \cdot \frac{1}{d} = \frac{c}{d}\end{aligned}$$

by applying essentially the same arguments as in the proof of '$\Rightarrow$'. As the the second implication is thus also true, the proposed equivalence holds.

Concerning (7.92), we observe first the truth of the equations

$$a \cdot (b \cdot d) = (a \cdot b) \cdot d = (b \cdot a) \cdot d = b \cdot (a \cdot d)$$

in light of the associativity and the commutativity of of the multiplication, so that

$$a \cdot (b \cdot d) = b \cdot (a \cdot d)$$

holds. Recalling the Zero-divisor freeness of fields, we have here that

$$b \cdot d \neq 0_X$$

is implied by the assumptions $b \neq 0$ and $d \neq 0$ because of the Criterion for zero-divisor freeness. In conjunction with the assumed $b \neq 0$ and the preceding equation, this yields with (7.91) the desired.

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d}.$$

Concerning (7.93), we can derive the equations

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= \left(a \cdot \frac{1}{b}\right) \cdot \left(d \cdot \frac{1}{d}\right) + \left(c \cdot \frac{1}{d}\right) \cdot \left(b \cdot \frac{1}{b}\right) \\ &= (a \cdot d) \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right) + (b \cdot c) \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right) \\ &= (a \cdot d + b \cdot c) \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right) = (a \cdot d + b \cdot c) \cdot \frac{1}{b \cdot d} \\ &= \frac{a \cdot d + b \cdot c}{b \cdot d}\end{aligned}$$

by using notation (7.86) alongside the definition of the unity element and (7.59), the associativity & commutativity of the multiplication, the distributivity of the multiplication over the addition, (7.70), and finally again notation (7.86).

The equation (7.94) can be proved by using most of the arguments in the proof of (7.94).

Concerning (7.95) we observe the truth of the equations

$$\frac{\frac{a}{b}}{\frac{d}{e}} = \frac{a}{b}\cdot\frac{1}{d\cdot\frac{1}{e}} = \frac{a}{b}\cdot\left(\frac{1}{d}\cdot\frac{1}{\frac{1}{e}}\right) = \frac{a}{b}\cdot\left(\frac{1}{d}\cdot e\right) = \frac{a}{b}\cdot\left(e\cdot\frac{1}{d}\right) = \frac{a}{b}\cdot\frac{e}{d} = \frac{a\cdot e}{b\cdot d}$$

in light of notation (7.86), (7.70), (7.68), the commutativity of the multiplication and (7.94).

Concerning (7.96) we obtain on the one hand

$$-\frac{a}{b} = -\left(a\cdot\frac{1}{b}\right) = -\left(\frac{1}{b}\cdot a\right) = \frac{1}{b}\cdot(-a) = (-a)\cdot\frac{1}{b} = \frac{-a}{b}$$

by applying (7.86), the commutativity of $\cdot$, the Sign Law (6.63) with respect to the ring $(X,+,\cdot,-)$, again the commutativity of $\cdot$, and again (7.62). On the other hand, the Sign Law (6.65) yields

$$a\cdot b = (-a)\cdot(-b),$$

which gives us

$$\frac{a}{-b} = \frac{-a}{b} \quad \left[= -\frac{a}{b}\right]$$

with (7.91) based on the fact that the assumption $b \neq 0_X$ implies $-b \neq 0_X$ according to (6.39).

Initially, the field and the elements were arbitrary, so that the previous findings imply the truth of the theorem. □

Exercise 7.24. Verify the following equation for any field $(X,+,\cdot,-,/)$ and any elements $a,b,c \in X$ with $b \neq 0_X$.

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \tag{7.97}$$

(Hint: Use (7.92) and (7.93).)

We now adjoin to a field a linear ordering which is compatible with the addition and multiplication in the sense that the usual monotony laws are satisfied.

Definition 7.8 (Ordered Field). For any field $(X, +, \cdot, -, /)$ and any linear ordering $<$ of X satisfying the monotony laws

$$\forall a, b, c\,(a, b, c \in X \Rightarrow [a < b \Rightarrow a + c < b + c]), \tag{7.98}$$

$$\forall a, b, c\,([a, b, c \in X \;\wedge\; 0_X < c] \Rightarrow [a < b \Rightarrow a \cdot c < b \cdot c]), \tag{7.99}$$

we say that

$$(X, +, \cdot, -, /, <) \tag{7.100}$$

an *ordered field.*

Note 7.12. An ordered field is thus characterized by the following *field axioms*:

1. X contains at least two elements,
2. the addition is associative,
3. the addition is commutative,
4. the multiplication is associative,
5. the multiplication is commutative,
6. the multiplication is distributive over the addition,
7. X contains the zero element,
8. X contains the negative of each of its elements,
9. X contains the unity element,
10. X contains the reciprocal of every nonzero element,
11. $<$ is comparable,
12. $<$ is transitive, and
13. $<$ is compatible with the addition and multiplication.

Note 7.13. Every ordered field is an ordered integral domain as $(X, +, \cdot, -)$ constitutes on the one hand a commutative, non-trivial, zero-divisor free ring with unity element 1_X. On the other hand, an ordered integral domain has a linear ordering $<$ and satisfies the same monotony laws as an ordered field. Consequently, all of the laws stated in Section 6.6 immediately carry over to ordered fields, including the Generalized Associative & Commutative Law for semirings and the formation of n-fold additions & multiplications (and in particular of multiples na and powers a^n).

Proposition 7.32. *For any ordered field $(X, +, \cdot, -, /, <)$, it is true that the reciprocal of a positive element of X is again positive, that is,*

$$\forall a\,([a \in X \wedge a > 0_X] \Rightarrow \frac{1}{a} > 0_X). \tag{7.101}$$

Proof. We let X, $+$, $\cdot$, $-$, $/$, $<$ and a be arbitrary, assume $(X, +, \cdot, -, /, <)$ to be an ordered field, assume a to be an element of X, and assume $a > 0_X$ to be true. Since the linear ordering $<$ satisfies the Characterization of comparability, the latter assumption implies that $a = 0_X$ is false, in other words, that $a \neq 0_X$ is true. Thus, $a \in X \setminus \{0_X\}$ follows to be true by definition of a set difference and (2.169). Then, the reciprocal $\frac{1}{a}$ is also in $X \setminus \{0_X\}$ due to (7.80), which finding evidently implies $\frac{1}{a} \in X$ and $\frac{1}{a} \neq 0_X$. Now, since the given ordered field constitutes an ordered integral domain, as mentioned in Note 7.13, we may obtain $\left(\frac{1}{a}\right)^2 > 0_X$ with (6.69). In conjunction with the assumed inequality $0_X < a$, this gives us with (7.99)

$$0_X \cdot \left(\frac{1}{a}\right)^2 < a \cdot \left(\frac{1}{a}\right)^2.$$

Here, we can write for the left-hand side

$$0_X \cdot \left(\frac{1}{a}\right)^2 = 0_X,$$

by using the Cancellation Law for 0_X (in rings), and for the right-hand side

$$a \cdot \left(\frac{1}{a}\right)^2 = a \cdot \left(\frac{1}{a} \cdot \frac{1}{a}\right) = \left(a \cdot \frac{1}{a}\right) \cdot \frac{1}{a} = 1_X \cdot \frac{1}{a} = \frac{1}{a}$$

by applying (5.478), the associativity of the multiplication, (7.59), and the definition of the unity element. Thus, substitutions yield the desired result $0_X < \frac{1}{a}$, proving the implication in (7.101). Because X, $+$, $\cdot$, $-$, $/$, $<$ and a were initially arbitrary, we conclude that the proposition holds. □

Theorem 7.33 (Denseness of ordered fields). *It is true for any ordered field $(X, +, \cdot, -, /, <)$ that the set $(X, <)$ is densely ordered.*

Proof. Letting $(X, +, \cdot, -, /, <)$ be an arbitrary ordered field, we note that $(X, <)$ is linearly ordered, by definition. Let us recall also that X contains the zero element 0_X, so that X is clearly nonempty. In addition, we can prove by contradiction that $\forall a\,(X \neq \{a\})$ holds. Assuming the negation of that universal sentence to be true, it follows with the Negation Law for universal sentences that there exists a constant, say $\bar{a}$, such that $\neg X \neq \{\bar{a}\}$

is true. This implies with the Double Negation Law $X = \{\bar{a}\}$, so that the uniquely existential sentence $\exists! y\,(y \in \{\bar{a}\})$ holds according to (2.180). Because X contains, besides the zero element 0_X, also the unity element 1_X, we find $1_X, 0_X \in \{\bar{a}\}$ to be true through substitutions. Therefore, the uniqueness part of the preceding uniquely existential sentence yields $1_X = 0_X$, in contradiction to the fact that $1_X \neq 0_X$ holds in fields in view of (7.88). We thus demonstrated that X is neither empty nor a singleton, as required by Property 1 of a densely ordered set.

To verify the remaining Property 2, we let a and b be arbitrary elements of X with $a < b$ and show that there exists an intermediate value of a and b. Because $1_X \neq 0_X$ implies $1_X \cdot 1_X > 0_X$ with (6.238), we obtain $0_X < 1_X$ by means of the definition of the unity element. Consequently, the monotony law (7.98) yields $0_X + 1_X < 1_X + 1_X$, which we can write equivalently as $1_X + 1_X > 1_X$ $[> 0_X]$ by definition of the zero element. Then, $1_X + 1_X > 0_X$ follows to be true with the transitivity of the linear ordering $<$, and this inequality further implies $\frac{1}{1_X+1_X} > 0_X$ with (7.101). With this, the assumed $a < b$ gives

$$a \cdot \frac{1}{1_X + 1_X} < b \cdot \frac{1}{1_X + 1_X}. \tag{7.102}$$

with the monotony law (7.99). The preceding inequality further implies with the monotony law (7.98)

$$a \cdot \frac{1}{1_X + 1_X} + a \cdot \frac{1}{1_X + 1_X} < b \cdot \frac{1}{1_X + 1_X} + a \cdot \frac{1}{1_X + 1_X} \tag{7.103}$$

where we can write for the left-hand side

$$\begin{aligned} a \cdot \frac{1}{1_X + 1_X} + a \cdot \frac{1}{1_X + 1_X} &= \left(a \cdot \frac{1}{1_X + 1_X}\right) \cdot 1_X + \left(a \cdot \frac{1}{1_X + 1_X}\right) \cdot 1_X \\ &= \left(a \cdot \frac{1}{1_X + 1_X}\right) \cdot (1_X + 1_X) \\ &= a \cdot \left[\frac{1}{1_X + 1_X} \cdot (1_X + 1_X)\right] \\ &= a \cdot 1_X \\ &= a, \end{aligned}$$

by applying the definition of the unity element, the distributivity of the multiplication over the addition, the associativity of the multiplication, and (7.63) in connection with the fact that the previously found $1_X + 1_X > 0_X$ implies $1_X + 1_X \neq 0_X$ with the comparability of the linear ordering $<$ and therefore $1_X + 1_X \in X \setminus \{0_X\}$ with the definition of a set difference and (2.169). Thus, we can write (7.103) as

$$a < \frac{a}{1_X + 1_X} + \frac{b}{1_X + 1_X}. \tag{7.104}$$

On the other hand, (7.102) implies with the monotony law (7.98)

$$a \cdot \frac{1}{1_X + 1_X} + b \cdot \frac{1}{1_X + 1_X} < b \cdot \frac{1}{1_X + 1_X} + b \cdot \frac{1}{1_X + 1_X} \tag{7.105}$$

where we can write for the right-hand side

$$\begin{aligned} b \cdot \frac{1}{1_X + 1_X} + b \cdot \frac{1}{1_X + 1_X} &= \left(b \cdot \frac{1}{1_X + 1_X}\right) \cdot 1_X + \left(b \cdot \frac{1}{1_X + 1_X}\right) \cdot 1_X \\ &= \left(b \cdot \frac{1}{1_X + 1_X}\right) \cdot (1_X + 1_X) \\ &= b \cdot \left[\frac{1}{1_X + 1_X} \cdot (1_X + 1_X)\right] \\ &= b \cdot 1_X \\ &= b, \end{aligned}$$

by using exactly the same arguments as in the simplification of the left-hand side of (7.103). Thus, (7.105) becomes after substitution

$$\frac{a}{1_X + 1_X} + \frac{b}{1_X + 1_X} < b. \tag{7.106}$$

Combining (7.104) with (7.106) yields now

$$a < \frac{a}{1_X + 1_X} + \frac{b}{1_X + 1_X} < b, \tag{7.107}$$

where the sum is a value of the binary addition operation on X, which thus constitutes an element of X. We thus proved the existential sentence $\exists z\,(z \in X \wedge a < z < b)$, and since a and b were arbitrary, we may therefore conclude that the linearly ordered set $(X, <)$ satisfies also Property 2 of a densely ordered set. The ordered field $(X, +, \cdot, -, /, <)$ was initially arbitrary, so that the stated theorem follows now to be true. □

Exercise 7.25. Show for any ordered field $(X, +, \cdot, -, /, <)$ and any elements $a, b \in X$ with $a < b$ that

$$\frac{a}{1_X + 1_X} + \frac{b}{1_X + 1_X} = \frac{a + b}{1_X + 1_X}. \tag{7.108}$$

(Hint: Apply (7.92) and (7.93).)

Note 7.14. Since (7.107) holds for arbitrary $a, b \in X$ with $a < b$, we see in light of (7.108) that

$$\forall a, b\,([a, b \in X \ \wedge\ a < b] \Rightarrow a < \frac{a+b}{1_X + 1_X} < b). \tag{7.109}$$

7.3. The Ordered Field $(\mathbb{Q}, +_\mathbb{Q}, \cdot_\mathbb{Q}, -_\mathbb{Q}, /_\mathbb{Q}, <_\mathbb{Q})$

Exercise 7.26. Show that $(\mathbb{Q}, +_\mathbb{Q}, \cdot_\mathbb{Q}, -_\mathbb{Q})$ is a nontrivial ring.
(Hint: Proceed as for Exercise 6.38.)

Theorem 7.34. *The ring* $(\mathbb{Q}, +_\mathbb{Q}, \cdot_\mathbb{Q}, -_\mathbb{Q})$ *allows for the definition of the division on* $\mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}$.

Proof. Let us begin with the observation that the unity element $[(1,1)]_{\sim_q}$ of $\mathbb{Q}$ exists according to Exercise 7.14. We can show now that this element satisfies

$$\forall p\,(p \in \mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\} \Rightarrow \exists p^{-1}\,(p \cdot p^{-1} = [(1,1)]_{\sim_q} \wedge p^{-1} \cdot p = [(1,1)]_{\sim_q})). \tag{7.110}$$

For this purpose, we let $p \in \mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}$ be arbitrary, which means by definition of a set difference and by definition of $\mathbb{Q}$ that $p \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]/\sim_q$ and $p \notin \{[(0,1)]_{\sim_q}\}$ are true. Here, the latter negation implies $p \neq [(0,1)]_{\sim_q}$ with (2.169). By definition of a quotient set, there exists then an element, say $\bar{z} \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$, for which $[\bar{z}]_{\sim_q} = p$ is satisfied. According to the definition of the Cartesian product of two sets, there exist now also elements, say $\bar{m} \in \mathbb{Z}$ and $\bar{n} \in \mathbb{Z} \setminus \{0\}$, such that $\bar{z} = (\bar{m}, \bar{n})$. Here, we note that $\bar{n} \in \mathbb{Z}$ and $\bar{n} \neq 0$ hold by definition of a set difference and (2.169). Applying substitution to the previous equation for p results now in $[(\bar{m}, \bar{n})]_{\sim_q} = p$, and this gives again by substitution $[(\bar{m}, \bar{n})]_{\sim_q} \neq [(0,1)]_{\sim_q}$. This negation in turn implies $\bar{m} \neq 0$ with (7.16), and consequently $\bar{m} \notin \{0\}$. In conjunction with $\bar{m} \in \mathbb{Z}$, this gives $\bar{m} \in \mathbb{Z} \setminus \{0\}$. Let us observe now that $\bar{n} \in \mathbb{Z} \setminus \{0\}$ implies with the definition of a set difference in especially $\bar{n} \in \mathbb{Z}$. Then, the ordered pair $(\bar{n}, \bar{m})$ is in the Cartesian product $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ and defines therefore the equivalence class $[(\bar{n}, \bar{m})]_{\sim_q}$. We obtain now the equations

$$\begin{aligned} p \cdot_\mathbb{Q} [(\bar{n}, \bar{m})]_{\sim_q} &= [(\bar{m}, \bar{n})]_{\sim_q} \cdot_\mathbb{Q} [(\bar{n}, \bar{m})]_{\sim_q} \\ &= [(\bar{m} \cdot_\mathbb{Z} \bar{n}, \bar{n} \cdot_\mathbb{Z} \bar{m})]_{\sim_q} \\ &= [(\bar{m} \cdot_\mathbb{Z} \bar{n}, \bar{m} \cdot_\mathbb{Z} \bar{n})]_{\sim_q} \\ &= [(1,1)]_{\sim_q} \end{aligned} \tag{7.111}$$

by applying substitution, (7.47), the Commutative Law for the multiplication on $\mathbb{Z}$, and finally (7.15) based on the fact that $\bar{m} \neq 0$ and $\bar{n} \neq 0$ imply $\bar{m} \cdot_\mathbb{Z} \bar{n} \neq 0$ with the Criterion for zero-divisor freeness – recalling the Zero-divisor freeness of $(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z})$. Since

$$p \cdot_\mathbb{Q} [(\bar{n}, \bar{m})]_{\sim_q} = [(\bar{n}, \bar{m})]_{\sim_q} \cdot_\mathbb{Q} p$$

also holds because of the Commutative Law for the multiplication on $\mathbb{Q}$, we may apply substitution to (7.111) to obtain

$$[(\bar{n},\bar{m})]_{\sim_q} \cdot_{\mathbb{Q}} p = [(1,1)]_{\sim_q}. \tag{7.112}$$

We thus found the particular rational number $[(\bar{n},\bar{m})]_{\sim_q}$ for which the two equations (7.111) and (7.112) are true, so that the existential sentence in (7.110) also holds. Because p is arbitrary, we may infer from this finding that the universal sentence (7.110) is true, which means that the multiplicative inverse of any element of $\mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}$ exists.

To summarize, we found $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, -_{\mathbb{Q}})$ to be a nontrivial, commutative ring (see Exercise 7.26 and Note 7.5) for which the unity element $[(1,1)]_{\sim_q}$ and the reciprocal of every element in $\mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}$ exist (with respect to the multiplication on $\mathbb{Q}$), where $[(0,1)]_{\sim_q}$ is the zero element of $\mathbb{Q}$ (see Proposition 7.9). According to Theorem 7.26f), the ordered pair

$$(\mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}, \cdot_{\mathbb{Q}} \restriction [\mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\} \times \mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}]) \tag{7.113}$$

constitutes therefore a group, so that the division

$$/_{\mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}} \tag{7.114}$$

is defined. □

Exercise 7.27. Establish the universal sentence

$$\forall m, n\,([m, n \in \mathbb{Z} \,\wedge\, n \neq 0] \Rightarrow [(n,m)]_{\sim_q} = [(m,n)]_{\sim_q}^{-1}). \tag{7.115}$$

We will use in the following abbreviations of (7.113) and (7.114) according to Notation 7.6.

Definition 7.9 (Division on the set of (nonzero) rational numbers). We say that

$$/_{\mathbb{Q}} : \mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\} \times \mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\} \to \mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}, \quad (p,q) \mapsto \frac{p}{q}. \tag{7.116}$$

is the *division on the set of (nonzero) rational numbers.*

Having thus established the division $/_{\mathbb{Q}}$ on $\mathbb{Q} \setminus \{[(0,1)]_{\sim_q}\}$, we can apply the definition of a field to the set of rational numbers.

Definition 7.10 (Field of rational numbers). We call

$$(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, -_{\mathbb{Q}}, /_{\mathbb{Q}}) \tag{7.117}$$

the *field of rational numbers.*

Note 7.15. The semiring $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}})$ underlying the field of rational numbers is zero-divisor free because of Theorem 7.29, and satisfies therefore according to the Criterion for zero-divisor freeness

$$\begin{aligned} &\forall p, q\, (p, q \in \mathbb{Q} \Rightarrow [(p \cdot_{\mathbb{Q}} q = [(0,1)]_{\sim_q} \vee q \cdot_{\mathbb{Q}} p = [(0,1)]_{\sim_q}) \\ &\quad \Rightarrow (p = [(0,1)]_{\sim_q} \vee q = [(0,1)]_{\sim_q})]). \end{aligned} \tag{7.118}$$

Note 7.16. The Notation 7.1 used for rational numbers is consistent with the notation for fractions if one treats the integers m, n forming a rational number $p = [(m,n)]_{\sim_q} = \frac{m}{n}$ as rational numbers, noting that we can write for its inverse on the one hand

$$p^{-1} = [(m,n)]_{\sim_q}^{-1} = [(n,m)]_{\sim_q} = \frac{n}{m}$$

in view of (7.115), and on the other hand

$$\frac{1}{p} = \frac{1}{\frac{m}{n}} = \frac{n}{m}$$

by means of the Laws for reciprocals. Observing for two given rational numbers $p = [(m,n)]_{\sim_q} = \frac{m}{n}$ and $q = [(M,N)]_{\sim_q} = \frac{M}{N}$ the truth of the equivalences

$$\begin{aligned} p = q \;&\Leftrightarrow\; [(m,n)]_{\sim_q} = [(M,N)]_{\sim_q} \\ &\Leftrightarrow\; (m,n) \sim_q (M,N) \\ &\Leftrightarrow\; m \cdot_{\mathbb{Z}} N = M \cdot_{\mathbb{Z}} n \end{aligned}$$

and (treating integers as rational numbers)

$$\begin{aligned} p = q \;&\Leftrightarrow\; \frac{m}{n} = \frac{M}{N} \\ &\Leftrightarrow\; m \cdot_{\mathbb{Q}} N = M \cdot_{\mathbb{Q}} n \end{aligned}$$

we also see that the the equivalence relation $\sim_q$ in connection with Notation 7.1 models the equivalence (7.91) for field elements (initially in terms of integers). Similarly, we see in light of the equations

$$\begin{aligned} p +_{\mathbb{Q}} q &= [(m,n)]_{\sim_q} +_{\mathbb{Q}} [(M,N)]_{\sim_q} = [(m \cdot_{\mathbb{Z}} N +_{\mathbb{Z}} n \cdot_{\mathbb{Z}} M, n \cdot_{\mathbb{Z}} N)]_{\sim_q} \\ &= \frac{m \cdot_{\mathbb{Z}} N +_{\mathbb{Z}} n \cdot_{\mathbb{Z}} M}{n \cdot_{\mathbb{Z}} N}, \end{aligned}$$

$$\begin{aligned} p +_{\mathbb{Q}} q &= \frac{m}{n} +_{\mathbb{Q}} \frac{M}{N} \\ &= \frac{m \cdot_{\mathbb{Q}} N +_{\mathbb{Q}} n \cdot_{\mathbb{Q}} M}{n \cdot_{\mathbb{Q}} N} \end{aligned}$$

that the definition of the addition on $\mathbb{Q}$ (and Notation 7.1) model the behavior (7.93) of the sum of two fractions formed by field elements. Concerning the product of two such fractions, which is determined on the one hand by the multiplication on $\mathbb{Q}$ and on the other hand characterized in fields by (7.95), we also find consistency in view of the equations

$$\begin{aligned} p \cdot_{\mathbb{Q}} q &= [(m,n)]_{\sim_q} \cdot_{\mathbb{Q}} [(M,N)]_{\sim_q} = [(m \cdot_{\mathbb{Z}} M, n \cdot_{\mathbb{Z}} N)]_{\sim_q} \\ &= \frac{m \cdot_{\mathbb{Z}} M}{n \cdot_{\mathbb{Z}} N}, \\ p \cdot_{\mathbb{Q}} q &= \frac{m}{n} \cdot_{\mathbb{Q}} \frac{M}{N} \\ &= \frac{m \cdot_{\mathbb{Q}} M}{n \cdot_{\mathbb{Q}} N}. \end{aligned}$$

Theorem 7.35. *The following sentences are true.*

a) There exists a unique set $\leq_{\mathbb{Q}}$ *such that*

$$\forall X (X \in \leq_{\mathbb{Q}} \Leftrightarrow [X \in \mathbb{Q} \times \mathbb{Q} \wedge \exists m,n,p,q (X = ([(m,n)]_{\sim_q}, [(p,q)]_{\sim_q}) \\ \wedge \; [n \cdot q > 0 \Rightarrow m \cdot q \leq n \cdot p] \wedge [n \cdot q < 0 \Rightarrow m \cdot q \geq n \cdot p])]) \quad (7.119)$$

and this set $\leq_{\mathbb{Q}}$ *is a binary relation on* $\mathbb{Q}$ *satisfying*

$$\forall X \, (X \in \leq_{\mathbb{Q}} \Leftrightarrow \exists m,n,p,q \, (X = ([(m,n)]_{\sim_q}, [(p,q)]_{\sim_q}) \quad (7.120) \\ \wedge \; [n \cdot q > 0 \Rightarrow m \cdot q \leq n \cdot p] \; \wedge \; [n \cdot q < 0 \Rightarrow m \cdot q \geq n \cdot p])).$$

b) Then, the binary relation $\leq_{\mathbb{Q}}$ *satisfies also*

$$\forall m,n,p,q \, ([(m,n)]_{\sim_q} \leq_{\mathbb{Q}} [(p,q)]_{\sim_q} \Leftrightarrow (n \neq 0 \; \wedge \; q \neq 0 \quad (7.121) \\ \wedge \; [n \cdot q > 0 \Rightarrow m \cdot q \leq n \cdot p] \; \wedge \; [n \cdot q < 0 \Rightarrow m \cdot q \geq n \cdot p])).$$

c) Furthermore, the binary relation $\leq_{\mathbb{Q}}$ *is a total ordering of* $\mathbb{Q}$.

Exercise 7.28. Prove (7.119), (7.120) and the implication '$\Leftarrow$' in (7.121).

Proof. Part a) can be established in analogy to the corresponding part in Proposition 6.78. We now prove b), letting $\bar{m}$, $\bar{n}$, $\bar{p}$ and $\bar{q}$ be arbitrary and assuming first that the inequality $[(\bar{m},\bar{n})]_{\sim_q} \leq_{\mathbb{Q}} [(\bar{p},\bar{q})]_{\sim_q}$ is true, which we may write also in the form $([(\bar{m},\bar{n})]_{\sim_q}, [(\bar{p},\bar{q})]_{\sim_q}) \in \leq_{\mathbb{Q}}$. The latter implies because of (7.120) that there exist constants, say $\bar{M}$, $\bar{N}$, $\bar{P}$ and $\bar{Q}$, such that equation

$$([(\bar{m},\bar{n})]_{\sim_q}, [(\bar{p},\bar{q})]_{\sim_q}) = ([(\bar{M},\bar{N})]_{\sim_q}, [(\bar{P},\bar{Q})]_{\sim_q})$$

and the implications

$$\bar{N} \cdot \bar{Q} > 0 \Rightarrow \bar{M} \cdot \bar{Q} \leq \bar{N} \cdot \bar{P} \tag{7.122}$$

$$\bar{N} \cdot \bar{Q} < 0 \Rightarrow \bar{M} \cdot \bar{Q} \geq \bar{N} \cdot \bar{P} \tag{7.123}$$

are satisfied. The former equation implies with the Equality Criterion for ordered pairs the truth of the two equations

$$[(\bar{m}, \bar{n})]_{\sim_q} = [(\bar{M}, \bar{N})]_{\sim_q},$$
$$[(\bar{p}, \bar{q})]_{\sim_q} = [(\bar{P}, \bar{Q})]_{\sim_q},$$

which in turn imply with the Equality Criterion for equivalence classes

$$(\bar{m}, \bar{n}) \sim_q (\bar{M}, \bar{N}),$$
$$(\bar{p}, \bar{q}) \sim_q (\bar{P}, \bar{Q}).$$

According to the characterization of the equivalence relation $\sim_q$ in (7.3), the equations

$$\bar{m} \cdot \bar{N} = \bar{M} \cdot \bar{n} \tag{7.124}$$

$$\bar{p} \cdot \bar{Q} = \bar{P} \cdot \bar{q} \tag{7.125}$$

are therefore true. Let us observe now that the ordered pairs $(\bar{m}, \bar{n})$, $(\bar{p}, \bar{q})$, $(\bar{M}, \bar{N})$, $(\bar{P}, \bar{Q})$ – which form the equivalence classes $[(\bar{m}, \bar{n})]_{\sim_q}$, $[(\bar{p}, \bar{q})]_{\sim_q}$, $[(\bar{M}, \bar{N})]_{\sim_q}$ and $[(\bar{P}, \bar{Q})]_{\sim_q}$ with respect to the equivalence relation $\sim_q$ on $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ – are thus elements of the Cartesian product $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$. By definition of the Cartesian product of two sets, we find in particular $\bar{n}, \bar{q}, \bar{N}, \bar{Q} \in \mathbb{Z} \setminus \{0\}$ to be true, and this implies $\bar{n} \neq 0$, $\bar{q} \neq 0$, $\bar{N} \neq 0$ and $\bar{Q} \neq 0$ with the definition of a set difference and with (2.169). Then, we obtain $\bar{n} \cdot \bar{q} \neq 0$ as well as $\bar{N} \cdot \bar{Q} \neq 0$ with the Criterion for zero-divisor freeness with respect to the semiring $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}})$. Due to the connexity of the linear ordering $<_{\mathbb{Z}}$, we therefore have the true disjunctions

$$\bar{n} \cdot \bar{q} > 0 \lor \bar{n} \cdot \bar{q} < 0,$$
$$\bar{N} \cdot \bar{Q} > 0 \lor \bar{N} \cdot \bar{Q} < 0$$

which we use now to prove the implications

$$\bar{n} \cdot \bar{q} > 0 \Rightarrow \bar{m} \cdot \bar{q} \leq \bar{n} \cdot \bar{p} \tag{7.126}$$

$$\bar{n} \cdot \bar{q} < 0 \Rightarrow \bar{m} \cdot \bar{q} \geq \bar{n} \cdot \bar{p} \tag{7.127}$$

by cases and subcases. The first case $\bar{n} \cdot \bar{q} > 0$ immediately implies the truth of the second desired implication (7.127), noting that the antecedent is false due to the Characterization of comparability with respect to the linear ordering $<_{\mathbb{Z}}$. In addition, the first subcase $\bar{N} \cdot \bar{Q} > 0$ implies

$$\bar{M} \cdot \bar{Q} \;\leq\; \bar{N} \cdot \bar{P} \tag{7.128}$$

with (7.122). Then, we can use the Monotony Law for $\cdot$ and $\leq$ for the ordered integral domain of integers to infer from the case assumption $0 < \bar{n} \cdot \bar{q}$ and (7.128) the truth of

$$(\bar{M} \cdot \bar{Q}) \cdot (\bar{n} \cdot \bar{q}) \;\leq\; (\bar{N} \cdot \bar{P}) \cdot (\bar{n} \cdot \bar{q}).$$

Rearranging the terms by means of the Associative & Commutative Law for the multiplication on $\mathbb{Z}$ gives us now

$$(\bar{M} \cdot \bar{n}) \cdot (\bar{Q} \cdot \bar{q}) \;\leq\; (\bar{P} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{n}).$$

Furthermore, we obtain via substitutions based on (7.124) and (7.125)

$$(\bar{m} \cdot \bar{N}) \cdot (\bar{Q} \cdot \bar{q}) \;\leq\; (\bar{p} \cdot \bar{Q}) \cdot (\bar{N} \cdot \bar{n}),$$

and after rearranging the terms (again by means of the Associative & the Commutative Law for the multiplication on $\mathbb{Z}$)

$$(\bar{m} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{Q}) \;\leq\; (\bar{n} \cdot \bar{p}) \cdot (\bar{N} \cdot \bar{Q}).$$

In view of the subcase assumption $0 < \bar{N} \cdot \bar{Q}$, this implies now

$$\bar{m} \cdot \bar{q} \;\leq\; \bar{n} \cdot \bar{p}.$$

Thus, the other desired implication (7.126) is also true (in the first subcase of the first case). The second subcase $\bar{N} \cdot \bar{Q} < 0$ implies now on the one hand evidently $0 < -(\bar{N} \cdot \bar{Q})$ with the Monotony Law for $+$ and $<$, and on the other hand

$$\bar{M} \cdot \bar{Q} \;\geq\; \bar{N} \cdot \bar{P} \tag{7.129}$$

with (7.123), where the current case assumption $0 < \bar{n} \cdot \bar{q}$ allows us to derive

$$(\bar{N} \cdot \bar{P}) \cdot (\bar{n} \cdot \bar{q}) \;\leq\; (\bar{M} \cdot \bar{Q}) \cdot (\bar{n} \cdot \bar{q})$$

with the Monotony Law for $\cdot$ and $\leq$. We can evidently rewrite this inequality in the form

$$(\bar{P} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{n}) \;\leq\; (\bar{M} \cdot \bar{n}) \cdot (\bar{Q} \cdot \bar{q}),$$

and then after substitutions based on (7.124) – (7.125) also as

$$(\bar{p} \cdot \bar{Q}) \cdot (\bar{N} \cdot \bar{n}) \;\leq\; (\bar{m} \cdot \bar{N}) \cdot (\bar{Q} \cdot \bar{q}).$$

After rearranging terms, we find

$$(\bar{n} \cdot \bar{p}) \cdot (\bar{N} \cdot \bar{Q}) \;\leq\; (\bar{m} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{Q}),$$

and this further implies

$$-\left[(\bar{m} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{Q})\right] \;\leq\; -\left[(\bar{n} \cdot \bar{p}) \cdot (\bar{N} \cdot \bar{Q})\right],$$

with (6.234), which we can rewrite as

$$(\bar{m} \cdot \bar{q}) \cdot \left[-(\bar{N} \cdot \bar{Q})\right] \;\leq\; (\bar{n} \cdot \bar{p}) \cdot \left[-(\bar{N} \cdot \bar{Q})\right],$$

by means of the Sign Law (6.63) for the ring of integers. Due to the previously established $0 < -(\bar{N} \cdot \bar{Q})$, the preceding inequality can be simplified to

$$\bar{m} \cdot \bar{q} \;\leq\; \bar{n} \cdot \bar{p}$$

by means of the Monotony Law for $\cdot$ and $\leq$. Consequently, the two desired implications (7.126) and (7.127) hold also for the second case, so that the proof of the first case is now complete.

In the second case $\bar{n} \cdot \bar{q} < 0$, the first desired implication (7.126) clearly has a false antecedent and is therefore true. Moreover, we see that $0 < -(\bar{n} \cdot \bar{q})$ follows to be true with the Monotony Law for $+$ and $<$. Let us consider now again the first subcase $\bar{N} \cdot \bar{Q} > 0$, which implies again the inequality (7.128). An application of the Monotony Law for $\cdot$ and $<$ results then in

$$(\bar{M} \cdot \bar{Q}) \cdot [-(\bar{n} \cdot \bar{q})] \;\leq\; (\bar{N} \cdot \bar{P}) \cdot [-(\bar{n} \cdot \bar{q})],$$

which yields

$$-[(\bar{M} \cdot \bar{Q}) \cdot (\bar{n} \cdot \bar{q})] \;\leq\; -[(\bar{N} \cdot \bar{P}) \cdot (\bar{n} \cdot \bar{q})],$$

with the Sign Law (6.63) and furthermore

$$(\bar{N} \cdot \bar{P}) \cdot (\bar{n} \cdot \bar{q}) \;\leq\; (\bar{M} \cdot \bar{Q}) \cdot (\bar{n} \cdot \bar{q}),$$

with (6.234). Clearly, we can write this inequality also as

$$(\bar{M} \cdot \bar{n}) \cdot (\bar{Q} \cdot \bar{q}) \;\geq\; (\bar{P} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{n}),$$

allowing us to carry out the same substitutions as before to obtain

$$(\bar{m} \cdot \bar{N}) \cdot (\bar{Q} \cdot \bar{q}) \;\geq\; (\bar{p} \cdot \bar{Q}) \cdot (\bar{N} \cdot \bar{n}).$$

Rewriting this inequality in the form of

$$(\bar{m} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{Q}) \;\geq\; (\bar{n} \cdot \bar{p}) \cdot (\bar{N} \cdot \bar{Q}),$$

the Monotony Law for $\cdot$ and $\leq$ becomes applicable, with the consequence that

$$\bar{m} \cdot \bar{q} \;\geq\; \bar{n} \cdot \bar{p}$$

is true. This finding proves the second desired implication (7.127), so that it remains for us to investigate the second subcase $\bar{N} \cdot \bar{Q} < 0$. Here, we have again the inequalities $0 < -(\bar{N} \cdot \bar{Q})$ and (7.129), where the latter evidently implies

$$(\bar{M} \cdot \bar{Q}) \cdot [-(\bar{n} \cdot \bar{q})] \;\geq\; (\bar{N} \cdot \bar{P}) \cdot [-(\bar{n} \cdot \bar{q})],$$

then

$$-[(\bar{M} \cdot \bar{Q}) \cdot (\bar{n} \cdot \bar{q})] \;\geq\; -[(\bar{N} \cdot \bar{P}) \cdot (\bar{n} \cdot \bar{q})],$$

and subsequently

$$-[(\bar{M} \cdot \bar{n}) \cdot (\bar{Q} \cdot \bar{q})] \;\geq\; -[(\bar{P} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{n})].$$

We now find through substitutions

$$-[(\bar{m} \cdot \bar{N}) \cdot (\bar{Q} \cdot \bar{q})] \;\geq\; -[(\bar{p} \cdot \bar{Q}) \cdot (\bar{N} \cdot \bar{n})]$$

and then

$$-[(\bar{m} \cdot \bar{q}) \cdot (\bar{N} \cdot \bar{Q})] \;\geq\; -[(\bar{n} \cdot \bar{p}) \cdot (\bar{N} \cdot \bar{Q})],$$

so that

$$(\bar{m} \cdot \bar{q}) \cdot [-(\bar{N} \cdot \bar{Q})] \;\geq\; (\bar{n} \cdot \bar{p}) \cdot [-(\bar{N} \cdot \bar{Q})],$$

holds. The Monotony Law for $\cdot$ and $\leq$ gives us therefore

$$\bar{m} \cdot \bar{q} \;\geq\; \bar{n} \cdot \bar{p}$$

and thus the true implication (7.127) also for the second subcase. This completes the proof of the implications (7.126) and (7.127) by cases, so that the first part of the equivalence in (7.121) holds in view of the previously established $\bar{n} \neq 0$ and $\bar{q} \neq 0$.

The proof of the second part ('$\Leftarrow$') is similar to the proof of the corresponding implication in (6.265). As $\bar{m}$, $\bar{n}$, $\bar{p}$ and $\bar{q}$ were arbitrary, we may therefore conclude that $\leq_{\mathbb{Q}}$ satisfies indeed (7.121).

Concerning c), we establish first the reflexivity of the binary relation $\leq_{\mathbb{Q}}$, by demonstrating the truth of

$$\forall p \, (p \in \mathbb{Q} \Rightarrow p \leq_{\mathbb{Q}} p). \tag{7.130}$$

To do this, we let p be an arbitrary rational number, so that p is in the quotient set $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]/ \sim_q$. Therefore, there is a particular element $\bar{z}$

in the Cartesian product $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ whose equivalence class $[\bar{z}]_{\sim_q}$ equals p. Here, $\bar{z}$ can be written as the ordered pair $(\bar{m}, \bar{n})$ for some particular integers $\bar{m} \in \mathbb{Z}$ and $\bar{n} \in \mathbb{Z} \setminus \{0\}$, so that the rational number/equivalence class can be written as

$$p = [(\bar{m}, \bar{n})]_{\sim_q}. \tag{7.131}$$

We note that $\bar{n} \in \mathbb{Z} \setminus \{0\}$ implies especially $\bar{n} \neq 0$, which gives $[n^2 =]\ n \cdot n > 0$ with (6.238). Now, the Commutative Law for the multiplication on $\mathbb{Z}$ yields also the true equation $\bar{m} \cdot \bar{n} = \bar{n} \cdot \bar{m}$, and the reflexivity of the standard total ordering $\leq_{\mathbb{Z}}$ gives the inequality $\bar{m} \cdot \bar{n} \leq \bar{m} \cdot \bar{n}$, resulting in $\bar{m} \cdot \bar{n} \leq \bar{n} \cdot \bar{m}$. Therefore, the implication

$$n \cdot n > 0 \Rightarrow \bar{m} \cdot \bar{n} \leq \bar{m} \cdot \bar{n}$$

is true, because it has a true antecedent and a true consequent. Due to the Characterization of comparability of the linear ordering of $\mathbb{Z}$, the implication

$$n \cdot n < 0 \Rightarrow \bar{m} \cdot \bar{n} \geq \bar{m} \cdot \bar{n}$$

has then a false antecedent, so that this implication also holds. We thus obtain the true conjunction

$$\bar{n} \neq 0 \wedge \bar{n} \neq 0 \wedge [n \cdot n > 0 \Rightarrow \bar{m} \cdot \bar{n} \leq \bar{m} \cdot \bar{n}] \wedge [n \cdot n < 0 \Rightarrow \bar{m} \cdot \bar{n} \geq \bar{m} \cdot \bar{n}],$$

which implies

$$[(\bar{m}, \bar{n})]_{\sim_q} \ \leq_{\mathbb{Q}}\ [(\bar{m}, \bar{n})]_{\sim_q}$$

with (7.121). In view of the equation (6.272), the desired inequality $p \leq_{\mathbb{Q}} p$ follows now to be true via substitution, so that the implication in (7.130) is true. Because p was arbitrary, we may therefore conclude that the universal sentence (7.130) holds, which means that $\leq_{\mathbb{Q}}$ is a reflexive binary relation on $\mathbb{Q}$.

We establish next the antisymmetry of $\leq_{\mathbb{Q}}$, by verifying

$$\forall p, q\, (p, q \in \mathbb{Q} \Rightarrow [(p \leq_{\mathbb{Q}} q \wedge q \leq_{\mathbb{Q}} p) \Rightarrow p = q]). \tag{7.132}$$

We let p and q be arbitrary in $\mathbb{Q}$ $[= \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]/ \sim_q]$, which rational numbers thus constitute the equivalence classes $[\bar{e}]_{\sim_q} = p$ and $[\bar{f}]_{\sim_q} = q$ for some particular elements $\bar{e}, \bar{f} \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$. These elements in turn constitute the ordered pairs $(\bar{m}, \bar{n}) = \bar{e}$ and $(\bar{M}, \bar{N}) = \bar{f}$ for some particular elements $\bar{m}, \bar{M} \in \mathbb{Z}$ and $\bar{n}, \bar{N} \in \mathbb{Z} \setminus \{0\}$. We may therefore write the rational numbers in the form $p = [(\bar{m}, \bar{n})]_{\sim_q}$ and $q = [(\bar{M}, \bar{N})]_{\sim_q}$, by carrying out substitutions. Next, we assume the conjunction in (7.132) to be true, and we show that $p = q$ is implied. Due to the preceding equations for p and q, we may write the desired consequent $p = q$ equivalently as

$$[(\bar{m}, \bar{n})]_{\sim_q} = [(\bar{M}, \bar{N})]_{\sim_q}. \tag{7.133}$$

Furthermore, the assumed conjunction $p \leq_{\mathbb{Q}} q \wedge q \leq_{\mathbb{Q}} p$ implies

$$[(\bar{m}, \bar{n})]_{\sim_q} \leq_{\mathbb{Q}} [(\bar{M}, \bar{N})]_{\sim_q} \wedge [(\bar{M}, \bar{N})]_{\sim_q} \leq_{\mathbb{Q}} [(\bar{m}, \bar{n})]_{\sim_q}.$$

In view of the equivalence in (7.121), this conjunction further implies $\bar{n} \neq 0$, $\bar{N} \neq 0$ and the implications

$$\bar{n} \cdot \bar{N} > 0 \Rightarrow \bar{m} \cdot \bar{N} \leq \bar{n} \cdot \bar{M} \tag{7.134}$$
$$\bar{n} \cdot \bar{N} < 0 \Rightarrow \bar{m} \cdot \bar{N} \geq \bar{n} \cdot \bar{M} \tag{7.135}$$
$$\bar{N} \cdot \bar{n} > 0 \Rightarrow \bar{M} \cdot \bar{n} \leq \bar{N} \cdot \bar{m} \tag{7.136}$$
$$\bar{N} \cdot \bar{n} < 0 \Rightarrow \bar{M} \cdot \bar{n} \geq \bar{N} \cdot \bar{m}. \tag{7.137}$$

Here, $\bar{n} \neq 0$ and $\bar{N} \neq 0$ imply $\bar{n} \cdot \bar{N} \neq 0$ with the Criterion for zero-divisor freeness, so that the connexity of the linear ordering of $\mathbb{Z}$ gives us the true disjunction $\bar{n} \cdot \bar{N} > 0 \vee \bar{n} \cdot \bar{N} < 0$. We use this disjunction to prove

$$\bar{m} \cdot \bar{N} = \bar{M} \cdot \bar{n}, \tag{7.138}$$

by cases. In case that $\bar{n} \cdot \bar{N} > 0$ is true, we obtain also $\bar{N} \cdot \bar{n} > 0$ with the commutativity of the multiplication on $\mathbb{Z}$, and these inequalities imply then, respectively, $\bar{m} \cdot \bar{N} \leq \bar{n} \cdot \bar{M}$ with (7.134) and $\bar{M} \cdot \bar{n} \leq \bar{N} \cdot \bar{m}$ with (7.136). Evidently, these can be written also as

$$\begin{aligned} \bar{m} \cdot \bar{N} &\leq \bar{M} \cdot \bar{n} \\ \bar{M} \cdot \bar{n} &\leq \bar{m} \cdot \bar{N}, \end{aligned}$$

so that (7.138) follows to be true with the antisymmetry of $\leq_{\mathbb{Z}}$. Similarly, the second case $\bar{n} \cdot \bar{N} < 0$ gives $\bar{N} \cdot \bar{n} < 0$, so that the inequalities $\bar{m} \cdot \bar{N} \geq \bar{n} \cdot \bar{M}$ and $\bar{M} \cdot \bar{n} \geq \bar{N} \cdot \bar{m}$ follow to be true with (7.135) and (7.137), respectively. Rearranging some of the factors, we obtain

$$\begin{aligned} \bar{m} \cdot \bar{N} &\geq \bar{M} \cdot \bar{n} \\ \bar{M} \cdot \bar{n} &\geq \bar{m} \cdot \bar{N}, \end{aligned}$$

which are evidently the same two inequalities as in the first case. Thus, the antisymmetry of $\leq_{\mathbb{Z}}$ yields (7.138) also for the second case. This equation demonstrates the equivalence $(\bar{m}, \bar{n}) \sim_q (\bar{M}, \bar{N})$ in view of (7.3). Therefore, the Equality Criterion for equivalence classes gives us (7.133) and then also the desired equation $p = q$ via substitutions. Because p and q were arbitrary, we can infer from this finding the truth of (7.132), which means that $\leq_{\mathbb{Q}}$ is indeed antisymmetric.

To establish the transitivity of $\leq_{\mathbb{Q}}$, we prove

$$\forall p, q, r\,(p, q, r \in \mathbb{Q} \Rightarrow [(p \leq_{\mathbb{Q}} q \wedge q \leq_{\mathbb{Q}} r) \Rightarrow p \leq_{\mathbb{Q}} r]), \tag{7.139}$$

letting p, q and r be arbitrary rational numbers such that $p \leq_{\mathbb{Q}} q$ and $q \leq_{\mathbb{Q}} r$ are both satisfied. Clearly, we can write these rational numbers as the equivalence classes $p = [(a,b)]_{\sim_q}$, $q = [(c,d)]_{\sim_q}$ and $r = [(e,f)]_{\sim_q}$ with $a, b, c, d, e, f \in \mathbb{Z}$ and with $b \neq 0$, $d \neq 0$, $f \neq 0$. The desired consequent takes then the form

$$[(a,b)]_{\sim_q} \;\leq_{\mathbb{Q}}\; [(e,f)]_{\sim_q}, \tag{7.140}$$

and the antecedent reads

$$[(a,b)]_{\sim_q} \leq_{\mathbb{Q}} [(c,d)]_{\sim_q} \,\wedge\, [(c,d)]_{\sim_q} \leq_{\mathbb{Q}} [(e,f)]_{\sim_q}. \tag{7.141}$$

Due to (7.121), the consequent is equivalent to the conjunction of $b \neq 0$, $f \neq 0$ and the implications

$$b \cdot f > 0 \;\Rightarrow\; a \cdot f \leq b \cdot e \tag{7.142}$$

$$b \cdot f < 0 \;\Rightarrow\; a \cdot f \geq b \cdot e, \tag{7.143}$$

whereas the antecedent gives us $b \neq 0$, $d \neq 0$, $f \neq 0$ and the implications

$$b \cdot d > 0 \;\Rightarrow\; a \cdot d \leq b \cdot c \tag{7.144}$$

$$b \cdot d < 0 \;\Rightarrow\; a \cdot d \geq b \cdot c \tag{7.145}$$

$$d \cdot f > 0 \;\Rightarrow\; c \cdot f \leq d \cdot e \tag{7.146}$$

$$d \cdot f < 0 \;\Rightarrow\; c \cdot f \geq d \cdot e. \tag{7.147}$$

Here, we observe that $b \neq 0$, $d \neq 0$ and $f \neq 0$ imply $b{\cdot}d \neq 0$ and $d{\cdot}f \neq 0$ with the Criterion for zero-divisor freeness, so that in particular the disjunction

$$b \cdot d > 0 \,\vee\, b \cdot d < 0$$

turns out to be true because of the connexity of $<_{\mathbb{Z}}$. We consider now corresponding cases to prove the implications (7.142) and (7.143) directly. Assuming first the antecedent $b \cdot f > 0$ to be true, we obtain in the first case $b \cdot d > 0$ the inequality $d \cdot f > 0$ with (6.252), and therefore the inequalities

$$a \cdot d \;\leq\; b \cdot c$$

$$c \cdot f \;\leq\; d \cdot e$$

with (7.144) and (7.146), respectively. Noting that the the assumed antecedent reads $0 < b \cdot f$ and observing that $b \neq 0$ gives $0 < b \cdot b$ with (6.238), we apply the Monotony Law for $\cdot$ and $\leq$ to obtain

$$\begin{aligned}(a \cdot d) \cdot (b \cdot f) &\leq (b \cdot c) \cdot (b \cdot f) \\ (c \cdot f) \cdot (b \cdot b) &\leq (d \cdot e) \cdot (b \cdot b).\end{aligned}$$

Rearranging the terms by means of the Associative Law and the Commutative Law for the multiplication on $\mathbb{Z}$, we get

$$\begin{aligned}(a \cdot f) \cdot (b \cdot d) &\leq (b \cdot c) \cdot (b \cdot f) \\ (b \cdot c) \cdot (b \cdot f) &\leq (b \cdot e) \cdot (b \cdot d).\end{aligned}$$

Since $\leq_{\mathbb{Z}}$ is transitive, we can infer from these two inequalities the truth of

$$(a \cdot f) \cdot (b \cdot d) \leq (b \cdot e) \cdot (b \cdot d),$$

where the current case assumption $0 < b \cdot d$ allows to apply again the Monotony Law for $\cdot$ and $\leq$ in order to obtain the desired inequality $a \cdot f \leq b \cdot e$. Thus the implication (7.142) holds in the first case.

In the second case $b \cdot d < 0$, we now find in connection with the assumed antecedent $b \cdot f > 0$ the inequality $d \cdot f < 0$ with (6.254), and consequently

$$\begin{aligned}a \cdot d &\geq b \cdot c \\ c \cdot f &\geq d \cdot e\end{aligned}$$

by virtue of (7.145) and (7.147). We can evidently use exactly the same arguments as in the first case to derive

$$(a \cdot f) \cdot (b \cdot d) \geq (b \cdot e) \cdot (b \cdot d),$$

which we can write also as

$$(b \cdot e) \cdot (b \cdot d) \leq (a \cdot f) \cdot (b \cdot d).$$

This implies now

$$-[(a \cdot f) \cdot (b \cdot d)] \leq -[(b \cdot e) \cdot (b \cdot d)].$$

with (6.234), and in addition

$$(a \cdot f) \cdot [-(b \cdot d)] \leq (b \cdot e) \cdot [-(b \cdot d)].$$

with the Sign Law (6.63). Since the current case assumption $b \cdot d < 0$ clearly implies $0 < -(b \cdot d)$, we are now in a position to apply the Monotony Law for $\cdot$ and $\leq$ to the preceding inequality, which gives $a \cdot f \leq b \cdot e$, as desired. Thus, the implication (7.142) holds in any case.

Let us now prove the other implication (7.143) directly and by considering as before the two cases $b \cdot d > 0$ and $b \cdot d < 0$. Assuming the antecedent $b \cdot f < 0$ to be true, this yields in connection with the first case $b \cdot d > 0$ the inequality $d \cdot f < 0$ because of (6.255), which in turn imply

$$\begin{aligned} a \cdot d &\leq b \cdot c \\ c \cdot f &\geq d \cdot e \end{aligned}$$

with (7.144) and (7.147). Clearly, the assumed antecedent $b \cdot f < 0$ can also be written as $0 < -(b \cdot f)$, so that the Monotony Law for $\cdot$ and $\leq$ yields

$$\begin{aligned} (a \cdot d) \cdot [-(b \cdot f)] &\leq (b \cdot c) \cdot [-(b \cdot f)] \\ (c \cdot f) \cdot (b \cdot b) &\geq (d \cdot e) \cdot (b \cdot b). \end{aligned}$$

These inequalities give us

$$\begin{aligned} (a \cdot f) \cdot (b \cdot d) &\geq (b \cdot c) \cdot (b \cdot f) \\ (b \cdot c) \cdot (b \cdot f) &\geq (b \cdot e) \cdot (b \cdot d) \end{aligned}$$

through applications of the Sign Law (6.63), (6.234), the Associative Law and the Commutative Law for the multiplication, and therefore

$$(a \cdot f) \cdot (b \cdot d) \geq (b \cdot e) \cdot (b \cdot d)$$

with the transitivity of $\leq_{\mathbb{Z}}$. Due to $b \cdot d > 0$, we can evidently simplify this inequality to $a \cdot f \geq b \cdot e$, proving the implication (7.143) for the first case. The second case $b \cdot d < 0$ gives in conjunction with the assumption $b \cdot f < 0$ the inequality $d \cdot f > 0$ with (6.253), so that we now find

$$\begin{aligned} a \cdot d &\geq b \cdot c \\ c \cdot f &\leq d \cdot e. \end{aligned}$$

This yields in analogy to the preceding first case

$$\begin{aligned} (a \cdot d) \cdot [-(b \cdot f)] &\geq (b \cdot c) \cdot [-(b \cdot f)] \\ (c \cdot f) \cdot (b \cdot b) &\leq (d \cdot e) \cdot (b \cdot b), \end{aligned}$$

for which the Sign Law (6.63) in connection with (6.234) gives

$$\begin{aligned} -[(a \cdot d) \cdot (b \cdot f)] &\geq -[(b \cdot c) \cdot (b \cdot f)] \\ -[(c \cdot f) \cdot (b \cdot b)] &\geq -[(d \cdot e) \cdot (b \cdot b)]. \end{aligned}$$

Now, rearranging terms by means of the Associative Law and the Commutative Law for the multiplication and applying then again the Sign Law (6.63), we obtain

$$(a \cdot f) \cdot [-(b \cdot d)] \geq (b \cdot c) \cdot [-(b \cdot f)]$$
$$(b \cdot c) \cdot [-(b \cdot f)] \geq (b \cdot e) \cdot [-(b \cdot d)].$$

The transitivity of $\leq_{\mathbb{Z}}$ yields therefore

$$(a \cdot f) \cdot [-(b \cdot d)] \geq (b \cdot e) \cdot [-(b \cdot d)],$$

which we can simplify to $a \cdot f \geq b \cdot e$ with the Monotony Law for $\cdot$ and $\leq$, noting that $0 < -(b \cdot d)$ holds in view of the current case assumption $b \cdot d < 0$. This proves the implication (7.143) also for the second case, so that the proofs of the two implication (7.142) – (7.143) by cases are now complete. Since the assumed antecedent gave us also $b \neq 0$ and $f \neq 0$, we can infer from these findings the truth of the equivalent inequality (7.140). Substitutions give us then $p \leq_{\mathbb{Q}} r$, and since p, q, r were initially arbitrary, (7.139) follows therefore to be true.

We thus established $\leq_{\mathbb{Q}}$ as a reflexive, antisymmetric and transitive binary relation on $\mathbb{Q}$, so that $\leq_{\mathbb{Q}}$ constitutes a reflexive partial ordering of $\mathbb{Q}$.

Our final task is to establish the totality of $\leq_{\mathbb{Q}}$, that is,

$$\forall p, q \, (p, q \in \mathbb{Q} \Rightarrow [p \leq_{\mathbb{Q}} q \,\vee\, q \leq_{\mathbb{Q}} p]). \tag{7.148}$$

For this purpose, we take arbitrary rational numbers p and q, which can then evidently be written as $p = [(\bar{m}, \bar{n})]_{\sim_q}$ and $q = [(\bar{M}, \bar{N})]_{\sim_q}$ for some particular integers $\bar{m}, \bar{n}, \bar{M}, \bar{N}$ with $\bar{n} \neq 0$ and $\bar{N} \neq 0$. Thus, the desired disjunction in (7.132) reads

$$[(\bar{m}, \bar{n})]_{\sim_q} \leq_{\mathbb{Q}} [(\bar{M}, \bar{N})]_{\sim_q} \,\vee\, [(\bar{M}, \bar{N})]_{\sim_q} \leq_{\mathbb{Q}} [(\bar{m}, \bar{n})]_{\sim_q}. \tag{7.149}$$

Let us observe now that the totality of $\leq_{\mathbb{Z}}$ gives rise to the truth of the disjunction

$$\bar{m} \cdot \bar{N} \leq \bar{n} \cdot \bar{M} \,\vee\, \bar{n} \cdot \bar{M} \leq \bar{m} \cdot \bar{N},$$

which we use in the following to prove (7.149) by cases. Furthermore, $\bar{n} \neq 0$ and $\bar{N} \neq 0$ imply with the Criterion for zero-divisor freeness $\bar{n} \cdot \bar{N} \neq 0$ and therefore the disjunction

$$\bar{n} \cdot \bar{N} > 0 \,\vee\, \bar{n} \cdot \bar{N} < 0$$

with the Characterization of comparability, which we use for a proof by (sub-)cases within each of the two cases.

Then, the first case $\bar{m} \cdot \bar{N} \leq \bar{n} \cdot \bar{M}$ in connection with the first sub-case $\bar{n} \cdot \bar{N} > 0$ shows in light of the truth of $\bar{n} \neq 0$ and $\bar{N} \neq 0$ and in light of the falsity of $\bar{n} \cdot \bar{N} < 0$ (using again the comparability of the linear ordering $<_{\mathbb{Z}}$) that the conjunctions

$$\bar{n} \neq 0 \wedge \bar{N} \neq 0 \wedge [\bar{n} \cdot \bar{N} > 0 \Rightarrow \bar{m} \cdot \bar{N} \leq \bar{n} \cdot \bar{M}] \wedge [\bar{n} \cdot \bar{N} < 0 \Rightarrow \bar{m} \cdot \bar{N} \geq \bar{n} \cdot \bar{M}] \tag{7.150}$$

are true, noting that the antecedent and the consequent of the first implication are both true, and that the antecedent of the second implication is false. This multiple conjunction implies $[(\bar{m}, \bar{n})]_{\sim_q} \leq_{\mathbb{Q}} [(\bar{M}, \bar{N})]_{\sim_q}$ with (7.121), and the desired disjunction (7.149) is then also true.

In the second sub-case $\bar{n} \cdot \bar{N} < 0$, the commutativity of the multiplication on $\mathbb{Z}$ gives us $\bar{N} \cdot \bar{n} < 0$ and in addition for the current case assumption $\bar{M} \cdot \bar{n} \geq \bar{N} \cdot \bar{m}$, so that we now have the true conjunctions

$$\bar{N} \neq 0 \wedge \bar{n} \neq 0 \wedge [\bar{N} \cdot \bar{n} > 0 \Rightarrow \bar{M} \cdot \bar{n} \leq \bar{N} \cdot \bar{m}] \wedge [\bar{N} \cdot \bar{n} < 0 \Rightarrow \bar{M} \cdot \bar{n} \geq \bar{N} \cdot \bar{m}], \tag{7.151}$$

because the first implication has a false antecedent, whereas the antecedent and the consequent of the second implication are both true. This in turn implies $[(\bar{M}, \bar{N})]_{\sim_q} \leq_{\mathbb{Q}} [(\bar{m}, \bar{n})]_{\sim_q}$ with (7.121), and therefore the disjunction (7.149) is true also for the second sub-case. We thus completed the verification of the first case.

Similarly, the assumptions $\bar{n} \cdot \bar{M} \leq \bar{m} \cdot \bar{N}$ and $\bar{n} \cdot \bar{N} > 0$ for the second case and the first sub-case can be written as $\bar{M} \cdot \bar{n} \leq \bar{N} \cdot \bar{m}$ and $\bar{N} \cdot \bar{n} > 0$ (by virtue of the commutativity of the multiplication on $\mathbb{Z}$), with the consequence that (7.151) holds again, observing the truth of the antecedent and the consequent of the first implication as well as the falsity of the antecedent of the second implication. Thus, the disjunction (7.149) follows to be also true, as explained in the proof of the first case.

Finally, we find in the second sub-case $\bar{n} \cdot \bar{N} < 0$ the multiple conjunction (7.150) to be true again, writing the current case assumption in the form $\bar{m} \cdot \bar{N} \geq \bar{n} \cdot \bar{M}$. Evidently, this finding gives us then the disjunction (7.149), which is thus true in any case.

Applying now substitutions based on the equations for p and q, we arrive at the true disjunction $p \leq_{\mathbb{Q}} q \vee q \leq_{\mathbb{Q}} p$, so that the proof of the implication (7.148) is complete. As p and q were initially arbitrary, we may therefore infer from this the truth of the universal sentence (7.148), and thus the totality of the reflexive partial ordering $\leq_{\mathbb{Q}}$. □

Note 7.17. The total ordering $\leq_{\mathbb{Q}}$ of $\mathbb{Q}$ induces then the linear ordering $<_{\mathbb{Q}}$ of $\mathbb{Q}$.

Definition 7.11 (Standard total & linear ordering of $\mathbb{Q}$). We call the total ordering

$$\leq_{\mathbb{Q}} \tag{7.152}$$

of the set of rational number $\mathbb{Q}$ the *standard total ordering of* $\mathbb{Q}$, and we call the induced linear ordering

$$<_{\mathbb{Q}} \tag{7.153}$$

of $\mathbb{Q}$ the *standard linear ordering of* $\mathbb{Q}$.

Theorem 7.36 (Characterization of the linear ordering of $\mathbb{Q}$). *The standard linear ordering of $\mathbb{Q}$ is characterized by*

$$\begin{aligned}\forall m, n, p, q\, ([(m,n)]_{\sim_q} <_{\mathbb{Q}} [(p,q)]_{\sim_q} \Leftrightarrow (n \neq 0 \,\wedge\, q \neq 0 \\ \wedge\, [n \cdot q > 0 \Rightarrow m \cdot q < n \cdot p] \,\wedge\, [n \cdot q < 0 \Rightarrow m \cdot q > n \cdot p])).\end{aligned} \tag{7.154}$$

Proof. We take arbitrary sets m, n, p, q and assume first

$$[(m,n)]_{\sim_q} <_{\mathbb{Q}} [(p,q)]_{\sim_q} \tag{7.155}$$

to be true. By definition of an induced irreflexive partial ordering, this inequality implies the conjunction

$$[(m,n)]_{\sim_q} \leq_{\mathbb{Q}} [(p,q)]_{\sim_q} \,\wedge\, [(m,n)]_{\sim_q} \neq [(p,q)]_{\sim_q}. \tag{7.156}$$

The first part of this conjunction implies with (7.121)

$$n \neq 0 \wedge q \neq 0 \wedge [n \cdot q > 0 \Rightarrow m \cdot q \leq n \cdot p] \wedge [n \cdot q < 0 \Rightarrow m \cdot q \geq n \cdot p], \tag{7.157}$$

whereas the second part implies with the Equality Criterion for equivalence classes $\neg(m,n) \sim_q (p,q)$. This negation in turn implies

$$\neg(n \neq 0 \,\wedge\, q \neq 0 \,\wedge\, m \cdot q = p \cdot n) \tag{7.158}$$

with (7.3), so that we obtain by means of De Morgan's Law for the conjunction

$$\neg(n \neq 0 \,\wedge\, q \neq 0) \,\vee\, m \cdot q \neq p \cdot n. \tag{7.159}$$

In view of (7.157), the conjunction $n \neq 0 \,\wedge\, q \neq 0$ is true, and therefore its negation is false. This means that the first part of the preceding disjunction is false, so that its second part $m \cdot q \neq p \cdot n$ must be true. We are now in a position to prove

$$n \neq 0 \wedge q \neq 0 \wedge [n \cdot q > 0 \Rightarrow m \cdot q < n \cdot p] \wedge [n \cdot q < 0 \Rightarrow m \cdot q > n \cdot p]. \tag{7.160}$$

Here, we already know that $n \neq 0$ and $q \neq 0$ are true. To establish the first implication, we assume $n \cdot q > 0$, so that $m \cdot q \leq n \cdot p$ follows to be true with (7.157). In conjunction with the previously established inequality $m \cdot q \neq p \cdot n$, this gives us now $m \cdot q < n \cdot p$ with the definition of an induced irreflexive partial ordering (applied now to the total/linear ordering of $\mathbb{Z}$), proving the first implication. Regarding the other implication, we assume conversely that $n \cdot q < 0$, which implies $m \cdot q \geq n \cdot p$ with (7.157). Since $m \cdot q \neq p \cdot n$, we therefore obtain $m \cdot q > n \cdot p$ (again by definition of an induced irreflexive partial ordering). Having thus proved the conjunctions (7.160), the first part ('$\Rightarrow$') of the equivalence in (7.154) is true.

To establish the second part ('$\Leftarrow$'), we assume now (7.160) to be true. Let us verify first the conjunctions (7.157). Because of the preceding assumption, the first two parts $n \neq 0$ and $q \neq 0$ are automatically true. Regarding the third part, we assume the antecedent $n \cdot q > 0$ to hold, so that the first implication in (7.160) yields $m \cdot q < n \cdot p$. Then, the disjunction

$$m \cdot q < n \cdot p \lor m \cdot q = n \cdot p$$

is also true, and this gives the desired consequent $m \cdot q < n \cdot p$ with the definition of an induced irreflexive partial ordering. Regarding the fourth part of the multiple conjunction (7.157), we assume now $n \cdot q < 0$, which assumption implies then $m \cdot q > n \cdot p$ by virtue of the second implication in (7.160). Therefore, the disjunction

$$m \cdot q > n \cdot p \lor m \cdot q = n \cdot p$$

is true as well, and this clearly shows that $m \cdot q \geq n \cdot p$ holds, completing the proof of (7.157). These conjunctions in turn imply the inequality $[(m, n)]_{\sim_q} \leq_{\mathbb{Q}} [(p, q)]_{\sim_q}$, that is, the first part of the conjunction (7.156). To establish its second part, we verify (7.159). For this purpose, we observe first that $n \neq 0$ and $q \neq 0$ imply $n \cdot q \neq 0$ with the Criterion for zero-divisor freeness, so that the disjunction $n \cdot q > 0 \lor n \cdot q < 0$ is true because of the connexity of the linear ordering $<_{\mathbb{Q}}$. We can use this disjunction to prove $m \cdot q \neq p \cdot n$. Indeed, the first case $n \cdot q > 0$ implies $m \cdot q < n \cdot p$ and the second case $n \cdot q < 0$ gives $m \cdot q > n \cdot p$ with (7.160), so that $m \cdot q \neq p \cdot n$ holds in any case due to the fact that the linear ordering $<_{\mathbb{Q}}$ satisfies the Characterization of comparability. Then, the disjunction (7.159) is true as well, and this disjunction further implies (7.158) with De Morgan's Law for the conjunction. Consequently, the negation $\neg(m, n) \sim_q (p, q)$ turns out to be true by virtue of (7.3), and this results in $[(m, n)]_{\sim_q} \neq [(p, q)]_{\sim_q}$ because of the the Equality Criterion for equivalence classes. We thus completed the proof of the conjunction (7.156), which yields then also (7.155), once again by means of the definition of an induced irreflexive partial ordering.

This finding proves the second part of the equivalence in (7.154), in which m, n, p and q were arbitrary, so that we my finally conclude that the stated universal is true. □

Definition 7.12 (Negative rational number, nonnegative rational number, positive rational number). We say that a reational number q is

(1) *negative* iff
$$q <_{\mathbb{Q}} [(0,1)]_{\sim_q}. \tag{7.161}$$

(2) *nonnegative* iff
$$[(0,1)]_{\sim_q} \leq_{\mathbb{Q}} q. \tag{7.162}$$

(3) *positive* iff
$$[(0,1)]_{\sim_q} <_{\mathbb{Q}} q. \tag{7.163}$$

Note 7.18. We see in light of the Axiom of Specification and the Equality Criterion for sets that there exist unique sets $\mathbb{Q}_-$, $\mathbb{Q}_+^0$ and $\mathbb{Q}_+$ consisting, respectively, of all rational numbers that are negative, nonnegative and positive, that is,

$$\forall q\,(q \in \mathbb{Q}_- \Leftrightarrow [q \in \mathbb{Q} \wedge q <_{\mathbb{Q}} [(0,1)]_{\sim_q}]), \tag{7.164}$$
$$\forall q\,(q \in \mathbb{Q}_+^0 \Leftrightarrow [q \in \mathbb{Q} \wedge [(0,1)]_{\sim_q} \leq_{\mathbb{Q}} q]), \tag{7.165}$$
$$\forall q\,(q \in \mathbb{Q}_+ \Leftrightarrow [q \in \mathbb{Q} \wedge [(0,1)]_{\sim_q} <_{\mathbb{Q}} q]). \tag{7.166}$$

Since $q \in \mathbb{Q}_-$, $q \in \mathbb{Q}_+^0$ and $q \in \mathbb{Q}_+$ all imply $q \in \mathbb{Q}$ for any q, the inclusions

$$\mathbb{Q}_- \subseteq \mathbb{Q}, \tag{7.167}$$
$$\mathbb{Q}_+^0 \subseteq \mathbb{Q}, \tag{7.168}$$
$$\mathbb{Q}_+ \subseteq \mathbb{Q} \tag{7.169}$$

are true by definition of a subset.

Definition 7.13 (Set of negative & of nonnegative & of positive rational numbers). We call
$$\mathbb{Q}_- \tag{7.170}$$
the *set of negative rational numbers*,
$$\mathbb{Q}_+^0 \tag{7.171}$$
the *set of nonnegative rational numbers*, and
$$\mathbb{Q}_+ \tag{7.172}$$
the *set of positive rational numbers*.

Exercise 7.29. Establish the following equivalences for any m and any n.

$$[(0,1)]_{\sim_q} \leq_{\mathbb{Q}} [(m,n)]_{\sim_q} \Leftrightarrow (n \neq 0 \wedge [n > 0 \Rightarrow m \geq 0] \wedge [n < 0 \Rightarrow m \leq 0]) \tag{7.173}$$

$$[(0,1)]_{\sim_q} <_{\mathbb{Q}} [(m,n)]_{\sim_q} \Leftrightarrow (n \neq 0 \wedge [n > 0 \Rightarrow m > 0] \wedge [n < 0 \Rightarrow m < 0]) \tag{7.174}$$

(Hint: Use (7.121) and (7.154) in connection with the Distinctness of 1_X and 0_X, the definition of the unity element and the Cancellation Law for 0.)

Theorem 7.37 (Characterization of positive rational numbers). *It is true that a rational number* $[(m,n)]_{\sim_q}$ *is positive iff the product of* m *and* n *is positive, that is,*

$$\forall m, n\,([(0,1)]_{\sim_q} <_{\mathbb{Q}} [(m,n)]_{\sim_q} \;\Leftrightarrow\; m \cdot n > 0). \tag{7.175}$$

Proof. We take two arbitrary sets m and n, and we assume first that

$$[(0,1)]_{\sim_q} <_{\mathbb{Q}} [(m,n)]_{\sim_q} \tag{7.176}$$

holds. Consequently, we obtain because of (7.174) the true conjunctions

$$n \neq 0 \wedge [n > 0 \Rightarrow m > 0] \wedge [n < 0 \Rightarrow m < 0], \tag{7.177}$$

where $n \neq 0$ implies the disjunction $n > 0 \vee n < 0$ because the linear ordering $<_{\mathbb{Z}}$ is connex. Based on this disjunction, we prove now

$$[m > 0 \wedge n > 0] \vee [m < 0 \wedge n < 0] \tag{7.178}$$

by cases. On the one hand, the first case $n > 0$ yields $m > 0$ with the first implication in (7.177), so that the conjunction $m > 0 \wedge n > 0$ is true; the disjunction (7.178) is then true as well. On the other hand, the second case $n < 0$ gives $m < 0$ with the second implication in (7.177), resulting in the true conjunction $m < 0 \wedge n < 0$ and thus again in the true disjunction (7.178). Having completed the proof by cases, we can infer now from the truth of the disjunction (7.178) the truth of $m \cdot n > 0$ because of (6.235) – applied to the ordered integral domain of integers. This finding proves the first part ('$\Rightarrow$') of the equivalence in (7.175).

To prove the second part ('$\Leftarrow$'), we assume now $m \cdot n > 0$ to be true, so that (7.178) follows to be true with (6.235). We use this true disjunction to prove (7.177) by cases. The first case $m > 0 \wedge n > 0$ implies on the one hand $n \neq 0$ with the fact that the linear ordering $<_{\mathbb{Z}}$ satisfies the Characterization of comparability (so that $n > 0$ implies the falseness of

$n = 0$). On the other hand, we see that the antecedent and the consequent of the implication $n > 0 \Rightarrow m > 0$ are both true, and that the antecedent of the implication $n < 0 \Rightarrow m < 0$ is false, which means that both implications are true. We thus proved that (7.177) holds for the first case.

Similarly, the second case $m < 0 \land n < 0$ implies on the one hand $n \neq 0$ (again with the comparability of $<_{\mathbb{Z}}$). On the other hand, the implication $n > 0 \Rightarrow m > 0$ evidently has a false antecedent, whereas the antecedent and the consequent of the other implication $n < 0 \Rightarrow m < 0$ are now both true. Thus, both implications are again true, and we see in light of these findings that (7.177) holds then also in the current second case. The truth of this multiple conjunction implies now the truth of the inequality (7.176) by virtue of (7.174). Thus, the second part of the equivalence in (7.175) holds, too, so that the proof of that equivalence is complete.

Here, m and n are arbitrary, so that the stated theorem is true indeed. □

Theorem 7.38 (Generation of positive rational numbers with positive numerators and denominators). *Every positive rational number can be written as a fraction with positive numerator and denominator, i.e.*

$$\forall p \, (p \in \mathbb{Q}_+ \Rightarrow \exists m, n \, (p = [(m, n)]_{\sim_q} \land 0 < m \land 0 < n). \tag{7.179}$$

Proof. Letting p be arbitrary and assuming $p \in \mathbb{Q}_+$ to be true, we find $p \in \mathbb{Q}$ and $[(0,1)]_{\sim_q} <_{\mathbb{Q}} p$ with the specification of the set of positive rational numbers in (7.166). Clearly, we may express the rational number p as the equivalence class $[(\bar{m}, \bar{n})]_{\sim_q}$ where $\bar{m}$ and $\bar{n}$ are some particular integers with $\bar{n} \neq 0$. We may therefore write for the preceding inequality also $[(0,1)]_{\sim_q} <_{\mathbb{Q}} [(\bar{m}, \bar{n})]_{\sim_q}$. This inequality implies

$$0 < \bar{m} \cdot \bar{n} \tag{7.180}$$

with the Characterization of positive rational numbers. Note that the negations $\bar{m} \cdot \bar{n} \neq 0$ and $\neg \bar{m} \cdot \bar{n} < 0$ is then true according to the Characterization of comparability (with respect to the linear ordering of $\mathbb{Z}$). Let us now prove $\neg \bar{m} = 0$ by contradiction, assuming the negation of that negation to be true. Therefore, $\bar{m} = 0$ holds by the Double Negation Law, with the consequence that $\bar{m} \cdot \bar{n} = 0 \cdot \bar{n} = 0$ (applying substitution and then the Cancellation for 0_X in rings). The resulting equation $\bar{m} \cdot \bar{n} = 0$ contradicts the previously established true negation $\bar{m} \cdot \bar{n} \neq 0$, so that the proof of $\neg \bar{m} = 0$ is now complete. As the linear ordering of $\mathbb{Z}$ is connex, we may infer from the truth of that negation the truth of the disjunction $0 < \bar{m} \lor \bar{m} < 0$, which we intend to use to prove the desired existential sentence in (7.179) by cases. As a preparation for this, we observe the previously found negation $\bar{n} \neq 0$ implies the truth of the disjunction

$$0 < \bar{n} \lor \bar{n} < 0 \tag{7.181}$$

(exploiting again the connexity of $<_{\mathbb{Z}}$).

Now, in the first case $0 < \bar{m}$, we may prove the negation $\neg \bar{n} < 0$ by contradiction. Indeed, $\bar{n} < 0$ yields due to the current case assumption $0 < \bar{m}$ the inequality $\bar{n} \cdot \bar{m} < 0 \cdot \bar{m}$ by virtue of the Monotony Law for $\cdot$ and $<$, evidently resulting in $\bar{m} \cdot \bar{n} < 0$ due to the commutativity of the multiplication on $\mathbb{Z}$. Since we previously found the negation $\neg \bar{m} \cdot \bar{n} < 0$ to be true as well, we arrived at a contradiction, so that $\neg \bar{n} < 0$ is indeed true. This means that the second part of the previous disjunction with respect to $\bar{n}$ is false, so that the first part $0 < \bar{n}$ of that disjunction is true. In conjunction with the case assumption $0 < \bar{m}$ and the previously found expression $p = [(\bar{m}, \bar{n})]_{\sim_q}$, we may conclude that p satisfies the existential sentence in (7.179), as desired.

The second case $\bar{m} < 0$ evidently implies $0 < -\bar{m}$ with the Monotony Law for $+$ and $<$. Let us therefore take the integer $\bar{M} = -\bar{m}$, which thus satisfies $0 < \bar{M}$. We now establish the negation $\neg 0 < \bar{n}$ by contradiction. If $0 < \bar{n}$ were true, then the current case assumption $\bar{m} < 0$ would imply $\bar{m} \cdot \bar{n} < 0 \cdot \bar{n}$ (using the Monotony Law for $\cdot$ and $<$) and therefore $\bar{m} \cdot \bar{n} < 0$, in contradiction to the previously established true negation $\neg \bar{m} \cdot \bar{n} < 0$. We thus proved $\neg 0 < \bar{n}$, so that the second part $\bar{n} < 0$ of the disjunction (7.181) is now true. The preceding inequality evidently implies $0 < -\bar{n}$, which suggests that we take the integer $\bar{N} = -\bar{n}$, as it satisfies $0 < \bar{N}$. Note that this inequality implies $\bar{N} \neq 0$, so that we may form the integer $\frac{\bar{M}}{\bar{N}} = [(\bar{M}, \bar{N})]_{\sim_q}$. In addition, we may derive the equations

$$\bar{m} \cdot \bar{N} = \bar{m} \cdot (-\bar{n}) = -(\bar{m} \cdot \bar{n}) = (-\bar{m}) \cdot \bar{n} = \bar{M} \cdot \bar{n}$$

by applying substitutions and the Sign Laws (6.63) – (6.64). The resulting equation $\bar{m} \cdot \bar{N} = \bar{M} \cdot \bar{n}$ means that $(\bar{m}, \bar{n}) \sim_q (\bar{M}, \bar{N})$, according to (7.3). By the Equality Criterion for equivalence classes, we therefore obtain $[(\bar{m}, \bar{n})]_{\sim_q} = [(\bar{M}, \bar{N})]_{\sim_q}$, so that $p = [(\bar{M}, \bar{N})]_{\sim_q}$ follows to be true via substitution. In view of $0 < \bar{M}$ and $0 < \bar{N}$, we thus see that the existential sentence in (7.179) holds also in the current second case. Since p was arbitrary, we may therefore conclude that the universal sentence (7.179) is true. □

Note 7.19. In view of the inclusion $\mathbb{Z}_{\mathbb{Q}} \subseteq \mathbb{Q}$ established in Exercise 7.7, the total ordering $\leq_{\mathbb{Q}}$ of the set of rational numbers induces also the total ordering $\leq_{\mathbb{Z}_{\mathbb{Q}}}$ of the set of integers in $\mathbb{Q}$ (according to the Total ordering of subsets), which satisfies (according to the Reflexive partial ordering of subsets)

$$\forall m, n \, (m, n \in \mathbb{Z}_{\mathbb{Q}} \Rightarrow [m \leq_{\mathbb{Z}_{\mathbb{Q}}} n \Leftrightarrow m \leq_{\mathbb{Q}} n]). \tag{7.182}$$

The total ordering $\leq_{\mathbb{Z}_{\mathbb{Q}}}$ induces then the linear ordering $<_{\mathbb{Z}_{\mathbb{Q}}}$.

Exercise 7.30. Show that the induced linear ordering $<_{\mathbb{Z}_\mathbb{Q}}$ satisfies

$$\forall m, n\,(m, n \in \mathbb{Z}_\mathbb{Q} \Rightarrow [m <_{\mathbb{Z}_\mathbb{Q}} n \Leftrightarrow m <_\mathbb{Q} n]). \tag{7.183}$$

(Hint: Proceed similarly as in the proof of Corollary 6.79.)

Theorem 7.39 (Order-embedding from $(\mathbb{Z}, \leq_\mathbb{Z})$ to $(\mathbb{Q}, \leq_\mathbb{Q})$ and order-isomorphism from $(\mathbb{Z}, \leq_\mathbb{Z})$ to $(\mathbb{Z}_\mathbb{Q}, \leq_{\mathbb{Z}_\mathbb{Q}})$). *It is true that the function $f_\mathbb{Z}^\mathbb{Q}$ defining the Identification of $\mathbb{Z}$ in $\mathbb{Q}$ constitutes*

a) an order-embedding from $(\mathbb{Z}, \leq_\mathbb{Z})$ to $(\mathbb{Q}, \leq_\mathbb{Q})$, that is,

$$f_\mathbb{Z}^\mathbb{Q} : (\mathbb{Z}, \leq_\mathbb{Z}) \hookrightarrow (\mathbb{Q}, \leq_\mathbb{Q}), \qquad m \mapsto [(m, 1)]_{\sim_q}. \tag{7.184}$$

b) an order-isomorphism from $(\mathbb{Z}, \leq_\mathbb{Z})$ to $(\mathbb{Z}_\mathbb{Q}, \leq_{\mathbb{Z}_\mathbb{Q}})$, that is,

$$f_\mathbb{Z}^\mathbb{Q} : (\mathbb{Z}, \leq_\mathbb{Z}) \rightleftarrows (\mathbb{Z}_\mathbb{Q}, \leq_{\mathbb{Z}_\mathbb{Q}}), \qquad m \mapsto [(m, 1)]_{\sim_q}. \tag{7.185}$$

Exercise 7.31. Prove Theorem 7.39.

(Hint: Proceed in analogy to the proof of Theorem 6.80, using here the tautologies $1 \neq 0$ and $1 \cdot 1 > 0$, and beginning with the verification that

$$\forall m, n\,(m, n \in \mathbb{Z} \Rightarrow [m \leq_\mathbb{Z} n \Leftrightarrow f_\mathbb{Z}^\mathbb{Q}(m) \leq_\mathbb{Q} f_\mathbb{Z}^\mathbb{Q}(n)]) \tag{7.186}$$

holds.)

Exercise 7.32. Verify

$$\forall m, n\,(m, n \in \mathbb{Z} \Rightarrow [m <_\mathbb{Z} n \Leftrightarrow f_\mathbb{Z}^\mathbb{Q}(m) <_\mathbb{Q} f_\mathbb{Z}^\mathbb{Q}(n)]). \tag{7.187}$$

(Hint: Proceed as in Exercise 6.39.)

Note 7.20. The inequality $0 <_\mathbb{Z} 1$ in (6.309) implies

$$f_\mathbb{Z}^\mathbb{Q}(0) <_\mathbb{Q} f_\mathbb{Z}^\mathbb{Q}(1), \tag{7.188}$$

and therefore

$$[(0, 1)]_{\sim_q} <_\mathbb{Q} [(1, 1)]_{\sim_q} \tag{7.189}$$

with the Identification of $\mathbb{Z}$ in $\mathbb{Q}$.

Theorem 7.40. *The set $(\mathbb{Q}, +_\mathbb{Q}, \cdot_\mathbb{Q}, -_\mathbb{Q}, /_\mathbb{Q}, <_\mathbb{Q})$ constitutes an ordered field.*

Proof. We must only demonstrate that the field $(\mathbb{Q}, +_\mathbb{Q}, \cdot_\mathbb{Q}, -_\mathbb{Q}, /_\mathbb{Q})$ satisfies in connection the linear ordering $<_\mathbb{Q}$ the monotony laws

$$\forall p, q, r\, (p, q, r \in \mathbb{Q} \Rightarrow [p <_\mathbb{Q} q \Rightarrow p +_\mathbb{Q} r <_\mathbb{Q} q +_\mathbb{Q} r]) \tag{7.190}$$

$$\forall p, q, r\, ([p, q, r \in \mathbb{Q} \wedge [(0,1)]_{\sim_q} <_\mathbb{Q} r] \Rightarrow [p <_\mathbb{Q} q \Rightarrow p \cdot_\mathbb{Q} r <_\mathbb{Q} q \cdot_\mathbb{Q} r]) \tag{7.191}$$

letting p, q and r in the following be arbitrary. Concerning (7.190), we assume $p, q, r \in \mathbb{Q}$ and prove the implication

$$p <_\mathbb{Q} q \;\Rightarrow\; p +_\mathbb{Q} r <_\mathbb{Q} q +_\mathbb{Q} r \tag{7.192}$$

by contraposition, by verifying instead

$$\neg p +_\mathbb{Q} r <_\mathbb{Q} q +_\mathbb{Q} r \;\Rightarrow\; \neg p <_\mathbb{Q} q. \tag{7.193}$$

To do this, we assume that the antecedent of the preceding implication to be true, so that the Negation Formula for $<$ yields

$$q +_\mathbb{Q} r \;\leq_\mathbb{Q}\; p +_\mathbb{Q} r. \tag{7.194}$$

As rational numbers, p, q and r are by definition elements of the quotient set $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}] / \sim_q$, so that there exist particular elements x, y and z in the Cartesian product $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ such that $[x]_{\sim_q} = p$, $[y]_{\sim_q} = q$ and $[z]_{\sim_q} = r$. Then, there exist also particular integers $a, c, e \in \mathbb{Z}$ and $b, d, f \in \mathbb{Z} \setminus \{0\}$ for which $x = (a, b)$, $y = (c, d)$ and $z = (e, f)$, so that the chosen integers can be written as $p = [(a, b)]_{\sim_q}$, $q = [(c, d)]_{\sim_q}$ and $r = [(e, f)]_{\sim_q}$. With these equations, (7.194) becomes

$$[(c, d)]_{\sim_q} +_\mathbb{Q} [(e, f)]_{\sim_q} \;\leq_\mathbb{Q}\; [(a, b)]_{\sim_q} +_\mathbb{Q} [(e, f)]_{\sim_q}.$$

Here, we can use (7.26) to rewrite the sums according to

$$[(c \cdot f + d \cdot e, d \cdot f)]_{\sim_q} \;\leq_\mathbb{Q}\; [(a \cdot f + b \cdot e, b \cdot f)]_{\sim_q}.$$

This inequality in turn implies with (7.121) the conjunction of $d \cdot f \neq 0$, of $b \cdot f \neq 0$, of

$$(d \cdot f) \cdot (b \cdot f) > 0 \;\Rightarrow\; (c \cdot f + d \cdot e) \cdot (b \cdot f) \leq (d \cdot f) \cdot (a \cdot f + b \cdot e), \tag{7.195}$$

and of

$$(d \cdot f) \cdot (b \cdot f) < 0 \;\Rightarrow\; (c \cdot f + d \cdot e) \cdot (b \cdot f) \geq (d \cdot f) \cdot (a \cdot f + b \cdot e). \tag{7.196}$$

Similarly, the Negation Formula for $<$ gives for the desired consequent of (7.193) the equivalent $q \leq_\mathbb{Q} p$, that is,

$$[(c, d)]_{\sim_q} \;\leq_\mathbb{Q}\; [(a, b)]_{\sim_q},$$

which is also equivalent to the multiple conjunction

$$d \neq 0 \wedge b \neq 0 \wedge [d \cdot b > 0 \Rightarrow c \cdot b \leq d \cdot a] \wedge [d \cdot b < 0 \Rightarrow c \cdot b \geq d \cdot a]. \quad (7.197)$$

Since the previously established $b, d, f \in \mathbb{Z} \setminus \{0\}$ implies $b \neq 0$, $d \neq 0$ and $f \neq 0$ with the definition of a set difference and (2.169), we see that the first and the second part of the multiple conjunction are true. We now prove the first implication directly, assuming $d \cdot b > 0$ to be true. Because $f \neq 0$ evidently implies $0 < f \cdot f$ with (6.238), we can apply the Monotony Law for $\cdot$ and $<$ to infer the truth of the inequality

$$(d \cdot b) \cdot (f \cdot f) \; > \; 0 \cdot (f \cdot f), \quad (7.198)$$

which implies

$$(d \cdot f) \cdot (b \cdot f) \; > \; 0 \quad (7.199)$$

with the Associative & Commutative Law for the multiplication on $\mathbb{Z}$ and with the Cancellation Law for 0. That inequality further implies

$$(c \cdot f + d \cdot e) \cdot (b \cdot f) \; \leq \; (d \cdot f) \cdot (a \cdot f + b \cdot e) \quad (7.200)$$

with (7.195), which gives by means of the Distributive Law for $\mathbb{Z}$

$$(c \cdot f) \cdot (b \cdot f) + (d \cdot e) \cdot (b \cdot f) \; \leq \; (d \cdot f) \cdot (a \cdot f) + (d \cdot f) \cdot (b \cdot e). \quad (7.201)$$

Further applications of the Associative & Commutative Law for the multiplication on $\mathbb{Z}$ result in

$$(c \cdot b) \cdot (f \cdot f) + (d \cdot e) \cdot (b \cdot f) \; \leq \; (d \cdot a) \cdot (f \cdot f) + (d \cdot e) \cdot (b \cdot f), \quad (7.202)$$

and this can be simplified by means of the Monotony Law for $+$ and $\leq$ to

$$(c \cdot b) \cdot (f \cdot f) \; \leq \; (d \cdot a) \cdot (f \cdot f). \quad (7.203)$$

Recalling now the truth of $0 < f \cdot f$, we can then apply the Monotony Law for $\cdot$ and $\leq$ to derive $c \cdot b \leq d \cdot a$, which finding completes the proof of the first implication in (7.197).

Regarding the second implication, we assume now $d \cdot b < 0$ to be true. Evidently, we can apply here essentially the same sequence of arguments as in the proof of the first implication to obtain the inequalities (7.198) – (7.203) with reversed inequality symbol (i.e., with '$<$' instead of '$>$' and subsequently with '$\geq$' instead of '$\leq$'), where we use the implication (7.196) instead of the implication (7.195). We thus find

$$(c \cdot b) \cdot (f \cdot f) \; \geq \; (d \cdot a) \cdot (f \cdot f)$$

to be true, which clearly simplifies to the desired consequent $c \cdot b \geq d \cdot a$ of the second implication in (7.197).

This completes the proof of the multiple conjunction (7.197), so that the equivalent inequality $q \leq_{\mathbb{Q}} p$ and the equivalent negation $\neg p \leq_{\mathbb{Q}} q$ are also true. Thus, the implication (7.193) holds, which means that the proof of (7.192) by contraposition is complete. Since p, q and r were arbitrary, we may therefore conclude that the monotony law (7.190) is satisfied.

The second monotony law (7.191) can be established similarly. Therefore, $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, -_{\mathbb{Q}}, /_{\mathbb{Q}}, <_{\mathbb{Q}})$ constitutes an ordered field, by definition. □

Exercise 7.33. Establish (7.191).

(Hint: Proceed as in the proof of (7.190) and use for the final simplification of the inequality the Monotony Law for $\cdot$ and $\leq$ now jointly with the Characterization of positive rational numbers.)

Definition 7.14 (Ordered field of rational numbers). We call

$$(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, -_{\mathbb{Q}}, /_{\mathbb{Q}}, <_{\mathbb{Q}}) \tag{7.204}$$

the *ordered field of rational numbers.*

The following fact follows immediately for the ordered field of rational numbers in view of Theorem 7.33.

Corollary 7.41. *The set* $(\mathbb{Q}, <_{\mathbb{Q}})$ *is densely ordered.*

Theorem 7.42 (Ordered integral domain of $\mathbb{Z}$ in $\mathbb{Q}$). *It is true that the set*

$$(\mathbb{Z}_{\mathbb{Q}}, +_{\mathbb{Z}_{\mathbb{Q}}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}}, -_{\mathbb{Z}_{\mathbb{Q}}}, <_{\mathbb{Z}_{\mathbb{Q}}}) \tag{7.205}$$

constitutes an ordered integral domain.

Proof. Let us recall Lemma 7.22 according to which $(\mathbb{Z}_{\mathbb{Q}}, +_{\mathbb{Z}_{\mathbb{Q}}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}}, -_{\mathbb{Z}_{\mathbb{Q}}})$ constitutes a commutative ring with zero element $f_{\mathbb{Z}}^{\mathbb{Q}}(0) = [(0,1)]_{\sim_q}$ and unity element $f_{\mathbb{Z}}^{\mathbb{Q}}(1) = [(1,1)]_{\sim_q}$. Thus, Property 1 and Property 4 of an ordered integral domain are satisfied.

Concerning Property 2, we observe that $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}})$ is zero-divisor free according to Theorem 7.29. We now apply the Criterion for zero-divisor freeness and establish the truth of

$$\begin{aligned} &\forall m, n\, (m, n \in \mathbb{Z}_{\mathbb{Q}} \Rightarrow [(m \cdot_{\mathbb{Z}_{\mathbb{Q}}} n = [(0,1)]_{\sim_q} \vee n \cdot_{\mathbb{Z}_{\mathbb{Q}}} m = [(0,1)]_{\sim_q}) \\ &\quad \Rightarrow (m = [(0,1)]_{\sim_q} \vee n = [(0,1)]_{\sim_q})]), \end{aligned} \tag{7.206}$$

letting $m, n \in \mathbb{Z}_{\mathbb{Q}}$ be arbitrary, assuming the disjunction

$$m \cdot_{\mathbb{Z}_{\mathbb{Q}}} n = [(0,1)]_{\sim_q} \vee n \cdot_{\mathbb{Z}_{\mathbb{Q}}} m = [(0,1)]_{\sim_q}$$

to be true. Here, we observe in light of (7.52) that the equations $m \cdot_{\mathbb{Z}_{\mathbb{Q}}} n = m \cdot_{\mathbb{Q}} n$ and $n \cdot_{\mathbb{Z}_{\mathbb{Q}}} m = n \cdot_{\mathbb{Q}}$ are true, so that we may apply substitutions to the preceding disjunction and write it equivalently as

$$m \cdot_{\mathbb{Q}} n = [(0,1)]_{\sim_q} \ \vee \ n \cdot_{\mathbb{Q}} m = [(0,1)]_{\sim_q}.$$

This disjunction implies now already the desired disjunction $m = [(0,1)]_{\sim_q} \vee n = [(0,1)]_{\sim_q}$ with (7.118). Since p and q were initially arbitrary, we may therefore conclude that (7.206) holds, so that $(\mathbb{Z}_{\mathbb{Q}}, +_{\mathbb{Z}_{\mathbb{Q}}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}})$ is indeed zero-divisor free.

Concerning Property 3, we now prove $\mathbb{Z}_{\mathbb{Q}} \neq \{[(0,1)]_{\sim_q}\}$ by contradiction, assuming the negation of that inequality to be true, so that $\mathbb{Z}_{\mathbb{Q}} = \{[(0,1)]_{\sim_q}\}$ follows to be true with the Double Negation Law. As $\mathbb{Z}_{\mathbb{Q}}$ is a singleton, we obtain with (2.180) the uniquely existential sentence $\exists! m\,(m \in \mathbb{Z}_{\mathbb{Q}})$, whose uniqueness part is given by

$$\forall m, m'([m \in \mathbb{Z}_{\mathbb{Q}} \ \wedge \ m' \in \mathbb{Z}_{\mathbb{Q}}] \Rightarrow m = m').$$

Therefore, $[(1,1)]_{\sim_q} \in \mathbb{Z}_{\mathbb{Q}}$ and $[(0,1)]_{\sim_q} \in \mathbb{Z}_{\mathbb{Q}}$ imply $[(1,1)]_{\sim_q} = [(0,1)]_{\sim_q}$, which contradicts the fact that $[(1,1)]_{\sim_q} \neq [(0,1)]_{\sim_q}$ holds according to (7.12). This completes the proof of $\mathbb{Z}_{\mathbb{Q}} \neq \{[(0,1)]_{\sim_q}\}$, so that the semiring $(\mathbb{Z}_{\mathbb{Q}}, +_{\mathbb{Z}_{\mathbb{Q}}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}})$ is nontrivial by definition. Thus, $(\mathbb{Z}_{\mathbb{Q}}, +_{\mathbb{Z}_{\mathbb{Q}}}, \cdot_{\mathbb{Z}_{\mathbb{Q}}}, -_{\mathbb{Z}_{\mathbb{Q}}}, <_{\mathbb{Z}_{\mathbb{Q}}})$ satisfies also Property 3 of an ordered integral domain.

Concerning Property 5, we show that this set satisfies the monotony laws

$$\forall m, n, p\,(m, n, p \in \mathbb{Z}_{\mathbb{Q}} \Rightarrow [m <_{\mathbb{Z}_{\mathbb{Q}}} n \Rightarrow m +_{\mathbb{Z}_{\mathbb{Q}}} p <_{\mathbb{Z}_{\mathbb{Q}}} n +_{\mathbb{Z}_{\mathbb{Q}}} p]) \tag{7.207}$$

$$\begin{aligned} &\forall m, n, p\,([m, n, p \in \mathbb{Z}_{\mathbb{Q}} \ \wedge \ [(0,1)]_{\sim_q} <_{\mathbb{Z}_{\mathbb{Q}}} p] \\ &\quad \Rightarrow [m <_{\mathbb{Z}_{\mathbb{Q}}} n \Rightarrow m \cdot_{\mathbb{Z}_{\mathbb{Q}}} p <_{\mathbb{Z}_{\mathbb{Q}}} n \cdot_{\mathbb{Z}_{\mathbb{Q}}} p]). \end{aligned} \tag{7.208}$$

Regarding (7.208), we let $m, n, p \in \mathbb{Z}_{\mathbb{Q}}$ be arbitrary and assume $m <_{\mathbb{Z}_{\mathbb{Q}}} n$ as well as $[(0,1)]_{\sim_q} <_{\mathbb{Z}_{\mathbb{Q}}} p$ to be true. We therefore obtain $m, n, p \in \mathbb{Q}$ with the inclusion (7.24) by definition of a subset, and moreover $[(0,1)]_{\sim_q} \in \mathbb{Q}$ with the fact that $[(0,1)]_{\sim_q} \in \mathbb{Q}$ is the zero element of $\mathbb{Q}$. Then, $[(0,1)]_{\sim_q} <_{\mathbb{Q}} p$ and $m <_{\mathbb{Q}} n$ follow to be true with (7.183). These findings imply

$$m \cdot_{\mathbb{Q}} p \ <_{\mathbb{Q}} \ n \cdot_{\mathbb{Q}} p$$

with (7.191), where the initial assumptions $m, n, p \in \mathbb{Z}_{\mathbb{Q}}$ give the equations

$$\begin{aligned} m \cdot_{\mathbb{Z}_{\mathbb{Q}}} p &= m \cdot_{\mathbb{Q}} p \\ n \cdot_{\mathbb{Z}_{\mathbb{Q}}} p &= n \cdot_{\mathbb{Q}} p \end{aligned}$$

with (7.52). Consequently, substitutions yield

$$m \cdot_{\mathbb{Z}_\mathbb{Q}} p \;<_\mathbb{Q}\; n \cdot_{\mathbb{Z}_\mathbb{Q}} p,$$

where the products $m \cdot_{\mathbb{Z}_\mathbb{Q}} p$ and $n \cdot_{\mathbb{Z}_\mathbb{Q}} p$ are values of the binary operation $\cdot_{\mathbb{Z}_\mathbb{Q}}$ and thus elements of $\mathbb{Z}_\mathbb{Q}$. Therefore, we can use again (7.183) to write the preceding inequality equivalently as

$$m \cdot_{\mathbb{Z}_\mathbb{Q}} p \;<_{\mathbb{Z}_\mathbb{Q}}\; n \cdot_{\mathbb{Z}_\mathbb{Q}} p,$$

thereby proving the implications in (7.208). As m, n and p are arbitrary, we may therefore conclude that (7.208) is satisfied.

The other universal sentence (7.207) can be established similarly. We thus showed that $(\mathbb{Z}_\mathbb{Q}, +_{\mathbb{Z}_\mathbb{Q}}, \cdot_{\mathbb{Z}_\mathbb{Q}}, -_{\mathbb{Z}_\mathbb{Q}}, <_{\mathbb{Z}_\mathbb{Q}})$ satisfies all of the defining properties of an ordered integral domain. □

Note 7.21. According to the preceding theorem, we can consider the ordered integral domain of integers in $\mathbb{Q}$

$$(\mathbb{Z}_\mathbb{Q}, +_{\mathbb{Z}_\mathbb{Q}}, \cdot_{\mathbb{Z}_\mathbb{Q}}, -_{\mathbb{Z}_\mathbb{Q}}, <_{\mathbb{Z}_\mathbb{Q}})$$

to be an 'isomorphic copy' of the ordered integral domain of integers

$$(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z}, -_\mathbb{Z}, <_\mathbb{Z})$$

since the former shares the essential algebraic properties of the latter. Rather than using the integers in their original form, it will in the sequel be more convenient to use their mapped versions in $\mathbb{Q}$ in connection with the binary operations on $\mathbb{Q}$ and with the linear/total ordering of $\mathbb{Q}$. We simply keep in mind that the isomorphism $f_\mathbb{Z}^\mathbb{Q}$ enables us to switch between both representations at any time, for instance after carrying out rational-number-based actions.

Notation 7.7. As the zero element $[(0,1)]_{\sim_q}$ and the unity element $[(1,1)]_{\sim_q}$ of $\mathbb{Q}$ are integers in $\mathbb{Q}$ and furthermore the 'isomorphic copies' of the zero element 0 and the unity element 1 of $\mathbb{Z}$ (under the mapping $f_\mathbb{Z}^\mathbb{Q}$), we abbreviate

$$0 = [(0,1)]_{\sim_q}, \tag{7.209}$$
$$1 = [(1,1)]_{\sim_q}, \tag{7.210}$$

further overloading the symbols '0' and '1'. Furthermore, we will usually write from now on $\frac{m}{n}$ instead of $[(m,n)]_{\sim_q}$, as indicated by Notation 7.1. Thus, we may write for (7.189), in analogy to $0 <_\mathbb{N} 1$ and $0 <_\mathbb{Z} 1$, also

$$0 <_\mathbb{Q} 1. \tag{7.211}$$

Proposition 7.43. *The set* $\mathbb{Q}$ *is*

a) neither bounded from above, that is,

$$\neg\exists u\,(u \in \mathbb{Q} \wedge \forall p\,(p \in \mathbb{Q} \Rightarrow p \leq_{\mathbb{Q}} u)), \tag{7.212}$$

b) nor bounded from below, that is,

$$\neg\exists a\,(a \in \mathbb{Q} \wedge \forall p\,(p \in \mathbb{Q} \Rightarrow a \leq_{\mathbb{Q}} p)). \tag{7.213}$$

Proof. We prove the sentence (7.213) by contradiction, assuming its negation to be true, so that the existential sentence in (7.213) follows to be true with the Double Negation Law. Thus, there exists an element of $\mathbb{Q}$, say $\bar{a}$, with

$$\forall p\,(p \in \mathbb{Q} \Rightarrow \bar{a} \leq_{\mathbb{Q}} p).$$

Let us now observe in light of the fact that $\mathbb{Q}$ is part of an ordered integral domain (see Note 7.13) that the inequality $0 <_{\mathbb{Q}} 1$ holds according to Corollary 6.70, which implies $-1 <_{\mathbb{Q}} 0$ with Proposition 6.67 and (6.38). Consequently, we obtain $-1 +_{\mathbb{Q}} \bar{a} <_{\mathbb{Q}} 0 +_{\mathbb{Q}} \bar{a}$ with (7.190), and then also $-1 +_{\mathbb{Q}} \bar{a} <_{\mathbb{Q}} \bar{a}$ with the definition of an inverse element. Now, since $-1 +_{\mathbb{Q}} \bar{a}$ is a rational number, we obtain with the preceding universal sentence the inequality $\bar{a} \leq_{\mathbb{Q}} -1 +_{\mathbb{Q}} \bar{a}$. In conjunction with with the previously established inequality $-1 +_{\mathbb{Q}} \bar{a} <_{\mathbb{Q}} \bar{a}$, this further implies $\bar{a} <_{\mathbb{Q}} \bar{a}$, in contradiction to the fact that $\neg \bar{a} <_{\mathbb{Q}} \bar{a}$ is true because of the irreflexivity of the induced linear ordering $<_{\mathbb{Q}}$. □

Exercise 7.34. Establish Part a) of Proposition 7.43.

(Hint: The proof is a slightly simpler version of the proof of Part b).)

We may also construct the following useful sequence of increasingly large rationals which are all greater than a given rational number.

Exercise 7.35. Show for any rational number q and any positive natural number m that there exists a unique sequence $s = (s_i)_{i \in \mathbb{N}}$ in $\mathbb{Q}$ with terms $s_i = q +_{\mathbb{Q}} \frac{i}{m}$, and demonstrate that this sequence is strictly increasing.

(Hint: Apply Function definition by replacement and prove in addition

$$\forall i, j\,([i, j \in \mathbb{N} \wedge i <_{\mathbb{N}} j] \Rightarrow s(i) <_{\mathbb{Q}} s(j)). \tag{7.214}$$

by means of (7.190) and (7.191).)

Corollary 7.44. *For any* $q \in \mathbb{Q}$ *and any* $m, n \in \mathbb{N}_+$*, there exists the unique sequence* $t = (t_i \,|\, i \in \{0, \ldots, n\})$ *in* $\mathbb{Q}$ *such that*

$$\forall i\,(i \in \{0, \ldots, n\} \Rightarrow t_i = q +_{\mathbb{Q}} \frac{i}{m}), \tag{7.215}$$

and this sequence is strictly increasing.

Proof. Letting $q \in \mathbb{Q}$ and $m, n \in \mathbb{N}_+$ be arbitrary, we observe in light of the definition of an initial segment of $\mathbb{N}$ that $\{0, \ldots, n\} \in \mathcal{P}(\mathbb{N})$ holds, so that the inclusion $\{0, \ldots, n\} \subseteq \mathbb{N}$ holds by definition of a power set. Therefore, the restriction $t = (t_i \,|\, i \in \{0, \ldots, n\})$ of the strictly increasing sequence $s = (s_i)_{i \in \mathbb{N}}$ in $\mathbb{Q}$ with terms $s_i = q +_{\mathbb{Q}} \frac{i}{m}$ (established in Exercise 7.35) to the subset $\{0, \ldots, n\}$ is itself strictly increasing in view of Exercise 3.116. The terms of the restriction satisfy $t_i = (s \restriction \{0, \ldots, n\})_i = s_i$ for any $i \in \{0, \ldots, n\}$ due to (3.567), which means that t satisfies (7.215). Since q, m and n are arbitrary, we may conclude that the corollary is true. □

Since $\mathbb{Q}$ has no upper and no lower bound, it is in particular true that the maximum and the minimum of that set do not exist.

Corollary 7.45. *The set* $\mathbb{Q}$

a) neither has a maximum, that is,

$$\neg \exists u \, (u \in \mathbb{Q} \wedge u = \overset{\leq_{\mathbb{Q}}}{\max} \mathbb{Q}), \tag{7.216}$$

a) nor a minimum, that is,

$$\neg \exists a \, (a \in \mathbb{Q} \wedge a = \overset{\leq_{\mathbb{Q}}}{\min} \mathbb{Q}). \tag{7.217}$$

Proposition 7.46. *It is true for any rational number that there exists a larger positive natural number, that is,*

$$\forall p \, (p \in \mathbb{Q} \Rightarrow \exists n \, (n \in \mathbb{N}_+ \wedge p <_{\mathbb{Q}} n)). \tag{7.218}$$

Proof. Letting $\frac{m}{n}$ be an arbitrary rational number, it is true by definition that $m, n \in \mathbb{Z}$ and $n \neq 0$ are true. We consider now the two exhaustive cases $\frac{m}{n} \leq_{\mathbb{Q}} 0$ and $\neg \frac{m}{n} \leq_{\mathbb{Q}} 0$ (using the Law of the Excluded Middle). Since $0 \leq_{\mathbb{Q}} 1$ is true according to Corollary 6.70 and Note 7.13 in connection with the ordered field of rational numbers, we find

$$[0 \geq_{\mathbb{Q}}] \quad \frac{m}{n} <_{\mathbb{Q}} 1 \tag{7.219}$$

with the Transitivity Formula for $\leq$ and $<$. As 1, as the beginning of $\mathbb{N}_+$, constitutes by definition an element of that set, we already found a positive natural number larger than $\frac{m}{n}$. Thus, the existential sentence to be proven holds in the first case.

In the second case $\neg \frac{m}{n} \leq_{\mathbb{Q}} 0$, we obtain with the Negation Formula for $\leq$ the inequality $0 <_{\mathbb{Q}} \frac{m}{n}$, and consequently $0 <_{\mathbb{Z}} m \cdot_{\mathbb{Z}} n$ with the Characterization of positive rational numbers. Thus, $m \cdot_{\mathbb{Z}} n$ constitutes a natural number (in $\mathbb{Z}$) due to Corollary 6.82, which we may also treat as a rational number. Next, we observe that $n \neq 0$ implies $n^2 >_{\mathbb{Z}} 0$ with Proposition 6.69, so that n^2 is also a natural number (in $\mathbb{Z}$ and $\mathbb{Q}$). Ignoring notational issues, the preceding inequality yields $1 \leq_{\mathbb{N}} n^2$ with (4.157), which we write as $1 \leq_{\mathbb{Q}} n^2$. Writing the previously established inequality $n^2 >_{\mathbb{Z}} 0$ as $n^2 >_{\mathbb{Q}} 0$, we obtain $\frac{1}{n^2} >_{\mathbb{Q}} 0$ with Proposition 7.32, which allows us to apply the Monotony Law for $\cdot$ and $\leq$ to infer from $1 \leq_{\mathbb{Q}} n^2$ the inequality $1 \cdot_{\mathbb{Q}} \frac{1}{n^2} \leq_{\mathbb{Q}} n^2 \cdot_{\mathbb{Q}} \frac{1}{n^2}$. This simplifies to $\frac{1}{n^2} \leq_{\mathbb{Q}} 1$ with the property of a unity element and (7.63). Because of $0 <_{\mathbb{Q}} m \cdot_{\mathbb{Q}} n$, another application of the Monotony Law for $\cdot$ and $\leq$ (alongside the definition of a square) gives us

$$\frac{1}{n \cdot_{\mathbb{Q}} n} \cdot_{\mathbb{Q}} (m \cdot_{\mathbb{Q}} n) \leq_{\mathbb{Q}} 1 \cdot_{\mathbb{Q}} (m \cdot_{\mathbb{Q}} n).$$

This simplifies to $\frac{m}{n} \leq_{\mathbb{Q}} m \cdot_{\mathbb{Q}} n$ by means of (7.92) and the property of the unity element. Since the inequality $m \cdot_{\mathbb{N}} n <_{\mathbb{N}} (m \cdot_{\mathbb{N}} n) +_{\mathbb{N}} 1$ is also true due to (4.153), which we write as $m \cdot_{\mathbb{Q}} n <_{\mathbb{Q}} m \cdot_{\mathbb{Q}} n +_{\mathbb{Q}} 1$, we obtain with the Transitivity Formula for $\leq$ and $<$

$$[0 <_{\mathbb{Q}}] \quad \frac{m}{n} <_{\mathbb{Q}} m \cdot_{\mathbb{Q}} n +_{\mathbb{Q}} 1. \tag{7.220}$$

The transitivity of the linear ordering $<_{\mathbb{Q}}$ yields $0 <_{\mathbb{Q}} m \cdot_{\mathbb{Q}} n +_{\mathbb{Q}} 1$, which we may write also as $0 <_{\mathbb{N}} m \cdot_{\mathbb{N}} n +_{\mathbb{N}} 1$, which clearly shows that $m \cdot_{\mathbb{N}} n +_{\mathbb{N}} 1$ is a positive natural number, whose image in $\mathbb{Q}$ is greater than $\frac{m}{n}$ Thus, the existential sentence to be proven holds also in the current second case.

Having completed the proof by case, we may now conclude that the universal sentence (7.218) is true since p was initially arbitrary. □

Proposition 7.47. *It is true for any positive rational number that there exists a positive natural number whose reciprocal is smaller, that is,*

$$\forall p \, (0 <_{\mathbb{Q}} p \Rightarrow \exists n \, (n \in \mathbb{N}_+ \land \frac{1}{n} <_{\mathbb{Q}} p)). \tag{7.221}$$

Proof. We take an arbitrary rational number p greater than 0, so that $0 <_{\mathbb{Q}} \frac{1}{p}$ follows to be true with Proposition 7.32. Then, (7.218) shows that there exists a positive natural number, say $\bar{n}$, with $\frac{1}{p} <_{\mathbb{Q}} \bar{n}$. As this number is clearly greater than 0, we obtain also $0 <_{\mathbb{Q}} \frac{1}{\bar{n}}$ with Proposition 7.32. Applying now the monotony law (7.99) to the ordered field of rational numbers, we find the inequality $\frac{1}{p} \cdot_{\mathbb{Q}} \frac{1}{\bar{n}} <_{\mathbb{Q}} \bar{n} \cdot_{\mathbb{Q}} \frac{1}{\bar{n}}$. This gives $\frac{1}{\bar{n}} \cdot_{\mathbb{Q}} \frac{1}{p} <_{\mathbb{Q}} 1$ with the commutativity of $\cdot_{\mathbb{Q}}$ and (7.63). Recalling the initial assumption $0 <_{\mathbb{Q}} p$, we now apply the previous monotony law a second time to obtain

$\frac{1}{\bar{n}} \cdot_{\mathbb{Q}} \frac{1}{p} \cdot_{\mathbb{Q}} p <_{\mathbb{Q}} 1 \cdot_{\mathbb{Q}} p$ (omitting brackets in view of the associativity of $\cdot_{\mathbb{Q}}$). We may evidently simplify this inequality to $\frac{1}{\bar{n}} <_{\mathbb{Q}} p$, which - in conjunction with the previous finding $\bar{n} \in \mathbb{N}_+$ - demonstrates the truth of the existential sentence to be proven. As p was arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Proposition 7.48. *Every fraction with positive numerator and positive denominator is less than or equal to its numerator, in the sense that*

$$\forall m, n\,([0 <_{\mathbb{Z}} m \,\wedge\, 0 <_{\mathbb{Z}} n] \Rightarrow \frac{m}{n} \leq_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(m)). \tag{7.222}$$

Proof. We take two arbitrary positive integers m and n, which define the rational number $\frac{m}{n}$ (according to the Generation of rational numbers). The evident fact $0 <_{\mathbb{Z}} 1$ and the previous assumption $0 <_{\mathbb{Z}} n$ imply $0 \cdot_{\mathbb{Z}} 1 <_{\mathbb{Z}} n \cdot_{\mathbb{Z}} 1$ with the Monotony Law for $\cdot$ and $<$. This inequality simplifies to

$$0 <_{\mathbb{Z}} n \cdot_{\mathbb{Z}} 1 \tag{7.223}$$

due to the Cancellation Law for 0_X in rings. Let us observe now that $0 <_{\mathbb{Z}} n$ implies also $0 +_{\mathbb{Z}} 1 \leq_{\mathbb{Z}} n$ with (6.314), where we write 1 for $f_{\mathbb{N}}^{\mathbb{Z}}(1)$. By the property of a zero element, we thus have the inequality $1 \leq_{\mathbb{Z}} n$. The conjunction of this inequality and the initial assumption $0 <_{\mathbb{Z}} m$ gives us now $1 \cdot_{\mathbb{Z}} m \leq_{\mathbb{Z}} n \cdot_{\mathbb{Z}} m$ with the Monotony Law for $\cdot$ and $\leq$. Due to the commutativity of the multiplication on $\mathbb{Z}$, we may write for this inequality

$$m \cdot_{\mathbb{Z}} 1 \leq_{\mathbb{Z}} n \cdot_{\mathbb{Z}} m.$$

Due to (7.223), that inequality further implies

$$[(m, n)]_{\sim_q} \leq_{\mathbb{Q}} [(m, 1)]_{\sim_q}$$

with (7.121), since $n \neq 0$ and $1 \neq 0$ are evidently both true. In view of Notation 7.1 and the Identification of $\mathbb{Z}$ in $\mathbb{Q}$, the preceding inequality can be written as the desired inequality in (7.222). As m and n were initially arbitrary, the proposed universal sentence follows therefore to be true. □

Proposition 7.49. *For every rational number p and every positive rational number q, there is some natural number n (in $\mathbb{Q}$) such that p is less than the product of n and q, that is,*

$$\forall p, q\,([p \in \mathbb{Q}_+ \,\wedge\, q \in \mathbb{Q}] \Rightarrow \exists n\,(n \in \mathbb{N} \,\wedge\, q <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(n)) \cdot_{\mathbb{Q}} p)). \tag{7.224}$$

Proof. We let $p \in \mathbb{Q}_+$ and $q \in \mathbb{Q}$ be arbitrary, and we prove the existential sentence by cases, based on the fact that $q \leq_{\mathbb{Q}} [(0, 1)]_{\sim_q}$ or $\neg q \leq_{\mathbb{Q}} [(0, 1)]_{\sim_q}$ holds by the Law of the Excluded Middle.

In the first case $q \leq_{\mathbb{Q}} [(0,1)]_{\sim_q}$, we observe that the initial assumption $p \in \mathbb{Q}_+$ allows us, on the one hand, to express p as the equivalence class $[(\bar{M}, \bar{N})]_{\sim_q}$ for some particular integers $0 < \bar{M}$ and $0 < \bar{N}$, according to the Generation of positive rational numbers with positive numerators and positive denominators. On the other hand, $p \in \mathbb{Q}_+$ implies $[(0,1)]_{\sim_q} <_{\mathbb{Q}} p$ by definition of the set $\mathbb{Q}_+$. In conjunction with the case assumption, this gives us $q <_{\mathbb{Q}} p$ with the Transitivity Formula for $\leq$ and $<$. Note that the fact $0 <_{\mathbb{N}} 1$ in (4.164) implies $[(0,0)]_{\sim_d} <_{\mathbb{Z}} f_{\mathbb{N}}^{\mathbb{Z}}(1)$ with (6.284) and the Identification of $\mathbb{N}$ in $\mathbb{Z}$. Writing 0 for $[(0,0)]_{\sim_d}$ as in (6.307), note also that the preceding inequality implies

$$[(0,1)]_{\sim_q} <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(1))$$

with (7.187) and the Identification of $\mathbb{Z}$ in $\mathbb{Q}$. In conjunction with the previously established inequality $q <_{\mathbb{Q}} p$, this gives us

$$q \cdot_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(1)) <_{\mathbb{Q}} p \cdot_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(1)),$$

with the Monotony Law for $\cdot$ and $<$. Writing $1 = [(1,0)]_{\sim_d} = f_{\mathbb{N}}^{\mathbb{Z}}(1)$, as shown in (6.203) and (6.308), we obtain

$$q \cdot_{\mathbb{Q}} [(1,1)]_{\sim_q} <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(1)) \cdot_{\mathbb{Q}} p$$

with the identification of $\mathbb{Z}$ in $\mathbb{Q}$ (using also the commutativity of the multiplication on $\mathbb{Q}$). By the property of the identity element $[(1,1)]_{\sim_d}$, we may simplify the inequality to

$$q <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(1)) \cdot_{\mathbb{Q}} p,$$

which demonstrates in light of $1 \in \mathbb{N}$ that the existential sentence in (7.224) is true in the current first case.

The second case $\neg q \leq_{\mathbb{Q}} [(0,1)]_{\sim_q}$ implies $[(0,1)]_{\sim_q} <_{\mathbb{Q}} q$ with the Negation Formula for $\leq$, so that we may express q as the equivalence class $q = [(\bar{m}, \bar{n})]_{\sim_q}$ for some particular integers $0 < \bar{m}$ and $0 < \bar{n}$ (according to the Generation of positive rational numbers with positive numerators and positive denominators). Observe now that $0 < \bar{N}, 0 < \bar{m}$ implies $0 < \bar{m} \cdot \bar{N}$ and that $0 < \bar{M}, 0 < \bar{n}$ implies $0 < \bar{n} \cdot \bar{M}$ by means of the Monotony Law for $\cdot$ and $<$ and by means of the Cancellation Law for 0_X in rings. These two inequalities further imply

$$\frac{\bar{m} \cdot \bar{N}}{\bar{n} \cdot \bar{M}} \leq_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m} \cdot \bar{N}) \tag{7.225}$$

because of (7.222). The evident fact $0 < 1$ implies $\bar{m} \cdot \bar{N} < \bar{m} \cdot \bar{N} + 1$ with the Monotony Law for $+$ and $<$ (using also the commutativity of the addition on $\mathbb{Z}$), so that

$$f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m} \cdot \bar{N}) <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m} \cdot \bar{N} + 1)$$

follows to be true with (7.187). In view of (7.225), this results in

$$\frac{\bar{m}}{\bar{n}} \cdot_{\mathbb{Q}} \frac{\bar{N}}{\bar{M}} <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m} \cdot \bar{N} + 1) \tag{7.226}$$

due to the Transitivity Formula for $\leq$ and $<$ (applying also the Calculus of fractions). Notice that the given inequalities $0 < \bar{M}$ and $0 < \bar{N}$ evidently imply $0 < \bar{M} \cdot \bar{N}$, so that $[(0,1)]_{\sim_q} <_{\mathbb{Q}} \frac{\bar{M}}{\bar{N}}$ is true according to the Characterization of positive rational numbers and in view of Notation 7.1. An application of the Monotony Law for $\cdot$ and $<$ to (7.226) gives us therefore

$$\left(\frac{\bar{m}}{\bar{n}} \cdot_{\mathbb{Q}} \frac{\bar{N}}{\bar{M}}\right) \cdot_{\mathbb{Q}} \frac{\bar{M}}{\bar{N}} <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m} \cdot \bar{N} + 1) \cdot_{\mathbb{Q}} \frac{\bar{M}}{\bar{N}}.$$

Applying now the associativity of the multiplication on $\mathbb{Q}$, the Calculus of fractions and substitutions based on the equations $p = [(\bar{M}, \bar{N})]_{\sim_q} = \frac{\bar{M}}{\bar{N}}$ as well as $q = [(\bar{m}, \bar{n})]_{\sim_q} = \frac{\bar{m}}{\bar{n}}$, we arrive at

$$q <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m} \cdot \bar{N} + 1) \cdot_{\mathbb{Q}} p. \tag{7.227}$$

Recalling now the truth of the inequalities $0 < \bar{m} \cdot \bar{N}$ and $\bar{m} \cdot \bar{N} < \bar{m} \cdot \bar{N} + 1$, we find $0 < \bar{m} \cdot \bar{N} + 1$ with the transitivity of the linear ordering $<_{\mathbb{Z}}$. This inequality implies $0 \leq \bar{m} \cdot \bar{N} + 1$ with the Characterization of induced irreflexive partial orderings, where we may write $[(0,0)]_{\sim_d}$ instead of 0, so that $\bar{m} \cdot \bar{N} + 1 \in \mathbb{N}_{\mathbb{Z}}$ follows to be true with (6.289). Recalling also that $f_{\mathbb{N}}^{\mathbb{Z}}$ constitutes a bijection from $\mathbb{N}$ to $\mathbb{N}_{\mathbb{Z}}$, we find $(f_{\mathbb{N}}^{\mathbb{Z}})^{-1}(\bar{m} \cdot \bar{N} + 1) \in \mathbb{N}$ due to (3.673). Denoting this natural number by n^*, we obtain $f_{\mathbb{N}}^{\mathbb{Z}}(n^*) = \bar{m} \cdot \bar{N} + 1$ by means of the Characterization of the function values of an inverse function. Using now this equation for a substitution in (7.227), we finally arrive at

$$q <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(n^*)) \cdot_{\mathbb{Q}} p.$$

Due to $n^* \in \mathbb{N}$, we see that the existential sentence in (7.224) holds also in the current second case. Since p and q were initially arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Definition 7.15 (Fraction in lowest terms). We say for any integers m, n with $m \neq 0$ $n \neq 0$ that the fraction $\frac{m}{n}$ is *in lowest terms* iff m and n are coprime.

Proposition 7.50. *It is true that the quotient $\frac{m}{n}$ of any two nonzero integers m and n can be written as*

$$\frac{m}{n} = \frac{\frac{m}{\mathrm{gcf}(m,n)}}{\frac{n}{\mathrm{gcf}(m,n)}}, \tag{7.228}$$

where the quotient

$$\frac{\frac{m}{\mathrm{gcf}(m,n)}}{\frac{n}{\mathrm{gcf}(m,n)}}, \tag{7.229}$$

constitutes a fraction in lowest terms.

Proof. We take two arbitrary nonzero integers m and n, so that Theorem 6.93 gives us unique coprime integers M and N which satisfy the equations

$$m = M \cdot_{\mathbb{Z}} \mathrm{gcf}(m,n), \tag{7.230}$$

$$n = N \cdot_{\mathbb{Z}} \mathrm{gcf}(m,n). \tag{7.231}$$

We therefore obtain through substitutions

$$\frac{m}{n} = \frac{M \cdot_{\mathbb{Z}} \mathrm{gcf}(m,n)}{N \cdot_{\mathbb{Z}} \mathrm{gcf}(m,n)}.$$

Treating all of the occurring integers as rational numbers and thus as field elements, we may apply then (7.94) to write the preceding equation as

$$\frac{m}{n} = \frac{M}{N} \cdot_{\mathbb{Q}} \frac{\mathrm{gcf}(m,n)}{\mathrm{gcf}(m,n)},$$

which we can subsequently simplify to

$$\frac{m}{n} = \frac{M}{N} \cdot_{\mathbb{Q}} 1 = \frac{M}{N} \tag{7.232}$$

by means of (7.63) and the definition of the unity element. Next, we observe the truth of the equations

$$\begin{aligned}
\frac{m}{\mathrm{gcf}(m,n)} &= m \cdot_{\mathbb{Q}} \frac{1}{\mathrm{gcf}(m,n)} = [M \cdot_{\mathbb{Q}} \mathrm{gcf}(m,n)] \cdot_{\mathbb{Q}} \frac{1}{\mathrm{gcf}(m,n)} \\
&= M \cdot_{\mathbb{Q}} \left[\mathrm{gcf}(m,n) \cdot_{\mathbb{Q}} \frac{1}{\mathrm{gcf}(m,n)}\right] = M \cdot_{\mathbb{Q}} 1 \\
&= M
\end{aligned}$$

in light of the definition of the (field) division, the equation (7.230) in terms of integers in $\mathbb{Q}$, the Associative Law for the multiplication on $\mathbb{Q}$, the definition of a multiplicative inverse, and the definition of the unity element. In view of (7.231), thus see also that the equations

$$\begin{aligned}\frac{n}{\mathrm{gcf}(m,n)} &= n \cdot_{\mathbb{Q}} \frac{1}{\mathrm{gcf}(m,n)} = [N \cdot_{\mathbb{Q}} \mathrm{gcf}(m,n)] \cdot_{\mathbb{Q}} \frac{1}{\mathrm{gcf}(m,n)} \\ &= N \cdot_{\mathbb{Q}} \left[\mathrm{gcf}(m,n) \cdot_{\mathbb{Q}} \frac{1}{\mathrm{gcf}(m,n)}\right] = N \cdot_{\mathbb{Q}} 1 \\ &= N\end{aligned}$$

are true. Applying now substitutions to (7.232) gives us (7.228). Viewed as integers, M and N are coprime (as mentioned earlier), which fact implies via substitution that the quotients $\frac{m}{\mathrm{gcf}(m,n)}$ and $\frac{n}{\mathrm{gcf}(m,n)}$ are coprime. Thus, the quotient (7.229) is indeed a fraction in lowest terms, by definition. Since m and n were initially arbitrary, we may therefore conclude that the proposed universal holds. □

We end this section with the observation that the set of rational numbers has 'gaps', in the sense of the following proposition.

Proposition 7.51. *It is true that there is no rational number whose square equals* 2*, that is,*

$$\neg\exists q\,(q \in \mathbb{Q} \wedge q^2 = 2). \tag{7.233}$$

Proof. We begin with the observation that the proposed negation is equivalent to the universal sentence

$$\forall q\,(q \in \mathbb{Q} \Rightarrow q^2 \neq 2) \tag{7.234}$$

because of the Negation Law for existential conjunctions. To prove this sentence, we take now an arbitrary rational number q, which we can write in the form

$$q = [(\bar{m}, \bar{n})]_{\sim_q} = \frac{\bar{m}}{\bar{n}}$$

for some particular integers $\bar{m} \in \mathbb{Z}$ and $\bar{n} \in \mathbb{Z} \setminus \{0\}$, according to the definition of a rational number and Notation 7.1. We therefore obtain $\bar{n} \neq 0$ with the definition of a set difference and with (2.169). We consider now the two cases based on the true disjunction $\bar{m} = 0 \vee \bar{m} \neq 0$, implied by the Law of the Excluded Middle. In the first case $\bar{m} = 0$, we obtain because of $\bar{n} \neq 0$

$$[q =] \quad [(\bar{m}, \bar{n})]_{\sim_q} = [(0,1)]_{\sim_q} \quad [= 0]$$

by applying (7.16) in connection with the Law of Contraposition and the notation (7.209). The resulting equation $q = 0$ gives us then

$$q^2 = 0^2 = 0 \cdot_{\mathbb{Q}} 0 = 0$$

by means of (5.478) and the Cancellation Law for 0. Since $0 \neq 2$ is evidently true, we obtain the desired $q^2 \neq 2$ via substitution.

In the second case $\bar{m} \neq 0$, we prove $q^2 \neq 2$ by contradiction, assuming the negation $\neg q^2 \neq 2$ to be true. The Double Negation Law yields then $q^2 = 2$. Due to $\bar{m} \neq 0$ and $\bar{n} \neq 0$, we can write for p

$$q = \frac{\bar{m}}{\bar{n}} = \frac{\frac{\bar{m}}{\operatorname{gcf}(\bar{m},\bar{n})}}{\frac{\bar{n}}{\operatorname{gcf}(\bar{m},\bar{n})}}$$

by virtue of Proposition 7.50, which fraction is in lowest terms. This means by definition that the integers $\bar{M} = \frac{\bar{m}}{\operatorname{gcf}(\bar{m},\bar{n})}$ and $\bar{N} = \frac{\bar{n}}{\operatorname{gcf}(\bar{m},\bar{n})}$ are coprime, so that their greatest common factor is equal to 1 by definition. Next, we derive the equations

$$\begin{aligned} 2 \cdot_{\mathbb{Q}} \bar{N}^2 &= q^2 \cdot_{\mathbb{Q}} \bar{N}^2 = (q \cdot_{\mathbb{Q}} q) \cdot_{\mathbb{Q}} (\bar{N} \cdot_{\mathbb{Q}} \bar{N}) = \left(\frac{\bar{M}}{\bar{N}} \cdot_{\mathbb{Q}} \frac{\bar{M}}{\bar{N}}\right) \cdot_{\mathbb{Q}} (\bar{N} \cdot_{\mathbb{Q}} \bar{N}) \\ &= \left(\frac{\bar{M} \cdot_{\mathbb{Q}} \bar{M}}{\bar{N} \cdot_{\mathbb{Q}} \bar{N}}\right) \cdot_{\mathbb{Q}} (\bar{N} \cdot_{\mathbb{Q}} \bar{N}) = \left[(\bar{M} \cdot_{\mathbb{Q}} \bar{M}) \cdot_{\mathbb{Q}} \frac{1}{\bar{N} \cdot_{\mathbb{Q}} \bar{N}}\right] \cdot_{\mathbb{Q}} (\bar{N} \cdot_{\mathbb{Q}} \bar{N}) \\ &= (\bar{M} \cdot_{\mathbb{Q}} \bar{M}) \cdot_{\mathbb{Q}} \left[\frac{1}{\bar{N} \cdot_{\mathbb{Q}} \bar{N}} \cdot_{\mathbb{Q}} (\bar{N} \cdot_{\mathbb{Q}} \bar{N})\right] = (\bar{M} \cdot_{\mathbb{Q}} \bar{M}) \cdot_{\mathbb{Q}} 1 = \bar{M}^2 \end{aligned}$$

by applying substitution based on the previously found equation $q^2 = 2$, (5.478), substitutions based on the equations for $\bar{M}$ and $\bar{N}$, the law (7.94) of the Calculus of fractions, the notation for quotients (7.86), the Associative Law for the multiplication on $\mathbb{Q}$, (7.63), and finally (5.478) together with the definition of the unity element. All numbers within the resulting equation $2 \cdot_{\mathbb{Q}} \bar{N}^2 = \bar{M}^2$ constitute integers, so that we can also write

$$\bar{M}^2 = 2 \cdot_{\mathbb{Z}} \bar{N}^2.$$

This equation shows that $\bar{M}^2$ is even, so that $\bar{M}$ is also even due to (6.394). This means that there exists an integer, say $\bar{p}$, for which $\bar{M} = 2 \cdot_{\mathbb{Z}} \bar{p}$. Consequently, we evidently have the true equations

$$\begin{aligned} \bar{N}^2 &= 1 \cdot_{\mathbb{Z}} \bar{N}^2 = \left(\frac{1}{2} \cdot_{\mathbb{Z}} 2\right) \cdot_{\mathbb{Z}} \bar{N}^2 = \frac{1}{2} \cdot_{\mathbb{Z}} (2 \cdot_{\mathbb{Z}} \bar{N}^2) = \frac{1}{2} \cdot_{\mathbb{Z}} \bar{M}^2 = \frac{1}{2} \cdot_{\mathbb{Z}} (2 \cdot_{\mathbb{Z}} \bar{p})^2 \\ &= \frac{1}{2} \cdot_{\mathbb{Z}} [(2 \cdot_{\mathbb{Z}} \bar{p}) \cdot_{\mathbb{Z}} (2 \cdot_{\mathbb{Z}} \bar{p})] = \left(\frac{1}{2} \cdot_{\mathbb{Z}} 2\right) \cdot_{\mathbb{Z}} (2 \cdot_{\mathbb{Z}} \bar{p}^2) = 1 \cdot_{\mathbb{Z}} (2 \cdot_{\mathbb{Z}} \bar{p}^2) \\ &= 2 \cdot_{\mathbb{Z}} \bar{p}^2 \end{aligned}$$

Thus, $\bar{N}^2$ is clearly an even integer, which fact implies that $\bar{N}$ is also even, in view of (6.394). This means that we can write $\bar{N}$ as $\bar{N} = 2 \cdot_{\mathbb{Z}} \bar{P}$ for some particular integer $\bar{P}$. Let us now rewrite the equations $\bar{M} = 2 \cdot_{\mathbb{Z}} \bar{p}$ and $\bar{N} = 2 \cdot_{\mathbb{Z}} \bar{P}$ as $\bar{M} = \bar{p} \cdot_{\mathbb{Z}} 2$ and $\bar{N} = \bar{P} \cdot_{\mathbb{Z}} 2$. Noting that the coprime integers $\bar{M}$ and $\bar{N}$ are by definition nonzero, we thus have that 2 is a common factor of these integers. Because 1 is the greatest common factor of $\bar{M}$ and $\bar{N}$, it follows that $1 \geq_{\mathbb{Z}} 2$, so that we find $\neg 1 <_{\mathbb{Z}} 2$ to be true with the Negation Formula for $<$. This finding contradicts the evident fact $1 <_{\mathbb{Z}} 2$, completing the proof of $q^2 \neq 2$ for the second case. As q was initially arbitrary, the universal sentence (7.234) follows therefore to be true, and the equivalent negation (7.233) holds then as well. □

7.4. Countability of $\mathbb{Q}$

Exercise 7.36. Show that the set $\mathbb{Z}_{\mathbb{Q}}$ of integers in $\mathbb{Q}$ is countably infinite.

Theorem 7.52 (Countable infinity of $\mathbb{Q}$). *$\mathbb{Q}$ is countably infinite.*

Proof. We first apply Function definition by replacement to establish a unique function f with domain $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ such that

$$\forall x \, (x \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}] \Rightarrow f(x) = [x]_{\sim_q}). \tag{7.235}$$

For this purpose, we verify

$$\forall x \, (x \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}] \Rightarrow \exists! y \, (y = [x]_{\sim_q})). \tag{7.236}$$

Letting $x \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ be arbitrary, we recall that $\sim_q$ is an equivalence relation on the preceding Cartesian product (see Exercise 7.1b)), so that the equivalence class $[x]_{\sim_q}$ is a uniquely specified set. Thus, the uniquely existential sentence $\exists! y \, (y = [x]_{\sim_q})$ is true according to (1.109), and as x was arbitrary, we may therefore conclude that the universal sentence (7.236) holds. Consequently, there exists indeed a unique function with domain $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ satisfying (7.236). We prove next that the range of this function is given by $\mathbb{Q}$, by applying the Equality Criterion for sets, i.e. by verifying

$$\forall y \, (y \in \mathrm{ran}(f) \Leftrightarrow y \in \mathbb{Q}). \tag{7.237}$$

We let y be arbitrary, and we assume first $y \in \mathrm{ran}(f)$ to be true. By definition of a range, there is then a constant, say $\bar{x}$, such that $(\bar{x}, y) \in f$. By definition of a domain, we thus find $\bar{x} \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ [$= \mathrm{dom}(f)$] to be true, so that (7.235) yields for the corresponding function value $y = f(\bar{x}) = [\bar{x}]_{\sim_q}$ in view of $(\bar{x}, y) \in f$. By definition of a quotient set, we obtain then $y \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}] / \sim_q$, and this implies $y \in \mathbb{Q}$ with the definition of the set of rational numbers. We thus established the first part ('$\Rightarrow$') of the equivalence in (7.237).

To establish the second part ('$\Leftarrow$'), we assume now $y \in \mathbb{Q}$ to be true, which means $y \in \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]/\sim_q$ by definition of $\mathbb{Q}$. According to the definition of a quotient set, there exists then an element of $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$, say $\bar{x}$, for which $[\bar{x}]_{\sim_q} = y$ holds. By definition of the function f, the value associated with $\bar{x}$ is given by $f(\bar{x}) = [\bar{x}]_{\sim_q}$, so that we obtain $y = f(\bar{x})$ through substitution. We can write this equation also in the form $(\bar{x}, y) \in f$, which shows in light of the definition of a range that $y \in \operatorname{ran}(f)$ holds, as desired.

Having thus completed the proof of the equivalence in (7.237), we may now infer from the truth of that equivalence the truth of the universal sentence (7.237), because y was arbitrary. This universal sentence implies then the truth of the suggested equality $\operatorname{ran}(f) = \mathbb{Q}$, which demonstrates that f is a surjection from $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ to $\mathbb{Q}$, that is,

$$f : \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}] \twoheadrightarrow \mathbb{Q}. \tag{7.238}$$

We now recall the Countable Infinity of $\mathbb{Z}$, which implies with Exercise 4.48 that $\mathbb{Z}$ is countable but not finite. This means that $\mathbb{Z}$ is a countable, infinite set. Next, we observe the truth of the inclusion $\mathbb{Z} \setminus \{0\} \subseteq \mathbb{Z}$ in light of (2.125), so that $\mathbb{Z} \setminus \{0\}$ is also a countable set according to (4.655). Furthermore, the fact $0 \in \mathbb{Z}$ implies that the singleton $\{0\}$ is a finite subset of $\mathbb{Z}$ in view of (4.528) and (2.184), so that the infinity of $\mathbb{Z}$ implies the infinity of the set difference $\mathbb{Z} \setminus \{0\}$ with (4.616). Being countable but not finite, $\mathbb{Z} \setminus \{0\}$ is thus countably infinite (again by virtue of the finding of Exercise 4.48). Consequently, the Cartesian product $\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]$ is also countably infinite, according to Exercise 5.62, where (5.624) shows in connection with the definition of equinumerous sets that there exists a bijection from $\mathbb{N}$ to the preceding Cartesian product, say

$$g : \mathbb{N} \rightleftarrows \mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]. \tag{7.239}$$

This means that g is especially a surjection, which fact allows us to apply the Surjectivity of the composition of two surjections in order to obtain also the surjection

$$f \circ g : \mathbb{N} \twoheadrightarrow \mathbb{Q}. \tag{7.240}$$

According to the Countability Criterion (4.653), the existence of such a surjection implies the countability of $\mathbb{Q}$. Let us recall now the truth of the inclusion $\mathbb{Z}_{\mathbb{Q}} \subseteq \mathbb{Q}$ (see Exercise 7.7), where $\mathbb{Z}_{\mathbb{Q}}$ is countably infinite (see Exercise 7.36) and thus infinite (again according to Exercise 4.48). These findings imply then that $\mathbb{Q}$ is infinite, because of (4.615). As a countable and infinite set, $\mathbb{Q}$ is therefore countably infinite. □

Chapter 8.

The Ordered Field of Real Numbers

8.1. The Linear Continuum $(\mathbb{R}, <_{\mathbb{R}})$

Definition 8.1 (Dense subset (of a densely ordered set), separably ordered set). We say for any densely ordered set $(X, <)$ that a set A is a *dense subset* of X (alternatively, that a set A is *dense in* X) iff

1. A is a subset of X, that is,

$$A \subseteq X, \tag{8.1}$$

   and

2. for any two elements x and y in X with $x < y$ there exists an element of A which is strictly between x and y, that is,

$$\forall x, y\,([x, y \in X \,\wedge\, x < y] \Rightarrow \exists z\,(z \in A \,\wedge\, x < z < y)). \tag{8.2}$$

We then say that a densely ordered set $(X, <)$ is *separably ordered* with respect to A iff A is a dense subset of X.

Note 8.1. Whereas the intermediate values of a densely ordered set $(X, <)$ are required to be in X, the intermediate values of a separably ordered set $(X, <)$ are more restrictively required to be in some (dense) subset of X.

Proposition 8.1. *It is true for any densely ordered set* $(X, <)$ *and for any dense subset* A *of* X *that* $(A, <_A)$ *constitutes a densely ordered set, where* $<_A$ *is specified by (3.210).*

Proof. We take arbitrary sets X, $<$ and A, assuming $(X, <)$ to be a densely ordered set and assuming A to be a dense subset of X. Thus, $<$ is a linear ordering, which gives rise to the unique linear ordering $<_A$ of A according to the Linear ordering of subsets and (3.210). Next, we show that A is neither empty nor a singleton, as required by Property 1 of a densely ordered set. For this purpose, we apply (2.183) and prove the equivalent existential sentence

$$\exists x, y\,(x \in A \,\wedge\, y \in A \,\wedge\, x \neq y). \tag{8.3}$$

Since $(X, <)$ is densely ordered, it is by definition true that X is neither empty nor a singleton, so that the existential sentence

$$\exists x, y\,(x \in X \,\wedge\, y \in X \,\wedge\, x \neq y). \tag{8.4}$$

follows to be true with (2.183). Thus, there exist particular elements $\bar{x} \in X$ and $\bar{y} \in X$ such that $\bar{x} \neq \bar{y}$. The dense ordering of $(X, <)$ means by definition also that $<$ is a linear ordering, so that $<$ is connex. Therefore, $\bar{x} \neq \bar{y}$ implies the truth of the disjunction $\bar{x} < \bar{y} \vee \bar{y} < \bar{x}$, which we use now to prove the desired existential sentence (8.3) by cases.

In the first case $\bar{x} < \bar{y}$, we obtain with Property 2 of a densely ordered set a particular element $\bar{z} \in X$ that satisfies $\bar{x} < \bar{z}$ and $\bar{z} < \bar{y}$. Consequently, Property 2 of a dense subset gives on the one hand a particular element $\bar{z}_1 \in A$ satisfying $\bar{x} < \bar{z}_1 < \bar{z}$, and on the other hand a particular element $\bar{z}_2 \in A$ with $\bar{z} < \bar{z}_2 < \bar{y}$. The truth of the inequalities $\bar{z}_1 < \bar{z}$ and $\bar{z} < \bar{z}_2$ implies now with the transitivity of the linear ordering $<$ the truth of $\bar{z}_1 < \bar{z}_2$. Then, because the linear ordering $<$ satisfies the Characterization of comparability, it follows that $\bar{z}_1 = \bar{z}_2$ must be false, so that $\bar{z}_1 \neq \bar{z}_2$ is true. In conjunction with $\bar{z}_1 \in A$ and $\bar{z}_2 \in A$, this demonstrates the truth of the existential sentence (8.3) in the first case.

In the second case $\bar{y} < \bar{x}$, we can use exactly the same line of arguments to establish (8.3). To begin with, $\bar{y} < \bar{x}$ implies the existence of a particular $\bar{z} \in X$ that satisfies $\bar{y} < \bar{z}$ and $\bar{z} < \bar{x}$. Consequently, there are also particular elements $\bar{z}_1 \in A$ and $\bar{z}_2 \in A$ such that $\bar{y} < \bar{z}_1 < \bar{z}$ and $\bar{z} < \bar{z}_2 < \bar{x}$ hold, where $\bar{z}_1 < \bar{z}$ and $\bar{z} < \bar{z}_2$ imply $\bar{z}_1 < \bar{z}_2$ and therefore $\bar{z}_1 \neq \bar{z}_2$. In view of $\bar{z}_1 \in A$ and $\bar{z}_2 \in A$, we thus see that the existential sentence (8.3) holds also in the second case.

Having completed the proof by cases, we can now infer from the truth of (8.3) the truth of the equivalent conjunction $A \neq \emptyset \wedge \forall a\,(A \neq \{a\})$, which means that A is neither empty nor a singleton. Thus, the ordered pair $(A, <_A)$ satisfies Property 1 of a densely ordered set.

Concerning Property 2, we establish the universal sentence

$$\forall x, y\,([x, y \in A \,\wedge\, x <_A y] \Rightarrow \exists z\,(z \in A \,\wedge\, x <_A z <_A y)), \tag{8.5}$$

letting $x, y \in A$ be arbitrary and assuming $x <_A y$ to be true. Because of Property 1 of a dense subset, we have the inclusion $A \subseteq X$, so that $x, y \in A$ implies $x, y \in X$ by definition of a subset. Furthermore, $x, y \in A$

and $x <_A y$ imply $x < y$ with the Irreflexive partial ordering of subsets. Then, the conjunction of $x, y \in X$ and $x < y$ implies with Property 2 of a dense subset that there exists an element in A, say $\bar{z}$, for which $x < \bar{z} < y$ is satisfied. Since x, $\bar{z}$ and y are elements of A, we can write these inequalities also as $x <_A \bar{z} <_A y$, according to the Irreflexive partial ordering of subsets. In connection with $\bar{z} \in A$, these inequalities demonstrate now the truth of the existential sentence in (8.5). As x and y were arbitrary, we can now infer from this the truth of the universal sentence (8.5), which means that $(A, <_A)$ satisfies also Property 2 of a densely ordered set.

Since X, $<$ and A were initially arbitrary, we therefore conclude that the proposition holds. □

Definition 8.2 (Linear continuum, completely ordered set, Supremum Property/Least Upper Bound Property). We say that a densely ordered set $(X, <)$ is a *linear continuum* or a *completely ordered set* iff every nonempty, bounded-from-above subset A of X has a supremum (with respect to the total ordering $\leq$ induced by the linear ordering $<$), i.e. iff

$$\forall A\,([A \subseteq X \ \wedge\ A \neq \emptyset \ \wedge\ \exists u\,(u \in X \ \wedge\ \forall x\,(x \in A \Rightarrow x \leq u))] \\ \Rightarrow \exists S\,(S = \sup A)). \tag{8.6}$$

We call (8.6) the *Supremum Property* or the *Least Upper Bound Property*.

Definition 8.3 (Infimum Property/Greatest Lower Bound Property). We say that a densely ordered set $(X, <)$ has the *Infimum Property* or the *Greatest Lower Bound Property* iff every nonempty, bounded-from-below subset A of X has an infimum (with respect to the total ordering $\leq$ induced by the linear ordering $<$), i.e. iff

$$\forall A\,([A \subseteq X \ \wedge\ A \neq \emptyset \ \wedge\ \exists a\,(a \in X \ \wedge\ \forall x\,(x \in A \Rightarrow a \leq x))] \\ \Rightarrow \exists I\,(I = \inf A)). \tag{8.7}$$

Definition 8.4 (Cut, Dedekind cut, gap). We say for any linearly ordered set $(X, <)$ and for any sets A, B that the ordered pair (A, B) is

(1) a *cut* (in X) iff

1. A and B are subsets of X, i.e.

$$A \subseteq X \ \wedge\ B \subseteq X, \tag{8.8}$$

2. A and B are nonempty, i.e.

$$A \neq \emptyset \ \wedge\ B \neq \emptyset, \tag{8.9}$$

3. A and B are disjoint, i.e.
$$A \cap B = \emptyset, \tag{8.10}$$
4. the union of A and B is identical with X, i.e.
$$A \cup B = X, \tag{8.11}$$
5. every element of A is less than every element of B, i.e.
$$\forall a, b\,([a \in A \,\wedge\, b \in B] \Rightarrow a < b), \tag{8.12}$$

(2) a *Dedekind cut* (in X) iff

1. (A, B) is a cut (in X) and
2. there is no element in X which is the maximum of A, i.e.
$$\neg\exists u\,(u \in X \,\wedge\, u = \max A). \tag{8.13}$$

(3) a *gap* (in X) iff

1. (A, B) is a Dedekind cut and
2. there is no element in X which is the minimum of B, i.e.
$$\neg\exists a\,(a \in X \,\wedge\, a = \min B). \tag{8.14}$$

Proposition 8.2. *It is true for any linearly ordered set $(X, <)$, for any cut (A, B) in X and for any element y in A that every x which is less than y is also contained in A, i.e.*
$$\forall x, y\,([x \in X \,\wedge\, y \in A] \Rightarrow [x < y \Rightarrow x \in A]). \tag{8.15}$$

Proof. We let X, $<$, A, B, x and y be arbitrary, assuming $(X, <)$ to be a linearly ordered set, assuming (A, B) to be a cut in X, assuming x to be an element of X, and assuming y to be an element of A. We prove now the implication $x < y \Rightarrow x \in A$ by contraposition, assuming the negation $x \notin A$ to be true. By Property 4 of a cut, we have $A \cup B = X$, so that the assumed $x \in X$ implies $x \in A \vee x \in B$ with the definition of the union of two sets. Since $x \notin A$ shows that $x \in A$ is false, the second part $x \in B$ of the disjunction must be true. Then, the truth of $y \in A$ and $x \in B$ implies $y < x$ with Property 5 of a cut. Because the linear ordering $<$ satisfies the Characterization of comparability, it follows that $x < y$ is false, which means that the negation $\neg x < y$ is true. This finding completes the proof of the implication $x < y \Rightarrow x \in A$ by contraposition, and as X, $<$, A, B, x and y were initially all arbitrary, we therefore conclude that the proposed universal sentence is true. □

Exercise 8.1. Show for any linearly ordered set $(X, <)$, for any cut (A, B) in X and for any element y in B that every x which is greater than y is also contained in B, i.e.

$$\forall x, y\,([x \in X \wedge y \in B] \Rightarrow [y < x \Rightarrow x \in B]). \tag{8.16}$$

(Hint: The proof is similar to the proof of Proposition 8.2.)

Proposition 8.3. *It is true for any linearly ordered set* $(X, <)$ *and any cut* (A, B) *in* X *that an element* x *of* X *is element of* A *iff* x *is not in* B*, i.e.*

$$\forall x\,(x \in X \Rightarrow [x \in A \Leftrightarrow x \notin B]). \tag{8.17}$$

Proof. Letting X, $<$, A, B and x be arbitrary such that $(X, <)$ is linearly ordered, such that (A, B) is a cut in X and such that $x \in X$ is true, we observe in light of Property 3 and Property 4 of a cut that the equations $A \cap B = \emptyset$ and $A \cup B = X$ hold. Thus, $x \in X$ implies $x \in A \cup B$ by means of substitution and then $x \in A \vee x \in B$ by definition of the union of two sets. Therefore, assuming $x \notin B$ to be true, the second part of the disjunction if false, so that its first part $x \in A$ must be true. Thus, the second part ('$\Leftarrow$') of the equivalence in (8.17) holds. We prove the first part ('$\Rightarrow$') of the equivalence by contradiction, assuming $x \in A$ and $\neg x \notin B$ to be true. The latter yields $x \in B$ with the Double Negation Law, so that $x \in A \wedge x \in B$ holds. Consequently, we obtain $x \in A \cap B$ with the definition of the intersection of two sets, and this finding shows that the intersection $A \cap B$ is nonempty, which is in contradiction to the true equation $A \cap B = \emptyset$. We thus completed the proof of the implication $x \in A \Rightarrow x \notin B$ and thus the proof of the equivalence. Since x was arbitrary, we may therefore conclude that the universal sentence (8.17) is true, and as the sets X, $<$, A and B were initially also arbitrary, we now further conclude that the proposed sentence holds. □

Exercise 8.2. Show for any linearly ordered set $(X, <)$ and any cut (A, B) in X that an element x of X is not element of A iff x is in B, i.e.

$$\forall x\,(x \in X \Rightarrow [x \notin A \Leftrightarrow x \in B]). \tag{8.18}$$

Proposition 8.4. *It is true for any linearly ordered set* $(X, <)$ *and for any cuts* (A, B) *and* (C, D) *in* X *that* D *is included in* B *if* A *is included in* C*, i.e.*

$$A \subseteq C \;\Rightarrow\; D \subseteq B. \tag{8.19}$$

Proof. We let $(X, <)$ be an arbitrary linearly ordered set, we let (A, B) and (C, D) be arbitrary cuts in X, we assume $A \subseteq C$ to be true, which means by definition of a subset

$$\forall x\,(X \in A \Rightarrow x \in C), \tag{8.20}$$

and we show that $D \subseteq B$ is implied, that is,

$$\forall x\,(X \in D \Rightarrow x \in B). \tag{8.21}$$

Letting $x \in D$ be arbitrary and observing the truth of the inclusion $D \subseteq X$ in light of Property 1 of a cut, we obtain $x \in X$ with the definition of a subset and then $x \notin C$ with (8.18). This finding implies with (8.20) and the Law of Contraposition $x \notin A$, which in turn yields $x \in B$ because of (8.18). Thus, the implication in (8.21) holds, in which x is arbitrary, so that the universal sentence (8.21) follows to be true. This gives us $D \subseteq B$ with the definition of a subset, which proves the implication (8.19). As $(X, <)$, (A, B) and (C, D) were arbitrary, we therefore conclude that the proposition holds, as claimed. □

Theorem 8.5 (Characterization of Dedekind cuts). *It is true for any linearly ordered set $(X, <)$, for any Dedekind cut (A, B) in X and for any element x in A that A contains an element y which is greater than x, i.e.*

$$\forall x\,(x \in A \Rightarrow \exists y\,(y \in A \,\wedge\, x < y)). \tag{8.22}$$

Proof. Letting X, $<$, A, B and x be arbitrary such that $(X, <)$ is a linearly ordered set, such that (A, B) is a Dedekind cut in X and such that x is an element of A, we observe that the inclusion $A \subseteq X$ holds according to Property 1 of a cut, so that $x \in A$ implies $x \in X$ by definition of a subset. Furthermore, Property 2 of a Dedekind cut implies with the Negation Law for existential conjunctions the true universal sentence

$$\forall u\,(u \in X \Rightarrow u \neq \max A), \tag{8.23}$$

so that $x \in X$ implies $\neg x = \max A$. This negation implies now with the definition of a maximum

$$\neg[\forall y\,(y \in A \Rightarrow y \leq x) \,\wedge\, x \in A],$$

and this gives us with De Morgan's Law for the conjunction and with the Negation Law for universal implications

$$\exists y\,(y \in A \,\wedge\, \neg y \leq x) \,\vee\, x \notin A.$$

The initial assumption $x \in A$ shows that the second part of this disjunction is false, so that its first part is true, which means that there is a particular element $\bar{y} \in A$ satisfying $\neg\bar{y} \leq x$. This yields by means of the Negation Formula for $\leq$ the true inequality $x < \bar{y}$, and since $\bar{y} \in A$ is also true, we see now that the existential sentence in (8.22) is true. As x was arbitrary, we may infer from this finding the truth of the universal sentence (8.22), and since X, $<$, A, B were initially arbitrary as well, the theorem follows therefore to be true. □

Exercise 8.3. Show for any linearly ordered set $(X, <)$ such that there is no element in X which is the maximum of X, for any Dedekind cut (A, B) in X and for any element x in B that B contains an element y which is greater than x, i.e.

$$\forall x\, (x \in B \Rightarrow \exists y\, (y \in B \,\wedge\, x < y)). \tag{8.24}$$

(Hint: Proceed similarly as in the proof of Theorem 8.5, using here in addition (8.16).)

Exercise 8.4. Establish for any linearly ordered set $(X, <)$ the unique set $\mathcal{D}$ consisting of all the Dedekind cuts in X, in the sense that

$$\forall D\, (D \in \mathcal{D} \Leftrightarrow \exists A, B\, (D = (A, B) \,\wedge\, (A, B) \text{ is a Dedekind cut in } X)). \tag{8.25}$$

Definition 8.5 (Set of real numbers, real number). We call the set of Dedekind cuts in $\mathbb{Q}$ the *set of real numbers*, which we symbolize by

$$\mathbb{R} \tag{8.26}$$

We call every element of $\mathbb{R}$ a *real number.*

With the set of real numbers at our disposal, we can define the following relevant type of function and matrix.

Definition 8.6 (Real-valued function). We call any function $f : X \to \mathbb{R}$ on any set X a *real-valued function* (*on* X).

Definition 8.7 (Real m-by-n matrix). For any positive natural numbers m and n, we call every element of $\mathbb{R}^{m\times n}$ a *real* (m-*by*-n) *matrix.*

Theorem 8.6 (Total ordering of the set of Dedekind cuts). *The following sentences are true for any linearly ordered set* $(X, <)$ *(giving rise to the set* $\mathcal{D}$ *of Dedekind cuts in* X*).*

a) There exists a unique set $\leq_{\mathcal{D}}$ *such that*

$$\begin{aligned}&\forall Z\, (Z \in {\leq_{\mathcal{D}}}\\ &\quad \Leftrightarrow [Z \in \mathcal{D} \times \mathcal{D} \,\wedge\, \exists A, B, C, D\, (Z = ((A, B), (C, D)) \,\wedge\, A \subseteq C)),\end{aligned} \tag{8.27}$$

and this set $\leq_{\mathcal{D}}$ *is a binary relation on* $\mathcal{D}$ *satisfying*

$$\forall A, B, C, D\, ((A, B), (C, D) \in \mathcal{D} \Rightarrow [(A, B) \leq_{\mathcal{D}} (C, D) \Leftrightarrow A \subseteq C]). \tag{8.28}$$

b) Moreover, the binary relation $\leq_{\mathcal{D}}$ is a total ordering of $\mathcal{D}$.

Proof. Letting X and $<$ be arbitrary sets such that $(X,<)$ is linearly ordered, we can prove by means of the Axiom of Specification and the Equality Criterion for sets that there exists a unique set (system) $\leq_{\mathcal{D}}$ such that (8.27). Letting now Z be arbitrary and assuming $Z \in \leq_{\mathcal{D}}$, it follows in particular that $Z \in \mathcal{D} \times \mathcal{D}$ holds. Since Z is arbitrary, we may infer from the truth of this implication the truth of $\leq_{\mathcal{D}} \subseteq \mathcal{D} \times \mathcal{D}$ (according to the definition of a subset), which inclusion reveals that $\leq_{\mathcal{D}}$ is a binary relation on $\mathcal{D}$ (see Note 3.3).

Next, we take arbitrary sets A, B, C and D, assuming the ordered pairs (A,B) and (C,D) to be Dedekind cuts in X, and assuming furthermore the antecedent $(A,B) \leq_{\mathcal{D}} (C,D)$ of the implication '$\Rightarrow$' in (8.28) to be true. Writing this assumption in the form $((A,B),(C,D)) \in \leq_{\mathcal{D}}$, it follows with (8.27) on the one hand that $((A,B),(C,D)) \in \mathcal{D} \times \mathcal{D}$ is true, and there exist on the other hand particular sets $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ such that the equation $((A,B),(C,D)) = ((\bar{A},\bar{B}),(\bar{C},\bar{D}))$ and the inclusion $\bar{A} \subseteq \bar{C}$ are satisfied. Because of the Equality Criterion for ordered pairs, the preceding equation yields first the equations $(A,B) = (\bar{A},\bar{B})$ and $(C,D) = (\bar{C},\bar{D})$, which in turn give $A = \bar{A}$, $B = \bar{B}$, $C = \bar{C}$ and $D = \bar{D}$. Therefore, the preceding inclusion $\bar{A} \subseteq \bar{C}$ implies $A \subseteq C$ by means of substitutions, so that the first part ('$\Rightarrow$') of the equivalence in (8.28) holds.

Regarding the second part ('$\Leftarrow$'), we assume now the inclusion $A \subseteq C$ to be true. Defining now the ordered pair $Z = ((A,B),(C,D))$, we obtain on the one hand from the previous assumptions $(A,B),(C,D) \in \mathcal{D}$ with the definition of the Cartesian product of two sets $Z \in \mathcal{D} \times \mathcal{D}$. On the other hand, we see that there exist constants A, B, C and D satisfying both $Z = ((A,B),(C,D))$ and $A \subseteq C$. These findings imply now $Z \in \leq_{\mathcal{D}}$ with (8.27), that is, $((A,B),(C,D)) \in \leq_{\mathcal{D}}$. Since $\leq_{\mathcal{D}}$ is a binary relation, we can also write $(A,B) \leq_{\mathcal{D}} (C,D)$, which proves the the second part ('$\Leftarrow$') of the equivalence in (8.28). As A, B, C and D were arbitrary, the universal sentence (8.28) follows then to be true, so that the proof of a) is now complete.

We begin the proof of b) with the verification of the reflexivity of the binary relation $\leq_{\mathcal{D}}$, by establishing the truth of the universal sentence

$$\forall a \, (a \in \mathcal{D} \Rightarrow a \leq_{\mathcal{D}} a). \tag{8.29}$$

To do this, we let a be arbitrary, and we assume $a \in \mathcal{D}$ to be true, so that a constitutes the Dedekind cut $a = (\bar{A},\bar{B})$ for some particular sets $\bar{A}$ and $\bar{B}$, according to (8.25). We thus have $(\bar{A},\bar{B}) \in \mathcal{D}$, and $(\bar{A},\bar{B}),(\bar{A},\bar{B}) \in \mathcal{D}$ is then also true. Observing now that $\bar{A} \subseteq \bar{A}$ is true in view of (2.10), we obtain $(\bar{A}, \bar{B}) \leq_{\mathcal{D}} (\bar{A}, \bar{B})$ with (8.28), and this gives via substitutions the desired consequent $a \leq_{\mathcal{D}} a$. Since a was arbitrary, we therefore conclude that the universal sentence (8.29) holds, which means that $\leq_{\mathcal{D}}$ is indeed reflexive.

To prove that $\leq_{\mathcal{D}}$ is antisymmetric, we establish the universal sentence

$$\forall a, b\,(a, b \in \mathcal{D} \Rightarrow [(a \leq_{\mathcal{D}} b \,\wedge\, b \leq_{\mathcal{D}} a) \Rightarrow a = b]), \tag{8.30}$$

letting a and b be arbitrary and assuming $a, b \in \mathcal{D}$ to be true. Consequently, there evidently exist sets, say $\bar{A}$ and $\bar{B}$, such that $a = (\bar{A}, \bar{B})$ is a Dedekind cut, and there are sets, say $\bar{C}$ and $\bar{D}$, such that $b = (\bar{C}, \bar{D})$ is also a Dedekind cut in X. We assume now the inequalities $a \leq_{\mathcal{D}} b$ and $b \leq_{\mathcal{D}} a$ to be true, which we can write after substitutions as $(\bar{A}, \bar{B}) \leq_{\mathcal{D}} (\bar{C}, \bar{D})$ and $(\bar{C}, \bar{D}) \leq_{\mathcal{D}} (\bar{A}, \bar{B})$. Clearly, these inequalities imply the inclusions $\bar{A} \subseteq \bar{C}$ and $\bar{C} \subseteq \bar{A}$, which further imply $\bar{A} = \bar{C}$ with the Axiom of Extension. Because the Dedekind cuts $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ constitute cuts, the elements forming these cuts satisfy $\bar{A} \cup \bar{B} = X$, $\bar{A} \cap \bar{B} = \emptyset$, $\bar{C} \cup \bar{D} = X$ and $\bar{C} \cap \bar{D} = \emptyset$. Due to the Commutative Law for the union of two sets, these equations can be also written as

$$\bar{B} \cup \bar{A} = X \,\wedge\, \bar{B} \cap \bar{A} = \emptyset, \tag{8.31}$$

$$\bar{D} \cup \bar{C} = X \,\wedge\, \bar{D} \cap \bar{C} = \emptyset, \tag{8.32}$$

and these conjunction imply now, respectively, $\bar{B} = X \setminus \bar{A}$ and $\bar{D} = X \setminus \bar{C}$ with (2.262). The previously established equation $\bar{A} = \bar{C}$ allows us to write the latter as $\bar{D} = X \setminus \bar{A}$ $[= \bar{B}]$, with the consequence that $\bar{B} = \bar{D}$ holds. In conjunction with $\bar{A} = \bar{C}$, this implies with the Equality Criterion for ordered pairs $(\bar{A}, \bar{B}) = (\bar{C}, \bar{D})$, which yields through substitutions $a = b$, as desired. Having thus proved the implications in (8.30), we may infer from the truth of these implications the truth of the universal sentence (8.30), since a and b were arbitrary. This finding demonstrates that $\leq_{\mathcal{D}}$ is antisymmetric.

To establish the transitivity of $\leq_{\mathcal{D}}$, we verify

$$\forall a, b, c\,(a, b, c \in \mathcal{D} \Rightarrow [(a \leq_{\mathcal{D}} b \,\wedge\, b \leq_{\mathcal{D}} c) \Rightarrow a \leq_{\mathcal{D}} c]), \tag{8.33}$$

taking arbitrary sets a, b, c such that $a, b, c \in \mathcal{D}$ as well as the inequalities $a \leq_{\mathcal{D}} b$ and $b \leq_{\mathcal{D}} c$ hold. Then, there are particular sets $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$ and $\bar{F}$ forming the Dedekind cuts $a = (\bar{A}, \bar{B})$, $b = (\bar{C}, \bar{D})$ and $c = (\bar{E}, \bar{F})$. The assumed inequalities can thus be written as $(\bar{A}, \bar{B}) \leq_{\mathcal{D}} (\bar{C}, \bar{D})$ and $(\bar{C}, \bar{D}) \leq_{\mathcal{D}} (\bar{E}, \bar{F})$, which imply now the inclusions $\bar{A} \subseteq \bar{C}$ and $\bar{C} \subseteq \bar{E}$. Therefore, the transitivity property (2.13) of '$\subset$' yields $\bar{A} \subseteq \bar{E}$. Since the initial assumption $a, b, c \in \mathcal{D}$ gives via substitutions in particular

$(\bar{A},\bar{B}),(\bar{E},\bar{F})\in\mathcal{D}$, it follows from the preceding inclusion with (8.28) that $(\bar{A},\bar{B})\leq_{\mathcal{D}}(\bar{E},\bar{F})$ is true, so that we obtain $a\leq_{\mathcal{D}}c$ by applying substitutions again. Consequently, the implications in (8.33) hold, in which the sets a, b and c are arbitrary, so that the universal sentence (8.33) turns out to be true, too. Having thus established the transitivity of $\leq_{\mathcal{D}}$, alongside the reflexivity and the antisymmetry, we found $\leq_{\mathcal{D}}$ to be a reflexive partial ordering of $\mathcal{D}$.

It remains for us to prove that $\leq_{\mathcal{D}}$ is total, that is,

$$\forall a,b\,(a,b\in\mathcal{D}\Rightarrow[a\leq_{\mathcal{D}}b\,\vee\,b\leq_{\mathcal{D}}a]). \tag{8.34}$$

Letting a and b be arbitrary and assuming that $a,b\in\mathcal{D}$ is true, we write a and b again in the form of Dedekind cuts $a=(\bar{A},\bar{B})$ and $b=(\bar{C},\bar{D})$, where $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are particular sets. We therefore have $(\bar{A},\bar{B}),(\bar{C},\bar{D})\in\mathcal{D}$, and we can write for the desired disjunction equivalently

$$(\bar{A},\bar{B})\leq_{\mathcal{D}}(\bar{C},\bar{D})\,\vee\,(\bar{C},\bar{D})\leq_{\mathcal{D}}(\bar{A},\bar{B}), \tag{8.35}$$

which is evidently also equivalent to the disjunction

$$\bar{A}\subseteq\bar{C}\,\vee\,\bar{C}\subseteq\bar{A} \tag{8.36}$$

in view of (8.28). We prove this disjunction by contradiction, assuming its negation to be true, so that we obtain the true conjunction

$$[\neg\bar{A}\subseteq\bar{C}]\,\wedge\,[\neg\bar{C}\subseteq\bar{A}]$$

with De Morgan's Law for the disjunction. This conjunction implies with the definition of a subset

$$[\neg\forall x\,(x\in\bar{A}\Rightarrow x\in\bar{C})]\,\wedge\,[\neg\forall y\,(y\in\bar{C}\Rightarrow y\in\bar{A})],$$

which further implies with the Negation Law for universal implications

$$\exists x\,(x\in\bar{A}\,\wedge\,x\notin\bar{C})\,\wedge\,\exists y\,(y\in\bar{C}\,\wedge\,y\notin\bar{A}).$$

Thus, there are constants, say $\bar{x}$ and $\bar{y}$, which satisfy $\bar{x}\in\bar{A}$, $\bar{x}\notin\bar{C}$, $\bar{y}\in\bar{C}$ and $\bar{y}\notin\bar{A}$. Because $(\bar{A},\bar{B})$ and $(\bar{C},\bar{D})$ are cuts, the inclusions $\bar{A}\subseteq X$ and $\bar{C}\subseteq X$ are true by definition, so that $\bar{x}\in\bar{A}$ and $\bar{y}\in\bar{C}$ imply $\bar{x}\in X$ and $\bar{y}\in X$ with the definition of a subset. These findings give us with the definition of a set difference $\bar{x}\in X\setminus\bar{C}$ and $\bar{y}\in X\setminus\bar{A}$. As shown in the verification of the antisymmetry of $\leq_{\mathcal{D}}$, the conjunctions (8.31) and (8.32) are true for the arbitrarily selected Dedekind cuts $(\bar{A},\bar{B})$ and $(\bar{C},\bar{D})$, so that we obtain $\bar{B}=X\setminus\bar{A}$ and $\bar{D}=X\setminus\bar{C}$ with (2.262) also here in the current verification of the totality of $\leq_{\mathcal{D}}$. Consequently, we obtain through substitutions $\bar{x} \in \bar{D}$ and $\bar{y} \in \bar{B}$. By definition of a cut, $\bar{x} \in \bar{A}$ and $\bar{y} \in \bar{B}$ imply $\bar{x} < \bar{y}$, whereas $\bar{y} \in \bar{C}$ and $\bar{x} \in \bar{D}$ imply $\bar{y} < \bar{x}$. However, since the linear ordering $<$ is comparable, $\bar{x} < \bar{y}$ and $\bar{y} < \bar{x}$ are not simultaneously true, so that we arrived at a contradiction. We thus proved the disjunction (8.36), and the equivalent disjunctions (8.35) and $a \leq_{\mathcal{D}} b \vee b \leq_{\mathcal{D}} a$ are then true as well. Since a and b were arbitrary, we therefore conclude that the universal sentence (8.34), which completes the proof that the reflexive partial ordering $\leq_{\mathcal{D}}$ is total. Thus, $\leq_{\mathcal{D}}$ constitutes by definition a total ordering of $\mathcal{D}$.

Initially, the sets X and $<$ were also arbitrary, so that the theorem is indeed true. □

Note 8.2. The totally ordered set $(\mathcal{D}, \leq_{\mathcal{D}})$ induces the linearly ordered set $(\mathcal{D}, <_{\mathcal{D}})$.

Theorem 8.7 (Characterization of the linear ordering of Dedekind cuts). *It is true for any linearly ordered set $(X,<)$ that a Dedekind cut (A,B) is less than a Dedekind cut (C,D) iff there is an element in C which is not in A, i.e.*

$$\exists x\,(x \notin A \wedge x \in C) \;\Leftrightarrow\; (A,B) <_{\mathcal{D}} (C,D) \tag{8.37}$$

Proof. We take arbitrary sets X, $<$, A, B, C and D such that $(X,<)$ is a linearly ordered set and such that (A,B) and (C,D) are Dedekind cuts in X. To prove the first part ('$\Rightarrow$') of the equivalence directly, we assume that there exists a constant, say $\bar{x}$, such that $\bar{x} \notin A$ and $\bar{x} \in C$ are both true. Then, the disjunction

$$[\bar{x} \in A \wedge \bar{x} \notin C] \vee [\bar{x} \in C \wedge \bar{x} \notin A]$$

is also true, and the existence of such an element $\bar{x}$ implies therefore $A \neq C$ with (2.23). Consequently, the disjunction $(\neg A = C) \vee (\neg B = D)$ holds then also, so that the negation $\neg(A = C \wedge B = D)$ follows to be true with De Morgan's Law for the conjunction. This further implies $(A,B) \neq (C,D)$ with the Equality Criterion for ordered pairs and the Law of Contraposition. Let us establish now the truth of the inclusion $A \subseteq C$, by proving the universal sentence

$$\forall y\,(y \in A \Rightarrow y \in C). \tag{8.38}$$

We take an arbitrary element $y \in A$ (noting that A is nonempty according to Property 2 of a cut), and we recall the truth of $\bar{x} \notin A$, so that $y \neq \bar{x}$ follows to be true with (2.4). Observing that $y \in A$ and $\bar{x} \in C$ imply $y \in X$ and $\bar{x} \in X$ with Property 1 of a cut, and noting that the linear ordering

$<$ is connex, it follows from $y \neq \bar{x}$ that the disjunction $y < \bar{x} \lor \bar{x} < y$ is true. Here, we can prove by contradiction that $\bar{x} < y$ is false, i.e. that the negation $\neg \bar{x} < y$ is true. Assuming the negation of that negation to be true, we obtain $\bar{x} < y$ with the Double Negation Law, and this inequality implies because of $\bar{x} \in X$ and $y \in A$ that $\bar{x} \in A$ holds, in view of (8.15). This finding contradicts the previously established negation $\bar{x} \notin A$, so that $\bar{x} < y$ is indeed false. Thus, $y < \bar{x}$ is true, and since $y \in X$ and $\bar{x} \in C$ also hold, we obtain now $y \in C$ by applying (8.15) again. This proves the implication in (8.38), in which y is arbitrary, so that the universal sentence (8.38) follows to be true. This in turn implies the suggested inclusion $A \subseteq C$ by definition of a subset. According to the specification of the Total ordering of the set of Dedekind cuts in (8.28), this inclusion implies $(A, B) \leq_{\mathcal{D}} (C, D)$. In conjunction with the true inequality $(A, B) \neq (C, D)$, this gives us $(A, B) <_{\mathcal{D}} (C, D)$ with the definition of an induced irreflexive partial ordering, so that the proof of the implication '$\Rightarrow$' in (8.37) is now complete.

We prove the converse implication '$\Leftarrow$' by contraposition, assuming the negation of the existential sentence in (8.37) to be true. Consequently, the Negation Law for existential sentences yields the true universal sentence

$$\forall x\,(\neg[x \notin A \,\land\, x \in C]), \tag{8.39}$$

which allows us to establish the truth of

$$\forall x\,(x \in C \Rightarrow x \in A). \tag{8.40}$$

Letting x be arbitrary, we obtain with (8.39) the true negation $\neg[x \notin A \land x \in C]$, and therefore the true disjunction $x \in A \lor x \notin C$ with De Morgan's Law for the conjunction and the Double Negation Law. We are now in a position to prove the implication $x \in C \Rightarrow x \in A$ directly. Assuming for this purpose $x \in C$ to be true, we see that the second part $x \notin C$ of the preceding true disjunction is false, so that its first part $x \in A$ must be true. Thus, the implication is true, in which x is arbitrary, so that (8.40) follows to be true. This universal sentence implies now the inclusion $C \subseteq A$ with the definition of a subset. As Dedekind cuts, (C, D) and (A, B) are elements of $\mathcal{D}$, so that the preceding inclusion implies $(C, D) \leq_{\mathcal{D}} (A, B)$ with (8.28). Since $<_{\mathcal{D}}$ is a linear ordering, we may apply now the Negation Formula for $<$ to obtain the negation $\neg(A, B) <_{\mathcal{D}} (C, D)$, which completes the proof of the implication '$\Leftarrow$' via contraposition.

Because the sets X, $<$, A, B, C and D were arbitrary, we may infer from the truth of that implication finally the truth of the proposed universal sentence. $\square$

Definition 8.8 (Standard total & standard linear ordering of $\mathbb{R}$). We call the total ordering

$$\leq_{\mathbb{R}} \tag{8.41}$$

of the set of real number $\mathbb{R}$ the *standard total ordering of* $\mathbb{R}$, and we call the induced linear ordering

$$<_{\mathbb{R}} \tag{8.42}$$

of $\mathbb{R}$ the *standard linear ordering of* $\mathbb{R}$.

Theorem 8.8. *For any densely ordered set $(X, <)$ such that there are no elements a and u in X with $a = \min X$ and $u = \max X$, it is true that there is a unique function $f_X^{\mathcal{D}}$ with domain X such that*

$$\forall x\,(x \in X \Rightarrow f_X^{\mathcal{D}}(x) = ((-\infty, x)_X, [x, +\infty)_X)\,), \tag{8.43}$$

and $f_X^{\mathcal{D}}$ is an injection from X to the set $\mathcal{D}$ of Dedekind cuts in X.

Proof. We let X and $<$ be arbitrary sets, assuming $(X, <)$ to be densely ordered and assuming that neither the minimum nor the maximum of X exist. We first apply Function definition by replacement and prove the universal sentence

$$\forall x\,(x \in X \Rightarrow \exists! y\,(y = ((-\infty, x)_X, [x, +\infty)_X)\,)). \tag{8.44}$$

Letting x be arbitrary and assuming $x \in X$ to be true, we can use the total ordering $\leq$ of X induced by $<$ to define the unbounded intervals $(-\infty, x)_X$ and $[x, +\infty)_X$. Forming then the the ordered pair $((-\infty, x)_X, [x, +\infty)_X)$, we see that the uniquely existential sentence in (8.44) holds according to (1.109). As x was arbitrary, we may therefore conclude that the universal sentence (8.44) is true, so that there exists indeed a unique function $f_X^{\mathcal{D}}$ with domain X such that (8.43) is satisfied.

To establish $\mathcal{D}$ as a codomain of that function, we show that the range of $f_X^{\mathcal{D}}$ is a subset of the set $\mathcal{D}$ of Dedekind cuts, and we prove accordingly

$$\forall y\,(y \in \operatorname{ran}(f_X^{\mathcal{D}}) \Rightarrow y \in \mathcal{D}). \tag{8.45}$$

We let y be arbitrary such that $y \in \operatorname{ran}(f_X^{\mathcal{D}})$ holds, which assumption implies with the definitions of a range and of a domain that there exists an element in $\operatorname{dom}(f_X^{\mathcal{D}}) = X$, say $\bar{x}$, such that $(\bar{x}, y) \in f_X^{\mathcal{D}}$. As $f_X^{\mathcal{D}}$ is a function defined by (8.43), this yields

$$y = f_X^{\mathcal{D}}(\bar{x}) = ((-\infty, \bar{x})_X, [\bar{x}, +\infty)_X).$$

We verify now that the sets $\bar{A} = (-\infty, \bar{x})_X$ and $\bar{B} = [\bar{x}, +\infty)_X$ have all the properties in order for $y = (\bar{A}, \bar{B})$ to be a Dedekind cut in X. To begin with, Exercise 3.56b) shows that the intervals $(-\infty, \bar{x})_X$ and $[\bar{x}, +\infty)_X$ are subsets of X, so that Property 1 of a cut in X is satisfied by $(\bar{A}, \bar{B})$. Furthermore, $(-\infty, \bar{x})_X \neq \emptyset$ and $[\bar{x}, +\infty)_X \neq \emptyset$ are true according to, respectively, Exercise 3.65b) and Exercise 3.56d), so that $(\bar{A}, \bar{B})$ satisfies also Property 2 of a cut. Moreover, we obtain the equations

$$\begin{aligned}(-\infty, \bar{x})_X \cap [\bar{x}, +\infty)_X &= (-\infty, \bar{x})_X \cap (-\infty, \bar{x})_X^c = \emptyset \\ (-\infty, \bar{x})_X \cup [\bar{x}, +\infty)_X &= (-\infty, \bar{x})_X \cup (-\infty, \bar{x})_X^c = X\end{aligned}$$

by applying (3.449), (2.135) and (2.257), which demonstrate that $(\bar{A}, \bar{B})$ has Property 3 and Property 4 of a cut in X. Regarding Property 5, we verify

$$\forall a, b\,([a \in (-\infty, \bar{x})_X \,\wedge\, b \in [\bar{x}, +\infty)_X] \Rightarrow a < b), \tag{8.46}$$

letting a be an arbitrary element of the interval $(-\infty, \bar{x})_X$ and b an arbitrary element of the interval $[\bar{x}, +\infty)_X$. According to the specifications of an open, left-unbounded interval and of a left-closed, right-unbounded interval in X (see Exercise 3.56c)), we therefore have the inequalities $a < \bar{x}$ and $\bar{x} \leq b$, which imply with the Transitivity Formula for $<$ and $\leq$ that the desired inequality $a < b$ is indeed true. Because a and b are arbitrary, we can infer from this finding the truth of the universal sentence (8.46), so that we established $y = (\bar{A}, \bar{B})$ as a cut in X. It remains for us to establish Property 2 of a Dedekind cut in X, which task requires the verification of

$$\neg\exists u\,(u \in X \,\wedge\, u = \max \bar{A}). \tag{8.47}$$

We prove this negation by contradiction, assuming the negation of that negation to be true. Because of the Double Negation Law, it is therefore true that there exists an element in X, say $\bar{u}$, which is the maximum of $\bar{A} = (-\infty, \bar{x})_X$. By definition of a maximum, this means that the conjunction

$$\forall y\,(y \in (-\infty, \bar{x})_X \Rightarrow y \leq \bar{u}) \,\wedge\, \bar{u} \in (-\infty, \bar{x})_X \tag{8.48}$$

holds, whose second part implies now $\bar{u} \in X$ and $\bar{u} < \bar{x}$ with the definition of an open and right-unbounded interval. In view of the assumption that $(X, <)$ is densely ordered, it follows from the preceding findings $\bar{u}, \bar{x} \in X$ and $\bar{u} < \bar{x}$ that there exists a particular element $\bar{z} \in X$ such that $\bar{u} < \bar{z} < \bar{x}$. On the one hand, $\bar{u} < \bar{z}$ yields $\neg \bar{z} \leq \bar{u}$ with the Negation Formula for $\leq$, on the other hand $\bar{z} < \bar{x}$ gives us evidently $\bar{z} \in (-\infty, \bar{x})_X$ and therefore $\bar{z} \leq \bar{u}$ by virtue of the first part of the conjunction (8.48). We thus arrived at a contradiction, so that the proof of (8.47) is now complete. This means that $y = (\bar{A}, \bar{B})$ is a Dedekind cut in X, which finding implies that y is an element of the set $\mathcal{D}$ specified by (8.25). Since y was arbitrary, we may therefore conclude that the universal sentence (8.45) holds, which in turn implies the truth of the inclusion $\mathrm{ran}(f_X^{\mathcal{D}}) \subseteq \mathcal{D}$ by means of the definition of a subset. This inclusion demonstrates by definition of a codomain that $f_X^{\mathcal{D}}$ constitutes a function from X to $\mathcal{D}$.

Finally, we verify that the function $f_X^{\mathcal{D}} : X \to \mathcal{D}$ constitutes an injection, that is,

$$\forall x, x' \, ([x, x' \in X \,\wedge\, f_X^{\mathcal{D}}(x) = f_X^{\mathcal{D}}(x')] \Rightarrow x = x'). \tag{8.49}$$

We let for this purpose $x, x' \in X$ be arbitrary and assume the equation $f_X^{\mathcal{D}}(x) = f_X^{\mathcal{D}}(x')$ to be true. The values occurring in this equation are given by $f_X^{\mathcal{D}}(x) = ((-\infty, x)_X, [x, +\infty)_X)$ and $f_X^{\mathcal{D}}(x') = ((-\infty, x')_X, [x', +\infty)_X)$, so that the assumed equation implies with the Equality Criterion for ordered pairs in particular

$$(-\infty, x)_X = (-\infty, x')_X. \tag{8.50}$$

Since the linear ordering $<$ is by definition connex, the disjunction $x < x' \vee x' < x \vee x = x'$ is true, for which we now show by two consecutive proofs by contradiction that $x < x'$ and $x' < x$ are false, in other words, that $\neg x < x'$ and $\neg x' < x$ are true. Regarding the first of these negations, we assume its negation to be true, so that $x < x'$ holds (due to the Double Negation Law). By definition of an open, left-unbounded interval, we therefore find $x \in (-\infty, x')$ to be true, and this implies via substitution based on (8.50) $x \in (-\infty, x)$. Consequently, we obtain $x < x$ (using again the preceding interval definition), in contradiction to the fact that the negation $\neg x < x$ holds by virtue of the irreflexivity of the linear ordering $<$. We thus proved $\neg x < x'$, and we can establish $\neg x' < x$ by using similar arguments. Indeed, assuming the negation of that negation to be true, we get $x' < x$ and therefore $x' \in (-\infty, x)_X$, so that substitution yields $x' \in (-\infty, x')_X$. This in turn gives us $x' < x'$, in contradiction to the true negation $\neg x' < x'$ (implied with the irreflexivity of $<$). Having established the two negations $\neg x < x'$ and $\neg x' < x$, we see now indeed that the first two parts of the true disjunction $x < x' \vee x' < x \vee x = x'$ are false, so that $x = x'$ must be true. This equation proves the implication in (8.49), in which x and x' are arbitrary, so that the universal sentence (8.49) follows now to be true. This means that $f_X^{\mathcal{D}} : X \to \mathcal{D}$ is an injection, as claimed.

Because X and $<$ were initially arbitrary sets, we may therefore conclude that the stated theorem is true. □

Note 8.3. Writing the range of the injection $f_X^{\mathcal{D}} : X \hookrightarrow \mathcal{D}$ as the image $X_{\mathcal{D}} = f_X^{\mathcal{D}}[X]$ of the domain X (as shown in Corollary 3.216), we obtain the bijection

$$f_X^{\mathcal{D}} : X \rightleftarrows X_{\mathcal{D}}$$

according to Corollary 3.204, which allows us to unambiguously identify the elements of X within the set of Dedekind cuts. The back-transformation of Dedekind cuts representing elements of X is accomplished by means of the inverse bijection (exploiting the Bijectivity of inverse functions)

$$[f_X^{\mathcal{D}}]^{-1} : f_X^{\mathcal{D}}[X] \rightleftarrows X. \tag{8.51}$$

We immediately find the following special case by recalling that $(\mathbb{Q}, <_{\mathbb{Q}})$ is densely ordered (see Corollary 7.41) and that $\mathbb{Q}$ neither has a minimum nor a maximum (see Corollary 7.45).

Corollary 8.9 (Identification of $\mathbb{Q}$ in $\mathbb{R}$). *it is true that there exists a unique function $f_{\mathbb{Q}}^{\mathbb{R}}$ with domain $\mathbb{Q}$ such that*

$$\forall p\,(p \in \mathbb{Q} \Rightarrow f_{\mathbb{Q}}^{\mathbb{R}}(p) = ((-\infty, p)_{\mathbb{Q}}, [p, +\infty)_{\mathbb{Q}})\,), \tag{8.52}$$

and $f_{\mathbb{Q}}^{\mathbb{R}}$ is an injection from $\mathbb{Q}$ to $\mathbb{R}$.

Notation 8.1. We thus have in particular that the image of $\mathbb{Q}$ under $f_{\mathbb{Q}}^{\mathbb{R}}$ is given by $\mathbb{Q}_{\mathbb{R}}$, that is,

$$\mathbb{Q}_{\mathbb{R}} = f_{\mathbb{Q}}^{\mathbb{R}}[\mathbb{Q}], \tag{8.53}$$

and we will call its elements *rational numbers in* $\mathbb{R}$. The bijection in Note 8.3 can then be written in the form

$$f_{\mathbb{Q}}^{\mathbb{R}} : \mathbb{Q} \rightleftarrows \mathbb{Q}_{\mathbb{R}}. \tag{8.54}$$

For simplicity, we will usually omit the explicit reference to $\mathbb{R}$ and set

$$\mathbb{Q} = \mathbb{Q}_{\mathbb{R}}. \tag{8.55}$$

Exercise 8.5. Establish the inclusion

$$\mathbb{Q}_{\mathbb{R}} \subseteq \mathbb{R}. \tag{8.56}$$

(Hint: Proceed as in Exercise 6.18.)

Note 8.4. Due to the inclusion $\mathbb{Q}_{\mathbb{R}} \subseteq \mathbb{R}$, the total ordering $\leq_{\mathbb{R}}$ of the set of real numbers induces also the total ordering $\leq_{\mathbb{Q}_{\mathbb{R}}}$ of the set of rational numbers in $\mathbb{R}$ (according to the Total ordering of subsets), which satisfies (according to the Reflexive partial ordering of subsets)

$$\forall p, q\,(p, q \in \mathbb{Q}_{\mathbb{R}} \Rightarrow [p \leq_{\mathbb{Q}_{\mathbb{R}}} q \Leftrightarrow p \leq_{\mathbb{R}} q]). \tag{8.57}$$

The total ordering $\leq_{\mathbb{Q}_{\mathbb{R}}}$ induces then the linear ordering $<_{\mathbb{Q}_{\mathbb{R}}}$.

Exercise 8.6. Show that the induced linear ordering $<_{\mathbb{Q}_\mathbb{R}}$ satisfies

$$\forall p, q\,(p, q \in \mathbb{Q}_\mathbb{R} \Rightarrow [p <_{\mathbb{Q}_\mathbb{R}} q \Leftrightarrow p <_\mathbb{R} q]). \tag{8.58}$$

(Hint: Proceed similarly as in Exercise 7.30.)

Theorem 8.10 (Order-embedding from $(\mathbb{Q}, \leq_\mathbb{Q})$ to $(\mathbb{R}, \leq_\mathbb{R})$ and order-isomorphism from $(\mathbb{Q}, \leq_\mathbb{Q})$ to $(\mathbb{Q}_\mathbb{R}, \leq_{\mathbb{Q}_\mathbb{R}})$). *It is true that the function $f_\mathbb{Q}^\mathbb{R}$ defining the Identification of $\mathbb{Q}$ in $\mathbb{R}$ constitutes*

a) an order-embedding from $(\mathbb{Q}, \leq_\mathbb{Q})$ to $(\mathbb{R}, \leq_\mathbb{R})$, that is,

$$f_\mathbb{Q}^\mathbb{R} : (\mathbb{Q}, \leq_\mathbb{Q}) \hookrightarrow (\mathbb{R}, \leq_\mathbb{R}), \qquad p \mapsto ((-\infty, p)_\mathbb{Q}, [p, +\infty)_\mathbb{Q}). \tag{8.59}$$

b) an order-isomorphism from $(\mathbb{Q}, \leq_\mathbb{Q})$ to $(\mathbb{Q}_\mathbb{R}, \leq_{\mathbb{Q}_\mathbb{R}})$, that is,

$$f_\mathbb{Q}^\mathbb{R} : (\mathbb{Q}, \leq_\mathbb{Q}) \rightleftarrows (\mathbb{Q}_\mathbb{R}, \leq_{\mathbb{Q}_\mathbb{R}}), \qquad p \mapsto ((-\infty, p)_\mathbb{Q}, [p, +\infty)_\mathbb{Q}). \tag{8.60}$$

Proof. We first prove the universal sentence

$$\forall p, q\,(p, q \in \mathbb{Q} \Rightarrow [p \leq_\mathbb{Q} q \Leftrightarrow f_\mathbb{Q}^\mathbb{R}(p) \leq_\mathbb{R} f_\mathbb{Q}^\mathbb{R}(q)]), \tag{8.61}$$

letting p and q be arbitrary rationals, so that the definition of $f_\mathbb{Q}^\mathbb{R}$ yields the corresponding values

$$\begin{aligned} f_\mathbb{Q}^\mathbb{R}(p) &= ((-\infty, p)_\mathbb{Q}, [p, +\infty)_\mathbb{Q}) \\ f_\mathbb{Q}^\mathbb{R}(q) &= ((-\infty, q)_\mathbb{Q}, [q, +\infty)_\mathbb{Q}). \end{aligned}$$

We obtain then the true equivalences

$$\begin{aligned} f_\mathbb{Q}^\mathbb{R}(p) \leq_\mathbb{R} f_\mathbb{Q}^\mathbb{R}(q) &\Leftrightarrow ((-\infty, p)_\mathbb{Q}, [p, +\infty)_\mathbb{Q}) \leq_\mathbb{R} ((-\infty, q)_\mathbb{Q}, [q, +\infty)_\mathbb{Q}) \\ &\Leftrightarrow (-\infty, p)_\mathbb{Q} \subseteq (-\infty, q)_\mathbb{Q} \\ &\Leftrightarrow p \leq_\mathbb{Q} q \end{aligned}$$

by applying substitutions, then the total ordering of the set of Dedekind cuts (applied to $\mathbb{R}$), and finally (3.457) in connection with the linearly ordered set $(\mathbb{Q}, <_\mathbb{Q})$. These equivalences prove the implication in (8.61), with the consequence that the universal sentence (8.61) is true, since p and q were arbitrary. Thus, $f_\mathbb{Q}^\mathbb{R}$ is indeed an order-embedding from $(\mathbb{Q}, \leq_\mathbb{Q})$ to $(\mathbb{R}, \leq_\mathbb{R})$.

Since $f_\mathbb{Q}^\mathbb{R}$ is a bijection from $\mathbb{Q}$ to $\mathbb{Q}_\mathbb{R}$ (see Notation 8.1), it is by definition a surjection. Next, we prove that $f_\mathbb{Q}^\mathbb{R}$ is an order-embedding from $(\mathbb{Q}, \leq_\mathbb{Q})$ to $(\mathbb{Q}_\mathbb{R}, \leq_{\mathbb{Q}_\mathbb{R}})$, by verifying

$$\forall p, q\,(p, q \in \mathbb{Q} \Rightarrow [p \leq_\mathbb{Q} q \Leftrightarrow f_\mathbb{Q}^\mathbb{R}(p) \leq_{\mathbb{Q}_\mathbb{R}} f_\mathbb{Q}^\mathbb{R}(q)]). \tag{8.62}$$

We let p and q be arbitrary rationals and observe that the corresponding values $f_{\mathbb{Q}}^{\mathbb{R}}(p)$ and $f_{\mathbb{Q}}^{\mathbb{R}}(q)$ are elements of the range $\mathbb{Q}_{\mathbb{R}}$ of the bijection/surjection $f_{\mathbb{Q}}^{\mathbb{R}} : \mathbb{Q} \rightleftarrows \mathbb{Q}_{\mathbb{R}}$. We obtain the true equivalences

$$\begin{aligned} p \leq_{\mathbb{Q}} q \; &\Leftrightarrow \; f_{\mathbb{Q}}^{\mathbb{R}}(p) \leq_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(q) \\ &\Leftrightarrow \; f_{\mathbb{Q}}^{\mathbb{R}}(p) \leq_{\mathbb{Q}_{\mathbb{R}}} f_{\mathbb{Q}}^{\mathbb{R}}(q) \end{aligned}$$

with (8.61) and (3.204) – according to the Reflexive partial ordering of subsets. Therefore, the equivalence in (8.62) follows to be true, and because p and q were initially arbitrary, we may infer from this the truth of the universal sentence (8.62). We thus proved that $f_{\mathbb{Q}}^{\mathbb{R}}$ is a surjective order-embedding from $(\mathbb{Q}, \leq_{\mathbb{Q}})$ to $(\mathbb{Q}_{\mathbb{R}}, \leq_{\mathbb{Q}_{\mathbb{R}}})$, and this is an order-isomorphism by definition. □

Exercise 8.7. Establish the truth of the universal sentence

$$\forall p, q\, (p, q \in \mathbb{Q} \Rightarrow [p <_{\mathbb{Q}} q \Leftrightarrow f_{\mathbb{Q}}^{\mathbb{R}}(p) <_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(q)]). \tag{8.63}$$

(Hint: Use Definition 3.24, the Injection Criterion, and Proposition 3.149.)

Note 8.5. The inequality $0 <_{\mathbb{Q}} 1$ from (7.211) implies

$$f_{\mathbb{Q}}^{\mathbb{R}}(0) <_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(1). \tag{8.64}$$

Theorem 8.11 (Completion of densely ordered sets). *For any densely ordered set $(X, <)$ such that there are no elements a and u in X with $a = \min X$ and $u = \max X$, it is true that*

a) the induced linearly ordered set $(\mathcal{D}, <_{\mathcal{D}})$ is densely ordered,

b) the densely ordered set $(\mathcal{D}, <_{\mathcal{D}})$ is separably ordered with respect to the image $X_{\mathcal{D}}$ of X under the function $f_X^{\mathcal{D}}$,

c) the separably ordered set $(\mathcal{D}, <_{\mathcal{D}})$ is a linear continuum,

d) the linear continuum $(\mathcal{D}, <_{\mathcal{D}})$ has the Infimum Property.

Proof. We let X and $<$ be arbitrary sets, we assume that $<$ is a linear ordering of X, we assume that $(X, <)$ is densely ordered, and we assume that neither the minimum nor the maximum of X exists. Here, the linearly ordered set $(X, <)$ gives rise to the Total ordering of the set of Dedekind cuts in X, and the resulting totally ordered set $(\mathcal{D}, \leq_{\mathcal{D}})$ induces the linearly ordered set $(\mathcal{D}, <_{\mathcal{D}})$.

Concerning a), we begin with the verification of the requirement that $\mathcal{D}$

is neither empty nor a singleton, which requirement is equivalent to the universal sentence

$$\exists D, E\, (D \in \mathcal{D} \wedge E \in \mathcal{D} \wedge D \neq E) \tag{8.65}$$

because of (2.183). Since $(X, <)$ is densely ordered, X is neither empty nor a singleton, so that the preceding argument gives us the true existential sentence

$$\exists x, y\, (x \in X \wedge y \in X \wedge x \neq y)$$

and therefore particular elements $\bar{x}, \bar{y} \in X$ with $\bar{x} \neq \bar{y}$. Thus, $\bar{x}$ and $\bar{y}$ are in the domain of the injection $f_X^{\mathcal{D}} : X \hookrightarrow \mathcal{D}$ defined by (8.43), so that the corresponding values

$$\bar{D} = f_X^{\mathcal{D}}(\bar{x}) = ((-\infty, \bar{x}), [\bar{x}, +\infty))$$
$$\bar{E} = f_X^{\mathcal{D}}(\bar{y}) = ((-\infty, \bar{y}), [\bar{y}, +\infty))$$

are elements of the codomain $\mathcal{D}$ due to the Function Criterion. In view of $\bar{x}, \bar{y} \in X$ and $\bar{x} \neq \bar{y}$, it follows with the Injection Criterion that $f_X^{\mathcal{D}}(\bar{x}) \neq f_X^{\mathcal{D}}(\bar{y})$ holds, so that substitutions give us $\bar{D} \neq \bar{E}$. In conjunction with the aforementioned fact $\bar{D}, \bar{E} \in \mathcal{D}$, this inequality proves the existential sentence (8.65), so that $\mathcal{D}$ is indeed neither empty nor a singleton. Thus, $(\mathcal{D}, <_{\mathcal{D}})$ satisfies Property 1 of a densely ordered set. To establish Property 2, we verify the universal sentence

$$\forall D, E\, ([D, E \in \mathcal{D} \wedge D <_{\mathcal{D}} E] \Rightarrow \exists Z\, (Z \in \mathcal{D} \wedge D <_{\mathcal{D}} Z <_{\mathcal{D}} E)), \tag{8.66}$$

letting D and E be arbitrary elements of $\mathcal{D}$ such that $D <_{\mathcal{D}} E$ is satisfied. Due to (8.25), there are then particular sets $\bar{a}$, $\bar{b}$, $\bar{A}$ and $\bar{B}$ such that $D = (\bar{a}, \bar{b})$ and $E = (\bar{A}, \bar{B})$ are Dedekind cuts in X. We can therefore rewrite the preceding inequality as $(\bar{a}, \bar{b}) <_{\mathcal{D}} (\bar{A}, \bar{B})$. We now see in light of the Characterization of the linear ordering of Dedekind cuts that there exists an elements, say $\bar{x}$, which is not in $\bar{a}$ but in $\bar{A}$. In view of the Characterization of Dedekind cuts, $\bar{x} \in \bar{A}$ implies the existence of another particular element $\bar{y} \in \bar{A}$ which is greater than $\bar{x}$. This finding $\bar{x} < \bar{y}$ implies with the definition of an open, left-unbounded interval $\bar{x} \in (-\infty, \bar{y})$. Furthermore, since the irreflexivity of $<$ yields $\neg \bar{y} < \bar{y}$, we have $\neg \bar{y} \in (-\infty, \bar{y})$. Let us observe now the truth of the inclusion $\bar{A} \subseteq X$ in light of Property 1 of a cut. We then see that $\bar{y} \in \bar{A}$ implies $\bar{y} \in X$ with the definition of a subset, so that $\bar{y}$ is in the domain of the function $f_X^{\mathcal{D}}$. Therefore, the interval $(-\infty, \bar{y})$ constitutes the first component of the Dedekind cut

$$f_X^{\mathcal{D}}(\bar{y}) = ((-\infty, \bar{y}), [\bar{y}, +\infty)). \tag{8.67}$$

The previous findings $\bar{x} \notin \bar{a}$ and $\bar{x} \in (-\infty, \bar{y})$ imply now with the Characterization of the linear ordering of Dedekind cuts

$$(\bar{a}, \bar{b}) <_{\mathcal{D}} ((-\infty, \bar{y}), [\bar{y}, +\infty)),$$

and for the same reason the previously established $\bar{y} \notin (-\infty, \bar{y})$ and $\bar{y} \in \bar{A}$ imply

$$((-\infty, \bar{y}), [\bar{y}, +\infty)) <_{\mathcal{D}} (\bar{A}, \bar{B}).$$

We obtain therefore after substitutions

$$D <_{\mathcal{D}} f_X^{\mathcal{D}}(\bar{y}) <_{\mathcal{D}} E, \tag{8.68}$$

so that there is indeed a Dedekind cut Z which is strictly between the two Dedekind cuts D and E. We thus proved the existential sentence in (8.66), and since D and E were arbitrary, we therefore conclude that the universal sentence (8.66) holds. In summary, $(\mathcal{D}, <_{\mathcal{D}})$ has all of the properties of a densely ordered set, so that the proof of a) is now complete.

Concerning b), we recall from Note 8.3 that $X_{\mathcal{D}}$ constitutes the range of the function $f_X^{\mathcal{D}} : X \to \mathcal{D}$. By definition of a codomain, the inclusion $X_{\mathcal{D}} \subseteq \mathcal{D}$ is therefore true, as required by Property 1 of a dense subset of $\mathcal{D}$. Property 2 requires the truth of

$$\forall D, E\,([D, E \in \mathcal{D} \,\wedge\, D <_{\mathcal{D}} E] \Rightarrow \exists Z\,(Z \in X_{\mathcal{D}} \,\wedge\, D <_{\mathcal{D}} Z <_{\mathcal{D}} E)), \tag{8.69}$$

which we establish by taking arbitrary Dedekind cuts $D, E \in \mathcal{D}$ and by assuming $D <_{\mathcal{D}} E$ to be true. This is the same antecedent as for the implication in (8.66), which we showed to imply the existence of a particular cut (8.67) in $\mathcal{D}$ satisfying (8.68). Denoting this Dedekind cut by $\bar{Z} = f_X^{\mathcal{D}}(\bar{y})$, we can also write $(\bar{y}, \bar{Z}) \in f_X^{\mathcal{D}}$, which shows in light of the definition of a range that $\bar{Z} \in \operatorname{ran}(f_X^{\mathcal{D}})$ is true. As this range is given by $X_{\mathcal{D}}$, we thus found a particular element $\bar{Z} \in X_{\mathcal{D}}$ with $D <_{\mathcal{D}} \bar{Z} <_{\mathcal{D}} E$, so that the existential sentence in (8.69) is true. Because D and E are arbitrary, we can infer from the truth of this existential sentence the truth of the universal sentence (8.69). We thus proved that $X_{\mathcal{D}}$ is a dense subset of $\mathcal{D}$, in other words, that $(\mathcal{D}, <_{\mathcal{D}})$ is separably ordered with respect to $X_{\mathcal{D}}$.

Concerning c), we prove that the densely ordered set $(\mathcal{D}, <_{\mathcal{D}})$ has the Supremum Property

$$\begin{aligned}&\forall \mathcal{A}\,([\mathcal{A} \subseteq \mathcal{D} \,\wedge\, \mathcal{A} \neq \emptyset \,\wedge\, \exists u\,(u \in \mathcal{D} \,\wedge\, \forall x\,(x \in \mathcal{A} \Rightarrow x \leq_{\mathcal{D}} u))] \\ &\qquad \Rightarrow \exists S\,(S = \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A})).\end{aligned} \tag{8.70}$$

We take an arbitrary set (system) $\mathcal{A}$, assume that $\mathcal{A}$ is a nonempty subset of $\mathcal{D}$, and assume in addition that there exists an upper bound for $\mathcal{A}$ with respect to $\leq_{\mathcal{D}}$, say $\bar{u}$. Clearly, we can write this element of $\mathcal{D}$ in the explicit form of a Dedekind cut $\bar{u} = (\bar{U}, \bar{V})$ for some particular sets $\bar{U}$ and $\bar{V}$. This upper bound satisfies thus

$$\forall x \,(x \in \mathcal{A} \Rightarrow x \leq_{\mathcal{D}} (\bar{U}, \bar{V})). \tag{8.71}$$

We can now use the Axiom of Specification in connection with the Equality Criterion for sets to establish the unique existence of sets $\mathcal{X}$ and $\mathcal{Y}$ consisting, respectively, of all the first components and of all the second components of the Dedekind cuts in $\mathcal{A}$, in the sense that

$$\forall A \,(A \in \mathcal{X} \Leftrightarrow [A \in \mathcal{P}(X) \,\wedge\, \exists B \,(\,(A, B) \in \mathcal{A}\,)]), \tag{8.72}$$

$$\forall B \,(B \in \mathcal{Y} \Leftrightarrow [B \in \mathcal{P}(X) \,\wedge\, \exists A \,(\,(A, B) \in \mathcal{A}\,)]). \tag{8.73}$$

Let us observe here that the assumed $\mathcal{A} \neq \emptyset$ implies the existence of a particular element $\bar{x} \in \mathcal{A}$, so that the assumed inclusion $\mathcal{A} \subseteq \mathcal{D}$ gives $\bar{x} \in \mathcal{D}$ (by definition of a subset). Thus, there are evidently particular sets $\bar{A}$ and $\bar{B}$ such that $\bar{x} = (\bar{A}, \bar{B})$ is a Dedekind cut. According to Property 1 of a cut, we then have the inclusions $\bar{A} \subseteq X$ and $\bar{B} \subseteq X$, which imply $\bar{A} \in \mathcal{P}(X)$ and $\bar{B} \in \mathcal{P}(X)$ with the definition of a power set. Furthermore, $\bar{x} \in \mathcal{A}$ and $\bar{x} = (\bar{A}, \bar{B})$ imply through substitution $(\bar{A}, \bar{B}) \in \mathcal{A}$, so that the existential sentences $\exists B \,(\,(\bar{A}, B) \in \mathcal{A}\,)$ and $\exists A \,(\,(A, \bar{B}) \in \mathcal{A}\,)$ are both true. In connection with $\bar{A} \in \mathcal{P}(X)$ and $\bar{B} \in \mathcal{P}(X)$, respectively, these yield now $\bar{A} \in \mathcal{X}$ and $\bar{B} \in \mathcal{Y}$ with (8.72) – (8.73), so that the set systems $\mathcal{X}$ and $\mathcal{Y}$ are clearly nonempty. In particular the finding $\mathcal{Y} \neq \emptyset$ allows us to form the intersection $\bigcap \mathcal{Y}$ (alongside the union $\bigcup \mathcal{X}$).

We demonstrate in the following that the ordered pair $\bar{S} = (\bigcup \mathcal{X}, \bigcap \mathcal{Y})$ constitutes a Dedekind cut in X, and subsequently that this cut is the supremum of $\mathcal{A}$. Regarding Property 1 of a cut, we observe in light of (8.72) and (8.73) that $A \in \mathcal{X}$ implies $A \in \mathcal{P}(X)$ for an arbitrary A and that $B \in \mathcal{Y}$ implies $B \in \mathcal{P}(X)$ for an arbitrary B, so that we may evidently infer from these implications the truth of the inclusions $\mathcal{X} \subseteq \mathcal{P}(X)$ and $\mathcal{Y} \subseteq \mathcal{P}(X)$, respectively. The former inclusion implies then $\bigcup \mathcal{X} \in \mathcal{P}(X)$ with (3.22), and the latter inclusion implies in conjunction with the previously established fact $\mathcal{Y} \neq \emptyset$ and (3.18) that $\bigcap \mathcal{Y} \in \mathcal{P}(X)$ holds. These findings in turn imply with the definition of a power set the inclusions

$$\bigcup \mathcal{X} \subseteq X \,\wedge\, \bigcap \mathcal{Y} \subseteq X, \tag{8.74}$$

as required for $(\bigcup \mathcal{X}, \bigcap \mathcal{Y})$ to satisfy Property 1 of a cut.

Regarding Property 2, we must show that the sets $\bigcup \mathcal{X}$ and $\bigcap \mathcal{Y}$ are both nonempty. On the one hand, the previously established nonemptiness of $\mathcal{X}$ implies the existence of a particular element $\bar{A} \in \mathcal{X}$, and there exists then also a particular set $\bar{B}$ with $(\bar{A}, \bar{B}) \in \mathcal{A}$, according to (8.72). Due to the assumed inclusion $\mathcal{A} \subseteq \mathcal{D}$, this implies that $(\bar{A}, \bar{B})$ is a Dedekind cut of $\mathcal{D}$, which means in particular that $\bar{A} \neq \emptyset$ holds, according to Property 2 of a cut. Thus, there exists a particular element $\bar{y}$ in $\bar{A}$, and this demonstrates in conjunction with $\bar{A} \in \mathcal{X}$ that there exists a set A in $\mathcal{X}$ containing $\bar{y}$, so that $\bar{y} \in \bigcup \mathcal{X}$ holds by definition of the union of a set system. We therefore see that $\bigcup \mathcal{X}$ is nonempty, as required. Next, we observe that the component $\bar{V}$ of the Dedekind cut $(\bar{U}, \bar{V})$ is nonempty (again by Property 2 of a cut), so that there is an element contained in it, say $\bar{y} \in \bar{V}$. To complete the proof that $\bigcap \mathcal{Y}$ is nonempty, we establish now the inclusion $\bar{V} \subseteq \bigcap \mathcal{Y}$, by verifying the equivalent universal sentence (using the definition of a subset)

$$\forall y \, (y \in \bar{V} \Rightarrow y \in \bigcap \mathcal{Y}). \tag{8.75}$$

We let $y \in \bar{V}$ be arbitrary, and we establish $y \in \bigcap \mathcal{Y}$ by means of the definition of the intersection of a set system, that is, by proving

$$\forall B \, (B \in \mathcal{Y} \Rightarrow y \in B). \tag{8.76}$$

Letting $B \in \mathcal{Y}$ be arbitrary, (8.73) gives us a particular set $\bar{A}$ that satisfies $(\bar{A}, B) \in \mathcal{A}$. Consequently, we obtain the inequality $(\bar{A}, B) \leq_{\mathcal{D}} (\bar{U}, \bar{V})$ with the upper bound property (8.71), and this inequality implies the inclusion $\bar{A} \subseteq \bar{U}$ according to the Total ordering of the set of Dedekind cuts $\mathcal{D}$. This inclusion further implies $\bar{V} \subseteq B$ with (8.19), so that the assumed $y \in \bar{V}$ gives the desired consequent $y \in B$ with the definition of subset. Since B was arbitrary, we may infer from this finding the truth of the universal sentence (8.76) and then also the truth of the equivalent $y \in \bigcap \mathcal{Y}$. This in turn proves the implication in (8.75), in which y is arbitrary, so that the universal sentence (8.75) follows to be true. This means that the inclusion $\bar{V} \subseteq \bigcap \mathcal{Y}$ is indeed true, and therefore the previously found $\bar{y} \in \bar{V}$ implies $\bar{y} \in \bigcap \mathcal{Y}$, demonstrating thereby the nonemptiness of the intersection $\bigcap \mathcal{Y}$. Having thus established the truth of

$$\bigcup \mathcal{X} \neq \emptyset \;\wedge\; \bigcap \mathcal{Y} \neq \emptyset,$$

it is clear that Property 2 of a cut is satisfied by $(\bigcup \mathcal{X}, \bigcap \mathcal{Y})$.

Then, Property 3 of a cut requires that $\bigcup \mathcal{X}$ and $\bigcap \mathcal{Y}$ are disjoint sets, which we check by considering the universal sentence

$$\forall y \, (y \notin \left[\bigcup \mathcal{X}\right] \cap \left[\bigcap \mathcal{Y}\right]). \tag{8.77}$$

Here, we let y be arbitrary, and we prove the negation by contradiction, assuming the negation of that negation to be true. In view of the Double

Negation Law, the sentence $y \in [\bigcup \mathcal{X}] \cap [\bigcap \mathcal{Y}]$ follows now to be true, so that $y \in \bigcup \mathcal{X}$ and $y \in \bigcap \mathcal{Y}$ both hold by definition of the intersection of two sets. Whereas the latter yields the true universal sentence

$$\forall B\,(B \in \mathcal{Y} \Rightarrow y \in B), \tag{8.78}$$

(by definition of the intersection of a set system), the former implies the existence of a particular set $\bar{A} \in \mathcal{X}$ with $y \in \bar{A}$ (by definition of the union of a set system). Here, $\bar{A} \in \mathcal{X}$ implies with (8.72) the existence also of a particular set $\bar{B}$ satisfying $(\bar{A}, \bar{B}) \in \mathcal{A}$. Since $\mathcal{A}$ was assumed to be a subset of $\mathcal{D}$, we evidently have that $(\bar{A}, \bar{B})$ is a Dedekind cut in X, so that the inclusion $\bar{B} \subseteq X$ as well as the disjointness property $\bar{A} \cap \bar{B} = \emptyset$ are fulfilled. The preceding inclusion implies $\bar{B} \in \mathcal{P}(X)$ with the definition of a power set, and this gives because of $(\bar{A}, \bar{B}) \in \mathcal{A}$ the true sentence $\bar{B} \in \mathcal{Y}$ with (8.73). This further implies $y \in \bar{B}$ with (8.78), and recalling now that $y \in \bar{A}$ is also true, we obtain with the definition of the intersection of two sets $y \in \bar{A} \cap \bar{B}$. Thus, the intersection $\bar{A} \cap \bar{B}$ is nonempty, but this finding contradicts the aforementioned disjointness property $\bar{A} \cap \bar{B} = \emptyset$, so that the proof of the negation in (8.77) is complete. Here, y is arbitrary, so that the universal sentence (8.77) holds, which means by definition of the empty set that

$$\left[\bigcup \mathcal{X}\right] \cap \left[\bigcap \mathcal{Y}\right] = \emptyset \tag{8.79}$$

is true.

According to Property 4 of a cut, the union of $\bigcup \mathcal{X}$ and $\bigcap \mathcal{Y}$ must be identical with X, which we verify in the following by means of the Equality Criterion for sets. We prove accordingly the universal sentence

$$\forall y\,(y \in \left[\bigcup \mathcal{X}\right] \cup \left[\bigcap \mathcal{Y}\right] \Leftrightarrow y \in X), \tag{8.80}$$

letting y be arbitrary. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $y \in [\bigcup \mathcal{X}] \cup [\bigcap \mathcal{Y}]$ to be true, so that $y \in \bigcup \mathcal{X}$ or $y \in \bigcap \mathcal{Y}$ holds (by definition of the union of two sets). We can use this disjunction to prove the desired consequent $y \in X$ by cases. Indeed, the first case $y \in \bigcup \mathcal{X}$ implies $y \in X$ with the first inclusion in (8.74), and the second case $y \in \bigcap \mathcal{Y}$ gives $y \in X$ with the second inclusion in (8.74). We prove the second part ('$\Leftarrow$') of the equivalence in (8.80) by contradiction, assuming $y \in X$ and the negation $\neg y \in [\bigcup \mathcal{X}] \cup [\bigcap \mathcal{Y}]$ to be true. The latter assumption implies by definition of the union of two sets and with De Morgan's Law for the disjunction that the negations $\neg y \in \bigcup \mathcal{X}$ and $\neg y \in \bigcap \mathcal{Y}$ are both true. Here, the former negation gives with the definition of the union of a set system and with the Negation Law for existential conjunctions the true universal sentence

$$\forall A\,(A \in \mathcal{X} \Rightarrow y \notin A), \tag{8.81}$$

whereas the other negation yields by means of the definition of the intersection of a set system and by means of the Negation Law for universal implications the true existential sentence

$$\exists B \, (B \in \mathcal{Y} \wedge y \notin B). \tag{8.82}$$

This existential sentence gives us a particular set $\bar{B} \in \mathcal{Y}$ with $y \notin \bar{B}$, and the definition of the set $\mathcal{Y}$ yields another particular set $\bar{A}$ for which $(\bar{A}, \bar{B}) \in \mathcal{A}$ is satisfied. As we assumed $\mathcal{A}$ to be a subset of $\mathcal{D}$, it is evidently true that $(\bar{A}, \bar{B})$ constitutes a Dedekind cut in X, which finding means in particular that the equation $\bar{A} \cup \bar{B} = X$ and the inclusion $\bar{A} \subseteq X$ hold. This inclusion implies $\bar{A} \in \mathcal{P}(X)$, and this implies in conjunction with $(\bar{A}, \bar{B}) \in \mathcal{A}$ the truth of $\bar{A} \in \mathcal{X}$, by definition of the set $\mathcal{X}$. This in turn implies with (8.81) $y \notin \bar{A}$, and since $y \notin \bar{B}$ is also true, we may apply De Morgan's Law for the disjunction to infer from these negations the negated disjunction $\neg(y \in \bar{A} \vee y \in \bar{B})$. Clearly, this means that the negation $\neg y \in \bar{A} \cup \bar{B}$ $[= X]$ holds as well, so that we find $\neg y \in X$ to be true. In view of the initial assumption $y \in X$, we arrived at a contradiction, and the proof of the implication '$\Leftarrow$' in (8.80) is therefore complete. Because y was arbitrary, we now further conclude that the universal sentence (8.80) is true, which in turn implies the truth of the required equality

$$\left[\bigcup \mathcal{X}\right] \cup \left[\bigcap \mathcal{Y}\right] = X.$$

To complete the proof that $\bar{S} = (\bigcup \mathcal{X}, \bigcap \mathcal{Y})$ is a cut in X, we prove now

$$\forall a, b \, ([a \in \bigcup \mathcal{X} \wedge b \in \bigcap \mathcal{Y}] \Rightarrow a < b). \tag{8.83}$$

For this purpose, we take an arbitrary element $a \in \bigcup \mathcal{X}$ and an arbitrary element $b \in \bigcap \mathcal{Y}$. Whereas the latter means

$$\forall B \, (B \in \mathcal{Y} \Rightarrow b \in B). \tag{8.84}$$

the former means that there exists a set in $\mathcal{X}$ containing a, say $\bar{A}$. By definition of the set $\mathcal{X}$, we find then also a particular set $\bar{B}$ such that the ordered pair $(\bar{A}, \bar{B})$ is in $\mathcal{A}$. Recalling that $\mathcal{A}$ is included in $\mathcal{D}$, we clearly have that $(\bar{A}, \bar{B})$ is also in $\mathcal{D}$, so that $(\bar{A}, \bar{B})$ is a Dedekind cut. Therefore, $\bar{B} \subseteq X$ is especially true (according to Property 1 of a cut), and this yields $\bar{B} \in \mathcal{P}(X)$ (by definition of a power set). Together with $(\bar{A}, \bar{B}) \in \mathcal{A}$, this gives evidently $\bar{B} \in \mathcal{Y}$, with the consequence that $b \in \bar{B}$ is true, according to (8.84). Let us recall that $a \in \bar{A}$ also holds, so that Property 5 of a cut allows us to infer the truth of $a < b$. This is the desired consequent of the implication in (8.83), in which a and b are arbitrary, so that the universal sentence (8.83) follows now to be true. This means that $\bar{S} = (\bigcup \mathcal{X}, \bigcap \mathcal{Y})$ satisfies also Property 5 of a cut.

We thus proved that $(\bigcup \mathcal{X}, \bigcap \mathcal{Y})$ is a cut, so that Property 1 of a Dedekind cut is satisfied. Concerning the remaining Property 2, we demonstrate the truth of

$$\neg \exists v \, (v \in X \;\wedge\; v = \max \bigcup \mathcal{X}), \tag{8.85}$$

which negation we can write equivalently as the universal sentence

$$\forall v \, (v \in X \Rightarrow v \neq \max \bigcup \mathcal{X}), \tag{8.86}$$

by applying the Negation Law for existential conjunctions. We prove this sentence by taking an arbitrary v and by proving the implication by contradiction, assuming $v \in X$ and $\neg v \neq \max \bigcup \mathcal{X}$ to be true, so that the equation $v = \max \bigcup \mathcal{X}$ turns out to be true according to the Double Negation Law. By definition of a maximum, v is then an upper bound for $\bigcup \mathcal{X}$ contained in that union, i.e.

$$\forall x \, (x \in \bigcup \mathcal{X} \Rightarrow x \leq v) \;\wedge\; v \in \bigcup \mathcal{X}. \tag{8.87}$$

Evidently, $v \in \bigcup \mathcal{X}$ implies that $v \in \bar{A}$ holds for some particular set $\bar{A} \in \mathcal{X}$. We thus have $(\bar{A}, \bar{B}) \in \mathcal{A}$ for some particular set $\bar{B}$, so that $(\bar{A}, \bar{B})$ is a Dedekind cut. According to Property 2 of a Dedekind cut, the negation

$$\neg \exists u \, (u \in X \;\wedge\; u = \max \bar{A})$$

holds, and the equivalent universal sentence

$$\forall u \, (u \in X \Rightarrow u \neq \max \bar{A})$$

is then also true. Therefore, the initial assumption $v \in X$ implies $v \neq \max \bar{A}$, which means by definition of a maximum that the negation

$$\neg \left[\forall x \, (x \in \bar{A} \Rightarrow x \leq v) \;\wedge\; v \in \bar{A} \right]$$

holds. Applying now De Morgan's Law for the conjunction in connection with the Negation Law for universal implications, and noting that the previously established $v \in \bar{A}$ implies the falseness of $\neg v \in \bar{A}$, we see that there exists some particular element $\bar{x} \in \bar{A}$ with $\neg \bar{x} \leq v$. Evidently, $x \in \bar{A}$ and the previously obtained $\bar{A} \in \mathcal{X}$ imply $\bar{x} \in \bigcup \mathcal{X}$, which gives $\bar{x} \leq v$ with the first part of the conjunction (8.87). Because we just found the negation $\neg \bar{x} \leq v$ also to be true, we managed to establish a contradiction, so that the proof of the implication in (8.86) is complete. Since v is arbitrary, we can infer from the truth of this implication the truth of the universal sentence

(8.86) and thus the truth of the equivalent negation (8.85). This finding completes the demonstration of the validity of Property 2 of a Dedekind cut with respect to $\bar{S} = (\bigcup \mathcal{X}, \bigcap \mathcal{Y})$, which ordered pair thus constitutes a Dedekind cut.

Our next task is to prove that the Dedekind cut $\bar{S}$ is an upper bound for $\mathcal{A}$, that is,

$$\forall x\,(x \in \mathcal{A} \Rightarrow x \leq_{\mathcal{D}} \bar{S}). \tag{8.88}$$

Taking an arbitrary element $x \in \mathcal{A}$, we see in light of the assumed inclusion $\mathcal{A} \subseteq \mathcal{D}$ and the definition of the set $\mathcal{D}$ that x can be written as the ordered pair $x = (\bar{A}, \bar{B})$ and that x is a Dedekind cut. Consequently, we obtain $\bar{A} \subseteq X$ with Property 1 of a Dedekind cut and subsequently $\bar{A} \subseteq \mathcal{P}(X)$ with the definition of a power set. Furthermore, $x \in \mathcal{A}$ yields via substitution $(\bar{A}, \bar{B}) \in \mathcal{A}$, and these findings imply $\bar{A} \in \mathcal{X}$ with (8.72). Then, the inclusion $\bar{A} \subseteq \bigcup \mathcal{X}$ follows to be true with (2.201), so that the inequality $(\bar{A}, \bar{B}) \leq_{\mathcal{D}} (\bigcup \mathcal{X}, \bigcap \mathcal{Y})$ holds according to the Total ordering of the set of Dedekind cuts. Substitutions gives us now indeed the desired consequent $x \leq_{\mathcal{D}} \bar{S}$ of the implication in (8.88), and as x was arbitrary, we may infer from this the truth of the universal sentence (8.88). This means that $\bar{S}$ is indeed an upper bound for $\mathcal{A}$. According to the Characterization of a supremum, it remains for us to demonstrate the truth of the universal sentence

$$\forall S'\,(\forall x\,(x \in \mathcal{A} \Rightarrow x \leq_{\mathcal{D}} S') \Rightarrow \bar{S} \leq_{\mathcal{D}} S') \tag{8.89}$$

To do this, we take an arbitrary set S' and assume the universal sentence

$$\forall x\,(x \in \mathcal{A} \Rightarrow x \leq_{\mathcal{D}} S') \tag{8.90}$$

to be true, which means that S' is also an upper bound for $\mathcal{A}$ with respect to $\leq_{\mathcal{D}}$. Thus, S' is evidently an element of $\mathcal{D}$, which element constitutes then a Dedekind cut (A', B') for some particular sets A' and B'. Let us apply now the definition of a subset to establish the inclusion $\bigcup \mathcal{X} \subseteq A'$, by proving accordingly

$$\forall y\,(y \in \bigcup \mathcal{X} \Rightarrow y \in A'). \tag{8.91}$$

We let $y \in \bigcup \mathcal{X}$ be arbitrary, so that there exists (by definition of the union of a set system) a particular set $\bar{A} \in \mathcal{X}$ with $y \in \bar{A}$. According to the definition of the set $\mathcal{X}$, there exists then also a particular set $\bar{B}$ with $(\bar{A}, \bar{B}) \in \mathcal{A}$. Due to the assumed universal sentence (8.90), this implies $(\bar{A}, \bar{B}) \leq_{\mathcal{D}} S'$, so that substitution based on the previously established $S' = (A', B')$ yields $(\bar{A}, \bar{B}) \leq_{\mathcal{D}} (A', B')$. Consequently, $\bar{A} \subseteq A'$ follows to be true with the specification of the Total ordering of the set of Dedekind cuts, and this inclusion allows us to infer from the previously found $y \in \bar{A}$ that $y \in A'$

also holds. As this is the desired consequent of the implication in (8.91) and since y was arbitrary, we can further conclude that the universal sentence (8.91) is true. We thus proved the inclusion $\bigcup \mathcal{X} \subseteq A'$, which gives us now the inequality $(\bigcup \mathcal{X}, \bigcap \mathcal{Y}) \leq_{\mathcal{D}} (A', B')$, using again the specification of $\leq_{\mathcal{D}}$. We therefore obtain after substitutions the desired consequent $\bar{S} \leq_{\mathcal{D}} S'$ of the implication in (8.89). Because S' was arbitrary, we can infer from the truth of that implication the truth of the universal sentence (8.89), which shows that the upper bound $\bar{S}$ is less than or equal to every upper bound for $\mathcal{A}$. Consequently, $\bar{S}$ constitutes the supremum of $\mathcal{A}$ (with respect to $\leq_{\mathcal{D}}$), and this fact proves the existence of a supremum of $\mathcal{A}$. As the set $\mathcal{A}$ was initially arbitrary, it follows from this that $(\mathcal{D}, <_{\mathcal{D}})$ has the Supremum Property (8.70). The densely ordered set $(\mathcal{D}, <_{\mathcal{D}})$ is therefore a linear continuum by definition.

Concerning d), we prove that the linear continuum $(\mathcal{D}, <_{\mathcal{D}})$ has the Infimum Property

$$\forall \mathcal{A}\,([\mathcal{A} \subseteq \mathcal{D} \wedge \mathcal{A} \neq \emptyset \wedge \exists a\,(a \in \mathcal{D} \wedge \forall x\,(x \in \mathcal{A} \Rightarrow a \leq_{\mathcal{D}} x))] \\ \Rightarrow \exists I\,(I = \overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A})). \tag{8.92}$$

We take an arbitrary set (system) $\mathcal{A}$, assume that $\mathcal{A}$ is a nonempty subset of $\mathcal{D}$, and assume in addition that there exists a lower bound for $\mathcal{A}$ with respect to $\leq_{\mathcal{D}}$, say $\bar{a}$. This lower bound satisfies by definition

$$\bar{a} \in \mathcal{D} \wedge \forall x\,(x \in \mathcal{A} \Rightarrow \bar{a} \leq_{\mathcal{D}} x). \tag{8.93}$$

Let us observe now in light of the Axiom of Specification and the Equality Criterion for sets that there exists a unique set $\mathcal{L}$ consisting of all Dedekind cuts in X which are lower bounds for $\mathcal{A}$, in the sense that

$$\forall a\,(a \in \mathcal{L} \Rightarrow [a \in \mathcal{D} \wedge \forall x\,(x \in \mathcal{A} \Rightarrow a \leq_{\mathcal{D}} x)]). \tag{8.94}$$

Because $a \in \mathcal{L}$ implies especially $a \in \mathcal{D}$ for any a, we obtain the inclusion $\mathcal{L} \subseteq \mathcal{D}$ by definition of a subset. Furthermore, since $\bar{a}$ satisfies (8.93), it follows with (8.94) that $\bar{a}$ is an element of the set $\mathcal{L}$, which finding shows that $\mathcal{L}$ is nonempty. We prove now by contradiction that $\mathcal{L}$ is bounded from above, assuming for this purpose the negation

$$\neg \exists u\,(u \in \mathcal{D} \wedge \forall a\,(a \in \mathcal{L} \Rightarrow a \leq_{\mathcal{D}} u)).$$

to be true. Then, the Negation Law for existential conjunction yields the true universal sentence

$$\forall u\,(u \in \mathcal{D} \Rightarrow \neg \forall a\,(a \in \mathcal{L} \Rightarrow a \leq_{\mathcal{D}} u)). \tag{8.95}$$

As we assumed $\mathcal{A}$ to be nonempty, there exists an element in $\mathcal{A}$, say $\bar{y}$. Due to the assumed inclusion $\mathcal{A} \subseteq \mathcal{D}$, we therefore obtain $\bar{y} \in \mathcal{D}$ by definition of a subset. This in turn implies with (8.95) the truth of the negation

$$\neg\forall a\,(a \in \mathcal{L} \Rightarrow a \leq_{\mathcal{D}} \bar{y}),$$

and this negation gives us with the Negation Law for universal implications the true existential sentence

$$\exists a\,(a \in \mathcal{L} \wedge \neg a \leq_{\mathcal{D}} \bar{y}).$$

This means that there exists a lower bound for $\mathcal{A}$ in $\mathcal{L}$, say $\bar{\bar{a}}$, for which the negation $\neg\bar{\bar{a}} \leq_{\mathcal{D}} \bar{y}$ holds. Thus, $\bar{\bar{a}} \in \mathcal{L}$ implies with (8.94) especially the truth of the universal sentence

$$\forall x\,(x \in \mathcal{A} \Rightarrow \bar{\bar{a}} \leq_{\mathcal{D}} x).$$

Consequently, the previously found $\bar{y} \in \mathcal{A}$ satisfies $\bar{\bar{a}} \leq_{\mathcal{D}} \bar{y}$, in contradiction to the previously established negation $\neg\bar{\bar{a}} \leq_{\mathcal{D}} \bar{y}$. We thus proved that $\mathcal{L}$ is bounded from above, that is,

$$\exists u\,(u \in \mathcal{D} \wedge \forall a\,(a \in \mathcal{L} \Rightarrow a \leq_{\mathcal{D}} u)).$$

In conjunction with the already proven inclusion $\mathcal{L} \subseteq \mathcal{D}$ and inequality $\mathcal{L} \neq \emptyset$, this implies the existence of the supremum of $\mathcal{L}$ because of the Supremum Property (8.70) of $(\mathcal{D}, <_{\mathcal{D}})$. This supremum $\sup^{\leq_{\mathcal{D}}} \mathcal{L}$ is by definition an upper bound for $\mathcal{L}$, which we show now to be an element of $\mathcal{L}$. Noting that this supremum with respect to $\leq_{\mathcal{D}}$ is by definition an element of $\mathcal{D}$, we must demonstrate – according to (8.94) – the truth of

$$\forall x\,(x \in \mathcal{A} \Rightarrow \sup^{\leq_{\mathcal{D}}} \mathcal{L} \leq_{\mathcal{D}} x). \tag{8.96}$$

We take an arbitrary constant x and prove the implication by contradiction, assuming $x \in \mathcal{A}$ and the negation $\neg \sup^{\leq_{\mathcal{D}}} \mathcal{L} \leq_{\mathcal{D}} x$ to be true. The former assumption gives $x \in \mathcal{D}$ with the assumed inclusion $\mathcal{A} \subseteq \mathcal{D}$, and the preceding negation implies with the Negation Formula for $\leq$ the inequality $x <_{\mathcal{D}} \sup^{\leq_{\mathcal{D}}} \mathcal{L}$. Consequently, it follows with the Supremum Criterion that there exists an element in $\mathcal{L}$, say $\bar{a}'$, such that $x <_{\mathcal{D}} \bar{a}'$. Then, $\bar{a}' \in \mathcal{L}$ implies in connection with the assumed $x \in \mathcal{A}$ also $\bar{a}' \leq_{\mathcal{D}} x$ by definition of the set $\mathcal{L}$ of lower bounds for $\mathcal{A}$, so that the negation $\neg x <_{\mathcal{D}} \bar{a}'$ follows to be true with the Negation Formula for $<$. As we found $x <_{\mathcal{D}} \bar{a}'$ to be true as well, we obtained a contradiction, so that the proof of the implication in (8.96) is complete. Since x is arbitrary, we may therefore conclude that the universal sentence (8.96) holds, which implies now in conjunction with the aforementioned fact $\sup^{\leq_{\mathcal{D}}} \mathcal{L} \in \mathcal{D}$ that $\sup^{\leq_{\mathcal{D}}} \mathcal{L} \in \mathcal{L}$ is true. We thus showed that this supremum is an upper bound for $\mathcal{L}$ contained in that set, and this means by definition that $\sup^{\leq_{\mathcal{D}}} \mathcal{L}$ is the greatest element of $\mathcal{L}$. In other words, $\sup^{\leq_{\mathcal{D}}} \mathcal{L}$ is the greatest element of the set $\mathcal{L}$ consisting of all lower bounds for $\mathcal{A}$ (with respect to $\leq_{\mathcal{D}}$), which means that $\sup^{\leq_{\mathcal{D}}} \mathcal{L}$ is the infimum of $\mathcal{A}$. This finding demonstrates the existence of an infimum $I = \inf^{\leq_{\mathcal{D}}} \mathcal{A}$, that is, the truth of the existential sentence in (8.92). Since $\mathcal{A}$ was an arbitrary set, the universal sentence (8.92) follows to be true.

The truth of a) – d) implies now the truth of the stated theorem as the sets X and $<$ were initially arbitrary. □

Because the set of rational numbers neither has a minimum nor a maximum, we can now carry out the completion of the densely ordered set $(\mathbb{Q}, <_{\mathbb{Q}})$.

Corollary 8.12. *It is true that the linearly ordered set of real numbers*

a) $(\mathbb{R}, <_{\mathbb{R}})$ *is densely ordered,*

b) $(\mathbb{R}, <_{\mathbb{R}})$ *is separably ordered with respect to the image* $\mathbb{Q}_{\mathbb{R}}$ *of* $\mathbb{Q}$ *under the function* $f_{\mathbb{Q}}^{\mathbb{R}}$*,*

c) $(\mathbb{R}, <_{\mathbb{R}})$ *is a linear continuum,*

d) $(\mathbb{R}, <_{\mathbb{R}})$ *has the Infimum Property.*

Note 8.6. The Supremum Property and the Infimum Property of $(\mathbb{R}, <_{\mathbb{R}})$ (i.e., the fact that every nonempty and bounded-from-above/below subset A of $\mathbb{R}$ has a supremum/an infimum with respect to the total ordering $\leq_{\mathbb{R}}$) read

$$\forall A\,([A \subseteq \mathbb{R} \wedge A \neq \emptyset \wedge \exists u\,(u \in \mathbb{R} \wedge \forall x\,(x \in A \Rightarrow x \leq_{\mathbb{R}} u))] \\ \Rightarrow \exists S\,(S = \overset{\leq_{\mathbb{R}}}{\sup} A)) \tag{8.97}$$

and

$$\forall A\,([A \subseteq \mathbb{R} \wedge A \neq \emptyset \wedge \exists a\,(a \in \mathbb{R} \wedge \forall x\,(x \in A \Rightarrow a \leq_{\mathbb{R}} x))] \\ \Rightarrow \exists I\,(I = \overset{\leq_{\mathbb{R}}}{\inf} A)). \tag{8.98}$$

Definition 8.9 (Linear continuum of real numbers). We call

$$(\mathbb{R}, <_{\mathbb{R}}) \tag{8.99}$$

the *linear continuum of real numbers*.

Theorem 8.13 (Lattices of Dedekind cuts). *For any densely ordered set* $(X, <)$ *such that there are no elements* a *and* u *in* X *with* $a = \min X$ *and* $u = \max X$*, it is true that*

a) the totally ordered set $(\mathcal{D}, \leq_{\mathcal{D}})$ *is a lattice,*

b) $\mathcal{D}$ *is unbounded from above (with respect to* $\leq_{\mathcal{D}}$*),*

c) $\mathcal{D}$ *is unbounded from below (with respect to* $\leq_{\mathcal{D}}$*),*

d) the lattice $(\mathcal{D}, \leq_{\mathcal{D}})$ *is not complete.*

Proof. Letting X and $<$ be arbitrary sets, we assume that $<$ is a linear ordering of X, we assume that $(X, <)$ is densely ordered, and we assume that neither the minimum nor the maximum of X exists.

Concerning a), we verify that the supremum and infimum of every pair of elements in $\mathcal{D}$ exists in $\mathcal{D}$, that is,

$$\forall x, y\,(x, y \in \mathcal{D} \Rightarrow \exists S, I\,(S, I \in \mathcal{D} \wedge S = \overset{\leq_{\mathcal{D}}}{\sup}\{x, y\} \wedge I = \overset{\leq_{\mathcal{D}}}{\inf}\{x, y\})), \tag{8.100}$$

For this purpose, we let x and y be arbitrary elements in $\mathcal{D}$. Consequently, the supremum $\sup\{x, y\}$ and the infimum $\inf\{x, y\}$ of the pair $\{x, y\}$ exist due to Proposition 3.113c) and Exercise 3.52c). Since x and y are arbitrary, we therefore conclude that $(\mathcal{D}, \leq_{\mathcal{D}})$ satisfies (8.100) and constitutes therefore indeed a lattice.

Concerning b), we demonstrate that there does not exist an upper bound for $\mathcal{D}$, that is,

$$\neg \exists u\,(u \in \mathcal{D} \wedge \forall x\,(x \in \mathcal{D} \Rightarrow x \leq_{\mathcal{D}} u)).$$

In view of the Negation Law for existential conjunctions, the preceding negation is equivalent to the universal sentence

$$\forall u\,(u \in \mathcal{D} \Rightarrow \neg \forall x\,(x \in \mathcal{D} \Rightarrow x \leq_{\mathcal{D}} u)), \tag{8.101}$$

which we now prove, taking an arbitrary element $u \in \mathcal{D}$. Therefore, u constitutes a Dedekind cut and thus by definition an ordered, so that we can write

$$u = (\bar{A}, \bar{B})$$

for particular sets $\bar{A}, \bar{B}$. According to the properties of a Dedekind cut, we have in particular $\bar{B} \neq \emptyset$ and $\bar{B} \subseteq X$, where the former evidently implies the existence of a particular element $\bar{y} \in \bar{B}$. Then, since the maximum of X does not exist, there is another particular element $\bar{z} \in \bar{B}$ such that

$\bar{y} < \bar{z}$, according to (8.24). The preceding inclusion yields now $\bar{y} \in X$ and $\bar{z} \in X$ by definition of a subset. Due to the latter finding, we obtain with the function $f_X^{\mathcal{D}} : X \to \mathcal{D}$ defined by (8.43) the Dedekind cut

$$f_X^{\mathcal{D}}(\bar{z}) = ((-\infty, \bar{z}), [\bar{z}, +\infty)).$$

Let us observe here that $\bar{y} < \bar{z}$ implies $\bar{y} \in (-\infty, \bar{z})$ with the definition of an open and left-unbounded interval in X. Furthermore, the previously established $\bar{y} \in X$ and $\bar{y} \in \bar{B}$ imply $\bar{y} \notin \bar{A}$ with (8.18). We see now that there exists a constant x which satisfies both $x \notin \bar{A}$ and $x \in (-\infty, \bar{z})$. This existential sentence implies then with the Characterization of the linear ordering of Dedekind cuts that

$$[u =] \quad (\bar{A}, \bar{B}) <_{\mathcal{D}} ((-\infty, \bar{z}), [\bar{z}, +\infty))$$

holds, and this inequality further implies

$$\neg((-\infty, \bar{z}), [\bar{z}, +\infty)) \leq_{\mathcal{D}} u$$

with the Negation Formula for $\leq$. In view of $u \in \mathcal{D}$, we thus proved the existential sentence

$$\exists x \, (x \in \mathcal{D} \wedge \neg((-\infty, \bar{z}), [\bar{z}, +\infty)) \leq_{\mathcal{D}} x),$$

which in turn implies the truth of the negated universal sentence in (8.101). Because u was arbitrary, we may therefore conclude that the universal sentence (8.101) holds, and this proves the equivalent assertion that $\mathcal{D}$ is unbounded from above.

The proof of Part c) is a slightly simpler version of the proof of Part b).

Concerning d), we verify the negation of the complete lattice property

$$\forall \mathcal{A} \, (\mathcal{A} \subseteq \mathcal{D} \Rightarrow \exists S, I \, (S, I \in \mathcal{D} \wedge S = \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A} \wedge I = \overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A})). \tag{8.102}$$

by contradiction, assuming the negation of that negation to be true. Thus, (8.102) holds because of the Double Negation Law. Since the inclusion $\mathcal{D} \subseteq \mathcal{D}$ is true in view of (2.10), it follows from this that there are elements $S, I \in \mathcal{D}$ such that S is the supremum and I the infimum of $\mathcal{D}$. Thus, S is an upper bound and I a lower bound for $\mathcal{D}$, which findings evidently contradict the facts in b) and c) that $\mathcal{D}$ neither has an upper bound nor a lower bound.

Because the sets X and $<$ were initially arbitrary, we may infer from the truth of a) – d) the truth of the entire theorem. □

Exercise 8.8. Prove Part c) of Theorem 8.13.

Corollary 8.14. *It is true that*

a) *the totally ordered set* $(\mathbb{R}, \leq_{\mathbb{R}})$ *is a lattice,*

b) $\mathbb{R}$ *is unbounded from above (with respect to* $\leq_{\mathbb{R}}$*),*

c) $\mathbb{R}$ *is unbounded from below (with respect to* $\leq_{\mathbb{R}}$*),*

d) *the lattice* $(\mathbb{R}, \leq_{\mathbb{R}})$ *is not complete.*

The fact that $\mathbb{R}$ has no upper and no lower bound implies, *a fortiori*, that the maximum and the minimum of that set do not exist.

Corollary 8.15. *The set* $\mathbb{R}$

a) *neither has a maximum, that is,*

$$\neg\exists u \, (u \in \mathbb{R} \, \wedge \, u = \overset{\leq_{\mathbb{R}}}{\max} \mathbb{R}), \tag{8.103}$$

a) *nor a minimum, that is,*

$$\neg\exists a \, (a \in \mathbb{R} \, \wedge \, a = \overset{\leq_{\mathbb{R}}}{\min} \mathbb{R}). \tag{8.104}$$

Exercise 8.9. Show for any real number that there is a smaller and a larger rational number in $\mathbb{R}$, that is,

$$\forall y \, (y \in \mathbb{R} \Rightarrow \exists p, q \, (p, q \in \mathbb{Q}_{\mathbb{R}} \, \wedge \, p <_{\mathbb{R}} y <_{\mathbb{R}} q)). \tag{8.105}$$

(Hint: Use Corollary 8.12b) and Corollary 8.14b,c) in connection with Definition 3.27 and Definition 8.1.)

Proposition 8.16. *It is true that*

$$\forall y \, (y \in \mathbb{R} \Rightarrow \exists p, q \, (p, q \in \mathbb{Q} \, \wedge \, f_{\mathbb{Q}}^{\mathbb{R}}(p) <_{\mathbb{R}} y <_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(q))). \tag{8.106}$$

Proof. Letting y be an arbitrary real number, we find with (8.105) particular numbers $\bar{p}, \bar{q} \in \mathbb{Q}_{\mathbb{R}}$ satisfying $\bar{p} <_{\mathbb{R}} y <_{\mathbb{R}} \bar{q}$. Since the inverse of the bijection $f_{\mathbb{Q}}^{\mathbb{R}} : \mathbb{Q} \rightleftarrows \mathbb{Q}_{\mathbb{R}}$ in (8.54) is given by $(f_{\mathbb{Q}}^{\mathbb{R}})^{-1} : \mathbb{Q}_{\mathbb{R}} \rightleftarrows \mathbb{Q}$ according to the Bijectivity of inverse functions, we find the genuine rationals $p^* = (f_{\mathbb{Q}}^{\mathbb{R}})^{-1}(\bar{p})$ and $q^* = (f_{\mathbb{Q}}^{\mathbb{R}})^{-1}(\bar{q})$. Furthermore, we obtain

$$f_{\mathbb{Q}}^{\mathbb{R}}(p^*) = f_{\mathbb{Q}}^{\mathbb{R}}((f_{\mathbb{Q}}^{\mathbb{R}})^{-1}(\bar{p})) = (f_{\mathbb{Q}}^{\mathbb{R}} \circ (f_{\mathbb{Q}}^{\mathbb{R}})^{-1})(\bar{p}) = \mathrm{id}_{\mathbb{Q}_{\mathbb{R}}}(\bar{p}) = \bar{p},$$
$$f_{\mathbb{Q}}^{\mathbb{R}}(q^*) = f_{\mathbb{Q}}^{\mathbb{R}}((f_{\mathbb{Q}}^{\mathbb{R}})^{-1}(\bar{q})) = (f_{\mathbb{Q}}^{\mathbb{R}} \circ (f_{\mathbb{Q}}^{\mathbb{R}})^{-1})(\bar{q}) = \mathrm{id}_{\mathbb{Q}_{\mathbb{R}}}(\bar{q}) = \bar{q}$$

by applying substitution, the notation for function compositions, (3.680) and the definition of the identity function. Therefore, the previously established inequality $\bar{p} <_{\mathbb{R}} y <_{\mathbb{R}} \bar{q}$ implies $f_{\mathbb{Q}}^{\mathbb{R}}(p^*) <_{\mathbb{R}} y <_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(q^*)$, which demonstrates in conjunction with $p^*, q^* \in \mathbb{Q}$ the truth of the existential sentence in (8.106). Consequently, since y was arbitrary, we may conclude that the proposed universal sentence holds. □

Notation 8.2. For brevity of expressions, we will usually omit $f_{\mathbb{Q}}^{\mathbb{R}}(.)$ and write, for instance, p instead of $f_{\mathbb{Q}}^{\mathbb{R}}(p)$.

Proposition 8.17. *It is true for any $x \in \mathbb{R}$ and any $m \in \mathbb{N}_+$ that x is enclosed by some rational numbers p, q which are no more than $\frac{1}{m}$ apart, in the sense that*

$$\exists p, q\,(p, q \in \mathbb{Q} \wedge x \in (p, q] \wedge q -_{\mathbb{Q}} p \leq_{\mathbb{Q}} \frac{1}{m}). \tag{8.107}$$

Proof. Letting $x \in \mathbb{R}$ and $m \in \mathbb{N}_+$ be arbitrary, we note that the former implies the existence of particular numbers $\bar{p}, \bar{q} \in \mathbb{Q}$ with $\bar{p} <_{\mathbb{R}} x <_{\mathbb{R}} \bar{q}$, according to Proposition 8.16 and Notation 8.2. Viewing now m as a rational number, we form the rational number $m \cdot_{\mathbb{Q}} (\bar{q} -_{\mathbb{Q}} \bar{p})$. Then, there exists a positive natural number, say $\bar{n}$, which is larger than that product, that is, we have $m \cdot_{\mathbb{Q}} (\bar{q} -_{\mathbb{Q}} \bar{p}) <_{\mathbb{Q}} \bar{n}$, by virtue of Proposition 7.46. Since $m \in \mathbb{N}_+$ evidently implies $m \neq 0$, the preceding inequality gives first $\bar{q} -_{\mathbb{Q}} \bar{p} <_{\mathbb{Q}} \frac{\bar{n}}{m}$ with the monotony law (7.190) and subsequently

$$[x <_{\mathbb{Q}}] \quad \bar{q} <_{\mathbb{Q}} \bar{p} +_{\mathbb{Q}} \frac{\bar{n}}{m}, \tag{8.108}$$

with the monotony law (7.191). Let us now define the strictly increasing sequence $t = (t_i \,|\, i \in \{0, \ldots, \bar{n}\})$ in $\mathbb{Q}$ with terms $t_i = \bar{p} +_{\mathbb{Q}} \frac{i}{m}$ for all $i \in \{0, \ldots, \bar{n}\}$, according to Corollary 7.44. We may then use the Axiom of Specification alongside the Equality Criterion for sets to prove the unique existence of a set I such that

$$\forall i\,(i \in I \Leftrightarrow [i \in \{0, \ldots, \bar{n}\} \wedge t_i <_{\mathbb{R}} x]). \tag{8.109}$$

Here, $i \in I$ evidently implies in particular $i \in \{0, \ldots, \bar{n}\}$ for all i, so that the inclusion $I \subseteq \{0, \ldots, \bar{n}\}$ is true by definition of a subset. We may prove by contradiction that $\bar{n}$ is not in I. Assuming for this purpose $\neg \bar{n} \notin I$ to be true, we find the true $\bar{n} \in I$ with the Double Negation Law. Thus, the corresponding term $t_{\bar{n}} = \bar{p} +_{\mathbb{Q}} \frac{\bar{n}}{m}$ satisfies $t_{\bar{n}} <_{\mathbb{R}} x$ according to (8.109). Since the linear ordering $<_{\mathbb{R}}$ is comparable, the negation $\neg x <_{\mathbb{R}} \bar{p} +_{\mathbb{Q}} \frac{\bar{n}}{m}$ is then true. This however contradicts the fact that (8.108) gives the true inequality $x <_{\mathbb{R}} \bar{p} +_{\mathbb{Q}} \frac{\bar{n}}{m}$ with the transitivity of the linear ordering $<_{\mathbb{R}}$. Thus, the proof of

$$\bar{n} \notin I \tag{8.110}$$

is complete. We now verify that the set I is not empty. For this purpose, we observe the evident truth of $0 \in \{0, \ldots, \bar{n}\}$, so that there is the associated term $t_0 = \bar{p} +_{\mathbb{Q}} \frac{0}{m}$. Clearly, this term is identical to $\bar{p}$, and since we already know that $\bar{p} <_{\mathbb{R}} x$ holds, we obtain $t_0 <_{\mathbb{R}} x$ via substitution. In conjunction with our previous observation $0 \in \{0, \ldots, \bar{n}\}$, this implies $0 \in I$ with the definition of the set I in (8.109). Thus, there exists an element in I, so that I is nonempty. Because every nonempty subset of the initial segment of $\mathbb{N}$ up to a natural number n has a greatest element (see Corollary 4.41a)), it follows from the previous findings $I \neq \emptyset$ and $I \subseteq \{0, \ldots, \bar{n}\}$ that the natural number $k = \max I$ exists. By definition of a maximum, this maximum is element of I, i.e.,

$$k \in I, \tag{8.111}$$

so that the preceding inclusion gives $k \in \{0, \ldots, \bar{n}\}$. This in turn implies $k \leq_{\mathbb{N}} \bar{n}$ with (4.180). Because $k \in I$ and (8.110) imply $k \neq \bar{n}$ with (2.4), the preceding inequality yields $k <_{\mathbb{N}} \bar{n}$ by definition of an induced irreflexive partial ordering. This inequality in turn implies $k +_{\mathbb{N}} 1 \leq_{\mathbb{N}} \bar{n}$ with (4.157), and therefore

$$k +_{\mathbb{N}} 1 \in \{0, \ldots, \bar{n}\} \tag{8.112}$$

with (4.180). Let us now prove $k +_{\mathbb{N}} 1 \notin I$ by contradiction, assuming the negation of that sentence to be true. Consequently, $k +_{\mathbb{N}} 1 \in I$ is clearly true, and since we found k to be the maximum of I, we therefore have $k +_{\mathbb{N}} 1 \leq_{\mathbb{N}} k$ by definition of an upper bound. The Negation Formula for $<_{\mathbb{N}}$ gives us then the negation $\neg k <_{\mathbb{N}} k +_{\mathbb{N}} 1$, in contradiction to the fact that $k <_{\mathbb{N}} k +_{\mathbb{N}} 1$ holds according to (4.153). Thus, the proof of $k +_{\mathbb{N}} 1 \notin I$ is complete, so that (8.109) yields the true negation $\neg[k +_{\mathbb{N}} 1 \in \{0, \ldots, \bar{n}\} \wedge t_{k+1} <_{\mathbb{R}} x]$. Due to De Morgan's Law for the conjunction, the disjunction of $\neg k +_{\mathbb{N}} 1 \in \{0, \ldots, \bar{n}\}$ and $\neg t_{k+1} <_{\mathbb{R}} x$ follows to be true, whose first part is false in view of the truth of (8.112). Thus, its second part $\neg t_{k+1} <_{\mathbb{R}} x$ is true, which in turn implies $x \leq_{\mathbb{R}} t_{k+1}$ with the Negation Formula for $<$. Let us observe that (8.111) implies $t_k <_{\mathbb{R}} x$ with (8.109), so that we found the particular numbers $t_k, t_{k+1} \in \mathbb{Q}$ that satisfy $x \in (t_k, t_{k+1}]$, according to the definition of a left-open and right-closed interval in $\mathbb{Q}$. Since we evidently obtain

$$\begin{aligned} t_{k+1} -_{\mathbb{Q}} t_k &= \left(\bar{p} +_{\mathbb{Q}} \frac{k +_{\mathbb{Q}} 1}{m}\right) -_{\mathbb{Q}} \left(\bar{p} +_{\mathbb{Q}} \frac{k}{m}\right) \\ &= \frac{k +_{\mathbb{Q}} 1}{m} -_{\mathbb{Q}} \frac{k}{m} \\ &= \frac{k +_{\mathbb{Q}} 1 -_{\mathbb{Q}} k}{m} \\ &= \frac{1}{m} \end{aligned}$$

with the computation rule for fields, the required inequality $t_{k+1} -_{\mathbb{Q}} t_k \leq_{\mathbb{Q}} \frac{1}{m}$ turns out to be true as well. These findings demonstrate the truth of the existential sentence (8.107), and as x and n were initially arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Definition 8.10 (Lattice of real numbers). We define according to Proposition 5.2 the binary operations

$$\sqcup_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto x \sqcup_{\mathbb{R}} y = \overset{\leq_{\mathbb{R}}}{\sup}\{x, y\}, \tag{8.113}$$

$$\sqcap_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto x \sqcap_{\mathbb{R}} y = \overset{\leq_{\mathbb{R}}}{\inf}\{x, y\}, \tag{8.114}$$

and we call

$$(\mathbb{R}, \sqcup_{\mathbb{R}}, \sqcap_{\mathbb{R}}, \leq_{\mathbb{R}}) \tag{8.115}$$

the *lattice of real numbers*.

Let us recall that Theorem 3.120 enables the Generation of lattices based on closed intervals.

Definition 8.11 ((Real) unit interval lattice). We call the lattice

$$([0, 1], \leq_{[0,1]}) = ([0, 1], \sqcup_{[0,1]}, \sqcap_{[0,1]}, \leq_{[0,1]}) \tag{8.116}$$

generated by the lattice of real numbers the (*real*) *unit interval lattice*.

Theorem 8.18 (Completeness of the real unit interval lattice). *The real unit interval lattice* $([0, 1], \leq_{[0,1]})$ *is a complete lattice.*

Proof. To show that $([0, 1], \leq_{[0,1]})$ is a complete lattice, we need to prove

$$\forall A\, (A \subseteq [0, 1] \Rightarrow \exists S, I\, (S, I \in [0, 1] \wedge S = \overset{\leq_{[0,1]}}{\sup} A \wedge I = \overset{\leq_{[0,1]}}{\inf} A)). \tag{8.117}$$

To do this, we take an arbitrary set A, assume $A \subseteq [0,1]$ to be true, so that that the definition of a subset yields

$$\forall x\,(x \in A \Rightarrow x \in [0,1]), \tag{8.118}$$

and we establish the truth of the desired consequent by considering the two cases $A = \emptyset$ and $A \neq \emptyset$. In the first case $A = \emptyset$, we use the evident fact $0 \leq_{\mathbb{R}} 1$, which implies with Corollary 3.118 that $0 = \min[0,1]$ and $1 = \max[0,1]$ are true, so that $0, 1 \in [0,1]$ holds according to the definitions of a minimum and maximum. This finding implies with Proposition 3.90 and Exercise 3.34 that $\bar{S} = 0$ is an upper bound and $\bar{I} = 1$ a lower bound for the empty set A (with respect to $\leq_{[0,1]}$). We now apply the Characterization of the supremum & infimum to demonstrate that $\bar{S}$ is the supremum and $\bar{I}$ the infimum of A (with respect to $\leq_{[0,1]}$). To do this, we let S' and I' be arbitrary such that S' is an upper bound and I' a lower bound for A (with respect to $\leq_{[0,1]}$). Thus, S' and I' are both elements of $[0,1]$, so that the inequalities $0 \leq_{\mathbb{R}} S'$ and $I' \leq_{\mathbb{R}} 1$ follow to be true with Proposition 3.117. Since the reflexive partial ordering $\leq_{[0,1]}$ was obtained from the reflexive partial ordering $\leq_{\mathbb{R}}$ according to the Reflexive partial ordering of subsets, we may write the previous two inequalities also as $0 \leq_{[0,1]} S'$ and $I' \leq_{[0,1]} 1$, recalling $0, S' \in [0,1]$ as well as $I', 1 \in [0,1]$. Since S' and I' are arbitrary, we thus see that the upper bound $\bar{S} = 0$ for A is the least one and that the lower bound $\bar{I} = 1$ for A is the greatest one (with respect to $\leq_{[0,1]}$). We thus proved the existential sentence in (8.117) for the case of $A = \emptyset$.

Regarding the second case $A \neq \emptyset$, Proposition 3.117b,c) shows that $[0,1]$ is a subset of $\mathbb{R}$ which is both bounded from below (by 0) and bounded from above (by 1) with respect to $\leq_{\mathbb{R}}$. Therefore, the subset A of $[0,1]$ is itself bounded from below (by 0) and bounded from above (by 1) with respect to $\leq_{\mathbb{R}}$, according to Exercise 3.38 and Proposition 3.94). Thus, the universal sentences

$$\forall x\,(x \in A \Rightarrow 0 \leq_{\mathbb{R}} x) \tag{8.119}$$

and

$$\forall x\,(x \in A \Rightarrow x \leq_{\mathbb{R}} 1) \tag{8.120}$$

are true. Recalling that the linear continuum $(\mathbb{R}, <_{\mathbb{R}})$ possesses both the Supremum Property and the Infimum Property, there are particular real numbers $\bar{S}$ and $\bar{I}$ such that $\bar{S} = \sup^{\leq_{\mathbb{R}}} A$ and $\bar{I} = \inf^{\leq_{\mathbb{R}}} A$ hold. As an upper bound for A, $\bar{S}$ satisfies

$$\forall x\,(x \in A \Rightarrow x \leq_{\mathbb{R}} \bar{S}), \tag{8.121}$$

and the (greatest) lower bound $\bar{I}$ for A satisfies

$$\forall x\,(x \in A \Rightarrow \bar{I} \leq_{\mathbb{R}} x). \tag{8.122}$$

Let us verify next that $\bar{S}$ and $\bar{I}$ are elements of $[0,1]$. To begin with, as we found 1 to be an upper bound for A and $\bar{S}$ to be the least upper bound for A with respect to $\leq_{\mathbb{R}}$, we obtain

$$\bar{S} \leq_{\mathbb{R}} 1 \tag{8.123}$$

with the Characterization of the supremum. Similarly, because 0 is a lower bound for A and $\bar{I}$ the greatest lower bound for A with respect to $\leq_{\mathbb{R}}$, we have

$$0 \leq_{\mathbb{R}} \bar{I} \tag{8.124}$$

according to the Characterization of the infimum. Then, the current case assumption $A \neq \emptyset$ implies the existence of a particular element $\bar{x} \in A$ because of (2.42), and this element satisfies

$$0 \leq_{\mathbb{R}} \bar{x} \wedge \bar{x} \leq_{\mathbb{R}} \bar{S}$$

in view of (8.119) and (8.121), as well as

$$\bar{I} \leq_{\mathbb{R}} \bar{x} \wedge \bar{x} \leq_{\mathbb{R}} 1$$

due to (8.122) and (8.120). Applying the transitivity of the total ordering $\leq_{\mathbb{R}}$ to the previous two conjunctions gives then

$$0 \leq_{\mathbb{R}} \bar{S} \tag{8.125}$$

as well as

$$\bar{I} \leq_{\mathbb{R}} 1. \tag{8.126}$$

Combining now (8.125) with (8.123) as well as (8.124) with (8.126), we then obtain indeed $\bar{S}, \bar{I} \in [0,1]$ with the definition of a closed interval.

In the next step, we may prove that $\bar{S}$ is an upper bound and $\bar{I}$ a lower bound for A with respect to $\leq_{[0,1]}$, i.e. that the universal sentences

$$\forall x\, (x \in A \Rightarrow x \leq_{[0,1]} \bar{S}) \tag{8.127}$$

and

$$\forall x\, (x \in A \Rightarrow \bar{I} \leq_{[0,1]} x) \tag{8.128}$$

are true. Letting $x \in A$ be arbitrary, we obtain $x \in [0,1]$ with (8.118), so that x, $\bar{S}$ and $\bar{I}$ are all in $[0,1]$. Furthermore, $x \in A$ implies $x \leq_{\mathbb{R}} \bar{S}$ with (8.121) and $\bar{I} \leq_{\mathbb{R}} x$ with (8.122). Because of $x, \bar{S} \in [0,1]$ and $\bar{I}, x \in [0,1]$, we may evidently write the former inequality also as $x \leq_{[0,1]} \bar{S}$ and the latter inequality as $\bar{I} \leq_{[0,1]} x$. As x is arbitrary, we may therefore conclude that the universal sentences (8.127) and (8.128) are true, so that $\bar{S}$ is indeed an upper bound and $\bar{I}$ a lower bound for A with respect to $\leq_{[0,1]}$.

It now remains for us to demonstrate that $\bar{S}$ is the least upper bound and $\bar{I}$ the greatest lower bound for A with respect to $\leq_{[0,1]}$. Letting S' and I' be arbitrary and assuming S' to be an upper bound as well as I' to be a lower bound for A with respect to $\leq_{[0,1]}$, we thus have the true universal sentences

$$\forall x\,(x \in A \Rightarrow x \leq_{[0,1]} S') \tag{8.129}$$

and

$$\forall x\,(x \in A \Rightarrow I' \leq_{[0,1]} x). \tag{8.130}$$

at our disposal. The task is now to establish $\bar{S} \leq_{[0,1]} S'$ and $I' \leq_{[0,1]} \bar{I}$. First, we verify that S' is an upper bound and I' a lower bound for A with respect to $\leq_{\mathbb{R}}$, i.e. that the universal sentences

$$\forall x\,(x \in A \Rightarrow x \leq_{\mathbb{R}} S') \tag{8.131}$$

and

$$\forall x\,(x \in A \Rightarrow I' \leq_{\mathbb{R}} x). \tag{8.132}$$

hold. Letting $x \in A$ be arbitrary (so that $x \in [0,1]$ is evidently true), it follows with (8.129) and (8.130) that the inequalities $x \leq_{[0,1]} S'$ and $I' \leq_{[0,1]} x$ hold. Clearly, x, S' and I' are all in $[0,1]$, so that we may write these inequalities also as $x \leq_{\mathbb{R}} S'$ and $I' \leq_{\mathbb{R}} x$, proving the implications in (8.131) and (8.132). As x was arbitrary, we may therefore conclude that the universal sentences (8.131) and (8.132) are true, so that S' indeed constitutes an upper bound and I' a lower bound for A with respect to $\leq_{\mathbb{R}}$. Because we already established $\bar{S}$ as the least upper bound and $\bar{I}$ as the greatest lower bound for A with respect to $\leq_{\mathbb{R}}$, we see that $\bar{S} \leq_{\mathbb{R}} S'$ and $I' \leq_{\mathbb{R}} \bar{I}$ are true. As mentioned before, the constants $\bar{S}$, S', $\bar{I}$ and I' are all elements of $[0,1]$, so that we may write the preceding inequalities in the desired form $\bar{S} \leq_{[0,1]} S'$ and $I' \leq_{[0,1]} \bar{I}$. Since S' and I' were arbitrary, we may infer from these findings that $\bar{S}$ is indeed the least upper bound and $\bar{I}$ the greatest lower bound for A with respect to $\leq_{[0,1]}$. Thus, the proof of the existential sentence in (8.117) is complete. Because A is arbitrary, we may therefore conclude that the universal sentence (8.117) is true, so that the real unit interval lattice $([0,1], \leq_{[0,1]})$ is complete, by definition. □

Note 8.7. The first case considered in the proof of the preceding theorem shows that the empty subset of the real unit interval has the supremum 0 and the infimum 1 with respect to $\leq_{[0,1]}$, that is,

$$\overset{\leq_{[0,1]}}{\sup}\,\emptyset = 0, \tag{8.133}$$

$$\overset{\leq_{[0,1]}}{\inf}\,\emptyset = 1. \tag{8.134}$$

We consider now functions whose codomains are specified by the set of real numbers.

Definition 8.12 (Real/real-valued function, real/real-valued sequence). We call

a) for any set X any function

$$f : X \to \mathbb{R} \tag{8.135}$$

a *real function* or a *real-valued function* (on X) and

$$\mathbb{R}^X \tag{8.136}$$

the *set of real functions* or the *set of real-valued functions* (on X).

b) in particular any sequence

$$(a_n)_{n\in\mathbb{N}_+}, \quad \text{or } (a_n)_{n\in\mathbb{N}} \tag{8.137}$$

in $\mathbb{R}$ a *real sequence* or a *real-valued sequence* and

$$\omega = \mathbb{R}^{\mathbb{N}_+}, \quad \text{or } \omega = \mathbb{R}^{\mathbb{N}} \tag{8.138}$$

the *set of real sequences* or the *set of real-valued sequences.*

Applying Corollary 5.5 to the lattice of real numbers, we immediately obtain corresponding lattices of functions.

Corollary 8.19. *The ordered pair* $(\mathbb{R}^X, \preceq_{\mathbb{R}^X})$ *is a lattice for any nonempty set* X.

Definition 8.13 (Lattice of real(-valued) functions). We define the join and meet on the set of real-valued functions by

$$\curlyvee_{\mathbb{R}^X} : \mathbb{R}^X \times \mathbb{R}^X \to \mathbb{R}^X, \quad (f, g) \mapsto f \curlyvee g = \sup^{\preceq_{\mathbb{R}^X}}\{f, g\}, \tag{8.139}$$

$$\curlywedge_{\mathbb{R}^X} : \mathbb{R}^X \times \mathbb{R}^X \to \mathbb{R}^X, \quad (f, g) \mapsto f \curlywedge g = \inf^{\preceq_{\mathbb{R}^X}}\{f, g\}, \tag{8.140}$$

and we call

$$(\mathbb{R}^X, \curlyvee_{\mathbb{R}^X}, \curlywedge_{\mathbb{R}^X}, \preceq_{\mathbb{R}^X}) \tag{8.141}$$

the *lattice of real-valued functions* (*on* X).

Proposition 8.20. *It is true for any nonempty set* X *and any functions* $f, g \in \mathbb{R}^X$ *that the pointwise supremum of* $\{f, g\}$ *satisfies*

$$\forall x\, (x \in X \Rightarrow (f \curlyvee_{\mathbb{R}^X} g)(x) = \max^{\leq_{\mathbb{R}}}\{f(x), g(x)\}), \tag{8.142}$$

$$\forall x\, (x \in X \Rightarrow (f \curlywedge_{\mathbb{R}^X} g)(x) = \min^{\leq_{\mathbb{R}}}\{f(x), g(x)\}). \tag{8.143}$$

Proof. Letting X be an arbitrary nonempty set, f and g arbitrary real functions on X, and x an arbitrary element of X, we obtain

$$(f \curlyvee_{\mathbb{R}^X} g)(x) = f(x) \sqcup_{\mathbb{R}} g(x) = \sup^{\leq_{\mathbb{R}}} \{f(x), g(x)\} = \max^{\leq_{\mathbb{R}}} \{f(x), g(x)\}$$

with the definition of $\curlyvee_{\mathbb{R}^X}$, the definition of $\sqcup_{\mathbb{R}}$, and Proposition 3.113 in connection with the fact that $(\mathbb{R}, \leq_{\mathbb{R}})$ is a totally ordered set. Similarly, we obtain

$$(f \curlywedge_{\mathbb{R}^X} g)(x) = f(x) \sqcap_{\mathbb{R}} g(x) = \inf^{\leq_{\mathbb{R}}} \{f(x), g(x)\} = \min^{\leq_{\mathbb{R}}} \{f(x), g(x)\}$$

with the definition of $\curlywedge_{\mathbb{R}^X}$, the definition of $\sqcap_{\mathbb{R}}$, and Exercise 3.52. As x is arbitrary, we conclude that (8.142) and (8.143) are true. As X, f and g were also arbitrary, we further conclude that the proposition holds. □

Proposition 8.21. *For any nonempty set X and any real-valued functions f, g on X, it is true that the ranges of the pointwise supremum and of the pointwise infimum of the pair $\{f, g\}$ are included in the union of the ranges of f and g, that is,*

$$\operatorname{ran}(f \curlyvee_{\mathbb{R}^X} g) \subseteq \operatorname{ran}(f) \cup \operatorname{ran}(g) \tag{8.144}$$

$$\operatorname{ran}(f \curlywedge_{\mathbb{R}^X} g) \subseteq \operatorname{ran}(f) \cup \operatorname{ran}(g) \tag{8.145}$$

Proof. Letting X be an arbitrary nonempty set, f and g arbitrary elements of $\mathbb{R}^X$, and y an arbitrary element of $\operatorname{ran}(f \curlyvee_{\mathbb{R}^X} g)$, it follows with the definition of a range that there is a particular constant $\bar{x}$ such that $(\bar{x}, y) \in f \curlyvee_{\mathbb{R}^X} g$. This yields

$$y = (f \curlyvee_{\mathbb{R}^X} g)(\bar{x}) = \max^{\leq_{\mathbb{R}}} \{f(\bar{x}), g(\bar{x})\}.$$

using function notation and (8.142). Because the maximum y of the set $\{f(\bar{x}), g(\bar{x})\}$ is by definition an element of that set, we have $y \in \{f(\bar{x}), g(\bar{x})\}$. By definition of a pair, we therefore find that $y = f(\bar{x})$ or $y = g(\bar{x})$ is true. We now use this true disjunction to prove the $y \in \operatorname{ran}(f) \lor y \in \operatorname{ran}(g)$ by cases. On the one hand, if $y = f(\bar{x})$ holds, which we may also write as $(\bar{x}, y) \in f$, we see that $y \in \operatorname{ran}(f)$ follows to be true with the definition of a range. Then, the disjunction $y \in \operatorname{ran}(f) \lor y \in \operatorname{ran}(g)$ also holds. On the other hand, if $y = g(\bar{x})$ holds, which we may also write as $(\bar{x}, y) \in g$, we clearly see that $y \in \operatorname{ran}(g)$ is now true. Consequently, the disjunction $y \in \operatorname{ran}(f) \lor y \in \operatorname{ran}(g)$ holds also in the second case. This implies now with the definition of the union of two sets $y \in \operatorname{ran}(f) \cup \operatorname{ran}(g)$. Since y is arbitrary, we may, according to the definition of a subset, infer from this finding the truth of the inclusion (8.144). The second inclusion can be proved similarly. These inclusions are then evidently true for any X and any real-valued functions f, g on X. □

Exercise 8.10. Prove the inclusion (8.145).

Proposition 8.22. *The following implications hold for any set X and any sequence $(f_n)_{n\in\mathbb{N}_+}$ of real-valued functions on X.*

a) If the sequence $(f_n)_{n\in\mathbb{N}_+}$ is bounded from above, then its pointwise supremum exists i.e.,

$$\exists g\,(g \in \mathbb{R}^X \,\wedge\, g = \sup_{n\in\mathbb{N}_+} f_n \,\text{pointwise}). \tag{8.146}$$

b) If the sequence $(f_n)_{n\in\mathbb{N}_+}$ is increasing and bounded from above, then it has a limit, i.e.,

$$\exists g\,(g \in \mathbb{R}^X \,\wedge\, g = \lim_{n\to\infty} f_n). \tag{8.147}$$

Proof. We let X be an arbitrary set and $f = (f_n)_{n\in\mathbb{N}_+}$ an arbitrary sequence of real functions on X. Concerning a), we assume that the range of that sequence $f : \mathbb{N}_+ \to \mathbb{R}^X$, which is evidently included in $\mathbb{R}^X$, has an upper bound (with respect to $\preceq_{\mathbb{R}^X}$), say $\bar{f}$. As this upper bound is by definition an element of $\mathbb{R}^X$, it is a function $\bar{f} : X \to \mathbb{R}$. This assumption implies with the Characterization of upper bounds for a family of functions that $\bar{f}(x)$ is an upper bound for the range of the sequence $f^{(x)} = (f_n(x))_{n\in\mathbb{N}_+}$ (with respect to $\leq_{\mathbb{R}}$) for any $x \in X$. Note that the domain $\mathbb{N}_+$ of these sequences is nonempty, so that their ranges are nonempty as well due to (3.119). Since, for any $x \in X$, the range of $(f_n(x))_{n\in\mathbb{N}_+}$ is a nonempty and bounded-from-above subset of $\mathbb{R}$, it has a supremum by virtue of the Supremum Property of the linear continuum $(\mathbb{R}, <)$. This finding demonstrates the truth of the universal sentence

$$\forall x\,(x \in X \Rightarrow \exists!y\,(y = \sup\nolimits^{\leq_{\mathbb{R}}} \operatorname{ran}(f^{(x)}))),$$

so that there exists (according to Function definition by replacement) a unique function g with domain X such that

$$\forall x\,(x \in X \Rightarrow g(x) = \sup\nolimits^{\leq_{\mathbb{R}}} \operatorname{ran}(f^{(x)})).$$

Next, we demonstrate that the range of g is included in $\mathbb{R}$. Letting for this purpose $y \in \operatorname{ran}(g)$ be arbitrary, there exists (by definition of a range) a particular constant $\bar{x}$ such that $(\bar{x}, y) \in g$. In view of the definition of a domain, we thus have $\bar{x} \in X$, so that the specification of the function g gives the associated value $y = g(\bar{x}) = \sup^{\leq_{\mathbb{R}}} \operatorname{ran}(f^{(\bar{x})})$. Because the supremum is taken with respect to $\leq_{\mathbb{R}}$, it is true that $y \in \mathbb{R}$. We thus showed that $y \in \operatorname{ran}(g)$ implies $y \in \mathbb{R}$, and since y was arbitrary, we may therefore conclude that the range of g is a subset of $\mathbb{R}$. The latter set therefore constitutes a codomain of g, so that g is indeed a function from X to Y. The previous findings imply with the Characterization of the supremum of a sequence of functions that g is the supremum of (the range of) the sequence $f = (f_n)_{n \in \mathbb{N}_+}$ with respect to $\preceq_{\mathbb{R}^X}$, that is, $g = \sup^{\preceq_{\mathbb{R}^X}} \mathrm{ran}(f)$. We now see in light of Notation 3.11 that g is the pointwise supremum of f, that is, $g = \sup_{n \in \mathbb{N}_+} f_n$ pointwise. The existence of such a function demonstrates the truth of a).

Concerning b), we assume that $(f_n)_{n \in \mathbb{N}_+}$ is now both increasing an bounded from above (with respect to $\preceq_{\mathbb{R}^X}$). The latter implies with a) that the pointwise supremum g of that sequence exists. Then, as $(f_n)_{n \in \mathbb{N}_+}$ is also increasing, the supremum g is by definition the limit of that sequence, that is, $g = \lim_{n \to \infty}^{\leq_{\mathbb{R}^X}} f_n$. This proves b).

Since X and $(f_n)_{n \in \mathbb{N}_+}$ are arbitrary, we may finally conclude that the proposition holds. □

Exercise 8.11. Verify the following implications for any set X and any sequence $(f_n)_{n \in \mathbb{N}_+}$ of real-valued functions on X.

a) If the sequence $(f_n)_{n \in \mathbb{N}_+}$ is bounded from below, then its pointwise infimum exists, i.e.,

$$\exists g\,(g \in \mathbb{R}^X \ \wedge\ g = \inf_{n \in \mathbb{N}_+} f_n \,\text{pointwise}). \tag{8.148}$$

b) If the sequence $(f_n)_{n \in \mathbb{N}_+}$ is decreasing and bounded from below, then it has a limit, i.e., (8.147) holds again.

In the remainder of the current section, we investigate certain intervals.

Lemma 8.23. *It is true for any densely ordered set $(X, <)$ for which the minimum and the maximum of X do not exist and for any nonempty convex set $\mathcal{A}$ in the set $\mathcal{D}$ of Dedekind cuts in X with respect to $<_{\mathcal{D}}$ that*

a) if $\mathcal{A}$ is bounded from above and from below, then

$$(\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}) \ \subseteq\ \mathcal{A}\ \subseteq\ [\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}]. \tag{8.149}$$

b) if $\mathcal{A}$ is bounded from above but not bounded from below, then

$$(-\infty, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}) \ \subseteq\ \mathcal{A}\ \subseteq\ (-\infty, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}]. \tag{8.150}$$

c) if $\mathcal{A}$ is not bounded from above but bounded from below, then

$$(\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, +\infty) \ \subseteq\ \mathcal{A}\ \subseteq\ [\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, +\infty). \tag{8.151}$$

d) if $\mathcal{A}$ is neither bounded from above nor bounded from below, then

$$\mathcal{A} = \mathcal{D}. \tag{8.152}$$

Proof. We let X, $<$ and $\mathcal{A}$ be arbitrary sets, assuming $(X, <)$ to be a densely ordered set such that the minimum and the maximum of X do not exist, and assuming $\mathcal{A}$ to be a nonempty convex set in $\mathcal{D}$ with respect to $<_{\mathcal{D}}$. According to the Completion of densely ordered sets, $(\mathcal{D}, <_{\mathcal{D}})$ satisfies the the denseness condition

$$\forall x, y\,([x, y \in \mathcal{D} \wedge x <_{\mathcal{D}} y] \Rightarrow \exists z\,(z \in \mathcal{D} \wedge x <_{\mathcal{D}} z <_{\mathcal{D}} y)), \tag{8.153}$$

the Supremum Property and the Infimum Property. We also note that $\mathcal{A}$ satisfies $\mathcal{A} \neq \emptyset$, $\mathcal{A} \subseteq \mathcal{D}$ as well as

$$\forall a, b\,(a, b \in \mathcal{A} \Rightarrow \forall x\,(a <_{\mathcal{D}} x <_{\mathcal{D}} b \Rightarrow x \in \mathcal{A})) \tag{8.154}$$

(see Exercise 3.67).

Concerning a), we assume that $\mathcal{A}$ is both bounded from above and bounded from below. Then, the preceding assumptions imply that the supremum and the infimum of $\mathcal{A}$ exist indeed. We establish now the first inclusion in (8.149) by proving (according to the definition of a subset)

$$\forall x\,(x \in (\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A})_{\mathcal{D}} \Rightarrow x \in \mathcal{A}). \tag{8.155}$$

We let x be arbitrary and assume $x \in (\inf^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ to be true, which means by definition of an open interval in $\mathcal{D}$ that the inequalities

$$\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A} <_{\mathcal{D}} x$$
$$x <_{\mathcal{D}} \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}$$

are true. Noting that x, $\inf^{\leq_{\mathcal{D}}} \mathcal{A}$ and $\sup^{\leq_{\mathcal{D}}} \mathcal{A}$ are elements of $\mathcal{D}$, the preceding inequalities imply with the denseness property (8.153) of $\mathcal{D}$ the existence of further particular elements $I', S' \in \mathcal{D}$ satisfying, respectively,

$$\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A} <_{\mathcal{D}} I' <_{\mathcal{D}} x,$$
$$x <_{\mathcal{D}} S' <_{\mathcal{D}} \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}.$$

According to the Infimum Criterion and the Supremum Criterion, there exist then particular elements $\bar{z}_1, \bar{z}_2 \in \mathcal{A}$ such that

$$\bar{z}_1 <_{\mathcal{D}} I',$$
$$S' <_{\mathcal{D}} \bar{z}_2.$$

The preceding inequalities imply now with the transitivity of the linear ordering $<_{\mathcal{D}}$ that

$$\bar{z}_1 <_{\mathcal{D}} x,$$
$$x <_{\mathcal{D}} \bar{z}_2.$$

are true, which inequalities we can combine to

$$\bar{z}_1 <_{\mathcal{D}} x <_{\mathcal{D}} \bar{z}_2.$$

In conjunction with $\bar{z}_1, \bar{z}_2 \in \mathcal{A}$, this gives us now with the convexity property (8.154) $x \in \mathcal{A}$, which is the desired consequent of the implication in (8.155). Since x was arbitrary, we may therefore conclude that the universal sentence (8.155) holds, so that the first inclusion $(\inf^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A}) \subseteq \mathcal{A}$ in (8.149) is indeed true.

To establish the other inclusion, we prove

$$\forall x\,(x \in \mathcal{A} \Rightarrow x \in [\inf^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A}]_{\mathcal{D}}), \tag{8.156}$$

letting $x \in \mathcal{A}$ be arbitrary. Since the infimum $\inf^{\leq_{\mathcal{D}}} \mathcal{A}$ is by definition a lower bound and the supremum $\sup^{\leq_{\mathcal{D}}} \mathcal{A}$ an upper bound for $\mathcal{A}$, we obtain then the inequalities

$$\inf^{\leq_{\mathcal{D}}} \mathcal{A} \leq_{\mathcal{D}} x \,\wedge\, x \leq_{\mathcal{D}} \sup^{\leq_{\mathcal{D}}} \mathcal{A},$$

which imply then the desired consequent

$$x \in [\inf^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A}]_{\mathcal{D}}$$

with the definition of a closed interval in $\mathcal{D}$. Because x is arbitrary, we can infer from this the truth of (8.156) and consequently the truth of the second inclusion in (8.149). We thus completed the proof of a).

Concerning b), we assume $\mathcal{A}$ still to be bounded from above, but we assume now that $\mathcal{A}$ is not bounded from below, that is,

$$\neg\exists a\,(a \in X \,\wedge\, \forall y\,(y \in \mathcal{A} \Rightarrow a \leq_{\mathcal{D}} y)).$$

This negation implies with the Negation Law for existential conjunctions that the universal sentence

$$\forall a\,(a \in X \Rightarrow \neg\forall y\,(y \in \mathcal{A} \Rightarrow a \leq_{\mathcal{D}} y)) \tag{8.157}$$

holds. Let us now establish the first inclusion in (8.150) via the proof of

$$\forall x\,(x \in (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A})_{\mathcal{D}} \Rightarrow x \in \mathcal{A}). \tag{8.158}$$

The assumption $x \in (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A})_{\mathcal{D}}$ implies by definition of an open and left-unbounded interval

$$x \;<_{\mathcal{D}}\; \sup^{\leq_{\mathcal{D}}} \mathcal{A}.$$

Here, x and the supremum are evidently elements of $\mathcal{D}$, so that (8.153) gives us a particular element S' of $\mathcal{D}$ strictly between x and the supremum, i.e.

$$x \;<_{\mathcal{D}}\; S' \;<_{\mathcal{D}}\; \sup^{\leq_{\mathcal{D}}} \mathcal{A}.$$

Because of the Supremum Criterion, the second inequality implies the existence of a particular element $\bar{z} \in \mathcal{A}$ greater than S', that is,

$$S' \;<_{\mathcal{D}}\; \bar{z}.$$

Combining the preceding inequalities and using the transitivity of $<_{\mathcal{D}}$, we obtain now

$$x \;<_{\mathcal{D}}\; \bar{z}. \tag{8.159}$$

On the other hand, (8.157) yields the true negation

$$\neg\forall y\,(y \in \mathcal{A} \Rightarrow x \leq_{\mathcal{D}} y),$$

which implies with the Negation Law for universal implications that there exists a constant, say $\bar{y}$, such that $\bar{y} \in \mathcal{A}$ and $\neg x \leq_{\mathcal{D}} \bar{y}$. The latter gives

$$\bar{y} \;<_{\mathcal{D}}\; x$$

with the Negation Formula for $\leq$. We can fuse this inequality with the inequality (8.159) and write

$$\bar{y} \;<_{\mathcal{D}}\; x \;<_{\mathcal{D}}\; \bar{z}.$$

As $\bar{y}, \bar{z} \in \mathcal{A}$ also holds, we obtain therefore $x \in \mathcal{A}$ by virtue of (8.154), which proves the implication in (8.158). Consequently, the first inclusion $(-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A}) \subseteq \mathcal{A}$ in (8.150) turns out to be true since x was arbitrary.

Regarding the second inclusion $\mathcal{A} \subseteq (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A}]$, we consider the equivalent universal sentence

$$\forall x\,(x \in \mathcal{A} \Rightarrow x \in (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A}]) \tag{8.160}$$

and observe that the antecedent $x \in \mathcal{A}$ implies $x \leq_{\mathcal{D}} \sup^{\leq_{\mathcal{D}}}$ as in a) because the supremum of $\mathcal{A}$ is an upper bound for that set. In view of the definition of a left-unbounded and right-closed interval, we therefore find $x \in (-\infty, \sup^{\leq_\mathcal{D}} \mathcal{A}]$ to be true. This proves the implication in (8.160), in which x is arbitrary, so that the universal sentence (8.160) follows to be true. We thus demonstrated the truth also of the second inclusion in (8.150), so that b) holds, too.

The proof of c) is carried out in analogy to the proof of b).

Finally, we assume with respect to d) that $\mathcal{A}$ is neither bounded from above nor bounded from below. We establish in the following the inclusion $\mathcal{D} \subseteq \mathcal{A}$ by means of the definition of a subset, by proving accordingly

$$\forall x \, (x \in \mathcal{D} \Rightarrow x \in \mathcal{A}). \tag{8.161}$$

To to this, we let x be arbitrary, and we assume $x \in \mathcal{D}$ to be true. Since $\mathcal{A}$ is not bounded from below, the preceding assumption implies the existence of a particular element $\bar{y}_1 \in \mathcal{A}$ such that $\bar{y}_1 <_\mathcal{D} x$, as shown in the proof of b). Similarly, the proof of c) showed that the assumption that $\mathcal{A}$ is not bounded from above gives rise to the existence of a particular element $\bar{y}_2 \in \mathcal{A}$ with $x <_\mathcal{D} \bar{y}_2$. We thus found $\bar{y}_1, \bar{y}_2 \in \mathcal{A}$ and $\bar{y}_1 <_\mathcal{D} x <_\mathcal{D} \bar{y}_2$ to be true, so that $x \in \mathcal{A}$ also holds according to (8.154). As x was arbitrary, we therefore conclude that the universal sentence (8.161), so that the inclusion $\mathcal{D} \subseteq \mathcal{A}$ holds indeed. Because the reversed inclusion $\mathcal{A} \subseteq \mathcal{D}$ is true as well, the proposed equation $\mathcal{A} = \mathcal{D}$ follows to be true with the Axiom of Extension.

Initially, the sets X, $<$ and $\mathcal{A}$ were all arbitrary, allowing us to infer from the truth of a) – d) the truth of the stated lemma. $\square$

Exercise 8.12. Establish Part c) of Lemma 8.23.

Because of the preceding lemma, the notions of an interval and of a convex turn out to be equivalent for linear continua.

Theorem 8.24 (Equivalence of intervals and convex sets in linear continua). *It is true for any densely ordered set $(X, <)$ for which the minimum and the maximum of X do not exist that a set $\mathcal{A}$ is convex in the set $\mathcal{D}$ of Dedekind cuts in X with respect to $<_\mathcal{D}$ iff $\mathcal{A}$ is an interval in $\mathcal{D}$. In particular,* *a) if $\mathcal{A}$ is nonempty, bounded from above and bounded from below, then*

$$\left[\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A} \in \mathcal{A} \wedge \overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A} \in \mathcal{A}\right] \Rightarrow \mathcal{A} = [\overset{\leq_{\mathcal{D}}}{\min}\mathcal{A}, \overset{\leq_{\mathcal{D}}}{\max}\mathcal{A}], \tag{8.162}$$

$$\left[\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A} \notin \mathcal{A} \wedge \overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A} \notin \mathcal{A}\right] \Rightarrow \mathcal{A} = (\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A}), \tag{8.163}$$

$$\left[\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A} \notin \mathcal{A} \wedge \overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A} \in \mathcal{A}\right] \Rightarrow \mathcal{A} = (\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A}, \overset{\leq_{\mathcal{D}}}{\max}\mathcal{A}], \tag{8.164}$$

$$\left[\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A} \in \mathcal{A} \wedge \overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A} \notin \mathcal{A}\right] \Rightarrow \mathcal{A} = [\overset{\leq_{\mathcal{D}}}{\min}\mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A}). \tag{8.165}$$

b) if $\mathcal{A}$ is nonempty, bounded from above and not bounded from below, then

$$\overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A} \in \mathcal{A} \Rightarrow \mathcal{A} = (-\infty, \overset{\leq_{\mathcal{D}}}{\max}\mathcal{A}], \tag{8.166}$$

$$\overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A} \notin \mathcal{A} \Rightarrow \mathcal{A} = (-\infty, \overset{\leq_{\mathcal{D}}}{\sup}\mathcal{A}). \tag{8.167}$$

c) if $\mathcal{A}$ is nonempty, not bounded from above and bounded from below, then

$$\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A} \in \mathcal{A} \Rightarrow \mathcal{A} = [\overset{\leq_{\mathcal{D}}}{\min}\mathcal{A}, +\infty), \tag{8.168}$$

$$\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A} \notin \mathcal{A} \Rightarrow \mathcal{A} = (\overset{\leq_{\mathcal{D}}}{\inf}\mathcal{A}, +\infty). \tag{8.169}$$

Proof. We take arbitrary sets X, $<$ and $\mathcal{A}$, we assume that $(X, <)$ is densely ordered such that the minimum and the maximum of X do not exist, and we assume for the first part ('$\Rightarrow$') of the stated equivalence that $\mathcal{A}$ is convex in $\mathcal{D}$ with respect to $<_{\mathcal{D}}$. We consider first the two cases $\mathcal{A} = \emptyset$ and $\mathcal{A} \neq \emptyset$, based on the fact that the Law of the Excluded Middle gives rise to the true disjunction $\mathcal{A} = \emptyset \vee \mathcal{A} \neq \emptyset$. In case of $\mathcal{A} = \emptyset$, we have an interval according to Definition 3.35. In the other case of $\mathcal{A} \neq \emptyset$, we can use Lemma 8.23 to show that $\mathcal{A}$ is an interval. To do this, we observe that $\mathcal{A}$ is bounded from above or not bounded from above, and we use this true disjunction for a proof by (sub-)cases.

In the first sub-case, we assume that $\mathcal{A}$ is bounded from above. Then, it is clearly also true that $\mathcal{A}$ is bounded from below or not bounded from below, so that we can consider two further sub-cases. If $\mathcal{A}$ is bounded from below, then, the inclusions (8.149) hold. Evidently, we have now the two conjunctions

$$\inf^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A} \ \vee \ \inf^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}, \tag{8.170}$$

$$\sup^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A} \ \vee \ \sup^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}, \tag{8.171}$$

which allows us to consider further sub-cases.

We assume first $\inf^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A}$ and $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A}$ to be true. Being thus a lower bound for $\mathcal{A}$ contained in that set, the infimum of $\mathcal{A}$ constitutes therefore the minimum of $\mathcal{A}$ by definition, that is,

$$\inf^{\leq_{\mathcal{D}}} \mathcal{A} = \min^{\leq_{\mathcal{D}}} \mathcal{A}. \tag{8.172}$$

Similarly, the fact that the upper bound $\sup^{\leq_{\mathcal{D}}} \mathcal{A}$ for $\mathcal{A}$ is contained in that set implies by definition that it is the maximum of $\mathcal{A}$, that is,

$$\sup^{\leq_{\mathcal{D}}} \mathcal{A} = \max^{\leq_{\mathcal{D}}} \mathcal{A}. \tag{8.173}$$

Next, we observe that Lemma 8.23a) gives, in connection with (8.172) and (8.173), the inclusion

$$\mathcal{A} \ \subseteq \ [\min^{\leq_{\mathcal{D}}} \mathcal{A}, \max^{\leq_{\mathcal{D}}} \mathcal{A}]. \tag{8.174}$$

We establish now the converse inclusion $[\min^{\leq_{\mathcal{D}}} \mathcal{A}, \max^{\leq_{\mathcal{D}}} \mathcal{A}] \subseteq \mathcal{A}$ by means of the definition of a subset, that is, by verifying

$$\forall x \, (x \in [\min^{\leq_{\mathcal{D}}} \mathcal{A}, \max^{\leq_{\mathcal{D}}} \mathcal{A}] \Rightarrow x \in \mathcal{A}). \tag{8.175}$$

Letting x be arbitrary and assuming the antecedent to be true, we obtain by definition of a closed interval

$$\min^{\leq_{\mathcal{D}}} \mathcal{A} \ \leq_{\mathcal{D}} \ x \ \leq_{\mathcal{D}} \ \max^{\leq_{\mathcal{D}}} \mathcal{A}.$$

By definition of an induced irreflexive partial ordering, these two inequalities imply the two disjunctions

$$\min^{\leq_{\mathcal{D}}} \mathcal{A} <_{\mathcal{D}} x \ \vee \ \min^{\leq_{\mathcal{D}}} \mathcal{A} = x, \tag{8.176}$$

$$x <_{\mathcal{D}} \max^{\leq_{\mathcal{D}}} \mathcal{A} \ \vee \ x = \max^{\leq_{\mathcal{D}}} \mathcal{A}. \tag{8.177}$$

In case of $\min^{\leq_{\mathcal{D}}} \mathcal{A} <_{\mathcal{D}} x$ and, on the one hand, in case of $x <_{\mathcal{D}} \max^{\leq_{\mathcal{D}}} \mathcal{A}$, we get

$$x \in (\min^{\leq_{\mathcal{D}}} \mathcal{A}, \max^{\leq_{\mathcal{D}}} \mathcal{A})$$

with the definition of an open interval. Because Lemma 8.23a) gives, in connection with (8.172) and (8.173) the inclusion

$$(\overset{\leq_{\mathcal{D}}}{\min} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\max} \mathcal{A}) \subseteq \mathcal{A},$$

$x \in \mathcal{A}$ follows to be true (by definition of a subset), as desired in the current proof of (8.175). On the other hand, the case of $x = \max^{\leq_{\mathcal{D}}} \mathcal{A}$ yields immediately $x \in \mathcal{A}$ with the aforementioned fact that the maximum of $\mathcal{A}$ is contained in $\mathcal{A}$. Similarly, the other case $\min^{\leq_{\mathcal{D}}} \mathcal{A} = x$ gives us directly $x \in \mathcal{A}$ because we already know that the minimum of $\mathcal{A}$ is contained in $\mathcal{A}$. We thus completed the proof of the implication in (8.175), in which x is arbitrary, so that the inclusion $[\min^{\leq_{\mathcal{D}}} \mathcal{A}, \max^{\leq_{\mathcal{D}}} \mathcal{A}] \subseteq \mathcal{A}$ follows to be true. In conjunction with the reversed inclusion (8.174), this implies with the Axiom of Extension the proposed equation $\mathcal{A} = [\min^{\leq_{\mathcal{D}}} \mathcal{A}, \max^{\leq_{\mathcal{D}}} \mathcal{A}]$, thereby proving the implication (8.162). This shows in light of Definition 3.35 that $\mathcal{A}$ is an interval.

Let us return to the disjunction (8.171) and let us assume now its second part $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$ to be true. Here, we note that (8.172) is still true, whereas the supremum of $\mathcal{A}$ cannot be the maximum of $\mathcal{A}$. Thus, Lemma 8.23a) gives us now the inclusions

$$(\overset{\leq_{\mathcal{D}}}{\min} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}) \subseteq \mathcal{A} \subseteq [\overset{\leq_{\mathcal{D}}}{\min} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}]. \tag{8.178}$$

We demonstrate in the sequel that the equation $\mathcal{A} = [\min^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ holds in this case. We apply for this purpose the Equality Criterion for sets and prove accordingly the universal sentence

$$\forall x\, (x \in \mathcal{A} \Leftrightarrow x \in [\overset{\leq_{\mathcal{D}}}{\min} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A})\,), \tag{8.179}$$

letting x be arbitrary. Regarding the implication '$\Rightarrow$', we assume $x \in \mathcal{A}$ to be true, so that the second inclusion in (8.178) yields in connection with the definition of a closed interval

$$\overset{\leq_{\mathcal{D}}}{\min} \mathcal{A} \leq_{\mathcal{D}} x \leq_{\mathcal{D}} \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}.$$

Here, the second inequality implies $x <_{\mathcal{D}} \sup^{\leq_{\mathcal{D}}} \mathcal{A}$ with the definition of an induced irreflexive partial ordering, because $x \in \mathcal{A}$ and $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$ gives $x \neq \sup^{\leq_{\mathcal{D}}} \mathcal{A}$ with (2.4). We therefore have the inequalities

$$\overset{\leq_{\mathcal{D}}}{\min} \mathcal{A} \leq_{\mathcal{D}} x <_{\mathcal{D}} \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}, \tag{8.180}$$

so that the desired

$$x \in [\overset{\leq_{\mathcal{D}}}{\min} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}) \tag{8.181}$$

follows to be true by definition of a left-closed and right-open interval. Regarding the implication '$\Leftarrow$' in (8.179), we assume (8.181) to be true, so that (8.180) also holds (by definition of a left-closed and right-open interval). The first inequality implies then evidently the disjunction

$$\overset{\leq_{\mathcal{D}}}{\min} \mathcal{A} <_{\mathcal{D}} x \;\vee\; \overset{\leq_{\mathcal{D}}}{\min} \mathcal{A} = x,$$

which we use to carry out a proof by cases. On the one hand, the case $\min^{\leq_{\mathcal{D}}} \mathcal{A} <_{\mathcal{D}} x$ implies in conjunction with the second inequality in (8.180) by definition of an open interval that $x \in (\min^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ holds; consequently, we obtain with the first inclusion in (8.178) the desired $x \in \mathcal{A}$. On the other hand, the case $\min^{\leq_{\mathcal{D}}} \mathcal{A} = x$ immediately gives $x \in \mathcal{A}$, recalling that $\min^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A}$ is true. We thus completed the proof of the equivalence in (8.179), where x is arbitrary, so that the equality $\mathcal{A} = [\min^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ follows to be true. In view of the current case assumptions, this equations demonstrates that the implication (8.165) is true. Thus, $\mathcal{A}$ is a left-closed and right-open interval.

Having dealt with the disjunction (8.171) in the current first case, we go back to the disjunction (8.170) and consider the second case $\inf^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$. This infimum can then not be the minimum of $\mathcal{A}$. We now consider again the first sub-case $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A}$ within the disjunction (8.171), so that (8.173) holds. Consequently, Lemma 8.23a) yields the inclusions

$$(\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\max} \mathcal{A}) \;\subseteq\; \mathcal{A} \;\subseteq\; [\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\max} \mathcal{A}], \tag{8.182}$$

and we can proceed in analogy to the preceding proof of (8.165) that the implication (8.164) follows to be true. This means that $\mathcal{A}$ turns out to be a left-open and right-closed interval.

Concerning the disjunctions (8.170) – (8.171), it remains for us to inspect the case of $\inf^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$ in connection with $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$. Thus, the infimum cannot be the minimum and the supremum cannot be the maximum of $\mathcal{A}$. We therefore take over the inclusions

$$(\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}) \;\subseteq\; \mathcal{A} \;\subseteq\; [\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}] \tag{8.183}$$

from Lemma 8.23a) without modifications. We establish now the inclusion $\mathcal{A} \subseteq (\inf^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ by verifying equivalently (using the definition of a subset)

$$\forall x\, (x \in \mathcal{A} \Rightarrow x \in (\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A})\,). \tag{8.184}$$

Letting x be arbitrary and assuming $x \in \mathcal{A}$ to be true, we obtain with the second inclusion in (8.183)

$$x \in [\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A}, \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}]$$

and therefore by definition of a closed interval

$$\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A} \leq_{\mathcal{D}} x \leq_{\mathcal{D}} \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A}.$$

Observing that $x \in \mathcal{A}$, $\inf^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$ and $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$ imply $\inf^{\leq_{\mathcal{D}}} \mathcal{A} \neq x$ and $x \neq \sup^{\leq_{\mathcal{D}}} \mathcal{A}$ with (2.4), the preceding inequalities evidently give

$$\overset{\leq_{\mathcal{D}}}{\inf} \mathcal{A} <_{\mathcal{D}} x <_{\mathcal{D}} \overset{\leq_{\mathcal{D}}}{\sup} \mathcal{A},$$

so that $x \in (\inf^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ holds by definition of an open interval. Because x was arbitrary, we can infer from this finding the truth of the universal sentence (8.184) and therefore the truth of the inclusion $\mathcal{A} \subseteq (\inf^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$. Together with the first inclusion in (8.183), this yields the equation $\mathcal{A} = (\inf^{\leq_{\mathcal{D}}} \mathcal{A}, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$, so that we established the implication (8.163). As this means that $\mathcal{A}$ is an open interval, we completed the proof of a), so that the nonempty set $\mathcal{A}$ is an interval in case $\mathcal{A}$ is bounded from above and bounded from below.

We consider now the other sub-case that $\mathcal{A}$ is not bounded from below (while being still bounded from above). Therefore, the inclusions (8.150) are true, and since the supremum of $\mathcal{A}$ thus exists, we obtain once again the true disjunction (8.171). In case of $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A}$, this supremum is an upper bound for $\mathcal{A}$ contained in $\mathcal{A}$, so that we also find (8.173) to be true again. This equation allows us to rewrite the inclusions (8.150) as

$$(-\infty, \overset{\leq_{\mathcal{D}}}{\max} \mathcal{A}) \subseteq \mathcal{A} \subseteq (-\infty, \overset{\leq_{\mathcal{D}}}{\max} \mathcal{A}]. \tag{8.185}$$

We establish now also the inclusion $(-\infty, \max^{\leq_{\mathcal{D}}} \mathcal{A}] \subseteq \mathcal{A}$ via the proof of the universal sentence

$$\forall x \, (x \in (-\infty, \overset{\leq_{\mathcal{D}}}{\max} \mathcal{A}] \Rightarrow x \in \mathcal{A}). \tag{8.186}$$

We take an arbitrary element $x \in (-\infty, \max^{\leq_{\mathcal{D}}} \mathcal{A}]$, for which we then have $x \leq_{\mathcal{D}} \max^{\leq_{\mathcal{D}}} \mathcal{A}$ according to the definition of a left-unbounded and right-closed interval. Clearly, that inequality implies that $x <_{\mathcal{D}} \max^{\leq_{\mathcal{D}}} \mathcal{A}$ or $x = \max^{\leq_{\mathcal{D}}} \mathcal{A}$ holds. On the one hand, if $x <_{\mathcal{D}} \max^{\leq_{\mathcal{D}}} \mathcal{A}$ is true, then the definition of an open and left-unbounded interval gives us $x \in$

$(-\infty, \max^{\leq_{\mathcal{D}}} \mathcal{A})$, which in turn implies the desired consequent $x \in \mathcal{A}$ with the first inclusion in (8.185). On the other hand, if $x = \max^{\leq_{\mathcal{D}}} \mathcal{A}$ is true, then the aforementioned fact $\max^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A}$ yields $x \in \mathcal{A}$ by means of substitution. Having thus proved the implication in (8.186), we can infer from the truth of that implication the truth of the universal sentence (8.186) and then also the truth of the suggested inclusion $(-\infty, \max^{\leq_{\mathcal{D}}} \mathcal{A}] \subseteq \mathcal{A}$ (using the definition of a subset). Together with the second inclusion in (8.185), this gives us now the equation $\mathcal{A} = (-\infty, \max^{\leq_{\mathcal{D}}} \mathcal{A}]$ by means of the Axiom of Extension. We notice that the truth of the assumption $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \in \mathcal{A}$ and the truth of the preceding equation turn the implication (8.166) into a true sentence, and we also see that $\mathcal{A}$ constitutes a left-unbounded and right-closed interval.

We consider now the second possibility that $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$ holds. In this case, we can demonstrate the truth of the inclusion $\mathcal{A} \subseteq (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$, by proving accordingly

$$\forall x \, (x \in \mathcal{A} \Rightarrow x \in (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A})\,). \tag{8.187}$$

Letting x be arbitrary in $\mathcal{A}$, we recall that $\sup^{\leq_{\mathcal{D}}} \mathcal{A}$ is an upper bound for $\mathcal{A}$, so that $x \leq_{\mathcal{D}} \sup^{\leq_{\mathcal{D}}} \mathcal{A}$ holds. Because of $x \in \mathcal{A}$ and $\sup^{\leq_{\mathcal{D}}} \mathcal{A} \notin \mathcal{A}$, we evidently have $x \neq \sup^{\leq_{\mathcal{D}}} \mathcal{A}$ and therefore $x <_{\mathcal{D}} \sup^{\leq_{\mathcal{D}}} \mathcal{A}$. This means by definition of an open and left-unbounded interval that $x \in (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ is true, so that the proof of the implication in (8.187) is complete. As x was arbitrary, the inclusion $\mathcal{A} \subseteq (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ follows then to be true (according to the definition of a subset). The preceding inclusion implies now in conjunction with the first inclusion in (8.150) the equation $\mathcal{A} = (-\infty, \sup^{\leq_{\mathcal{D}}} \mathcal{A})$ by means of the Axiom of Extension. Consequently, the implication (8.167) is true, and we thereby showed that $\mathcal{A}$ is again an interval.

We thus completed the proof by cases based on the disjunction (8.171) and also the proof of b).

The next sub-case to consider is that the set $\mathcal{A}$ is not bounded from above. Then, $\mathcal{A}$ is bounded from below or not bounded from below. The situation that $\mathcal{A}$ is bounded from below (besides being unbounded from above) can be proved in analogy to the preceding proof of the case that $\mathcal{A}$ is bounded from above and not bounded from below. This proof is based on the true disjunction (8.170) and shows that the two implications in c) are true. Thus, $\mathcal{A}$ is an interval again.

Finally, in case $\mathcal{A}$ is neither bounded from above nor bounded from below, we see in light of Lemma 8.23d) that the set $\mathcal{A}$ is identical with the set $\mathcal{D}$, so that $\mathcal{A}$ constitutes an interval according to Definition 3.35. This finding completes the proof of the case $\mathcal{A} \neq \emptyset$, so that $\mathcal{A}$ is an interval in any case.

Regarding the second part ('$\Leftarrow$') of the stated equivalence, we assume that $\mathcal{A}$ is an interval in $\mathcal{D}$. It then follows immediately with Proposition 3.143 that $\mathcal{A}$ is convex in $\mathcal{D}$ with respect to $<_{\mathcal{D}}$.

As X, $<$ and $\mathcal{A}$ were initially arbitrary sets, we therefore conclude that the stated theorem is indeed true. □

Exercise 8.13. Give detailed proofs of the implications (8.164), (8.168) and (8.169).)

Corollary 8.25 (Equivalence of intervals and convex sets in $\mathbb{R}$). *It is true that a subset I of $\mathbb{R}$ is an interval iff I is convex. In particular,*

a) if I is nonempty, bounded from above and bounded from below, then

$$\left[\overset{\leq_{\mathbb{R}}}{\inf} I \in I \ \wedge\ \overset{\leq_{\mathbb{R}}}{\sup} I \in I\right] \ \Rightarrow\ I = [\overset{\leq_{\mathbb{R}}}{\min} I, \overset{\leq_{\mathbb{R}}}{\max} I], \tag{8.188}$$

$$\left[\overset{\leq_{\mathbb{R}}}{\inf} I \notin I \ \wedge\ \overset{\leq_{\mathbb{R}}}{\sup} I \notin I\right] \ \Rightarrow\ I = (\overset{\leq_{\mathbb{R}}}{\inf} I, \overset{\leq_{\mathbb{R}}}{\sup} I), \tag{8.189}$$

$$\left[\overset{\leq_{\mathbb{R}}}{\inf} I \notin I \ \wedge\ \overset{\leq_{\mathbb{R}}}{\sup} I \in I\right] \ \Rightarrow\ I = (\overset{\leq_{\mathbb{R}}}{\inf} I, \overset{\leq_{\mathbb{R}}}{\max} I], \tag{8.190}$$

$$\left[\overset{\leq_{\mathbb{R}}}{\inf} I \in I \ \wedge\ \overset{\leq_{\mathbb{R}}}{\sup} I \notin I\right] \ \Rightarrow\ I = [\overset{\leq_{\mathbb{R}}}{\min} I, \overset{\leq_{\mathbb{R}}}{\sup} I). \tag{8.191}$$

b) if I is nonempty, bounded from above and not bounded from below, then

$$\overset{\leq_{\mathbb{R}}}{\sup} I \in I \ \Rightarrow\ I = (-\infty, \overset{\leq_{\mathbb{R}}}{\max} I], \tag{8.192}$$

$$\overset{\leq_{\mathbb{R}}}{\sup} I \notin I \ \Rightarrow\ I = (-\infty, \overset{\leq_{\mathbb{R}}}{\sup} I). \tag{8.193}$$

c) if I is nonempty, not bounded from above and bounded from below, then

$$\overset{\leq_{\mathbb{R}}}{\inf} I \in I \ \Rightarrow\ I = [\overset{\leq_{\mathbb{R}}}{\min} I, +\infty), \tag{8.194}$$

$$\overset{\leq_{\mathbb{R}}}{\inf} I \notin I \ \Rightarrow\ I = (\overset{\leq_{\mathbb{R}}}{\inf} I, +\infty). \tag{8.195}$$

d) if I is nonempty and neither bounded from above nor bounded from below, then

$$I = \mathbb{R}. \tag{8.196}$$

8.2. The Ordered Field $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, -_{\mathbb{R}}, /_{\mathbb{R}}, <_{\mathbb{R}})$

Theorem 8.26 (Addition on the set of real numbers). *There exists a unique function* $+_{\mathbb{R}}$ *on* $\mathbb{R} \times \mathbb{R}$ *that satisfies*

$$\begin{aligned} \forall z\, (z \in \mathbb{R} \times \mathbb{R} \Rightarrow \exists A, B, C, D\, (z = ((A, B), (C, D)) \\ \wedge\ +_{\mathbb{R}}(z) = (A \oplus C, \mathbb{Q} \setminus A \oplus C))), \end{aligned} \tag{8.197}$$

and this function is a binary operation on $\mathbb{R}$.

Proof. We apply Function definition by replacement and prove accordingly

$$\begin{aligned} \forall z\, (z \in \mathbb{R} \times \mathbb{R} \Rightarrow \exists! y\, (\exists A, B, C, D\, (z = ((A, B), (C, D)) \\ \wedge\ y = (A \oplus C, \mathbb{Q} \setminus A \oplus C)))), \end{aligned} \tag{8.198}$$

For this purpose, we let $z \in \mathbb{R} \times \mathbb{R}$ be arbitrary, so that there exist particular real numbers $\bar{x}$ and $\bar{y}$ with $(\bar{x}, \bar{y}) = z$, according to Exercise 3.4. Then, there exist also particular sets $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ such that $\bar{x} = (\bar{A}, \bar{B})$ and $\bar{y} = (\bar{C}, \bar{D})$ constitute Dedekind cuts in $\mathbb{Q}$, according to Exercise 8.4. These equations allow us to rewrite the previous equation for z as $z = ((\bar{A}, \bar{B}), (\bar{C}, \bar{D}))$. Furthermore, $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are subsets of $\mathbb{Q}$ in view of Property 1 of a cut (in $\mathbb{Q}$). Consequently, the set $\bar{A} \oplus \bar{C}$ (in the sense of the Addition of sets) is specified, and this set gives rise to the set difference $\mathbb{Q} \setminus \bar{A} \oplus \bar{C}$. Defining now the ordered pair $\bar{y} = (\bar{A} \oplus \bar{C}, \mathbb{Q} \setminus \bar{A} \oplus \bar{C})$, we see that the given set $\bar{y}$ satisfies the existential sentence

$$\exists A, B, C, D\, (z = ((A, B), (C, D)) \wedge \bar{y} = (A \oplus C, \mathbb{Q} \setminus A \oplus C)),$$

proving the uniqueness part of the uniquely existential sentence in (8.198). To establish the uniqueness part, we take arbitrary sets y, y' and assume the existential sentences

$$\exists A, B, C, D\, (z = ((A, B), (C, D)) \wedge y = (A \oplus C, \mathbb{Q} \setminus A \oplus C))$$
$$\exists A, B, C, D\, (z = ((A, B), (C, D)) \wedge y' = (A \oplus C, \mathbb{Q} \setminus A \oplus C))$$

to be true. Consequently, there are particular sets A, B, C, D satisfying $z = ((A, B), (C, D))$ and $y = (A \oplus C, \mathbb{Q} \setminus A \oplus C)$, as well as particular sets A', B', C', D' satisfying $z = ((A', B'), (C', D'))$ and $y' = (A' \oplus C', \mathbb{Q} \setminus A' \oplus C')$. Combining the two equations for z yields with the Equality Criterion for sets first $(A, B) = (A', B')$ and $(C, D) = (C', D')$, subsequently $A = A'$, $B = B'$, $C = C'$ and $D = D'$. Carrying out substitutions in the previous equation for y gives therefore $y = (A' \oplus C', \mathbb{Q} \setminus A' \oplus C') = y'$, so that the desired equation $y = y'$ follows to be true. As y and y' were arbitrary, we conclude that the uniqueness part also holds. This finding completes the proof of the uniquely existential sentence in (8.198). As z was arbitrary, we may further conclude that the universal sentence (8.198) is true, so that there indeed exists a unique function $+_{\mathbb{R}}$ on $\mathbb{R} \times \mathbb{R}$ satisfying (8.197).

To prove that the function $+_{\mathbb{R}}$ is a binary operation on $\mathbb{R}$, we need to demonstrate that $\mathbb{R}$ is a codomain of that function, i.e., that the inclusion $\mathrm{ran}(+_{\mathbb{R}}) \subseteq \mathbb{R}$ holds. To do this, we let $\bar{y} \in \mathrm{ran}(+_{\mathbb{R}})$ be arbitrary. By definition of a range, we then have $(\bar{x}, \bar{y}) \in +_{\mathbb{R}}$ for a particular constant $\bar{x}$. We already showed that $+_{\mathbb{R}}$ is a function, so that we may write $\bar{y} = +_{\mathbb{R}}(\bar{x})$. Moreover, we see in light of the definition of a domain that $\bar{x} \in \mathbb{R} \times \mathbb{R}$ [$= \mathrm{dom}(+_{\mathbb{R}})$] holds. Evidently, we may therefore express $\bar{x}$ as the ordered pair $\bar{x} = ((\bar{A}, \bar{B}), (\bar{C}, \bar{D}))$ formed by the Dedekind cuts $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ in $\mathbb{Q}$ for particular sets $\bar{A}, \bar{B}, \bar{C}, \bar{D}$. The associated value is then $\bar{y} = +_{\mathbb{R}}(\bar{x}) = (\bar{A} \oplus \bar{C}, \mathbb{Q} \setminus \bar{A} \oplus \bar{C})$, by definition of the function $+_{\mathbb{R}}$.

According to the Addition of sets, $\bar{A} \oplus \bar{C}$ constitutes a subset of $\mathbb{Q}$, and so is the set difference $\mathbb{Q} \setminus \bar{A} \oplus \bar{C}$ in view of (2.125). Thus, the ordered pair $(\bar{A} \oplus \bar{C}, \mathbb{Q} \setminus \bar{A} \oplus \bar{C})$ has Property 1 of a cut.

Next, we address Property 2 and show that the two sets forming that ordered pair are nonempty. On the one hand, since $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ are Dedekind cuts in $\mathbb{Q}$, we have in particular $\bar{A} \neq \emptyset$ and $\bar{C} \neq \emptyset$, so that these sets evidently have elements, say $\bar{a} \in \bar{A}$ and $\bar{c} \in \bar{C}$. Furthermore, $\bar{A}$ and $\bar{C}$ are clearly included in $\mathbb{Q}$, so that $\bar{a} \in \mathbb{Q}$ and $\bar{c} \in \mathbb{Q}$ follow to be by definition of a subset. Therefore, we may form the sum $\bar{s} = \bar{a} +_{\mathbb{Q}} \bar{c}$, and this implies in conjunction with $\bar{a} \in \bar{A}$ and $\bar{c} \in \bar{C}$ that $\bar{s} \in \bar{A} \oplus \bar{C}$ holds (according to the Addition of sets). Thus, $\bar{A} \oplus \bar{C}$ has an element, so that $\bar{A} \oplus \bar{C} \neq \emptyset$. On the other hand, we also have $\bar{B} \neq \emptyset$ and $\bar{D} \neq \emptyset$, so that these sets evidently have elements, say $\bar{b} \in \bar{B}$ and $\bar{d} \in \bar{D}$. Since $\bar{B}$ and $\bar{D}$ are clearly included in $\mathbb{Q}$, it follows that $\bar{b} \in \mathbb{Q}$ and $\bar{d} \in \mathbb{Q}$. Let us form the corresponding sum $\bar{t} = \bar{b} +_{\mathbb{Q}} \bar{d}$ and prove the universal sentence

$$\forall a, c\,([a \in \bar{A} \wedge c \in \bar{C}] \Rightarrow \bar{t} \neq a +_{\mathbb{Q}} c). \tag{8.199}$$

Letting $a^* \in \bar{A}$ and $c^* \in \bar{C}$ be arbitrary, we find due to $\bar{b} \in \bar{B}$ and $\bar{d} \in \bar{D}$ by virtue of Property 5 of the Dedekind cuts $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ the inequalities $a^* <_{\mathbb{Q}} \bar{b}$ and $c^* <_{\mathbb{Q}} \bar{d}$. In view of the Additivity of $<$-inequalities with respect to the ordered integral domain $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, -_{\mathbb{Q}}, <_{\mathbb{Q}})$, we obtain then $a^* +_{\mathbb{Q}} c^* <_{\mathbb{Q}} \bar{b} +_{\mathbb{Q}} \bar{d}$ [$= \bar{t}$]. According to the Characterization of comparability with respect to the linear ordering $<_{\mathbb{Q}}$, this implies the truth of the inequality $\bar{t} \neq a^* +_{\mathbb{Q}} c^*$. Since a^* and c^* were arbitrary, we may infer from this finding the truth of the universal sentence (8.199), which in turn implies the truth of the negation

$$\neg\exists a, c\,([a \in \bar{A} \wedge c \in \bar{c}] \wedge \bar{t} = a +_{\mathbb{Q}} c)$$

by means of the Negation Law for existential conjunctions. According to the Addition of sets, this negation further implies $\bar{t} \notin \bar{A} \oplus \bar{C}$. As the sum $\bar{t} = \bar{b} +_{\mathbb{Q}} \bar{d}$ clearly constitutes a rational number, we thus have $\bar{t} \in \mathbb{Q} \setminus \bar{A} \oplus \bar{C}$ by definition of a set difference. This demonstrates that the preceding set difference has an element, so that $\mathbb{Q} \setminus \bar{A} \oplus \bar{C} \neq \emptyset$. Alongside the previous finding $\bar{A} \oplus \bar{C} \neq \emptyset$, this shows that $(\bar{A} \oplus \bar{C}, \mathbb{Q} \setminus \bar{A} \oplus \bar{C})$ also has Property 2 of a cut.

According to the Generation of two disjoint sets, we have

$$\bar{A} \oplus \bar{C} \cap (\mathbb{Q} \setminus \bar{A} \oplus \bar{C}) = \emptyset.$$

In addition, as we already know that the inclusion $\bar{A} \oplus \bar{C} \subseteq \mathbb{Q}$ is true, we find

$$\bar{A} \oplus \bar{C} \cup (\mathbb{Q} \setminus \bar{A} \oplus \bar{C}) = \mathbb{Q}$$

with (2.263). These two equations show that Property 3 and Property 4 are also satisfied by $(\bar{A} \oplus \bar{C}, \mathbb{Q} \setminus \bar{A} \oplus \bar{C})$.

Regarding Property 5 of a cut, we establish

$$\forall s, t\,([s \in \bar{A} \oplus \bar{C} \,\wedge\, t \in \mathbb{Q} \setminus \bar{A} \oplus \bar{C}] \Rightarrow s <_{\mathbb{Q}} t). \tag{8.200}$$

Letting $s \in \bar{A} \oplus \bar{C}$ and $t \in \mathbb{Q} \setminus \bar{A} \oplus \bar{C}$ be arbitrary, we may on the one hand express s as the sum $\bar{a} +_{\mathbb{Q}} \bar{c}$ for some particular constants $\bar{a} \in \bar{A}$ and $\bar{c} \in \bar{C}$. On the other hand, it is clearly true that $t \in \mathbb{Q}$ and $t \notin \bar{A} \oplus \bar{C}$. Since s is in $\bar{A} \oplus \bar{C}$ but t is not, we find $s \neq t$ with (2.4). As the linear ordering $<_{\mathbb{Q}}$ is in particular connex, that inequality implies the truth of the disjunction

$$s <_{\mathbb{Q}} t \,\vee\, t <_{\mathbb{Q}} s. \tag{8.201}$$

We now prove the negation $\neg t <_{\mathbb{Q}} s$ by contradiction, assuming the negation of that negation to be true. By the Double Negation Law, $t <_{\mathbb{Q}} s\ [= \bar{a} +_{\mathbb{Q}} \bar{c}]$ is then true, and this inequality implies $t -_{\mathbb{Q}} \bar{c} <_{\mathbb{Q}} \bar{a}$ with the Monotony Law for $+$ and $<$. Due to $t -_{\mathbb{Q}} \bar{c} \in \mathbb{Q}$ and $\bar{a} \in A$, the preceding inequality implies $t -_{\mathbb{Q}} \bar{c} \in \bar{A}$ with Proposition 8.2. Since $\bar{c} \in \bar{C}$ also holds and since t can evidently be written as $t = (t -_{\mathbb{Q}} \bar{c}) +_{\mathbb{Q}} \bar{c}$, we now find $t \in \bar{A} \oplus \bar{C}$, in contradiction to the previously established negation $t \notin \bar{A} \oplus \bar{C}$. We thus completed the proof of $\neg t <_{\mathbb{Q}} s$ by contradiction, so that the first part $s <_{\mathbb{Q}} t$ of the true disjunction (8.201) is true. This in turn proves the implication in (8.200), in which s and t are arbitrary, so that the universal sentence (8.200) holds as well. Thus, Property 5 of a cut is satisfied by $(\bar{A} \oplus \bar{C}, \mathbb{Q} \setminus \bar{A} \oplus \bar{C})$. Therefore, the previous findings show that this ordered pair is a cut in $\mathbb{Q}$.

To show that this is a Dedekind cut in $\mathbb{Q}$, we prove the negation

$$\neg \exists u\,(u \in \mathbb{Q} \,\wedge\, u = \overset{\leq_{\mathbb{Q}}}{\max}\, \bar{A} \oplus \bar{C}) \tag{8.202}$$

by contradiction. Assuming the negation of that negation, there is then a particular rational number $\bar{u} = \max^{\leq_{\mathbb{Q}}} \bar{A} \oplus \bar{C}$. By definition of a maximum, we thus have

$$\forall s\,(s \in \bar{A} \oplus \bar{C} \Rightarrow s \leq_{\mathbb{Q}} \bar{u}) \tag{8.203}$$

and $\bar{u} \in \bar{A} \oplus \bar{C}$. The latter means that $\bar{u} = \bar{a} +_{\mathbb{Q}} \bar{c}$ holds for two particular constants $\bar{a} \in \bar{A}$ and $\bar{c} \in \bar{C}$. According to the Characterization of Dedekind cuts, there is then also particular constant $a^* \in \bar{A}$ with $\bar{a} <_{\mathbb{Q}} a^*$. This inequality evidently implies $[\bar{u} =]\ \bar{a} +_{\mathbb{Q}} \bar{c} <_{\mathbb{Q}} a^* +_{\mathbb{Q}} \bar{c}$, so that $\bar{u} <_{\mathbb{Q}} a^* +_{\mathbb{Q}} \bar{c}$. Denoting the sum $a^* +_{\mathbb{Q}} \bar{c}$ by s, we thus have $\bar{u} <_{\mathbb{Q}} s$. Moreover, the conjunction of $a^* \in \bar{A}$, $\bar{c} \in \bar{C}$ and $s = a^* +_{\mathbb{Q}} \bar{c}$ clearly implies $s \in \bar{A} \oplus \bar{C}$, and therefore $s \leq_{\mathbb{Q}} \bar{u}$ in view of (8.203). As this inequality yields the true negation $\neg \bar{u} <_{\mathbb{Q}} s$ with the Negation Formula for $<$, we arrived at a contradiction (recalling the truth of $\bar{u} <_{\mathbb{Q}} s$), so that the proof of (8.202) is now complete. This finding in turn completes the proof that $\bar{y} = (\bar{A} \oplus \bar{C}, \mathbb{Q} \setminus \bar{A} \oplus \bar{C})$ is a Dedekind cut in $\mathbb{Q}$. Thus, $\bar{y}$ constitutes a real number by definition of $\mathbb{R}$. As $\bar{y}$ was arbitrary, we may therefore conclude that the range of the function $+_{\mathbb{R}}$ is indeed a subset of $\mathbb{R}$, which means that $\mathbb{R}$ is a codomain of $+_{\mathbb{R}}$. Because the Cartesian product $\mathbb{R} \times \mathbb{R}$ is the domain of this function, it constitutes a binary operation on $\mathbb{R}$. □

Definition 8.14 (Addition on the set of real numbers). We call

$$+_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto x +_{\mathbb{R}} y \tag{8.204}$$

the *addition on the set of real numbers*.

Exercise 8.14. Establish the truth of the universal sentences

$$\begin{aligned} \forall x, y\,(x, y \in \mathbb{R} \Rightarrow \exists A, B, C, D\,(x = (A, B) \,\wedge\, y = (C, D) \\ \wedge\, x +_{\mathbb{R}} y = (A \oplus C, \mathbb{Q} \setminus A \oplus C))) \end{aligned} \tag{8.205}$$

and

$$\forall A, B, C, D\,((A, B), (C, D) \in \mathbb{R} \Rightarrow (A, B) +_{\mathbb{R}} (C, D) = (A \oplus C, \mathbb{Q} \setminus A \oplus C)). \tag{8.206}$$

(Hint: Apply Exercise 8.4, the definition of the Cartesian product of two sets, and the Equality Criterion for ordered pairs.)

Let us inspect the inner workings for the particular case of a sum of two rational numbers in $\mathbb{R}$.

Lemma 8.27 (Addition of rational numbers in $\mathbb{R}$). *It is true for any rational numbers p and q that*

$$f_{\mathbb{Q}}^{\mathbb{R}}(p) +_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(q) = f_{\mathbb{Q}}^{\mathbb{R}}(p +_{\mathbb{Q}} q). \tag{8.207}$$

Proof. Letting $p, q \in \mathbb{R}$ be arbitrary, we observe in light if the Identification of $\mathbb{Q}$ in $\mathbb{R}$ that $f_{\mathbb{Q}}^{\mathbb{R}}(p) = ((-\infty, p), [p, +\infty))$ and $f_{\mathbb{Q}}^{\mathbb{R}}(q) = ((-\infty, q), [q, +\infty))$ indeed constitute Dedekind cuts in $\mathbb{Q}$, that is, real numbers. We obtain for their sum

$$\begin{aligned} f_{\mathbb{Q}}^{\mathbb{R}}(p) +_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(q) &= ((-\infty, p), [p, +\infty)) +_{\mathbb{R}} ((-\infty, q), [q, +\infty)) \\ &= ((-\infty, p) \oplus (-\infty, q), \mathbb{Q} \setminus (-\infty, p) \oplus (-\infty, q)), \end{aligned} \tag{8.208}$$

according to (8.206). Similarly, the rational number $p +_{\mathbb{Q}} q$ is associated with the Dedekind cut/real number

$$f_{\mathbb{Q}}^{\mathbb{R}}(p +_{\mathbb{Q}} q) = ((-\infty, p +_{\mathbb{Q}} q), [p +_{\mathbb{Q}} q, +\infty)). \tag{8.209}$$

To verify the equality of (8.208) and (8.209), we may apply the Equality Criterion for ordered pairs and prove the conjunction

$$(-\infty, p) \oplus (-\infty, q) = (-\infty, p +_{\mathbb{Q}} q) \land \mathbb{Q} \setminus (-\infty, p) \oplus (-\infty, q) = [p +_{\mathbb{Q}} q, +\infty). \tag{8.210}$$

We establish both equations successively by means of the Equality Criterion for sets. Regarding the first equation, we let s be arbitrary, and we assume first $s \in (-\infty, p) \oplus (-\infty, q)$ to be true. According to the Addition of sets, we may therefore express s as the sum $s = p^* +_{\mathbb{Q}} q^*$ for some particular elements $p^* \in (-\infty, p)$ and $q^* \in (-\infty, q)$. Consequently, the inequalities $p^* <_{\mathbb{Q}} p$ and $q^* <_{\mathbb{Q}} q$ are true by definition of an open and left-unbounded interval in $\mathbb{Q}$. These give us $[s =]\ p^* +_{\mathbb{Q}} q^* <_{\mathbb{Q}} p +_{\mathbb{Q}} q$ by the Additivity of $<$-inequalities, with the evident and desired consequence that $s \in (-\infty, p +_{\mathbb{Q}} q)$.

Assuming conversely $s \in (-\infty, p +_{\mathbb{Q}} q)$ to be true, we evidently obtain $s <_{\mathbb{Q}} p +_{\mathbb{Q}} q$, and subsequently also $0 <_{\mathbb{Q}} p +_{\mathbb{Q}} q -_{\mathbb{Q}} s$ due to the Monotony Law for $+$ and $<$ (omitting brackets in view of the Associative Law for the addition on $\mathbb{Q}$). Since $0 <_{\mathbb{Q}} \frac{1}{2}$ is clearly true as well, we may apply the Monotony Law for $\cdot$ and $<$ to these inequalities to obtain $0 <_{\mathbb{Q}} \frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2}$ (using also the Cancellation Law for 0_X in rings). Two more applications of the Monotony Law for $+$ and $<$ gives us now first $-\frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} <_{\mathbb{Q}} 0$ and then

$$\begin{aligned} p -_{\mathbb{Q}} \frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} &<_{\mathbb{Q}} p, \\ q -_{\mathbb{Q}} \frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} &<_{\mathbb{Q}} q. \end{aligned}$$

These inequalities mean that

$$p -_{\mathbb{Q}} \frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} \in (-\infty, p), \tag{8.211}$$

$$q -_{\mathbb{Q}} \frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} \in (-\infty, q). \tag{8.212}$$

Moreover, we can derive the equations

$$\begin{aligned}
s &= 0 +_{\mathbb{Q}} [-(-s)] \\
&= (p +_{\mathbb{Q}} q) -_{\mathbb{Q}} (p +_{\mathbb{Q}} q) -_{\mathbb{Q}} (-s) \\
&= p +_{\mathbb{Q}} q -_{\mathbb{Q}} (p +_{\mathbb{Q}} q -_{\mathbb{Q}} s) \\
&= p +_{\mathbb{Q}} q -_{\mathbb{Q}} \left(\frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} +_{\mathbb{Q}} \frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} \right) \\
&= \left(p -_{\mathbb{Q}} \frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} \right) +_{\mathbb{Q}} \left(q -_{\mathbb{Q}} \frac{p +_{\mathbb{Q}} q -_{\mathbb{Q}} s}{2} \right)
\end{aligned}$$

by using the property of a zero element, the Sign Law (6.50), the Associative Law for the addition on $\mathbb{Q}$, the Sign Law (6.52), the Commutative Law for the addition on $\mathbb{Q}$, and the Calculus of fractions. In light of (8.211) – (8.212), we therefore see that $s \in (-\infty, p) \oplus (-\infty, q)$ holds, as desired. We thus proved that $s \in (-\infty, p) \oplus (-\infty, q)$ and $s \in (-\infty, p +_{\mathbb{Q}} q)$ are equivalent, where s is arbitrary, so that the first equation in (8.210) is true indeed.

We now prove the second equation, letting again s be arbitrary. We obtain the true equivalences

$$\begin{aligned}
s \in \mathbb{Q} \setminus (-\infty, p) \oplus (-\infty, q) \;&\Leftrightarrow\; s \in \mathbb{Q} \,\wedge\, s \notin (-\infty, p) \oplus (-\infty, q) \\
&\Leftrightarrow\; s \in \mathbb{Q} \,\wedge\, s \notin (-\infty, p +_{\mathbb{Q}} q) \\
&\Leftrightarrow\; s \in \mathbb{Q} \,\wedge\, \neg s <_{\mathbb{Q}} p +_{\mathbb{Q}} q \\
&\Leftrightarrow\; s \in \mathbb{Q} \,\wedge\, p +_{\mathbb{Q}} q \leq_{\mathbb{Q}} s \\
&\Leftrightarrow\; s \in \mathbb{Q} \,\wedge\, s \in [p +_{\mathbb{Q}} q, +\infty)
\end{aligned}$$

by applying the definition of a set differences, the first equation in (8.210), the definition of an open and left-unbounded interval in $\mathbb{Q}$, the negation Formula for $<$, and the definition of a left-closed and right-unbounded interval in $\mathbb{Q}$. Consequently, $s \in \mathbb{Q} \setminus (-\infty, p) \oplus (-\infty, q)$ and $s \in [p +_{\mathbb{Q}} q, +\infty)$ are equivalent, so that the second equation in (8.210) follows to be true (since s was arbitrary). Having proved the conjunction (8.210), we may now infer from this the truth of the equivalent equation (8.207). As p and q are arbitrary rationals, we may therefore conclude that the stated lemma holds. □

Theorem 8.28 (Associative Law for the addition on $\mathbb{R}$). *It is true that the addition $+_{\mathbb{R}}$ on $\mathbb{R}$ is associative.*

Proof. Letting x, y and z be arbitrary real numbers, we find $x = (A, B)$, $y = (C, D)$ and $z = (E, F)$ for particular sets A, B, C, D, E and F such that all three ordered pairs constitute Dedekind cuts in $\mathbb{Q}$ (according to Exercise 8.4). We then find

$$\begin{aligned} x +_{\mathbb{R}} y &= (A, B) +_{\mathbb{R}} (C, D) = (A \oplus C, \mathbb{Q} \setminus A \oplus C), \\ y +_{\mathbb{R}} z &= (C, D) +_{\mathbb{R}} (E, F) = (C \oplus E, \mathbb{Q} \setminus C \oplus E). \end{aligned}$$

by applying substitutions and (8.206). Consequently, we obtain for the same reason

$$\begin{aligned} (x +_{\mathbb{R}} y) +_{\mathbb{R}} z &= ((A, B) +_{\mathbb{R}} (C, D)) +_{\mathbb{R}} (E, F) \\ &= ((A \oplus C) \oplus E, \mathbb{Q} \setminus (A \oplus C) \oplus E), \qquad (8.213) \\ x +_{\mathbb{R}} (y +_{\mathbb{R}} z) &= (A, B) +_{\mathbb{R}} ((C, D) +_{\mathbb{R}} (E, F)) \\ &= (A \oplus (C \oplus E), \mathbb{Q} \setminus A \oplus (C \oplus E)). \qquad (8.214) \end{aligned}$$

Recalling that (A, B), (C, D) and (E, F) are Dedekind cuts in $\mathbb{Q}$, we have by Property 1 of a cut in $\mathbb{Q}$ in particular that A, C and E are subsets of $\mathbb{Q}$, and therefore elements of the power set of $\mathbb{Q}$. As the addition $\bigoplus$ of sets in $\mathcal{P}(\mathbb{Q})$ is associative (see Proposition 5.36), it follows from $A, C, E \in \mathcal{P}(\mathbb{Q})$ that

$$(A \oplus C) \oplus E = A \oplus (B \oplus C)$$

is true. Consequently, carrying out substitutions in (8.214) based on these equations and (8.213) yields

$$\begin{aligned} (x +_{\mathbb{R}} y) +_{\mathbb{R}} z &= ((A \oplus C) \oplus E, \mathbb{Q} \setminus (A \oplus C) \oplus E) \\ &= (A \oplus (C \oplus E), \mathbb{Q} \setminus A \oplus (C \oplus E)) \\ &= x +_{\mathbb{R}} (y +_{\mathbb{R}} z). \end{aligned}$$

This gives us the desired equation

$$(x +_{\mathbb{R}} y) +_{\mathbb{R}} z = x +_{\mathbb{R}} (y +_{\mathbb{R}} z),$$

where x, y and z are arbitrary (real numbers), so that $+_{\mathbb{R}}$ is itself associative, by definition. □

Theorem 8.29 (Commutative Law for the addition on $\mathbb{R}$). *It is true that the addition $+_{\mathbb{R}}$ on $\mathbb{R}$ is commutative.*

Exercise 8.15. Establish the Commutative Law for the addition on $\mathbb{R}$.

(Hint: Proceed similarly as in the proof of the Associative Law for the addition on $\mathbb{R}$.)

Note 8.8. We see in light of the Commutative and Associative Law for the addition on $\mathbb{R}$ that $(\mathbb{R}, +_{\mathbb{R}})$ is a commutative semigroup.

Proposition 8.30. *It is true that*

$$f_{\mathbb{Q}}^{\mathbb{R}}(0) = ((-\infty, 0), [0, +\infty)) \tag{8.215}$$

constitutes the neutral element of $\mathbb{R}$ *with respect to the addition* $+_{\mathbb{R}}$ *on* $\mathbb{R}$.

Proof. We prove

$$\forall x\, (x \in \mathbb{R} \Rightarrow [f_{\mathbb{Q}}^{\mathbb{R}}(0) +_{\mathbb{R}} x = x \,\wedge\, x +_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(0) = x]), \tag{8.216}$$

taking first an arbitrary real number x, which we may write as the Dedekind cut (A, B) for particular sets A and B. We first observe the truth of the equations

$$\begin{aligned} f_{\mathbb{Q}}^{\mathbb{R}}(0) +_{\mathbb{R}} x &= ((-\infty, 0), [0, +\infty)) +_{\mathbb{R}} (A, B) \\ &= ((-\infty, 0) \oplus A, \mathbb{Q} \setminus (-\infty, 0) \oplus A) \end{aligned} \tag{8.217}$$

in light of the definition of the Addition of sets. We now apply the Equality Criterion for sets to prove the equality of the sets $(-\infty, 0) \oplus A$ and A. To do this, we let s be arbitrary, assuming first $s \in (-\infty, 0) \oplus A$ to be true. This evidently means that s can be written as a sum $s = z +_{\mathbb{Q}} a$ with $z \in (-\infty, 0)$ and $a \in A$. By definition of an open and left-unbounded interval in $\mathbb{Q}$, the inequality $z <_{\mathbb{Q}} 0$ is then true, which in turn implies $[s =]\ z +_{\mathbb{Q}} a <_{\mathbb{Q}} a$ with the Monotony Law for $+$ and $<$. The resulting inequality $s <_{\mathbb{Q}} a$, in conjunction with the evident $s \in \mathbb{Q}$ and the fact $a \in A$, further implies the truth of the desired consequent $s \in A$ by virtue of Proposition 8.2.

Let us conversely assume $s \in A$ to be true. According to the Characterization of Dedekind cuts, there exists then some element $t \in A$ greater than s. This inequality $s <_{\mathbb{Q}} t$ evidently yields $s -_{\mathbb{Q}} t <_{\mathbb{Q}} 0$ and therefore $s -_{\mathbb{Q}} t \in (-\infty, 0)$. Since we can express s by $s = (s -_{\mathbb{Q}} t) +_{\mathbb{Q}} t$, we now see in light of $t \in A$ that $s \in (-\infty, 0) \oplus A$ holds, as desired.

We thus proved that $s \in (-\infty, 0) \oplus A$ and $s \in A$ are equivalent, where s is arbitrary, so that the sets $(-\infty, 0) \oplus A$ and A are indeed identical. Applying now substitutions to (8.217) based on this equation, we find

$$f_{\mathbb{Q}}^{\mathbb{R}}(0) +_{\mathbb{R}} x = (A, \mathbb{Q} \setminus A). \tag{8.218}$$

By definition of a cut in $\mathbb{Q}$, the given sets A and B forming the Dedekind cut $x = (A, B)$ satisfy $A \cap B = \emptyset$ as well as $A \cup B = \mathbb{Q}$. Due to the Commutative Law for the intersection and for the union of two sets, we may write these equations also as $B \cup A = \mathbb{Q}$ and $B \cap A = \emptyset$. These imply $B = \mathbb{Q} \setminus A$ with (2.262), so that (8.218) becomes after substitution

$$f_{\mathbb{Q}}^{\mathbb{R}}(0) +_{\mathbb{R}} x = (A, B) = x.$$

As the addition $+_{\mathbb{R}}$ is commutative, the resulting equation $f_{\mathbb{Q}}^{\mathbb{R}}(0) +_{\mathbb{R}} x = x$ also gives us $x +_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(0) = x$. As x is an arbitrary real number, we may infer from these findings the truth of the universal sentence (8.216), which shows that (8.215) is indeed the zero element of $\mathbb{R}$. □

Definition 8.15 (Negative real number, nonnegative real number, positive real number). We say that a real number x is

(1) *negative* iff

$$x < f_{\mathbb{Q}}^{\mathbb{R}}(0). \tag{8.219}$$

(2) *nonnegative* iff

$$f_{\mathbb{Q}}^{\mathbb{R}}(0) \leq x. \tag{8.220}$$

(3) *positive* iff

$$f_{\mathbb{Q}}^{\mathbb{R}}(0) < x. \tag{8.221}$$

Note 8.9. We see in light of the Axiom of Specification and the Equality Criterion for sets that there exist unique sets $\mathbb{R}_-$, $\mathbb{R}_+^0$ and $\mathbb{R}_+$ consisting, respectively, of all real numbers that are negative, nonnegative and positive, that is,

$$\forall x\,(x \in \mathbb{R}_- \Leftrightarrow [x \in \mathbb{R} \,\wedge\, x <_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(0)]), \tag{8.222}$$

$$\forall x\,(x \in \mathbb{R}_+^0 \Leftrightarrow [x \in \mathbb{R} \,\wedge\, f_{\mathbb{Q}}^{\mathbb{R}}(0) \leq_{\mathbb{R}} x]), \tag{8.223}$$

$$\forall x\,(x \in \mathbb{R}_+ \Leftrightarrow [x \in \mathbb{R} \,\wedge\, f_{\mathbb{Q}}^{\mathbb{R}}(0) <_{\mathbb{R}} x]). \tag{8.224}$$

Since $x \in \mathbb{R}_-$, $x \in \mathbb{R}_+^0$ and $x \in \mathbb{R}_+$ all imply $x \in \mathbb{R}$ for any x, the inclusions

$$\mathbb{R}_- \subseteq \mathbb{R}, \tag{8.225}$$

$$\mathbb{R}_+^0 \subseteq \mathbb{R}, \tag{8.226}$$

$$\mathbb{R}_+ \subseteq \mathbb{R} \tag{8.227}$$

are true by definition of a subset.

Definition 8.16 (Set of negative & of nonnegative & of positive real numbers). We call

$$\mathbb{R}_- \tag{8.228}$$

the *set of negative real numbers*,

$$\mathbb{R}_+^0 \tag{8.229}$$

the *set of nonnegative real numbers*, and

$$\mathbb{R}_+ \tag{8.230}$$

the *set of positive real numbers*.

Definition 8.17 (Nonnegative real function). We call for any set X any function

$$f : X \to \mathbb{R}_+^0 \tag{8.231}$$

a *nonnegative real function* (on X) and

$$[\mathbb{R}_+^0]^X \tag{8.232}$$

the *set of nonnegative real functions* (on X).

Note 8.10. In view of the definition of a codomain and (8.226), we have for any nonnegative real function $f : X \to \mathbb{R}_+^0$ the inclusions $\mathrm{ran}(f) \subseteq \mathbb{R}_+^0 \subseteq \mathbb{R}$, so that $\mathbb{R}$ follows to be a codomain of f as well. Thus, any nonnegative real function f on X is indeed a real function on X.

Next, we prepare the definition of negatives for real numbers.

Lemma 8.31. *The following universal sentence holds for any real number* (A, B)*:*

$$\forall p \, (p \in \mathbb{Q}_+ \Rightarrow \exists q \, (q \in A \wedge p +_{\mathbb{Q}} q \in B)). \tag{8.233}$$

Proof. First we let A and B be arbitrary sets such that (A, B) constitutes a Dedekind cut in $\mathbb{Q}$, thus a real number. Consequently, A and B are by definition nonempty subsets of $\mathbb{Q}$. Then, $A \neq \emptyset$ and $B \neq \emptyset$ imply the existence of some elements, say $\bar{a} \in A$ and $\bar{b} \in B$. Due to the inclusions $A \subseteq \mathbb{Q}$ and $B \subseteq \mathbb{Q}$, we therefore find $\bar{a}, \bar{b} \in \mathbb{Q}$ with the definition of a subset. Evidently, $-\bar{a} \in \mathbb{Q}$ is then also true. Letting now $p \in \mathbb{Q}_+$ be arbitrary, there are particular natural numbers n^* and $\bar{n}$ satisfying

$$-\bar{a} <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(n^*)) \cdot_{\mathbb{Q}} p, \tag{8.234}$$

$$\bar{b} <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})) \cdot_{\mathbb{Q}} p. \tag{8.235}$$

Here, (8.235) implies due to $\bar{b} \in B$ and the evident fact $f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})) \cdot_{\mathbb{Q}} p \in \mathbb{Q}$ that

$$f_{\mathbb{Z}}^{\mathbb{Q}}(f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})) \cdot_{\mathbb{Q}} p \in B \tag{8.236}$$

is true, according to Exercise 8.1. Since $f_{\mathbb{N}}^{\mathbb{Z}}(n^*)$ clearly constitutes an integer, we may take the integer $\bar{m} = -f_{\mathbb{N}}^{\mathbb{Z}}(n^*)$. Then, $-\bar{m} = f_{\mathbb{N}}^{\mathbb{Z}}(n^*)$ follows to be true with the Sign Law (6.50). Applying a substitution to (8.234) based on that equation yields

$$-\bar{a} <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(-\bar{m}) \cdot_{\mathbb{Q}} p,$$

which yields

$$-\bar{a} <_{\mathbb{Q}} -f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m}) \cdot_{\mathbb{Q}} p$$

with (7.43). By the Monotony Law for $+$ and $<$, we therefore find

$$f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m}) \cdot_{\mathbb{Q}} p <_{\mathbb{Q}} \bar{a},$$

and this yields

$$f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m}) \cdot_{\mathbb{Q}} p \in A \tag{8.237}$$

due to $\bar{a} \in A$ and the evident fact $f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m}) \cdot_{\mathbb{Q}} p \in \mathbb{Q}$, according to Proposition 8.2.

Now, we may evidently apply the Axiom of Specification and the Equality Criterion for sets to prove that there exists a unique set X such that

$$\forall m\,(m \in X \Leftrightarrow [m \in \mathbb{Z} \wedge f_{\mathbb{Z}}^{\mathbb{Q}}(m) \cdot_{\mathbb{Q}} p \in B]). \tag{8.238}$$

In view of (8.236), where $f_{\mathbb{N}}^{\mathbb{Z}}(\bar{n})$ is evidently an integer, we see that X has an element, so that $X \neq \emptyset$ is clearly true. We may now show also that $\bar{m}$ is a lower bound for X, that is,

$$\forall m\,(m \in X \Rightarrow \bar{m} \leq_{\mathbb{Z}} m). \tag{8.239}$$

Letting for this purpose $m \in X$ be arbitrary, we find $m \in \mathbb{Z}$ and $f_{\mathbb{Z}}^{\mathbb{Q}}(m) \cdot_{\mathbb{Q}} p \in B$ with the specification of the set X in (8.238). This finding and (8.237) imply the inequality

$$f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m}) \cdot_{\mathbb{Q}} p <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(m) \cdot_{\mathbb{Q}} p$$

since (A, B) is a Dedekind cut. As the initial assumption $p \in \mathbb{Q}_+$ implies $[(0,1)]_{\sim_q} <_{\mathbb{Q}} p$ (by definition of the set $\mathbb{Q}_+$), it follows with the Monotony Law for $\cdot$ and $<$ that $f_{\mathbb{Z}}^{\mathbb{Q}}(\bar{m}) <_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(m)$ holds. This inequality in turn gives us $\bar{m} <_{\mathbb{Z}} m$ with (7.187), and $\bar{m} \leq_{\mathbb{Z}} m$ is then also true according to the Characterization of induced irreflexive partial orderings. The truth of this inequality proves the implication in (8.239), in which m is arbitrary, so that the universal sentence (8.239) holds as well. This means that $\bar{m}$ is a lower bound for X, so that the nonempty set X is bounded from below. Thus, X has a least element a^* because of Proposition 6.87, i.e., the minimum $a^* = \min^{\leq_{\mathbb{Z}}} X$ exists. By definition then, $a^* \in X$ and the lower bound property

$$\forall m\,(m \in X \Rightarrow a^* \leq_{\mathbb{Z}} m) \tag{8.240}$$

hold. Here, $a^* \in X$ implies $a^* \in \mathbb{Z}$ and

$$f_{\mathbb{Z}}^{\mathbb{Q}}(a^*) \cdot_{\mathbb{Q}} p \in B \tag{8.241}$$

by definition of the set X. Furthermore, as the evident fact $a^* -_{\mathbb{Z}} 1 <_{\mathbb{Z}} a^*$ yields $\neg a^* \leq_{\mathbb{Z}} a^* -_{\mathbb{Z}} 1$ with the Negation Formula for $\leq$, we obtain because of the implication in (8.240) and the Law of Contraposition the true negation $a^* -_{\mathbb{Z}} 1 \notin X$. This allows us to prove the negation

$$f_{\mathbb{Z}}^{\mathbb{Q}}(a^* -_{\mathbb{Z}} 1) \cdot_{\mathbb{Q}} p \notin B \tag{8.242}$$

by contradiction. To this end, we assume the negation of that negation to be true, so that $f_{\mathbb{Z}}^{\mathbb{Q}}(a^* -_{\mathbb{Z}} 1) \cdot_{\mathbb{Q}} p \in B$ follows to be true with the Double Negation Law; this and the evident fact $a^* -_{\mathbb{Z}} 1 \in \mathbb{Z}$ imply $a^* -_{\mathbb{Z}} 1 \in X$, in contradiction to the previously found negation $a^* -_{\mathbb{Z}} 1 \notin X$. We thus completed the proof of (8.242), and this further implies $f_{\mathbb{Z}}^{\mathbb{Q}}(a^* -_{\mathbb{Z}} 1) \cdot_{\mathbb{Q}} p \in A$ with (8.17). Denoting the preceding product by $\bar{q}$, we thus have $\bar{q} \in A$. Let us now derive the equations

$$\begin{aligned}
p +_{\mathbb{Q}} \bar{q} &= p +_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(a^* -_{\mathbb{Z}} 1) \cdot_{\mathbb{Q}} p \\
&= [(1,1)]_{\sim_q} \cdot_{\mathbb{Q}} p +_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(a^* -_{\mathbb{Z}} 1) \cdot_{\mathbb{Q}} p \\
&= ([(1,1)]_{\sim_q} +_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(a^* -_{\mathbb{Z}} 1)) \cdot_{\mathbb{Q}} p \\
&= (f_{\mathbb{Z}}^{\mathbb{Q}}(1) +_{\mathbb{Q}} f_{\mathbb{Z}}^{\mathbb{Q}}(a^* -_{\mathbb{Z}} 1)) \cdot_{\mathbb{Q}} p \\
&= f_{\mathbb{Z}}^{\mathbb{Q}}(1 +_{\mathbb{Z}} [a^* -_{\mathbb{Z}} 1]) \cdot_{\mathbb{Q}} p \\
&= f_{\mathbb{Z}}^{\mathbb{Q}}(a^*) \cdot_{\mathbb{Q}} p
\end{aligned}$$

by applying substitution based on the definition of $\bar{q}$, the property of an identity, the distributivity of $\cdot_{\mathbb{Q}}$ over $+_{\mathbb{Q}}$, the Identification of $\mathbb{Z}$ in $\mathbb{Q}$, (7.38), and the Associative & Commutative Law for the addition on $\mathbb{Z}$ alongside the property of an identity. In view of (8.241), we therefore see that $p +_{\mathbb{Q}} \bar{q} \in B$ holds. Since $\bar{q} \in A$ is also true, the existential sentence in (8.233) holds then as well. As p was initially arbitrary, we may therefore conclude that the stated universal sentence (8.233) is true. □

Theorem 8.32. *The following sentences are true for any real number* (A, B).

a) *There exists a unique set* A^- *such that*

$$\forall p^- \, (p^- \in A^- \Leftrightarrow [p^- \in \mathbb{Q} \wedge \exists p \, (0 <_{\mathbb{Q}} p \wedge -p^- -_{\mathbb{Q}} p \in B)]). \tag{8.243}$$

b) *The set* $(A^-, \mathbb{Q} \setminus A^-)$ *is a Dedekind cut in* $\mathbb{Q}$.

c) *The set* $(A^-, \mathbb{Q} \setminus A^-)$ *is the negative of* (A, B).

Proof. Letting (A, B) be an arbitrary real number, we may evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the existence of a unique set $(A, B)^-$ such that (8.243) holds.

Concerning b), we observe first that $p^- \in A^-$ implies $p^- \in \mathbb{Q}$ for any p, so that A^- is a subset of $\mathbb{Q}$, by definition. In addition, $\mathbb{Q} \setminus A^-$ is also a subset of $\mathbb{Q}$ because of (2.125). Thus, the ordered pair $(A^-, \mathbb{Q} \setminus A^-)$ possesses Property 1 of a cut in $\mathbb{Q}$.

Regarding Property 2, we note that $B \neq \emptyset$ and $B \subseteq \mathbb{Q}$ since (A, B) is a cut. Consequently, B has some element, say $\bar{b}$, which is in $\mathbb{Q}$ by definition of a subset. Taking the rational number $\bar{p} = 1$, we clearly have $0 <_{\mathbb{Q}} \bar{p}$. Furthermore, we may define the rational number $p^- = -\bar{b} -_{\mathbb{Q}} \bar{p}$. Now, we evidently obtain

$$-p^- -_{\mathbb{Q}} \bar{p} = -(-\bar{b} -_{\mathbb{Q}} \bar{p}) -_{\mathbb{Q}} \bar{p} = (\bar{b} +_{\mathbb{Q}} \bar{p}) -_{\mathbb{Q}} \bar{p} = \bar{b},$$

so that the previously found $\bar{b} \in B$ implies $-p^- -_{\mathbb{Q}} \bar{p} \in B$. These findings demonstrate the truth of the existential sentence in (8.243), so that $p^- \in A^-$ follows to be true; thus, $A^- \neq \emptyset$. To show that $\mathbb{Q} \setminus A^-$ is nonempty as well, we use the fact that $A \neq \emptyset$ and $A \subseteq \mathbb{Q}$ hold (recalling that (A, B) is a cut), so that A contains some rational number, say $\bar{a}$. Let us denote the negative of that number by $p^- = -\bar{a}$ and prove the universal sentence

$$\forall p \, (0 <_{\mathbb{Q}} p \Rightarrow -p^- -_{\mathbb{Q}} p \notin B). \tag{8.244}$$

Letting p be arbitrary, the assumption $0 <_{\mathbb{Q}} p$ implies $\bar{a} -_{\mathbb{Q}} p <_{\mathbb{Q}} \bar{a}$ with the Monotony Law for $+$ and $<$. Here, $\bar{a} = -(-\bar{a}) = -p^-$ is evidently true, so that substitution yields $-p^- -_{\mathbb{Q}} p <_{\mathbb{Q}} \bar{a}$. In conjunction with the facts $-p^- -_{\mathbb{Q}} p \in \mathbb{Q}$ and $\bar{a} \in A$, this further implies $-p^- -_{\mathbb{Q}} p \in A$ due to (8.15). Consequently, $-p^- -_{\mathbb{Q}} p \notin B$ holds because of (8.17), which is the desired consequent of the implication in (8.244). As p was arbitrary, we may therefore conclude that (8.244) is true, and this universal sentence implies now the negation $\neg \exists p \, (0 <_{\mathbb{Q}} p \wedge -p^- -_{\mathbb{Q}} p \in B)$ with the Negation Law for existential conjunctions. Therefore, the conjunction in (8.243) is false, so that the negation $p^- \notin A^-$ follows to be true. Since p^-, as the negative of the rational number $\bar{a}$, is itself an element of $\mathbb{Q}$, we finally obtain $p^- \in \mathbb{Q} \setminus A^-$ with the definition of a set difference; thus, $\mathbb{Q} \setminus A^- \neq \emptyset$. This finding completes the proof that $(A^-, \mathbb{Q} \setminus A^-)$ possesses Property 2 of a cut in $\mathbb{Q}$.

Then, as $A^- \cap (\mathbb{Q} \setminus A^-) = \emptyset$ holds by the Generation of two disjoint sets, and since the previously established inclusion $A^- \subseteq \mathbb{Q}$ implies $A^- \cup (\mathbb{Q} \setminus A^-) = \mathbb{Q}$ with (2.263), we see that Property 3 and Property 4 of a cut in $\mathbb{Q}$ are also satisfied by the ordered pair $(A^-, \mathbb{Q} \setminus A^-)$.

To establish Property 5, we let $p^- \in A^-$ and $q \in \mathbb{Q} \setminus A^-$ be arbitrary. On the one hand, $p^- \in A^-$ implies (by definition of the set A^-) that $0 <_{\mathbb{Q}} \bar{p}$ and $-p^- - \bar{p} \in B$ are satisfied by some particular rational number $\bar{p}$. On the other hand, $q \in \mathbb{Q} \setminus A^-$ implies (besides $q \in \mathbb{Q}$) the negation $q \notin A^-$.

Due to $p^- \in A^-$, this gives inequality $p^- \neq q$ with (2.4), because of the connexity of the linear ordering $<_\mathbb{Q}$, we thus have that $p^- <_\mathbb{Q} q$ or $q <_\mathbb{Q} p^-$ is true. Here, we may prove by contradiction that the second part $q <_\mathbb{Q} p^-$ is false, i.e., that $\neg q <_\mathbb{Q} p^-$ is true. To this end, we assume the negation of that negation to be true, so that $q <_\mathbb{Q} p^-$ follows to be true with the Double Negation Law. This inequality gives us $-p^- -_\mathbb{Q} \bar{p} <_\mathbb{Q} q -_\mathbb{Q} \bar{p}$. In view of $-p^- - \bar{p} \in B$, this inequality implies $q -_\mathbb{Q} \bar{p} \in B$ by virtue of (8.16). Recalling that $0 <_\mathbb{Q} \bar{p}$ also holds, this demonstrates the existence of some rational number p satisfying both $0 <_\mathbb{Q} p$ and $q -_\mathbb{Q} p \in B$. It evidently follows from this that q is an element of A^-, which contradicts the true negation $q \notin A^-$. This completes the proof of $\neg q <_\mathbb{Q} p^-$, which means that $q <_\mathbb{Q} p^-$ is false. Consequently, the first part $p^- <_\mathbb{Q} q$ of the preceding true disjunction is true. Since p^- and q were initially arbitrary, we may therefore conclude that Property 5 of a cut in $\mathbb{Q}$ is satisfied by $(A^-, \mathbb{Q} \setminus A^-)$.

Finally, we prove by contradiction that there is no maximum of A^- with respect to $\leq_\mathbb{Q}$. Assuming the negation of the proposed negation to be true, it follows that there exists a rational number, say u^*, such that $u^* = \max A^-$. By definition of a maximum, u^* is then an upper bound for A^- (with respect to $\leq_\mathbb{Q}$) in that set. Consequently, be definition of A^-, there is a particular rational number $0 <_\mathbb{Q} p^*$ with $-u^* -_\mathbb{Q} p^* \in B$. The inequality evidently implies $0 <_\mathbb{Q} p^*/2$ with the Monotony Law for $\cdot$ and $<$ and then $u^* <_\mathbb{Q} u^* +_\mathbb{Q} p^*/2$ with the Monotony Law for $+$ and $<$. Let us denote the right-hand side of the latter inequality by p^-. On the one hand, this yields $u^* <_\mathbb{Q} p^-$. On the other hand, we evidently obtain the true equations

$$-p^- -_\mathbb{Q} p^*/2 = -(u^* +_\mathbb{Q} p^*/2) -_\mathbb{Q} p^*/2 = -u^* -_\mathbb{Q} p^*.$$

Recalling the previous finding $-u^* -_\mathbb{Q} p^* \in B$, we therefore find also $-p^- -_\mathbb{Q} p^*/2 \in B$. Since $0 <_\mathbb{Q} p^*/2$ also holds, we see that there exists a rational number p satisfying both $0 <_\mathbb{Q} p$ and $-p^- -_\mathbb{Q} p \in B$, so that $p^- \in A^-$ follows to be true. As we previously found u^* to be an upper bound for A^-, we therefore obtain $p^- \leq_\mathbb{Q} u^*$, and this implies the truth of the negation $\neg u^* <_\mathbb{Q} p^-$ by virtue of the Negation Formula for $<$. As we found $u^* <_\mathbb{Q} p^-$ to be true as well, we arrived at a contradiction, so that the proof by contradiction is now complete. Thus, the set A^- indeed has no maximum (with respect to $\leq_\mathbb{Q}$), which means that the cut $(A^-, \mathbb{Q} \setminus A^-)$ in $\mathbb{Q}$ constitutes a Dedekind cut in $\mathbb{Q}$.

Concerning c), we first show that

$$(A, B) +_\mathbb{R} (A^-, \mathbb{Q} \setminus A^-) = f_\mathbb{Q}^\mathbb{R}(0) \tag{8.245}$$

is true. In view of (8.206) and (8.215), this equation is equivalent to

$$(A \oplus A^-, \mathbb{Q} \setminus A \oplus A^-) = ((-\infty, 0), [0, +\infty)). \tag{8.246}$$

In addition, the preceding equation is equivalent to the conjunction

$$A \oplus A^- = (-\infty, 0) \ \wedge \ \mathbb{Q} \setminus A \oplus A^- = [0, +\infty) \tag{8.247}$$

due to the Equality Criterion for ordered pairs. We demonstrate the truth of the first part of this conjunction by means of the Equality Criterion for sets, by verifying

$$\forall s\, (s \in A \oplus A^- \Leftrightarrow s \in (-\infty, 0)). \tag{8.248}$$

Letting s be arbitrary, we first assume $s \in A \oplus A^-$ to be true. By definition of the Addition of sets, there are then particular elements $\bar{a} \in A$ and $\bar{a}^- \in A^-$ such that $s = \bar{a} +_{\mathbb{Q}} \bar{a}^-$. Here, $\bar{a}^- \in A^-$ implies with (8.243) that there is also a particular element $\bar{p} \in \mathbb{Q}$ satisfying $0 <_{\mathbb{Q}} \bar{p}$ and $-\bar{a}^- -_{\mathbb{Q}} \bar{p} \in B$. The inequality yields first $-\bar{p} <_{\mathbb{Q}} 0$ and then $-\bar{a}^- -_{\mathbb{Q}} \bar{p} <_{\mathbb{Q}} -\bar{a}^-$ with the Monotony Law for $+$ and $<$. Due to $-\bar{a}^- -_{\mathbb{Q}} \bar{p} \in B$, it follows from the preceding inequality that $-\bar{a}^- \in B$ is true, according to Exercise 8.1. This finding and the previously found $\bar{a} \in A$ imply $\bar{a} <_{\mathbb{Q}} -\bar{a}^-$ because (A, B) is a Dedekind cut in $\mathbb{Q}$. This in turn implies $[s =]\ \bar{a} +_{\mathbb{Q}} \bar{a}^- <_{\mathbb{Q}} 0$ with the Monotony Law for $+$ and $<$. The resulting inequality $s <_{\mathbb{Q}} 0$ gives us now $s \in (-\infty, 0)$ with the definition of an open and left-unbounded interval in $\mathbb{Q}$. We thus established the first part ('$\Rightarrow$') of the equivalence in (8.248).

To establish the second part ('$\Leftarrow$'), we assume now $s \in (-\infty, 0)$ to be true, with the evident consequence that $s <_{\mathbb{Q}} 0$. Clearly, $0 <_{\mathbb{Q}} -s$ holds as well. Due to the evident fact $0 <_{\mathbb{Q}} \frac{1}{2}$, we find on the one hand $0 <_{\mathbb{Q}} -\frac{s}{2}$ with the Monotony Law for $\cdot$ and $<$ and with the Cancellation for 0_X in rings, on the other hand $-\frac{s}{2} \in \mathbb{Q}_+$ by definition of the set $\mathbb{Q}_+$. We now see in light of Lemma 8.31 that there is an element of A, say $\bar{q}$, such that $-\frac{s}{2} +_{\mathbb{Q}} \bar{q} \in B$. Let us take the rational number $p^- = s -_{\mathbb{Q}} \bar{q}$. We evidently obtain then

$$\begin{aligned}
-p^- -_{\mathbb{Q}} \left(-\frac{s}{2}\right) &= -(s -_{\mathbb{Q}} \bar{q}) -_{\mathbb{Q}} \left(-\frac{s}{2}\right) \\
&= (\bar{q} -_{\mathbb{Q}} s) +_{\mathbb{Q}} \frac{s}{2} \\
&= \bar{q} -_{\mathbb{Q}} (s +_{\mathbb{Q}} \frac{s}{2}) \\
&= \bar{q} -_{\mathbb{Q}} \frac{2 \cdot s - s}{2} \\
&= \bar{q} -_{\mathbb{Q}} \frac{(2-1) \cdot s}{2} \\
&= \bar{q} -_{\mathbb{Q}} \frac{s}{2} \\
&= -\frac{s}{2} +_{\mathbb{Q}} \bar{q}
\end{aligned}$$

by applying substitution, the Sign Laws (6.50) & (6.53), the Associative Law for the addition on $\mathbb{Q}$, the Calculus of fractions, the Distributive law for $\mathbb{Z}$, the property of the identity, and the Commutative Law for the addition on $\mathbb{Q}$. Therefore, the previously established $-\frac{s}{2} +_{\mathbb{Q}} \bar{q} \in B$ yields $-p^- -_{\mathbb{Q}} \left(-\frac{s}{2}\right)$ via substitution. Together with the previous findings $p^- \in \mathbb{Q}$ and $0 <_{\mathbb{Q}} -\frac{s}{2}$, this evidently gives us $p^- \in A^-$ due to (8.243). Recalling now that $\bar{q} \in A$ holds, and observing the truth of the equations

$$s = s +_{\mathbb{Q}} 0 = s +_{\mathbb{Q}} (\bar{q} -_{\mathbb{Q}} \bar{q}) = \bar{q} +_{\mathbb{Q}} (s -_{\mathbb{Q}} \bar{q}) = \bar{q} +_{\mathbb{Q}} p^-,$$

we see in light of the Addition of sets that $s \in A \oplus A^-$ follows to be true, too. This completes the proof of the equivalence in (8.248), where s is arbitrary, so that we may conclude that the universal sentence (8.248) holds. Thus, the equivalent first part of the conjunction (8.247) is true.

To prove the second part of the conjunction, we apply again the Equality Criterion for sets by verifying

$$\forall s\, (s \in \mathbb{Q} \setminus A \oplus A^- \Leftrightarrow s \in [0, +\infty). \tag{8.249}$$

To do this, we take an arbitrary set s, and we assume first $s \in \mathbb{Q} \setminus A \oplus A^-$. By definition of a set difference, $s \in \mathbb{Q}$ and $s \notin A \oplus A^-$ are then both true. The latter implies due to the equivalence in (8.248) that $s \notin (-\infty, 0)$. Now, since $((-\infty, 0), [0, +\infty))$ constitutes a Dedekind cut, the preceding negation implies $s \in [0, +\infty)$ due to Proposition 8.3, as desired. The converse assumption $s \in [0, +\infty)$ implies for the same reason $s \notin (-\infty, 0)$, and this negation yields $s \notin A \oplus A^-$ with (8.248). In addition, since $[0, +\infty)$ constitutes an interval in $\mathbb{Q}$, $s \in \mathbb{Q}$ is also true. Thus, $s \in \mathbb{Q} \setminus A \oplus A^-$ holds by definition of a set difference. We thus proved the equivalence in (8.249), in which s is arbitrary, so that the universal sentence (8.249) follows to be true as well. Thus, the second part of the conjunction (8.247) is true, too. Consequently, the equivalent equation (8.245) also holds. It remains to observe the truth of the equation

$$(A^-, \mathbb{Q} \setminus A^-) +_{\mathbb{R}} (A, B) = (A, B) +_{\mathbb{R}} (A^-, \mathbb{Q} \setminus A^-) \quad \left[= f_{\mathbb{Q}}^{\mathbb{R}}(0)\right]$$

in light of the Commutative Law for the addition on $\mathbb{R}$, with the consequence that

$$(A^-, \mathbb{Q} \setminus A^-) +_{\mathbb{R}} (A, B) = f_{\mathbb{Q}}^{\mathbb{R}}(0).$$

The truth of this equation and of (8.245) allows us to infer that $(A^-, \mathbb{Q} \setminus A^-)$ is the negative of (A, B), by definition. Since (A, B) was initially arbitrary, we may therefore conclude that the stated theorem is true indeed. □

Corollary 8.33. *The semigroup* $(\mathbb{R}, +_{\mathbb{R}})$ *is a commutative group.*

Proof. According to Proposition 8.30, $f_{\mathbb{Q}}^{\mathbb{R}}(0)$ is the identity element of $\mathbb{R}$ with respect to $+_{\mathbb{R}}$. Thus, $(\mathbb{R}, +_{\mathbb{R}})$ has the Property 1 of a group. Regarding Property 2, we prove that the inverse element of any element of $\mathbb{R}$ with respect to $+_{\mathbb{R}}$ exists. To do this, we let $x \in \mathbb{R}$ be arbitrary, so that there are sets, say $\bar{A}$ and $\bar{B}$, such that $x = (\bar{A}, \bar{B})$ constitutes a Dedekind cut in $\mathbb{Q}$ (see Exercise 8.4). Since the negative of this real number $(\bar{A}, \bar{B})$ is given by $(A^-, \mathbb{Q} \setminus A^-)$, as defined and shown in Theorem 8.32, we may conclude that the negative of x exists. Since x was arbitrary, we may further conclude that $(\mathbb{R}, +_{\mathbb{R}})$ also possesses Property 2 of a group. In view of the Commutative Law for the addition on $\mathbb{R}$, this is a commutative group. □

Proposition 8.34. *The negative of a rational in* $\mathbb{R}$ *is also a rational in* $\mathbb{R}$.

Proof. We take an arbitrary element $x \in \mathbb{Q}_{\mathbb{R}}$. Recalling from the Identification of $\mathbb{Q}$ in $\mathbb{R}$ that $f_{\mathbb{Q}}^{\mathbb{R}}$ constitutes a bijection from $\mathbb{Q}$ to $\mathbb{Q}_{\mathbb{R}}$, we see on the one hand that $q = (f_{\mathbb{Q}}^{\mathbb{R}})^{-1}(x)$ is a rational number (using Lemma 3.208), so that $f_{\mathbb{Q}}^{\mathbb{R}}(q) = ((-\infty, q), [q, +\infty))$. On the other hand, the former equation for q implies $x = f_{\mathbb{Q}}^{\mathbb{R}}(q)$ due to the Characterization of the function values of an inverse function. Consequently, substitution gives us $x = ((-\infty, q), [q, +\infty))$, whose negative we denote by $((-\infty, q)^-, \mathbb{Q} \setminus (-\infty, q)^-)$ in view of Theorem 8.32c). We may now establish the equations

$$(-\infty, q)^- = (-\infty, -q) \ \wedge \ \mathbb{Q} \setminus (-\infty, q)^- = [q, +\infty) \tag{8.250}$$

by means of the Equality Criterion for sets. To do this, we let p^- be arbitrary, assuming first $p^- \in (-\infty, q)^-$ to hold. According to Theorem 8.32a), it follows that p^- is a rational, and there is a particular constant $0 < \bar{p}$ such that $-p^- - \bar{p} \in [q, +\infty)$. By definition of a left-closed and right-unbounded interval in $\mathbb{Q}$, the inequality $q \leq -p^- - \bar{p}$ is then true. The Monotony Law for $+$ and $\leq$ gives us then $\bar{p} \leq -p^- - q$. In conjunction with the given inequality $0 < \bar{p}$, this implies $0 < -p^- - q$ with the Transitivity Formula for $<$ and $\leq$. Another application of the aforementioned monotony law yields $p^- < -q$, so that $p^- \in (-\infty, -q)$ turns out to be true by definition of an open and left-unbounded interval in $\mathbb{Q}$. This finding concludes the first part ('⇒') of the application of the Equality Criterion for sets.

Regarding its second part ('⇐'), we conversely assume $p^- \in (-\infty, -q)$ to hold, so that $p^- < -q$ and $0 < -p^- - q$ are evidently true as well. This inequality further implies

$$0 < \frac{0 + (-p^- - q)}{1 + 1} < -p^- - q \tag{8.251}$$

with (7.109), where we may evidently simplify the middle part. Subsequently, the second inequality can be written as

$$q < -p^- - \frac{-p^- - q}{2} \tag{8.252}$$

by virtue of the Monotony Law for $+$ and $<$. Thus, the rational number $p^* = \frac{-p^- - q}{2}$ satisfies $q < -p^- - p^*$, which implies the truth also of the inequality $q \leq -p^- - p^*$ because of the Characterization of induced irreflexive partial orderings. Clearly, this yields $-p^- - p^* \in [q, +\infty)$, and since $0 < p^*$ also holds in view of (8.251), we now see that the existential sentence

$$\exists p \, (0 < p \,\wedge\, -p^- - p \in [q, +\infty))$$

is true. Together with the evident fact $p^- \in \mathbb{Q}$, this gives us $p^- \in (-\infty, q)^-$, as desired. Since p^- was arbitrary, we may therefore conclude that the first equation (8.250) is true indeed. To establish the second one, we let p be arbitrary, and we observe the truth of the equivalences

$$\begin{aligned} p \in \mathbb{Q} \setminus (-\infty, q)^- \;&\Leftrightarrow\; p \in \mathbb{Q} \,\wedge\, p \notin (-\infty, q)^- \\ &\Leftrightarrow\; p \in \mathbb{Q} \,\wedge\, p \notin (-\infty, -q) \\ &\Leftrightarrow\; p \in \mathbb{Q} \,\wedge\, \neg p < -q \\ &\Leftrightarrow\; p \in \mathbb{Q} \,\wedge\, -q \leq p \\ &\Leftrightarrow\; p \in \mathbb{Q} \,\wedge\, p \in [-q, +\infty) \end{aligned}$$

(where we applied the Negation Formula for $<$). As p is arbitrary, we may evidently infer from the the truth of the second equation (8.250). Thus, substitutions based on the previous findings give us

$$\begin{aligned} -x &= -f_{\mathbb{Q}}^{\mathbb{R}}(q) \\ &= -((-\infty, q), [q, +\infty)) \\ &= ((-\infty, q)^-, \mathbb{Q} \setminus (-\infty, q)^-) \\ &= ((-\infty, -q), [q, +\infty)) \\ &= f_{\mathbb{Q}}^{\mathbb{R}}(-q). \end{aligned}$$

Here, the value $f_{\mathbb{Q}}^{\mathbb{R}}(-q)$ is clearly in the range $\mathbb{Q}_{\mathbb{R}}$ of the bijection $f_{\mathbb{Q}}^{\mathbb{R}}$, so that $-x \in f_{\mathbb{Q}}^{\mathbb{R}}$ holds as well due to the preceding equations. Since x is an arbitrary rational in $\mathbb{R}$, we may therefore conclude that the negative of every rational in $\mathbb{R}$ constitutes a rational in $\mathbb{R}$. □

Definition 8.18 (Subtraction on the set of real numbers). We say that

$$-_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto x -_{\mathbb{R}} y = x +_{\mathbb{R}} (-y) \tag{8.253}$$

is the *subtraction on the set of real numbers*.

Theorem 8.35 (Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$). *It is true that the linearly ordered set* $(\mathbb{R}, <_{\mathbb{R}})$ *and the addition* $+_{\mathbb{R}}$ *satisfy the monotony law*

$$\forall x, y, z\,(x, y, z \in \mathbb{R} \Rightarrow [x <_{\mathbb{R}} y \Rightarrow x +_{\mathbb{R}} z <_{\mathbb{R}} y +_{\mathbb{R}} z]). \tag{8.254}$$

Proof. We take arbitrary real numbers x, y and z, which can evidently be written as the Dedekind cuts $x = (A, B)$, $y = (C, D)$ and $z = (E, F)$ in $\mathbb{Q}$. Next, we assume $x <_{\mathbb{R}} y$ to be true, which implies on the one hand $x \neq y$ with the Characterization of comparability with respect to the standard linear ordering of $\mathbb{R}$, on the other hand $x \leq_{\mathbb{R}} y$ with the Characterization of induced irreflexive partial orderings. Substitutions give us then $(A, B) \leq_{\mathbb{R}} (C, D)$, so that the inclusion $A \subseteq C$ follows to be true with the Total ordering of the set of Dedekind cuts ($\mathbb{R}$). Let us observe now the truth of the equations

$$x +_{\mathbb{R}} z = (A, B) +_{\mathbb{R}} (E, F) = (A \oplus E, \mathbb{Q} \setminus A \oplus E), \tag{8.255}$$
$$y +_{\mathbb{R}} z = (C, D) +_{\mathbb{R}} (E, F) = (C \oplus E, \mathbb{Q} \setminus C \oplus E) \tag{8.256}$$

in light of (8.206). Here, we may prove the inclusion $A \oplus E \subseteq C \oplus E$. To this end, we apply the definition of a subset, letting $s \in A \oplus E$ be arbitrary. According to the Addition of sets, s therefore constitutes the sum

$$s = \bar{a} +_{\mathbb{Q}} \bar{e} \tag{8.257}$$

with $\bar{a} \in A$ and $\bar{e} \in E$. In view of the previously established inclusion $A \subseteq C$, it follows from $\bar{a} \in A$ that $\bar{a} \in C$ holds as well. The findings (8.257), $\bar{a} \in C$ and $\bar{e} \in E$ clearly demonstrate that $s \in C \oplus E$ holds. Having proved that $s \in A \oplus E$ implies $s \in C \oplus E$, where s is arbitrary, we may now conclude that the inclusion $A \oplus E \subseteq C \oplus E$ holds indeed. According to the Total ordering of the set of Dedekind cuts, the inequality

$$(A \oplus E, \mathbb{Q} \setminus A \oplus E) \;\leq_{\mathbb{R}}\; (C \oplus E, \mathbb{Q} \setminus C \oplus E)$$

is therefore true. In view of (8.255) – (8.256), this inequality can also be written as

$$x +_{\mathbb{R}} z \;\leq_{\mathbb{R}}\; y +_{\mathbb{R}} z. \tag{8.258}$$

By the Characterization of induced irreflexive partial orderings, we thus find the true disjunction

$$x +_{\mathbb{R}} z <_{\mathbb{R}} y +_{\mathbb{R}} z \;\vee\; x +_{\mathbb{R}} z = y +_{\mathbb{R}} z, \tag{8.259}$$

whose second part we may now prove to be wrong by contradiction. Thus, to establish the negation $x +_{\mathbb{R}} z \neq y +_{\mathbb{R}} z$, we assume its negation to be true. Consequently, the equation $x +_{\mathbb{R}} z = y +_{\mathbb{R}} z$ holds by virtue of the Double Negation Law. Since $(\mathbb{R}, +_{\mathbb{R}})$ is a group (see Corollary 8.33), we may apply the Cancellation Law for groups to simplify the preceding equation to $x = y$. As we previously found $x \neq y$ to be true, too, we arrived at a contradiction. We thus proved that the second part of the disjunction (8.259) is false; hence, its first part is true. This is the desired consequent of the implications in (8.254), where x, y and z are arbitrary, so that the universal sentence (8.254) follows now to be true. □

As a preparation for the definition of the multiplication on the set of real numbers, we study the following special function in some detail. For this purpose, we introduce the following convenient notation.

Notation 8.3. As the zero element $f_{\mathbb{Q}}^{\mathbb{R}}(0)$ is a natural/integer/rational number in $\mathbb{R}$, we write

$$0 = f_{\mathbb{Q}}^{\mathbb{R}}(0). \tag{8.260}$$

Lemma 8.36. *It is true that the restriction of the addition on* $\mathbb{R}$ *to* $\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}}$ *is a binary operation on* $\mathbb{Q}_{\mathbb{R}}$*, that is,*

$$+_{\mathbb{R}} \upharpoonright (\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}}) : \mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}} \to \mathbb{Q}_{\mathbb{R}}. \tag{8.261}$$

Proof. Since the inclusion $\mathbb{Q}_{\mathbb{R}} \subseteq \mathbb{R}$ is true in view of (8.56), we obtain the inclusion

$$\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}} \subseteq \mathbb{R} \times \mathbb{R} \tag{8.262}$$

by means of the Idempotent Law for the conjunction and (3.40). Because the addition on $\mathbb{R}$ is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, we obtain for its restriction to $\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}}$

$$+_{\mathbb{R}} \upharpoonright (\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}}) : \mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}} \to \mathbb{R}. \tag{8.263}$$

because of (3.566). It remains for us to demonstrate that $\mathbb{Q}_{\mathbb{R}}$ is also a codomain of the restriction, i.e. that the range of the restriction is included in $\mathbb{Q}_{\mathbb{R}}$. For this purpose, we apply the definition of a subset and verify accordingly

$$\forall y\,(y \in \operatorname{ran}(+_{\mathbb{R}} \upharpoonright (\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}})) \Rightarrow y \in \mathbb{Q}_{\mathbb{R}}). \tag{8.264}$$

Letting y be arbitrary and assuming the antecedent to be true, we obtain by definition of a range a particular constant $\bar{z}$ such that

$$(\bar{z}, y) \in +_{\mathbb{R}} \upharpoonright (\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}}). \tag{8.265}$$

As the restricted addition is a function with domain $\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}}$, we have (by definition) that $\bar{z}$ is an element of that Cartesian product, so that there exist (by definition) particular elements $a \in \mathbb{Q}_{\mathbb{R}}$ and $b \in \mathbb{Q}_{\mathbb{R}}$ with $(a, b) = \bar{z}$. We may therefore apply substitution to rewrite (8.265) as

$$((a, b), y) \in +_{\mathbb{R}} \upharpoonright (\mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}}),$$

which implies with the definition of a restriction $((a, b), y) \in +_{\mathbb{R}}$. Using now the notation for binary operations, we may write this also as $y = a +_{\mathbb{R}} b$. Let us observe next that $a \in \mathbb{Q}_{\mathbb{R}}$ and $b \in \mathbb{Q}_{\mathbb{R}}$ give the unique values $p = (f_{\mathbb{Q}}^{\mathbb{R}})^{-1}(a)$ and $q = (f_{\mathbb{Q}}^{\mathbb{R}})^{-1}(b)$ in $\mathbb{Q}$ under the inverse $(f_{\mathbb{Q}}^{\mathbb{R}})^{-1} : \mathbb{Q}_{\mathbb{R}} \rightleftarrows \mathbb{Q}$ of the bijection $f_{\mathbb{Q}}^{\mathbb{R}} : \mathbb{Q} \rightleftarrows \mathbb{Q}_{\mathbb{R}}$ in (8.54). According to the Characterization of the function values of an inverse function, the previous two equations for p and q give us $a = f_{\mathbb{Q}}^{\mathbb{R}}(p)$ and $b = f_{\mathbb{Q}}^{\mathbb{R}}(q)$. Now, according to the Addition of rational numbers in $\mathbb{R}$, the sum of a and b is given by

$$[y = a +_{\mathbb{R}} b =] \quad f_{\mathbb{Q}}^{\mathbb{R}}(p) +_{\mathbb{R}} f_{\mathbb{Q}}^{\mathbb{R}}(q) = f_{\mathbb{Q}}^{\mathbb{R}}(p +_{\mathbb{Q}} q),$$

where $f_{\mathbb{Q}}^{\mathbb{R}}(p +_{\mathbb{Q}} q)$ is evidently in the range $\mathbb{Q}_{\mathbb{R}}$ of the bijection $f_{\mathbb{Q}}^{\mathbb{R}}$. Consequently, $y \in \mathbb{Q}_{\mathbb{R}}$ turns out be true as well, which is the desired consequent of the implication in (8.264). As y was arbitrary, we may infer from the truth of that implication the truth of the universal sentence (8.264) and therefore the truth of the inclusion the restricted addition's range in $\mathbb{Q}_{\mathbb{R}}$. We thus completed the proof of (8.261). □

Notation 8.4. We symbolize the restricted binary operation (8.261) also by

$$+_{\mathbb{Q}_{\mathbb{R}}} : \mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}} \to \mathbb{Q}_{\mathbb{R}}. \tag{8.266}$$

Exercise 8.16. Show that the sum of two rationals in $\mathbb{R}$ with respect to the addition on $\mathbb{Q}_{\mathbb{R}}$ is identical with the sum of these numbers with respect to the addition on $\mathbb{R}$, that is,

$$\forall p, q\, (p, q \in \mathbb{Q}_{\mathbb{R}} \Rightarrow p +_{\mathbb{Q}_{\mathbb{R}}} q = p +_{\mathbb{R}} q). \tag{8.267}$$

(Hint: Recall the proof of Corollary 7.11.)

Theorem 8.37 (Isomorphism from $(\mathbb{Q}, +_{\mathbb{Q}})$ to $(\mathbb{Q}_{\mathbb{R}}, +_{\mathbb{Q}_{\mathbb{R}}})$). *It is true that $f_{\mathbb{Q}}^{\mathbb{R}}$ constitutes an isomorphism from $(\mathbb{Q}, +_{\mathbb{Q}})$ to $(\mathbb{Q}_{\mathbb{R}}, +_{\mathbb{Q}_{\mathbb{R}}})$, that is,*

$$f_{\mathbb{Q}}^{\mathbb{R}} : (\mathbb{Q}, +_{\mathbb{Q}}) \rightleftarrows (\mathbb{Q}_{\mathbb{R}}, +_{\mathbb{Q}_{\mathbb{R}}}). \tag{8.268}$$

Exercise 8.17. Establish Theorem 8.37 by proving

$$\forall p, q\, (p, q \in \mathbb{Q} \Rightarrow f_{\mathbb{Q}}^{\mathbb{R}}(p +_{\mathbb{Q}} q) = f_{\mathbb{Q}}^{\mathbb{R}}(p) +_{\mathbb{Q}_{\mathbb{R}}} f_{\mathbb{Q}}^{\mathbb{R}}(q)). \tag{8.269}$$

We prepare now the specification of the multiplication on $\mathbb{R}$.

Lemma 8.38 (Multiplication on $\mathbb{R}_+$). *The following sentences are true:*

a) For any sets $A \subseteq \mathbb{Q}$ and $B \subseteq \mathbb{Q}$, there exists a unique set $A \otimes B$ containing precisely every rational number p that is less than or equal to the product of some positive element of A and some positive element of B. This set satisfies then

$$\forall p\,(p \in A \otimes B \Leftrightarrow \exists a, b\,(a \in A \wedge b \in B \wedge 0 <_{\mathbb{Q}} a \wedge 0 <_{\mathbb{Q}} b \wedge p \leq_{\mathbb{Q}} a \cdot_{\mathbb{Q}} b)), \tag{8.270}$$

and constitutes itself a subset of $\mathbb{Q}$.

b) There exists a unique function $\bigotimes$ on $\mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q})$ such that

$$\begin{aligned}\forall x\,(x \in \mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q}) \Rightarrow \exists A, B\,(A \subseteq \mathbb{Q} \wedge B \subseteq \mathbb{Q} \wedge (A, B) = x \\ \wedge \bigotimes(x) = A \otimes B)),\end{aligned} \tag{8.271}$$

and this function is a binary operation on $\mathcal{P}(\mathbb{Q})$, which satisfies

$$\forall A, B\,([A \subseteq \mathbb{Q} \wedge B \subseteq \mathbb{Q}] \Rightarrow \bigotimes((A, B)) = A \otimes B). \tag{8.272}$$

c) The binary operation $\bigotimes$ on $\mathcal{P}(\mathbb{Q})$ is commutative.

d) The binary operation $\bigotimes$ is associative.

e) There exists a unique function $\cdot_{\mathbb{R}_+}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ which satisfies

$$\begin{aligned}\forall z\,(z \in \mathbb{R}_+ \times \mathbb{R}_+ \Rightarrow \exists A, B, C, D\,(z = ((A, B), (C, D)) \\ \wedge \cdot_{\mathbb{R}_+}(z) = (A \otimes C, \mathbb{Q} \setminus A \otimes C))),\end{aligned} \tag{8.273}$$

and this function is a binary operation on $\mathbb{R}_+$, satisfying also

$$\begin{aligned}\forall A, B, C, D\,((A, B), (C, D) \in \mathbb{R}_+ \\ \Rightarrow (A, B) \cdot_{\mathbb{R}_+} (C, D) = (A \otimes C, \mathbb{Q} \setminus A \otimes C)).\end{aligned} \tag{8.274}$$

f) The binary operation $\cdot_{\mathbb{R}_+}$ on $\mathbb{R}_+$ is commutative and associative.

g) The rational number

$$f_{\mathbb{Q}}^{\mathbb{R}}(1) = ((-\infty, 1), [1, +\infty)) \tag{8.275}$$

in $\mathbb{R}$ constitutes the neutral element of $\mathbb{R}_+$ with respect to the multiplication $\cdot_{\mathbb{R}_+}$ on $\mathbb{R}_+$.

Proof. The proof of a) – c) are accomplished through Exercise 8.18. Concerning d), we take arbitrary sets A, B and C in $\mathcal{P}(\mathbb{Q})$ and demonstrate the truth of the equation $(A \otimes B) \otimes C = A \otimes (B \otimes C)$. For this purpose, we apply the Equality Criterion for sets, letting p be arbitrary and assuming first $p \in (A \otimes B) \otimes C$ to hold. According to the definition of that set, we then have $p \leq \bar{f} \cdot \bar{c}$ for some particular constants $\bar{f} \in A \otimes B$ and $\bar{c} \in C$ satisfying $0 <_{\mathbb{Q}} \bar{f}$ and $0 <_{\mathbb{Q}} \bar{c}$. Here, $\bar{f} \in A \otimes B$ implies the existence of particular constants $\bar{a} \in A$ and $\bar{b} \in B$ such that $0 <_{\mathbb{Q}} \bar{a}$, $0 <_{\mathbb{Q}} \bar{b}$ and $\bar{f} \leq_{\mathbb{Q}} \bar{a} \cdot_{\mathbb{Q}} \bar{b}$. Taking now the product $\bar{g} = \bar{b} \cdot_{\mathbb{Q}} \bar{c}$, it follows with the Characterization of induced irreflexive partial orderings that $\bar{g} \leq_{\mathbb{Q}} \bar{b} \cdot_{\mathbb{Q}} \bar{c}$ holds, too. Since $0 <_{\mathbb{Q}} \bar{b}$ and $0 <_{\mathbb{Q}} \bar{c}$ imply $0 <_{\mathbb{Q}} \bar{b} \cdot_{\mathbb{Q}} \bar{c}$ with the Monotony Law for $\cdot$ and $<$ and with the Cancellation Law for 0_X in rings, we thus have $0 <_{\mathbb{Q}} \bar{g}$. These findings clearly show that $\bar{g} \in B \otimes C$ is true. Let us observe next that the previous inequality $\bar{f} \leq_{\mathbb{Q}} \bar{a} \cdot_{\mathbb{Q}} \bar{b}$ yields (due to $0 <_{\mathbb{Q}} \bar{c}$) the inequality $\bar{f} \cdot_{\mathbb{Q}} \bar{c} \leq_{\mathbb{Q}} (\bar{a} \cdot_{\mathbb{Q}} \bar{b}) \cdot_{\mathbb{Q}} \bar{c}$ by virtue of the Monotony Law for $\cdot$ and $\leq$. In view of the Associative Law for the multiplication on $\mathbb{Q}$, the preceding inequality gives us $\bar{f} \cdot_{\mathbb{Q}} \bar{c} \leq_{\mathbb{Q}} \bar{a} \cdot_{\mathbb{Q}} (\bar{b} \cdot_{\mathbb{Q}} \bar{c})$, and then also $\bar{f} \cdot_{\mathbb{Q}} \bar{c} \leq_{\mathbb{Q}} \bar{a} \cdot_{\mathbb{Q}} \bar{g}$ after substitution. In conjunction with the previous inequality $p \leq_{\mathbb{Q}} \bar{f} \cdot_{\mathbb{Q}} \bar{c}$, this further implies $p \leq_{\mathbb{Q}} \bar{a} \cdot_{\mathbb{Q}} \bar{g}$ with the transitivity of the standard total ordering of $\mathbb{Q}$. That inequality, alongside the previous findings $\bar{a} \in A$, $\bar{g} \in B \otimes C$, $0 <_{\mathbb{Q}} \bar{a}$ and $0 <_{\mathbb{Q}} \bar{g}$, evidently implies now $p \in A \otimes (B \otimes C)$, as desired. The converse direction can be proven similarly. Thus, $p \in A \otimes (B \otimes C)$ and $p \in (A \otimes B) \otimes C$ are equivalent, where p is arbitrary, so that the sets $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ are identical. As A, B and C were arbitrary, it follows that the binary operation $\bigotimes$ on $\mathbb{Q}$ is associative, by definition.

Concerning e), it is a straightforward exercise establish the function $\cdot_{\mathbb{R}_+}$ via Function definition by replacement. To prove that this function is a binary operation on $\mathbb{R}_+$, we need to demonstrate that $\mathbb{R}_+$ is a codomain of that function, i.e., that the inclusion $\operatorname{ran}(\cdot_{\mathbb{R}_+}) \subseteq \mathbb{R}_+$ holds. To do this, we let $\bar{y} \in \operatorname{ran}(\cdot_{\mathbb{R}_+})$ be arbitrary. By definition of a range, we then have $(\bar{x}, \bar{y}) \in \cdot_{\mathbb{R}_+}$ for a particular constant $\bar{x}$. We already showed that $\cdot_{\mathbb{R}_+}$ is a function, so that we may write $\bar{y} = \cdot_{\mathbb{R}_+}(\bar{x})$. Moreover, we see in light of the definition of a domain that $\bar{x} \in \mathbb{R}_+ \times \mathbb{R}_+$ $[= \operatorname{dom}(\cdot_{\mathbb{R}_+})]$ holds. Evidently, we may therefore express $\bar{x}$ as the ordered pair $\bar{x} = ((\bar{A}, \bar{B}), (\bar{C}, \bar{D}))$ formed by the positive real numbers $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ for particular sets $\bar{A}, \bar{B}, \bar{C}, \bar{D}$. The associated value is then $\bar{y} = \cdot_{\mathbb{R}_+}(\bar{x}) = (\bar{A} \otimes \bar{C}, \mathbb{Q} \setminus \bar{A} \otimes \bar{C})$, by definition of the function $\cdot_{\mathbb{R}_+}$.

According to a), $\bar{A} \otimes \bar{C}$ constitutes a subset of $\mathbb{Q}$, and so is the set difference $\mathbb{Q} \setminus \bar{A} \otimes \bar{C}$ in view of (2.125). Thus, the ordered pair $\bar{y}$ has Property 1 of a cut in $\mathbb{Q}$.

Next, we show that the $\bar{A} \otimes \bar{C}$ and $\mathbb{Q} \setminus \bar{A} \otimes \bar{C}$ are nonempty sets. Recalling that $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ are positive real numbers, we obtain the inequalities $((-\infty, 0)_{\mathbb{Q}}, [0, +\infty)_{\mathbb{Q}}) <_{\mathbb{R}} (\bar{A}, \bar{B})$ and $((-\infty, 0)_{\mathbb{Q}}, [0, +\infty)_{\mathbb{Q}}) <_{\mathbb{R}} (\bar{C}, \bar{D})$ with the definition of the set $\mathbb{R}_+$ and the Identification of $\mathbb{Q}$ in $\mathbb{R}$. Due to the Characterization of the linear ordering of Dedekind cuts, these inequalities imply the existence of particular constants $\bar{a} \in \bar{A}$ and $\bar{c} \in \bar{C}$ with, respectively,

$$\bar{a} \notin (-\infty, 0)_{\mathbb{Q}}, \tag{8.276}$$
$$\bar{c} \notin (-\infty, 0)_{\mathbb{Q}}. \tag{8.277}$$

By the Characterization of Dedekind cuts, the findings $\bar{a} \in \bar{A}$ and $\bar{c} \in \bar{C}$ furthermore imply the existence of particular constants $a^* \in \bar{A}$ and $c^* \in \bar{C}$ satisfying, respectively,

$$\bar{a} <_{\mathbb{Q}} a^*, \tag{8.278}$$
$$\bar{c} <_{\mathbb{Q}} c^*. \tag{8.279}$$

The negations (8.276) and (8.277) give us the negations $\neg \bar{a} <_{\mathbb{Q}} 0$ and $\neg \bar{c} <_{\mathbb{Q}} 0$ by definition of an open and left-unbounded interval in $\mathbb{Q}$. These in turn yield

$$0 \leq_{\mathbb{Q}} \bar{a}, \tag{8.280}$$
$$0 \leq_{\mathbb{Q}} \bar{c}, \tag{8.281}$$

by virtue of the Negation Formula for $<$. The conjunction of (8.280) & (8.278) and the conjunction of (8.281) & (8.279) result in the two inequalities $0 <_{\mathbb{Q}} a^*$ and $0 <_{\mathbb{Q}} c^*$ by means of the Transitivity Formula for $\leq$ and $<$. We may therefore apply the Monotony Law for $\cdot$ and $<$ to derive from these the inequality $0 \cdot_{\mathbb{Q}} c^* <_{\mathbb{Q}} a^* \cdot_{\mathbb{Q}} c^*$, which in turn implies $0 \leq_{\mathbb{Q}} a^* \cdot_{\mathbb{Q}} c^*$ with the Cancellation Law for 0_X in rings and with the Characterization of induced irreflexive partial orderings. In summary, we found

$$a^* \in \bar{A} \wedge c^* \in \bar{C} \wedge 0 <_{\mathbb{Q}} a^* \wedge 0 <_{\mathbb{Q}} c^* \wedge 0 \leq_{\mathbb{Q}} a^* \cdot_{\mathbb{Q}} c^*$$

to be true, so that $0 \in \bar{A} \otimes \bar{C}$ evidently holds by definition of that set. This finding clearly demonstrates that $\bar{A} \otimes \bar{C}$ is not empty. To show that $\mathbb{Q} \setminus \bar{A} \otimes \bar{C}$ is not empty either, we first observe that the sets $\bar{B}$ and $\bar{D}$ are nonempty subsets of $\mathbb{Q}$ since $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ are (Dedekind) cuts. Therefore, these nonempty sets have some elements, say $\bar{b} \in \bar{B}$ and $\bar{d} \in \bar{D}$, which are then also elements of $\mathbb{Q}$ by definition of a subset. Thus, we may form the product $\bar{q} = \bar{b} \cdot_{\mathbb{Q}} \bar{d}$. We may now prove that $\bar{q}$ is an element of $\mathbb{Q} \setminus \bar{A} \otimes \bar{C}$, which means by definition of a set difference that $\bar{q} \in \mathbb{Q}$ and $\bar{q} \notin \bar{A} \otimes \bar{C}$ are both true. Clearly, $\bar{q}$ is a rational, being defined as the product of the two rational numbers $\bar{b}$ and $\bar{d}$. To establish the required negation, we prove the universal sentence

$$\forall a, c\,([a \in \bar{A} \,\wedge\, c \in \bar{C} \,\wedge\, 0 <_{\mathbb{Q}} a \,\wedge\, 0 <_{\mathbb{Q}} c] \Rightarrow \neg \bar{q} \leq_{\mathbb{Q}} a \cdot_{\mathbb{Q}} c), \tag{8.282}$$

letting a be an arbitrary element of $\bar{A}$ satisfying $0 <_{\mathbb{Q}} a$ and c an arbitrary element of $\bar{C}$ such that $0 <_{\mathbb{Q}} c$. As $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ are cuts, it follows on the one hand from $a \in \bar{A}$ and $\bar{b} \in \bar{B}$ that $a <_{\mathbb{Q}} \bar{b}$ holds; on the other hand, it follows from $c \in \bar{C}$ and $\bar{d} \in \bar{D}$ that $c <_{\mathbb{Q}} \bar{d}$. Due to $0 <_{\mathbb{Q}} c$, we may apply the Monotony Law for $\cdot$ and $<$ to the former inequality $a <_{\mathbb{Q}} \bar{b}$ to obtain

$$a \cdot_{\mathbb{Q}} c \;<_{\mathbb{Q}}\; \bar{b} \cdot_{\mathbb{Q}} c. \tag{8.283}$$

Let us now observe that the previous inequalities $0 <_{\mathbb{Q}} a$ and $a <_{\mathbb{Q}} \bar{b}$ imply $0 <_{\mathbb{Q}} \bar{b}$ with the transitivity of the standard linear ordering of $\mathbb{Q}$. Consequently, we may apply the Monotony Law for $\cdot$ and $<$ to the previous inequality $c <_{\mathbb{Q}} \bar{d}$ to derive the new inequality

$$\bar{b} \cdot_{\mathbb{Q}} c \;<_{\mathbb{Q}}\; \bar{b} \cdot_{\mathbb{Q}} \bar{d}$$

(applying also the Commutative Law for the multiplication on $\mathbb{Q}$). Combining that inequality with the inequality (8.283) gives us then

$$a \cdot_{\mathbb{Q}} c \;<_{\mathbb{Q}}\; \bar{b} \cdot_{\mathbb{Q}} \bar{d}\, [= \bar{q}],$$

so that $\neg \bar{q} \leq_{\mathbb{Q}} a \cdot_{\mathbb{Q}} c$ follows to be true with the Negation Formula for $\leq$. This negation is the desired consequent of the implication in (8.282), in which a and b are arbitrary, so that we may infer from the truth of this inequality the truth of the universal sentence (8.282). Consequently, the negation

$$\neg \exists a, b\,(a \in \bar{A} \,\wedge\, c \in \bar{C} \,\wedge\, 0 <_{\mathbb{Q}} a \,\wedge\, 0 <_{\mathbb{Q}} c \,\wedge\, \bar{q} \leq_{\mathbb{Q}} a \cdot_{\mathbb{Q}} c)$$

holds by the Negation Law for existential conjunctions, and this negation further implies the negation $\bar{q} \notin \bar{A} \otimes \bar{C}$ by virtue of (8.270). As mentioned earlier, this means that $\bar{q} \in \mathbb{Q} \setminus \bar{A} \otimes \bar{C}$, so that $\mathbb{Q} \setminus \bar{A} \otimes \bar{C}$ is clearly not empty. We thus completed the verification of Property 2 of a cut for $\bar{y}$.

Property 3 and Property 4 are immediately seen to hold since the equations

$$\bar{A} \otimes \bar{C} \cap (\mathbb{Q} \setminus \bar{A} \otimes \bar{C}) = \emptyset.$$

and

$$\bar{A} \otimes \bar{C} \cup (\mathbb{Q} \setminus \bar{A} \otimes \bar{C}) = \mathbb{Q}$$

are true according to the Generation of two disjoint sets and (2.263), using the already established fact that $\bar{A} \otimes \bar{C}$ is a subset of $\mathbb{Q}$.

To establish Property 5, we let $p \in \bar{A} \otimes \bar{C}$ and $q \in \mathbb{Q} \setminus \bar{A} \otimes \bar{C}$ be arbitrary. The latter evidently gives $q \in \mathbb{Q}$ and $q \notin \bar{A} \otimes \bar{C}$, and the former implies the existence of particular constants $\bar{a}^*$ and $\bar{c}^*$ satisfying $\bar{a}^* \in \bar{A}$, $\bar{c}^* \in \bar{C}$, $0 <_\mathbb{Q} \bar{a}^*$, $0 <_\mathbb{Q} \bar{c}^*$, and $p \leq_\mathbb{Q} \bar{a}^* \cdot_\mathbb{Q} \bar{c}^*$. The two findings $p \in \bar{A} \otimes \bar{C}$ and $q \notin \bar{A} \otimes \bar{C}$ imply $p \neq q$ because of (2.4). Due to the connexity of the standard linear ordering of $\mathbb{Q}$, it therefore follows that $p <_\mathbb{Q} q$ or $q <_\mathbb{Q} p$ is true. We may prove by contradiction that the second part of that disjunction is false. Indeed, $q <_\mathbb{Q} p$ implies, in conjunction with $p \leq_\mathbb{Q} \bar{a}^* \cdot_\mathbb{Q} \bar{c}^*$ and due to the Transitivity Formula for $<$ and $\leq$, that $q <_\mathbb{Q} \bar{a}^* \cdot_\mathbb{Q} \bar{c}^*$, so that evidently $q \leq_\mathbb{Q} \bar{a}^* \cdot_\mathbb{Q} \bar{c}^*$, and therefore $q \in \bar{A} \otimes \bar{C}$, in contradiction to the previous finding $q \notin \bar{A} \otimes \bar{C}$. Thus, the first part $p <_\mathbb{Q} q$ of the preceding disjunction must be true. Since p and q were arbitrary, we may now infer from the truth of this inequality that $\bar{y} = (\bar{A} \otimes \bar{C}, \mathbb{Q} \setminus \bar{A} \otimes \bar{C})$ indeed possesses Property 5 of a cut.

It remains to show that this cut is a Dedekind cut. To do this, we prove the required definite property ba contradiction, assuming

$$\neg\neg\exists u\,(u \in \mathbb{Q} \wedge u = \max^{\leq_\mathbb{Q}} \bar{A} \otimes \bar{C}) \tag{8.284}$$

to be true. It follows from this with the Double Negation Law that $u = \max^{\leq_\mathbb{Q}} \bar{A} \otimes \bar{C}$ holds for a particular rational number u. According to the definition of a maximum, it is therefore true that u is an upper bound for $\bar{A} \otimes \bar{C}$ and an element of that set. The latter fact implies with the definition of that set that there are particular constants a and c such that $a \in \bar{A}$, $c \in \bar{C}$, $0 <_\mathbb{Q} a$, $0 <_\mathbb{Q} c$ and $u \leq_\mathbb{Q} a \cdot_\mathbb{Q} c$. Evidently, $a \in \bar{A}$ implies the existence of some $a^* \in \bar{A}$ with $a <_\mathbb{Q} a^*$; together with $0 <_\mathbb{Q} a$, this clearly yields $0 <_\mathbb{Q} a^*$. In addition, $a <_\mathbb{Q} a^*$ and $0 <_\mathbb{Q} c$ gives us $a \cdot_\mathbb{Q} c <_\mathbb{Q} a^* \cdot_\mathbb{Q} c$. The conjunction of this inequality and the previously found inequality $u \leq_\mathbb{Q} a \cdot_\mathbb{Q} c$ gives us $u <_\mathbb{Q} a^* \cdot_\mathbb{Q} c$ with the Transitivity Formula for $\leq$ and $<$. Denoting the latter product by $s = a^* \cdot_\mathbb{Q} c$, we obtain via substitution $u <_\mathbb{Q} s$, as well as $s \leq_\mathbb{Q} a^* \cdot_\mathbb{Q} c$ by using the Characterization of induced irreflexive partial orderings. The latter evidently implies $s \in \bar{A} \otimes \bar{C}$ in view of $a^* \in \bar{A}$, $c \in \bar{C}$, $0 <_\mathbb{Q} a^*$ and $0 <_\mathbb{Q} c$. Recalling that u is an upper bound for $\bar{A} \otimes \bar{C}$, it follows that $s \leq_\mathbb{Q} u$ is true. But this inequality implies the truth of the negation $\neg u <_\mathbb{Q} s$ with the Negation Formula for $<$, in contradiction to the previously established fact $u <_\mathbb{Q} s$. We thus completed the proof by contradiction, so that the cut $\bar{y} = (\bar{A} \otimes \bar{C}, \mathbb{Q} \setminus \bar{A} \otimes \bar{C})$ constitutes a Dedekind cut in $\mathbb{Q}$, which means that $\bar{y} \in \mathbb{R}$.

To complete the proof that $\bar{y} \in \mathbb{R}_+$, we need to demonstrate the truth of

$$((-\infty, 0)_\mathbb{Q}, [0, +\infty)_\mathbb{Q}) \ <_\mathbb{R} \ (\bar{A} \otimes \bar{C}, \mathbb{Q} \setminus \bar{A} \otimes \bar{C}) \quad [= \bar{y}]. \tag{8.285}$$

Let us recall from the proof that $\bar{A} \otimes \bar{C}$ constitutes a nonempty set that $0 \in \bar{A} \otimes \bar{C}$ holds. In view of the irreflexivity of the standard linear ordering of

$\mathbb{Q}$, the negation $\neg 0 <_{\mathbb{Q}} 0$ is true, and therefore the negation $\neg 0 \in (-\infty, 0)_{\mathbb{Q}}$ evidently holds as well. This proves the existential sentence

$$\exists p \, (p \notin (-\infty, 0)_{\mathbb{Q}} \wedge p \in \bar{A} \otimes \bar{C}),$$

which implies the truth of the inequality (8.285) with the Characterization of the linear ordering of Dedekind cuts. Therefore, $\bar{y} \in \mathbb{R}_+$ is true by definition of the set $\mathbb{R}_+$. We thus showed that $\bar{y} \in \operatorname{ran}(\cdot_{\mathbb{R}_+})$ implies $\bar{y} \in \mathbb{R}_+$, where $\bar{y}$ is arbitrary, so that the range of $\cdot_{\mathbb{R}_+}$ is indeed included in $\mathbb{R}_+$. Thus, the latter set constitutes a codomain of the function $\cdot_{\mathbb{R}_+}$ on $\mathbb{R}_+ \times \mathbb{R}_+$, so that this function is a binary operation on $\mathbb{R}_+$. To show that this binary operation satisfies also (8.274), we may proceed as in the proof of (8.206) within Exercise 8.14.

The proofs required in f) are similar to the proofs of the Commutative Law and of the Associative Law for the addition on $\mathbb{R}$.

Concerning g), we prove

$$\forall x \, (x \in \mathbb{R}_+ \Rightarrow [f_{\mathbb{Q}}^{\mathbb{R}}(1) \cdot_{\mathbb{R}_+} x = x \wedge x \cdot_{\mathbb{R}_+} f_{\mathbb{Q}}^{\mathbb{R}}(1) = x]), \tag{8.286}$$

taking first an arbitrary positive real number x, which we may write as the Dedekind cut (A, B) for particular sets A and B. We first observe the truth of the equations

$$\begin{aligned} f_{\mathbb{Q}}^{\mathbb{R}}(1) \cdot_{\mathbb{R}_+} x &= ((-\infty, 1), [1, +\infty)) \cdot_{\mathbb{R}_+} (A, B) \\ &= ((-\infty, 1) \otimes A, \mathbb{Q} \setminus (-\infty, 1) \oplus A) \end{aligned} \tag{8.287}$$

in light of e). We now apply the Equality Criterion for sets to prove the equality of the sets $(-\infty, 1) \otimes A$ and A. To do this, we let p be arbitrary, assuming first $p \in (-\infty, 1) \otimes A$ to be true. This evidently means that $s \leq_{\mathbb{Q}} z \cdot_{\mathbb{Q}} a$ with $z \in (-\infty, 1)$, $a \in A$, $0 <_{\mathbb{Q}} z$ and $0 <_{\mathbb{Q}} a$. By definition of an open and left-unbounded interval in $\mathbb{Q}$, the inequality $z <_{\mathbb{Q}} 1$ is then true, which in turn implies $[p \leq_{\mathbb{Q}}] \; z \cdot_{\mathbb{Q}} a <_{\mathbb{Q}} a$ with the Monotony Law for $\cdot$ and $<$ and with the property of an identity element. These inequalities result in the inequality $p <_{\mathbb{Q}} a$ by means of the Transitivity Formula for $\leq$ and $<$. In conjunction with the evident $p \in \mathbb{Q}$ and the fact $a \in A$, this further implies the truth of the desired consequent $p \in A$ by virtue of Proposition 8.2.

We now conversely assume $p \in A$ to be true, and we prove the desired consequent $p \in (-\infty, 1) \otimes A$ by cases, based on the fact that $p <_{\mathbb{R}} 0$, $p = 0$ or $0 <_{\mathbb{R}} p$ holds due to the connexity of the standard linear ordering of $\mathbb{R}$. In the first case $p <_{\mathbb{R}} 0$, we observe that the initial assumption $x \in \mathbb{R}_+$ implies $0 <_{\mathbb{R}} x$ with the definition of the set of positive real numbers, and therefore

$((-\infty, 0), [0, +\infty)) <_\mathbb{R} (A, B)$ because of (8.260) and the Identification of $\mathbb{Q}$ in $\mathbb{R}$. According to the Characterization of the linear ordering of Dedekind cuts, there is then an element of A, say q^*, which is not in $(-\infty, 0)$. On the one, hand, $q^* \in A$ implies with the Characterization of Dedekind cuts that there is a greater element in A, say r^*. On the other hand, the preceding negation $\neg q^* \in (-\infty, 0)$ implies $\neg q^* <_\mathbb{R} 0$ with the definition of an open and left-unbounded interval in $<_\mathbb{Q}$, and then $0 \leq_\mathbb{R} q^*$ with the Negation Formula for $<$. The conjunction of this inequality and the previous finding $q^* <_\mathbb{R} r^*$ gives us now $0 <_\mathbb{R} r^*$ with the Transitivity Formula for $\leq$ and $<$. Due to the evident fact $0 <_\mathbb{R} \frac{1}{2}$, the inequality $0 <_\mathbb{R} \frac{1}{2} \cdot_\mathbb{Q} r^*$ follows to be true with the Monotony Law for $\cdot$ and $<$, with the Cancellation Law for 0_X in rings and with the Commutative Law for the multiplication on $\mathbb{Q}$. In conjunction with the current case assumption $p <_\mathbb{R} 0$, this gives us $p <_\mathbb{R} \frac{1}{2} \cdot_\mathbb{Q} r^*$, and then also

$$p \leq_\mathbb{R} \frac{1}{2} \cdot_\mathbb{Q} r^* \tag{8.288}$$

due to the Characterization of induced irreflexive partial orderings. Evidently, the inequality $\frac{1}{2} <_\mathbb{Q} 1$ is true, so that $\frac{1}{2} \in (-\infty, 1)$. Together with the previous findings $r^* \in A$, $0 <_\mathbb{R} \frac{1}{2}$ and $0 <_\mathbb{R} r^*$, the inequality (8.288) implies now the truth of the desired consequent $p \in (-\infty, 1) \otimes A$, by definition of the preceding set.

In the second case $p = 0$, we observe in light of the Characterization of Dedekind cuts that there exists an element of A, say $\bar{q}$, greater than $p = 0$. Thus, $0 <_\mathbb{R} \bar{q}$ holds, with the evident consequence that $p <_\mathbb{R} \frac{1}{2} \cdot_\mathbb{Q} \bar{q}$. Clearly, $p \leq_\mathbb{R} \frac{1}{2} \cdot_\mathbb{Q} \bar{q}$ is then also true, so that $p \in (-\infty, 1) \otimes A$ turns out to be true also for the second case.

In the third case $0 <_\mathbb{R} p$, the previously found constant $\bar{q} \in A$ with $p <_\mathbb{R} \bar{q}$ satisfies then $0 <_\mathbb{R} \bar{q}$, by the transitivity of the standard linear ordering of $\mathbb{R}$. Due to the Characterization of comparability with respect to that linear ordering, $\bar{q} \neq 0$ holds then, too. Thus, the fraction $\frac{p}{\bar{q}}$ is defined, and we may evidently express p as the product $p = \frac{p}{\bar{q}} \cdot_\mathbb{Q} \bar{q}$. In view of the Characterization of induced irreflexive partial orderings, the inequality

$$p \leq_\mathbb{Q} \frac{p}{\bar{q}} \cdot_\mathbb{Q} \bar{q} \tag{8.289}$$

follows therefore to be true. Let us observe here that the previously established inequality $0 <_\mathbb{R} \bar{q}$ implies $0 <_\mathbb{R} \frac{1}{\bar{q}}$ by virtue of Proposition 7.32. This evidently implies $0 <_\mathbb{R} \frac{p}{\bar{q}}$ (due to the current case assumption) as well as $\frac{p}{\bar{q}} <_\mathbb{R} \bar{q} \cdot_\mathbb{Q} \frac{1}{\bar{q}}$ $[= 1]$ (due to $p <_\mathbb{R} \bar{q}$). The latter yields $\frac{p}{\bar{q}} <_\mathbb{R} 1$ and therefore $\frac{p}{\bar{q}} \in (-\infty, 1)$. Alongside the previous findings $\bar{q} \in A$, $0 <_\mathbb{R} \frac{p}{\bar{q}}$, $0 <_\mathbb{R} \bar{q}$ and

(8.289), this further implies $p \in (-\infty, 1) \otimes A$, which sentence is thus true in all three cases.

This completes the proof that $p \in (-\infty, 1) \otimes A$ and $p \in A$ are equivalent, where p is arbitrary, so that the sets $(-\infty, 1) \oplus A$ and A are indeed identical. Applying now substitutions to (8.287) based on this equation, we find

$$f_{\mathbb{Q}}^{\mathbb{R}}(1) \cdot_{\mathbb{R}_+} x = (A, \mathbb{Q} \setminus A). \tag{8.290}$$

By definition of a cut in $\mathbb{Q}$, the given sets A and B forming the Dedekind cut $x = (A, B)$ satisfy $A \cap B = \emptyset$ as well as $A \cup B = \mathbb{Q}$. Due to the Commutative Law for the intersection and for the union of two sets, we may write these equations also as $B \cup A = \mathbb{Q}$ and $B \cap A = \emptyset$. These imply $B = \mathbb{Q} \setminus A$ with (2.262), so that (8.290) becomes after substitution

$$f_{\mathbb{Q}}^{\mathbb{R}}(1) \cdot_{\mathbb{R}_+} x = (A, B) = x.$$

As the multiplication $\cdot_{\mathbb{R}_+}$ is commutative, the resulting equation $f_{\mathbb{Q}}^{\mathbb{R}}(1) \cdot_{\mathbb{R}_+} x = x$ also gives us $x \cdot_{\mathbb{R}_+} f_{\mathbb{Q}}^{\mathbb{R}}(1) = x$. As x is an arbitrary real number, we may infer from these findings the truth of the universal sentence (8.286), which shows that (8.275) is indeed the identity element of $\mathbb{R}$. □

Exercise 8.18. Establish the missing parts in the proof of Lemma 8.38.

(Hint: The proofs are similar to the corresponding proofs regarding the Addition of sets and the Addition on the set of real numbers.)

Lemma 8.39. *The following sentences are true for any positive real number* (A, B)*.*

a) *There exists a unique set* A' *such that*

$$\forall q' \, (q' \in A' \Leftrightarrow [q' \in \mathbb{Q} \wedge (0 <_{\mathbb{Q}} q' \Rightarrow \exists p \, (0 <_{\mathbb{Q}} q' <_{\mathbb{Q}} p \wedge \frac{1}{p} \in B))]). \tag{8.291}$$

b) *The set* $(A', \mathbb{Q} \setminus A')$ *is a Dedekind cut in* $\mathbb{Q}$*.*

c) *The set* $(A', \mathbb{Q} \setminus A')$ *is the reciprocal of* (A, B)*.*

Proof. To do!

□

Theorem 8.40 (Multiplication on $\mathbb{R}$). *There exists a unique function $\cdot_\mathbb{R}$ on $\mathbb{R} \times \mathbb{R}$ such that*

$$\begin{aligned}
&\forall z\,(z \in \mathbb{R} \times \mathbb{R} \Rightarrow \exists x, y\,(x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge (x, y) = z && (8.292)\\
&\quad \wedge [([x < 0 \wedge y < 0] \Rightarrow \cdot_\mathbb{R}(z) = [-x] \cdot_{\mathbb{R}_+} [-y]) \wedge \\
&\quad\quad ([x < 0 \wedge y = 0] \Rightarrow \cdot_\mathbb{R}(z) = 0) \wedge \\
&\quad\quad ([x < 0 \wedge 0 < y] \Rightarrow \cdot_\mathbb{R}(z) = -([-x] \cdot_{\mathbb{R}_+} y)) \wedge \\
&\quad\quad ([x = 0 \wedge y < 0] \Rightarrow \cdot_\mathbb{R}(z) = 0) \wedge \\
&\quad\quad ([x = 0 \wedge y = 0] \Rightarrow \cdot_\mathbb{R}(z) = 0) \wedge \\
&\quad\quad ([x = 0 \wedge 0 < y] \Rightarrow \cdot_\mathbb{R}(z) = 0) \wedge \\
&\quad\quad ([0 < x \wedge y < 0] \Rightarrow \cdot_\mathbb{R}(z) = -(x \cdot_{\mathbb{R}_+} [-y])) \wedge \\
&\quad\quad ([0 < x \wedge y = 0] \Rightarrow \cdot_\mathbb{R}(z) = 0) \wedge \\
&\quad\quad ([0 < x \wedge 0 < y] \Rightarrow \cdot_\mathbb{R}(z) = x \cdot_{\mathbb{R}_+} y)])), && (8.293)
\end{aligned}$$

and this function is a binary operation on $\mathbb{R}$, which satisfies for any $x, y \in \mathbb{R}$

$$x \cdot_\mathbb{R} y = \begin{cases} [-x] \cdot_{\mathbb{R}_+} [-y] & \text{if } x < 0 \wedge y < 0, \\ -([-x] \cdot_{\mathbb{R}_+} y) & \text{if } x < 0 \wedge 0 < y, \\ -(x \cdot_{\mathbb{R}_+} [-y]) & \text{if } 0 < x \wedge y < 0, \\ 0 & \text{if } x = 0 \vee y = 0, \\ x \cdot_{\mathbb{R}_+} y & \text{if } 0 < x \wedge 0 < y. \end{cases} \tag{8.294}$$

Proof. We apply Function definition by replacement and verify accordingly

$$\begin{aligned}
&\forall z\,(z \in \mathbb{R} \times \mathbb{R} \Rightarrow \exists! p\,(\exists x, y\,(x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge (x, y) = z && (8.295)\\
&\quad \wedge [([x < 0 \wedge y < 0] \Rightarrow p = [-x] \cdot_{\mathbb{R}_+} [-y]) \wedge \\
&\quad\quad ([x < 0 \wedge y = 0] \Rightarrow p = 0) \wedge \\
&\quad\quad ([x < 0 \wedge 0 < y] \Rightarrow p = -([-x] \cdot_{\mathbb{R}_+} y)) \wedge \\
&\quad\quad ([x = 0 \wedge y < 0] \Rightarrow p = 0) \wedge \\
&\quad\quad ([x = 0 \wedge y = 0] \Rightarrow p = 0) \wedge \\
&\quad\quad ([x = 0 \wedge 0 < y] \Rightarrow p = 0) \wedge \\
&\quad\quad ([0 < x \wedge y < 0] \Rightarrow p = -(x \cdot_{\mathbb{R}_+} [-y])) \wedge \\
&\quad\quad ([0 < x \wedge y = 0] \Rightarrow p = 0) \wedge \\
&\quad\quad ([0 < x \wedge 0 < y] \Rightarrow p = x \cdot_{\mathbb{R}_+} y)]))), && (8.296)
\end{aligned}$$

letting $\bar{z} \in \mathbb{R} \times \mathbb{R}$ be arbitrary. It follows from this (by Exercise 3.4) that there are particular elements $\bar{x} \in \mathbb{R}$ and $\bar{y} \in \mathbb{R}$ such that $(\bar{x}, \bar{y}) = \bar{z}$. As the standard linear ordering of $\mathbb{R}$ is connex, the disjunctions

$$\bar{x} < 0 \vee \bar{x} = 0 \vee 0 < \bar{x},$$
$$\bar{y} < 0 \vee \bar{y} = 0 \vee 0 < \bar{y}$$

are true. We use these disjunctions to prove the uniquely existential sentence by cases and subcases.

The first case $\bar{x} < 0$ and the first subcase $\bar{y} < 0$ imply, respectively, $0 < -\bar{x}$ and $0 < -\bar{y}$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$. Consequently, we find $-\bar{x} \in \mathbb{R}_+$ as well as $-\bar{y} \in \mathbb{R}_+$ according to the definition of the set $\mathbb{R}_+$. By definition of the Cartesian product of two sets, the ordered pair $(-\bar{x}, -\bar{y})$ is therefore in the domain $\mathbb{R}_+ \times \mathbb{R}_+$ of the binary operation $\cdot_{\mathbb{R}_+}$, which yields the product $\bar{p} = [-\bar{x}] \cdot_{\mathbb{R}_+} [-\bar{y}]$. Thus, the numbers $\bar{p}$, $\bar{x}$ and $\bar{y}$ satisfy the implication

$$[\bar{x} < 0 \ \wedge \ \bar{y} < 0] \ \Rightarrow \ \bar{p} = [-\bar{x}] \cdot_{\mathbb{R}_+} [-\bar{y}]. \tag{8.297}$$

Since $\bar{x} = 0$, $0 < \bar{x}$, $\bar{y} = 0$ and $0 < \bar{y}$ are all false in view of the Characterization of comparability (with respect to the standard linear ordering of $\mathbb{R}$), we see that the antecedents of the implications

$$[\bar{x} < 0 \ \wedge \ \bar{y} = 0] \ \Rightarrow \ \bar{p} = 0, \tag{8.298}$$
$$[\bar{x} < 0 \ \wedge \ 0 < \bar{y}] \ \Rightarrow \ \bar{p} = -([-\bar{x}] \cdot_{\mathbb{R}_+} \bar{y}), \tag{8.299}$$
$$[\bar{x} = 0 \ \wedge \ \bar{y} < 0] \ \Rightarrow \ \bar{p} = 0, \tag{8.300}$$
$$[\bar{x} = 0 \ \wedge \ \bar{y} = 0] \ \Rightarrow \ \bar{p} = 0, \tag{8.301}$$
$$[\bar{x} = 0 \ \wedge \ 0 < \bar{y}] \ \Rightarrow \ \bar{p} = 0, \tag{8.302}$$
$$[0 < \bar{x} \ \wedge \ \bar{y} < 0] \ \Rightarrow \ \bar{p} = -(\bar{x} \cdot_{\mathbb{R}_+} [-\bar{y}]), \tag{8.303}$$
$$[0 < \bar{x} \ \wedge \ \bar{y} = 0] \ \Rightarrow \ \bar{p} = 0, \tag{8.304}$$
$$[0 < \bar{x} \ \wedge \ 0 < \bar{y}] \ \Rightarrow \ \bar{p} = \bar{x} \cdot_{\mathbb{R}_+} \bar{y} \tag{8.305}$$

are all false, so that these implications are all true. Thus, the existential part of the uniquely existential sentence in (8.296) clearly holds in the first subcase (of the first case). To establish the uniqueness part, we take arbitrary sets p', p'' satisfying the existential sentence with respect to x and y in (8.296). Thus, there are particular constants $x', y', x'', y'' \in \mathbb{R}$

satisfying $(x', y') = \bar{z} = (x'', y'')$ and the implications

$$[x' < 0 \wedge y' < 0] \Rightarrow p' = [-x'] \cdot_{\mathbb{R}_+} [-y'], \tag{8.306}$$
$$[x' < 0 \wedge y' = 0] \Rightarrow p' = 0, \tag{8.307}$$
$$[x' < 0 \wedge 0 < y'] \Rightarrow p' = -([-x'] \cdot_{\mathbb{R}_+} y'), \tag{8.308}$$
$$[x' = 0 \wedge y' < 0] \Rightarrow p' = 0, \tag{8.309}$$
$$[x' = 0 \wedge y' = 0] \Rightarrow p' = 0, \tag{8.310}$$
$$[x' = 0 \wedge 0 < y'] \Rightarrow p' = 0, \tag{8.311}$$
$$[0 < x' \wedge y' < 0] \Rightarrow p' = -(x' \cdot_{\mathbb{R}_+} [-y']), \tag{8.312}$$
$$[0 < x' \wedge y' = 0] \Rightarrow p' = 0, \tag{8.313}$$
$$[0 < x' \wedge 0 < y'] \Rightarrow p' = x' \cdot_{\mathbb{R}_+} y', \tag{8.314}$$

as well as

$$[x'' < 0 \wedge y'' < 0] \Rightarrow p'' = [-x''] \cdot_{\mathbb{R}_+} [-y''], \tag{8.315}$$
$$[x'' < 0 \wedge y'' = 0] \Rightarrow p'' = 0, \tag{8.316}$$
$$[x'' < 0 \wedge 0 < y''] \Rightarrow p'' = -([-x''] \cdot_{\mathbb{R}_+} y''), \tag{8.317}$$
$$[x'' = 0 \wedge y'' < 0] \Rightarrow p'' = 0, \tag{8.318}$$
$$[x'' = 0 \wedge y'' = 0] \Rightarrow p'' = 0, \tag{8.319}$$
$$[x'' = 0 \wedge 0 < y''] \Rightarrow p'' = 0, \tag{8.320}$$
$$[0 < x'' \wedge y'' < 0] \Rightarrow p'' = -(x'' \cdot_{\mathbb{R}_+} [-y'']), \tag{8.321}$$
$$[0 < x'' \wedge y'' = 0] \Rightarrow p'' = 0, \tag{8.322}$$
$$[0 < x'' \wedge 0 < y''] \Rightarrow p'' = x'' \cdot_{\mathbb{R}_+} y''. \tag{8.323}$$

Recalling the previous finding $(\bar{x}, \bar{y}) = \bar{z}$, we see in light of the Equality Criterion for ordered pairs that the equations $\bar{x} = x' = x''$ and $\bar{y} = y' = y''$ hold. Therefore, the current case assumptions $\bar{x} < 0$ and $\bar{y} < 0$ gives us $x' < 0$, $x'' < 0$, $y' < 0$ and $y'' < 0$ via substitutions. These findings imply because of the implications (8.306) and (8.315) that $p' = [-x'] \cdot_{\mathbb{R}_+} [-y']$ and $p'' = [-x''] \cdot_{\mathbb{R}_+} [-y'']$ hold. Further substitutions then lead evidently to the desired equality $p' = p''$. As p' and p'' were arbitrary, we may therefore conclude that the uniqueness part of the uniquely existential sentence in (8.296) holds indeed (besides the existential part). We thus completed the verification of that uniquely existential sentence in the first subcase of the first case.

In the second subcase $\bar{y} = 0$ of the first case $\bar{x} < 0$, the choice $\bar{p} = 0$ clearly results in the true implication (8.298); since now $\bar{x} = 0$, $0 < \bar{x}$, $\bar{y} < 0$ and $0 < \bar{y}$ are all false, the antecedents of the implications (8.297) and (8.299) – (8.305) are false, causing these implications to be true as well.

This means that the existential part of the uniquely existential sentence in (8.296) also holds for the second subcase (within the first case). Regarding the uniqueness part, we let p' and p'' again be arbitrary such that each satisfies the corresponding existential sentence with respect to x and y in (8.296). Then, there are again particular real numbers x', y', x'' and y'' such that $(x', y') = \bar{z} = (x'', y'')$ and the implications (8.306) – (8.323) are satisfied. As in the first subcase, we find $\bar{x} = x' = x''$ and $\bar{y} = y' = y''$ to be true, so that the current case assumptions $\bar{x} < 0$ and $\bar{y} = 0$ result in $x' < 0$, $x'' < 0$, $y' = 0$ and $y'' = 0$. Because of the implications (8.307) and (8.316), we therefore find $p' = 0$ and $p'' = 0$, thus $p' = p''$, as desired. We may infer from the truth of this equation that the uniquely existential sentence in (8.296) is true also in the second subcase.

In the third subcase $0 < \bar{y}$ of the first case $\bar{x} < 0$, we have $\bar{y} \in \mathbb{R}_+$. Recalling that $-\bar{x} \in \mathbb{R}_+$ holds, too, we evidently have $(-\bar{x}, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$. These elements of the domain of the multiplication on $\mathbb{R}_+$ are associated with the product $[-\bar{x}] \cdot_{\mathbb{R}_+} \bar{y}$. Taking the negative $\bar{p} = -([-\bar{x}] \cdot_{\mathbb{R}_+} \bar{y})$, the implication (8.299) is true for these numbers, and the implications (8.297), (8.298) and (8.300) – (8.305) also hold since $\bar{x} = 0$, $0 < \bar{x}$, $\bar{y} < 0$ and $\bar{y} = 0$ are false. This completes the proof of the existential part, and the uniqueness part can be proved similarly to the previous subcases. Moreover, the other two cases $\bar{x} = 0$ and $0 < \bar{x}$ can be proven in analogy to the first case. Since $\bar{z}$ was arbitrary, we may therefore conclude that the universal sentence (8.296) holds, so that there exists a unique function $\cdot_{\mathbb{R}}$ on $\mathbb{R} \times \mathbb{R}$ such that

$$
\begin{aligned}
&\forall z\, (z \in \mathbb{R} \times \mathbb{R} \Rightarrow \exists x, y\, (x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge (x, y) = z && (8.324)\\
&\quad \wedge [([x < 0 \wedge y < 0] \Rightarrow \cdot_{\mathbb{R}}(z) = [-x] \cdot_{\mathbb{R}_+} [-y]) \wedge \\
&\qquad ([x < 0 \wedge y = 0] \Rightarrow \cdot_{\mathbb{R}}(z) = 0) \wedge \\
&\qquad ([x < 0 \wedge 0 < y] \Rightarrow \cdot_{\mathbb{R}}(z) = -([-x] \cdot_{\mathbb{R}_+} y)) \wedge \\
&\qquad ([x = 0 \wedge y < 0] \Rightarrow \cdot_{\mathbb{R}}(z) = 0) \wedge \\
&\qquad ([x = 0 \wedge y = 0] \Rightarrow \cdot_{\mathbb{R}}(z) = 0) \wedge \\
&\qquad ([x = 0 \wedge 0 < y] \Rightarrow \cdot_{\mathbb{R}}(z) = 0) \wedge \\
&\qquad ([0 < x \wedge y < 0] \Rightarrow \cdot_{\mathbb{R}}(z) = -(x \cdot_{\mathbb{R}_+} [-y])) \wedge \\
&\qquad ([0 < x \wedge y = 0] \Rightarrow \cdot_{\mathbb{R}}(z) = 0) \wedge \\
&\qquad ([0 < x \wedge 0 < y] \Rightarrow \cdot_{\mathbb{R}}(z) = x \cdot_{\mathbb{R}_+} y)])). && (8.325)
\end{aligned}
$$

Let us observe here, that the value $\cdot_{\mathbb{R}}(z)$ is either a product in $\mathbb{R}_+$, or the zero element 0 of $\mathbb{R}$, or the negative of a product in $\mathbb{R}_+$. Thus, the proof that the set of real numbers constitutes a codomain of the function $\cdot_{\mathbb{R}}$ is straightforward. For this purpose, we establish the inclusion $\mathrm{ran}(\cdot_{\mathbb{R}}) \subseteq \mathbb{R}$, letting $p^* \in \mathrm{ran}(\cdot_\mathbb{R})$ be arbitrary. By definition of a range there is then a particular set z^* such that $(z^*, p^*) \in \cdot_\mathbb{R}$. Since $\cdot_\mathbb{R}$ is a function, we may write $p^* = \cdot_\mathbb{R}(z^*)$. Furthermore, $z^* \in \mathbb{R} \times \mathbb{R}$ $[= \mathrm{dom}(\cdot_\mathbb{R})]$ holds by definition of a domain. Due to (8.325), there are particular real number x^* and y^* that satisfy $(x^*, y^*) = z^*$ and the listed implications. In the first case $x^* < 0 \wedge y^* < 0$, we find $p^* = [-x^*] \cdot_{\mathbb{R}_+} [-y^*]$, which is an element of $\mathbb{R}_+$ since $\cdot_{\mathbb{R}_+}$ constitutes a binary operation $\mathbb{R}_+$. In view of the inclusion $\mathbb{R}_+ \subseteq \mathbb{R}$, we therefore obtain $p^* \in \mathbb{R}$. For the same reason, the number $p^* = x \cdot_{\mathbb{R}_+} y$ found in case of $0 < x^* \wedge 0 < y^*$ is in $\mathbb{R}$. In the five cases $x^* < 0 \wedge y^* = 0$, $x^* = 0 \wedge y^* < 0$, $x^* = 0 \wedge y^* = 0$, $x^* = 0 \wedge 0 < y^*$ and $0 < x^* \wedge y^* = 0$, we find $p^* = 0$ and consequently $p^* \in \mathbb{R}$ because the zero element 0 of $\mathbb{R}$ by definition is an element of that set. In the case $x^* < 0 \wedge 0 < y^*$, we obtain $p^* = -([-x^*] \cdot_{\mathbb{R}_+} y^*)$; here, $[-x^*] \cdot_{\mathbb{R}_+} y^*$ is clearly an element of $\mathbb{R}$, so that the negative p^* of the preceding real number is by definition also in that set. Likewise, the number $p^* = -(x^* \cdot_{\mathbb{R}_+} [-y^*])$ obtained in case of $0 < x^* \wedge y^* < 0$ turns out to be real. We thus found $p^* \in \mathbb{R}$ to be true in any case, and as p^* was initially arbitrary, we may infer from the truth of this finding the truth of the proposed inclusion, which means that $\mathbb{R}$ is a codomain of the function $\cdot_\mathbb{R}(z)$ on $\mathbb{R} \times \mathbb{R}$. Therefore, $\cdot_\mathbb{R}(z)$ is a binary operation on $\mathbb{R}$, and this allows us to write $x \cdot_\mathbb{R} y$ instead of $\cdot_\mathbb{R}((x, y))$ for any $x, y \in \mathbb{R}$. Letting now x and y be arbitrary real numbers, we may form the ordered pair $z = (x, y)$, which clearly constitutes an element of the domain $\mathbb{R} \times \mathbb{R}$ of the function $\cdot_\mathbb{R}$. Then, the first, second, third and fifth case in (8.294) follow immediately from (8.325). It remains to verify that the disjunction $x = 0 \vee y = 0$ implies $x \cdot_\mathbb{R} y = 0$. If $x = 0$ holds, then we may have $y < 0$, $y = 0$ or $0 < y$, and if $y = 0$ is true, then either $x < 0$, $x = 0$ or $0 < x$ holds. It is evident from the implications in (8.325) that the desired equation turns out to be true in any case. As x and y are arbitrary real numbers, we may infer from this that $\cdot_\mathbb{R}$ satisfied (8.294) indeed for any $x, y \in \mathbb{R}$. □

Definition 8.19 (Multiplication on the set of real numbers). We call

$$\cdot_\mathbb{R} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto x \cdot_\mathbb{R} y \tag{8.326}$$

the *multiplication on the set of real numbers*.

Theorem 8.41 (Commutative Law for the multiplication on $\mathbb{R}$). *The multiplication $\cdot_\mathbb{R}$ on $\mathbb{R}$ is commutative.*

Proof. We let x and y be arbitrary real numbers. Let us recall the fact that the standard linear ordering of $\mathbb{R}$ is connex, so that the disjunctions

$$x < 0 \lor x = 0 \lor 0 < x,$$
$$y < 0 \lor y = 0 \lor 0 < y$$

are true. We use them to prove

$$x \cdot_{\mathbb{R}} y = y \cdot_{\mathbb{R}} x \tag{8.327}$$

by cases and subcases. We consider first the case of $x < 0$ in connection with the first subcase $y < 0$, from which assumptions we obtain on the one hand

$$x \cdot_{\mathbb{R}} y = [-x] \cdot_{\mathbb{R}_+} [-y] = [-y] \cdot_{\mathbb{R}_+} [-x]$$

due to the definition of $\cdot_{\mathbb{R}_+}$ and the commutativity of $\cdot_{\mathbb{R}_+}$. On the other hand, $y < 0$ and $x < 0$ imply then (using the preceding equations)

$$y \cdot_{\mathbb{R}} x = [-y] \cdot_{\mathbb{R}_+} [-x] = x \cdot_{\mathbb{R}} y,$$

so that the desired equality (8.327) holds under the current assumptions. In the second subcase $y = 0$, we obtain $x \cdot_{\mathbb{R}} y = 0$, and the evidently true conjunction $y = 0 \land x < 0$ yields $y \cdot_{\mathbb{R}} x = 0$, so that (8.327) holds again. Then, the third subcase $0 < y$ yields

$$x \cdot_{\mathbb{R}} y = -([-x] \cdot_{\mathbb{R}_+} y) = -(y \cdot_{\mathbb{R}_+} [-x]),$$

and the true conjunction $0 < y \land x < 0$ results in (using the preceding equations)

$$y \cdot_{\mathbb{R}} x = -(y \cdot_{\mathbb{R}_+} [-x]) = x \cdot_{\mathbb{R}} y,$$

again with the desired consequent that (8.327). In the second case $x = 0$, the disjunction $x = 0 \lor y = 0$ is true, so that $x \cdot_{\mathbb{R}} y = 0$; as the disjunction $y = 0 \lor x = 0$ holds as well, it follows that $y \cdot_{\mathbb{R}} x = 0$; thus, the desired equality holds in this case, too. This equation is found to be true also in the third case by applying the arguments used within the first case. Since x and y are arbitrary real numbers, we may infer from these findings that the binary operation $\cdot_{\mathbb{R}}$ is commutative, by definition. □

Exercise 8.19. Establish the third case within the proof of the Commutative Law for the multiplication on $\mathbb{R}$.

Theorem 8.42 (Associative Law for the multiplication on $\mathbb{R}$). *The multiplication $\cdot_{\mathbb{R}}$ on $\mathbb{R}$ is associative.*

Proof. We let $x, y, z \in \mathbb{R}$ be arbitrary, so that the disjunctions

$$\begin{aligned} &x < 0 \vee x = 0 \vee 0 < x, \\ &y < 0 \vee y = 0 \vee 0 < y, \\ &z < 0 \vee z = 0 \vee 0 < z \end{aligned}$$

are true because of the connexity of the standard linear ordering of $\mathbb{R}$. Let us use these disjunction to prove

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{8.328}$$

by cases, subcases and sub-subcases (where we write $\cdot$ instead of $\cdot_{\mathbb{R}}$ for brevity of expression). In the first case $x < 0$ and the first subcase $y < 0$, we obtain

$$x \cdot y = [-x] \cdot_{\mathbb{R}_+} [-y] \tag{8.329}$$

by definition of the multiplication on $\mathbb{R}$. This product constitutes a value of the binary operation $\cdot_{\mathbb{R}_+}$, so that $x \cdot y \in \mathbb{R}_+$, and therefore $0 < x \cdot y$ by definition of the set of positive real numbers. In the first sub-subcase $z < 0$, we obtain for the same reason

$$y \cdot z = [-y] \cdot_{\mathbb{R}_+} [-z] \tag{8.330}$$

and consequently $0 < y \cdot z$. Then, the definition of the multiplication on $\mathbb{R}$ gives us on the one hand

$$\begin{aligned} x \cdot (y \cdot z) &= -([-x] \cdot_{\mathbb{R}_+} ([-y] \cdot_{\mathbb{R}_+} [-z])) \\ &= -(([-x] \cdot_{\mathbb{R}_+} [-y]) \cdot_{\mathbb{R}_+} [-z]) \end{aligned}$$

due to (8.330) and the associativity of the multiplication on $\mathbb{R}_+$, on the other hand

$$(x \cdot y) \cdot z = -(([-x] \cdot_{\mathbb{R}_+} [-y]) \cdot_{\mathbb{R}_+} [-z])$$

with (8.330), so that (8.328) is clearly true. In the second sub-subcase $z = 0$, the definition of the multiplication on $\mathbb{R}$ yields the three zero products $y \cdot z = 0$, $x \cdot (y \cdot z)$ and $(x \cdot y) \cdot z = 0$, resulting again in the true equation (8.328). In the third subcase $0 < z$, we now evidently find

$$y \cdot z = -([-y] \cdot_{\mathbb{R}_+} z), \tag{8.331}$$

where $0 < [-y] \cdot_{\mathbb{R}_+} z$, which implies $y \cdot z < 0$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$. Consequently, we find

$$\begin{aligned} x \cdot (y \cdot z) &= [-x] \cdot_{\mathbb{R}_+} -[-([-y] \cdot_{\mathbb{R}_+} z)] \\ &= [-x] \cdot_{\mathbb{R}_+} ([-y] \cdot_{\mathbb{R}_+} z) \\ &= ([-x] \cdot_{\mathbb{R}_+} [-y]) \cdot_{\mathbb{R}_+} z \end{aligned}$$

by applying the Sign Law (6.50) with respect to the group $(\mathbb{R}, +_{\mathbb{R}})$ (using the fact that the positive real number $[-y] \cdot_{\mathbb{R}_+} z$ is more generally an element of $\mathbb{R}$) and the associativity of the multiplication on $\mathbb{R}_+$. As the equation

$$(x \cdot y) \cdot z = ([-x] \cdot_{\mathbb{R}_+} [-y]) \cdot_{\mathbb{R}_+} z$$

follows to be also true by means of (8.329), the desired equality (8.328) holds again. We thus completed the proof of the first subcase. The remaining subcases and cases can be established with the same arguments as before. Since x, y and z are arbitrary real numbers, we may infer from the truth of the equality (8.328) that the associative law applies to the multiplication on $\mathbb{R}$. □

Exercise 8.20. Verify the remaining subcases and cases within the proof of Theorem 8.42.

Note 8.11. In view of the Commutative and Associative Law for the multiplication on $\mathbb{R}$, it is clear that $(\mathbb{R}, \cdot_{\mathbb{R}})$ is a commutative semigroup.

Theorem 8.43 (Distributive Law for $\mathbb{R}$). *The multiplication $\cdot_{\mathbb{R}}$ on the set of real numbers is distributive over the addition $+_{\mathbb{R}}$ on $\mathbb{R}$.*

Proof. To do!

□

Note 8.12. The set $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, -_{\mathbb{R}})$ constitutes a commutative ring because

1. $(\mathbb{R}, +_{\mathbb{R}})$ is a commutative group (see Corollary 8.33),
2. $(\mathbb{R}, \cdot_{\mathbb{R}})$ is a commutative semigroup (see Note 8.11), and
3. the Distributive Law for $\mathbb{R}$ holds.

Exercise 8.21. Show that the rational number

$$f_{\mathbb{Q}}^{\mathbb{R}}(1) = ((-\infty, 1), [1, +\infty)) \tag{8.332}$$

in $\mathbb{R}$ constitutes the neutral element of $\mathbb{R}$ with respect to the multiplication $\cdot_{\mathbb{R}}$ on $\mathbb{R}$.

(Hint: Carry out a proof by cases based on the connexity of $<_{\mathbb{R}}$, using (8.64), (8.294), (6.50) and Theorem 8.41.)

Notation 8.5. As the unity element $f_{\mathbb{Q}}^{\mathbb{R}}(1)$ of $\mathbb{R}$ is a natural/integer/rational number in $\mathbb{R}$, we abbreviate

$$1 = f_{\mathbb{Q}}^{\mathbb{R}}(1). \tag{8.333}$$

Thus, we may write for (8.64), in analogy to $0 <_{\mathbb{N}} 1$, $0 <_{\mathbb{Z}} 1$ and $0 <_{\mathbb{Q}} 1$, also

$$0 <_{\mathbb{R}} 1. \tag{8.334}$$

Exercise 8.22. Show that $(\mathbb{R}, +_\mathbb{R}, \cdot_\mathbb{R}, -_\mathbb{R})$ is a nontrivial ring.
(Hint: Proceed as for Exercise 7.26, using (8.334).)

Theorem 8.44. *The reciprocal of every element of* $\mathbb{R} \setminus \{0\}$ *with respect to* $\cdot_\mathbb{R}$ *exists.*

Proof. To do! □

Definition 8.20 (Division on the set of (nonzero) real numbers). We say that

$$/_\mathbb{R} : \mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}, \quad (x, y) \mapsto \frac{x}{y}. \tag{8.335}$$

is the *division on the set of (nonzero) real numbers.*

Having thus established the existence of the identity element and of the reciprocals of all non-zero real numbers, we may apply the definition of a field to the non-trivial commutative ring of real numbers.

Definition 8.21 (Field of real numbers). We call

$$(\mathbb{R}, +_\mathbb{R}, \cdot_\mathbb{R}, -_\mathbb{R}, /_\mathbb{R}) \tag{8.336}$$

the *field of real numbers.*

Theorem 8.45 (Monotony Law for $\cdot_\mathbb{R}$ and $<_\mathbb{R}$). *The linearly ordered set* $(\mathbb{R}, <_\mathbb{R})$ *and the multiplication* $\cdot_\mathbb{R}$ *satisfy the monotony law*

$$\forall x, y, z\, ([x, y, z \in \mathbb{R} \land 0 <_\mathbb{R} z] \Rightarrow [x <_\mathbb{R} y \Rightarrow x \cdot_\mathbb{R} z <_\mathbb{R} y \cdot_\mathbb{R} z]). \tag{8.337}$$

Proof. To do! □

As the Monotony Law for $+_\mathbb{R}$ and $<_\mathbb{R}$ as well as the Monotony Law for $\cdot_\mathbb{R}$ and $<_\mathbb{R}$ holds in connection with the Field of real numbers, we may immediately define its ordered version.

Definition 8.22 (Ordered field of real numbers). We call

$$(\mathbb{R}, +_\mathbb{R}, \cdot_\mathbb{R}, -_\mathbb{R}, /_\mathbb{R}, <_\mathbb{R}) \tag{8.338}$$

the *ordered field of real numbers.*

Note 8.13. In view of Note 7.13, the ordered field of real numbers constitutes an ordered integral domain.

The observation in Note 8.13 allows us to apply Theorem 6.75 in order to establish the following definition.

Definition 8.23 (Ordered elementary domain of nonnegative real numbers). We call the induced ordered elementary domain

$$(\mathbb{R}_+^0, +_{\mathbb{R}_+^0}, \cdot_{\mathbb{R}_+^0}, <_{\mathbb{R}_+^0}) \tag{8.339}$$

the *ordered elementary domain of nonnegative real numbers.*

Note 8.14. Since the zero and unity element exist in the set defining any ordered elementary domain, we have here

$$0, 1 \in \mathbb{R}_+^0. \tag{8.340}$$

Note 6.21 and Corollary 6.76 show us that the linear ordering $<_{\mathbb{R}_+^0}$ and the corresponding induced total ordering $\leq_{\mathbb{R}_+^0}$ satisfy

$$\forall x, y \, (x, y \in \mathbb{R}_+^0 \Rightarrow [x <_{\mathbb{R}_+^0} y \Leftrightarrow x <_{\mathbb{R}} y]), \tag{8.341}$$

$$\forall x, y \, (x, y \in \mathbb{R}_+^0 \Rightarrow [x \leq_{\mathbb{R}_+^0} y \Leftrightarrow x \leq_{\mathbb{R}} y]). \tag{8.342}$$

Corollary 8.46. *The set of positive rational numbers is included in the set of nonnegative real numbers, that is,*

$$\mathbb{Q}_+ \subseteq \mathbb{R}_+^0. \tag{8.343}$$

Proof. Letting $x \in \mathbb{Q}_+$ be arbitrary, we thus have $0 <_{\mathbb{Q}} x$ by definition of $\mathbb{Q}_+$, which inequality implies $0 <_{\mathbb{R}} x$ with (8.58). This in turn yields $0 \leq_{\mathbb{R}} x$ according to the Characterization of induced irreflexive partial orderings; thus, $x \in \mathbb{R}_+^0$ by definition of $\mathbb{R}_+^0$. This finding is then true for any $x \in \mathbb{Q}_+$, so that the inclusion follows to be true by definition of a subset. □

Theorem 8.47. *It is true that the linearly ordered set of nonnegative real numbers*

a) $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0})$ *is densely ordered,*

b) $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0})$ *is separably ordered with respect to* $\mathbb{Q}_+$*,*

c) $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0})$ *is a linear continuum,*

d) $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0})$ *has the Infimum Property.*

Proof. We first observe in light of (8.340) and the distinctness of 0 and 1 that $\mathbb{R}^0_+$ is neither empty nor a singleton. Let us now establish the universal sentence

$$\forall x, y\,([x, y \in \mathbb{R}^0_+ \,\wedge\, x <_{\mathbb{R}^0_+} y] \Rightarrow \exists z\,(z \in \mathbb{R}^0_+ \,\wedge\, x <_{\mathbb{R}^0_+} z <_{\mathbb{R}^0_+} y)), \quad (8.344)$$

$$\forall x, y\,([x, y \in \mathbb{R}^0_+ \,\wedge\, x <_{\mathbb{R}^0_+} y] \Rightarrow \exists z\,(z \in \mathbb{Q}_+ \,\wedge\, x <_{\mathbb{R}^0_+} z <_{\mathbb{R}^0_+} y)) \quad (8.345)$$

We take arbitrary nonnegative real numbers x, y and assume $x <_{\mathbb{R}^0_+} y$ to be true. This inequality implies $x <_{\mathbb{R}} y$ with (8.341). Now, as $(\mathbb{R}, <_{\mathbb{R}})$ is densely ordered, there exists a particular number $\bar{z} \in \mathbb{R}$ such that $x <_{\mathbb{R}} \bar{z} <_{\mathbb{R}} y$. The fact that x is a nonnegative real number means $0 \leq_{\mathbb{R}} x$, so that the preceding inequalities imply $0 <_{\mathbb{R}} \bar{z}$ with the Transitivity Formula for $\leq$ and $<$. This inequality in turn implies $0 <_{\mathbb{Q}} \bar{z}$ with (8.58), which shows that $\bar{z}$ is a positive rational number. Since $0 <_{\mathbb{R}} \bar{z}$ implies also $0 \leq_{\mathbb{R}} \bar{z}$ with the Characterization of induced irreflexive partial orderings, we have $\bar{z} \in \mathbb{R}^0_+$ (besides $x, y \in \mathbb{R}^0_+$). Therefore, we may write the inequalities $x <_{\mathbb{R}} \bar{z} <_{\mathbb{R}} y$ equivalently as $x <_{\mathbb{R}^0_+} \bar{z} <_{\mathbb{R}^0_+} y$. These findings clearly demonstrate the truth of the existential sentences in (8.344) and (8.345). As x and y were initially arbitrary, we may therefore conclude that the universal sentences (8.344) and (8.345) are true. Consequently, $(\mathbb{R}^0_+, <_{\mathbb{R}^0_+})$ is densely ordered, and in view of the inclusion (8.343), we furthermore find that $\mathbb{Q}_+$ is a dense subset of $\mathbb{R}^0_+$.

Concerning c), it remains for us to show that the densely ordered set $(\mathbb{R}^0_+, <_{\mathbb{R}^0_+})$ has the Supremum Property. For this purpose, we take an arbitrary nonempty and bounded-from-above (with respect to $\leq_{\mathbb{R}^0_+}$) subset $\bar{A}$ of $\mathbb{R}^0_+$, and we show that its supremum with respect to $\leq_{\mathbb{R}^0_+}$ exists. Since $\mathbb{R}^0_+$ is a subset of $\mathbb{R}$, it follows with the transitivity of $\subseteq$ that $\bar{A}$ is a (nonempty) subset also of $\mathbb{R}$. We now verify that $\bar{A}$ is bounded from above also with respect to $\leq_{\mathbb{R}}$. Since $\bar{A}$ is bounded from above with respect to $\leq_{\mathbb{R}^0_+}$, there exists a constant $\bar{u} \in \mathbb{R}^0_+$ such that

$$\forall x\,(x \in \bar{A} \Rightarrow x \leq_{\mathbb{R}^0_+} \bar{u}). \quad (8.346)$$

We may show that

$$\forall x\,(x \in \bar{A} \Rightarrow x \leq_{\mathbb{R}} \bar{u}) \quad (8.347)$$

holds as well. Letting $x \in \bar{A}$ be arbitrary, we obtain $x \leq_{\mathbb{R}^0_+} \bar{u}$ with (8.346). This inequality in turn implies $x \leq_{\mathbb{R}} \bar{u}$ with (8.342), which is the desired consequent of the implication in (8.347). Here, x is arbitrary, so that the universal sentence (8.347) follows to be true. This shows that $\bar{u}$ is an upper bound for $\bar{A}$ with respect to $\leq_{\mathbb{R}}$, and this implies that $\bar{A}$ is bounded from above with respect to $\leq_{\mathbb{R}}$. Thus, $\bar{A}$ is a nonempty and bounded from above subset of $\mathbb{R}$. Since $(\mathbb{R}, <_{\mathbb{R}})$ has the Supremum Property, it follows from this that its supremum $S = \sup^{\leq_{\mathbb{R}}}$ exists, which thus satisfies the conjunction

$$\forall x\,(x \in \bar{A} \Rightarrow x \leq_{\mathbb{R}} S) \;\wedge\; \forall S'\,(\forall x\,(x \in \bar{A} \Rightarrow x \leq_{\mathbb{R}} S') \Rightarrow S \leq_{\mathbb{R}} S'). \quad (8.348)$$

We may now prove that S is also the supremum of $\bar{A}$ with respect to $\leq_{\mathbb{R}_+^0}$. For this purpose, we demonstrate the truth of

$$\forall x\,(x \in \bar{A} \Rightarrow x \leq_{\mathbb{R}_+^0} S) \;\wedge\; \forall S'\,(\forall x\,(x \in \bar{A} \Rightarrow x \leq_{\mathbb{R}_+^0} S') \Rightarrow S \leq_{\mathbb{R}_+^0} S'). \quad (8.349)$$

Letting $x \in \bar{A}$ be arbitrary, we find $x \leq_{\mathbb{R}} S$ with the first part of the conjunction (8.348). Due to the inclusion $\bar{A} \subseteq \mathbb{R}_+^0$, $x \in \bar{A}$ implies $x \in \mathbb{R}_+^0$ and therefore $0 \leq_{\mathbb{R}} x$ (by definition of $\mathbb{R}_+^0$). Combining the previous two inequalities gives us now $0 \leq_{\mathbb{R}} S$ with the transitivity of the total ordering $\leq_{\mathbb{R}}$, so that $S \in \mathbb{R}_+^0$ also holds. Since x and S are both elements of $\mathbb{R}_+^0$, the inequality $x \leq_{\mathbb{R}} S$ can evidently be written as $x \leq_{\mathbb{R}_+^0} S$. As x was arbitrary, we may therefore conclude that the first part of the conjunction (8.349) holds.

To establish the second part, we let S' be arbitrary, and we assume

$$\forall x\,(x \in \bar{A} \Rightarrow x \leq_{\mathbb{R}_+^0} S') \quad (8.350)$$

to be true. To prove the corresponding universal sentence

$$\forall x\,(x \in \bar{A} \Rightarrow x \leq_{\mathbb{R}} S'), \quad (8.351)$$

we let $x \in \bar{A}$ be arbitrary. This yields $x \leq_{\mathbb{R}_+^0} S'$ with (8.350), which inequality can evidently be written also as $x \leq_{\mathbb{R}} S'$ (noting that x and S' are both elements of $\mathbb{R}_+^0$). This proves the implication in (8.351), in which x is arbitrary, so that the universal sentence (8.351) follows to be true as well. This universal sentence implies now $S \leq_{\mathbb{R}} S'$ because of the second part of the conjunction (8.348). As we previously found that S and S' are both elements of $\mathbb{R}_+^0$, we may write the preceding inequality as $S \leq_{\mathbb{R}_+^0} S'$. Since S' was initially arbitrary, we may infer from this finding the truth of the second part of the conjunction (8.349). We thus completed the proof that S is the supremum of $\bar{A}$ with respect to $\leq_{\mathbb{R}_+^0}$. The existence of this supremum in $\mathbb{R}_+^0$, together with the fact that $\bar{A}$ was initially arbitrary, allows us to further conclude that $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0})$ has the Supremum Property. In conjunction with a), this means that $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0})$ is a linear continuum. The Infimum Property of $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0})$ can be established in analogy to the Supremum Property, by using the fact that 0 is a lower bound (with respect to $\leq_{\mathbb{R}}$) for any nonempty and bounded-from-below subset A of $\mathbb{R}_+^0$. □

Exercise 8.23. Establish the Infimum Property for $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0})$.

Note 8.15. The Supremum Property and the Infimum Property of $(\mathbb{R}_+^0, <_{\mathbb{R}_+^0}$) (i.e., the fact that every nonempty and bounded-from-above/below subset A of $\mathbb{R}_+^0$ has a supremum/an infimum with respect to the total ordering $\leq_{\mathbb{R}_+^0}$) read

$$\forall A\,([A \subseteq \mathbb{R}_+^0 \;\wedge\; A \neq \emptyset \;\wedge\; \exists u\,(u \in \mathbb{R}_+^0 \;\wedge\; \forall x\,(x \in A \Rightarrow x \leq_{\mathbb{R}_+^0} u))]$$
$$\Rightarrow \exists S\,(S = \sup^{\leq_{\mathbb{R}_+^0}} A)) \tag{8.352}$$

and

$$\forall A\,([A \subseteq \mathbb{R}_+^0 \;\wedge\; A \neq \emptyset \;\wedge\; \exists a\,(a \in \mathbb{R}_+^0 \;\wedge\; \forall x\,(x \in A \Rightarrow a \leq_{\mathbb{R}_+^0} x))]$$
$$\Rightarrow \exists I\,(I = \inf^{\leq_{\mathbb{R}_+^0}} A)). \tag{8.353}$$

Definition 8.24 (Linear continuum of nonnegative real numbers). We call

$$(\mathbb{R}_+^0, <_{\mathbb{R}_+^0}) \tag{8.354}$$

the *linear continuum of nonnegative real numbers*.

As $(\mathbb{R}_+^0, +_{\mathbb{R}_+^0}, \cdot_{\mathbb{R}_+^0})$ constitutes a commutative semiring (by definition of an ordered elementary domain), we may immediately define the semiring of related functions by virtue of Corollary 5.39.

Definition 8.25 (Semiring of nonnegative real functions on X). For any set X, we call

$$([\mathbb{R}_+^0]^X, +_{[\mathbb{R}_+^0]^X}, \cdot_{[\mathbb{R}_+^0]^X}) \tag{8.355}$$

(containing the pointwise addition and multiplication of nonnegative real functions) the *semiring of nonnegative real functions on X*.

Since we showed that $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, -_{\mathbb{R}})$ is a (commutative) ring, we may now define corresponding (commutative) rings of functions and rings of real-valued matrices.

Definition 8.26 (Ring of real-valued functions on X). For any set X, we call

$$(\mathbb{R}^X, +_{\mathbb{R}^X}, \cdot_{\mathbb{R}^X}, -_{\mathbb{R}^X}) \tag{8.356}$$

(containing the pointwise addition and multiplication of functions in $\mathbb{R}^X$) the *ring of real-valued functions on X*.

Definition 8.27 (Ring of real n-by-n matrices). For any positive natural number n, we call

$$(\mathbb{R}^{n\times n}, +_{\mathbb{R}^{n\times n}}, \cdot_{\mathbb{R}^{n\times n}}, -_{\mathbb{R}^{n\times n}}) \tag{8.357}$$

the *ring of real n-by-n matrices.*

The monotony laws for the ordered field of real numbers allow us to establish a number of useful facts.

Proposition 8.48. *For any nonempty and bounded-from-below subset A of $\mathbb{R}$ and any positive real number ε there exists an $x \in A$ which is less than the sum of ε and the infimum of A, that is,*

$$\forall \varepsilon\,(\varepsilon \in \mathbb{R}_+ \Rightarrow \exists x\,(x \in A \wedge x <_{\mathbb{R}} \inf^{\leq_{\mathbb{R}}} A +_{\mathbb{R}} \varepsilon)). \tag{8.358}$$

Proof. We let A be an arbitrary nonempty and bounded-from-below subset of $\mathbb{R}$, so that $\inf^{\leq_{\mathbb{R}}} A$ exists (in $\mathbb{R}$) in view of the Infimum Property of $(\mathbb{R}, <_{\mathbb{R}}$). Next, we let ε be arbitrary and prove the implication by contradiction, assuming that ε is a positive real number (so that $\varepsilon \in \mathbb{R}$ and $0 <_{\mathbb{R}} \varepsilon$ hold by definition of $\mathbb{R}_+$), and that the negation of the existential sentence in (8.358) is true. We may write the latter also as

$$\forall x\,(x \in A \Rightarrow \neg x <_{\mathbb{R}} \inf^{\leq_{\mathbb{R}}} A +_{\mathbb{R}} \varepsilon) \tag{8.359}$$

by applying the Negation Formula for existential conjunctions. This allows us to prove the universal sentence

$$\forall x\,(x \in A \Rightarrow \inf^{\leq_{\mathbb{R}}} A +_{\mathbb{R}} \varepsilon \leq_{\mathbb{R}} x).$$

Indeed, letting $x \in A$ be arbitrary, we obtain with (8.359) the true negation $\neg x <_{\mathbb{R}} \inf^{\leq_{\mathbb{R}}} A +_{\mathbb{R}} \varepsilon$, and therefore the inequality $\inf^{\leq_{\mathbb{R}}} A +_{\mathbb{R}} \varepsilon \leq_{\mathbb{R}} x$ with the Negation Formula for $<$. As x is arbitrary, we may infer from this the truth of the preceding universal sentence. Thus, $\inf^{\leq_{\mathbb{R}}} A +_{\mathbb{R}} \varepsilon$ is a lower bound for A. By definition of an infimum, we then obtain $\inf^{\leq_{\mathbb{R}}} A +_{\mathbb{R}} \varepsilon \leq_{\mathbb{R}} \inf^{\leq_{\mathbb{R}}} A$, which inequality evidently implies $\varepsilon \leq_{\mathbb{R}} 0$ with the Monotony Law (7.98) (adding $-\inf^{\leq_{\mathbb{R}}} A$ to both sides), in connection with the Commutative & the Associative Law for the addition on the set of real numbers and with the property of a negative. Now, the preceding inequality implies $\neg 0 <_{\mathbb{R}} \varepsilon$ with the Negation Formula for $<$, in contradiction to the initially found inequality $0 <_{\mathbb{R}} \varepsilon$. This completes the proof of the implication in (8.358), and since A and ε were arbitrary, we therefore conclude that the proposed universal sentence is true. □

Exercise 8.24. Show for any nonempty and bounded-from-above subset A of $\mathbb{R}$ and any positive real number ε that there exists an $x \in A$ which is greater than the difference of the supremum of A and ε, that is,

$$\forall \varepsilon\, (\varepsilon \in \mathbb{R}_+ \Rightarrow \exists x\, (x \in A \,\wedge\, \sup^{\leq_{\mathbb{R}}} A -_{\mathbb{R}} \varepsilon <_{\mathbb{R}} x)). \tag{8.360}$$

(Hint: The proof is similar to that of Proposition 8.48.)

Theorem 8.49 (Archimedean Property). *It is true for any real number x that there exists a positive natural number greater than x, that is,*

$$\forall x\, (x \in \mathbb{R} \Rightarrow \exists n\, (n \in \mathbb{N}_+ \,\wedge\, x <_{\mathbb{R}} n)). \tag{8.361}$$

Proof. We let x be arbitrary, and we prove the implication by contradiction, assuming $x \in \mathbb{R}$ and the negation of the existential sentence in (8.361) to be true. Consequently, we obtain with the Negation Law for existential conjunctions the true universal sentence

$$\forall n\, (n \in \mathbb{N}_+ \Rightarrow \neg x <_{\mathbb{R}} n),$$

which evidently implies

$$\forall n\, (n \in \mathbb{N}_+ \Rightarrow n \leq_{\mathbb{R}} x). \tag{8.362}$$

with the Negation Formula for $<$. This means that that x is an upper bound for $\mathbb{N}_+$ with respect to $\leq_{\mathbb{R}}$. Viewing positive natural numbers as real numbers via the mappings $f_{\mathbb{N}}^{\mathbb{Z}}$, $f_{\mathbb{Z}}^{\mathbb{Q}}$ and $f_{\mathbb{Q}}^{\mathbb{R}}$, and noting that $\mathbb{N}_+$ is nonempty, we thus evidently have that $\mathbb{N}_+$ is a non-empty and bounded-from-above subset of $\mathbb{R}$. Because of the Supremum Property with respect to the linear continuum $(\mathbb{R}, <_{\mathbb{R}})$, the supremum S of $\mathbb{N}_+$ exists then. Recalling from (4.164) the truth of $0 <_{\mathbb{N}} 1$, which we may write as $0 <_{\mathbb{R}} 1$, we evidently obtain $S -_{\mathbb{R}} 1 <_{\mathbb{R}} S$ with the Monotony Law (7.98). We now demonstrate that $S -_{\mathbb{R}} 1$ is not an upper bound for $\mathbb{N}_+$, i.e., that the negation

$$\neg \forall n\, (n \in \mathbb{N}_+ \Rightarrow n \leq_{\mathbb{R}} S -_{\mathbb{R}} 1) \tag{8.363}$$

is true. For this purpose, we apply a proof by contradiction, assuming the negation of that negation to be true. Then, the universal sentence in (8.363) follows to be true with the Double Negation Law. Let us observe in light of the Supremum Criterion that the previously established inequality $S -_{\mathbb{R}} 1 <_{\mathbb{R}} S$ implies the existence of a particular number $\bar{n} \in \mathbb{N}_+$ satisfying $S -_{\mathbb{R}} 1 <_{\mathbb{R}} \bar{n}$. This implies $\neg \bar{n} \leq_{\mathbb{R}} S -_{\mathbb{R}} 1$ with the Negation Formula for $\leq$; since $\bar{n} \in \mathbb{N}_+$ implies with the aforementioned universal sentence the truth also of $\bar{n} \leq_{\mathbb{R}} S -_{\mathbb{R}} 1$, we arrived at a contradiction. Thus, the negation

(8.363) is indeed true. This negation evidently yields the true existential sentence

$$\exists n\,(n \in \mathbb{N}_+ \,\wedge\, \neg n \leq_{\mathbb{R}} S -_{\mathbb{R}} 1)$$

with the Negation Formula for universal implications. Thus, there is a positive natural number, say $\bar{m}$, such that $\neg\bar{m} \leq_{\mathbb{R}} S -_{\mathbb{R}} 1$ holds. Then, $S -_{\mathbb{R}} 1 <_{\mathbb{R}} \bar{m}$ follows to be true as well with the Negation Formula for $\leq$. Consequently, the aforementioned monotony law gives us evidently the true inequality $S <_{\mathbb{R}} \bar{m} +_{\mathbb{R}} 1$, and the Negation Formula for $\leq$ yields in addition $\neg\bar{m} +_{\mathbb{R}} 1 \leq_{\mathbb{R}} S$. Here, the successor $\bar{m} +_{\mathbb{R}} 1$ of the positive natural number $\bar{m}$ is clearly a positive natural number itself. This demonstrates the truth of the existential sentence

$$\exists n\,(n \in \mathbb{N}_+ \,\wedge\, \neg n \leq_{\mathbb{R}} S).$$

Then,

$$\neg\forall n\,(n \in \mathbb{N}_+ \,\wedge\, n \leq_{\mathbb{R}} S)$$

follows to be true with the Negation Law for universal implications and the Double Negation Law. This means that S is not an upper bound for $\mathbb{N}_+$, which finding contradicts the fact that the supremum S of $\mathbb{N}_+$ is an upper bound for that set. Thus, the proof of the implication in (8.361) is complete. As x was initially arbitrary, we may therefore conclude that the universal sentence (8.361) is true. □

Proposition 8.50. *It is for any positive real number x that there is some positive natural n whose reciprocal is less than x, that is,*

$$\forall x\,(x \in \mathbb{R}_+ \Rightarrow \exists n\,(n \in \mathbb{N}_+ \,\wedge\, \frac{1}{n} <_{\mathbb{R}} x)). \tag{8.364}$$

Proof. We take an arbitrary $x \in \mathbb{R}$, so that $x \in \mathbb{R}$ and $0 <_{\mathbb{R}} x$ are true by definition of the set of positive real numbers. Furthermore, the reciprocal of x is again a real number, so that there exists a particular positive natural number $\bar{n}$ greater than $\frac{1}{x}$, in view of the Archimedean Property. Due to $0 <_{\mathbb{R}} x$, we may now apply the Monotony Law (7.99) to infer from $\frac{1}{x} <_{\mathbb{R}} \bar{n}$ the truth of the inequality $\frac{1}{x} \cdot_{\mathbb{R}} x <_{\mathbb{R}} \bar{n} \cdot_{\mathbb{R}} x$. We may write this also as

$$1 <_{\mathbb{R}} x \cdot_{\mathbb{R}} \bar{n}, \tag{8.365}$$

using (7.59) and the Commutative Law for the multiplication of real numbers. Let us observe now that $\bar{n} \in \mathbb{N}_+$ evidently gives $[0 <_{\mathbb{R}}]\ 1 \leq_{\mathbb{R}} \bar{n}$ with (4.278), and therefore $0 <_{\mathbb{R}} \bar{n}$ with the Transitivity Formula for $<$ and $\leq$. Consequently, the inequality $0 <_{\mathbb{R}} \frac{1}{\bar{n}}$ also holds according to (7.101).

Another application of the aforementioned monotony law to (8.365) yields now

$$1 \cdot_{\mathbb{R}} \frac{1}{\bar{n}} <_{\mathbb{R}} (x \cdot_{\mathbb{R}} \bar{n}) \cdot_{\mathbb{R}} \frac{1}{\bar{n}},$$

which we can simplify to $\frac{1}{\bar{n}} <_{\mathbb{R}} x$ by applying the property of the identity element, the Associative Law for the multiplication of real numbers, and again (7.59). This inequality demonstrates the truth of the existential sentence in (8.364), and because x is arbitrary, we may infer from this the truth of the proposed universal sentence. □

Exercise 8.25. Establish the truth of the universal sentence

$$\forall x, y\, (x, y \in \mathbb{R}_+ \Rightarrow \exists n\, (n \in \mathbb{N}_+ \wedge n \cdot_{\mathbb{R}} x >_{\mathbb{R}} y)). \tag{8.366}$$

(Hint: Use some of the arguments within the proof of Proposition 8.50.)

An increasing convergent sequence is characterized by 'arbitrarily small' differences of its limit and all of its terms with indexes surpassing a certain threshold.

Lemma 8.51 (Limit Criterion for increasingly convergent real sequences). *For any increasingly convergent sequence $(x_n)_{n\in\mathbb{N}_+}$ as well as any number L in $\mathbb{R}$, it is true that L is the limit of the sequence iff, for any $\varepsilon \in \mathbb{R}_+$, there exists a natural number N such that for all $n \geq N$ the difference of L and the nth term of the sequence is less than ε, that is,*

$$\begin{aligned} &L = \lim_{n\to\infty} x_n \qquad (8.367)\\ &\quad \Leftrightarrow \forall \varepsilon (\varepsilon \in \mathbb{R}_+ \Rightarrow \exists N\, (N \in \mathbb{N}_+ \wedge \forall n\, (n \geq_{\mathbb{N}_+} N \Rightarrow 0 \leq L - x_n < \varepsilon))). \end{aligned}$$

Proof. We let $s = (x_n)_{n\in\mathbb{N}_+}$ be an arbitrary increasing sequence in $\mathbb{R}$ with limit $\lim_{n\to\infty} x_n$ and L an arbitrary real number. To prove the first part ('$\Rightarrow$') of the proposed equivalence, we assume $L = \lim_{n\to\infty} x_n$ and then let $\varepsilon \in \mathbb{R}_+$ be arbitrary (so that $0 < \varepsilon$ holds by definition of the set of positive real numbers). As the limit L is by definition the supremum of the range of s, and since $0 < \varepsilon$ evidently implies $L - \varepsilon < L$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$, we see in light of the Supremum Criterion that there exists a particular constant $\bar{y} \in \mathrm{ran}(s)$ satisfying $L - \varepsilon < \bar{y}$. By definition of a range, there exists then a particular constant $\bar{n}$ such that $(\bar{n}, \bar{y}) \in s$. By definition of a domain, it is therefore true that $\bar{n} \in \mathbb{N}_+$ [$= \mathrm{dom}(s)$]. Using the notations for functions and sequences, we can write $(\bar{n}, \bar{y}) \in s$ also as $\bar{y} = s(\bar{n}) = x_{\bar{n}}$. Thus, substitution into the preceding inequality gives us

$$L - \varepsilon < x_{\bar{n}}. \tag{8.368}$$

We establish now the truth of the universal sentence

$$\forall n\,(n \geq_{\mathbb{N}_+} \bar{n} \Rightarrow 0 \leq L - x_n < \varepsilon), \tag{8.369}$$

letting $n \geq_{\mathbb{N}_+} \bar{n}$ be arbitrary. As the sequence $(x_n)_{n\in\mathbb{N}_+}$ is increasing, that inequality implies $x_{\bar{n}} \leq x_n$ with (3.257). The conjunction of that inequality and (8.368) further implies $L - \varepsilon < x_n$ with the Transitivity Formula for $<$ and $\leq$. Consequently, we obtain $L - x_n < \varepsilon$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$; because the supremum L is an upper bound for $(x_n)_{n\in\mathbb{N}_+}$, we find with the Criterion for upper bounds for a family also the inequality $x_n \leq L$ and therefore evidently $0 \leq L - x_n$. We thus have $0 \leq L - x_n < \varepsilon$, and since n was arbitrary, we therefore conclude that (8.369) is indeed true. In view of $\bar{n} \in \mathbb{N}_+$, we now see that the existential sentence in (8.367) holds. Then, since ε was arbitrary, we conclude that the right-hand side of the equivalence (8.367) is true.

To prove the second part ('$\Rightarrow$') of the equivalence, we show that assuming the right-hand side of (8.367) implies $L = \lim_{n\to\infty} x_n$, i.e., that L is the supremum of the range of the increasing sequence $(x_n)_{n\in\mathbb{N}}$. We first show that L is an upper bound for (the range) of s, that is,

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow x_n \leq L). \tag{8.370}$$

For this purpose, we take an arbitrary $n \in \mathbb{N}_+$. Let us observe now that the choice of the positive real number $\varepsilon = 1$ implies the existence of a particular positive natural number $\bar{N}$ satisfying

$$\forall n\,(n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow 0 \leq L - x_n < 1). \tag{8.371}$$

We consider the two cases $n \geq_{\mathbb{N}_+} \bar{N}$ and $\neg n \geq_{\mathbb{N}_+} \bar{N}$ to prove $x_n \leq L$ by cases. The first case implies in particular $0 \leq L - x_n$ with (8.371) and therefore the desired $x_n \leq L$ (using the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$). The second case yields $n <_{\mathbb{N}_+} \bar{N}$ with the Negation Formula for $\leq$. Because of the reflexivity of $\leq_{\mathbb{N}_+}$, the inequality $\bar{N} \geq_{\mathbb{N}_+} \bar{N}$ is true, with the consequence that $0 \leq L - x_{\bar{N}}$ holds in view of (8.371); consequently, $x_{\bar{N}} \leq L$. Furthermore, as the sequence s is increasing, the previously established $n <_{\mathbb{N}_+} \bar{N}$ implies $x_n < x_{\bar{N}}$. The conjunction of these two inequalities gives us now $x_n < L$ with the Transitivity Formula for $<$ and $\leq$, and therefore evidently $x_n \leq L$, as desired. We thus completed the proof by cases, and since n was arbitrary, we may infer from this finding the truth of the universal sentence (8.370). This proves that L is an upper bound for the given sequence. To complete the application of the Supremum Criterion, we prove

$$\forall S'\,([S' \in \mathbb{R} \wedge S' < L] \Rightarrow \exists y\,(y \in \mathrm{ran}(s) \wedge S' < y)), \tag{8.372}$$

letting S' be arbitrary in $\mathbb{R}$ such that $S' < L$ holds. This implies evidently $0 < L - S'$ and consequently $L - S' \in \mathbb{R}_+$. The latter then further implies with the assumed right-hand side of (8.367) that there exists a particular positive natural number $\bar{n}$ such that

$$\forall n\, (n \geq_{\mathbb{N}_+} \bar{n} \Rightarrow 0 \leq L - x_n < L - S').$$

This universal sentence implies in particular $L - x_{\bar{n}} < L - S'$ and therefore $S' < x_{\bar{n}}$. Here, $x_{\bar{n}} = s(\bar{n})$, so that $(\bar{n}, x_{\bar{n}}) \in s$. By definition of a range, we then find $x_{\bar{n}} \in \mathrm{ran}(s)$. In conjunction with the preceding inequality, this demonstrates the truth of the existential sentence in (8.372). Since S' was arbitrary, we then conclude that the universal sentence (8.372) is true. Thus, the proof that L is the supremum of the range of $(x_n)_{n\in\mathbb{N}}$ is complete; as that sequence is increasing, we obtain by definition of a limit $L = \lim_{n\to\infty} x_n$. This finding completes the proof of the proposed equivalence (8.367).

As $(x_n)_{n\in\mathbb{N}_+}$ and L were arbitrary, we may finally conclude that the lemma is true. □

Theorem 8.52 (Additivity & Monotonicity of limits of increasingly convergent real sequences). *The following sentences are true for any increasingly convergent sequences $(x_n)_{n\in\mathbb{N}_+}$ and $(y_n)_{n\in\mathbb{N}_+}$ in $\mathbb{R}$.*

a) The sequence $(x_n + y_n)_{n\in\mathbb{N}_+}$ converges increasingly to the sum of the limits of $(x_n)_{n\in\mathbb{N}_+}$ and $(y_n)_{n\in\mathbb{N}_+}$, that is,

$$\lim_{n\to\infty} (x_n +_{\mathbb{R}} y_n) = \lim_{n\to\infty} x_n +_{\mathbb{R}} \lim_{n\to\infty} y_n. \tag{8.373}$$

b) If every term of $(x_n)_{n\in\mathbb{N}_+}$ is less than or equal to the corresponding term of $(y_n)_{n\in\mathbb{N}_+}$, then the limit of the former sequence is less than or equal to the limit of the latter, that is,

$$\forall n\, (n \in \mathbb{N}_+ \Rightarrow x_n \leq_{\mathbb{R}} y_n) \;\Rightarrow\; \lim_{n\to\infty} x_n \leq_{\mathbb{R}} \lim_{n\to\infty} y_n. \tag{8.374}$$

Proof. We let $f = (x_n)_{n\in\mathbb{N}_+}$ and $g = (y_n)_{n\in\mathbb{N}_+}$ be arbitrary increasingly convergent sequences in $\mathbb{R}$, so that the limits $L_f = \lim_{n\to\infty} x_n$ and $L_g = \lim_{n\to\infty} y_n$ exist. For brevity of expressions, we write $+$ instead of $+_{\mathbb{R}}$ and $\leq$ instead of $\leq_{\mathbb{R}}$ in the following.

Concerning a), we first observe in light of the Pointwise addition of functions that $h = f +_{\mathbb{R}^{\mathbb{N}_+}} g$ constitutes the real sequence $h = (z_n)_{n\in\mathbb{N}_+}$ with terms $z_n = f_n + g_n$.

To prove that this sequence is increasing, we apply the Monotony Criterion for increasing sequences (taking Exercise 4.20 into account) and accordingly let $n \in \mathbb{N}_+$ be arbitrary. We now observe that $x_n \leq x_{n+1}$ and

$y_n \leq y_{n+1}$ are both true since f and g are increasing by assumption. Then, we notice that these inequalities imply $x_n + y_n \leq x_{n+1} + y_{n+1}$ with the Additivity of $\leq$-inequalities for ordered integral domains. Since n was arbitrary, we therefore conclude that the sequence h is increasing.

Next, we apply the Criterion for upper bounds for a family to show that the sequence h is bounded from above by the real number $L = L_f + L_g$, i.e.

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow z_n \leq L). \tag{8.375}$$

Letting n be arbitrary in $\mathbb{N}_+$, we observe that the limits L_f and L_g are by definition suprema and thus upper bounds for f and g, respectively. so that $x_n \leq L_f$ and $y_n \leq L_g$ hold. An application of the Additivity of $\leq$-equations for ordered integral domains gives us then $x_n + y_n \leq L_f + L_g$, thus $z_n \leq L$ as desired. As n was arbitrary, we therefore conclude that the universal sentence (8.375) holds, so that L is indeed an upper bound for the sequence h.

Then, as the domain $\mathbb{N}_+$ of h is clearly a nonempty set, the range of h follows to be nonempty with (3.118). Recalling that h is a sequence in $\mathbb{R}$, we have that the range of h is included in the codomain $\mathbb{R}$ of h. In summary, $\mathrm{ran}(h)$ a nonempty and bounded-from-above subset of $\mathbb{R}$, so that the supremum of $\mathrm{ran}(h)$ exists by virtue of the Supremum Property of the linear continuum $(\mathbb{R}, <_{\mathbb{R}})$. Because h is increasing, that supremum is the limit of h by definition, and h is increasingly convergent. We now demonstrate that the upper bound L is that limit, i.e.

$$[L_f + L_g =] \quad L = \lim_{n \to \infty} (x_n + y_n). \tag{8.376}$$

For this purpose, we apply the Limit Criterion for increasingly convergent real sequences and prove accordingly the equivalent sentence

$$\forall \varepsilon (\varepsilon \in \mathbb{R}_+ \Rightarrow \exists N \, (N \in \mathbb{N}_+ \, \wedge \, \forall n \, (n \geq_{\mathbb{N}_+} N \Rightarrow 0 \leq L - (x_n + y_n) < \varepsilon))). \tag{8.377}$$

To do this, we let ε be arbitrary in $\mathbb{R}_+$. Then, we observe in light of the Limit Criterion for increasingly convergent real sequences that $L_f = \lim_{n \to \infty} x_n$, $L_g = \lim_{n \to \infty} y_n$ and the evident fact $0.5 \cdot \varepsilon \in \mathbb{R}_+$ imply the existence of particular positive natural numbers N_f and N_g such that

$$\forall n \, (n \geq_{\mathbb{N}_+} N_f \Rightarrow 0 \leq L_f - x_n < 0.5 \cdot \varepsilon), \tag{8.378}$$

$$\forall n \, (n \geq_{\mathbb{N}_+} N_g \Rightarrow 0 \leq L_g - x_n < 0.5 \cdot \varepsilon). \tag{8.379}$$

Now, because $<_{\mathbb{N}_+}$ is a linear ordering, it follows with Proposition 3.113 that $M = \max\{N_f, N_g\}$ exists uniquely, where $M = N_f$ or $M = N_g$. Thus, $M \in \mathbb{N}_+$ holds in any case. Moreover, the maximum M is by definition an upper bound for that set, so that $N_f, N_g \leq_{\mathbb{N}_+} N$ holds. We are now in a position to prove

$$\forall n \, (n \geq_{\mathbb{N}_+} M \Rightarrow 0 \leq L - (x_n + y_n) < \varepsilon), \tag{8.380}$$

letting $n \geq_{\mathbb{N}_+} M$ be arbitrary. Due to $N_f, N_g \leq_{\mathbb{N}_+} N$, we obtain from (8.378) – (8.379) the inequalities

$$\begin{aligned} 0 &\leq L_f - x_n < 0.5 \cdot \varepsilon, \\ 0 &\leq L_g - y_n \; < 0.5 \cdot \varepsilon. \end{aligned}$$

The Additivity of $\leq$/$<$-equations for ordered integral domains gives us therefore

$$0 \leq (L_f - x_n) + (L_g - y_n) < 0.5 \cdot \varepsilon + 0.5 \cdot \varepsilon \quad [= \varepsilon],$$

which we can rearrange to

$$0 \leq (L_f + L_g) - (x_n + y_n) < \varepsilon,$$

or equivalently

$$0 \leq L - (x_n + y_n) < \varepsilon.$$

As n was arbitrary, we may therefore conclude that the universal sentence (8.380) is true. Recalling that M is a positive natural number, we now see that the existential sentence in (8.377) holds. Since ε was arbitrary, we may then conclude that the universal sentence (8.377) is true, which in turn implies (8.376) with the aforementioned Limit Criterion. Substitution of the limit expressions for L_f and L_g gives the proposed equation (8.373). Since $(x_n)_{n \in \mathbb{N}_+}$ and $(y_n)_{n \in \mathbb{N}_+}$ were also arbitrary, we therefore conclude that a) is true.

Concerning b), we prove the implication by contraposition, assuming $\neg L_f \leq L_g$ to hold, and showing that

$$\neg \forall n \, (n \in \mathbb{N}_+ \Rightarrow x_n \leq y_n) \tag{8.381}$$

is implied. The assumed inequality implies $L_f > L_g$ with the negation formula for $\leq$ and then evidently $0 < L_f - L_g$, so that $L_f - L_g \in \mathbb{R}_+$; consequently, $0.5 \cdot (L_f - L_g) \in \mathbb{R}_+$. Since L_f is the limit of the increasingly convergent sequence f, it evidently follows that there exists a particular positive natural number $\bar{N}$ such that

$$\forall n \, (n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow 0 \leq L_f - x_n < 0.5 \cdot (L_f - L_g)).$$

In view of the fact $\bar{N} \geq_{\mathbb{N}_+} \bar{N}$, we therefore find in particular

$$0 \leq L_f - x_{\bar{N}} < 0.5 \cdot (L_f - L_g),$$

so that we obtain on the one hand with the Transitivity of $\leq$- and $<$-inequalities

$$0 < 0.5 \cdot (L_f - L_g), \tag{8.382}$$

on the other hand evidently

$$0.5 \cdot L_f + 0.5 \cdot L_g < x_{\bar{N}}. \tag{8.383}$$

Here, (8.382) implies evidently

$$y_{\bar{N}} < 0.5 \cdot (L_f - L_g) + y_{\bar{N}}. \tag{8.384}$$

Now since the supremum L_g is an upper bound for g, it follows with the Criterion for upper bounds for a family in particular $y_{\bar{N}} \leq L_g$, so that

$$y_{\bar{N}} + 0.5 \cdot (L_f - L_g) \leq L_g + 0.5 \cdot (L_f - L_g).$$

The conjunction of the preceding inequality with the inequality (8.384) yields

$$y_{\bar{N}} < L_g + 0.5 \cdot (L_f - L_g) \quad [= 0.5 \cdot L_f + 0.5 \cdot L_g]$$

with the Transitivity of $<$- and $\leq$-inequalities. In view of (8.383), the transitivity of $<_{\mathbb{R}}$ gives us therefore $y_{\bar{N}} < x_{\bar{N}}$. Applying now the Negation Formula for $\leq$, we find $\neg x_{\bar{N}} \leq y_{\bar{N}}$. Recalling that $\bar{N} \in \mathbb{N}_+$, this demonstrates the truth of the existential sentence

$$\exists n \, (n \in \mathbb{N}_+ \wedge \neg x_n \leq y_n),$$

so that (8.381) follows to be true with the Negation Law for universal implications and the Double Negation Law. Thus, the proof of the implication (8.374) is complete. As $(x_n)_{n \in \mathbb{N}_+}$ and $(y_n)_{n \in \mathbb{N}_+}$ were initially arbitrary, we may therefore conclude that b) also holds. □

Theorem 8.53 (Monotonicity of limits of increasingly convergent sequences of real functions). *The following implication is true for any set X and any increasingly convergent sequences $(f_n)_{n \in \mathbb{N}_+}$ and $(g_n)_{n \in \mathbb{N}_+}$ of real functions on X: If every term of $(f_n)_{n \in \mathbb{N}_+}$ is less than or equal to the corresponding term of $(g_n)_{n \in \mathbb{N}_+}$, then the pointwise limit of the former sequence is less than or equal to the pointwise limit of the latter, that is,*

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow f_n \preceq g_n) \;\Rightarrow\; \lim_{n \to \infty} f_n \,\text{pointwise} \preceq \lim_{n \to \infty} g_n \,\text{pointwise}. \tag{8.385}$$

Proof. We let X be an arbitrary set and $(f_n)_{n\in\mathbb{N}_+}$, $(g_n)_{n\in\mathbb{N}_+}$ arbitrary increasingly convergent sequences in $\mathbb{R}^X$. By definition, there are then particular functions $f, g \in Y^X$ such that

$$\bar{f} = \lim_{n\to\infty} f_n = \lim_{n\to\infty} f_n \,\text{pointwise}, \tag{8.386}$$

$$\bar{g} = \lim_{n\to\infty} g_n = \lim_{n\to\infty} g_n \,\text{pointwise}. \tag{8.387}$$

By the Characterization of the pointwise limit of an increasing sequence of functions, we thus have

$$\forall x\,(x \in X \Rightarrow \bar{f}(x) = \lim_{n\to\infty} f_n(x)), \tag{8.388}$$

$$\forall x\,(x \in X \Rightarrow \bar{g}(x) = \lim_{n\to\infty} g_n(x)). \tag{8.389}$$

Now, to prove the proposed implication directly, we assume that

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow f_n \preceq g_n) \tag{8.390}$$

holds, and we show that this implies

$$\forall x\,(x \in X \Rightarrow \bar{f}(x) \leq \bar{g}(x)). \tag{8.391}$$

For this purpose, we let $\bar{x} \in X$ be arbitrary, so that we obtain from (8.388) and (8.389) the two equations

$$\bar{f}(\bar{x}) = \lim_{n\to\infty} f_n(\bar{x}), \tag{8.392}$$

$$\bar{g}(\bar{x}) = \lim_{n\to\infty} g_n(\bar{x}). \tag{8.393}$$

Let us establish now the truth of the universal sentence

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow f_n(\bar{x}) \leq g_n(\bar{x})). \tag{8.394}$$

Letting $\bar{n} \in \mathbb{N}_+$ be arbitrary, it follows with (8.390) that $f_{\bar{n}} \preceq g_{\bar{n}}$ holds. This implies the truth of

$$\forall x\,(x \in X \Rightarrow f_{\bar{n}}(x) \leq g_{\bar{n}}(x))$$

by virtue of (3.903). We thus have in particular $f_{\bar{n}}(\bar{x}) \leq g_{\bar{n}}(\bar{x})$. Since $\bar{n}$ was arbitrary, we may therefore conclude that (8.394) is indeed true. This universal sentence implies now

$$\lim_{n\to\infty} f_n(\bar{x}) \leq \lim_{n\to\infty} g_n(\bar{x})$$

with (8.374). Substitutions based on (8.392) – (8.393) give us therefore $\bar{f}(\bar{x}) \leq \bar{g}(\bar{x})$, which is the desired consequent in (8.391). As $\bar{x}$ was arbitrary, we may therefore conclude that the universal sentence (8.391) holds, which further implies $\bar{f} \preceq \bar{g}$ in view of (3.903). Substitutions based on (8.386) – (8.387) then yield the desired consequent of the implication (8.385). Since X, $(f_n)_{n\in\mathbb{N}_+}$ and $(g_n)_{n\in\mathbb{N}_+}$ were initially arbitrary, we infer from this the truth of the stated proposition. □

8.3. Some Special Functions

In this section we will establish some important special functions on the set of real numbers, which we also intend to represent graphically. As a preparation, we define the following two essential constituents of a function plot. Depending on whether the context of a function is purely mathematical or geodetical, the meaning of these concepts is inverted.

Definition 8.28 (Mathematical x-axis & y-axis of $\mathbb{R}\times\mathbb{R}$, geodetical x-axis & y-axis of $\mathbb{R}\times\mathbb{R}$). We will call

1. the Cartesian product

$$\mathbb{R}\times\{0\} = \{(x,y) : x\in\mathbb{R} \wedge y\in\{0\}\} \tag{8.395}$$

the *mathematical x-axis of* $\mathbb{R}\times\mathbb{R}$ and the Cartesian product

$$\{0\}\times\mathbb{R} = \{(x,y) : x\in\{0\} \wedge y\in\mathbb{R}\} \tag{8.396}$$

the *mathematical y-axis of* $\mathbb{R}\times\mathbb{R}$.

2. the Cartesian product

$$\{0\}\times\mathbb{R} = \{(x,y) : x\in\{0\} \wedge y\in\mathbb{R}\} \tag{8.397}$$

the *geodetical x-axis of* $\mathbb{R}\times\mathbb{R}$ and the Cartesian product

$$\mathbb{R}\times\{0\} = \{(x,y) : x\in\mathbb{R} \wedge y\in\{0\}\} \tag{8.398}$$

the *geodetical y-axis of* $\mathbb{R}\times\mathbb{R}$.

Note 8.16. The values of the mathematical x-axis are viewed as being aligned 'horizontally' with values increasing 'to the right', and the values of the corresponding y-axis as being aligned 'vertically' with values increasing 'upward'. The geodetical x-axis is oriented exactly as the mathematical y-axis, and the geodetical y-axis exactly as the mathematical x-axis. We will usually use the mathematical axes to display special functions. To plot a given function f, certain elements $(x,y)\in f$ are determined, and the x-/y-coordinate of each 'point' (x,y) is compared – in the sense of the linear ordering $<_{\mathbb{R}}$ – to the values of the two axes to locate the point's plot location.

As a first example, the identity function $\mathrm{id}_{\mathbb{R}}$ is shown in Fig. 8.1.

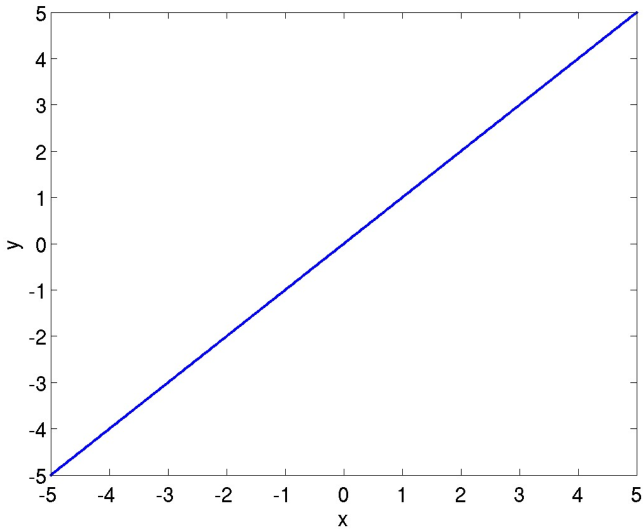

Figure 8.1.: Values of $\mathrm{id}_{\mathbb{R}}$ for $x \in \{-5, -4.99, \ldots, 4.99, 5\}$.

8.3.1. The absolute value function on $\mathbb{R}$

Proposition 8.54. *There is a unique function $|.|_{\mathbb{R}}$ with domain $\mathbb{R}$ such that*

$$\forall x\,(x \in \mathbb{R} \Rightarrow |.|_{\mathbb{R}}(x) = \overset{\leq_{\mathbb{R}}}{\sup}\{x, -x\}), \tag{8.399}$$

and $\mathbb{R}_+^0$ is a codomain of $|.|_{\mathbb{R}}$.

Proof. We may apply Function definition by replacement and verify for this purpose the universal sentence

$$\forall x\,(x \in \mathbb{R} \Rightarrow \exists! y\,(y = \overset{\leq_{\mathbb{R}}}{\sup}\{x, -x\})). \tag{8.400}$$

Letting x be arbitrary and assuming $x \in \mathbb{R}$ to be true, we observe that the supremum of $\{x, -x\}$ is a uniquely specified real number for the lattice of real numbers. As x is arbitrary, we therefore conclude that (8.400) holds, so that there exists a unique function $|.|_{\mathbb{R}}$ with domain $\mathbb{R}$ such that (8.399) holds. It now remains for us to show that $\mathbb{R}_+^0$ is a codomain of $|.|_{\mathbb{R}}$, i.e. that the range of $|.|_{\mathbb{R}}$ is included in $\mathbb{R}_+^0$. To do this, we verify

$$\forall y\,(y \in \mathrm{ran}(|.|_{\mathbb{R}}) \Rightarrow y \in \mathbb{R}_+^0). \tag{8.401}$$

We take an arbitrary $\bar{y}$ and assume that $\bar{y} \in \mathrm{ran}(|.|_{\mathbb{R}})$ is true. It then follows with the definition of a range that there is a constant, say $\bar{x}$, with

$(\bar{x}, \bar{y}) \in |.|_{\mathbb{R}}$. Therefore, we obtain $\bar{x} \in \mathbb{R}$ $[= \mathrm{dom}(|.|_{\mathbb{R}})]$ with the definition of a domain, so that (8.399) gives the function value $|.|_{\mathbb{R}}(\bar{x}) = \sup^{\leq_{\mathbb{R}}}\{\bar{x}, -\bar{x}\}$. Since the linear ordering $<_{\mathbb{R}}$ is connex, the disjunction

$$\bar{x} <_{\mathbb{R}} 0 \vee 0 <_{\mathbb{R}} \bar{x} \vee \bar{x} = 0$$

is true, which we now use to prove $0 \leq_{\mathbb{R}} |.|_{\mathbb{R}}(\bar{x})$ by cases.

If $\bar{x} <_{\mathbb{R}} 0$ holds, then we obtain

$$\bar{x} +_{\mathbb{R}} (-\bar{x}) <_{\mathbb{R}} 0 +_{\mathbb{R}} (-\bar{x})$$

with the monotony law (8.254), which we may simplify to $0 <_{\mathbb{R}} -\bar{x}$ by means of the properties of a negative. and of a zero element. In combination with the case assumption, this yields $\bar{x} <_{\mathbb{R}} -\bar{x}$ with the transitivity of the linear ordering $<_{\mathbb{R}}$. This in turn gives us $\bar{x} \leq_{\mathbb{R}} -\bar{x}$ with the Characterization of induced reflexive partial orderings, so that $\sup^{\leq_{\mathbb{R}}}\{\bar{x}, -\bar{x}\} = -\bar{x}$ follows to be true by virtue of Proposition 3.109. Substitution into $0 <_{\mathbb{R}} -\bar{x}$ yields

$$0 <_{\mathbb{R}} \overset{\leq_{\mathbb{R}}}{\sup}\{\bar{x}, -\bar{x}\} \quad [= |.|_{\mathbb{R}}(\bar{x})], \tag{8.402}$$

consequently $0 \leq_{\mathbb{R}} |.|_{\mathbb{R}}(\bar{x})$ with the Characterization of induced reflexive partial orderings, as desired.

If $0 <_{\mathbb{R}} \bar{x}$ holds, then evidently also

$$0 +_{\mathbb{R}} (-\bar{x}) <_{\mathbb{R}} \bar{x} +_{\mathbb{R}} (-\bar{x})$$

and therefore $-\bar{x} <_{\mathbb{R}} 0$. In connection with the current case assumption, this clearly implies $-\bar{x} <_{\mathbb{R}} \bar{x}$ and thus $-\bar{x} \leq_{\mathbb{R}} \bar{x}$, with the consequence that $\sup^{\leq_{\mathbb{R}}}\{\bar{x}, -\bar{x}\} = \sup^{\leq_{\mathbb{R}}}\{-\bar{x}, \bar{x}\} = \bar{x}$, using (2.161). With this, the case assumption $0 <_{\mathbb{R}} \bar{x}$ can be written as (8.402), so that $0 \leq_{\mathbb{R}} |.|_{\mathbb{R}}(\bar{x})$ is true also in the second case.

Finally, if $\bar{x} = 0$ is true, then we may observe the truth of the equations

$$\begin{aligned} |.|_{\mathbb{R}}(\bar{x}) &= \overset{\leq_{\mathbb{R}}}{\sup}\{\bar{x}, -\bar{x}\} \\ &= \overset{\leq_{\mathbb{R}}}{\sup}\{0, -0\} \\ &= \overset{\leq_{\mathbb{R}}}{\sup}\{0, 0\} \\ &= \overset{\leq_{\mathbb{R}}}{\sup}\{0\} \\ &= 0 \\ &\geq_{\mathbb{R}} 0 \end{aligned}$$

in light of (6.38), (2.152), (3.329), and the Characterization of induced reflexive partial orderings. Thus, the of $0 \leq_{\mathbb{R}} |.|_{\mathbb{R}}(\bar{x})$ by cases is complete,

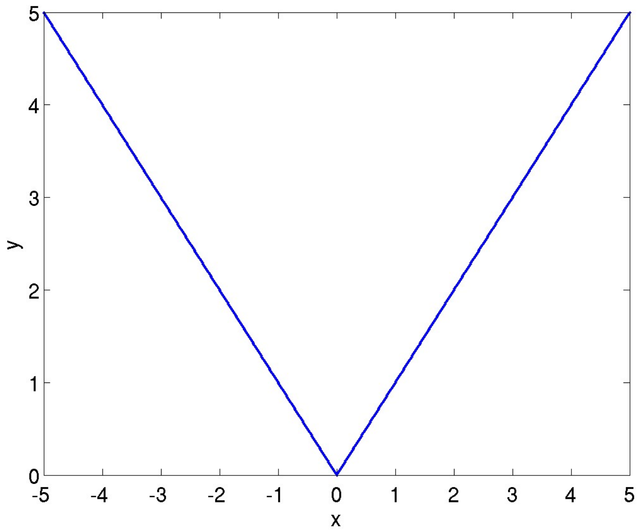

Figure 8.2.: Values of the absolute value function $|.|_{\mathbb{R}}$ for $-5, -4.99, \ldots, 4.99, 5$.

and since $\sup^{\leq_{\mathbb{R}}}\{\bar{x}, -\bar{x}\}$ is an element of $\mathbb{R}$, it follows by definition of the set of nonnegative real numbers that $\bar{y} = |.|_{\mathbb{R}}(\bar{x})$ is an element of $\mathbb{R}_+^0$. Because $\bar{y}$ is arbitrary, we may infer from this finding the truth of the universal sentence (8.401), which means that the inclusion $\mathrm{ran}(|.|_{\mathbb{R}}) \subseteq \mathbb{R}_+^0$ is indeed true. Thus, $\mathbb{R}_+^0$ is a codomain of the function $|.|_{\mathbb{R}}$. $\square$

Notation 8.6. We abbreviate

$$|x| = |.|_{\mathbb{R}}(x). \tag{8.403}$$

Definition 8.29 (Absolute value function on $\mathbb{R}$, absolute value of a real number). We call

$$|.|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}_+^0, \tag{8.404}$$

the *absolute value function on* $\mathbb{R}$. Furthermore, we say for any $x \in \mathbb{R}$ that the corresponding function value $|x|$ is the *absolute value of* x.

This definition yields two basic properties of the function $|.|$.

Corollary 8.55. *The absolute value function on $\mathbb{R}$ is* *a) nonnegative in the sense that*

$$\forall x\,(x \in \mathbb{R} \Rightarrow 0 \leq_{\mathbb{R}} |x|). \tag{8.405}$$

b) positive definite in the sense that

$$\forall x\,(x \in \mathbb{R} \Rightarrow |x| = 0 \Leftrightarrow x = 0). \tag{8.406}$$

c) even, that is,

$$\forall x\,(x \in \mathbb{R} \Rightarrow |x| = |-x|). \tag{8.407}$$

Proof. We let x be arbitrary in the domain $\mathbb{R}$ of $|.|_{\mathbb{R}}$. The proof of Proposition 8.54 already showed that $0 \leq_{\mathbb{R}} |x|$ is true. We prove the first part ('$\Rightarrow$') of the equivalence in (8.406) by contraposition, assuming $x \neq 0$ to be true. As the linear ordering $<_{\mathbb{R}}$ is connex, the disjunction $x <_{\mathbb{R}} 0 \vee 0 <_{\mathbb{R}} x$ is then true. The proof of Proposition 8.54 also shows that $0 <_{\mathbb{R}} |x|$ turns out to be true in both cases. According to the Characterization of comparability, this inequality implies then the desired consequent $|x| \neq 0$. We may prove the second part ('$\Leftarrow$') of the equivalence directly, assuming $x = 0$ to be true. As argued within the proof of Proposition 8.54, this equation implies $|x| = 0$, as desired. Now, concerning c), we observe that the equations

$$|-x| = \sup\{-x, -(-x)\} = \sup\{-x, x\} = \sup\{x, -x\} = |x|$$

hold due to the definition of the absolute value function on $\mathbb{R}$, the Sign Law (6.50) and (2.161). Since x was arbitrary, we therefore conclude that the universal sentences a), b) and c) hold, as claimed. □

Exercise 8.26. Establish the truth of the universal sentence

$$\forall x\,(x \in \mathbb{R} \Rightarrow [(x \geq_{\mathbb{R}} 0 \Rightarrow |x| = x) \wedge (x \leq_{\mathbb{R}} 0 \Rightarrow |x| = -x)]). \tag{8.408}$$

(Hint: Rewrite the inequalities by means of the Characterization of induced irreflexive partial orderings and use then some of the arguments within the proof of Proposition 8.54 to carry out the proofs by cases.)

Notation 8.7. We will also write (8.408) as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \tag{8.409}$$

In view of the proof of Exercise 8.26, it suffices to include the occurrence of $x = 0$ in the first case of (8.409) with respect to $x \leq 0$.

We now state three fundamental inequalities for absolute values, culminating in the widely applied 'triangle inequality'.

Exercise 8.27. Show that every real number x lies between the negative of its absolute value and its absolute value, that is,

$$\forall x\,(x \in \mathbb{R} \Rightarrow -|x| \leq x \leq |x|). \tag{8.410}$$

(Hint: Apply a proof by three cases as for Proposition 8.54.)

Proposition 8.56. *For any $x \in \mathbb{R}$ and any $y \in \mathbb{R}_+^0$, it is true that*

$$|x| \geq y \;\Leftrightarrow\; (x \leq -y \,\vee\, x \geq y), \tag{8.411}$$
$$|x| < y \;\Leftrightarrow\; -y < x < y. \tag{8.412}$$

Proof. We let x in $\mathbb{R}$ and $y \in \mathbb{R}_+^0$ be arbitrary. To prove the first part ('$\Rightarrow$') of the first equivalence, we assume $|x| \geq_{\mathbb{R}} y$ and consider the two exhaustive cases $x \geq_{\mathbb{R}} 0$ and $\neg x \geq_{\mathbb{R}} 0$. The first case implies $|x| = x$ with (8.409) and thus $x \geq_{\mathbb{R}} y$, which further implies the disjunction $x \leq_{\mathbb{R}} -y \vee x \geq_{\mathbb{R}} y$. Similarly, the second case $\neg x \geq_{\mathbb{R}} 0$ implies $x <_{\mathbb{R}} 0$ with the Negation Formula for $\leq$ and therefore $|x| = -x$ according to (8.409), so that $-x \geq_{\mathbb{R}} y$ follows to be true by virtue of the initial assumption. Thus, $-x >_{\mathbb{R}} y \vee -x = y$ holds according to the Characterization of induced irreflexive partial orderings. On the one hand, if $-x >_{\mathbb{R}} y$ holds, then we obtain $x <_{\mathbb{R}} -y$ with the Monotony Law (7.98) and therefore evidently $x \leq_{\mathbb{R}} -y$. On the other hand, if $-x = y$ holds, then we obtain $x = -(-x) = -y$ by applying substitution and the Sign Law (6.50), with the evident consequence that $x \leq_{\mathbb{R}} -y$ holds again. Then, the disjunction $x \leq_{\mathbb{R}} -y \vee x \geq_{\mathbb{R}} y$ is true as well.

To prove the second part ('$\Leftarrow$') of the first equivalence, we assume $x \leq_{\mathbb{R}} -y \vee x \geq_{\mathbb{R}} y$ and consider again the two cases $x \geq_{\mathbb{R}} 0$ and $\neg x \geq_{\mathbb{R}} 0$. As before, the first case implies $|x| = x$. On the one hand, if the first part $x \leq_{\mathbb{R}} -y$ of the assumed disjunction holds, then the conjunction of this and the case assumption $0 \leq_{\mathbb{R}} x$ implies $0 \leq_{\mathbb{R}} -y$ with the transitivity of the linear ordering $<_{\mathbb{R}}$. Thus, the disjunction $0 <_{\mathbb{R}} -y \vee 0 = -y$ holds according to the Characterization of induced irreflexive partial orderings. If $0 <_{\mathbb{R}} -y$, then $y <_{\mathbb{R}} 0$ $[\leq_{\mathbb{R}} x]$ due to the aforementioned monotony law, and therefore $y <_{\mathbb{R}} x$ $[= |x|]$ due to the Transitivity Formula for $<$ and $\leq$; thus, $|x| >_{\mathbb{R}} y$ holds, which evidently implies $|x| \geq_{\mathbb{R}} y$. If $0 = -y$, then clearly $y = 0$ and $y \leq_{\mathbb{R}} 0$ $[\leq_{\mathbb{R}} x]$, and therefore $y \leq_{\mathbb{R}} x$ $[= |x|]$ due to the transitivity of the total ordering $\leq_{\mathbb{R}}$; thus, the desired consequent $|x| \geq_{\mathbb{R}} y$ holds again. On the other hand, if the second part $x \geq_{\mathbb{R}} y$ of the assumed disjunction holds, then the previously established equation $|x| = x$ yields $|x| \geq_{\mathbb{R}} y$, as desired. As before, the second case $\neg x \geq_{\mathbb{R}} 0$ gives first $x <_{\mathbb{R}} 0$ and therefore $|x| = -x$. Furthermore, the preceding inequality gives evidently $0 <_{\mathbb{R}} -x$ and then also $0 \leq_{\mathbb{R}} -x$. As the initial assumption $y \in \mathbb{R}_+^0$

shows that $0 \leq_{\mathbb{R}} y$ is true, we may now establish the truth of $\neg x \geq_{\mathbb{R}} y$ via contradiction. Assuming for this purpose the negation of that negation to be true, we obtain $y \leq_{\mathbb{R}} x$ with the Double Negation Law. In conjunction with the previously found inequality $x <_{\mathbb{R}} 0$, this implies $y <_{\mathbb{R}} 0$ with the Transitivity Formula for $\leq$ and $<$. The Negation Formula for $\leq$ gives us then the true negation $\neg 0 \leq_{\mathbb{R}} y$, which contradicts the previously shown $0 \leq_{\mathbb{R}} y$. Thus, the proof of $\neg x \geq_{\mathbb{R}} y$ by contradiction is complete, which means that the second part of the assumed disjunction is false. Thus, the first part $x \leq_{\mathbb{R}} -y$ of that disjunction is true, which clearly means that the disjunction $x <_{\mathbb{R}} -y \vee x = -y$ holds. If $x <_{\mathbb{R}} -y$, then evidently $y <_{\mathbb{R}} -x$ $[= |x|]$ and therefore $|x| \geq_{\mathbb{R}} y$. If $x = -y$, then $y = -x$ $[= |x|]$, and consequently $|x| \geq_{\mathbb{R}} y$. Thus, the proof of the equivalence (8.411) is now complete.

Next, we observe that the equivalence

$$\neg(|x| \geq y) \;\Leftrightarrow\; \neg(x \leq -y \,\vee\, x \geq y)$$

is true as a consequence of (8.411). An application of De Morgan's Law (1.52) yields then

$$\neg(|x| \geq y) \;\Leftrightarrow\; (\neg x \leq -y \,\wedge\, \neg x \geq y).$$

Finally, three applications of the Negation Formula for $\leq$ give us then (8.412). Since x and y are arbitrary, we may therefore conclude that the proposed universal sentence holds.

As x and y were initially arbitrary, we then conclude that the proposition is true. □

Exercise 8.28. Establish for any $x \in \mathbb{R}$ and any $y \in \mathbb{R}_+^0$ the truth of the equivalences

$$|x| > y \;\Leftrightarrow\; (x < -y \,\vee\, x > y), \tag{8.413}$$

$$|x| \leq y \;\Leftrightarrow\; -y \leq x \leq y. \tag{8.414}$$

(Hint: The proof of the first equivalence is a slightly simpler version of the proof of the first equivalence in Proposition 8.56.)

Theorem 8.57 (Multiplicativity of the absolute value function on $\mathbb{R}$). *The absolute value of the product of two real numbers is identical with the product of their absolute values, that is,*

$$\forall x, y\,(x, y \in \mathbb{R} \Rightarrow |x \cdot y| = |x| \cdot |y|). \tag{8.415}$$

Proof. We let x and y be arbitrary in $\mathbb{R}$ and consider the three cases $x > 0$, $x < 0$ and $x = 0$.

In the first case, $x > 0$ implies $x \geq 0$ with the definition of an induced irreflexive partial ordering and then $|x| = x$ with (8.409). We now consider the two subcases $y \geq 0$ and $\neg y \geq 0$. If $y \geq 0$ holds (alongside $x > 0$), then we evidently obtain on the one hand $|y| = y$ and on the other hand $x \cdot y \geq 0$ with the Monotony Law (6.230) for the ordered integral domain $(\mathbb{R}, +, \cdot, -, <)$; as the latter evidently implies $|x \cdot y| = x \cdot y$, we obtain with the previous findings via substitutions $|x \cdot y| = |x| \cdot |y|$, as desired. If the other subcase $\neg y \geq 0$ holds, then $y < 0$ is true according to the Negation Formula for $\leq$. The preceding inequality evidently implies $|y| = -y$ as well as $x \cdot y < 0$ with the Monotony Law (6.229), so that an application of the Sign Law 6.63 gives us

$$|x \cdot y| = -(x \cdot y) = x \cdot (-y) = |x| \cdot |y|,$$

as desired.

The second case $x < 0$ implies $|x| = -x$. Here, we consider the three subcases $y > 0$, $y < 0$ and $y = 0$. If $y > 0$ holds, so that $|y| = y$ and $x \cdot y < 0$, then

$$|x \cdot y| = -(x \cdot y) = -(y \cdot x) = y \cdot (-x) = (-x) \cdot y = |x| \cdot |y|.$$

If $y < 0$ is true, then $|y| = -y$ and $0 < -y$; since $x < 0$ gives also $0 < -x$, we obtain $0 < (-x) \cdot (-y)$, and therefore $0 < x \cdot y$ with the Sign Law (6.65). Let us also observe that the Sign Law (6.50) yields the two further equations $x = -(-x)\,[= -|x|]$ and $y = -(-y)\,[= -|y|]$. With these findings, we get

$$|x \cdot y| = x \cdot y = (-|x|) \cdot (-|y|) = |x| \cdot |y|$$

(applying again the Sign Law (6.65)). If $y = 0$, then $|y| = y = 0$ and

$$|x \cdot y| = |x \cdot 0| = |0| = 0 = |x| \cdot 0 = |x| \cdot |y|.$$

Finally, the third case $x = 0$ gives $|x| = x = 0$ and

$$|x \cdot y| = |0 \cdot y| = |0| = 0 = 0 \cdot |y| = |x| \cdot |y|.$$

Thus, the proposed equation holds in any case, and since x and y were arbitrary, we conclude that the theorem is indeed true. □

8.3.2. The squaring function

Exercise 8.29. Define the function $\mathrm{id}_{\mathbb{R}}^2 : \mathbb{R} \to \mathbb{R}_+^0, \quad x \mapsto x^2$.

(Hint: Apply Function definition by replacement and recall Notation 5.21.)

Definition 8.30 (Squaring function on $\mathbb{R}$). We call the function

$$\mathrm{id}_{\mathbb{R}}^2 : \mathbb{R} \to \mathbb{R}_+^0, \quad x \mapsto x^2 \tag{8.416}$$

the *squaring function on* $\mathbb{R}$.

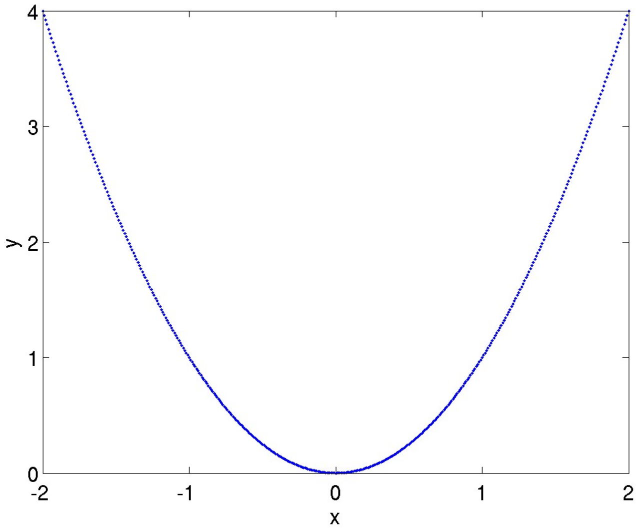

Figure 8.3.: Values of $\mathrm{id}_{\mathbb{R}}^2$ for $x \in \{-2, -1.99, \ldots, 1.99, 2\}$.

Exercise 8.30. Prove that the restriction $\mathrm{id}_{\mathbb{R}}^2 \upharpoonright \mathbb{R}_+^0$ of the squaring function (to $\mathbb{R}_+^0$) is strictly increasing.

(Hint: Use (5.554).)

Proposition 8.58. *The composition of the squaring and the absolute value function on* $\mathbb{R}$ *is identical with the squaring function, i.e.*

$$\mathrm{id}_{\mathbb{R}}^2 \circ |.|_{\mathbb{R}} \; = \; \mathrm{id}_{\mathbb{R}}^2. \tag{8.417}$$

Proof. We first observe that $\mathbb{R}$ is also a codomain of $|.|_{\mathbb{R}}$ due to the inclusions $\mathrm{ran}(|.|_{\mathbb{R}}) \subseteq \mathbb{R}_+^0 \subseteq \mathbb{R}$, so that $|.|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$. Consequently, the composition $\mathrm{id}_{\mathbb{R}}^2 \circ |.|_{\mathbb{R}}$ is a function with domain $\mathbb{R}$ and codomain $\mathbb{R}_+^0$ according to Proposition 3.178). We may therefore apply the Equality Criterion for functions to verify that this composition equals $\mathrm{id}_{\mathbb{R}}^2$. For this purpose, we let x be arbitrary in $\mathbb{R}$. We then obtain the equations

$$(\mathrm{id}_{\mathbb{R}}^2 \circ |.|_{\mathbb{R}})(x) = \mathrm{id}_{\mathbb{R}}^2(|x|) = |x|^2 = |x| \cdot |x| = |x \cdot x| = |x^2| = x^2 = \mathrm{id}_{\mathbb{R}}^2(x)$$

using Notation 3.6 together with the definition of the absolute value function, the definition of the squaring function, the notation for squares (with respect to the semigroup $(\mathbb{R}, \cdot_{\mathbb{R}})$), then the Multiplicativity of the absolute value function on $\mathbb{R}$, again the notation for squares, then (8.409) with the fact that $x^2 \geq 0$ holds due to (6.243), and finally once more the definition of the squaring function. Since x is arbitrary, we conclude that the proposed equation holds. □

8.3.3. The square root function

Theorem 8.59 (Existence of square roots). *The following sentences are true.*

a) For any positive real number x there exists a unique positive real number y such that y^2 equals x, that is,

$$\forall x\,(x \in \mathbb{R}_+ \Rightarrow \exists! y\,(y \in \mathbb{R}_+ \wedge y^2 = x)). \tag{8.418}$$

b) For any positive real number x there exists a unique negative real number y such that y^2 equals x, that is,

$$\forall x\,(x \in \mathbb{R}_+ \Rightarrow \exists! y\,(y <_{\mathbb{R}} 0 \wedge y^2 = x)). \tag{8.419}$$

Proof. Concerning a), we let $x \in \mathbb{R}_+$ be arbitrary, so that $0 < x$ holds by definition of the set of positive real numbers. We begin by verifying the existential part, for which we consider the two exhaustive cases $x \leq 1$ and $\neg x \leq 1$. Assuming the first case, we may evidently apply the Axiom of Specification and the Axiom of Extension to define the unique set X consisting of all positive real numbers c satisfying $c^2 \leq x$, i.e.

$$\exists! X \forall c\,(c \in X \Leftrightarrow [c \in \mathbb{R}_+ \wedge c^2 \leq x]).$$

As the assumptions $0 < x$ and $x \leq 1$ imply $x \cdot x \leq 1 \cdot x$ with the Monotony Law for $\cdot_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$, so that $x^2 \leq x$. Furthermore, $0 < x$ implies $x \in \mathbb{R}_+$ by definition of the set of positive real numbers. Therefore, $x \in X$ follows to be true by the preceding specification of the set X; consequently, X is clearly nonempty. Next, we show that X is bounded-from-above, letting $c \in X$ be arbitrary. Then, $c \in \mathbb{R}_+$ and $c^2 \leq x\,[\leq 1]$ follow to be true by definition of X, so that we obtain $c^2 \leq 1$ with the transitivity of the total ordering $\leq_{\mathbb{R}}$, and therefore $c^2 \leq 1 \cdot 1\ [= 1^2]$. The resulting inequality $c^2 \leq 1^2$ yields now $c \leq 1$ with the Monotony Law for the base and $\leq$, which we can apply here since $c \in \mathbb{R}_+$ evidently implies $c \in \mathbb{R}_+^0$ where $\mathbb{R}_+^0$ defines the ordered elementary domain of nonnegative real numbers. Since c is arbitrary, we conclude that $c^2 \leq 1$ holds for all $c \in X$, so that 1 is by definition an upper bound for X. We thus showed that X is a nonempty and bounded-from-above subset of $\mathbb{R}$; then, as $(\mathbb{R}, <)$ is a linear continuum, the supremum (s) of X exists. Here, s is a particular upper bound for X, so that $[0 <]\, c \leq s$ holds for any $c \in X$, which implies $0 < s$ with the Transitivity Formula for $<$ and $\leq$; thus, $s \in \mathbb{R}_+$ and also $s \in \mathbb{R}_+^0$.

In the following, we prove $s^2 = x$ by demonstrating that $s^2 \leq x$ and $x \leq s^2$ are both true. To prove $s^2 \leq x$, we first consider the real number

$$d = \frac{1}{2} \cdot \left(s + \frac{x}{s}\right), \tag{8.420}$$

which is evidently also in $\mathbb{R}_+$ due to $\frac{1}{2}, s, x \in \mathbb{R}_+$. Then, we obtain with the binomial formulae (6.387 and (6.388) the equations

$$\begin{aligned}
d^2 - x &= \left[\frac{1}{2} \cdot \left(s + \frac{x}{s}\right)\right]^2 - x \\
&= \left(\frac{1}{2}\right)^2 \cdot \left(s + \frac{x}{s}\right)^2 - x \\
&= \frac{1}{4} \cdot \left(s^2 + 2 \cdot s \cdot \frac{x}{s} + \left(\frac{x}{s}\right)^2\right) - \frac{1}{4} \cdot 4 \cdot s \cdot \frac{x}{s} \\
&= \frac{1}{4} \cdot \left(s^2 + 2 \cdot s \cdot \frac{x}{s} + \left(\frac{x}{s}\right)^2 - 4 \cdot s \cdot \frac{x}{s}\right) \\
&= \left(\frac{1}{2}\right)^2 \cdot \left(s^2 - 2 \cdot s \cdot \frac{x}{s} + \left(\frac{x}{s}\right)^2\right) \\
&= \left(\frac{1}{2}\right)^2 \cdot \left(s - \frac{x}{s}\right)^2 \\
&= \left[\frac{1}{2} \cdot \left(s - \frac{x}{s}\right)\right]^2 .
\end{aligned}$$

Here, $0 \leq \left[\frac{1}{2}\left(s - \frac{x}{s}\right)\right]^2$ holds due to (6.243), so that $0 \leq d^2 - x$; this inequality implies $x \leq d^2$ with the Monotony Law for $+_\mathbb{R}$ and $\leq_\mathbb{R}$. Recalling the specification of X, we thus see that $c^2 \leq x \leq d^2$ holds for any $c \in X$, which implies $c^2 \leq d^2$ with the transitivity of $\leq_\mathbb{R}$ for any $c \in X$. This inequality then implies $c \leq d$ with the Monotony Law for the base and $\leq$ and the fact that $c, d \in \mathbb{R}_+^0$. As this is true for any $c \in X$, we have that d is (by definition) an upper bound for X. Now, as s is the least upper bound for X, we see that $s \leq d$ holds. This further implies $2s^2 \leq d \cdot 2s$ with the Monotony Law for $\cdot_\mathbb{R}$ and $\leq_\mathbb{R}$ in combination with the fact that $0 < s$ and $0 < 2$ imply $0 < 2s$ with the Monotony Law for $\cdot_\mathbb{R}$ and $<_\mathbb{R}$. Substitution of

d by the expression on the right-hand side of (8.420) then evidently gives

$$2s^2 \leq 2s \cdot \frac{1}{2} \cdot \left(s + \frac{x}{s}\right) \; [= s^2 + x],$$

i.e. $2s^2 \leq s^2 + x$, which evidently further implies $s^2 \leq x$, as desired.

To prove that $x \leq s^2$ is also true, we consider the real number

$$b = \frac{x}{d}, \tag{8.421}$$

which is an element of $\mathbb{R}_+$ as a consequence of $x, d \in \mathbb{R}_+$. The previously established $x \leq d^2$ clearly implies $x^2 \leq d^2 \cdot x$ and then $[b^2 =] \frac{x^2}{d^2} \leq x$. Consequently, $b^2 \leq x$ holds, so that $b \in X$ is true by specification of X. Recalling that the supremum s of X is an upper bound for that set, we therefore find $b \leq s$. Thus, substitution of (8.421) gives us $\frac{x}{d} \leq s$, which implies $\frac{x}{d} \cdot 2d \leq s \cdot 2d$. Using (8.420) and simplifying evidently leads to

$$2x \leq 2s \cdot \frac{1}{2} \cdot \left(s + \frac{x}{s}\right),$$

i.e. $2x \leq s^2 + x$. This further implies $x \leq s^2$, as desired. We thus verified that $s^2 \leq x$ and $x \leq s^2$ are both true; the conjunction of these two inequalities then implies the desired $s^2 = x$ with the antisymmetry of the reflexive/total ordering $\leq_{\mathbb{R}}$. We thus established the existence of some $y \in \mathbb{R}_+$ with $y^2 = x$ in the first case $x \leq 1$. Let us observe that, since x is an arbitrary real number such that $0 < x$ and $x \leq 1$, we may infer the truth of the universal sentence

$$\forall x' \, (0 < x' \leq 1 \Rightarrow \exists y \, (y \in \mathbb{R}_+ \, \wedge \, y^2 = x')).$$

The second case $\neg x \leq 1$ implies $x > 1$ with the Negation Formula for $\leq$ and therefore $1 \cdot \frac{1}{x} < x \cdot \frac{1}{x}$, thus $\frac{1}{x} < 1$, and consequently also $\frac{1}{x} \leq 1$. Furthermore, the case assumption $0 < x$ implies $0 < \frac{1}{x}$ with (7.101), so that the inequalities $0 < \frac{1}{x} \leq 1$ hold. Because of the preceding universal sentence, there then exists a positive real number, say $\bar{r}$, such that $\bar{r}^2 = \frac{1}{x}$. The number $\bar{y} = \frac{1}{\bar{r}}$ is then also positive, and we obtain the equations

$$\bar{y}^2 = \left(\frac{1}{\bar{r}}\right)^2 = \frac{1}{\bar{r}^2} = \frac{1}{\frac{1}{x}} = x.$$

This finding proves the existence of some $y \in \mathbb{R}_+$ satisfying $y^2 = x$ also in the second case $\neg x \leq 1$; thus, we established the existential part in (8.418) for $0 < x$. We now verify the uniqueness part. To do this, we let y and y' be arbitrary elements of $\mathbb{R}_+$ such that $y^2 = x$ and $y'^2 = x$ are true.

Substitution yields then $y^2 = y'^2$. Since $y, y' \in \mathbb{R}_+$ implies $y, y' \in \mathbb{R}_+^0$ where $\mathbb{R}_+^0$ defines an ordered elementary domain, we may apply (5.559) to infer from the preceding equation the truth of $y = y'$. This finding completes the proof of the uniqueness part, so that the uniquely existential sentence in (8.418) holds. As x is arbitrary, we may therefore conclude that the universal sentence (8.418) is true.

To prove b), we let again $x \in \mathbb{R}_+$ be arbitrary and consider the existential part first. Because of a), there exists a unique element y of $\mathbb{R}_+$ such that $y^2 = x$ holds. Let us observe that $y \in \mathbb{R}_+$ implies $0 < y$ and then $-y < 0$. Furthermore, we obtain

$$(-y)^2 = (-y) \cdot (-y) = y \cdot y = y^2 = x$$

using in particular the Sign Law (6.65). Regarding the uniqueness part, we let $y < 0$ and $y' < 0$ be arbitrary such that $y^2 = x = y'^2$. Consequently, $0 < -y$ and $0 < -y'$, so that $-y, -y' \in \mathbb{R}_+$ and also $-y, -y' \in \mathbb{R}_+^0$; moreover,

$$\begin{aligned} x &= y^2 \; = (-y)^2, \\ x &= y'^2 = (-y')^2, \end{aligned}$$

so that $(-y)^2 = (-y')^2$, which implies $-y = -y'$ with (5.559). Substitutions then give

$$y = -(-y) = -(-y') = y',$$

which completes the proof of the uniqueness part. As x was arbitrary, we therefore conclude that (8.419) is true as well. □

Exercise 8.31. Verify the following uniquely existential sentence.

$$\exists! y \, (y \in \mathbb{R}_+^0 \, \wedge \, y^2 = 0). \tag{8.422}$$

(Hint: Consider $y = 0$ for the existential part; concerning the uniqueness part, apply the Criterion for zero-divisor freeness.)

Exercise 8.32. Show that there exists a unique function s such that

$$\forall x \, (x \in \mathbb{R}_+^0 \Rightarrow [s(x) \in \mathbb{R}_+^0 \, \wedge \, s(x)^2 = x]). \tag{8.423}$$

Verify also that $\mathbb{R}_+^0$ is a codomain of s.

Definition 8.31 (Square root function, principal square root, square root). We call the function s defined in Exercise 8.32, which we also symbolize by

$$\sqrt{\cdot} : \mathbb{R}_+^0 \to \mathbb{R}_+^0, \quad x \mapsto \sqrt{x}, \tag{8.424}$$

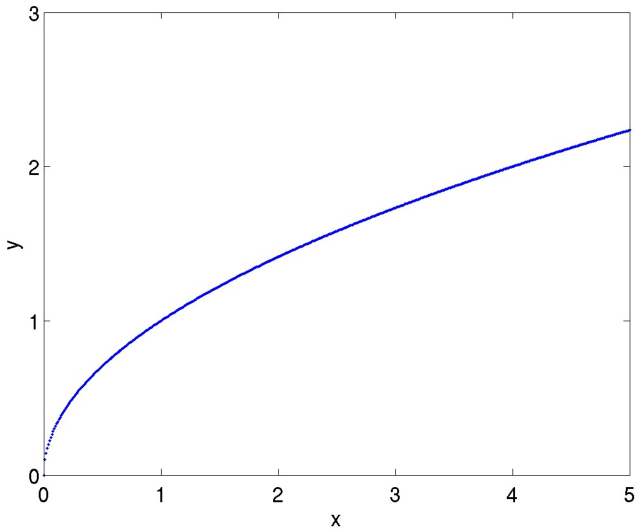

Figure 8.4.: Values of the square root function $\sqrt{\cdot}$ for $0, 0.01, \ldots, 4.99, 5$.

the *square root function*. We then call the function value $\sqrt{x}$ (at any $x \in \mathbb{R}_+^0$) the *principal square root* of x. We also say that $\sqrt{x}$ and $-\sqrt{x}$ are the *square roots* of x.

Note 8.17. As the value $y = \sqrt{x}$ of the square root function at x satisfies $y^2 = x$, we have that

$$\sqrt{x} \cdot \sqrt{x} = \sqrt{x}^2 = x. \tag{8.425}$$

Exercise 8.33. Verify

$$\forall x \, (\sqrt{x} = 0 \Leftrightarrow x = 0). \tag{8.426}$$

Note 8.18. The preceding exercise shows that, when $x = 0$ is true, then $\sqrt{x} = 0$ also holds. Thus, substitution yields the basic fact

$$\sqrt{0} = 0. \tag{8.427}$$

Proposition 8.60. *The square root function $\sqrt{\cdot}$ is strictly increasing.*

Proof. To verify (3.962) with respect to $f = \sqrt{\cdot}$, we let x and y be arbitrary, assume $x, y \in \mathbb{R}_+^0$ as well as $x < y$, and show that this implies $\sqrt{x} < \sqrt{y}$.

Let us first observe that

$$\begin{aligned}\sqrt{y} - \sqrt{x} = 1 \cdot (\sqrt{y} - \sqrt{x}) &= \frac{\sqrt{y} + \sqrt{x}}{\sqrt{y} + \sqrt{x}} \cdot \frac{\sqrt{y} - \sqrt{x}}{1} \\ &= \frac{(\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x})}{(\sqrt{y} + \sqrt{x}) \cdot 1} = \frac{\sqrt{y}^2 - \sqrt{x}^2}{\sqrt{y} + \sqrt{x}} \\ &= \frac{y - x}{\sqrt{y} + \sqrt{x}}\end{aligned}$$

holds with the property of the unity element 1, Note 7.8, (7.94), (6.389), and (8.425). Now, on the one hand, $x < y$ implies $x - x < y - x$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$, or equivalently $0 < y - x$. On the other hand, the conjunction of $x \in \mathbb{R}_+^0$ (i.e., $0 \leq x$) and $x < y$ implies $0 < y$ with the Transitivity Formula for $\leq$ and $<$, and therefore $0 < \sqrt{y}$ with Theorem 8.59a); from the latter inequality, it follows with the Monotony Laws that $\sqrt{x} < \sqrt{y} + \sqrt{x}$, where $0 \leq \sqrt{x}$ holds by definition of the square root function. These two inequalities then imply $0 < \sqrt{y} + \sqrt{x}$ (again with the Transitivity Formula for $\leq$ and $<$), so that $0 < \frac{1}{\sqrt{y}+\sqrt{x}}$ is true due to (7.101). This together with the previously established $0 < y - x$ implies $0 < (y - x) \cdot \frac{1}{\sqrt{y}+\sqrt{x}}$ with the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$, that is,

$$0 < \frac{(\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x})}{\sqrt{y} + \sqrt{x}} \ \left[= \sqrt{y} - \sqrt{x}\right].$$

Finally, $0 < \sqrt{y} - \sqrt{x}$ implies evidently $\sqrt{x} < \sqrt{y}$. Since x and y are arbitrary, we therefore conclude that $\sqrt{\cdot}$ is indeed strictly increasing. □

Exercise 8.34. Show that a nonnegative real number x is smaller than a nonnegative real number y iff the square root of x is smaller than the square root of y, that is,

$$\forall x, y\,(x, y \in \mathbb{R}_+^0 \Rightarrow [x < y \Leftrightarrow \sqrt{x} < \sqrt{y}]) \tag{8.428}$$

(Hint: Apply Proposition 8.60, Definition 3.63, Exercise 8.30, Corollary 3.165, and Note 8.17.)

Proposition 8.61. *The square root of the square of any real number x exists and is identical with the absolute value of x, that is,*

$$\forall x\,(x \in \mathbb{R} \Rightarrow \sqrt{x^2} = |x|). \tag{8.429}$$

Proof. We let $x \in \mathbb{R}$ be arbitrary and observe first that this implies $0 \leq x \cdot x$ $[= x^2]$ with (6.243); thus, $x^2 \in \mathbb{R}_+^0$ holds by definition of the set of nonnegative real numbers, so that $s = \sqrt{x^2}$ is a specified element of $\mathbb{R}_+^0$ according to (8.424), which value satisfies $s^2 = x^2$. We now consider the two exhaustive cases $0 \leq x$ and $\neg 0 \leq x$.

In the first case, we have $x \in \mathbb{R}_+^0$, where $\mathbb{R}_+^0$ defines the ordered elementary domain of nonnegative real numbers. Therefore, $s^2 = x^2$ implies $s = x$ with (5.559). Furthermore, the current case assumption $0 \leq x$ implies with (8.409)

$$|x| = x \; [= s = \sqrt{x^2}],$$

which gives the desired equation in the first case.

The second case $\neg 0 \leq x$ implies first $x < 0$ with the Negation Formula for $\leq$ and then $0 < -x$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$. Thus, $-x \in \mathbb{R}_+$ and therefore also $-x \in \mathbb{R}_+^0$. Observing in light of the Sign Law (6.65) the truth of

$$(-x)^2 = (-x) \cdot (-x) = x \cdot x = x^2 = s^2,$$

we find $s^2 = (-x)^2$, which implies $s = -x$ with (5.559). Furthermore, $x < 0$ yields

$$|x| = -x = s = \sqrt{x^2},$$

so that the proposed equation holds also in the second case. As x was arbitrary, we conclude that proposition holds, as claimed. □

8.4. Metric spaces

Definition 8.32 (Metric, triangle inequality, point, distance, metric space). For any set X we say that a function

$$d_X : X \times X \to \mathbb{R} \tag{8.430}$$

is a *metric* on X iff

1. d_X has only nonnegative function values, that is,
$$\forall x, y\, (x, y \in X \Rightarrow d_X(x, y) \geq 0), \tag{8.431}$$

2. d_X takes the value 0 precisely for identical arguments, that is,
$$\forall x, y\, (x, y \in X \Rightarrow [d_X(x, y) = 0 \Leftrightarrow x = y]), \tag{8.432}$$

3. d_X is symmetrical in the sense that
$$\forall x, y\, (x, y \in X \Rightarrow d_X(x, y) = d_X(y, x)), \tag{8.433}$$

4. d_X satisfies the *Triangle Inequality*, that is,
$$\forall x, y, z\, (x, y, z \in X \Rightarrow d_X(x, y) \leq d_X(x, z) + d_X(z, y)). \tag{8.434}$$

Then, we call each element of X a *point* (of X), and $d_X(x, y)$ the *distance* from x to y. Moreover, we call the ordered pair

$$(X, d_X) \tag{8.435}$$

a *metric space*.

Note 8.19. Recalling that the subtraction on $\mathbb{R}$ is a function $-_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the absolute value function on $\mathbb{R}$ a function $|.|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}_+^0$, we see in light of Proposition 3.178 that the composition $d_{\mathbb{R}}$ of $|.|_{\mathbb{R}}$ and $-_{\mathbb{R}}$ is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}_+^0$.

Definition 8.33 (Absolute difference function on $\mathbb{R}$, absolute difference of two real numbers). We call the composition

$$d_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+^0, \quad (x, y) \mapsto |x - y| \tag{8.436}$$

the *absolute difference function on* $\mathbb{R}$. We then call $|x - y|$ the *absolute difference* of x and y for any $x, y \in \mathbb{R}$.

Before we show that $d_{\mathbb{R}}$ is actually a metric, we establish the following useful inequalities as a preparation.

Theorem 8.62 (Subadditivity of the absolute value function on $\mathbb{R}$). *The following inequality holds for any real numbers x and y.*

$$|x + y| \leq |x| + |y|. \tag{8.437}$$

Proof. We let x and y be arbitrary in $\mathbb{R}$ and observe first that the inequalities

$$-|x| \leq x \wedge x \leq |x| \tag{8.438}$$

$$-|y| \leq y \wedge y \leq |y| \tag{8.439}$$

hold with Exercise 8.27. Applying the Monotony Law for $\leq_{\mathbb{R}}$ and $+_{\mathbb{R}}$ to $-|x| \leq x$ (adding $-|y|$) and then to $-|y| \leq y$ (adding x), we obtain

$$-|x| - |y| \leq x - |y|$$

$$-|y| + x \leq y + x$$

and therefore

$$-|x| - |y| \leq x + y \tag{8.440}$$

with the commutativity of $+_{\mathbb{R}}$ and the transitivity of $\leq_{\mathbb{R}}$. Similarly, applying the Monotony Law for $\leq_{\mathbb{R}}$ and $+_{\mathbb{R}}$ in (8.438) to $x \leq |x|$ (adding y) and in (8.439) to $y \leq |y|$ (adding $|x|$) we obtain

$$x + y \leq |x| + y$$

$$y + |x| \leq |y| + |x|$$

and therefore evidently

$$x + y \leq |x| + |y|. \tag{8.441}$$

Then, the combination of (8.440) and (8.441) reads

$$-|x| - |y| \leq x + y \leq |x| + |y|.$$

We may write these inequalities equivalently as

$$-(|x| + |y|) \leq x + y \leq (|x| + |y|), \tag{8.442}$$

using the Sign Law (6.52) in connection with the commutativity of $+_{\mathbb{R}}$. Now, the fact that $|x|$ and $|y|$ are elements of the codomain $\mathbb{R}_+^0$ of $|.|_{\mathbb{R}}$ implies $0 \leq |x|$ and $0 \leq |y|$ by definition of the set of nonnegative real numbers. These inequalities clearly imply $0 \leq |x| + |y|$ with the Additivity of $\leq$-inequalities, so that $|x| + |y|$ is evidently also an element of $\mathbb{R}_+^0$. We may now use the facts $x + y \in \mathbb{R}$ and $|x| + |y| \in \mathbb{R}_+^0$ to infer from (8.442) the truth of (8.437) by means of Exercise 8.28. Since x and y were arbitrary, we therefore conclude that the theorem is true. □

Theorem 8.63 (Triangle Inequality for the absolute value function on $\mathbb{R}$). *The following inequality holds for any real numbers x, y and z.*

$$|x-y| \leq |x-z| + |z-y|. \tag{8.443}$$

Proof. Letting $x, y, z \in \mathbb{R}$ be arbitrary, we define $x' = x - z$ and $y' = z - y$, so that evidently

$$x = x + 0 = x + (-z + z) = (x - z) + z = x' + z$$

and

$$-y = -y + 0 = -y + (z - z) = (z - y) - z = y' - z,$$

and therefore

$$x - y = x + (-y) = (x' + z) + (y' - z) = x' + y'.$$

Since the inequality

$$|x' + y'| \leq |x'| + |y'|,$$

is true because of Theorem 8.62, it follows through substitutions based on $x' = x - z$ and $y' = z - y$ that

$$[|x - y| =] \quad |(x - z) + (z - y)| \leq |x'| + |y'|,$$

so that the inequality (8.443) follows to be true. As x, y and z were arbitrary, we conclude that the theorem is true. □

Proposition 8.64. *The absolute difference function $d_{\mathbb{R}}$ on $\mathbb{R}$ is a metric (on $\mathbb{R}$).*

Proof. We let $x, y, z \in \mathbb{R}$ be arbitrary. Then, the value $d_{\mathbb{R}}(x, y)$ is an element of the codomain $\mathbb{R}_+^0$ of $d_{\mathbb{R}}$ (according to Corollary 3.147), which implies

$$d_{\mathbb{R}}(x, y) \geq 0 \tag{8.444}$$

with the definition of the set of nonnegative real numbers.

To prove that $d_{\mathbb{R}}$ satisfies the equivalence in (8.432), we notice that $d_{\mathbb{R}}(x, y) = 0$ is equivalent to $|x - y| = 0$ (by definition of $d_{\mathbb{R}}$) and then also to $x - y = 0$ due to the positive-definiteness of $|.|_{\mathbb{R}}$. Then, on the one hand, $x - y = 0$ evidently implies

$$x = x + 0 = x + (-y + y) = (x - y) + y = 0 + y = y,$$

thus $x = y$; on the other hand, $x = y$ implies

$$x - y = x - x = 0,$$

thus $x - y = 0$. Therefore, $x - y = 0$ is equivalent to $x = y$, which implies with the previously established equivalence of $d_{\mathbb{R}}(x, y) = 0$ and $x - y = 0$ also the equivalence of $d_{\mathbb{R}}(x, y) = 0$ and $x = y$, that is,

$$d_{\mathbb{R}}(x, y) = 0 \;\Leftrightarrow\; x = y. \tag{8.445}$$

Next, we observe the truth of the equations

$$d_{\mathbb{R}}(x, y) = |x - y| = |-(y - x)| = |y - x| = d_{\mathbb{R}}(y, x)$$

in light of (6.53) and (8.407), so that

$$d_{\mathbb{R}}(x, y) = d_{\mathbb{R}}(y, x). \tag{8.446}$$

Finally, we recall that the Triangle Inequality $|x-y| \leq |x-z|+|z-y|$ holds according to (8.443); as the definition of $d_{\mathbb{R}}$ gives the equation $d_{\mathbb{R}}(x, y) = |x-y|$, $d_{\mathbb{R}}(x, z) = |x-z|$ and $d_{\mathbb{R}}(z, y) = |z-y|$, we obtain after substitutions

$$d_{\mathbb{R}}(x, y) \leq d_{\mathbb{R}}(x, z) + d_{\mathbb{R}}(z, y). \tag{8.447}$$

Because x, y and z are arbitrary, we therefore conclude that $d_{\mathbb{R}}$ satisfies all of the properties of a metric on $\mathbb{R}$. $\square$

Proposition 8.65. *It is true for any metric space (X, d_X) and any subset $A \subseteq X$ that*

$$(A, d_A) = (A, d_X \restriction [A \times A]) \tag{8.448}$$

is also a metric space.

Proof. Letting (X, d_X) be an arbitrary metric space and A an arbitrary subset of X, we obtain the restriction $d_A = d_X \restriction (A \times A)$ as a function $d_A : A \times A \to \mathbb{R}$ by means of Proposition 3.164, using the fact that $A \subseteq X$ implies $A \times A \subseteq X \times X$ with Proposition 3.8. This restriction satisfies then

$$\forall z \, (z \in A \times A \Rightarrow d_A(z) = d_X(z)) \tag{8.449}$$

in view of Corollary 3.567. Next, we establish

$$\forall a, b \, (a, b \in A \Rightarrow d_A(a, b) = d_X(a, b)), \tag{8.450}$$

letting $a, b \in A$ be arbitrary. Let us define the ordered pair $z = (a, b)$, which is by definition element of the Cartesian product $A \times A$. Then, (8.449) gives $d_A((a, b)) = d_X((a, b))$, which we may also write as $d_A(a, b) = d_X(a, b)$. Since z is arbitrary, we may therefore conclude that the universal sentence (8.450) is true. We now verify that d_A is a metric on A. For this purpose, we let $x, y, z \in A$ be arbitrary, so that the assumed inclusion $A \subseteq X$ yields $x, y, z \in X$ by definition of a subset. Since d_X is a metric on X, we find

$$\begin{aligned} d_X(x,y) &\geq 0, \\ d_X(x,y) = 0 &\Leftrightarrow x = y, \\ d_X(x,y) &= d_X(y,x), \\ d_X(x,y) &\leq d_X(x,z) + d_X(z,y). \end{aligned}$$

Furthermore, $x, y, z \in A$ yields $d_X(x,y) = d_A(x,y)$, $d_X(y,x) = d_A(y,x)$, $d_X(x,z) = d_A(x,z)$ and $d_X(z,y) = d_A(z,y)$ with (8.450), so that substitutions into the previous four findings give us

$$\begin{aligned} d_A(x,y) &\geq 0, \\ d_A(x,y) = 0 &\Leftrightarrow x = y, \\ d_A(x,y) &= d_A(y,x), \\ d_A(x,y) &\leq d_A(x,z) + d_A(z,y). \end{aligned}$$

Since x, y and z are arbitrary, we may therefore conclude that (A, d_A) satisfies all four properties (8.431) – (8.434) of a metric space. Because (X, d_X) and A were also arbitrary, the proposed universal sentence follows to be true. □

We continue now with some additional basic definitions.

Note 8.20. For any metric space (X, d), any point x_0 and any positive real number r, we see in light of the Axiom of Specification and the Equality Criterion for sets that

a) there exists a unique set $B_d(x_0, r)$ consisting of all the points x whose distance to x_0 is less than r, in the sense that

$$\forall x\,(x \in B_d(x_0, r) \Leftrightarrow [x \in X \,\wedge\, d(x, x_0) < r]). \tag{8.451}$$

b) there exists a unique set $S_d(x_0, r)$ consisting of all the points x whose distance to x_0 equals r, in the sense that

$$\forall x\,(x \in S_d(x_0, r) \Leftrightarrow [x \in X \,\wedge\, d(x, x_0) = r]). \tag{8.452}$$

Exercise 8.35. Verify for any metric space (X, d), any point x_0 and any positive real number r that

a) there exists a unique set $B_d(x_0, r)$ consisting of all the points x whose distance to x_0 is less than r, in the sense that

$$\forall x\,(x \in B_d(x_0, r) \Leftrightarrow d(x, x_0) < r). \tag{8.453}$$

b) there exists a unique set $S_d(x_0, r)$ consisting of all the points x whose distance to x_0 equals r, in the sense that

$$\forall x\,(x \in S_d(x_0, r) \Leftrightarrow d(x, x_0) = r). \tag{8.454}$$

(Hint: Apply (3.28).)

Definition 8.34 (Open ball, sphere, center, radius). For any metric space (X, d), any point x_0 and any positive real number r

(1) we call the set
$$B_d(x_0, r) = \{x : d(x, x_0) < r\} \tag{8.455}$$
the *open ball* with *center* x_0 and *radius* r (with respect to d).

(2) we call the set
$$S_d(x_0, r) = \{x : d(x, x_0) = r\} \tag{8.456}$$
the *sphere* with *center* x_0 and *radius* r (with respect to d).

Exercise 8.36. Show for any metric space (X, d) that

a) there exists a unique set (system)

$$\mathcal{B}_X^{(d)} = \{B_d(x_0, r) \,|\, x_0 \in X \,\wedge\, r >_{\mathbb{R}} 0\} \tag{8.457}$$

consisting of all the open balls in $\mathcal{P}(X)$ with respect to d_X.

b) Then, this set satisfies

$$\forall A\,(A \in \mathcal{B}_X^{(d)} \Leftrightarrow \exists x_0, r\,(x_0 \in X \wedge r >_{\mathbb{R}} 0 \wedge B_d(x_0, r) = A)). \tag{8.458}$$

Definition 8.35 (Set of open balls). For any metric space (X, d), we call
$$\mathcal{B}_X^{(d)} = \{B_d(x_0, r) \,|\, x_0 \in X \,\wedge\, r >_{\mathbb{R}} 0\}$$
the *set of open balls* (with respect to d).

Corollary 8.66. *It is true for any metric space (X, d) that every open ball contains its center, that is,*

$$\forall x_0, r\,([x_0 \in X \,\wedge\, r \in \mathbb{R}_+] \Rightarrow x_0 \in B_d(x_0, r)). \tag{8.459}$$

Proof. We take an arbitrary metric space (X, d), an arbitrary point x_0 in X, and an arbitrary positive real number r. Then, due to Property 2 of a metric, the evident truth of $x_0 = x_0$ implies $d(x_0, x_0) = 0$. Since the assumption $r \in \mathbb{R}_+$ implies $0 < r$ with the definition of the set of positive real numbers, we obtain with the preceding equation $d(x_0, x_0) < r$ via substitution. In conjunction with $x_0 \in X$, this gives $x_0 \in B_d(x_0, r)$ by virtue of the specification of an open ball in (8.451). Thus, the implication in (8.459) holds. Since (X, d), x_0 and r are arbitrary, we therefore conclude that the stated universal sentence is true. □

Note 8.21. The preceding corollary clearly shows that every open ball in any metrix space (X, d) is nonempty, that is,

$$\forall x_0, r\,([x_0 \in X \,\wedge\, r \in \mathbb{R}_+] \Rightarrow B_d(x_0, r) \neq \emptyset). \tag{8.460}$$

For real numbers, the concepts of an open ball and of a nonempty open interval coincide.

Proposition 8.67. *The set of open balls with respect to the metric $d_{\mathbb{R}}$ and the set of open intervals in $\mathbb{R}$ without the empty set are identical, that is,*

$$\mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})} = \{(a, b) : a, b \in \mathbb{R}\} \setminus \{\emptyset\}. \tag{8.461}$$

Proof. We apply the Equality Criterion for sets and verify accordingly

$$\forall A \left(A \in \mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})} \Leftrightarrow A \in \{(a, b) : a, b \in \mathbb{R}\} \setminus \{\emptyset\}\right), \tag{8.462}$$

letting A be an arbitrary set. To establish the first part ('⇒') of the equivalence, we assume $A \in \mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}$ to be true. In view of (8.458), it is then true that $A = B_{d_{\mathbb{R}}}(\bar{x}_0, \bar{r})$ for a particular center $\bar{x}_0$ and radius $\bar{r}$. We now show that this open ball is identical with the open interval $(\bar{x}_0 - \bar{r}, \bar{x}_0 + \bar{r})$. For this purpose, we apply again the Equality Criterion for sets and demonstrate the truth of

$$\forall x\ (x \in B_{d_{\mathbb{R}}}(\bar{x}_0, \bar{r}) \Leftrightarrow x \in (\bar{x}_0 - \bar{r}, \bar{x}_0 + \bar{r}))\,. \tag{8.463}$$

Letting x be arbitrary, we observe the truth of the equivalences

$$\begin{aligned} x \in B_{d_{\mathbb{R}}}(\bar{x}_0, \bar{r}) &\Leftrightarrow d_{\mathbb{R}}(x, \bar{x}_0) < \bar{r} \\ &\Leftrightarrow |x - \bar{x}_0| < \bar{r} \\ &\Leftrightarrow -\bar{r} < x - \bar{x}_0 < \bar{r} \\ &\Leftrightarrow \bar{x}_0 - \bar{r} < x < \bar{x}_0 + \bar{r} \\ &\Leftrightarrow x \in (\bar{x}_0 - \bar{r}, \bar{x}_0 + \bar{r}) \end{aligned}$$

in light of the (8.453), the definition of the metric $d_{\mathbb{R}}$, (8.412) and the fact that the radius $\bar{r}$ is nonnegative, the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$, and (3.379). Consequently, the equivalence in (8.463) holds, and since x

is arbitrary, we infer from this the truth of the universal sentence. This implies

$$[A =] \quad B_{d_{\mathbb{R}}}(\bar{x}_0, \bar{r}) = (\bar{x}_0 - \bar{r}, \bar{x}_0 + \bar{r}),$$

and as $\bar{x}_0 - \bar{r}$ and $\bar{x}_0 + \bar{r}$ are both real numbers, we therefore find $A \in \{(a,b) : a, b \in \mathbb{R}\}$ with (3.385). Moreover, since the open ball is nonempty (see Note 8.21), we have $A \neq \emptyset$ and therefore $A \notin \{\emptyset\}$ by virtue of (2.169). These two findings show that A is in the set difference of $\{(a,b) : a, b \in \mathbb{R}\}$ and $\{\emptyset\}$, as desired.

To establish the second part ('$\Leftarrow$') of the equivalence in (8.462), we now assume $A \in \{(a,b) : a, b \in \mathbb{R}\} \setminus \{\emptyset\}$ to be true; we thus have $A \in \{(a,b) : a, b \in \mathbb{R}\}$ and $A \neq \emptyset$. Consequently, there are particular real numbers $\bar{a}$ and $\bar{b}$ such that $A = (\bar{a}, \bar{b})$ due to (3.385). Thus, $(\bar{a}, \bar{b}) \neq \emptyset$ holds, and this implies $\bar{a} < \bar{b}$ with (3.382) and the Double Negation Law. Noting that this inequality yields $\frac{\bar{b}-\bar{a}}{2} > 0$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$ and with the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$, we may show that the open interval $(\bar{a}, \bar{b})$ is identical with the open ball $B_{d_{\mathbb{R}}}\left(\frac{\bar{a}+\bar{b}}{2}, \frac{\bar{b}-\bar{a}}{2}\right)$. To do this, we apply once again the Equality Criterion for sets and prove

$$\forall x \left(x \in (\bar{a}, \bar{b}) \Leftrightarrow x \in x \in B_{d_{\mathbb{R}}}\left(\frac{\bar{a}+\bar{b}}{2}, \frac{\bar{b}-\bar{a}}{2}\right)\right). \tag{8.464}$$

Taking an arbitrary x, we evidently obtain the true equivalences

$$\begin{aligned} x \in (\bar{a}, \bar{b}) &\Leftrightarrow \bar{a} < x < \bar{b} \\ &\Leftrightarrow -\frac{\bar{b}-\bar{a}}{2} < x - \frac{\bar{a}+\bar{b}}{2} < \frac{\bar{b}-\bar{a}}{2} \\ &\Leftrightarrow \left| x - \frac{\bar{a}+\bar{b}}{2} \right| < \frac{\bar{b}-\bar{a}}{2} \\ &\Leftrightarrow d_{\mathbb{R}}\left(x, \frac{\bar{a}+\bar{b}}{2}\right) < \frac{\bar{b}-\bar{a}}{2} \\ &\Leftrightarrow x \in B_{d_{\mathbb{R}}}\left(\frac{\bar{a}+\bar{b}}{2}, \frac{\bar{b}-\bar{a}}{2}\right) \end{aligned}$$

with the previously used arguments and the laws of fractional calculus. These give us the equivalence in (8.464), where x is arbitrary, so that the universal sentence (8.464) holds as well. Therefore,

$$[A =] \quad (\bar{a}, \bar{b}) = B_{d_{\mathbb{R}}}\left(\frac{\bar{a}+\bar{b}}{2}, \frac{\bar{b}-\bar{a}}{2}\right),$$

which in turn implies $A \in \mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}$, as desired. This completes the proof of the equivalence in (8.462), in which A is arbitrary, so that the universal sentence (8.462) follows also to be true. This in turn completes the proof of the proposed equality (8.461). □

Exercise 8.37. Prove for any metric space (X, d) and any two open balls with center x_0 that the one with the greater radius includes the other one, that is,

$$\forall x_0, r, r' \left([x_0 \in X \wedge r, r' \in \mathbb{R}_+] \Rightarrow [r \leq r' \Rightarrow B_d(x_0, r) \subseteq B_d(x_0, r')]\right). \tag{8.465}$$

(Hint: Apply the Transitivity Formula for $<$ and $\leq$.)

Proposition 8.68. *For any metric space (X, d), any point x_0, any positive real number r and any element y in the open ball $B_d(x_0, r)$, there exists an open ball with center y and radius $r' = r - d(x_0, y)$ which is included in the open ball $B_d(x_0, r)$.*

Proof. We let (X, d) be an arbitrary metric space, x_0 an arbitrary point in X, r an arbitrary element of $\mathbb{R}_+$, and y an arbitrary point in $B_d(x_0, r)$. The latter means $d(y, x_0) < r$ by definition of an open ball, and this inequality implies $r - d(y, x_0) > 0$ with the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$. Thus, the real number $r' = r - d(y, x_0)$ is a positive natural number, so that the open ball $B_d(y, r')$ with center y and radius r' is defined. We may now establish the inclusion $B_d(y, r') \subseteq B_d(x_0, r)$, by verifying

$$\forall z \left(z \in B_d(y, r') \Rightarrow z \in B_d(x_0, r)\right). \tag{8.466}$$

Letting z be arbitrary and assuming $z \in B_d(y, r')$, it follows with the definition of an open ball that $d(z, y) < r'$ and therefore $d(y, z) < r - d(x_0, y)$ (applying substitution and the symmetry property of a metric). The latter inequality in turn implies $d(x_0, y) + d(y, z) < r$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$. This together with the Triangle Inequality $d(x_0, z) \leq d(x_0, y) + d(y, z)$ (which holds by virtue of Property 4 of a metric) implies $d(z, x_0) < r$ with the Transitivity Formula for $\leq$ and $<$ and with the symmetry property of the metric d. Consequently, $z \in B_d(x_0, r)$ holds by definition of an open ball. Since z was arbitrary, we conclude that the universal sentence (8.466) is true, which means that the open ball $B_d(y, r')$ is indeed included in the open ball $B_d(x_0, r)$. As y, x_0, r and (X, d) were initially arbitrary, we may further conclude that the proposition holds. □

We already encountered the concept of the limit of an increasing/decreasing sequence defined by means of a partial ordering. We introduce now a new concept of limit that applies also to sequences that are not necessarily increasing/decreasing.

Definition 8.36 (Limit of a sequence with respect to a metric, convergent sequence in a metric space). For any metric space (X, d_X), any sequence $(a_n)_{n\in\mathbb{N}_+}$ in X and any $L \in X$, we say that L is a *limit* of $(a_n)_{n\in\mathbb{N}_+}$ (with respect to d_X), symbolically

$$L = \lim_{n\to\infty} a_n = \overset{d_X}{\lim_{n\to\infty}} a_n, \tag{8.467}$$

iff for any positive real number ε there exists an $N \in \mathbb{N}_+$ such that for any positive natural number n greater than N the distance of a_n to L is less than ε, i.e., iff L satisfies

$$\forall \varepsilon\,(\varepsilon \in \mathbb{R}_+ \Rightarrow \exists N\,(N \in \mathbb{N}_+ \,\wedge\, \forall n\,(n \geq_{\mathbb{N}_+} N \Rightarrow d_X(a_n, L) < \varepsilon))). \tag{8.468}$$

We then say that $(a_n)_{n\in\mathbb{N}_+}$ converges to L (with respect to d_X). Furthermore, we say for any metric space (X, d_X) that a sequence $(a_n)_{n\in\mathbb{N}_+}$ in X is *convergent* (with respect to d_X) iff there exists an element in X which is a limit of $(a_n)_{n\in\mathbb{N}_+}$ (with respect to d_X).

Note 8.22. Applying the metric $d_{\mathbb{R}}$ in the context of real sequences, we may say that a sequence $(a_n)_{n\in\mathbb{N}_+}$ in $\mathbb{R}$ converges to L iff

$$\forall \varepsilon\,(\varepsilon \in \mathbb{R}_+ \Rightarrow \exists N\,(N \in \mathbb{N}_+ \,\wedge\, \forall n\,(n \geq_{\mathbb{N}} N \Rightarrow |a_n - L| < \varepsilon))). \tag{8.469}$$

Theorem 8.69 (Open-ball criterion for convergence of a sequence). *For any metric space (X, d), any sequence $(a_n)_{n\in\mathbb{N}_+}$ in X and any $L \in X$, it is true that L is a limit of $(a_n)_{n\in\mathbb{N}_+}$ iff every open ball with center L contains all except for finitely many terms of that sequence, i.e. iff*

$$\begin{aligned} L = \lim_{n\to\infty} a_n \Leftrightarrow \forall r\,(r \in \mathbb{R}_+ \Rightarrow \exists M\,(M \subseteq \mathbb{N}_+ \,\wedge\, M \text{ is finite} \\ \wedge\, \forall n\,(n \in \mathbb{N}_+ \setminus M \Rightarrow a_n \in B_d(L, r)))). \end{aligned} \tag{8.470}$$

Proof. We let (X, d) be an arbitrary metric space, $(a_n)_{n\in\mathbb{N}_+}$ an arbitrary sequence in X, and L an arbitrary element of X. To prove the first part ('$\Rightarrow$') of the equivalence, we assume that L is a limit of the sequence and let then r be an arbitrary element of $\mathbb{R}_+$. By definition of a limit, $r \in \mathbb{R}_+$ implies that there exists an element of $\mathbb{N}_+$, say $\bar{N}$, such that

$$\forall n\,(n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow d(a_n, L) < r) \tag{8.471}$$

Let us define the initial segment $\bar{M} = \{1, \ldots, \bar{N} - 1\}$, which is evidently empty in case of $\bar{N} = 1$. Since $\bar{N} - 1 \in \mathbb{N}$ is clearly true, we have $\bar{M} \subseteq \mathbb{N}_+$ by virtue of (4.240). Furthermore, $\bar{M}$ is a finite set in view of Exercise 4.34. Letting now $n \in \mathbb{N}_+ \setminus \bar{M}$ be arbitrary, we find $n \in \mathbb{N}_+$ and $\neg n \in \bar{M}$ with the definition of a set difference, and thus $\neg n \in \{1, \ldots, \bar{N} - 1\}$. We now prove $\bar{N} \leq_{\mathbb{N}_+} n$ by cases, recalling the fact $\bar{N} - 1 \geq_{\mathbb{N}} 0$ from (4.187), so that $\bar{N} - 1 >_{\mathbb{N}} 0$ or $\bar{N} - 1 = 0$ holds.

In case of $\bar{N} - 1 >_{\mathbb{N}} 0$, we find $1 \leq_{\mathbb{N}} \bar{N} - 1$ with (4.157), which clearly shows that $\bar{N} - 1 \in \mathbb{N}_+$ holds. Then, $\neg n \in \{1, \ldots, \bar{N} - 1\}$ implies $\neg n \leq_{\mathbb{N}_+} \bar{N} - 1$ with the Characterization of initial segments and moreover $\bar{N} - 1 <_{\mathbb{N}_+} n$ with the Negation Formula for $\leq$. This inequality further implies $\bar{N} \leq_{\mathbb{N}_+} n$ with (4.270). In the other case $\bar{N} - 1 = 0$, which evidently gives $\bar{N} = 1$, we note that $n \in \mathbb{N}_+$ implies $n \geq_{\mathbb{N}_+} 1$ and therefore the desired $n \geq_{\mathbb{N}_+} \bar{N}$ via substitution. Thus, the proof by cases is complete.

This finding implies $d(a_n, L) < r$ with (8.471), so that $a_n \in B_d(L, r)$ holds by definition of an open ball. The previous findings demonstrate the truth of the existential sentence in (8.470), and since r was arbitrary, the right-hand side of the equivalence (8.470) follows to be also true.

To prove the second part ('$\Leftarrow$') of the proposed equivalence, we now assume the right-hand side. To show that L is then a limit of the sequence, we prove (8.468). For this purpose, we let ε be arbitrary in $\mathbb{R}_+$, and we demonstrate the truth of the existential sentence

$$\exists N\,(N \in \mathbb{N}_+ \wedge \forall n\,(n \geq_{\mathbb{N}_+} N \Rightarrow d(a_n, L) < \varepsilon)). \tag{8.472}$$

$\varepsilon \in \mathbb{R}_+$ implies with the assumed right-hand side that there exists a finite subset of $\mathbb{N}_+$, say $\bar{M}$, such that

$$\forall n\,(n \in \mathbb{N}_+ \setminus \bar{M} \Rightarrow a_n \in B_d(L, \varepsilon)). \tag{8.473}$$

We consider now the two cases $\bar{M} = \emptyset$ and $\bar{M} \neq \emptyset$ to prove (8.472).

In case of $\bar{M} = \emptyset$, we obtain $\mathbb{N}_+ \setminus \bar{M} = \mathbb{N}_+$ with (2.102), so that (8.473) yields

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow a_n \in B_d(L, \varepsilon)). \tag{8.474}$$

This allows us to prove

$$\forall n\,(n \geq_{\mathbb{N}_+} 1 \Rightarrow d(a_n, L) < \varepsilon). \tag{8.475}$$

Letting $n \geq_{\mathbb{N}_+} 1$ be arbitrary, we thus have evidently $n \in \mathbb{N}_+$, which in turn implies $a_n \in B_d(L, \varepsilon)$ with (8.474). By definition of an open ball, we therefore find $d(a_n, L) < \varepsilon$, as desired. As n was arbitrary, we therefore conclude that the universal sentence (8.475) is true, and this demonstrates the truth of the existential sentence (8.472)

In the other case $\bar{M} \neq \emptyset$, we observe the truth of the inclusions $\bar{M} \subseteq \mathbb{N}_+ \subseteq \mathbb{N}$ in light of (2.308), so that $\bar{M}$ is a finite, nonempty subset of $\mathbb{N}$. Therefore, the maximum of $\bar{M}$ exists (see Corollary 4.119), which constitutes an upper bound for that set in that set. Thus, the previously established inclusion $\bar{M} \subseteq \mathbb{N}_+$ yields $\max \bar{M} \in \mathbb{N}_+$. We now show that the positive natural number $\max \bar{M} + 1$ satisfies

$$\forall n \, (n \geq_{\mathbb{N}_+} \max \bar{M} + 1 \Rightarrow d(a_n, L) < \varepsilon), \tag{8.476}$$

letting $n \geq_{\mathbb{N}_+} \max \bar{M} + 1$ be arbitrary. Due to the evident fact $\max \bar{M} \leq_{\mathbb{N}_+} \max \bar{M} + 1$, we clearly see that $\max \bar{M} + 1$ is not in $\bar{M}$ (since it exceeds the upper bound $\max \bar{M}$ for $\bar{M}$), i.e., $\neg \max \bar{M} + 1 \in \bar{M}$. In conjunction with the fact $\max \bar{M} + 1 \in \mathbb{N}_+$, this implies $\max \bar{M} + 1 \in \mathbb{N}_+ \setminus \bar{M}$. This in turn gives us $a_n \in B_d(L, \epsilon)$ with (8.473), so that evidently $a_n \in B_d(L, \epsilon)$. As n was arbitrary, we infer from this the truth of the universal sentence (8.476), and therefore the truth of the existential sentence (8.472).

Thus, the proof by cases is complete. Since ε was arbitrary, we therefore conclude that L is a limit of the given sequence, so that the proof of the equivalence is also complete. As (X, d), $(a_n)_{n \in \mathbb{N}_+}$ and L were initially also arbitrary, we finally conclude that the theorem holds. □

Theorem 8.70 (Uniqueness of the limit of a sequence with respect to a metric). *The following sentence holds for any metric space (X, d) and any sequence $(a_n)_{n \in \mathbb{N}_+}$ in X such that there exists an element L in X to which the sequence converges. There exists no limit L' of $(a_n)_{n \in \mathbb{N}_+}$ (in X) with $L' \neq L$.*

Proof. We let (X, d) be an arbitrary metric space, $(a_n)_{n \in \mathbb{N}_+}$ an arbitrary sequence in X converging to some $L \in X$. The sentence to be proven may evidently be written as

$$\neg \exists L' \, ([L' \in X \, \wedge \, L' = \lim_{n \to \infty} a_n] \, \wedge \, L' \neq L),$$

which is equivalent to

$$\forall L' \, ([L' \in X \, \wedge \, L' = \lim_{n \to \infty} a_n] \Rightarrow L' = L),$$

because of (1.81). Letting now L' be arbitrary in X with $L' = \lim_{n \to \infty} a_n$, we have by definition of a limit of a sequence that

$$\forall \varepsilon \, (\varepsilon \in \mathbb{R}_+ \Rightarrow \exists N_1 \, (N_1 \in \mathbb{N}_+ \, \wedge \, \forall n \, (n \geq_{\mathbb{N}_+} N_1 \Rightarrow d(a_n, L) < \varepsilon))), \tag{8.477}$$
$$\forall \varepsilon \, (\varepsilon \in \mathbb{R}_+ \Rightarrow \exists N_2 \, (N_2 \in \mathbb{N}_+ \, \wedge \, \forall n \, (n \geq_{\mathbb{N}_+} N_2 \Rightarrow d(a_n, L') < \varepsilon))) \tag{8.478}$$

are true. We first demonstrate that

$$\forall \varepsilon \, (\varepsilon \in \mathbb{R}_+ \Rightarrow d(L, L') < \varepsilon) \tag{8.479}$$

holds. For this purpose, we let ε be arbitrary in $\mathbb{R}_+$, so that evidently $0.5 \cdot \varepsilon \in \mathbb{R}_+$; due to (8.477) – (8.478) there exist then elements of $\mathbb{N}_+$, say $\bar{N}_1$ and $\bar{N}_2$, such that

$$\forall n\,(n \geq_{\mathbb{N}_+} \bar{N}_1 \Rightarrow d(a_n, L) < \varepsilon), \tag{8.480}$$

$$\forall n\,(n \geq_{\mathbb{N}_+} \bar{N}_2 \Rightarrow d(a_n, L') < \varepsilon). \tag{8.481}$$

Since $<_{\mathbb{N}_+}$ is a linear ordering, it follows with Proposition 3.113 that $N = \max^{\leq_{\mathbb{N}_+}} \{\bar{N}_1, \bar{N}_2\}$ exists uniquely and is an upper bound for that set, so that $N \geq_{\mathbb{N}_+} \bar{N}_1$ and $N \geq_{\mathbb{N}_+} \bar{N}_2$ hold; therefore, (8.480) – (8.481) give us $d(a_N, L) < 0.5 \cdot \varepsilon$ and $d(a_N, L') < 0.5 \cdot \varepsilon$. Since the left- and right-hand sides of these inequalities are evidently elements of the set of nonnegative real numbers, which defines an ordered elementary domain, we may apply the Additivity of $<$-inequalities to obtain

$$d(a_N, L) + d(a_N, L') < 0.5 \cdot \varepsilon + 0.5 \cdot \varepsilon \quad [= \varepsilon]$$

Applying Property 3 of a metric, we get

$$d(L, a_N) + d(a_N, L') < \varepsilon.$$

Let us now observe that

$$d(L, L') \leq d(L, a_N) + d(a_N, L') \quad [< \varepsilon]$$

holds with Property 4 of a metric, and therefore $d(L, L') < \varepsilon$ because of the Transitivity Formula for $\leq$ and $<$. Since ε was arbitrary, we conclude that (8.479) is true.

We now prove via contradiction that (8.479) implies $d(L, L') = 0$. Assuming for this purpose $d(L, L') \neq 0$, we then see that $d(L, L') > 0$ holds by Property 1 of a metric in connection with the definition of an induced irreflexive partial ordering. Now, the conjunction of $0 < d(L, L')$ with the evident fact $0 < 0.5$ and $0.5 < 1$, respectively, implies $0 < 0.5 \cdot d(L, L')$ and $0.5 \cdot d(L, L') < d(L, L')$ with the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$. Thus, the former inequality implies $0.5 \cdot d(L, L') \in \mathbb{R}_+$. With this finding, (8.479) gives $d(L, L') < 0.5 \cdot d(L, L')$, which contradicts the previously established inequality $0.5 \cdot d(L, L') < d(L, L')$ in view of the Characterization of comparability, completing the proof of $d(L, L') = 0$. This equation finally implies $L = L'$ with Property 2 of a metric. □

Lemma 8.71. *The following sentences are true for any real sequence $s = (x_n)_{n \in \mathbb{N}_+}$ and any real number L.*

a) If $(x_n)_{n\in\mathbb{N}_+}$ converges increasingly to L, then $(x_n)_{n\in\mathbb{N}_+}$ converges to L with respect to the metric $d_{\mathbb{R}}$, i.e.

$$\lim_{n\to\infty}^{\leq_{\mathbb{R}}} x_n = L \ \Rightarrow\ \lim_{n\to\infty}^{d_{\mathbb{R}}} x_n = L. \tag{8.482}$$

b) If $(x_n)_{n\in\mathbb{N}_+}$ converges decreasingly to L, then $(x_n)_{n\in\mathbb{N}_+}$ converges to L with respect to the metric $d_{\mathbb{R}}$, i.e.

$$\lim_{n\to\infty}^{\leq_{\mathbb{R}}} x_n = L \ \Rightarrow\ \lim_{n\to\infty}^{d_{\mathbb{R}}} x_n = L. \tag{8.483}$$

Proof. We let $s = (x_n)_{n\in\mathbb{N}_+}$ be an arbitrary sequence in $\mathbb{R}$ and L an arbitrary real number. Concerning a), we assume that s converges increasingly to $L = \lim_{n\to\infty}^{\leq_{\mathbb{R}}} x_n$ and show that

$$\forall \varepsilon\,(\varepsilon \in \mathbb{R}_+ \Rightarrow \exists N\,(N \in \mathbb{N}_+ \,\wedge\, \forall n\,(n \geq_{\mathbb{N}_+} N \Rightarrow d_{\mathbb{R}}(x_n, L) < \varepsilon))) \tag{8.484}$$

is satisfied. To prove that universal sentence, we let $\bar{\varepsilon} \in \mathbb{R}_+$ be arbitrary. It then follows with the Limit Criterion for increasingly convergent real sequences that there exists an element of $\mathbb{N}_+$, say $\bar{N}$, such that

$$\forall n\,(n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow 0 \leq L - x_n < \bar{\varepsilon}) \tag{8.485}$$

holds. We now demonstrate that

$$\forall n\,(n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow d_{\mathbb{R}}(x_n, L) < \bar{\varepsilon}) \tag{8.486}$$

is true. Letting $n \geq_{\mathbb{N}_+} \bar{N}$ be arbitrary, we see from (8.485) that $0 \leq L - x_n < \bar{\varepsilon}$ holds; here, the first inequality implies $|L - x_n| = L - x_n$ with (8.409), and therefore with the second inequality $|L - x_n| < \bar{\varepsilon}$. Observing that

$$|L - x_n| = d_{\mathbb{R}}(L, x_n) = d_{\mathbb{R}}(x_n, L)$$

holds with the definition and the symmetry of the metric $d_{\mathbb{R}}$, the preceding inequality implies $d_{\mathbb{R}}(x_n, L) < \bar{\varepsilon}$. As n was arbitrary, we therefore conclude that (8.486) holds. This finding demonstrates the truth of the existential sentence

$$\exists N\,(N \in \mathbb{N}_+ \,\wedge\, \forall n\,(n \geq_{\mathbb{N}_+} N \Rightarrow d_{\mathbb{R}}(x_n, L) < \bar{\varepsilon})).$$

Here, $\bar{\varepsilon}$ was also arbitrary, so that the universal sentence (8.484) follows also to be true. Thus, $(x_n)_{n\in\mathbb{N}_+}$ by definition converges to L with respect to the metric $d_{\mathbb{R}}$, so that the proof of Part a) is now complete. Part b) can be proved similarly. Because $(x_n)_{n\in\mathbb{N}_+}$ was initially arbitrary, we therefore conclude that the lemma holds. □

Exercise 8.38. Prove Part b) of Lemma 8.71.

The assumptions of the preceding lemma may be slightly generalized.

Theorem 8.72 (Monotone Convergence Theorem for real sequences). *The following implications hold for any real sequence $s = (x_n)_{n \in \mathbb{N}_+}$.*

a) *If that sequence is increasing and bounded from above, then it converges with respect to the metric $d_{\mathbb{R}}$ to the supremum of its range, i.e.*

$$\forall s \, ([s \in \mathbb{R}^{\mathbb{N}_+} \wedge s \text{ is increasing } \wedge s \text{ is bounded from above}] \Rightarrow \overset{d_{\mathbb{R}}}{\lim_{n \to \infty}} x_n = \overset{\leq_{\mathbb{R}}}{\sup} \operatorname{ran}(s)). \tag{8.487}$$

b) *If that sequence is decreasing and bounded from below, then it converges with respect to the metric $d_{\mathbb{R}}$ to the infimum of its range, i.e.*

$$\forall s \, ([s \in \mathbb{R}^{\mathbb{N}_+} \wedge s \text{ is decreasing } \wedge s \text{ is bounded from below}] \Rightarrow \overset{d_{\mathbb{R}}}{\lim_{n \to \infty}} x_n = \overset{\leq_{\mathbb{R}}}{\inf} \operatorname{ran}(s)). \tag{8.488}$$

Proof. We let $s = (x_n)_{n \in \mathbb{N}_+}$ be an arbitrary sequence in $\mathbb{R}$. Concerning a), we assume that this sequence is increasing and bounded from above. Consequently, the supremum L of its range exists in light of the Supremum Property of the linear continuum $(\mathbb{R}, <_{\mathbb{R}})$. Then, this sequence converges increasingly to $L = \sup \operatorname{ran}(s)$ by Definition 4.6, which implies with Lemma 8.71a) that it converges to this supremum L with respect to the metric $d_{\mathbb{R}}$. Thus, Part a) holds, and Part b) is proved similarly.

Since s is arbitrary, we conclude that the theorem holds. □

Exercise 8.39. Write down the short proof of Part b) of the preceding theorem.

Lemma 8.73 (Boundedness of convergent sequences in $\mathbb{R}$). *Any convergent sequence $s = (x_n)_{n \in \mathbb{N}_+}$ in the metric space $(\mathbb{R}, d_{\mathbb{R}})$ is bounded.*

Proof. We let $s = (x_n)_{n \in \mathbb{N}_+}$ be an arbitrary convergent sequence in the metric space $(\mathbb{R}, d_{\mathbb{R}})$, so that there exists a particular real number $\bar{L}$ which is the limit of s with respect to $d_{\mathbb{R}}$, that is, $\bar{L} = \lim_{n \to \infty} x_n$. According to Note 8.22, it then follows from the evident $1 \in \mathbb{R}_+$ that there exists a positive natural number, say $\bar{N}$, such that

$$\forall n \, (n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow |x_n - \bar{L}| < 1). \tag{8.489}$$

To show that all terms of the sequence s are between some lower and upper bound for s, let us first verify

$$\forall n\,(n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow -(1+|\bar{L}|) \leq x_n \leq 1+|\bar{L}|). \tag{8.490}$$

For this purpose, we let $n \geq_{\mathbb{N}_+} \bar{N}$ be arbitrary, which implies with (8.489)

$$|x_n - \bar{L}| < 1. \tag{8.491}$$

Clearly,

$$|x_n| = |x_n - \bar{L} + \bar{L}| \tag{8.492}$$

is also true, and moreover

$$[|x_n| =] \quad |(x_n - \bar{L}) + \bar{L}| \leq |x_n - \bar{L}| + |\bar{L}| \tag{8.493}$$

holds because of the Subadditivity of the absolute value function on $\mathbb{R}$. Also, (8.491) implies with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$

$$|x_n - \bar{L}| + |\bar{L}| < 1 + |\bar{L}|. \tag{8.494}$$

The conjunction of (8.493) and (8.494) yields now $|x_n| < 1 + |\bar{L}|$ with the Transitivity Formula for $\leq$ and $<$, and therefore evidently $|x_n| \leq 1 + |\bar{L}|$. The latter inequality implies

$$-(1+|\bar{L}|) \leq x_n \leq 1+|\bar{L}|,$$

with (8.414), which finding proves the consequent in (8.490). Since n was arbitrary, we therefore conclude that the universal sentence (8.490) is true.

Next, we recall that the given sequence is a function $s : \mathbb{N}_+ \to \mathbb{R}$ and that the absolute value function on $\mathbb{R}$ is a function $|.|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}_+^0$, so that the composition $|.|_{\mathbb{R}} \circ s$ is a function from $\mathbb{N}_+$ to $\mathbb{R}_+^0$ because of Proposition 3.178. As the initial segment $\{1, \ldots, \bar{N}-1\}$ of $\mathbb{N}_+$ is evidently a subset of $\mathbb{N}_+$, the restriction of $|.|_{\mathbb{R}} \circ s$ to $\{1, \ldots, \bar{N}-1\}$ is a function from $\{1, \ldots, \bar{N}-1\}$ to $\mathbb{R}_+^0$ according to Proposition 3.164. Then, the range of restriction is a finite set in view of Corollary 4.125. As the singleton $\{1+|\bar{L}|\}$ is also finite in view of Proposition 4.111a), it follows that the union

$$Y = \operatorname{ran}(|.|_{\mathbb{R}} \circ D] \upharpoonright \{1, \ldots, \bar{N}-1\}) \cup \{1+|\bar{L}|\} \tag{8.495}$$

is again finite due to Proposition 4.99. Recalling that the given restriction is a nonnegative real function, we have that $\mathbb{R}$ is also a codomain of that restriction (see Note 8.10), so that the range of the restriction is a subset of $\mathbb{R}$. Furthermore, $1+|\bar{L}|$ is an element of $\mathbb{R}$, so that $\{1+|\bar{L}|\}$ is another subset of $\mathbb{R}$ in view of (2.184). Consequently, the union Y is also a subset of $\mathbb{R}$ because of (2.252). Considering now the lattice of real numbers, the fact that Y is a finite subset of $\mathbb{R}$ implies the existence of the supremum $\sup Y$ (in $\mathbb{R}$) because of Proposition 5.108. We now prove that $-\sup Y$ is a lower bound and that $\sup Y$ an upper bound for the sequence $(x_n)_{n\in\mathbb{N}_+}$. To do this, we apply the both the Criterion for lower & upper bounds for a family and verify accordingly

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow -\sup Y \leq x_n), \tag{8.496}$$

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow x_n \leq \sup Y), \tag{8.497}$$

letting $\bar{n} \in \mathbb{N}_+$. We now consider the two exhaustive cases $\bar{n} \geq_{\mathbb{N}_+} \bar{N}$ and $\neg \bar{n} \geq_{\mathbb{N}_+} \bar{N}$ to prove $-\sup Y \leq x_{\bar{n}}$ as well as $x_{\bar{n}} \leq \sup Y$.

The first case $\bar{n} \geq_{\mathbb{N}_+} \bar{N}$ implies with (8.490)

$$-(1 + |\bar{L}|) \leq x_{\bar{n}} \leq 1 + |\bar{L}|. \tag{8.498}$$

Moreover, the evident fact $1 + |\bar{L}| \in \{1 + |\bar{L}|\}$ implies $1 + |\bar{L}| \in Y$ in view of (8.495) and the definition of the union of a pair. Since the supremum $\sup Y$ is, by definition, an upper bound for Y, we then have in particular

$$1 + |\bar{L}| \leq \sup Y. \tag{8.499}$$

An application of the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ then gives also

$$-\sup Y \leq -(1 + |\bar{L}|). \tag{8.500}$$

Combining the inequalities (8.498) – (8.500), we obtain $-\sup Y \leq x_{\bar{n}}$ as well as $x_{\bar{n}} \leq \sup Y$ with the transitivity of $\leq_{\mathbb{R}}$. Thus, the desired inequalities hold in the first case.

The second case $\neg \bar{n} \geq_{\mathbb{N}_+} \bar{N}$ implies $\bar{n} <_{\mathbb{N}_+} \bar{N}$ with the Negation Formula for $\leq$, so that the conjunction of $\bar{n} \leq_{\mathbb{N}_+} \bar{N}$ and $\bar{n} \neq \bar{N}$ holds by definition of an induced irreflexive partial ordering. As $\bar{n}$ and $\bar{N}$ are elements of $\mathbb{N}_+$, we have on the one hand that $n \leq_{\mathbb{N}_+} \bar{N}$ implies $\bar{n} \in \{1, \ldots, \bar{N}\}$ with (4.275). On the other hand, $\bar{n} \neq \bar{N}$ implies $\bar{n} \notin \{\bar{N}\}$ with (2.169). The conjunction of these two findings means by definition of a set difference that $\bar{n} \in \{1, \ldots, \bar{N}\} \setminus \{\bar{N}\}$ holds, which gives $\bar{n} \in \{1, \ldots, \bar{N} - 1\}$ with (4.256) and the evident fact that $\bar{N}$ is the successor of $\bar{N} - 1$. Thus, $\bar{n}$ is an element of the domain of the restriction $[|.|_{\mathbb{R}} \circ s] \upharpoonright \{1, \ldots, \bar{N} - 1\}$, which element is associated with the unique function value

$$[|.|_{\mathbb{R}} \circ s] \upharpoonright \{1, \ldots, \bar{N} - 1\}(\bar{n}) = (|.|_{\mathbb{R}} \circ s)(\bar{n}) = |x_{\bar{n}}|$$

using (3.567) and the notation for the values of a composition of the absolute value function. Clearly, that value is in the range of the restriction. In view of (8.495), we now see that $|x_{\bar{n}}| \in Y$ holds (by definition of the union of a pair). Since $\sup Y$ is an upper bound for Y, we therefore find $|x_{\bar{n}}| \leq \sup Y$, which yields $-\sup Y \leq x_{\bar{n}} \leq \sup Y$ with (8.414). Thus, we found the desired inequalities also in the second case.

Since $\bar{n}$ was arbitrary, we may therefore conclude that the universal sentences (8.496) and (8.497) both hold. Thus, $-\sup Y$ is indeed a lower bound and $\sup Y$ an upper bound for (the range of) the sequence $s = (x_n)_{n \in \mathbb{N}_+}$. The existence of these bounds then implies that s is bounded. Because s was initially arbitrary, we may now finally conclude that the lemma is true. □

Definition 8.37 (Cauchy/fundamental sequence, complete metric space). For any metric space (X, d)

(1) we say that a sequence $f = (a_n)_{n \in \mathbb{N}_+}$ in X is *Cauchy* or *fundamental* iff, for any positive real number ε, there exists a positive natural number N such that the distance of the term a_m to the term a_n is less than ε for any indexes m, n greater than N, i.e., iff

$$\forall \varepsilon \, (\varepsilon \in \mathbb{R}_+ \Rightarrow \exists N \, (N \in \mathbb{N}_+ \wedge \forall m, n \, (m, n \geq_{\mathbb{N}_+} N \Rightarrow d(a_m, a_n) < \varepsilon))). \tag{8.501}$$

(2) we furthermore say that (X, d) is *complete* iff any Cauchy sequence in X converges to some element of X.

Proposition 8.74. *It is true for any metric space (X, d_X) and any sequence $f = (a_n)_{n \in \mathbb{N}_+}$ in X that f is Cauchy if f is convergent (with respect to d_X).*

Proof. We let (X, d_X) be an arbitrary metric space and $f = (a_n)_{n \in \mathbb{N}_+}$ an arbitrary sequence in X. To prove the implication directly, we assume that there exists an element of X, say $\bar{a}$, with $\bar{a} = \lim_{n \to \infty} a_n$. We now verify that $(a_n)_{n \in \mathbb{N}_+}$ satisfies (8.501). To do this, we let ε be an arbitrary positive real number. Now, since $\bar{a}$ is the limit of f, it follows with (8.468) from the evident $0.5 \cdot \varepsilon \in \mathbb{R}_+$ that there exists an element of $\mathbb{N}_+$, say $\bar{N}$, such that

$$\forall n \, (n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow d_X(a_n, \bar{a}) < 0.5 \cdot \varepsilon) \tag{8.502}$$

holds. We now demonstrate that this implies

$$\forall m, n \, (m, n \geq_{\mathbb{N}_+} \bar{N} \Rightarrow d_X(a_m, a_n) < \varepsilon), \tag{8.503}$$

which will prove (8.501). For this purpose, we let m and n be arbitrary positive natural numbers greater than $\bar{N}$. This assumption implies with

(8.502) the inequalities

$$d_X(a_m, \bar{a}) < 0.5 \cdot \varepsilon,$$
$$d_X(a_n, \bar{a}) < 0.5 \cdot \varepsilon.$$

Applying now the Additivity of $<$-inequalities for ordered integral domains and Property 3 (i.e., the symmetry) of a metric, we obtain

$$d_X(a_m, \bar{a}) + d_X(\bar{a}, a_n) < \varepsilon \cdot 0.5 + \varepsilon \cdot 0.5 \quad [= \varepsilon]. \tag{8.504}$$

Let us now observe that

$$d_X(a_m, a_n) \leq d_X(a_m, \bar{a}) + d_X(\bar{a}, a_n) \tag{8.505}$$

using Property 4 (i.e., the triangle inequality) of a metric. Then, the conjunction of (8.505) and (8.504) gives the desired $d_X(a_m, a_n) < \varepsilon$ with the Transitivity Formula for $\leq$ and $<$. As m and n are arbitrary, we therefore conclude that (8.503), and thus the existential sentence in (8.501) holds. Furthermore, since ϵ is arbitrary, we conclude that the universal sentence (8.501) is true, which means that the sequence $(a_n)_{n \in \mathbb{N}_+}$ is Cauchy. This proves the proposed implication, and as (X, d_X) as well as f were also arbitrary, it follows that the proposition itself is true. $\square$

Chapter 9.

Extended Real Numbers

Axiom 9.1 (Axiom of Foundation/Regularity). It is true for any nonempty set X that, for some $x \in X$, there is no y which is both in X and in x, that is,

$$\forall X \, (X \neq \emptyset \Rightarrow \exists x \, (x \in X \; \wedge \; \neg \exists y \, (y \in X \; \wedge \; y \in x))). \tag{9.1}$$

Definition 9.1 (Zermelo-Fraenkel Axioms including the Axiom of Choice (ZFC)). We call the

1. Axiom of Extension,
2. Axiom of Specification,
3. Axiom of Pairing,
4. Axiom of Union,
5. Axiom of Infinity,
6. Axiom of Powers,
7. Axiom of Replacement,
8. Axiom of Choice,
9. Axiom of Foundation

in total the *Zermelo-Fraenkel Axioms including the Axiom of Choice*, or simply *ZFC*.

Proposition 9.1. *It is true that no set is an element of itself, that is,*

$$\forall Y \, (Y \notin Y). \tag{9.2}$$

Proof. Letting Y be an arbitrary set, we clearly see that the singleton $\{Y\}$ is nonempty due to $Y \in \{Y\}$. In view of the Axiom of Foundation, there is then a constant, say $\bar{X}$, such that $\bar{X} \in \{Y\}$ and the negation

$$\neg \exists y \, (y \in \{Y\} \wedge y \in \bar{X})$$

are satisfied. The former implies $\bar{X} = Y$ with (2.169), so that the latter negation reads after substitution

$$\neg \exists y \, (y \in \{Y\} \wedge y \in Y),$$

which may also be written as

$$\forall y \, (y \in \{Y\} \Rightarrow y \notin Y)$$

by using the Negation Formula for existential conjunctions. Thus, $Y \in \{Y\}$ implies $Y \notin Y$, and since Y is arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Corollary 9.2. *It is true for any set X that there is some x which is not included in Y, that is,*

$$\forall Y \, \exists x \, (x \notin Y). \tag{9.3}$$

Proof. Letting Y be an arbitrary set, we obtain $Y \notin Y$ with (9.2), which demonstrates the existence of a set which is not included in Y. This is then evidently true for any set Y. □

9.1. The Linearly Ordered Set $(\overline{\mathbb{R}}, <_{\overline{\mathbb{R}}})$

Note 9.1. In view of (9.3), there is some $x \notin \mathbb{R}$, say $+\infty$. As the evident fact $+\infty \in \{+\infty\}$ implies the truth of the disjunction $+\infty \in \mathbb{R} \vee +\infty \in \{+\infty\}$, we see in light of the definition of the union of two sets that

$$+\infty \in \mathbb{R} \cup \{+\infty\} \tag{9.4}$$

holds. Then, there is also some $x \notin \mathbb{R} \cup \{+\infty\}$, say $-\infty$. Clearly,

$$-\infty \neq +\infty, \tag{9.5}$$

since $-\infty = +\infty$ would imply the contradiction $-\infty \in \mathbb{R} \cup \{+\infty\}$ and $-\infty \notin \mathbb{R} \cup \{+\infty\}$ via substitutions into the previous findings. Furthermore, the fact $-\infty \in \{-\infty\}$ implies that $-\infty \in \mathbb{R} \cup \{+\infty\}$ or $-\infty \in \{-\infty\}$ holds, so that

$$-\infty \in (\mathbb{R} \cup \{+\infty\}) \cup \{-\infty\}.$$

Definition 9.2 (Projectively extended real line, set of extended real numbers, extended real number). We call the set

$$\widehat{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \tag{9.6}$$

the *projectively extended real line* and

$$\overline{\mathbb{R}} = \widehat{\mathbb{R}} \cup \{-\infty\} \tag{9.7}$$

the *set of extended real numbers*. Thus, we call every element of $\overline{\mathbb{R}}$ an *extended real number*.

Exercise 9.1. Establish the following facts.

$$-\infty \notin \mathbb{R}, \tag{9.8}$$
$$+\infty \notin \mathbb{R}, \tag{9.9}$$
$$-\infty \in \overline{\mathbb{R}}, \tag{9.10}$$
$$+\infty \in \overline{\mathbb{R}}, \tag{9.11}$$
$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}, \tag{9.12}$$
$$\mathbb{R} \subseteq \overline{\mathbb{R}}, \tag{9.13}$$
$$\mathbb{R} = \overline{\mathbb{R}} \setminus \{-\infty, +\infty\}. \tag{9.14}$$

(Hint: Concerning (9.14), prove first $\mathbb{R} \cap \{-\infty, +\infty\} = \emptyset$ by contradiction and apply then (2.262).)

Corollary 9.3. *Every real number is different from both $-\infty$ and $+\infty$, i.e.,*

$$\forall x\, (x \in \mathbb{R} \Rightarrow [x \neq -\infty \;\wedge\; x \neq +\infty]). \tag{9.15}$$

Proof. We take an arbitrary real number, so that (9.14) yields in particular $x \notin \{-\infty, +\infty\}$. This implies $\neg(x = -\infty \vee x = +\infty)$ with the definition of a pair, so that the conjunction in (9.15) follows to be true with De Morgan's Law for the disjunction. Since x is arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Proposition 9.4. *Every extended real number is precisely real, $-\infty$ or $+\infty$, i.e.,*

$$\begin{aligned}\forall x\, (x \in \overline{\mathbb{R}} \Rightarrow [&(x \in \mathbb{R} \;\wedge\; x \neq -\infty \;\wedge\; x = +\infty)\\ &\vee (x \notin \mathbb{R} \;\wedge\; x = -\infty \;\wedge\; x \neq +\infty)\\ &\vee (x \notin \mathbb{R} \;\wedge\; x \neq -\infty \;\wedge\; x = +\infty)]).\end{aligned} \tag{9.16}$$

Proof. We let $x \in \overline{\mathbb{R}}$ be arbitrary, which implies $x \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ with (9.7) – (9.6). We obtain then the true multiple disjunction

$$x \in \mathbb{R} \vee x \in \{-\infty\} \vee x \in \{+\infty\}$$

by using the definition of the union of a pair and the Associative Law for the union of two sets. Since $x \in \{-\infty\}$ is equivalent to $x = -\infty$ and $x \in \{+\infty\}$ equivalent to $x = +\infty$ due to (2.169), we may rewrite the preceding disjunction as

$$x \in \mathbb{R} \vee x = -\infty \vee x = +\infty, \tag{9.17}$$

so that the first part of the conjunction in (9.16) holds. To show that precisely one part of the disjunction is true, we consider first the case $x \in \mathbb{R}$, which implies the required $x \neq -\infty$ and $x \neq +\infty$ with (9.15). Thus, the multiple conjunction

$$x \in \mathbb{R} \wedge x \neq -\infty \wedge x = +\infty$$

holds, and the multiple disjunction in (9.16) is then also true. In the second case $x = -\infty$, we obtain $x \notin \mathbb{R}$ and $x \neq +\infty$ with (9.8) and (9.5) through substitutions. This means that the multiple conjunction

$$x \notin \mathbb{R} \wedge x = -\infty \wedge x \neq +\infty$$

holds in this case, so that the multiple disjunction in (9.16) is again true. Finally, the third case $x = +\infty$ gives $x \notin \mathbb{R}$ and $x \neq -\infty$ (9.9) and (9.5) via substitutions. We thus have

$$x \notin \mathbb{R} \wedge x \neq -\infty \wedge x = +\infty,$$

with the consequence that the multiple disjunction in (9.16) is true again. Because x was initially arbitrary, we may infer from these findings the truth of the proposed universal sentence. □

Definition 9.3 (Numerical function, extended real function, set of numerical/extended real functions). For any set X, we call every function

$$f : X \to \overline{\mathbb{R}} \tag{9.18}$$

a *numerical function* or an *extended real function* (on X) and

$$\overline{\mathbb{R}}^X \tag{9.19}$$

the *set of numerical functions* or the *set of extended real functions* (on X).

Lemma 9.5. *It is true that the union*

$$<_{\overline{\mathbb{R}}} = <_{\mathbb{R}} \cup \{-\infty\} \times \mathbb{R} \cup \mathbb{R} \times \{+\infty\} \cup \{(-\infty, +\infty)\} \tag{9.20}$$

constitutes a binary relation on $\overline{\mathbb{R}}$.

Proof. To show that $<_{\overline{\mathbb{R}}}$ satisfies the definition of a binary relation on $\overline{\mathbb{R}}$, we need to show that the inclusion

$$<_{\overline{\mathbb{R}}} \subseteq \overline{\mathbb{R}} \times \overline{\mathbb{R}}. \tag{9.21}$$

holds. For this purpose, we apply the definition of a subset and verify accordingly

$$\forall Z \, (Z \in <_{\overline{\mathbb{R}}} \Rightarrow Z \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}). \tag{9.22}$$

To do this we let Z be an arbitrary set in $<_{\overline{\mathbb{R}}}$. Substitution based on (9.20) yields then

$$Z \in <_{\mathbb{R}} \cup \{-\infty\} \times \mathbb{R} \cup \mathbb{R} \times \{+\infty\} \cup \{(-\infty, +\infty)\}.$$

Applying now the definition of the union of two sets (multiple times) and also the Associative Law for the union of two sets (multiple times to place or omit brackets whenever needed), we obtain the pair of disjunctions

$$(Z \in <_{\mathbb{R}} \vee Z \in \{-\infty\} \times \mathbb{R}) \vee (Z \in \mathbb{R} \times \{+\infty\} \vee Z \in \{(-\infty, +\infty)\}), \tag{9.23}$$

which we now use to prove the desired consequent $Z \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by cases.

In the first case, we assume that $Z \in <_{\mathbb{R}}$ or $Z \in \{-\infty\} \times \mathbb{R}$ is true. On the one hand, if $Z \in <_{\mathbb{R}}$ holds, then we find $Z \in \mathbb{R} \times \mathbb{R}$ with the fact that the linear ordering $<_{\mathbb{R}}$ of $\mathbb{R}$ is by definition a binary relation on $\mathbb{R}$, which is by definition a subset of $\mathbb{R} \times \mathbb{R}$. Here, the inclusion $\mathbb{R} \subseteq \overline{\mathbb{R}}$ holds according to (9.13), so that the inclusion $\mathbb{R} \times \mathbb{R} \subseteq \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ follows to be true with Proposition 3.8. Thus, the previously established $Z \in \mathbb{R} \times \mathbb{R}$ clearly implies the desired consequent $Z \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$. On the other hand, if $Z \in \{-\infty\} \times \mathbb{R}$ is true, we recall the truth of $-\infty \in \overline{\mathbb{R}}$ in view of (9.10). Therefore, $\{-\infty\} \subseteq \overline{\mathbb{R}}$ is also true according to (2.184). In conjunction with the known inclusion $\mathbb{R} \subseteq \overline{\mathbb{R}}$, this evidently implies $\{-\infty\} \times \mathbb{R} \subseteq \overline{\mathbb{R}} \times \overline{\mathbb{R}}$. Due to this inclusion, the currently assumed $Z \in \{-\infty\} \times \mathbb{R}$ implies again the desired consequent.

In the second case that $Z \in \mathbb{R} \times \{+\infty\}$ or $Z \in \{(-\infty, +\infty)\}$ is true, we consider first the subcase that the first part of that disjunction holds. Since $+\infty \in \overline{\mathbb{R}}$ is true as shown by (9.11), we evidently find $\{+\infty\} \subseteq \overline{\mathbb{R}}$. In conjunction with the known inclusion $\mathbb{R} \subseteq \overline{\mathbb{R}}$, this evidently gives us now $\mathbb{R} \times \{-\infty\} \subseteq \overline{\mathbb{R}} \times \overline{\mathbb{R}}$, so that the assumed $Z \in \mathbb{R} \times \{+\infty\}$ implies

$Z \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$, as desired. Finally, we consider the second case that the second part $Z \in \{(-\infty, +\infty)\}$ of the currently assumed disjunction holds. Here, the aforementioned facts $-\infty \in \overline{\mathbb{R}}$ and $+\infty \in \overline{\mathbb{R}}$ imply $(-\infty, +\infty) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ with the definition of the Cartesian product of two sets, so that the inclusion $\{(-\infty, +\infty)\} \subseteq \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ follows to be true. Thus, $Z \in \{(-\infty, +\infty)\}$ implies once again the desired consequent of the implication in (9.22).

Since Z is arbitrary, we may therefore conclude that the universal sentence (9.22) is true, so that the inclusion (9.21) holds as well. Consequently, $<_{\overline{\mathbb{R}}}$ is a binary relation on $\overline{\mathbb{R}}$, by definition. □

Corollary 9.6. *The following sentences are true:*

$$\begin{aligned} <_{\mathbb{R}} \; &\subseteq <_{\overline{\mathbb{R}}}, && (9.24)\\ \{-\infty\} \times \mathbb{R} \; &\subseteq <_{\overline{\mathbb{R}}}, && (9.25)\\ \mathbb{R} \times \{+\infty\} \; &\subseteq <_{\overline{\mathbb{R}}}, && (9.26)\\ -\infty \; &<_{\overline{\mathbb{R}}} \; +\infty. && (9.27) \end{aligned}$$

Proof. Observing that we may write (9.20) with brackets as

$$<_{\overline{\mathbb{R}}} = <_{\mathbb{R}} \; \cup (\{-\infty\} \times \mathbb{R} \; \cup \; \mathbb{R} \times \{+\infty\} \; \cup \; \{(-\infty, +\infty)\})$$

(using the Associative Law for the union of two sets), we find

$$<_{\mathbb{R}} \subseteq <_{\mathbb{R}} \; \cup (\{-\infty\} \times \mathbb{R} \; \cup \; \mathbb{R} \times \{+\infty\} \; \cup \; \{(-\infty, +\infty)\})$$

with (2.245), and this inclusion yields the inclusion in (9.24) via substitution. Applying now both the Associative Law and the Commutative Law for the union of two sets, we may write (9.20) also as

$$\begin{aligned} <_{\overline{\mathbb{R}}} &= \{-\infty\} \times \mathbb{R} \; \cup \; (<_{\mathbb{R}} \; \cup \mathbb{R} \times \{+\infty\} \; \cup \; \{(-\infty, +\infty)\}),\\ <_{\overline{\mathbb{R}}} &= \mathbb{R} \times \{+\infty\} \; \cup \; (<_{\mathbb{R}} \; \cup \{-\infty\} \times \mathbb{R} \; \cup \; \{(-\infty, +\infty)\}),\\ <_{\overline{\mathbb{R}}} &= \{(-\infty, +\infty)\} \; \cup \; (<_{\mathbb{R}} \; \cup \{-\infty\} \times \mathbb{R} \; \cup \; \mathbb{R} \times \{+\infty\}), \end{aligned}$$

from which equations we obtain the desired inclusions (9.25) – (9.26) as well as $\{(-\infty, +\infty)\} \subseteq <_{\overline{\mathbb{R}}}$ by means of (2.245). The latter implies with (2.169) $(-\infty, +\infty) \in <_{\overline{\mathbb{R}}}$, which we may then write also as (9.27) since $<_{\overline{\mathbb{R}}}$ is a binary relation (see Lemma 9.5). □

Corollary 9.7. *Every real number is strictly between $-\infty$ and $+\infty$, i.e.,*

$$\forall x \, (x \in \mathbb{R} \Rightarrow -\infty <_{\overline{\mathbb{R}}} x <_{\overline{\mathbb{R}}} +\infty). \tag{9.28}$$

Proof. Letting x be an arbitrary real number, we first observe the truth of the basic facts $-\infty \in \{-\infty\}$ and $+\infty \in \{+\infty\}$. By definition of the Cartesian product of two sets, we therefore find $(-\infty, x) \in \{-\infty\} \times \mathbb{R}$ and $(x, +\infty) \in \mathbb{R} \times \{+\infty\}$. Due to the inclusions (9.25) and (9.26), we therefore find $(-\infty, x) \in <_{\overline{\mathbb{R}}}$ as well as $(x, +\infty) \in <_{\overline{\mathbb{R}}}$. We can write these findings also as $-\infty <_{\overline{\mathbb{R}}} x$ and $x <_{\overline{\mathbb{R}}} +\infty$, which are the desired inequalities. As x was initially arbitrary, we therefore conclude that the proposed universal sentence is true. □

Proposition 9.8. *It is true that any extended real number D is not less than $-\infty$, i.e.*

$$\forall D\,(D \in \overline{\mathbb{R}} \Rightarrow \neg D <_{\overline{\mathbb{R}}} -\infty). \tag{9.29}$$

Proof. We let D be arbitrary, and we prove the implication by contradiction, assuming $D \in \overline{\mathbb{R}}$ and the negation of $\neg D <_{\overline{\mathbb{R}}} -\infty$ to be both true. The latter assumption implies with the Double Negation Law the truth of $D <_{\overline{\mathbb{R}}} -\infty$. Since $<_{\overline{\mathbb{R}}}$ is a binary relation (recall Lemma 9.5), we may write this inequality also as $(D, -\infty) \in <_{\overline{\mathbb{R}}}$. As shown by (9.23), we therefore find the true disjunctions

$$\begin{aligned} (D, -\infty) \in <_{\mathbb{R}} \ &\vee\ (D, -\infty) \in \{-\infty\} \times \mathbb{R} \\ &\vee\ (D, -\infty) \in \mathbb{R} \times \{+\infty\} \\ &\vee\ (D, -\infty) \in \{(-\infty, +\infty)\}, \end{aligned} \tag{9.30}$$

which we now show to be a contradiction.

Firstly, since $-\infty \notin \mathbb{R}$ holds according to (9.8), the disjunction $D \notin \mathbb{R} \vee -\infty \notin \mathbb{R}$ is then also true. Then, the negation $\neg(D \in \mathbb{R} \wedge -\infty \in \mathbb{R})$ follows to be true with De Morgan's Law for the conjunction. By definition of the Cartesian product of two sets, we therefore find $(D, -\infty) \notin \mathbb{R} \times \mathbb{R}$. Since the inclusion $<_{\mathbb{R}} \subseteq \mathbb{R} \times \mathbb{R}$ is also true (recalling that $<_{\mathbb{R}}$ is a binary relation on $\mathbb{R}$), we may apply (2.9) to infer from the preceding negation the truth of $(D, -\infty) \notin <_{\mathbb{R}}$. Thus, the first part of the multiple disjunction (9.30) is false.

Secondly, as $-\infty \notin \mathbb{R}$ was shown to be true, the disjunction $D \notin \{-\infty\} \vee -\infty \notin \mathbb{R}$ holds then, too. This evidently implies $\neg(D \in \{-\infty\} \wedge -\infty \in \mathbb{R})$ and therefore $(D, -\infty) \notin \{-\infty\} \times \mathbb{R}$. Thus, the second part of the multiple disjunction (9.30) is also false.

Thirdly, because $-\infty \neq +\infty$ holds (see Note 9.1), we obtain $-\infty \notin \{+\infty\}$ with (2.169), and the disjunction $D \notin \mathbb{R} \vee -\infty \notin \{+\infty\}$ is then also true. Clearly, this disjunction yields $\neg(D \in \mathbb{R} \wedge -\infty \in \{+\infty\})$ and subsequently $(D, -\infty) \notin \mathbb{R} \times \{+\infty\}$. This finding shows that the third part of the multiple disjunction to be disproven is indeed false.

Finally, we observe that the fact $-\infty \neq +\infty$ implies also the truth of the disjunction $D \neq -\infty \lor -\infty \neq +\infty$, with the evident consequence that $\neg(D = -\infty \land -\infty = +\infty)$ holds. The Equality Criterion for ordered pairs gives us therefore $(D, -\infty) \neq (-\infty, +\infty)$, so that $(D, -\infty) \notin \{(-\infty, +\infty)\}$ evidently follows to be true. Thus, all four parts of the multiple disjunction (9.30) are false, which means that we arrived indeed at a contradiction.

This completes the proof of the implication in (9.29), in which D is arbitrary, so that the proposed universal sentence holds, as claimed. □

Exercise 9.2. Show that any extended real number D is not greater than $+\infty$, i.e.

$$\forall D\,(D \in \overline{\mathbb{R}} \Rightarrow \neg + \infty <_{\overline{\mathbb{R}}} D). \tag{9.31}$$

(Hint: Proceed similarly as in the proof of Proposition 9.8.)

Proposition 9.9. *The following universal sentence is true:*

$$\forall x, y\,(x, y \in \mathbb{R} \Rightarrow [x <_{\mathbb{R}} y \Leftrightarrow x <_{\overline{\mathbb{R}}} y]). \tag{9.32}$$

Proof. We let $\bar{x}$ and $\bar{y}$ be arbitrary real numbers. To prove the first part ('$\Rightarrow$') of the equivalence, we assume $\bar{x} <_{\mathbb{R}} \bar{y}$, which we may write as $(\bar{x}, \bar{y}) \in\, <_{\mathbb{R}}$. This implies $(\bar{x}, \bar{y}) \in\, <_{\overline{\mathbb{R}}}$ with (9.24) and the definition of a subset, which finding we may then write as the desired inequality $x <_{\overline{\mathbb{R}}} y$. To prove the second part ('$\Leftarrow$') of the proposed equivalence, we now assume $x <_{\overline{\mathbb{R}}} y$, that is, $(\bar{x}, \bar{y}) \in\, <_{\overline{\mathbb{R}}}$. We thus find the multiple disjunction

$$\begin{aligned} (\bar{x}, \bar{y}) \in\, <_{\mathbb{R}} \ &\lor\ (\bar{x}, \bar{y}) \in \{-\infty\} \times \mathbb{R} \\ &\lor\ (\bar{x}, \bar{y}) \in \mathbb{R} \times \{+\infty\} \\ &\lor\ (\bar{x}, \bar{y}) \in \{(-\infty, +\infty)\} \end{aligned} \tag{9.33}$$

with (9.20) and the definition of the union of a pair (applied multiple times). Since the initial assumption of $\bar{x}, \bar{y} \in \mathbb{R}$ implies $\bar{x} \neq -\infty$ and $\bar{y} \neq +\infty$ with Corollary 9.3, it follows on the one hand that $\bar{x} \notin \{-\infty\}$ as well as $\bar{y} \notin \{+\infty\}$ holds by virtue of (2.169). Then, the disjunctions

$$\begin{aligned} \bar{x} \notin \{-\infty\} \ &\lor\ \bar{y} \notin \mathbb{R} \\ \bar{x} \notin \mathbb{R} \ &\lor\ \bar{y} \notin \{+\infty\} \end{aligned}$$

also hold. Applying now De Morgan's Law for the conjunction, we obtain the true negations

$$\begin{aligned} &\neg(\bar{x} \in \{-\infty\} \ \land\ \bar{y} \in \mathbb{R}), \\ &\neg(\bar{x} \in \mathbb{R} \ \land\ \bar{y} \in \{+\infty\}), \end{aligned}$$

which in turn imply

$$\neg(\bar{x}, \bar{y}) \in \{-\infty\} \times \mathbb{R} \tag{9.34}$$
$$(\bar{x}, \bar{y}) \in \mathbb{R} \times \{+\infty\} \tag{9.35}$$

with the definition of the Cartesian product of two sets. On the other hand, the previously established negations $\bar{x} \neq -\infty$ and $\bar{y} \neq +\infty$ imply the truth of the disjunction $\bar{x} \neq -\infty \lor \bar{y} \neq +\infty$, so that De Morgan's Law for the conjunction yields $\neg(\bar{x} = -\infty \land \bar{y} = +\infty)$. This negation implies with the Equality Criterion for ordered pairs $(\bar{x}, \bar{y}) \neq (-\infty, +\infty)$, and therefore

$$(\bar{x}, \bar{y}) \notin \{(-\infty, +\infty)\} \tag{9.36}$$

using (2.169) once again. The findings (9.34) – (9.36) show that only the first part of the multiple disjunction (9.33) is true. That we may write as $\bar{x} <_{\mathbb{R}} \bar{y}$, proving the second part of the equivalence in (9.32). Since $\bar{x}$ and $\bar{y}$ were initially arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Theorem 9.10. *It is true that the binary relation (9.20) constitutes a linear ordering of* $\overline{\mathbb{R}}$.

Proof. To show that $<_{\overline{\mathbb{R}}}$ is irreflexive, we verify

$$\forall D\, (D \in \overline{\mathbb{R}} \Rightarrow \neg D <_{\overline{\mathbb{R}}} D), \tag{9.37}$$

letting D be an arbitrary extended real number. We thus have

$$D \in \mathbb{R} \lor D = -\infty \lor D = +\infty \tag{9.38}$$

in view of (9.17), which disjunctions we may now use to prove $\neg D <_{\overline{\mathbb{R}}} D$ by three cases. In case of $D \in \mathbb{R}$, we recall that the standard linear ordering of $\mathbb{R}$ is irreflexive, which means that $\neg D <_{\mathbb{R}} D$ is true. This in turn implies the desired negation $\neg D <_{\overline{\mathbb{R}}} D$ with (9.32). In case of $D = -\infty$, we observe the truth of $\neg D <_{\overline{\mathbb{R}}} -\infty$ in light of (9.29), so that substitution yields again $\neg D <_{\overline{\mathbb{R}}} D$. Finally, in case of $D = +\infty$, we note that $\neg +\infty <_{\overline{\mathbb{R}}} D$ holds according to (9.31), so that substitution gives us once again the desired negation. Since D was initially arbitrary, we therefore conclude that the universal sentence (9.37) holds, which means that $<_{\overline{\mathbb{R}}}$ is irreflexive, by definition.

Next, we demonstrate that $<_{\overline{\mathbb{R}}}$ is connex, i.e., that $<_{\overline{\mathbb{R}}}$ satisfies

$$\forall D, E\, (D, E \in \overline{\mathbb{R}} \Rightarrow [D <_{\overline{\mathbb{R}}} E \lor E <_{\overline{\mathbb{R}}} D \lor D = E]). \tag{9.39}$$

Letting $D, E \in \overline{\mathbb{R}}$, we thus have that D satisfies again the disjunctions (9.38), which we use for a proof by cases. In the first case, we assume D to be real. Since the extended real number E evidently satisfies the disjunctions

$$E \in \mathbb{R} \lor E = -\infty \lor E = +\infty, \tag{9.40}$$

we have three subcases to consider. In the first subcase, E is also real. Since the linear ordering $<_{\mathbb{R}}$ is connex, the disjunctions $D <_{\mathbb{R}} E \lor E <_{\mathbb{R}} D \lor D = E$ hold. Since $D <_{\mathbb{R}} E$ is evidently equivalent to $D <_{\overline{\mathbb{R}}} E$ and $E <_{\mathbb{R}} D$ equivalent to $E <_{\overline{\mathbb{R}}} D$ for the real numbers D, E, these conjunctions imply the desired disjunctions in (9.39). In the second subcase, we assume $E = -\infty$. As the real number D satisfies

$$-\infty <_{\overline{\mathbb{R}}} D <_{\overline{\mathbb{R}}} +\infty \tag{9.41}$$

because of (9.28), we find $E <_{\overline{\mathbb{R}}} D$ via substitution. Thus, the desired multiple disjunction also holds. In the third subcase $E = +\infty$, we obtain $D <_{\overline{\mathbb{R}}} E$ with the second inequality in (9.41), so that the desired multiple disjunction is again true. We thus completed the first case.

The second case is based on the assumption $D = -\infty$. We consider again the three subcases based on (9.40). If E is real, then we evidently obtain the inequalities

$$-\infty <_{\overline{\mathbb{R}}} E <_{\overline{\mathbb{R}}} +\infty, \tag{9.42}$$

so that $D <_{\overline{\mathbb{R}}} E$ follows to be true via substitution. If $E = -\infty$, then $D = E$ holds. Finally, if $E = +\infty$, then the truth of $-\infty <_{\overline{\mathbb{R}}} +\infty$ (see Corollary 9.6) clearly implies the truth of $D <_{\overline{\mathbb{R}}} E$. Thus, each of the three subcases implies that one part of the (and thus the entire) multiple disjunction in (9.39) is true.

In the third, we assume now $D = +\infty$ to hold. Then, $E \in \mathbb{R}$ yields with the second inequality in (9.42) $E <_{\overline{\mathbb{R}}} D$. The latter inequality is evidently implies also by $E = -\infty$. Finally, $E = +\infty$ gives $D = E$, so that the desired multiple disjunction follows again to be true for each subcase.

We thus completed the proof(s) by cases, and since D and E were arbitrary, we may therefore conclude that $<_{\overline{\mathbb{R}}}$ is indeed connex.

It remains for us to verify that $<_{\overline{\mathbb{R}}}$ is transitive, that is,

$$\forall D, E, F\, (D, E, F \in \overline{\mathbb{R}} \Rightarrow [(D <_{\overline{\mathbb{R}}} E \land E <_{\overline{\mathbb{R}}} F) \Rightarrow D <_{\overline{\mathbb{R}}} F]). \tag{9.43}$$

Letting $D, E, F \in \overline{\mathbb{R}}$ such that the inequalities $D <_{\overline{\mathbb{R}}} E$ and $E <_{\overline{\mathbb{R}}} F$ are satisfied, we observe the truth of (9.38), (9.40) and

$$F \in \mathbb{R} \lor F = -\infty \lor F = +\infty. \tag{9.44}$$

First, we prove $D \neq +\infty$ by contradiction. Assuming the negation of that negation to be true, $D = +\infty$ follows to be true with the Double Negation Law. Then, $D <_{\overline{\mathbb{R}}} E$ implies $+\infty <_{\overline{\mathbb{R}}} E$, in contradiction to (9.31). Therefore, $D \neq +\infty$ holds indeed, which means that the first or the second part of (9.38) holds.

Next, we show via contradictions that $E \neq \pm\infty$. Indeed, if $E = -\infty$, then $D <_{\overline{\mathbb{R}}} E$ implies $D <_{\overline{\mathbb{R}}} -\infty$ in contradiction to (9.29), and if if $E = +\infty$, then $E <_{\overline{\mathbb{R}}} F$ gives $+\infty <_{\overline{\mathbb{R}}} F$ in contradiction to (9.31). Thus, E must be real.

Similarly, we may demonstrate the truth of $F \neq -\infty$. Indeed, the assumption $F = -\infty$ causes $E <_{\overline{\mathbb{R}}} F$ to become $E <_{\overline{\mathbb{R}}} -\infty$, which contradicts (9.29). Thus, the second part of the multiple disjunction (9.44) is false, so that $F \in \mathbb{R}$ or $F = +\infty$ holds.

We now use the true disjunctions $D \in \mathbb{R} \vee D = -\infty$ and $F \in \mathbb{R} \vee F = +\infty$ to prove $D <_{\overline{\mathbb{R}}} F$ by cases (and subcases).

In the first case $D \in \mathbb{R}$ and the first subcase $F \in \mathbb{R}$, we now have the situation that D, E and F are all real. Therefore, the assumed inequalities imply $D <_{\mathbb{R}} E$ and $E <_{\mathbb{R}} F$ with (9.32). As the linear ordering $<_{\mathbb{R}}$ is transitive, we therefore obtain $D <_{\mathbb{R}} F$, which in turn implies $D <_{\overline{\mathbb{R}}} F$, using (9.32) again. In the second subcase $F = +\infty$, we use the true inequality $D <_{\overline{\mathbb{R}}} +\infty$ in (9.41) to infer the truth of $D <_{\overline{\mathbb{R}}} F$.

In the second case $D = -\infty$ and the first subcase $F \in \mathbb{R}$, we observe the truth of $-\infty <_{\overline{\mathbb{R}}} F$ in light of (9.28), with the consequence that $D <_{\overline{\mathbb{R}}} F$ holds, as desired. In the second subcase $F = +\infty$, we recall the truth of the equation $-\infty <_{\overline{\mathbb{R}}} +\infty$, so that $D <_{\overline{\mathbb{R}}} F$ once again.

We thus completed the proof(s) by cases, and since D, E, F were arbitrary, we therefore conclude that $<_{\overline{\mathbb{R}}}$ is also transitive. Consequently, the binary relation $<_{\overline{\mathbb{R}}}$ on $\overline{\mathbb{R}}$ is a linear ordering of $\overline{\mathbb{R}}$, by definition. □

Note 9.2. The linear ordering $<_{\overline{\mathbb{R}}}$ induces the total ordering $\leq_{\overline{\mathbb{R}}}$.

Definition 9.4 (Standard linear & total ordering of $\overline{\mathbb{R}}$). We call $<_{\overline{\mathbb{R}}}$ the *standard linear ordering of* $\overline{\mathbb{R}}$ and $\leq_{\overline{\mathbb{R}}}$ the *standard total ordering of* $\overline{\mathbb{R}}$.

Proposition 9.11. *The set of real numbers is identical with the open interval in $\overline{\mathbb{R}}$ from $-\infty$ to $+\infty$, that is,*

$$\mathbb{R} = (-\infty, +\infty)_{\overline{\mathbb{R}}}. \tag{9.45}$$

Proof. We prove the equation by means of the Equality Criterion for sets, i.e. by verifying the equivalent universal sentence

$$\forall x\, (x \in \mathbb{R} \Leftrightarrow x \in (-\infty, +\infty)_{\overline{\mathbb{R}}}). \tag{9.46}$$

We take an arbitrary set $\bar{x}$ and assume first $\bar{x} \in \mathbb{R}$ to be true. Thus, the ordered pairs $(-\infty, \bar{x})$ and $(\bar{x}, +\infty)$ are evidently elements of the Cartesian products $\{-\infty\} \times \mathbb{R}$ and $\mathbb{R} \times \{+\infty\}$, respectively, and therefore elements of the union (9.20), i.e.

$$(-\infty, \bar{x}), (\bar{x}, +\infty) \in <_{\overline{\mathbb{R}}} .$$

As $<_{\overline{\mathbb{R}}}$ constitutes a binary relation, we may write this also in the form of $-\infty <_{\overline{\mathbb{R}}} \bar{x}$ and $\bar{x} <_{\overline{\mathbb{R}}} +\infty$, which inequalities give (according to the definition of an open interval in $\overline{\mathbb{R}}$) the desired consequent $\bar{x} \in (-\infty, +\infty)_{\overline{\mathbb{R}}}$ of the first part ('$\Rightarrow$') of the equivalence in (9.46).

To establish the second part ('$\Leftarrow$'), we now assume $\bar{x} \in (-\infty, +\infty)_{\overline{\mathbb{R}}}$ to be true, so that the inequalities $-\infty <_{\overline{\mathbb{R}}} \bar{x}$ and $\bar{x} <_{\overline{\mathbb{R}}} +\infty$ hold (by definition of an open interval in $\overline{\mathbb{R}}$). Since the linear ordering $<_{\overline{\mathbb{R}}}$ satisfies that Characterization of comparability, the truth of the former inequality implies $\bar{x} \neq -\infty$ and the truth of the latter $\bar{x} \neq +\infty$. Due to these inequalities, $\bar{x} \in \overline{\mathbb{R}}$ implies $\bar{x} \in \mathbb{R}$ with (9.17), which finding proves the second part of the equivalence.

As $\bar{x}$ was arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (9.46) and consequently the truth of the proposed equation (9.45). □

Proposition 9.12. *It is true that the standard total ordering $\leq_{\mathbb{R}}$ of $\mathbb{R}$ is included in the standard total ordering $\leq_{\overline{\mathbb{R}}}$ of $\overline{\mathbb{R}}$, that is,*

$$\leq_{\mathbb{R}} \; \subseteq \; \leq_{\overline{\mathbb{R}}} . \tag{9.47}$$

Proof. To prove the inclusion, we verify

$$\forall Z \, (Z \in \, \leq_{\mathbb{R}} \Rightarrow Z \in \, \leq_{\overline{\mathbb{R}}}). \tag{9.48}$$

For this purpose, we let Z be arbitrary in $\leq_{\mathbb{R}}$, which implies by definition of a binary relation relation that there exist elements, say $\bar{x}$ and $\bar{y}$ such that $Z = (\bar{x}, \bar{y})$. This yields $(\bar{x}, \bar{y}) \in \, \leq_{\mathbb{R}}$, which we may also write as $\bar{x} \leq_{\mathbb{R}} \bar{y}$. In view of the Characterization of induced irreflexive partial orderings, this implies the true disjunction $\bar{x} <_{\mathbb{R}} \bar{y} \vee \bar{x} = \bar{y}$, which we use to prove $\bar{x} \leq_{\overline{\mathbb{R}}} \bar{y}$ by cases.

On the one hand, if $\bar{x} <_{\mathbb{R}} \bar{y}$ holds, then we may also write this as $(\bar{x}, \bar{y}) \in \, <_{\mathbb{R}}$, which then implies $(\bar{x}, \bar{y}) \in \, <_{\overline{\mathbb{R}}}$ with Corollary 9.6 and the definition of a subset. We may write this also as $\bar{x} <_{\overline{\mathbb{R}}} \bar{y}$. According to the Characterization of induced reflexive partial orderings, this inequality implies $\bar{x} \leq_{\overline{\mathbb{R}}} \bar{y}$. On the other hand, if $\bar{x} = \bar{y}$ holds, then we obtain immediately $\bar{x} \leq_{\overline{\mathbb{R}}} \bar{y}$ with the Characterization of induced reflexive partial orderings, which inequality thus holds in both cases.

We may write that inequality also as $[Z =]\ (\bar{x}, \bar{y}) \in\ \leq_{\overline{\mathbb{R}}}$, which is the desired consequent of the implication in (9.48). As Z was arbitrary, we may therefore conclude that the universal sentence (9.48) and thus the proposed inclusion holds. $\square$

Exercise 9.3. Show that a real number x is less than or equal to a real number y with respect to the standard total ordering of $\mathbb{R}$ if x is less than or equal to y with respect to the standard total ordering of $\overline{\mathbb{R}}$, that is,

$$\forall x, y\,(x, y \in \mathbb{R} \Rightarrow [x \leq_{\mathbb{R}} y \Leftrightarrow x \leq_{\overline{\mathbb{R}}} y]). \tag{9.49}$$

(Hint: Use some of the arguments in the proof of Proposition 9.12.)

The definition of $<_{\overline{\mathbb{R}}}$ and its property of being a linear ordering implies the following inequalities.

Corollary 9.13. *It is true that any extended real number D is*

a) greater than or equal to $-\infty$, i.e.

$$\forall D\,(D \in \overline{\mathbb{R}} \Rightarrow -\infty \leq_{\overline{\mathbb{R}}} D). \tag{9.50}$$

b) less than or equal to $+\infty$, i.e.

$$\forall D\,(D \in \overline{\mathbb{R}} \Rightarrow D \leq_{\overline{\mathbb{R}}} +\infty). \tag{9.51}$$

Furthermore, it is true that $-\infty$ is the minimum and $+\infty$ the maximum of the set of extended real numbers, that is,

$$\overset{\leq_{\overline{\mathbb{R}}}}{\min}\,\overline{\mathbb{R}} = -\infty, \tag{9.52}$$

$$\overset{\leq_{\overline{\mathbb{R}}}}{\max}\,\overline{\mathbb{R}} = +\infty. \tag{9.53}$$

Proof. We obtain from the negations (9.29) and (9.31) the inequalities (9.50) and (9.51), respectively, by applying the Negation Formula for $<$ (applied to the linear ordering of $\overline{\mathbb{R}}$). As D was arbitrary, we may therefore conclude that the stated sentences are true for any D.

Then, as the universal sentences (9.50) and (9.51) mean that $-\infty$ is a lower and $+\infty$ an upper bound for $\overline{\mathbb{R}}$, which bounds are both contained in $\overline{\mathbb{R}}$ according to (9.10) and (9.11), it follows by definition that $-\infty$ is the minimum and $+\infty$ the maximum of $\overline{\mathbb{R}}$. $\square$

Definition 9.5 (Negative extended real number, nonnegative extended real number, positive extended real number). We say that an extended real number D is

(1) *negative* iff

$$D <_{\overline{\mathbb{R}}} 0. \tag{9.54}$$

(2) *nonnegative* iff

$$0 \leq_{\overline{\mathbb{R}}} D. \tag{9.55}$$

(3) *positive* iff

$$0 <_{\overline{\mathbb{R}}} D. \tag{9.56}$$

Note 9.3. We see in light of the Axiom of Specification and the Equality Criterion for sets that there exist unique sets $\overline{\mathbb{R}}_-$, $\overline{\mathbb{R}}_+^0$ and $\overline{\mathbb{R}}_+$ consisting, respectively, of all extended real numbers that are negative, nonnegative and positive, that is,

$$\forall D\,(D \in \overline{\mathbb{R}}_- \Leftrightarrow [D \in \overline{\mathbb{R}} \wedge D <_{\overline{\mathbb{R}}} 0]), \tag{9.57}$$

$$\forall D\,(D \in \overline{\mathbb{R}}_+^0 \Leftrightarrow [D \in \overline{\mathbb{R}} \wedge 0 \leq_{\overline{\mathbb{R}}} D]), \tag{9.58}$$

$$\forall D\,(D \in \overline{\mathbb{R}}_+ \Leftrightarrow [D \in \overline{\mathbb{R}} \wedge 0 <_{\overline{\mathbb{R}}} D]). \tag{9.59}$$

Since $D \in \overline{\mathbb{R}}_-$, $D \in \overline{\mathbb{R}}_+^0$ and $D \in \overline{\mathbb{R}}_+$ all imply $x \in \overline{\mathbb{R}}$ for any D, the inclusions

$$\overline{\mathbb{R}}_- \subseteq \overline{\mathbb{R}}, \tag{9.60}$$

$$\overline{\mathbb{R}}_+^0 \subseteq \overline{\mathbb{R}}, \tag{9.61}$$

$$\overline{\mathbb{R}}_+ \subseteq \overline{\mathbb{R}} \tag{9.62}$$

are true by definition of a subset.

Definition 9.6 (Set of negative & of nonnegative & of positive extended real numbers). We call

$$\overline{\mathbb{R}}_- \tag{9.63}$$

the *set of negative extended real numbers*,

$$\overline{\mathbb{R}}_+^0 \tag{9.64}$$

the *set of nonnegative extended real numbers*, and

$$\overline{\mathbb{R}}_+ \tag{9.65}$$

the *set of positive extended real numbers.*

Of these three sets, the set $\overline{\mathbb{R}}_+^0$ will be used most often in the sequel.

Note 9.4. Noting that the real number 0 is also an extended real number by virtue of the inclusion $\mathbb{R} \subseteq \overline{\mathbb{R}}$ in (9.13), Corollary 9.51 shows in particular that $0 \leq_{\overline{\mathbb{R}}} +\infty$ holds, so that

$$+\infty \in \overline{\mathbb{R}}_+^0 \tag{9.66}$$

is true according to the specification (9.58) of $\overline{\mathbb{R}}_+^0$. Thus,

$$\{+\infty\} \subseteq \overline{\mathbb{R}}_+^0 \tag{9.67}$$

also holds by virtue of (2.184). Moreover, the fact $+\infty \notin \mathbb{R}$ in (9.9) implies

$$+\infty \notin \mathbb{R}_+^0 \tag{9.68}$$

with the known inclusion $\mathbb{R}_+^0 \subseteq \mathbb{R}$ in (8.226) and (2.9). Because the total ordering $\leq_{\overline{\mathbb{R}}}$ is reflexive, we also have $0 \leq_{\overline{\mathbb{R}}} 0$, so that

$$0 \in \overline{\mathbb{R}}_+^0. \tag{9.69}$$

Exercise 9.4. Show that the set of nonnegative real numbers is included in the set of nonnegative extended real numbers, that is,

$$\mathbb{R}_+^0 \subseteq \overline{\mathbb{R}}_+^0. \tag{9.70}$$

(Hint: Use (8.220), (9.49) and (9.58).)

Proposition 9.14. *The set of nonnegative extended real numbers is obtained by the union of the set of nonnegative real numbers and the singleton formed by* $+\infty$*, that is,*

$$\overline{\mathbb{R}}_+^0 = \mathbb{R}_+^0 \cup \{+\infty\}. \tag{9.71}$$

Proof. We apply the Equality Criterion for sets and take for this purpose an arbitrary $D \in \overline{\mathbb{R}}_+^0$ first. Therefore, $D \in \overline{\mathbb{R}}$ and $0 \leq_{\overline{\mathbb{R}}} D$ are true according to (9.58). The former implies that $D \in \mathbb{R}$, $D = -\infty$ or $D = +\infty$ holds, as shown by (9.17). Here, we may prove by contradiction that $D \neq -\infty$ holds. Assuming for this purpose the negation of that negation to be true, so that the Double Negation Law yields the true equation $D = -\infty$, we obtain from the previous inequality $0 \leq_{\overline{\mathbb{R}}} -\infty$. The Negation Formula for $<$ gives us then the true negation $\neg -\infty <_{\overline{\mathbb{R}}} 0$, in contradiction to the fact $-\infty <_{\overline{\mathbb{R}}} 0$ obtained from (9.28). Thus, the negation $D \neq -\infty$ holds indeed, which means that the second part of the previous multiple disjunction is false. Consequently, only $D \in \mathbb{R}$ or $D = +\infty$ can be true. We now use this disjunction to prove the disjunction

$$D \in \mathbb{R}_+^0 \ \vee \ D \in \{+\infty\} \tag{9.72}$$

by cases. In the first case $D \in \mathbb{R}$, we can write $0 \leq_{\overline{\mathbb{R}}} D$ also as $0 \leq_{\mathbb{R}} D$ because of (9.49). This means $D \in \mathbb{R}_+^0$ in view of (8.220), with the consequence that the desired disjunction (9.72) is also true. The second case $D = +\infty$ implies $D \in \{+\infty\}$ with (2.169), so that the desired disjunction holds once again. Applying now the definition of the union of two sets, we therefore obtain $D \in \mathbb{R}_+^0 \cup \{+\infty\}$, as desired.

Assuming now conversely that D is an element of that union, we thus have that the disjunction (9.72) holds. On the one hand, $D \in \mathbb{R}_+^0$ implies $D \in \overline{\mathbb{R}}_+^0$ with the inclusion (9.70). On the other hand, $D \in \{+\infty\}$ implies evidently $D = +\infty$ and therefore $D \in \overline{\mathbb{R}}_+^0$ in view of (9.66). We thus obtain $D \in \overline{\mathbb{R}}_+^0$ in any case.

Since D was arbitrary, we may therefore conclude that the equality (9.71) holds indeed. □

The equation (9.71) immediately yields with the definition of the union of two sets and (2.169) the following characterization of nonnegative extended real numbers.

Corollary 9.15. *A constant is a nonnegative extended real number iff it is a nonnegative real number or* $+\infty$*, that is,*

$$\forall D\,(D \in \overline{\mathbb{R}}_+^0 \Leftrightarrow [D \in \mathbb{R}_+^0 \ \vee \ D = +\infty]). \tag{9.73}$$

Definition 9.7 (Nonnegative numerical function). We call for any set X any function

$$f : X \to \overline{\mathbb{R}}_+^0 \tag{9.74}$$

a *nonnegative numerical function* (on X) and

$$[\overline{\mathbb{R}}_+^0]^X \tag{9.75}$$

the *set of nonnegative numerical functions* (on X).

Note 9.5. In view of the definition of a codomain and (9.61), we have for any nonnegative numerical function $f : X \to \overline{\mathbb{R}}_+^0$ the inclusions

$$\operatorname{ran}(f) \subseteq \overline{\mathbb{R}}_+^0 \subseteq \overline{\mathbb{R}}, \tag{9.76}$$

so that $\overline{\mathbb{R}}$ follows to be a codomain of f as well (applying (2.13)). Thus, any nonnegative numerical function f on X is indeed a numerical function on X. In addition, any nonnegative real function $f : X \to \mathbb{R}_+^0$ turns out to be a (nonnegative) numerical function due to the inclusions

$$\operatorname{ran}(f) \subseteq \mathbb{R}_+^0 \subseteq \overline{\mathbb{R}}_+^0 \subseteq \overline{\mathbb{R}}, \tag{9.77}$$

according to (9.70) and the previous inclusions.

Proposition 9.16. *It is true that the closed interval in* $\overline{\mathbb{R}}$ *from a real number* x *to a real number* y *is identical with the closed interval in* $\mathbb{R}$ *from* x *to* y*, that is,*

$$\forall x, y\,(x, y \in \mathbb{R} \Rightarrow [x, y]_{\overline{\mathbb{R}}} = [x, y]_{\mathbb{R}}). \tag{9.78}$$

Proof. Letting x and y be arbitrary real numbers, with then also have $x, y \in \overline{\mathbb{R}}$ due to the inclusion (9.13) and the definition of a subset. Thus, the closed interval $[x, y]_{\overline{\mathbb{R}}}$ in $\overline{\mathbb{R}}$ from x to y is indeed defined. Letting now also a be arbitrary, we obtain the equivalences

$$\begin{aligned} a \in [x, y]_{\overline{\mathbb{R}}} &\Leftrightarrow x \leq_{\overline{\mathbb{R}}} a \wedge a \leq_{\overline{\mathbb{R}}} y \\ &\Leftrightarrow x \leq_{\mathbb{R}} a \wedge a \leq_{\mathbb{R}} y \\ &\Leftrightarrow a \in [x, y]_{\mathbb{R}} \end{aligned}$$

by applying the definition of a closed interval in $\overline{\mathbb{R}}$, (9.49) in connection with the initial assumption $x, y \in \mathbb{R}$, and the definition of a closed interval in $\mathbb{R}$. Since a is arbitrary, we may infer from the resulting equivalence $a \in [x, y]_{\overline{\mathbb{R}}} \Leftrightarrow a \in [x, y]_{\mathbb{R}}$ the truth of the equation $[x, y]_{\overline{\mathbb{R}}} = [x, y]_{\mathbb{R}}$ by means of the Equality Criterion for sets. Here, x and y were also arbitrary, so that the proposed universal sentence (9.78) follows now to be true. □

Exercise 9.5. Prove that

a) the open interval in $\overline{\mathbb{R}}$ from a real number x to a real number y is identical with the open interval in $\mathbb{R}$ from x to y, that is,

$$\forall x, y\,(x, y \in \mathbb{R} \Rightarrow (x, y)_{\overline{\mathbb{R}}} = (x, y)_{\mathbb{R}}). \tag{9.79}$$

b) the left-open and right-closed interval in $\overline{\mathbb{R}}$ from a real number x to a real number y is identical with the left-open and right-closed interval in $\mathbb{R}$ from x to y, that is,

$$\forall x, y\,(x, y \in \mathbb{R} \Rightarrow (x, y]_{\overline{\mathbb{R}}} = (x, y]_{\mathbb{R}}). \tag{9.80}$$

c) the left-closed and right-open interval in $\overline{\mathbb{R}}$ from a real number x to a real number y is identical with the left-closed and right-open interval in $\mathbb{R}$ from x to y, that is,

$$\forall x, y\,(x, y \in \mathbb{R} \Rightarrow [x, y)_{\overline{\mathbb{R}}} = [x, y)_{\mathbb{R}}). \tag{9.81}$$

Proposition 9.17. *It is true that* $\mathbb{R}$ *is a convex set in* $\overline{\mathbb{R}}$ *with respect to* $<_{\overline{\mathbb{R}}}$.

Proof. In view of the inclusion (9.13), $\mathbb{R}$ satisfies Property 1 of a convex set in $\overline{\mathbb{R}}$ with respect to $<_{\overline{\mathbb{R}}}$. To establish Property 2, we prove the universal sentence

$$\forall x, y\,(x, y \in \mathbb{R} \Rightarrow (x, y)_{\overline{\mathbb{R}}} \subseteq \mathbb{R}), \tag{9.82}$$

letting x and y be arbitrary real numbers; because of the preceding inclusion, we then also have $x \in \overline{\mathbb{R}}$ and $y \in \overline{\mathbb{R}}$, by definition of a subset. To establish now the desired inclusion $(x, y)_{\overline{\mathbb{R}}} \subseteq \mathbb{R}$, we apply again the definition of a subset and prove the equivalent universal sentence

$$\forall a\,(a \in (x, y)_{\overline{\mathbb{R}}} \Rightarrow a \in \mathbb{R}), \tag{9.83}$$

letting a be arbitrary and assuming a to be an element of the open interval in $\overline{\mathbb{R}}$ from x to y. The definition of that interval gives us then $a \in \overline{\mathbb{R}}$ and the inequalities $x <_{\overline{\mathbb{R}}} a <_{\overline{\mathbb{R}}} y$. The former finding implies now the truth of the disjunction

$$a = -\infty \lor a \in \mathbb{R} \lor a = +\infty \tag{9.84}$$

with (9.17). We now prove $\neg a = -\infty$ by contradiction, assuming the negation of that sentence to be true, so that the Double Negation Law yields $a = -\infty$. Due to this equation, the previously established $x <_{\overline{\mathbb{R}}} a$ gives $x <_{\overline{\mathbb{R}}} -\infty$ via substitution; since the negation $\neg x <_{\overline{\mathbb{R}}} -\infty$ also holds due to (9.29), we obtained a contradiction, so that the proof of $a \neq -\infty$ is now complete. Next, we prove in a similar way $a \neq +\infty$ by contradiction, assuming the negation of that negation to be true, so that $a = +\infty$ holds. With this equation, the previously obtained inequality $a <_{\overline{\mathbb{R}}} y$ yields $+\infty <_{\overline{\mathbb{R}}} y$, which contradicts (9.31) and which therefore proves $\neg a = +\infty$. We thus showed that the first and the third part of the true disjunction (9.84) are false, so that the second part $a \in \mathbb{R}$ must be true. Because a is arbitrary, we may now infer from this finding the truth of the universal sentence (9.83) and therefore the truth of the inclusion $(x, y)_{\overline{\mathbb{R}}} \subseteq \mathbb{R}$. Since x and y are also arbitrary, we may then further conclude that the universal sentence (9.82) is true, which demonstrates that $\mathbb{R}$ satisfies Property 1 of a convex set in $\overline{\mathbb{R}}$ with respect to $<_{\overline{\mathbb{R}}}$. □

Proposition 9.18. *Any numerical function $f : X \to \overline{\mathbb{R}}$ is bounded from below (by $-\infty$).*

Proof. We let X be any set, f any numerical function, and we verify that $-\infty$ is a lower bound for the range of f, that is,

$$\forall y \, (y \in \operatorname{ran}(f) \Rightarrow -\infty \leq_{\overline{\mathbb{R}}} y). \tag{9.85}$$

For this purpose, we let $\bar{y}$ be arbitrary in $\operatorname{ran}(f)$. By definition of a range, there then exists an element, say $\bar{x}$, such that $(\bar{x}, \bar{y}) \in f$ holds, which we may also write as $\bar{y} = f(\bar{x})$; thus, $f(\bar{x}) \in \operatorname{ran}(f)$. As f is by assumption a function with codomain $\overline{\mathbb{R}}$, we have the inclusion $\operatorname{ran}(f) \subseteq \overline{\mathbb{R}}$, so that $f(\bar{x}) \in \operatorname{ran}(f)$ implies $f(\bar{x}) \in \overline{\mathbb{R}}$ by definition of a subset. This in turn implies $-\infty \leq_{\overline{\mathbb{R}}} f(\bar{x})$ with (9.50), so that the previously obtained equation $\bar{y} = f(\bar{x})$ gives $-\infty \leq_{\overline{\mathbb{R}}} \bar{y}$, as desired. Since $\bar{y}$ was arbitrary, we therefore conclude that the universal sentence (9.85) holds, which shows that $-\infty$ is a lower bound for (the range of) f. Because f was also arbitrary, it then follows that the proposition is true. □

Exercise 9.6. Show that any numerical function $f : X \to \overline{\mathbb{R}}$ is bounded from above (by $+\infty$).

Note 9.6. Proposition 9.18 and Exercise 9.6 show that every numerical function $f : X \to \overline{\mathbb{R}}$, and thus in particular every sequence $(D_i)_{i \in \mathbb{N}_+}$ or $(D_i)_{i \in \mathbb{N}}$ in $\overline{\mathbb{R}}$, is bounded.

Proposition 9.19. *It is true that $-\infty$ is the least upper and $+\infty$ the greatest lower bound for the empty set with respect to the standard total ordering of $\overline{\mathbb{R}}$, that is,*

$$\overset{\leq_{\overline{\mathbb{R}}}}{\sup} \emptyset = -\infty, \tag{9.86}$$

$$\overset{\leq_{\overline{\mathbb{R}}}}{\inf} \emptyset = +\infty. \tag{9.87}$$

Proof. Recalling the fact $-\infty, +\infty \in \overline{\mathbb{R}}$ from (9.10) – (9.11), we have in particular that $\bar{S} = -\infty$ is an upper and $\bar{I} = +\infty$ a lower bound for $\emptyset$ (with respect to $\leq_{\overline{\mathbb{R}}}$), according to Proposition 3.90 and Exercise 3.34. We now apply the Characterization of the supremum & infimum to demonstrate that $\bar{S}$ is the supremum and $\bar{I}$ the infimum of $\emptyset$ (with respect to $\leq_{\overline{\mathbb{R}}}$). To do this, we let S' and I' be arbitrary such that S' is an upper and I' a lower bound for $\emptyset$ (with respect to $\leq_{\overline{\mathbb{R}}}$), and we show that $\bar{S} \leq_{\overline{\mathbb{R}}} S'$ and $I' \leq_{\overline{\mathbb{R}}} \bar{I}$ follow to be true. By definition the bounds S' and I' are both elements of $\overline{\mathbb{R}}$, so that the inequalities $-\infty \leq_{\overline{\mathbb{R}}} S'$ and $I' \leq_{\overline{\mathbb{R}}} +\infty$ follow to be true with (9.50) and (9.51). We therefore obtain the desired inequalities via substitutions. Since S' and I' were arbitrary, we thus see that the upper bound $\bar{S} = -\infty$ for $\emptyset$ is the least one and that the lower bound $\bar{I} = +\infty$ for $\emptyset$ is the greatest one. $\square$

Lemma 9.20. *It is true for any two extended real numbers D and E with $D <_{\overline{\mathbb{R}}} E$ that there exists a rational number which is strictly between D and E, that is,*

$$\forall D, E\,([D, E \in \overline{\mathbb{R}} \wedge D <_{\overline{\mathbb{R}}} E] \Rightarrow \exists q\,(q \in \mathbb{Q} \wedge D <_{\overline{\mathbb{R}}} q <_{\overline{\mathbb{R}}} E)). \tag{9.88}$$

Proof. We let D and E be arbitrary extended real numbers such that D is smaller than E. Therefore, $D \neq +\infty$, because $D = +\infty$ implies $+\infty <_{\overline{\mathbb{R}}} E$ in contradiction to (9.31). Thus, $D \in \mathbb{R}$ or $D = -\infty$ holds in view of (9.38). In addition, we find $E \neq -\infty$ because $E = -\infty$ implies $D <_{\overline{\mathbb{R}}} -\infty$, which contradicts (9.31). Thus, $E \in \mathbb{R}$ or $E = +\infty$ holds in view of (9.40).

In the first case $D \in \mathbb{R}$ and the first subcase $E \in \mathbb{R}$, the initial assumption $D <_{\overline{\mathbb{R}}} E$ implies $D <_{\mathbb{R}} E$ with (9.32) since D and E are both real. Recalling that $(\mathbb{R}, <_{\mathbb{R}})$ is separably ordered with respect to (the image $\mathbb{Q}_{\mathbb{R}}$) of $\mathbb{Q}$ (see Corollary 8.12), there exists a particular rational number $\bar{q}$ (in $\mathbb{R}$) such that $D <_{\mathbb{R}} \bar{q} <_{\mathbb{R}} E$. We may evidently write these inequalities also as $D <_{\overline{\mathbb{R}}} \bar{q} <_{\overline{\mathbb{R}}} E$, so that the existential sentence in (9.88) holds in the first subcase. On the other hand, if $E = +\infty$ is true, we observe that the current case assumption $D \in \mathbb{R}$ implies the existence of a particular rational number $\bar{q} >_{\mathbb{R}} D$ (in $\mathbb{R}$) by virtue of (8.105). Clearly, we may write this inequality also as $D <_{\overline{\mathbb{R}}} \bar{q}$. Furthermore, the fact that $\bar{q}$ is real implies $\bar{q} <_{\overline{\mathbb{R}}} +\infty$ with (9.28), and therefore $\bar{q} <_{\overline{\mathbb{R}}} E$ via substitution. We thus showed that there is a rational number strictly between D and E (with respect to $<_{\overline{\mathbb{R}}}$) also in the second subcase.

In the second case $D = -\infty$ and the first subcase $E \in \mathbb{R}$, we now see that there exists a particular rational number $\bar{p} <_{\mathbb{R}} E$, again due to (8.105); thus, $\bar{p} <_{\overline{\mathbb{R}}} E$. We also find $-\infty <_{\overline{\mathbb{R}}} \bar{p}$ because of (9.28), so that substitution yields $D <_{\overline{\mathbb{R}}} \bar{p}$. Having found a particular rational number strictly between D and E, we conclude that the desired existential sentence holds again. It remains for us to observe that the second subcase $E = +\infty$ allows us to choose, for instance, the rational number 0 (in $\mathbb{R}$) since it evidently satisfies $-\infty <_{\overline{\mathbb{R}}} 0 <_{\overline{\mathbb{R}}} +\infty$, and therefore $D <_{\overline{\mathbb{R}}} 0 <_{\overline{\mathbb{R}}} E$. The existence of such a rational number completes the proof(s) by cases.

As D and E were initially arbitrary, we may therefore conclude that the universal sentence (9.88) is true. $\square$

Theorem 9.21. *It is true that the linearly ordered set of extended real numbers*

a) $(\overline{\mathbb{R}}, <_{\overline{\mathbb{R}}})$ *is densely ordered.*

b) $(\overline{\mathbb{R}}, <_{\overline{\mathbb{R}}})$ *is separably ordered with respect to* $\mathbb{Q}$*.*

Proof. We already verified that $(\overline{\mathbb{R}}, <_{\overline{\mathbb{R}}})$ is a linearly ordered set (see Theorem 9.10). Next, we show that $(\overline{\mathbb{R}}, <_{\overline{\mathbb{R}}})$ is a densely ordered set. Since $\overline{\mathbb{R}}$ is evidently neither empty nor a singleton, it remains for us to verify that there exists an intermediate value for any $D, E \in \overline{\mathbb{R}}$ with $D <_{\overline{\mathbb{R}}} E$. We let D and E be arbitrary extended real numbers such that $D <_{\overline{\mathbb{R}}} E$ is satisfied. According to Lemma 9.20, there exists a particular rational number $\bar{q}$ strictly between D and E. Since rationals are contained in the reals and all real numbers contained in the extended real numbers (i.e., $\mathbb{Q} \subseteq \mathbb{R} \subseteq \overline{\mathbb{R}}$) according to (8.55) – (8.56) and (9.13), we may infer from these findings that $(\overline{\mathbb{R}}, <_{\overline{\mathbb{R}}})$ is densely ordered, having $\mathbb{Q}$ as a dense subset. □

9.2. The Complete Lattice $(\overline{\mathbb{R}}_+^0, \leq_{\overline{\mathbb{R}}_+^0})$

Note 9.7. According to the Total ordering of subsets, the total ordering $\leq_{\overline{\mathbb{R}}}$ gives rise to the total ordering $\leq_{\overline{\mathbb{R}}_+^0}$ of the subset $\overline{\mathbb{R}}_+^0$ of $\overline{\mathbb{R}}$, such that

$$\forall D, E\,(D, E \in \overline{\mathbb{R}}_+^0 \Rightarrow [D \leq_{\overline{\mathbb{R}}_+^0} E \Leftrightarrow D \leq_{\overline{\mathbb{R}}} E]). \tag{9.89}$$

This total ordering $\leq_{\overline{\mathbb{R}}_+^0}$ induces then the corresponding linear ordering $<_{\overline{\mathbb{R}}_+^0}$.

Corollary 9.22. *The linear ordering $<_{\overline{\mathbb{R}}_+^0}$ satisfies*

$$\forall D, E\,(D, E \in \overline{\mathbb{R}}_+^0 \Rightarrow [D <_{\overline{\mathbb{R}}_+^0} E \Leftrightarrow D <_{\overline{\mathbb{R}}} E]). \tag{9.90}$$

Proof. Letting $D, E \in \overline{\mathbb{R}}_+^0$ be arbitrary, we find the equivalences

$$\begin{aligned} D <_{\overline{\mathbb{R}}_+^0} E \;&\Leftrightarrow\; [D \leq_{\overline{\mathbb{R}}_+^0} E \,\wedge\, D \neq E] \\ &\Leftrightarrow\; [D \leq_{\overline{\mathbb{R}}} E \,\wedge\, D \neq E] \\ &\Leftrightarrow\; D <_{\overline{\mathbb{R}}} E \end{aligned}$$

by means of the Characterization of induced irreflexive partial orderings and the equivalence in (9.89). □

Definition 9.8 (Standard linear & total ordering of $\overline{\mathbb{R}}_+^0$). We call $<_{\overline{\mathbb{R}}_+^0}$ the *standard linear ordering of* $\overline{\mathbb{R}}_+^0$ and $\leq_{\overline{\mathbb{R}}_+^0}$ the *standard total ordering of* $\overline{\mathbb{R}}_+^0$.

Proposition 9.23. *There exists a positive rational number strictly between any nonnegative extended real number D and any larger nonnegative extended real number E, that is,*

$$\forall D, E\,([D, E \in \overline{\mathbb{R}}_+^0 \wedge D <_{\overline{\mathbb{R}}_+^0} E] \Rightarrow \exists q\,(q \in \mathbb{Q}_+ \wedge D <_{\overline{\mathbb{R}}_+^0} q <_{\overline{\mathbb{R}}_+^0} E)). \tag{9.91}$$

Proof. We take arbitrary numbers $D, E \in \overline{\mathbb{R}}_+^0$, so that $0 \leq_{\overline{\mathbb{R}}} D$ and $0 \leq_{\overline{\mathbb{R}}} E$ holds by definition of the set of nonnegative extended real numbers. Assuming now $D <_{\overline{\mathbb{R}}_+^0} E$ to be true, we may write this inequality also as $D <_{\overline{\mathbb{R}}} E$ by virtue of (9.89). Since $\mathbb{Q}$ is a dense subset of $\overline{\mathbb{R}}$ (see Theorem 9.21), there exists a rational number, say $\bar{q}$, such that

$$[0 \leq_{\overline{\mathbb{R}}}] \quad D <_{\overline{\mathbb{R}}} \bar{q} <_{\overline{\mathbb{R}}} E. \tag{9.92}$$

The Transitivity Formula for $\leq$ and $<$ yields $0 <_{\overline{\mathbb{R}}} \bar{q}$. According to the Characterization of induced reflexive partial orderings, $0 \leq_{\overline{\mathbb{R}}} \bar{q}$ is then also true, which shows that $\bar{q} \in \overline{\mathbb{R}}^0_+$. Therefore, the inequalities (9.92) imply

$$D <_{\overline{\mathbb{R}}^0_+} \bar{q} <_{\overline{\mathbb{R}}^0_+} E \tag{9.93}$$

with (9.90). Clearly, the rational number $\bar{q}$ may be treated as a real number, so that the previously established inequality $0 <_{\overline{\mathbb{R}}} \bar{q}$ may evidently be written as $0 <_{\mathbb{R}} \bar{q}$, or as $0 <_{\mathbb{Q}} \bar{q}$ in view of (8.58). The latter inequality shows that $\bar{q}$ is a positive rational number (by definition). In conjunction with (9.93), this finding demonstrates the truth of the existential sentence (9.91), and since D and E were initially arbitrary, we may therefore conclude that the universal sentence (9.91) holds. □

Corollary 9.24. *The total ordering $\leq_{\overline{\mathbb{R}}^0_+}$ and the linear ordering $<_{\overline{\mathbb{R}}^0_+}$ satisfy*

$$\forall x, y\,(x, y \in \mathbb{R}^0_+ \Rightarrow [x \leq_{\mathbb{R}^0_+} y \Leftrightarrow x \leq_{\overline{\mathbb{R}}^0_+} y]) \tag{9.94}$$

and

$$\forall x, y\,(x, y \in \mathbb{R}^0_+ \Rightarrow [x <_{\mathbb{R}^0_+} y \Leftrightarrow x <_{\overline{\mathbb{R}}^0_+} y]). \tag{9.95}$$

Proof. Letting x and y be arbitrary, we first observe that the assumption $x, y \in \mathbb{R}^0_+$ implies $x, y \in \mathbb{R}$, $x, y \in \overline{\mathbb{R}}^0_+$ and $x, y \in \overline{\mathbb{R}}$ with the inclusions (8.226), (9.70) and (9.61). We obtain therefore the true equivalences

$$\begin{array}{ccccccc} x <_{\mathbb{R}^0_+} y & \Leftrightarrow & x <_{\mathbb{R}} y & \Leftrightarrow & x <_{\overline{\mathbb{R}}} y & \Leftrightarrow & x <_{\overline{\mathbb{R}}^0_+} y, \\ x \leq_{\mathbb{R}^0_+} y & \Leftrightarrow & x \leq_{\mathbb{R}} y & \Leftrightarrow & x \leq_{\overline{\mathbb{R}}} y & \Leftrightarrow & x \leq_{\overline{\mathbb{R}}^0_+} y \end{array}$$

by means of (8.341) & (8.342), (9.32) & (9.49), and (9.90) & (9.89). □

Proposition 9.25. *A sequence $f = (x_n)_{n \in \mathbb{N}_+}$ in $\mathbb{R}^0_+$ is increasing with respect to $\leq_{\mathbb{R}^0_+}$ iff f is increasing with respect to $\leq_{\overline{\mathbb{R}}^0_+}$.*

Proof. We let $f = (x_n)_{n \in \mathbb{N}_+}$ be an arbitrary sequence in $\mathbb{R}^0_+$. Thus, the range of f is clearly a subset of $\mathbb{R}^0_+$. Due to the inclusion $\mathbb{R}^0_+ \subseteq \overline{\mathbb{R}}^0_+$ in (9.70), f is also a sequence in $\overline{\mathbb{R}}^0_+$ due to (3.519). By the Function Criterion, the terms $D_n = f(n)$ and $D_{n+1} = f(n+1)$ are therefore elements both of $\mathbb{R}^0_+$ and $\overline{\mathbb{R}}^0_+$ for any $n \in \mathbb{N}_+$. In view of the Monotony Criterion for increasing sequences, the stated equivalence is equivalent to the equivalence of

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow D_n \leq_{\mathbb{R}^0_+} D_{n+1}) \tag{9.96}$$

and

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow D_n \leq_{\overline{\mathbb{R}}^0_+} D_{n+1}). \tag{9.97}$$

Letting $n \in \mathbb{N}_+$ be arbitrary, we see in light of (9.94) on the one hand that $D_n \leq_{\mathbb{R}^0_+} D_{n+1}$ implies $D_n \leq_{\overline{\mathbb{R}}^0_+} D_{n+1}$ (which proves the implication in (9.97)), and on the other hand that $D_n \leq_{\overline{\mathbb{R}}^0_+} D_{n+1}$ implies $D_n \leq_{\mathbb{R}^0_+} D_{n+1}$ (which proves the implication in (9.96)). We may therefore conclude that (9.96) implies (9.97) and vice versa, so that the proposed equivalence is true. As f was initially arbitrary, we may now further conclude that the proposed universal sentence holds. □

Corollary 9.26. *It is true that every nonnegative extended real number D is*

a) greater than or equal to 0 *(with respect to $\leq_{\overline{\mathbb{R}}^0_+}$), i.e.*

$$\forall D\,(D \in \overline{\mathbb{R}}^0_+ \Rightarrow 0 \leq_{\overline{\mathbb{R}}^0_+} D). \tag{9.98}$$

b) less than or equal to $+\infty$ *(with respect to $\leq_{\overline{\mathbb{R}}^0_+}$), i.e.*

$$\forall D\,(D \in \overline{\mathbb{R}}^0_+ \Rightarrow D \leq_{\overline{\mathbb{R}}^0_+} +\infty). \tag{9.99}$$

Furthermore, it is true that 0 *is the minimum and* $+\infty$ *the maximum of the set of nonnegative extended real numbers (with respect to $\leq_{\overline{\mathbb{R}}^0_+}$), that is,*

$$\overset{\leq_{\overline{\mathbb{R}}^0_+}}{\min}\,\overline{\mathbb{R}}^0_+ = 0, \tag{9.100}$$

$$\overset{\leq_{\overline{\mathbb{R}}^0_+}}{\max}\,\overline{\mathbb{R}}^0_+ = +\infty. \tag{9.101}$$

Proof. Letting $D \in \overline{\mathbb{R}}^0_+$ be arbitrary, we have $0 \leq_{\overline{\mathbb{R}}} D$ by definition of the set of nonnegative extended real numbers, and this inequality implies $0 \leq_{\overline{\mathbb{R}}^0_+} D$ with (9.89). Furthermore, observing that D is an element of $\overline{\mathbb{R}}$, the inequality $D \leq_{\overline{\mathbb{R}}} +\infty$ is true according to (9.51), and this gives $D \leq_{\overline{\mathbb{R}}^0_+} +\infty$ again with (9.89). Since D was arbitrary, we therefore conclude that the universal sentences (9.98) and (9.99) both hold. Thus, 0 is a lower and $+\infty$ an upper bound for $\overline{\mathbb{R}}^0_+$ with respect to $\leq_{\overline{\mathbb{R}}^0_+}$. These bounds are both elements of $\overline{\mathbb{R}}^0_+$, as shown by (9.69) and (9.66). Consequently, 0 is the minimum and $+\infty$ the maximum of $\overline{\mathbb{R}}^0_+$ with respect to $\leq_{\overline{\mathbb{R}}^0_+}$, by definition. □

Exercise 9.7. Prove that any positive real number is less than $+\infty$, i.e,

$$\forall x\,(x \in \mathbb{R}^0_+ \Rightarrow x <_{\overline{\mathbb{R}}^0_+} +\infty). \tag{9.102}$$

(Hint: Use (9.68), (2.4), (9.70), (9.99) and (3.252).)

Theorem 9.27. *It is true that the linearly ordered set of nonnegative extended real numbers*

a) $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ *is densely ordered.*

b) $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ *is separably ordered with respect to* $\mathbb{Q}_+$*.*

c) $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ *is a linear continuum.*

d) $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ *has the Infimum Property.*

Proof. Since $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ is linearly ordered, it remains to prove that this set is densely ordered set. As the set $\overline{\mathbb{R}}^0_+$ contains, for instance, the numbers 0 and $+\infty$, it is clear that this set is neither empty nor a singleton. Letting now D and E be arbitrary elements of $\overline{\mathbb{R}}^0_+$ satisfying $D <_{\overline{\mathbb{R}}^0_+} E$, we see in light of (9.91) that there exists a particular number $\bar{q} \in \mathbb{Q}_+$ strictly between D and E. The proof of Proposition 9.23 showed that $\bar{q} \in \overline{\mathbb{R}}^0_+$ also holds, so that $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ is indeed densely ordered, with $\mathbb{Q}_+$ being a dense subset.

To establish the Supremum Property for $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$, we let A be an arbitrary nonempty and bounded-from-above subset of $\overline{\mathbb{R}}^0_+$. We consider now the two cases $+\infty \in A$ and $+\infty \notin A$. In the first case $+\infty \in A$, we recall from (9.101) that $+\infty$ is an upper bound for $\overline{\mathbb{R}}^0_+$; evidently, the subset A has then the same upper bound $(+\infty)$ in view of Proposition 3.94. As this upper bound is contained in A, it constitutes the maximum of A, and this implies $+\infty = \sup^{\leq_{\overline{\mathbb{R}}^0_+}}$ with Theorem 3.105a). Thus, the supremum of A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$ exists in the first case.

In the second case $+\infty \notin A$, we first establish the inclusion $A \subseteq \mathbb{R}^0_+$. Letting $x \in A$ be arbitrary, we obtain $x \in \overline{\mathbb{R}}^0_+$ with the assumed inclusion $A \subseteq \overline{\mathbb{R}}^0_+$ (using the definition of a subset), and therefore $x \in \mathbb{R}^0_+ \vee x = +\infty$ because of (9.73). Here, $x = +\infty$ is false since the truth of that equation would imply $x \notin A$ with the current case assumption, in contradiction to the fact $x \in A$. Thus, the first part $x \in \mathbb{R}^0_+$ of the preceding disjunction is true, and as x was arbitrary, we may therefore conclude that the proposed inclusion holds indeed. We now consider the two subcases that A is bounded from above with respect to $\leq_{\mathbb{R}^0_+}$ or not.

In the first subcase, A is a nonempty and bounded from above subset of $\mathbb{R}^0_+$. Since $(\mathbb{R}^0_+, <_{\mathbb{R}^0_+})$ has the Supremum Property, it follows from this that its supremum $S = \sup^{\leq_{\mathbb{R}^0_+}}$ exists, which thus satisfies the conjunction

$$\begin{aligned}&\forall x\,(x \in A \Rightarrow x \leq_{\mathbb{R}^0_+} S)\\ &\quad\wedge\ \forall S'\,([S' \in \mathbb{R}^0_+ \,\wedge\, S' <_{\mathbb{R}^0_+} S] \Rightarrow \exists x\,(x \in A \,\wedge\, S' <_{\mathbb{R}^0_+} x)\end{aligned} \tag{9.103}$$

according to the Supremum Criterion. We may now prove that S is also the supremum of A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$. For this purpose, we demonstrate the truth of

$$\begin{aligned}&\forall x\,(x \in A \Rightarrow x \leq_{\overline{\mathbb{R}}^0_+} S)\\ &\quad\wedge\ \forall S'\,([S' \in \overline{\mathbb{R}}^0_+ \,\wedge\, S' <_{\overline{\mathbb{R}}^0_+} S] \Rightarrow \exists x\,(x \in A \,\wedge\, S' <_{\overline{\mathbb{R}}^0_+} x).\end{aligned} \tag{9.104}$$

Letting $x \in A$ be arbitrary, we find $x \leq_{\mathbb{R}^0_+} S$ with the first part of the conjunction (9.103). Evidently, we may write this inequality as $x \leq_{\overline{\mathbb{R}}^0_+} S$. As x was arbitrary, we may therefore conclude that the first part of the conjunction (9.104) holds.

To establish the second part, we let $S' \in \overline{\mathbb{R}}^0_+$ be arbitrary such that $S' <_{\overline{\mathbb{R}}^0_+} S$. The former evidently implies that $S' \in \mathbb{R}^0_+$ or $S' = +\infty$ is true. Here, we may prove by contradiction that $S' \neq +\infty$ holds. Assuming for this purpose the negation of that negation to be true, we have $S' = +\infty$ according to the Double Negation Law. Then, the assumed inequality yields $+\infty <_{\overline{\mathbb{R}}^0_+} S$ via substitution, and the negation $\neg + \infty \leq_{\overline{\mathbb{R}}^0_+} S$ follows to be true with the Negation Formula for $\leq$. Since $+\infty \leq_{\overline{\mathbb{R}}^0_+} S$ also holds according to (9.99), we arrived at a contradiction, so that the second part of the preceding disjunction is indeed false. Thus, its first part $S' \in \mathbb{R}^0_+$ is true. As the supremum S of A with respect to $\leq_{\mathbb{R}^0_+}$ is also an element of $\mathbb{R}^0_+$ (by definition), we may write the assumed inequality $S' <_{\overline{\mathbb{R}}^0_+} S$ also as $S' <_{\mathbb{R}^0_+} S$. In conjunction with $S' \in \mathbb{R}^0_+$, this implies with the second part of the conjunction (9.103) that there exists a particular element $\bar{x} \in A$ satisfying $S' <_{\mathbb{R}^0_+} \bar{x}$. Clearly, we may write this inequality also as $S' <_{\overline{\mathbb{R}}^0_+} \bar{x}$, which shows in conjunction with the fact $\bar{x} \in A$ that the existential sentence in (9.104) is true. Since S' was arbitrary, we may therefore conclude that the second part of the conjunction (9.104) also holds. We thus completed the proof that S is the supremum of A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$. This demonstration of the existence of such a supremum also completes the proof of the first subcase.

The second subcase that A is not bounded from above with respect to $\leq_{\mathbb{R}^0_+}$ means that

$$\neg \exists u\,(u \in \mathbb{R}^0_+ \wedge \forall x\,(x \in A \Rightarrow x \leq_{\mathbb{R}^0_+} u)).$$

This negation implies with the Negation Law for existential conjunctions, the Negation Law for universal implications and the Negation Formula for $\leq$ that

$$\forall u\,(u \in \mathbb{R}^0_+ \Rightarrow \exists x\,(x \in A \wedge u <_{\mathbb{R}^0_+} x)). \tag{9.105}$$

We may now show that the previously established upper bound $+\infty$ for A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$ is the greatest one. For this purpose, we apply the Supremum Criterion and verify accordingly

$$\forall S'\,([S' \in \overline{\mathbb{R}}^0_+ \wedge S' <_{\overline{\mathbb{R}}^0_+} +\infty] \Rightarrow \exists x\,(x \in A \wedge S' <_{\overline{\mathbb{R}}^0_+} x)). \tag{9.106}$$

We let S' be arbitrary, assuming $S' \in \overline{\mathbb{R}}^0_+$ and $S' <_{\overline{\mathbb{R}}^0_+} +\infty$ to be both true. The latter implies $S' \neq +\infty$ with the Characterization of comparability with respect to the linear ordering $<_{\overline{\mathbb{R}}^0_+}$. Therefore, $S' \in \overline{\mathbb{R}}^0_+$ implies $S' \in \mathbb{R}^0_+$ with (9.73). This finding implies now with (9.105) that there exists a particular element $\bar{x} \in A$ with $S' <_{\mathbb{R}^0_+} \bar{x}$. Thus, S' and $\bar{x}$ are both elements of $\mathbb{R}^0_+$, so that we may write the preceding inequality as $S' <_{\overline{\mathbb{R}}^0_+} \bar{x}$, using (9.94). These findings demonstrate the truth of the existential sentence in (9.106), and since S' was arbitrary, we may therefore conclude that the universal sentence (9.106). Together with the fact that $+\infty$ is an upper bound for A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$, this implies that $+\infty$ is the supremum of A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$. Thus, a supremum of A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$ exists in both subcases, and consequently in both cases.

Together with the fact that A was initially arbitrary, this allows us to further conclude that $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ has the Supremum Property. In conjunction with a), this means that $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ is a linear continuum. The Infimum Property of $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ can be established in analogy to the Supremum Property.

Regarding d), we first prove the universal sentence

$$\forall A\,([A \subseteq \overline{\mathbb{R}}_{+}^{0} \wedge A \neq \emptyset \wedge \exists a\,(a \in \overline{\mathbb{R}}_{+}^{0} \wedge \forall x\,(x \in A \Rightarrow a \leq_{\overline{\mathbb{R}}_{+}^{0}} x))$$
$$\wedge +\infty \notin A] \Rightarrow \exists I\,(I = \inf^{\leq_{\overline{\mathbb{R}}_{+}^{0}}} A)). \tag{9.107}$$

For this purpose, we let A be an arbitrary nonempty and bounded-from-below subset of $\overline{\mathbb{R}}_{+}^{0}$ such that $+\infty \notin A$ holds. Here, we can establish the inclusion $A \subseteq \mathbb{R}_{+}^{0}$ in exactly the same way as in the previous proof concerning the Supremum Property. We now verify that A is bounded from below also with respect to $\leq_{\mathbb{R}_{+}^{0}}$. Since 0 is a lower bound for A with respect to $\leq_{\overline{\mathbb{R}}_{+}^{0}}$, as shown by (9.100), we have

$$\forall x\,(x \in A \Rightarrow 0 \leq_{\overline{\mathbb{R}}_{+}^{0}} x). \tag{9.108}$$

We may show that

$$\forall x\,(x \in A \Rightarrow \bar{a} \leq_{\mathbb{R}_{+}^{0}} x) \tag{9.109}$$

holds as well. Letting $x \in A$ be arbitrary, we obtain $0 \leq_{\overline{\mathbb{R}}_{+}^{0}} x$ with (9.108). Since 0 is a nonnegative real number and since $x \in A$ implies also $x \in \mathbb{R}_{+}^{0}$ with the previously established inclusion $A \subseteq \mathbb{R}_{+}^{0}$, we may write the preceding inequality as $0 \leq_{\mathbb{R}_{+}^{0}} x$ by using (9.94). This inequality is the desired consequent of the implication in (9.109), in which x is arbitrary, so that the universal sentence (9.109) follows to be true. This shows that 0 is a lower bound for A with respect to $\leq_{\mathbb{R}_{+}^{0}}$, and this implies that A is bounded from below with respect to $\leq_{\mathbb{R}_{+}^{0}}$. Thus, A is a nonempty and bounded-from-below subset of $\mathbb{R}_{+}^{0}$. Since $(\mathbb{R}_{+}^{0}, <_{\mathbb{R}_{+}^{0}})$ has the Infimum Property, it follows from this that the infimum $I = \inf^{\leq_{\mathbb{R}_{+}^{0}}}$ exists, which thus satisfies the conjunction

$$\forall x\,(x \in A \Rightarrow I \leq_{\mathbb{R}_{+}^{0}} x)$$
$$\wedge\ \forall I'\,(\forall x\,(x \in A \Rightarrow I' \leq_{\mathbb{R}_{+}^{0}} x) \Rightarrow I' \leq_{\mathbb{R}_{+}^{0}} I) \tag{9.110}$$

according to the Characterization of the infimum. We may now prove that I is also the infimum of A with respect to $\leq_{\overline{\mathbb{R}}_{+}^{0}}$. For this purpose, we demonstrate the truth of

$$\forall x\,(x \in A \Rightarrow I \leq_{\overline{\mathbb{R}}_{+}^{0}} x)$$
$$\wedge\ \forall I'\,(\forall x\,(x \in A \Rightarrow I' \leq_{\overline{\mathbb{R}}_{+}^{0}} x) \Rightarrow I' \leq_{\overline{\mathbb{R}}_{+}^{0}} I). \tag{9.111}$$

Letting $x \in A$ be arbitrary, we find $I \leq_{\mathbb{R}^0_+} x$ with the first part of the conjunction (9.110). Evidently, we may write this inequality as $I \leq_{\overline{\mathbb{R}}^0_+} x$. As x was arbitrary, we may therefore conclude that the first part of the conjunction (9.111) holds.

To establish the second part, we let I' be arbitrary, assume

$$\forall x\,(x \in A \Rightarrow I' \leq_{\overline{\mathbb{R}}^0_+} x), \tag{9.112}$$

and show that

$$\forall x\,(x \in A \Rightarrow I' \leq_{\mathbb{R}^0_+} x), \tag{9.113}$$

also holds. Letting $x \in A$ be arbitrary, we obtain on the one hand the inequality $I' \leq_{\overline{\mathbb{R}}^0_+} x$ with (9.112), on the other hand $x \in \mathbb{R}^0_+$ with the previously established inclusion $A \subseteq \mathbb{R}^0_+$. The latter gives $x <_{\overline{\mathbb{R}}^0_+} +\infty$ with (9.102). This and the previously established inequality $I' \leq_{\overline{\mathbb{R}}^0_+} x$ implies now $I' <_{\overline{\mathbb{R}}^0_+} +\infty$ with the Transitivity Formula for $\leq$ and $<$, so that $I' \neq +\infty$ is clearly true. Evidently, the fact $I' \in \overline{\mathbb{R}}^0_+$ implies therefore $I' \in \mathbb{R}^0_+$. Since $x \in \mathbb{R}^0_+$ also holds, we may now evidently write the inequality $I' \leq_{\overline{\mathbb{R}}^0_+} x$ as $I' \leq_{\mathbb{R}^0_+} x$. As x was arbitrary, we may therefore conclude that the universal sentence (9.113) holds. Due to (9.110), the inequality $I' \leq_{\mathbb{R}^0_+} I$ follows now to be true, which we may evidently write equivalently as $I' \leq_{\overline{\mathbb{R}}^0_+} I$. Since I' was arbitrary, we may therefore conclude that the corresponding universal sentence (9.111) holds, so that the proof of the conjunction (9.111) is complete. Consequently, I is indeed the infimum of A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$. The existence of such an infimum proves the implication in (9.107), in which A is arbitrary, so that the universal sentence (9.107) is true.

We are now in a position to establish the Infimum Property for $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$. We let A be an arbitrary nonempty and bounded-from-below subset of $\overline{\mathbb{R}}^0_+$, and we consider the two cases $+\infty \in A$ and $+\infty \notin A$. In case of $+\infty \notin A$, we see in light of (9.107) that the infimum of A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$ exists. In the other case $+\infty \in A$, we consider the two subcases $\exists x\,(A = \{x\})$ and $\neg\exists x\,(A = \{x\})$. In the first subcase, there exists a particular constant $\bar{x}$ such that $A = \{\bar{x}\}$. Therefore, the current case assumption $+\infty \in A$ implies $+\infty \in \{\bar{x}\}$ via substitution, so that $+\infty = \bar{x}$ follows to be true with (2.169). Then, $A = \{\bar{x}\}$ yields $A = \{+\infty\}$ via another substitution. This finding in turn implies $\min^{\leq_{\overline{\mathbb{R}}^0_+}} A = +\infty$ with Exercise (3.289), and therefore $\inf^{\leq_{\overline{\mathbb{R}}^0_+}} A = +\infty$ by virtue of Theorem

3.105b). Thus, the infimum of A with respect to $\leq_{\overline{\mathbb{R}}_+^0}$ exists in the first subcase. The second subcase $\neg\exists x\,(A = \{x\})$ gives us with the Quantifier Negation Laws $\forall x\,(A \neq \{x\})$, and this implies in conjunction with the initial assumption $A \neq \emptyset$ that there are particular elements $\bar{x}, \bar{y} \in A$ with $\bar{x} \neq \bar{y}$, according to (2.183). We now establish a number of facts for the set $A \setminus \{+\infty\}$.

Firstly, we observe the truth of the inclusions $A \setminus \{+\infty\} \subseteq A \subseteq \overline{\mathbb{R}}_+^0$ in light of (2.125) and the initial assumption, so that $A \setminus \{+\infty\}$ turns out to be a subset of $\overline{\mathbb{R}}_+^0$ in view of the transitivity property (2.13). Secondly, we may prove the existential sentence $\exists x\,(x \in A \setminus \{+\infty\})$ by cases, based on the evidently true disjunction $\bar{x} = +\infty$ and $\bar{x} \neq +\infty$. On the one hand, $\bar{x} = +\infty$ and $\bar{x} \neq \bar{y}$ give us $\bar{y} \neq +\infty$ through substitution, and therefore $\bar{y} \notin \{+\infty\}$ with (2.169). Then, the previous finding $\bar{y} \in A$ and the preceding negation yields $\bar{y} \in A \setminus \{+\infty\}$ by definition of a subset. We thus showed that there exists an element of $A \setminus \{+\infty\}$, as desired. On the other hand, $\bar{x} \neq +\infty$ implies evidently $\bar{x} \notin \{+\infty\}$ and then – in conjunction with the fact $\bar{x} \in A$ – also $\bar{x} \in A \setminus \{+\infty\}$. Thus, $A \setminus \{+\infty\}$ again has some element, so that the proof by cases is now complete. The finding that $A \setminus \{+\infty\}$ has some element implies now $A \setminus \{+\infty\} \neq \emptyset$ with (2.42). Thirdly, the fact that $\overline{\mathbb{R}}_+^0$ is bounded from below (by 0 and with respect to $\leq_{\overline{\mathbb{R}}_+^0}$), as shown by (9.100), that the subset $A \setminus \{+\infty\}$ of $\overline{\mathbb{R}}_+^0$ is also bounded from below with respect to $\leq_{\overline{\mathbb{R}}_+^0}$, according to Exercise (3.38. Fourthly, we have $+\infty \notin A \setminus \{+\infty\}$ because of (2.179). These four facts imply now with (9.107) that the infimum I of $A \setminus \{+\infty\}$ with respect to $\leq_{\overline{\mathbb{R}}_+^0}$ exists. According to the Characterization of the infimum, I satisfies therefore

$$\forall x\,(x \in A \setminus \{+\infty\} \Rightarrow I \leq_{\overline{\mathbb{R}}_+^0} x) \\ \wedge\ \forall I'\,(\forall x\,(x \in A \setminus \{+\infty\} \Rightarrow I' \leq_{\overline{\mathbb{R}}_+^0} x) \Rightarrow I' \leq_{\overline{\mathbb{R}}_+^0} I). \tag{9.114}$$

We now prove that I is also the infimum of A with respect to $\leq_{\overline{\mathbb{R}}_+^0}$, i.e., that I satisfies

$$\forall x\,(x \in A \Rightarrow I \leq_{\overline{\mathbb{R}}_+^0} x) \\ \wedge\ \forall I'\,(\forall x\,(x \in A \Rightarrow I' \leq_{\overline{\mathbb{R}}_+^0} x) \Rightarrow I' \leq_{\overline{\mathbb{R}}_+^0} I). \tag{9.115}$$

To establish the first part of the conjunction, we let $x \in A$ be arbitrary. Since $+\infty \in A$ holds under the current case assumption, the inclusion $\{+\infty\} \subseteq A$ follows to be true with (2.184), and this implies the truth of the equation $A = (A \setminus \{+\infty\}) \cup \{+\infty\}$ because of (2.263). Consequently, $x \in A$ implies with the definition of the union of two sets that $x \in A \setminus \{+\infty\}$ or $x \in \{+\infty\}$ is true. We now use this true disjunction to prove the inequality $I' \leq_{\overline{\mathbb{R}}^0_+} x$ by cases. On the one hand, $x \in A \setminus \{+\infty\}$ implies that inequality with the first part of the true conjunction (9.114). On the other hand, $x \in \{+\infty\}$ evidently yields $x = +\infty$, and since $I' \leq_{\overline{\mathbb{R}}^0_+} +\infty$ holds according to (9.99), we therefore obtain the desired inequality $I' \leq_{\overline{\mathbb{R}}^0_+} x$ again, by applying substitution. As x was arbitrary, we may therefore conclude that the first part of the conjunction (9.115) holds. To prove the second part, we let I' be arbitrary, assume the universal sentence

$$\forall x \, (x \in A \Rightarrow I' \leq_{\overline{\mathbb{R}}^0_+} x) \tag{9.116}$$

to be true, and show that the universal sentence

$$\forall x \, (x \in A \setminus \{+\infty\} \Rightarrow I' \leq_{\overline{\mathbb{R}}^0_+} x) \tag{9.117}$$

also holds. Letting for this purpose $x \in A \setminus \{+\infty\}$ be arbitrary, we immediately find $x \in A$ by means of the previously established inclusion $A \setminus \{+\infty\} \subseteq A$. This finding further implies the desired inequality $I' \leq_{\overline{\mathbb{R}}^0_+} x$ with the assumed universal sentence (9.116). As x was arbitrary, we may infer from the truth of this inequality the truth of (9.117), and then also the truth of the second part of the conjunction (9.115) since I' was also arbitrary. According to the Characterization of the infimum, this means that I is the infimum of A with respect to $\leq_{\overline{\mathbb{R}}^0_+}$. We thus completed the proofs by subcases and cases, so that $(\overline{\mathbb{R}}^0_+, <_{\overline{\mathbb{R}}^0_+})$ has indeed the Infimum Property. □

Exercise 9.8. Show that 0 is the least upper bound and $+\infty$ the greatest lower bound for the empty set with respect to the standard total ordering of $\overline{\mathbb{R}}^0_+$, that is,

$$\overset{\leq_{\overline{\mathbb{R}}^0_+}}{\sup} \emptyset = 0, \tag{9.118}$$

$$\overset{\leq_{\overline{\mathbb{R}}^0_+}}{\inf} \emptyset = +\infty. \tag{9.119}$$

(Hint: Recall the proof of Proposition 9.19 and use (9.98) – (9.99).)

Corollary 9.28. *It is true that* $(\overline{\mathbb{R}}^0_+, \leq_{\overline{\mathbb{R}}^0_+})$ *is a complete lattice.*

Proof. To prove that $(\overline{\mathbb{R}}_{+}^{0}, \leq_{\overline{\mathbb{R}}_{+}^{0}})$ satisfies

$$\forall A\,(A \subseteq \overline{\mathbb{R}}_{+}^{0} \Rightarrow \exists S, I\,(S, I \in \overline{\mathbb{R}}_{+}^{0} \,\wedge\, S = \overset{\leq_{\overline{\mathbb{R}}_{+}^{0}}}{\sup} A \,\wedge\, I = \overset{\leq_{\overline{\mathbb{R}}_{+}^{0}}}{\inf} A)), \tag{9.120}$$

we let A be an arbitrary subset of $\overline{\mathbb{R}}_{+}^{0}$, and we consider the two cases $A = \emptyset$ and $A \neq \emptyset$. In the first case, the supremum and the infimum of A exist with respect to $\overline{\mathbb{R}}_{+}^{0}$, as shown in Exercise 9.8. In the second case, we first recall the fact that $\overline{\mathbb{R}}_{+}^{0}$ is bounded from above (by $+\infty$) and bounded from below (by 0) with respect to $\leq_{\overline{\mathbb{R}}_{+}^{0}}$, as shown by (9.99) & (9.98). Consequently, the subset A of $\overline{\mathbb{R}}_{+}^{0}$ is also bounded from above (by $+\infty$) and bounded from below (by 0) with respect to $\leq_{\overline{\mathbb{R}}_{+}^{0}}$, because of Proposition 3.94 and Exercise 3.38. Thus, A is a nonempty, bounded-from-above and bounded-from-below subset of $\overline{\mathbb{R}}_{+}^{0}$. Since $\overline{\mathbb{R}}_{+}^{0}$ has both the Supremum and the Infimum Property (see Theorem 9.27), both the supremum and the infimum of A exist with respect to $\overline{\mathbb{R}}_{+}^{0}$. Thus, the proof by cases is complete. Since A was initially arbitrary, we may therefore conclude that the universal sentence (9.120) is true, which means that $(\overline{\mathbb{R}}_{+}^{0}, \leq_{\overline{\mathbb{R}}_{+}^{0}})$ is a complete lattice. □

Note 9.8. The complete lattice $(\overline{\mathbb{R}}_{+}^{0}, \leq_{\overline{\mathbb{R}}_{+}^{0}})$, being a lattice in view of Corollary 3.99, gives rise to the binary (join and meet) operations

$$\sqcup_{\overline{\mathbb{R}}_{+}^{0}} : \overline{\mathbb{R}}_{+}^{0} \times \overline{\mathbb{R}}_{+}^{0} \to \overline{\mathbb{R}}_{+}^{0}, \quad (D, E) \mapsto D \sqcup_{\overline{\mathbb{R}}_{+}^{0}} E = \overset{\leq_{\overline{\mathbb{R}}_{+}^{0}}}{\sup}\{D, E\}, \tag{9.121}$$

$$\sqcap_{\overline{\mathbb{R}}_{+}^{0}} : \overline{\mathbb{R}}_{+}^{0} \times \overline{\mathbb{R}}_{+}^{0} \to \overline{\mathbb{R}}_{+}^{0}, \quad (D, E) \mapsto D \sqcap_{\overline{\mathbb{R}}_{+}^{0}} E = \overset{\leq_{\overline{\mathbb{R}}_{+}^{0}}}{\inf}\{D, E\}. \tag{9.122}$$

Definition 9.9 (Complete lattice of nonnegative extended real numbers). We call

$$(\overline{\mathbb{R}}_{+}^{0}, \sqcup_{\overline{\mathbb{R}}_{+}^{0}}, \sqcap_{\overline{\mathbb{R}}_{+}^{0}}, \leq_{\overline{\mathbb{R}}_{+}^{0}}) \tag{9.123}$$

the *complete lattice of nonnegative extended real numbers*.

Note 9.9. Whereas a subset of $\mathbb{R}_{+}^{0}$ must be assumed to be bounded from above in order for the Supremum Property to hold, that assumption is unnecessary for subsets of $\overline{\mathbb{R}}_{+}^{0}$ as these are always bounded from above by $+\infty$.

Corollary 9.29. *Every increasing/decreasing sequence $(D_n)_{n \in \mathbb{N}_{+}}$ in $\overline{\mathbb{R}}_{+}^{0}$ is increasingly/decreasingly convergent with respect to $\leq_{\overline{\mathbb{R}}_{+}^{0}}$.*

Proof. We let $f = (D_n)_{n\in\mathbb{N}_+}$ be an arbitrary increasing/decreasing sequence in the set of nonnegative extended real numbers. Thus, $\overline{\mathbb{R}}_+^0$ is a codomain of f, which means that the range of f is a subset of $\overline{\mathbb{R}}_+^0$. Since $(\overline{\mathbb{R}}_+^0, \leq_{\overline{\mathbb{R}}_+^0})$ is a complete lattice, the supremum/infimum of that subset $\mathrm{ran}(f)$ (with respect to $\leq_{\overline{\mathbb{R}}_+^0}$) exists. By definition of the limit of an increasing/decreasing sequence, this supremum/infimum is the limit of f (with respect to $\leq_{\overline{\mathbb{R}}_+^0}$). Thus, f is increasingly/decreasingly convergent (with respect to $\leq_{\overline{\mathbb{R}}_+^0}$). Since f was arbitrary, we therefore conclude that the stated universal sentence holds. □

Theorem 9.30 (Characterization of sequences that converge increasingly to $+\infty$). *It is true for any increasing sequence $f = (D_n)_{n\in\mathbb{N}_+}$ in $\overline{\mathbb{R}}_+^0$ that f converges increasingly to $+\infty$ iff*

1) the range of $(D_n)_{n\in\mathbb{N}_+}$ contains $+\infty$ or

2) f is a sequence in $\mathbb{R}_+^0$ that does not have an upper bound with respect to $\leq_{\mathbb{R}_+^0}$.

Proof. We take an arbitrary increasing sequence $f = (D_n)_{n\in\mathbb{N}_+}$ in $\overline{\mathbb{R}}_+^0$. To prove the first part ('⇒') of the proposed equivalence, we assume that f converges increasingly to $+\infty$, which means that $+\infty$ is the supremum of the range of f with respect to $\leq_{\overline{\mathbb{R}}_+^0}$. Thus,

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow D_n \leq_{\overline{\mathbb{R}}_+^0} D_{n+1}). \tag{9.124}$$

holds according to the Monotony Criterion for increasing sequences, and

$$\forall S'\,(\forall y\,(y \in \mathrm{ran}(f) \Rightarrow y \leq_{\overline{\mathbb{R}}_+^0} S') \Rightarrow +\infty \leq_{\overline{\mathbb{R}}_+^0} S'). \tag{9.125}$$

is true according to the Characterization of suprema. We now consider the two cases $+\infty \in \mathrm{ran}(f)$ and $+\infty \notin \mathrm{ran}(f)$. The first case means that 1) is true, which implies the disjunction of 1) and 2) to be proven. In the other case $+\infty \notin \mathrm{ran}(f)$, we may show that f is a sequence in $\mathbb{R}_+^0$, i.e., that the inclusion $\mathrm{ran}(f) \subseteq \mathbb{R}_+^0$ holds. Letting $y \in \mathrm{ran}(f)$ be arbitrary, we see in light of the current case assumption $+\infty \notin \mathrm{ran}(f)$ and (2.4) that $y \neq +\infty$ holds. Since the inclusion $\mathrm{ran}(f) \subseteq \overline{\mathbb{R}}_+^0$ also holds, $y \in \mathrm{ran}(f)$ implies $y \in \overline{\mathbb{R}}_+^0$ with the definition of a subset. This finding and $y \neq +\infty$ further imply $y \in \mathbb{R}_+^0$ with (9.73). We thus showed that $y \in \mathrm{ran}(f)$ implies $y \in \mathbb{R}_+^0$ for an arbitrary y, so that the inclusion $\mathrm{ran}(f) \subseteq \mathbb{R}_+^0$ holds indeed by definition of subset. Thus, f is a sequence in $\mathbb{R}_+^0$. Next, we prove by contradiction that there is no upper bound for that sequence with respect to $\leq_{\mathbb{R}_+^0}$. To do this, we assume the negation of that negation to be true. By the Double Negation Law and by definition of an upper bound, there exists then a particular element $\bar{u} \in \mathbb{R}_+^0$ such that

$$\forall y \, (y \in \operatorname{ran}(f) \Rightarrow y \leq_{\mathbb{R}_+^0} \bar{u}). \tag{9.126}$$

We note that $\bar{u} \in \mathbb{R}_+^0$ implies $\bar{u} <_{\overline{\mathbb{R}}_+^0} +\infty$ due to (9.102). Let us now establish the universal sentence

$$\forall y \, (y \in \operatorname{ran}(f) \Rightarrow y \leq_{\overline{\mathbb{R}}_+^0} \bar{u}). \tag{9.127}$$

Letting $y \in \operatorname{ran}(f)$ be arbitrary, we obtain the inequality $y \leq_{\mathbb{R}_+^0} \bar{u}$ with (9.126), which we may also write as $y \leq_{\overline{\mathbb{R}}_+^0} \bar{u}$ in view of (9.94). As y was arbitrary, we may therefore conclude that the universal sentence (9.127) is indeed true. Then, $+\infty \leq_{\overline{\mathbb{R}}_+^0} \bar{u}$ follows to be true with (9.125), and this yields the true negation $\neg \bar{u} <_{\overline{\mathbb{R}}_+^0} +\infty$ with the Negation Formula for $<$. Since we previously found $\bar{u} <_{\overline{\mathbb{R}}_+^0} +\infty$ to be true, too, we arrived at a contradiction. Thus, there exists no upper bound for (the range of) f with respect to $\leq_{\mathbb{R}_+^0}$. We thus showed that 2) is true, so that the disjunction of 1) and 2) holds also in the current second case.

To prove the second part ('$\Leftarrow$') of the equivalence, we now assume that 1) or 2) is true. In case 1) holds, we may use the Supremum Criterion to prove that $+\infty$ is the supremum of the range of f with respect to $\leq_{\overline{\mathbb{R}}_+^0}$. On the one hand, since (9.101) shows that $+\infty$ is an upper bound for $\overline{\mathbb{R}}_+^0$ with respect to $\leq_{\overline{\mathbb{R}}_+^0}$, it follows with Proposition 3.94 that $+\infty$ is also an upper bound for the subset $\operatorname{ran}(f)$ of $\overline{\mathbb{R}}_+^0$ (with respect to $\leq_{\overline{\mathbb{R}}_+^0}$). On the other hand, assuming now that S' is an arbitrary element of $\overline{\mathbb{R}}_+^0$ satisfying $S' <_{\overline{\mathbb{R}}_+^0} +\infty$, we see in light of the current case assumption $+\infty \in \operatorname{ran}(f)$ that there exists a constant $y \in \operatorname{ran}(f)$ such that $S' <_{\overline{\mathbb{R}}_+^0} y$. Since S' was arbitrary, we may therefore conclude that $+\infty$ is indeed the supremum of the range of f with respect to $\leq_{\overline{\mathbb{R}}_+^0}$. As f is an increasing sequence in $\overline{\mathbb{R}}_+^0$, it follows from this that f converges increasingly to $+\infty$.

In the other case 2), there is no upper bound in $\mathbb{R}_+^0$ with respect to $\leq_{\mathbb{R}_+^0}$, that is,

$$\neg \exists \, (u \in \mathbb{R}_+^0 \, \wedge \, \forall y \, (y \in \operatorname{ran}(f) \Rightarrow y \leq_{\mathbb{R}_+^0} u)).$$

This negation implies

$$\forall \, (u \in \mathbb{R}_+^0 \Rightarrow \neg \forall y \, (y \in \operatorname{ran}(f) \Rightarrow y \leq_{\mathbb{R}_+^0} u)).$$

with the Negation Law for existential conjunctions. We now apply the Supremum Criterion again to establish $\lim_{n \to +\infty} D_n = +\infty$. On the one hand, we recall the previous finding that $+\infty$ is an upper bound for $\mathrm{ran}(f)$ (with respect to $\leq_{\overline{\mathbb{R}}_+^0}$). On the other hand, letting $S' \in \overline{\mathbb{R}}_+^0$ be arbitrary such that $S' <_{\overline{\mathbb{R}}_+^0} +\infty$ holds, we observe in light of (9.73) and the Characterization of comparability with respect to the linear ordering $<_{\overline{\mathbb{R}}_+^0}$ that $S' \in \mathbb{R}_+^0$ holds. In view of the preceding universal sentence, the Negation Law for universal implications and the Negation Formula for $\leq$, there exists then a particular element $\bar{y} \in \mathrm{ran}(f)$ such that $S' <_{\overline{\mathbb{R}}_+^0} \bar{y}$. Because S' is arbitrary, we may infer from these findings that $+\infty$ is again the supremum of the range of f with respect to $\leq_{\overline{\mathbb{R}}_+^0}$. Consequently, f converges increasingly to $+\infty$, as in the first case.

Since f was initially arbitrary, we may therefore conclude that the theorem is indeed true. □

9.3. A Calculus for Extended Real Numbers

9.3.1. The commutative semiring $(\overline{\mathbb{R}}^0_+, +_{\overline{\mathbb{R}}^0_+}, \cdot_{\overline{\mathbb{R}}^0_+})$

Theorem 9.31. *The set*

$$\begin{aligned} +_{\overline{\mathbb{R}}^0_+} \; = \; +_{\mathbb{R}^0_+} \; & \cup \; \left(\mathbb{R}^0_+ \times \{+\infty\}\right) \times \{+\infty\} \\ & \cup \; \left(\{+\infty\} \times \mathbb{R}^0_+\right) \times \{+\infty\} \\ & \cup \; \{((+\infty, +\infty), +\infty)\} \end{aligned} \tag{9.128}$$

constitutes a binary operation on $\overline{\mathbb{R}}$.

Proof. We first establish the inclusion

$$+_{\overline{\mathbb{R}}^0_+} \; \subseteq \; \left(\overline{\mathbb{R}}^0_+ \times \overline{\mathbb{R}}^0_+\right) \times \overline{\mathbb{R}}^0_+ \tag{9.129}$$

by applying the definition of a subset. For this purpose, we take an arbitrary element $Z \in +_{\overline{\mathbb{R}}^0_+}$, so that the multiple disjunction

$$\begin{aligned} Z \in +_{\mathbb{R}^0_+} \; & \vee \; Z \in \left(\mathbb{R}^0_+ \times \{+\infty\}\right) \times \{+\infty\} \\ & \vee \; Z \in \left(\{+\infty\} \times \mathbb{R}^0_+\right) \times \{+\infty\} \\ & \vee \; Z \in \{((+\infty, +\infty), +\infty)\} \end{aligned} \tag{9.130}$$

follows to be true with the definition of the union of two sets (omitting all brackets in view of the associativity of the disjunction and the union). Thus, (at least) one of the four parts of this disjunction must be true:

1. In case of $Z \in +_{\mathbb{R}^0_+}$, we recall that the binary operation $+_{\mathbb{R}^0_+}$ (established as part of the Ordered elementary domain of nonnegative real numbers) is a function from $\mathbb{R}^0_+ \times \mathbb{R}^0_+$ to $\mathbb{R}^0_+$, so that

   $$+_{\mathbb{R}^0_+} \; \subseteq \; \left(\mathbb{R}^0_+ \times \mathbb{R}^0_+\right) \times \mathbb{R}^0_+. \tag{9.131}$$

   holds according to (3.514). Recalling now the inclusion $\mathbb{R}^0_+ \subseteq \overline{\mathbb{R}}^0_+$ from (9.70), we find the further inclusions

   $$\begin{aligned} \mathbb{R}^0_+ \times \mathbb{R}^0_+ \; & \subseteq \; \overline{\mathbb{R}}^0_+ \times \overline{\mathbb{R}}^0_+, \\ \left(\mathbb{R}^0_+ \times \mathbb{R}^0_+\right) \times \mathbb{R}^0_+ \; & \subseteq \; \left(\overline{\mathbb{R}}^0_+ \times \overline{\mathbb{R}}^0_+\right) \times \overline{\mathbb{R}}^0_+ \end{aligned} \tag{9.132}$$

   by means of (3.40). Thus, $Z \in +_{\mathbb{R}^0_+}$ implies $Z \in (\mathbb{R}^0_+ \times \mathbb{R}^0_+) \times \mathbb{R}^0_+$ with the two inclusions (9.131) and (9.132).

2. In case of $Z \in \left(\mathbb{R}_+^0 \times \{+\infty\}\right) \times \{+\infty\}$, we use the fact $\{+\infty\} \subseteq \overline{\mathbb{R}}_+^0$ shown in (9.67) – jointly with $\mathbb{R}_+^0 \subseteq \overline{\mathbb{R}}_+^0$ – to obtain the inclusions

$$\begin{aligned} \mathbb{R}_+^0 \times \{+\infty\} \ &\subseteq \ \overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0, \\ \left(\mathbb{R}_+^0 \times \{+\infty\}\right) \times \{+\infty\} \ &\subseteq \ \left(\overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0\right) \times \overline{\mathbb{R}}_+^0, \end{aligned} \tag{9.133}$$

applying again (3.40). Therefore, the current case assumption evidently implies $Z \in (\mathbb{R}_+^0 \times \mathbb{R}_+^0) \times \mathbb{R}_+^0$, as desired.

3. The third case $Z \in \left(\{+\infty\} \times \mathbb{R}_+^0\right) \times \{+\infty\}$ gives the same result by virtue of the evidently true inclusions

$$\begin{aligned} \{+\infty\} \times \mathbb{R}_+^0 \ &\subseteq \ \overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0, \\ \left(\{+\infty\} \times \mathbb{R}_+^0\right) \times \{+\infty\} \ &\subseteq \ \left(\overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0\right) \times \overline{\mathbb{R}}_+^0. \end{aligned} \tag{9.134}$$

4. Finally, we observe that the fact $+\infty \in \overline{\mathbb{R}}_+^0$ shown in (9.66) yields

$$\begin{aligned} (+\infty, +\infty) &\in \overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0, \\ ((+\infty, +\infty), +\infty) &\in \left(\overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0\right) \times \overline{\mathbb{R}}_+^0 \end{aligned} \tag{9.135}$$

with the definition of the Cartesian product of two sets. Since the assumption $Z \in \{((+\infty, +\infty), +\infty)\}$ yields $Z = ((+\infty, +\infty), +\infty)$ with (2.169), we thus find the desired $Z \in (\mathbb{R}_+^0 \times \mathbb{R}_+^0) \times \mathbb{R}_+^0$ via substitution.

Since Z was arbitrary, we may therefore conclude that the inclusion (9.129) holds indeed. Defining $A = \overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0$ and $B = \overline{\mathbb{R}}_+^0$, we thus have $+_{\overline{\mathbb{R}}_+^0} \subseteq A \times B$, so that $+_{\overline{\mathbb{R}}_+^0}$ is a binary relation in view of (3.73).

To show that the binary relation $+_{\overline{\mathbb{R}}_+^0}$ is a function from $\overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0$ to $\overline{\mathbb{R}}_+^0$, we apply the Function Criterion and establish accordingly the universal sentence

$$\forall X \,(X \in \overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0 \Rightarrow \exists! Y \,(Y \in \overline{\mathbb{R}}_+^0 \,\wedge\, (X, Y) \in +_{\overline{\mathbb{R}}_+^0})). \tag{9.136}$$

To do this, we let $X \in \overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0$ be arbitrary, so that $X = (\bar{D}, \bar{E})$ holds for particular numbers $\bar{D}, \bar{E} \in \overline{\mathbb{R}}_+^0$ in view of (3.38). Since $\overline{\mathbb{R}}_+^0 = \mathbb{R}_+^0 \cup \{+\infty\}$ holds according to (9.71), we evidently find the true disjunctions

$$\bar{D} \in \mathbb{R}_+^0 \ \vee \ \bar{D} \in \{+\infty\}, \tag{9.137}$$

$$\bar{E} \in \mathbb{R}_+^0 \ \vee \ \bar{E} \in \{+\infty\}. \tag{9.138}$$

We consider first the case of $\bar{D} \in \mathbb{R}_+^0$ and the first subcase of $\bar{E} \in \mathbb{R}_+^0$. Then, the sum $\bar{Y} = \bar{D} +_{\mathbb{R}_+^0} \bar{E}$ is clearly in $\mathbb{R}_+^0$ as well, and therefore evidently also in $\overline{\mathbb{R}}_+^0$. Since $+_{\mathbb{R}_+^0}$ is a binary relation, we may write the preceding equation as $((\bar{D}, \bar{E}), \bar{Y}) \in +_{\mathbb{R}_+^0}$. Moreover, as the inclusion

$$+_{\mathbb{R}_+^0} \subseteq +_{\overline{\mathbb{R}}_+^0} \tag{9.139}$$

can be shown to be true from (9.128) by virtue of (2.245), we find $(X, \bar{Y}) \in +_{\overline{\mathbb{R}}_+^0}$ to be true (besides $\bar{Y} \in \overline{\mathbb{R}}_+^0$), which results demonstrate the truth of the existential part in (9.136) for the first subcase. Concerning the uniqueness part, we now take arbitrary numbers $Y, Y' \in \overline{\mathbb{R}}_+^0$ with $(X, Y), (X, Y') \in +_{\overline{\mathbb{R}}_+^0}$. We thus have $((\bar{D}, \bar{E}), Y) \in +_{\overline{\mathbb{R}}_+^0}$ and $((\bar{D}, \bar{E}), Y') \in +_{\overline{\mathbb{R}}_+^0}$. Clearly, $\bar{D}$ and $\bar{E}$ are real numbers, so that $\bar{D} \neq +\infty$ and $\bar{E} \neq +\infty$ are true because of (9.15). These inequalities give us $\bar{D} \notin \{+\infty\}$ as well as $\bar{E} \notin \{+\infty\}$ with (2.169). Then, the disjunctions

$$\bar{D} \notin \mathbb{R}_+^0 \;\vee\; \bar{E} \notin \{+\infty\},$$
$$\bar{D} \notin \{+\infty\} \;\vee\; \bar{E} \notin \mathbb{R}_+^0$$

also hold, with the evident consequence that $(\bar{D}, \bar{E}) \notin \mathbb{R}_+^0 \times \{+\infty\}$ and $(\bar{D}, \bar{E}) \notin \{+\infty\} \times \mathbb{R}_+^0$. Furthermore, $\bar{D} \neq +\infty$ implies $(\bar{D}, \bar{E}) \neq (+\infty, +\infty)$ with the Equality Criterion for ordered pairs. As these three negations are true, the disjunctions

$$(\bar{D}, \bar{E}) \notin \mathbb{R}_+^0 \times \{+\infty\} \;\vee\; Y \notin \{+\infty\},$$
$$(\bar{D}, \bar{E}) \notin \mathbb{R}_+^0 \times \{+\infty\} \;\vee\; Y' \notin \{+\infty\},$$
$$(\bar{D}, \bar{E}) \notin \{+\infty\} \times \mathbb{R}_+^0 \;\vee\; Y \notin \{+\infty\},$$
$$(\bar{D}, \bar{E}) \notin \{+\infty\} \times \mathbb{R}_+^0 \;\vee\; Y' \notin \{+\infty\},$$
$$(\bar{D}, \bar{E}) \neq (+\infty, +\infty) \;\vee\; Y \neq +\infty,$$
$$(\bar{D}, \bar{E}) \neq (+\infty, +\infty) \;\vee\; Y' \neq +\infty$$

hold then as well. Consequently, we obtain the negations

$$\begin{aligned}
((\bar{D}, \bar{E}), Y) &\notin \big(\mathbb{R}_+^0 \times \{+\infty\}\big) \times \{+\infty\}, \\
((\bar{D}, \bar{E}), Y') &\notin \big(\mathbb{R}_+^0 \times \{+\infty\}\big) \times \{+\infty\}, \\
((\bar{D}, \bar{E}), Y) &\notin \big(\{+\infty\} \times \mathbb{R}_+^0\big) \times \{+\infty\}, \\
((\bar{D}, \bar{E}), Y') &\notin \big(\{+\infty\} \times \mathbb{R}_+^0\big) \times \{+\infty\}, \\
((\bar{D}, \bar{E}), Y) &\neq ((+\infty, +\infty), +\infty), \\
((\bar{D}, \bar{E}), Y') &\neq ((+\infty, +\infty), +\infty).
\end{aligned}$$

Since an arbitrary $Z \in +_{\overline{\mathbb{R}}^0_+}$ satisfies the multiple disjunction (9.138), we see now for the particular elements $((\bar{D}, \bar{E}), Y) \in +_{\overline{\mathbb{R}}^0_+}$ and $((\bar{D}, \bar{E}), Y') \in +_{\overline{\mathbb{R}}^0_+}$ in light of the preceding negations that the second, third and fourth parts of the corresponding disjunctions are false. Therefore, the corresponding first parts $((\bar{D}, \bar{E}), Y) \in +_{\mathbb{R}^0_+}$ and $((\bar{D}, \bar{E}), Y') \in +_{\mathbb{R}^0_+}$ must be true. As the binary operation $+_{\mathbb{R}^0_+}$ is a function, these two findings imply now $Y = Y'$. Since Y and Y' were arbitrary, the proof of the uniqueness part is now complete. We thus established the uniquely existential sentence in (9.136) for the first subcase.

The second subcase $\bar{E} \in \{+\infty\}$ with respect to the disjunction (9.138) implies, in conjunction with the current case assumption $\bar{D} \in \mathbb{R}^0_+$, the truth of $(\bar{D}, \bar{E}) \in \mathbb{R}^0_+ \times \{+\infty\}$, and this yields with the basic fact $+\infty \in \{+\infty\}$

$$((\bar{D}, \bar{E}), +\infty) \in \big(\mathbb{R}^0_+ \times \{+\infty\}\big) \times \{+\infty\}.$$

Since the inclusion

$$\big(\mathbb{R}^0_+ \times \{+\infty\}\big) \times \{+\infty\} \ \subseteq \ +_{\overline{\mathbb{R}}^0_+} \tag{9.140}$$

can be established from (9.128) by applying (2.245) again, it follows that $((\bar{D}, \bar{E}), +\infty) \in +_{\overline{\mathbb{R}}^0_+}$; we thus have $(X, +\infty) \in +_{\overline{\mathbb{R}}^0_+}$, and $+\infty \in \overline{\mathbb{R}}^0_+$ is also true, so that the existential part in (9.136) holds. To establish the uniqueness part, we take again arbitrary numbers $Y, Y' \in \overline{\mathbb{R}}^0_+$ satisfying $(X, Y), (X, Y') \in +_{\overline{\mathbb{R}}^0_+}$, so that $((\bar{D}, \bar{E}), Y) \in +_{\overline{\mathbb{R}}^0_+}$ and $((\bar{D}, \bar{E}), Y') \in +_{\overline{\mathbb{R}}^0_+}$. The current assumption $\bar{E} \in \{+\infty\}$ implies $\bar{E} = +\infty$ with (2.169), so that the fact $+\infty \notin \mathbb{R}^0_+$ given in (9.66) yields $\bar{E} \notin \mathbb{R}^0_+$ via substitution. Then, the disjunctions

$$\bar{D} \notin \mathbb{R}^0_+ \ \vee \ \bar{E} \notin \mathbb{R}^0_+,$$
$$\bar{D} \notin \{+\infty\} \ \vee \ \bar{E} \notin \mathbb{R}^0_+$$

also hold, so that $(\bar{D}, \bar{E}) \notin \mathbb{R}^0_+ \times \mathbb{R}^0_+$ and $(\bar{D}, \bar{E}) \notin \{+\infty\} \times \mathbb{R}^0_+$. In the first subcase, we already found $(\bar{D}, \bar{E}) \notin \{+\infty, +\infty\}$, and these three negations give rise to the true disjunctions

$$(\bar{D}, \bar{E}) \notin \mathbb{R}^0_+ \times \mathbb{R}^0_+ \ \vee \ Y \notin \{+\infty\},$$
$$(\bar{D}, \bar{E}) \notin \mathbb{R}^0_+ \times \mathbb{R}^0_+ \ \vee \ Y' \notin \{+\infty\},$$
$$(\bar{D}, \bar{E}) \notin \{+\infty\} \times \mathbb{R}^0_+ \ \vee \ Y \notin \{+\infty\},$$
$$(\bar{D}, \bar{E}) \notin \{+\infty\} \times \mathbb{R}^0_+ \ \vee \ Y' \notin \{+\infty\},$$
$$(\bar{D}, \bar{E}) \neq (+\infty, +\infty) \ \vee \ Y \neq +\infty,$$
$$(\bar{D}, \bar{E}) \neq (+\infty, +\infty) \ \vee \ Y' \neq +\infty.$$

Consequently, we obtain the negations

$$\begin{aligned}((\bar{D},\bar{E}),Y) &\notin \left(\mathbb{R}_+^0 \times \mathbb{R}_+^0\right) \times \{+\infty\},\\ ((\bar{D},\bar{E}),Y') &\notin \left(\mathbb{R}_+^0 \times \mathbb{R}_+^0\right) \times \{+\infty\},\\ ((\bar{D},\bar{E}),Y) &\notin \left(\{+\infty\} \times \mathbb{R}_+^0\right) \times \{+\infty\},\\ ((\bar{D},\bar{E}),Y') &\notin \left(\{+\infty\} \times \mathbb{R}_+^0\right) \times \{+\infty\},\\ ((\bar{D},\bar{E}),Y) &\neq ((+\infty,+\infty),+\infty),\\ ((\bar{D},\bar{E}),Y') &\neq ((+\infty,+\infty),+\infty),\end{aligned}$$

which show that $Z = ((\bar{D},\bar{E}),Y)$ and $Z' = ((\bar{D},\bar{E}),Y')$ do not satisfy the first, third an fourth part of the disjunction (9.130). Therefore, the corresponding second parts

$$\begin{aligned}((\bar{D},\bar{E}),Y) \in \left(\mathbb{R}_+^0 \times \{+\infty\}\right) \times \{+\infty\},\\ ((\bar{D},\bar{E}),Y') \in \left(\mathbb{R}_+^0 \times \{+\infty\}\right) \times \{+\infty\}\end{aligned}$$

must be true. By definition of the Cartesian product of two sets, we then find in particular $Y \in \{+\infty\}$ as well as $Y' \in \{+\infty\}$, with the desired consequence that $Y = Y'$. This proves the uniqueness part, so that the uniquely existential sentence to be proven holds also for the second subcase.

Switching now to the second case $\bar{D} \in \{+\infty\}$ regrading the disjunction (9.137), we see that the first subcase $\bar{E} \in \mathbb{R}_+^0$ can be established in analogy to the preceding second subcase of the first case, simply by interchanging $\bar{D}$ and $\bar{E}$, as well as $\{+\infty\}$ and $\mathbb{R}_+^0$ within some arguments of the derivation. Here, we use the fact that the inclusion

$$\left(\{+\infty\} \times \mathbb{R}_+^0\right) \times \{+\infty\} \subseteq +_{\overline{\mathbb{R}}_+^0} \tag{9.141}$$

can be established by applying (2.245) again. In the second subcase $\bar{E} \in \{+\infty\}$ of the second case $\bar{D} \in \{+\infty\}$, we thus have $\bar{D} = \bar{E} = +\infty$, and therefore $((\bar{D},\bar{E}),+\infty) \in \{((+\infty,+\infty),+\infty)\}$. Now, (9.128) evidently yields also the inclusion

$$\{((+\infty,+\infty),+\infty)\} \subseteq +_{\overline{\mathbb{R}}_+^0}, \tag{9.142}$$

which shows that $((\bar{D},\bar{E}),+\infty) \in +_{\overline{\mathbb{R}}_+^0}$. Thus, $(X,+\infty) \in +_{\overline{\mathbb{R}}_+^0}$ holds besides the aforementioned fact $+\infty \in \overline{\mathbb{R}}_+^0$, proving the existential part. Letting $Y, Y' \in \overline{\mathbb{R}}_+^0$ be arbitrary such that $(X,Y),(X,Y') \in +_{\overline{\mathbb{R}}_+^0}$, we have once again $((\bar{D},\bar{E}),Y) \in +_{\overline{\mathbb{R}}_+^0}$ and $((\bar{D},\bar{E}),Y') \in +_{\overline{\mathbb{R}}_+^0}$. We previously showed that $\bar{D} = +\infty$ and $\bar{E} = +\infty$ imply $\bar{D} \notin \mathbb{R}^0_+$ and $\bar{E} \notin \mathbb{R}^0_+$, so that the disjunctions

$$\begin{aligned} \bar{D} \notin \mathbb{R}^0_+ \ &\vee \ \bar{E} \notin \mathbb{R}^0_+, \\ \bar{D} \notin \mathbb{R}^0_+ \ &\vee \ \bar{D} \notin \{+\infty\}, \\ \bar{D} \notin \{+\infty\} \ &\vee \ \bar{E} \notin \mathbb{R}^0_+ \end{aligned}$$

are all true, so that

$$\begin{aligned} (\bar{D}, \bar{E}) \ &\notin \ \mathbb{R}^0_+ \times \mathbb{R}^0_+, \\ (\bar{D}, \bar{E}) \ &\notin \ \mathbb{R}^0_+ \times \{+\infty\}, \\ (\bar{D}, \bar{E}) \ &\notin \ \{+\infty\} \times \mathbb{R}^0_+. \end{aligned}$$

Clearly, the disjunctions

$$\begin{aligned} (\bar{D}, \bar{E}) \notin \mathbb{R}^0_+ \times \mathbb{R}^0_+ \ &\vee \ Y \notin \{+\infty\}, \\ (\bar{D}, \bar{E}) \notin \mathbb{R}^0_+ \times \mathbb{R}^0_+ \ &\vee \ Y' \notin \{+\infty\}, \\ (\bar{D}, \bar{E}) \notin \mathbb{R}^0_+ \times \{+\infty\} \ &\vee \ Y \notin \{+\infty\}, \\ (\bar{D}, \bar{E}) \notin \mathbb{R}^0_+ \times \{+\infty\} \ &\vee \ Y' \notin \{+\infty\}, \\ (\bar{D}, \bar{E}) \notin \{+\infty\} \times \mathbb{R}^0_+ \ &\vee \ Y \notin \{+\infty\}, \\ (\bar{D}, \bar{E}) \notin \{+\infty\} \times \mathbb{R}^0_+ \ &\vee \ Y' \notin \{+\infty\} \end{aligned}$$

hold then as well, resulting in the negations

$$\begin{aligned} ((\bar{D}, \bar{E}), Y) \ &\notin \ \left(\mathbb{R}^0_+ \times \mathbb{R}^0_+\right) \times \{+\infty\}, \\ ((\bar{D}, \bar{E}), Y') \ &\notin \ \left(\mathbb{R}^0_+ \times \mathbb{R}^0_+\right) \times \{+\infty\}, \\ ((\bar{D}, \bar{E}), Y) \ &\notin \ \left(\mathbb{R}^0_+ \times \{+\infty\}\right) \times \{+\infty\}, \\ ((\bar{D}, \bar{E}), Y') \ &\notin \ \left(\mathbb{R}^0_+ \times \{+\infty\}\right) \times \{+\infty\}, \\ ((\bar{D}, \bar{E}), Y) \ &\notin \ \left(\{+\infty\} \times \mathbb{R}^0_+\right) \times \{+\infty\}, \\ ((\bar{D}, \bar{E}), Y') \ &\notin \ \left(\{+\infty\} \times \mathbb{R}^0_+\right) \times \{+\infty\}. \end{aligned}$$

In view of the multiple disjunction (9.130), the remaining possibility is

$$\begin{aligned} ((\bar{D}, \bar{E}), Y) &\in \{((+\infty, +\infty), +\infty)\}, \\ ((\bar{D}, \bar{E}), Y') &\in \{((+\infty, +\infty), +\infty)\}, \end{aligned}$$

or equivalently

$$\begin{aligned} ((\bar{D}, \bar{E}), Y) &= ((+\infty, +\infty), +\infty), \\ ((\bar{D}, \bar{E}), Y') &= ((+\infty, +\infty), +\infty). \end{aligned}$$

By the Equality Criterion for ordered pairs, $Y = +\infty$ and $Y' = +\infty$ follow to be true, so that we find $Y = Y'$, as desired. We may therefore conclude that the uniquely existential sentence in (9.136) holds again, and thus in any of the given cases and subcases. As X was initially arbitrary, we may now further conclude that the binary relation $+_{\overline{\mathbb{R}}_+^0}$ satisfies the universal sentence (9.136) and constitutes therefore a function from $\overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0$ to $\overline{\mathbb{R}}_+^0$. This finding shows also that $+_{\overline{\mathbb{R}}_+^0}$ is a binary operation on $\overline{\mathbb{R}}_+^0$. □

Exercise 9.9. Establish the first subcase of the second case in proof of Theorem 9.31 in detail.

Definition 9.10 (Addition on $\overline{\mathbb{R}}_+^0$). We call the binary operation

$$+_{\overline{\mathbb{R}}_+^0} : \overline{\mathbb{R}}_+^0 \times \overline{\mathbb{R}}_+^0 \to \overline{\mathbb{R}}_+^0, \quad (D, E) \mapsto D +_{\overline{\mathbb{R}}_+^0} E \tag{9.143}$$

the *addition on the set of nonnegative extended real numbers.*

Proposition 9.32. *The sum of any two nonnegative real numbers with respect to the addition on $\mathbb{R}_+^0$ equals their sum when using the addition on $\overline{\mathbb{R}}_+^0$, that is,*

$$\forall x, y\, (x, y \in \mathbb{R}_+^0 \Rightarrow x +_{\overline{\mathbb{R}}_+^0} y = x +_{\mathbb{R}_+^0} y). \tag{9.144}$$

Proof. Letting x and y be arbitrary nonnegative real numbers, we obtain with the definition of the Cartesian product of two sets $(x, y) \in \mathbb{R}_+^0 \times \mathbb{R}_+^0$, so that the ordered pair (x, y) is in the domain of the addition on $\mathbb{R}_+^0$. Consequently, there exists a constant, say $\bar{z}$, such that $((x, y), \bar{z} \in +_{\mathbb{R}_+^0}$. Applying now the notations for functions and binary operations, we may write the preceding finding as $\bar{z} = x +_{\mathbb{R}_+^0} y$. Furthermore, the previous finding also implies $((x, y), \bar{z} \in +_{\overline{\mathbb{R}}_+^0}$ with the inclusion (9.139). Since the preceding theorem shows that $+_{\overline{\mathbb{R}}_+^0}$ is a binary operation, we may write equivalently $\bar{z} = x +_{\overline{\mathbb{R}}_+^0} y$. Combining the two equations for $\bar{z}$ now yields the desired equation in (9.144). Since x and y were arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Proposition 9.33. *The sum of two nonnegative extended real numbers is $+\infty$ if one or the other number is $+\infty$, that is,*

$$\forall D, E\, (D, E \in \overline{\mathbb{R}}_+^0 \Rightarrow [(D = +\infty \lor E = +\infty) \Rightarrow D +_{\overline{\mathbb{R}}_+^0} E = +\infty]). \tag{9.145}$$

Proof. We take two arbitrary nonnegative extended real numbers D and E, and we prove the second implication by contraposition, assuming $D +_{\overline{\mathbb{R}}^0_+} E \neq +\infty$ to be true. Denoting that sum by S, have $S \neq +\infty$; as the latter implies $S \notin \{+\infty\}$ with (2.169), the disjunctions

$$\begin{gathered} (D, E) \notin \mathbb{R}^0_+ \times \{+\infty\} \ \vee \ S \notin \{+\infty\}, \\ (D, E) \notin \{+\infty\} \times \mathbb{R}^0_+ \ \vee \ S \notin \{+\infty\}, \\ (D, E) \neq (+\infty, +\infty) \ \vee \ S \neq +\infty \end{gathered}$$

are then also true. We therefore obtain the true negations

$$\begin{aligned} ((D, E), S) &\notin \big(\mathbb{R}^0_+ \times \{+\infty\}\big) \times \notin \{+\infty\}, \\ ((D, E), S) &\notin \big(\{+\infty\} \times \mathbb{R}^0_+\big) \times \notin \{+\infty\}, \\ ((D, E), S) &\neq ((+\infty, +\infty), +\infty), \end{aligned}$$

using the definition of the Cartesian products of two sets and the Equality Criterion for ordered pairs. Since we may write $S = D +_{\overline{\mathbb{R}}^0_+} E$ also as $S = +_{\overline{\mathbb{R}}^0_+}((D, E))$, and then also in the form of $((D, E), S) \in +_{\overline{\mathbb{R}}^0_+}$, we now see clearly that $((D, E), S)$ satisfies the multiple disjunction (9.130). The preceding negations show that the second, third and fourth part of that disjunction are false, so that the remaining first part $((D, E), S) \in +_{\mathbb{R}^0_+}$ is true. Thus, the ordered pair (D, E) is in the domain $\mathbb{R}^0_+ \times \mathbb{R}^0_+$ of that binary operation, and this implies evidently $D, E \in \mathbb{R}^0_+$. In view of the fact $+\infty \notin \mathbb{R}^0_+$ shown in (9.68), this further implies $D \neq +\infty$ and $E \neq +\infty$ with (2.4). Consequently, the negation $\neg(D = +\infty \vee E = +\infty)$ holds by De Morgan's Law for the disjunction, so that the proof of the implication in (9.145) by transposition is now complete. Since D and E were initially arbitrary, we may now finally conclude that the proposition holds, as claimed. $\square$

Proposition 9.34. *The addition on $\overline{\mathbb{R}}^0_+$ is commutative.*

Proof. We verify

$$\forall D, E\, (D, E \in \overline{\mathbb{R}}^0_+ \Rightarrow D +_{\overline{\mathbb{R}}^0_+} E = E +_{\overline{\mathbb{R}}^0_+} D), \tag{9.146}$$

letting $D, E \in \overline{\mathbb{R}}^0_+$ be arbitrary. Therefore, the disjunctions

$$\begin{gathered} D \in \mathbb{R}^0_+ \ \vee \ D = +\infty, \\ E \in \mathbb{R}^0_+ \ \vee \ E = +\infty \end{gathered}$$

are true in view of (9.73), which we now use to prove the desired equation by two cases and two subcases. In the first case $D \in \mathbb{R}_+^0$ and the first subcase $E \in \mathbb{R}_+^0$, we obtain

$$D +_{\overline{\mathbb{R}}_+^0} E = D +_{\mathbb{R}_+^0} E = E +_{\mathbb{R}_+^0} D = E +_{\overline{\mathbb{R}}_+^0} D$$

with (9.144) and the commutativity of the addition on $\mathbb{R}_+^0$, and consequently $D +_{\overline{\mathbb{R}}_+^0} E = E +_{\overline{\mathbb{R}}_+^0} D$, as desired. The second subcase $E = +\infty$ yields

$$D +_{\overline{\mathbb{R}}_+^0} E = D +_{\overline{\mathbb{R}}_+^0} +\infty = +\infty,$$
$$E +_{\overline{\mathbb{R}}_+^0} D = +\infty +_{\overline{\mathbb{R}}_+^0} D = +\infty$$

with (9.145), so that the desired equation follows to be true again. In the second case $D = +\infty$, we obtain for the same reason

$$D +_{\overline{\mathbb{R}}_+^0} E = +\infty +_{\overline{\mathbb{R}}_+^0} E = +\infty,$$
$$E +_{\overline{\mathbb{R}}_+^0} D = E +_{\overline{\mathbb{R}}_+^0} +\infty = +\infty,$$

so that $D +_{\overline{\mathbb{R}}_+^0} E = E +_{\overline{\mathbb{R}}_+^0} D$ follows to be true again. We thus completed the proof by cases. Since D and E were arbitrary, we may infer from the truth of the preceding equation the truth of the universal sentence (9.146), which means that $+_{\overline{\mathbb{R}}_+^0}$ is indeed commutative. □

Exercise 9.10. Show that the addition on $\overline{\mathbb{R}}_+^0$ is associative.

(Hint: Use similar arguments as in the proof of Proposition 9.34 and consider here two further sub-subcases.)

Note 9.10. The previous findings show that the ordered pair

$$(\overline{\mathbb{R}}_+^0, +_{\overline{\mathbb{R}}_+^0}) \tag{9.147}$$

constitutes a commutative semigroup.

Exercise 9.11. Show that the (nonnegative) real number 0 constitutes the zero element of $\overline{\mathbb{R}}_+^0$ (with respect to the addition $+_{\overline{\mathbb{R}}_+^0}$).

(Hint: Apply a proof by cases based on (9.73), using (9.144) and (9.145).)

Exercise 9.12. Prove that the addition $+_{\overline{\mathbb{R}}_+^0}$ and the standard total ordering $\leq_{\overline{\mathbb{R}}_+^0}$ satisfy the following monotony law:

$$\forall D, E, F\,(D, E, F \in \overline{\mathbb{R}}_+^0 \Rightarrow [D \leq_{\overline{\mathbb{R}}_+^0} E \Rightarrow D +_{\overline{\mathbb{R}}_+^0} F \leq_{\overline{\mathbb{R}}_+^0} E +_{\overline{\mathbb{R}}_+^0} F]). \tag{9.148}$$

(Hint: Use (9.73), (9.145), (9.99), (9.102), (3.156), (9.94), (9.144), and the Monotony Law for $+$ and $\leq$ with respect to the ordered integral domain of nonnegative real numbers.)

Note 9.11. Having defined the addition $+_{\overline{\mathbb{R}}^0_+}$ on $\overline{\mathbb{R}}^0_+$ in such a way that the zero element 0 exists, we may now form n-fold repeated additions $\sum_{i=1}^n$ for any n-tuple $(D_i \,|\, i \in \{1, \ldots, n\})$ in $\overline{\mathbb{R}}^0_+$. In addition, we may generate the series $(\sum_{i=1}^n D_i)_{n\in\mathbb{N}_+}$ from any sequence $(D_n)_{n\in\mathbb{N}_+}$ in $\overline{\mathbb{R}}^0_+$. Each such series in increasingly convergent with respect to $\leq_{\overline{\mathbb{R}}^0_+}$ according to Corollary 5.136 since

- the addition $+_{\overline{\mathbb{R}}^0_+}$ is commutative,
- $\overline{\mathbb{R}}^0_+$ contains the zero element (0),
- and $(\overline{\mathbb{R}}^0_+, \leq_{\overline{\mathbb{R}}^0_+})$ constitutes a complete lattice, where $0 \leq_{\overline{\mathbb{R}}^0_+} D$ holds for any $D \in \overline{\mathbb{R}}^0_+$, and where $+_{\overline{\mathbb{R}}^0_+}$ and $\leq_{\overline{\mathbb{R}}^0_+}$ satisfy the monotony law.

Chapter to be expanded!

Chapter 10.

Real Vector Spaces

10.1. Vector Spaces $(V, +, \cdot)$

Definition 10.1 (Vector space/linear space, vector, scalar, vector addition, zero vector, scalar multiplication). For any field $(X, +, \cdot, -, /)$, any set V, any addition $+_V$ on V and any function $\cdot_V : X \times V \to V$ we say that a set

$$(V, +_V, \cdot_V) \tag{10.1}$$

is a *vector space* or a *linear space* over $(X, +, \cdot, -, /)$ (or over X) iff

1. $(V, +_V)$ is a commutative group,

2. $\cdot_V$ is distributive over the addition $+_V$ in the sense that

$$\forall a, v, w\, ([a \in X \;\wedge\; v, w \in V] \Rightarrow a \cdot_V (v +_V w) = (a \cdot_V v) +_V (a \cdot_V w)), \tag{10.2}$$

3. $\cdot_V$ is distributive over the addition $+$ in the sense that

$$\forall a, b, v\, ([a, b \in X \wedge v \in V] \Rightarrow (a+b) \cdot_V v = (a \cdot_V v) +_V (b \cdot_V v)), \tag{10.3}$$

4. the multiplication $\cdot_V$ is associative in the sense that

$$\forall a, b, v\, ([a, b \in X \;\wedge\; v \in V] \Rightarrow a \cdot_V (b \cdot_V v) = (a \cdot b) \cdot_V v), \tag{10.4}$$

5. the unity element 1_X in X is neutral with respect to the multiplication $\cdot_V$, that is,

$$\forall v\, (v \in V \Rightarrow 1_X \cdot_V v = v). \tag{10.5}$$

We then call the elements of V *vectors* and the elements of X *scalars*. Furthermore, we call $+_V$ the *vector addition*, the corresponding zero element 0_V the *zero vector*, and $\cdot_V$ the *scalar multiplication* on V.

Note 10.1. The group property of $(V, +_V)$ immediately gives rise to the subtraction $-_V : V \times V \to V$ (see Exercise 6.4).

Exercise 10.1. Verify for any field $(X, +, \cdot, -, /)$ that the ordered triple $(X, +, \cdot)$ is a vector space over X (i.e., over 'itself').

Theorem 10.1 (Cancellation Laws for the zero vector and scalar). *The following laws hold for any field $(X, +, \cdot, -, /)$ and any vector space $(V, +_V, \cdot_V)$ over X.*

a) **Cancellation Law for the zero vector***:*

$$\forall a\,(a \in X \Rightarrow a \cdot_V 0_V = 0_V). \tag{10.6}$$

b) **Cancellation Law for the zero scalar***:*

$$\forall v\,(v \in V \Rightarrow 0_X \cdot_V v = 0_V). \tag{10.7}$$

Exercise 10.2. Prove the Cancellation Laws for the zero vector and scalar. (Hint: Proceed similarly as for (5.300) and apply Corollary 6.20.)

A vector space satisfies a property which is similar to zero-divisor-freeness in the context of semirings.

Proposition 10.2. *The following sentence is true for any field $(X, +, \cdot, -, /)$ and any vector space $(V, +_V, \cdot_V)$ over X.*

$$\forall a, v\,([a \in X \,\wedge\, v \in V \,\wedge\, a \cdot_V v = 0_V] \Rightarrow [a = 0_X \,\vee\, v = 0_V]). \tag{10.8}$$

Proof. We let $(X, +, \cdot, -, /)$ be an arbitrary field, $(V, +_V, \cdot_V)$ an arbitrary vector space over X, a an arbitrary scalar in X, v an arbitrary vector in V, and we assume that $a \cdot_V v = 0_V$ holds. To show that $a = 0_X \vee v = 0_V$ is implied, we consider the two exhaustive cases $a = 0_X$ and $a \neq 0_X$. The first case $a = 0_X$ implies immediately $a = 0_X \vee v = 0_V$. In the second case of $a \neq 0_X$, we note that the reciprocal a^{-1} of a exists by virtue of the field property. We then obtain

$$v = 1_X \cdot_V v = (a^{-1} \cdot a) \cdot_V v = a^{-1} \cdot_V (a \cdot_V v) = a^{-1} \cdot_V 0_V = 0_V$$

by using (10.5), (6.1), (10.4), the assumption $a \cdot_V v = 0_V$, and finally the Cancellation Law for the zero vector. Thus, $v = 0_V$, which again implies $a = 0 \vee v = 0_V$. Since a and v were arbitrary, (10.8) follows to be true. As the field $(X, +, \cdot, -, /)$ and the vector space $(V, +_V, \cdot_V)$ were also arbitrary, the proposition follows to be true. □

Theorem 10.3 (Sign Laws for $-$ and $\cdot_V$). *The following equations are true for any field $(X, +, \cdot, -, /)$, any vector space $(V, +_V, \cdot_V)$ over X, any scalar $a \in X$ and any vector $v \in V$.*

$$a \cdot_V (-v) = -(a \cdot_V v) \tag{10.9}$$
$$(-a) \cdot_V v = -(a \cdot_V v) \tag{10.10}$$
$$(-a) \cdot_V (-v) = a \cdot_V v \tag{10.11}$$

Proof. We let $(X, +, \cdot, -, /)$ be an arbitrary field, $(V, +_V, \cdot_V)$ an arbitrary vector space over X, a an arbitrary scalar in X, and v an arbitrary vector in V. Concerning the equation in (10.9), we show that $a \cdot_V (-v)$ is the additive inverse of $a \cdot_V v$. We obtain

$$(a \cdot_V (-v)) +_V (a \cdot_V v) = a \cdot_V (-v +_V v) = a \cdot_V 0_V = 0_V$$

by applying Property 2 of a vector space, the property of an additive inverse and the Cancellation Law for the zero vector. Due to the commutativity of the vector addition,

$$(a \cdot_V v) +_V (a \cdot_V (-v)) = 0_V$$

holds, too, so that $a \cdot_V (-v)$ is by definition an inverse element of $a \cdot_V v$. Because $-(a \cdot_V v)$ is the unique inverse element of $a \cdot_V v$, the equation (10.9) follows to be true, as desired.

The second law (10.10) can be proved similarly by exploiting Property 3 of a vector space.

Concerning (6.65), we observe the truth of

$$(-a) \cdot_V (-v) = -((-a) \cdot_V v) = -(-(a \cdot_V v)) = a \cdot_V v.$$

in light of (10.9), (10.10) and the law (6.50) with respect to the group $(V, +_V)$.

Since a, v, $(X, +, \cdot, -, /)$ and $(V, +_V, \cdot_V)$ were arbitrary, we may now infer from these findings the truth of the stated laws. □

Exercise 10.3. Prove the Sign Law (10.10).

Corollary 10.4. *For any field $(X, +, \cdot, -, /)$ and any vector space $(V, +_V, \cdot_V)$ over X, it is true that the negative of any vector can be expressed as a multiplication of the negative of the unity element of X and the vector, that is,*

$$\forall v \, (v \in V \Rightarrow (-1_X) \cdot_V v = -v). \tag{10.12}$$

Letting $(X,+,\cdot,-,/)$ be an arbitrary field, $(V,+_V,\cdot_V)$ an arbitrary vector space over X and v an arbitrary vector in V, we find

$$(-1_X)\cdot_V v = -(1_X\cdot v) = -v$$

by applying the Sign Law (10.10) and Property 5 of a vector space. As $(X,+,\cdot,-,/)$, $(V,+_V,\cdot_V)$ and v were arbitrary, we therefore conclude that the stated universal sentence is true.

Exercise 10.4. Establish for any field $(X,+,\cdot,-,/)$, any vector space $(V,+_V,\cdot_V)$ over X, any $n\in\mathbb{N}$, and

a) any scalar $a\in X$ and any sequence $(v_i\,|\,i\in\{1,\ldots,n\})$ of vectors in V the sequence $(a\cdot_V v_i\,|\,i\in\{1,\ldots,n\})$ in V.

b) any vector $v\in V$ and any sequence $(a_i\,|\,i\in\{1,\ldots,n\})$ of scalars in X the sequence $(a_i\cdot_V v\,|\,i\in\{1,\ldots,n\})$ in V.

c) any sequence $(a_i\,|\,i\in\{1,\ldots,n\})$ of scalars in X and any sequence $(v_i\,|\,i\in\{1,\ldots,n\})$ of vectors in V the sequence $(a_i\cdot_V v_i\,|\,i\in\{1,\ldots,n\})$ in V.

(Hint: Recall the proof of Proposition 5.129.)

We use the sequences of the preceding exercise to establish the following two generalized distributive laws and subsequently the notion of *linear combination*.

Theorem 10.5 (Generalized Distributive Laws for vector spaces). *For any field $(X,+,\cdot,-,/)$, any vector space $(V,+_V,\cdot_V)$ over X,*

a) and any scalar $a\in X$, it is true for any $n\in\mathbb{N}$ that

$$\forall v_1,\ldots,v_n\,(v_1,\ldots,v_n\in V\Rightarrow a\cdot_V\sum_{i=1}^n v_i=\sum_{i=1}^n(a\cdot_V v_i)). \tag{10.13}$$

b) and any vector $v\in V$, it is true for any $n\in\mathbb{N}$ that

$$\forall a_1,\ldots,a_n\,(a_1,\ldots,a_n\in X\Rightarrow \left(\sum_{i=1}^n a_i\right)\cdot_V v=\sum_{i=1}^n(a_i\cdot_V v)). \tag{10.14}$$

Exercise 10.5. Prove the Generalized Distributive Law for vector spaces.

(Hint: Proceed as in the proof of the Generalized Distributive Law for semirings, using now the vector space properties.)

Before we continue with further fundamental concepts about vector spaces, we take a first look at some slightly more structured vector spaces. Having already encountered the notion of a vector as part of the definition of a matrix, it is now a straightforward task to establish a special kind of vector space on the ground of that definition. We begin more generally with functions, for which we now define a suitable function $\cdot_V : X \times V \to V$, as required by a vector space.

Exercise 10.6 (Scalar multiplication for functions). The following sentences are true for any sets X, Y and for any multiplication $\cdot_Y$ on Y.

a) For any element $a \in Y$ and any function $f : X \to Y$ there exists a unique function $h : X \to Y$ satisfying

$$\forall x \, (x \in X \Rightarrow h(x) = a \cdot_Y f(x)). \tag{10.15}$$

b) There exists a unique set $\boxdot_{Y^X}$ such that an element Z is in $\boxdot_{Y^X}$ iff Z is in $(Y \times Y^X) \times Y^X$ and moreover if there are an element $a \in Y$ and functions f, h from X to Y satisfying (10.15) and $((a, f), h) = Z$, i.e. such that

$$\begin{aligned}\forall Z \, (Z \in \boxdot_{Y^X} \Leftrightarrow [Z \in (Y \times Y^X) \times Y^X \, \wedge \exists a, f, h \, (a \in Y \, \wedge \, f, h \in Y^X \\ \wedge \, \forall x \, (x \in X \Rightarrow h(x) = a \cdot_Y f(x)) \, \wedge \, ((a, f), h) = Z)]).\end{aligned} \tag{10.16}$$

The set $\boxdot$ is a function from $Y \times Y^X$ to Y^X satisfying

$$\begin{aligned}&\forall a, f, h \, ([a \in Y \, \wedge \, f, h \in Y^X \\ &\quad \Rightarrow [h = a \boxdot_{Y^X} f \Leftrightarrow \forall x \, (x \in X \Rightarrow h(x) = a \cdot_Y f(x))]).\end{aligned} \tag{10.17}$$

(Hint: Proceed in analogy to the proof of Theorem 5.6.)

Notation 10.1. We usually employ a similar notation as for binary operations and write $h = a \boxdot_{Y^X} f$ instead of $h = \boxdot_{Y^X}((a, f))$.

Exercise 10.7. Establish for any sets X, Y, any function $f : X \to Y$, any element c of Y and any multiplication $\cdot_Y$ on Y the identity

$$g_c \cdot_{Y^X} f = c \boxdot_{Y^X} f \tag{10.18}$$

with respect to the constant function g_c on X (with value c) and the pointwise multiplication $\cdot_{Y^X}$.

(Hint: Apply the Equality Criterion for functions.)

Proof. Letting X and Y be arbitrary sets, f an arbitrary function from X to Y, c an arbitrary element of Y and $\cdot_Y$ an arbitrary multiplication on Y, we first observe that $g_c \cdot_{Y^X} f$ and $c \boxdot_{Y^X} f$ are both elements of Y^X, so that we may apply the Equality Criterion for functions. For this purpose, we prove the universal sentence

$$\forall x\,(x \in X \Rightarrow (g_c \cdot_{Y^X} f)(x) = (c \boxdot_{Y^X} f)(x)), \tag{10.19}$$

letting $x \in X$ be arbitrary. We obtain then the true equations

$$(g_c \cdot_{Y^X} f)(x) = g_c(x) \cdot_Y f(x) = c \cdot_Y f(x) = (c \boxdot_{Y^X} f)(x)$$

by the Pointwise multiplication of functions, (3.534) and the scalar multiplication of functions. As x was arbitrary, we conclude that the universal sentence (10.19) is true, so that the functions $g_c \cdot_{Y^X} f$ and $c \boxdot_{Y^X} f$ are indeed equal. Since X, Y, f, c and $\cdot_Y$ were initially all arbitrary, we may further conclude that the proposed sentence holds. □

Theorem 10.6 (Specification of vector spaces of functions). *For any set X and any field $(Y, +, \cdot, -, /)$, it is true that the ordered triple $(Y^X, +_{Y^X}, \boxdot_{Y^X})$ containing the pointwise addition of functions (in Y^X) and the scalar multiplication for functions (in Y^X) constitutes a vector space over Y.*

Proof. We let X be an arbitrary set and $(Y, +, \cdot, -, /)$ an arbitrary field, which give rise to the pointwise addition $+_{Y^X}$ as well as the scalar multiplication $\boxdot_{Y^X} : Y \times Y^X \to Y^X$. Concerning Property 1 of a vector space, we recall that $(Y, +, \cdot, -)$ is a ring and thus $(Y, +)$ a commutative group according to the definitions of a field and of a ring, respectively. Therefore, Corollary 6.15 ensures that $(Y^X, +_{Y^X})$ is a commutative group. To establish Property 2, we take arbitrary $a \in Y$ and $f, g \in Y^X$ and apply the Equality Criterion for functions to prove the equality of the functions $h = a \boxdot_{Y^X} (f +_{Y^X} g)$ and $h' = (a \boxdot_{Y^X} f) +_{Y^X} (a \boxdot_{Y^X} g)$. For this purpose, we let $x \in X$ be arbitrary and obtain the equations

$$\begin{aligned}
h(x) &= a \cdot (f +_{Y^X} g)(x) \\
&= a \cdot (f(x) + g(x)) \\
&= (a \cdot f(x)) + (a \cdot g(x)) \\
&= (a \boxdot_{Y^X} f)(x) + (a \boxdot_{Y^X} g)(x) \\
&= h'(x)
\end{aligned}$$

by using the specification of the scalar multiplication for functions, the specification of the pointwise addition of functions, and the distributivity of $\cdot$ over $+$. Since x is arbitrary, we may conclude that $h(x) = h'(x)$ holds for all x, so that the functions h and h' are indeed identical. As a, f and g were arbitrary as well, we may further conclude that Property 2 of a vector space is satisfied by the given sets. Property 3 is verified accordingly. To verify Property 4, we now let $a, b \in Y$, $f \in Y^X$ and $x \in Y$ be arbitrary, so that we obtain

$$\begin{aligned}[a \boxdot_{Y^X} (b \boxdot_{Y^X} f)](x) &= a \cdot (b \boxdot_{Y^X} f)(x) \\ &= a \cdot (b \cdot f(x)) \\ &= (a \cdot b) \cdot f(x) \\ &= [(a \cdot b) \boxdot_{Y^X} f](x)\end{aligned}$$

with the specification of the scalar multiplication for functions and the associativity of $\cdot$. As x, a, b and f were arbitrary, we may conclude that Property 4 of a vector space is satisfied, too. Finally, letting $f \in Y^X$ and $x \in X$ be arbitrary, we find with the definition of the neutral element 1_Y with respect to the multiplication on Y

$$\begin{aligned}(1_Y \boxdot_{Y^X} f)(x) &= 1_Y \cdot f(x) \\ &= f(x).\end{aligned}$$

Consequently, we may evidently infer from this the equality of the functions $1_Y \boxdot_{Y^X} f$ and f. Since f is arbitrary, we therefore conclude that $(Y^X, +_{Y^X}, \boxdot_{Y^X})$ satisfies also Property 5, so that it constitutes a vector space over Y. Because X and $(Y^X, +_{Y^X}, \boxdot_{Y^X})$ were initially arbitrary, we finally conclude that the stated theorem is true. $\square$

Exercise 10.8. Establish Property 4 in the preceding proof of Theorem 10.6.

Definition 10.2 (Vector space of functions from X to Y). For any set X and any field $(Y, +, \cdot, -, /)$, we call the ordered triple

$$(Y^X, +, \cdot) = (Y^X, +_{Y^X}, \boxdot_{Y^X}) \tag{10.20}$$

the *vector space of functions from X to Y*.

Definition 10.3 (Vector space of n-tuples). For any positive natural number n and any field $(Y, +, \cdot, -, /)$, we call the ordered triple

$$(Y^n, +, \cdot) = (Y^n, +_{Y^n}, \boxdot_{Y^n}) \tag{10.21}$$

the *vector space of n-tuples* in Y.

Exercise 10.9 (Scalar multiplication for matrices). Show for any positive natural numbers m, n, any set Y and any multiplication $\cdot_Y$ on Y that the scalar multiplication in the sense of (10.16) exists uniquely and satisfies

$$\begin{aligned}&\forall c, \boldsymbol{A}, \boldsymbol{B}\,([c \in Y \,\wedge\, \boldsymbol{A}, \boldsymbol{B} \in Y^{m\times n} \Rightarrow [\boldsymbol{B} = c\boldsymbol{A} \\ &\quad\Leftrightarrow \forall i, j\,([i \in \{1, \ldots, m\} \,\wedge\, j \in \{1, \ldots, n\}] \Rightarrow b_{i,j} = c \cdot_Y a_{i,j})]).\end{aligned} \tag{10.22}$$

(Hint: Recall Exercise 5.4.)

Notation 10.2. The multiplication of a scalar with a matrix can be expressed conveniently by

$$c\boldsymbol{A} = c\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n}\end{bmatrix} = \begin{bmatrix} c \cdot_Y a_{1,1} & \cdots & c \cdot_Y a_{1,n} \\ \vdots & \ddots & \vdots \\ c \cdot_Y a_{m,1} & \cdots & c \cdot_Y a_{m,n}\end{bmatrix}. \tag{10.23}$$

Corollary 10.7. *It is true for any positive natural numbers m, n and any field $(Y, +, \cdot, -, /)$ that the ordered triple $(Y^{m\times n}, +_{Y^{m\times n}}, \boxdot_{Y^X})$ containing the pointwise addition of matrices (in $Y^{m\times n}$) and the scalar multiplication for matrices (in $Y^{m\times n}$) constitutes a vector space over Y.*

Definition 10.4 (Vector space of m-by-n matrices). For any positive natural numbers m, n and any field $(Y, +, \cdot, -, /)$, we call the ordered triple

$$(Y^{m\times n}, +, \cdot) = (Y^{m\times n}, +_{Y^{m\times n}}, \boxdot_{Y^X}) \tag{10.24}$$

the *vector space of m-by-n-matrices* over Y.

We now introduce some further fundamental concepts in the context of vector spaces in general.

Definition 10.5 (Linear combination, coefficient, trivial linear combination, linearly independent & dependent vectors). For any field $(X, +, \cdot, -, /)$, any vector space $(V, +_V, \cdot_V)$ over X, any positive natural number n and any n-tuple of vectors $(v_i \,|\, i \in \{1, \ldots, n\})$,

(1) we call for any n-tuple of scalars $(a_i \,|\, i \in \{1, \ldots, n\})$ the sum

$$\sum_{i=1}^{n} a_i \cdot_V v_i \tag{10.25}$$

a *linear combination* of the vectors $v_1, \ldots, v_n$, and each of the scalars $a_1, \ldots, a_n$ a *coefficient* of the linear combination.

(2) we say that a vector w in V is a linear combination of the vectors $v_1, \ldots, v_n$ iff there exist scalars $a_1, \ldots, a_n$ such that $w = \sum_{i=1}^{n} a_i \cdot_V v_i$.

(3) we say that a linear combination of the vectors v_1, …, v_n is *trivial* iff all of the coefficients are equal to zero.

(4) we say that the vectors $v_1, \ldots, v_n$ are *linearly independent* iff the coefficients of any linear combination of $v_1, \ldots, v_n$ resulting in the zero vector are all equal to the zero scalar, that is, iff

$$\forall a \left(a \in X^{\{1,\ldots,n\}} \Rightarrow \left[\sum_{i=1}^{n} a_i \cdot_V v_i = 0_V \Rightarrow a = \{1, \ldots, n\} \times \{0_X\} \right] \right). \tag{10.26}$$

(5) we say that the vectors $v_1, \ldots, v_n$ are *linearly dependent* iff $v_1, \ldots, v_n$ are not linearly independent.

Definition 10.6 (Linearly independent & dependent subset). For any vector space $(V, +_V, \cdot_V)$ over any field $(X, +, \cdot, -, /)$ we say that a subset $M \subseteq V$ is

(1) *linearly independent* iff

$$\begin{aligned} M = \emptyset \ \vee \ \forall n, v\, ([n \in \mathbb{N}_+ \ \wedge \ v : \{1, \ldots, n\} \to M] \\ \Rightarrow v_1, \ldots, v_n \text{ are linearly independent}). \end{aligned} \tag{10.27}$$

(2) *linearly dependent* iff M is not linearly independent

Note 10.2. For any vector space $(V, +_V, \cdot_V)$ over any field $(X, +, \cdot, -, /)$ and for any nonempty subset $M \subseteq V$ we may evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of a set $\mathrm{Lin}(M)$ such that

$$\begin{aligned} \forall w\, (w \in \mathrm{Lin}(M) \Leftrightarrow [w \in V \ \wedge \exists n, a, v\, (n \in \mathbb{N}_+ \ \wedge \ a : \{1, \ldots, n\} \to X \\ \wedge \ v : \{1, \ldots, n\} \to M \ \wedge \ w = \sum_{i=1}^{n} a_i \cdot_V v_i)]). \end{aligned} \tag{10.28}$$

As $w \in \mathrm{Lin}(M)$ implies especially $w \in V$ for any w, we see in light of the definition of a subset that the inclusion

$$\mathrm{Lin}(M) \ \subseteq \ V \tag{10.29}$$

holds. Since V contains the zero vector 0_V, the preceding inclusion holds also for the singleton $\{0_V\}$ due to (2.184), which we may denote by $\mathrm{Lin}(\emptyset)$ to allow for $M = \emptyset$.

Definition 10.7 (Linear hull, (linear) span, generating system for a vector space, finitely generated vector space, basis). For any vector space $(V, +_V, \cdot_V)$ over any field $(X, +, \cdot, -, /)$

(1) and for any nonempty subset $M \subseteq V$ we call the set

$$\operatorname{Lin}(M) = \left\{ \sum_{i=1}^{n} a_i \cdot_V v_i : n \in \mathbb{N}_+,\ a_i \in X,\ v_i \in M\ (i = 1, \ldots, n) \right\} \tag{10.30}$$

the *linear hull* or the (*linear*) *span* of M. Furthermore, we define

$$\operatorname{Lin}(\emptyset) = \{0_V\} \tag{10.31}$$

to be the *linear hull/span of* $\emptyset$.

(2) we call any subset $M \subseteq V$ a *generating system for* $(V, +_V, \cdot_V)$ iff the linear hull of M coincides with V, that is, iff

$$\operatorname{Lin}(M) = V. \tag{10.32}$$

In this case we also say that M *generates* $(V, +_V, \cdot_V)$.

(3) and for any generating system M for $(V, +_V, \cdot_V)$ we say that the vector space is *finitely generated* iff M is a finite set. In this case, we say for any denumeration $v : \{1, \ldots, n\} \rightleftarrows M$ of M that $(V, +_V, \cdot_V)$ *is generated by* $v_1, \ldots, v_n$.

(4) we call any generating system M for $(V, +_V, \cdot_V)$ a *basis of* $(V, +_V, \cdot_V)$ iff M is linearly independent.

Definition 10.8 (Vector subspace/linear subspace). For any field $(X, +, \cdot, -, /)$ and any vector space $(V, +_V, \cdot_V)$ over X we say that an ordered triple $(U, +_U, \cdot_U)$ is a *vector subspace* or a *linear subspace of* $(V, +_V, \cdot_V)$ iff

1. U is a subset of V, that is,

$$U \subseteq V, \tag{10.33}$$

2. U is non-empty, that is,

$$U \neq \emptyset, \tag{10.34}$$

3. $+_U$ is a binary operation on U included in $+_V$, that is,

$$+_U : U \times U \to U \quad \text{and} \quad +_U \subseteq +_V, \tag{10.35}$$

and

4. $\cdot_U$ is a function from $X \times U$ to U included in $\cdot_V$, that is,

$$\cdot_U : X \times U \to U \quad \text{and} \quad \cdot_U \subseteq \cdot_V. \tag{10.36}$$

Note 10.3. Adopting the notation for binary operations, Property 3 shows for any $u, v \in U$ that

$$u +_U v \in U. \tag{10.37}$$

Similarly, employing Notation 10.1, Property 4 demonstrates that we have for any $a \in X$ and any $v \in U$

$$a \cdot_U v \in U. \tag{10.38}$$

The following exercise shows that suitable functions $+_U$ and $\cdot_U$ can be constructed from the corresponding vector space functions as restrictions.

Exercise 10.10. Demonstrate for any vector space $(V, +_V, \cdot_V)$ over any field $(X, +, \cdot, -, /)$ and for any nonempty subset U of V that the restrictions

$$+_U = +_V \restriction (U \times U) \tag{10.39}$$

$$\cdot_U = \cdot_V \restriction (X \times U) \tag{10.40}$$

constitute functions

$$+_U : U \times U \to V \tag{10.41}$$

$$\cdot_U : X \times U \to V \tag{10.42}$$

satisfying

$$\forall u, v\, (u, v \in U \Rightarrow u +_U v = u +_V v), \tag{10.43}$$

$$\forall a, v\, ([a \in X \ \wedge \ v \in U] \Rightarrow a \cdot_U v = a \cdot_V v). \tag{10.44}$$

(Hint: Use Proposition 3.164, Proposition 3.8 and Corollary 3.567.)

Theorem 10.8 (Vector Subspace Criterion). *It is true for any vector space $(V, +_V, \cdot_V)$ over any field $(X, +, \cdot, -, /)$ and for any nonempty subset U of V that the ordered triple $(U, +_U, \cdot_U)$ (where $+_U$ is the restriction of $+_V$ to $U \times U$ and $\cdot_U$ the restriction of $\cdot_V$ to $X \times U$) constitutes a vector subspace of $(V, +_V, \cdot_V)$ if*

1. *the vector sum of any two elements of U is again in U, that is,*

$$\forall v, w\, (v, w \in U \Rightarrow v +_V w \in U), \tag{10.45}$$

2. *and the scalar multiplication of any scalar in X and any element of U is again in U, that is,*

$$\forall a, v\,([a \in X \,\wedge\, v \in U] \Rightarrow a \cdot_V v \in U). \tag{10.46}$$

Proof. We let $(X, +, \cdot, -, /)$ be an arbitrary field, $(V, +_V, \cdot_V)$ an arbitrary vector space over that field, and U an arbitrary nonempty subset of V. We may therefore define the restrictions (10.39) and (10.40), and thus the ordered triple $(U, +_U, \cdot_U)$. Moreover, we assume the universal sentences (10.45) and (10.46) to hold. In view of the preceding assumptions $U \subseteq V$ and $U \neq \emptyset$, we have that $(U, +_U, \cdot_U)$ satisfies already Property 1 and Property 2 of a vector subspace.

Regarding Property 3, we first need to show that U is a codomain of $+_U$, that is, $\mathrm{ran}(+_U) \subseteq U$. For this purpose, we apply the definition of a subset and let first $y \in \mathrm{ran}(+_U)$ be arbitrary. By definition of a range, there exists then a particular constant $\bar{z}$ such that $(\bar{z}, y) \in +_U$ holds. In function notation, this reads $y = +_U(\bar{z})$, and we find $\bar{z} \in U \times U$ $[= \mathrm{dom}(+_U)]$ by definition of a domain. The latter implies that there exist particular constants $\bar{v}, \bar{w} \in U$ such that $(\bar{v}, \bar{w}) = \bar{z}$, in view of Exercise 3.4. We may therefore rewrite $y = +_U(\bar{z})$ as

$$y = +_U((\bar{v}, \bar{w})) = \bar{v} +_U \bar{w} = \bar{v} +_V \bar{w} \quad [\in U]$$

using also the notation for binary operations, (10.43) and the assumption (10.45). Thus, we obtain $y \in U$, and since y is arbitrary, we may therefore conclude that the inclusion $\mathrm{ran}(+_U) \subseteq U$ is indeed true. This means that U is a codomain of the function $+_U : U \times U \to V$, which we may therefore write as $+_U : U \times U \to U$. Thus, $+_U$ is a binary operation on U. To prove that it is included in the binary operation $+_V$, we let $t \in +_U$ be arbitrary. Let us observe here that the inclusion $+_U \subseteq (U \times U) \times U$ holds by virtue of (3.514). Evidently, $t \in +_U$ implies then that there exist particular constants $z^* \in U \times U$ and $y^* \in U$ with $(z^*, y^*) = t$; consequently, there are also particular constants $v^*, w^* \in U$ with $(v^*, w^*) = z^*$, so that $t \in +_U$ yields $((v^*, w^*), y^*) \in +_U$ via substitutions. Writing this in the form of $y^* = v^* +_U w^*$, we therefore obtain $y^* = v^* +_V w^*$ with (10.45). Rewriting this in the form of $((v^*, w^*), y^*) \in +_V$, we thus find $t \in +_V$ via substitution. Since $t \in +_U$ implies $t \in +_V$ where t is arbitrary, we may infer from this that $+_U$ is indeed included in $+_V$. Thus, $(U, +_U, \cdot_U)$ satisfies also Property 3 of a vector subspace.

The proof that $(U, +_U, \cdot_U)$ satisfies also Property 4 of a vector subspace is similar to that concerning Property 3 and is left as an exercise. As $(X, +, \cdot, -, /)$, $(V, +_V, \cdot_V)$ and U were initially arbitrary, we may therefore conclude that the stated theorem is indeed true. □

Proposition 10.9. *For any field $(X, +, \cdot, -, /)$, any vector space $(V, +_V, \cdot_V)$ over X and any vector space subspace U of V, the zero vector 0_V in V is also in U, i.e.*

$$0_V \in U. \tag{10.47}$$

Proof. We let $(X, +, \cdot, -, /)$ be an arbitrary field, $(V, +_V, \cdot_V)$ an arbitrary vector space over X and $(U, +_U, \cdot_U)$ an arbitrary vector subspace of V. As the field $(X, +, \cdot, -, /)$ contains the zero element (0_X) and U at least one element (say, $\bar{u}$) because of (10.34), we have on the one hand

$$0_X \cdot_U \bar{u} \in U$$

with (10.38). On the other hand, we obtain the true equations

$$0_X \cdot_U \bar{u} = 0_X \cdot_V \bar{u} = 0_V$$

by means of (10.44) and (10.7), so that $0_V \in U$ follows to be true. □

Theorem 10.10 (Vector space property of vector subspaces). *Any vector subspace $(U, +_U, \cdot_U)$ of any vector space $(V, +_V, \cdot_V)$ on any field $(X, +, \cdot, -, /)$ is itself a vector space over X, where*

- *$0_U = 0_V$ is the zero element and*
- *$-u$ the negative of u (for any $u \in U$)*

with respect to both $+_U$ and $+_V$.

Proof. We let $(X, +, \cdot, -, /)$ be an arbitrary field, $(V, +_V, \cdot_V)$ an arbitrary vector space over X, and U an arbitrary vector subspace of V. It is a straightforward exercise to prove that $(U, +_U)$ with (10.39) is a commutative semigroup. We now verify that the zero element 0_V with respect to $+_V$ (which is an element of U because of Proposition 10.9) is also the zero element with respect to $+_U$. For this purpose, we observe for an arbitrary $u \in U$ that the equations

$$u +_U 0_V = u +_V 0_V = u$$
$$0_V +_U u = 0_V +_V u = u$$

hold in view of (10.43) and the property of the zero element 0_V with respect to $+_V$. The existence of $0_V \in U$ with these properties implies that Property 1 of a group is satisfied by $(U, +_U)$.

Next, we verify that every element of $u \in U$ has a negative in U (with respect to $+_U$). On the one hand, X contains the unity element (1_X) as well as its negative -1_X by definition of a field. Letting $u \in U$ be arbitrary, we therefore obtain

$$-1_X \cdot_U u \in U$$

because of (10.38). On the other hand,

$$-1_X \cdot_U u = -1_X \cdot_V u = 1_X \cdot_V -u = -(1_X \cdot_V u) = -u$$

due to (10.44), (10.9) and (10.5), which shows that the negative of u with respect to $+_V$ is also an element of U. Then, we obtain for arbitrary $u \in U$

$$u +_U (-u) = -u +_U u = -u +_V u = 0_V = 0_U$$

using the commutativity of $+_U$, (10.43), the property of the negative with respect $+_V$, and the already established fact that $0_V = 0_U$. Thus, the element $-u$ of U satisfies the property of the negative of u also with respect to $+_U$. Since u is arbitrary, we may conclude that $(U, +_U, \cdot_U)$ is a (commutative) group. To show that $(U, +_U, \cdot_U)$ satisfies Property 2 - Property 5 of a vector space, we basically exploit the corresponding properties of $(V, +_V, \cdot_V)$ in connection with (10.43) and (10.44). □

Exercise 10.11. Complete the proof of Theorem 10.10.

Definition 10.9 (Linear map/mapping/function, linear operator, endomorphism of a vector space, linear functional/form). For any vector spaces $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ over any field $(X, +, \cdot, -, /)$ we say that

(1) a function $f : V \to W$ is a *linear map* or a *linear mapping* or a *linear function* iff

1. f is *additive* in the sense that

$$\forall u, v\, (u, v \in V \Rightarrow f(u +_V v) = f(u) +_W f(v)) \tag{10.48}$$

and

2. f is *homogeneous* in the sense that

$$\forall c, v\, ([c \in X \,\wedge\, v \in V] \Rightarrow f(c \cdot_V v) = c \cdot_W f(v)). \tag{10.49}$$

(2) a linear map $f : V \to W$ is a *linear operator (on V)* or an *endomorphism of V* iff $V = W$.

(3) a linear map $f : V \to W$ is a *linear functional* or *linear form (on V)* iff $W = X$.

We now look at a useful mechanism for defining a linear map of one vector space of n-tuples into a vector space of m-tuples.

Theorem 10.11 (Definition of linear maps by matrices). *For any $m, n \in \mathbb{N}_+$, any field $(Y, +, \cdot, -, /)$ and any m-by-n matrix $\boldsymbol{A}$ with values in Y, there exists a unique function f on Y^n such that*

$$\forall \boldsymbol{x}\,(\boldsymbol{x} \in Y^n \Rightarrow f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}), \tag{10.50}$$

and this function is a linear map $f : Y^n \to Y^m$.

Proof. We take two arbitrary positive natural numbers m and n, an arbitrary field $(Y, +, \cdot, -, /)$, and an arbitrary m-by-n matrix $\boldsymbol{A}$ with values in Y. To establish the desired function f, we apply Function definition by replacement and prove accordingly

$$\forall \boldsymbol{x}\,(\boldsymbol{x} \in Y^n \Rightarrow \exists! \boldsymbol{y}\,(\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x})). \tag{10.51}$$

Letting $\boldsymbol{x} \in Y^n$ be arbitrary, we observe that the standard matrix-vector product $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ is a uniquely specified m-vector in Y. Since $\boldsymbol{x}$ is arbitrary, we may therefore conclude that there exists a unique function f on Y^n such that (10.50). Since the value $\boldsymbol{y}$ is an element of Y^m for every $\boldsymbol{x} \in Y^n$, the set Y^n evidently constitutes a codomain of f. Using the sets Y^n and Y^m to define the vector spaces $(Y^n, +_{Y^n}, \boxdot_{Y^n})$ and $(Y^m, +_{Y^m}, \boxdot_{Y^m})$ over Y, it remains for us to prove that this function $f : Y^n \to Y^m$ is a linear map. For this purpose, we first take arbitrary n-vectors/-tuples $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in Y^n$, and we observe that their pointwise sum $\boldsymbol{x}^{(1)} +_{Y^n} \boldsymbol{x}^{(2)}$ is also in Y^n since the addition $+_{Y^n}$ constitutes a binary operation on Y^n. Furthermore, we obtain the associated values

$$\boldsymbol{y}^{(1)} = f(\boldsymbol{x}^{(1)}) = \boldsymbol{A}\boldsymbol{x}^{(1)}, \tag{10.52}$$

$$\boldsymbol{y}^{(2)} = f(\boldsymbol{x}^{(2)}) = \boldsymbol{A}\boldsymbol{x}^{(2)} \tag{10.53}$$

and

$$\boldsymbol{y} = f(\boldsymbol{x}^{(1)} +_{Y^n} \boldsymbol{x}^{(2)}) = \boldsymbol{A}(\boldsymbol{x}^{(1)} +_{Y^n} \boldsymbol{x}^{(2)}) \tag{10.54}$$

by virtue of (10.50). Clearly, all three values are elements of the codomain Y^m of f. Therefore, the pointwise sum

$$\boldsymbol{y}^{(1)} +_{Y^m} \boldsymbol{y}^{(2)} = f(\boldsymbol{x}^{(1)}) +_{Y^m} f(\boldsymbol{x}^{(2)}) \tag{10.55}$$

is also in Y^m as the addition $+_{Y^m}$ is a binary operation on Y^m. Moreover, we may apply then the Equality Criterion for functions to prove that $\boldsymbol{y}$ and

$\boldsymbol{y}^{(1)} +_{Y^m} \boldsymbol{y}^{(2)}$ are identical. To do this, we let $i \in \{1, \ldots, m\}$ be arbitrary. We obtain then the true equations

$$\begin{aligned}
y_i &= \sum_{k=1}^{n} \Big[\boldsymbol{A}((i,k)) \cdot_Y (\boldsymbol{x}^{(1)} +_{Y^n} \boldsymbol{x}^{(2)})_k\Big] \\
&= \sum_{k=1}^{n} \Big[\boldsymbol{A}((i,k)) \cdot_Y (\boldsymbol{x}_k^{(1)} +_Y \boldsymbol{x}_k^{(2)})\Big] \\
&= \sum_{k=1}^{n} \Big[\boldsymbol{A}((i,k)) \cdot_Y \boldsymbol{x}_k^{(1)} +_Y \boldsymbol{A}((i,k)) \cdot_Y \boldsymbol{x}_k^{(2)}\Big] \\
&= \left[\sum_{k=1}^{n} \boldsymbol{A}((i,k)) \cdot_Y \boldsymbol{x}_k^{(1)}\right] +_Y \left[\sum_{k=1}^{n} \boldsymbol{A}((i,k)) \cdot_Y \boldsymbol{x}_k^{(2)}\right] \\
&= y_i^{(1)} +_Y y_i^{(2)} \\
&= (\boldsymbol{y}^{(1)} +_{Y^m} \boldsymbol{y}^{(2)})_i
\end{aligned}$$

by applying (5.420) with respect to (10.54), the definition of the pointwise addition $+_{Y^n}$, the distributivity of the field multiplication $\cdot_Y$ over the field addition $+_Y$, (5.424), again (5.420) now to (10.52) – (10.53), and finally the definition of the pointwise addition $+_{Y^m}$. Since i was arbitrary, we may therefore conclude that the m-vectors $\boldsymbol{y}$ and $\boldsymbol{y}^{(1)} +_{Y^m} \boldsymbol{y}^{(2)}$ are identical indeed, which implies with (10.54) – (10.55) that

$$f(\boldsymbol{x}^{(1)} +_{Y^n} \boldsymbol{x}^{(2)}) = f(\boldsymbol{x}^{(1)}) +_{Y^m} f(\boldsymbol{x}^{(2)}).$$

As the n-vectors $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ were arbitrary, we may infer from the preceding equation that f possesses Property 1 of a linear map.

To establish Property 2, we take an arbitrary scalar $c \in Y$ and an arbitrary n-vector/-tuple $\boldsymbol{x} \in Y^n$, and we note that the scalar multiplication of c and $\boldsymbol{x}$ yields another n-vector, that is, $c \boxdot_{Y^n} \boldsymbol{x} \in Y^n$. With these two elements of Y^n, we may associate the values

$$\boldsymbol{y} = f(\boldsymbol{x}) \qquad = \boldsymbol{A}\boldsymbol{x}, \tag{10.56}$$

$$\boldsymbol{y}^{(c)} = f(c \boxdot_{Y^n} \boldsymbol{x}) = \boldsymbol{A}(c \boxdot_{Y^n} \boldsymbol{x}). \tag{10.57}$$

of f in Y^m. Consequently, the scalar multiplication of c and $\boldsymbol{y}$ yields the vector

$$c \boxdot_{Y^m} \boldsymbol{y} = c \boxdot_{Y^m} f(\boldsymbol{x}) \tag{10.58}$$

in Y^m, and we may apply then the Equality Criterion for functions to demonstrate that $\boldsymbol{y}^{(c)}$ and $c \boxdot_{Y^m} \boldsymbol{y}$ are equal. Letting $i \in \{1, \ldots, m\}$ be arbitrary, we obtain

$$\begin{aligned}
y_i^{(c)} &= \sum_{k=1}^{n} \boldsymbol{A}((i,k)) \cdot_Y (c \boxdot_{Y^n} \boldsymbol{x})_k \\
&= \sum_{k=1}^{n} \boldsymbol{A}((i,k)) \cdot_Y (g_c \cdot_{Y^n} \boldsymbol{x})_k \\
&= \sum_{k=1}^{n} \boldsymbol{A}((i,k)) \cdot_Y (g_c(k) \cdot_{Y^n} \boldsymbol{x}_k) \\
&= \sum_{k=1}^{n} c \cdot_Y (\boldsymbol{A}((i,k)) \cdot_Y \boldsymbol{x}_k) \\
&= c \cdot_Y \sum_{k=1}^{n} \boldsymbol{A}((i,k)) \cdot_Y \boldsymbol{x}_k \\
&= h_c(i) \cdot_Y y_i \\
&= (h_c \cdot_{Y^m} \boldsymbol{y})_i \\
&= (c \boxdot_{Y^m} \boldsymbol{y})_i
\end{aligned}$$

using (5.420) with respect to (10.57), (10.18) where g_c is the constant function on Y^n with value c, the definition of the pointwise multiplication $\cdot_{Y^n}$, (3.534) jointly with the associativity and commutativity of the field multiplication $\cdot_Y$, (5.499), (3.534) alongside (5.420) with respect to (10.56) where h_c is the constant function on Y^m with value c, the definition of the pointwise multiplication $\cdot_{Y^m}$, and finally again (10.18). As i was arbitrary, we may therefore conclude that the m-vectors $\boldsymbol{y}^{(c)}$ and $c \boxdot_{Y^m} \boldsymbol{y}$ are equal, and this implies with (10.56) – (10.57) that

$$f(c \boxdot_{Y^n} \boldsymbol{x}) = c \boxdot_{Y^m} f(\boldsymbol{x}).$$

Since the scalar c and the n-vector $\boldsymbol{x}$ were arbitrary, we may conclude that f also possesses Property 2 of a linear map. As m, n, $(Y, +, \cdot, -, /)$ and $\boldsymbol{A}$ were initially arbitrary, we may infer from these findings the truth of the stated theorem. $\square$

Note 10.4. The proof of Theorem 10.11 shows that every m-by-n matrix $\boldsymbol{A}$ with values in any field set Y, defining a linear map, is

1. additive in the sense that

$$\forall \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in Y^n \Rightarrow \boldsymbol{A}(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}) = \boldsymbol{A}\boldsymbol{x}^{(1)} + \boldsymbol{A}\boldsymbol{x}^{(2)}\right) \tag{10.59}$$

and

2. homogeneous in the sense that

$$\forall c, \boldsymbol{x} \, ([c \in Y \, \wedge \, \boldsymbol{x} \in Y^n] \Rightarrow \boldsymbol{A}(c\boldsymbol{x}) = c(\boldsymbol{A}\boldsymbol{x})), \tag{10.60}$$

omitting for brevity of expressions the symbols of the occurring vector additions and scalar multiplications.

Corollary 10.12. *For any vector spaces* $(V, +_V, \cdot_V)$ *and* $(W, +_W, \cdot_W)$ *over any field* $(X, +, \cdot, -, /)$ *it is true that every linear map* $f : V \to W$ *maps the zero vector* 0_V *onto the zero vector* 0_W*, that is,*

$$f(0_V) = 0_W. \tag{10.61}$$

Proof. Letting $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ be arbitrary vector spaces over any field $(X, +, \cdot, -, /)$ and $f : V \to W$ an arbitrary linear map, we obtain the true equations

$$f(0_V) = f(0_X \cdot_V 0_V) = 0_X \cdot_W f(0_V) = 0_W$$

by using the Cancellation Law for the zero vector with respect to $(V, +_V, \cdot_V)$, then (10.49), and subsequently the Cancellation Law for the zero vector with respect to $(W, +_W, \cdot_W)$. Since $(V, +_V, \cdot_V)$, $(W, +_W, \cdot_W)$, $(X, +, \cdot, -, /)$ and $f : V \to W$ were arbitrary, we therefore conclude that the stated universal sentence holds. □

Note 10.5. We see in light of Corollary 10.12 and the Definition of linear maps by matrices that every m-by-n matrix $\boldsymbol{A}$ with values in any field set Y maps the zero vector $\mathbf{0}_n \in Y^n$ onto the zero vector $\mathbf{0}_m \in Y^m$, that is,

$$\boldsymbol{A}\mathbf{0}_n = \mathbf{0}_m. \tag{10.62}$$

Proposition 10.13. *For any vector spaces* $(V, +_V, \cdot_V)$ *and* $(W, +_W, \cdot_W)$ *over any field* $(X, +, \cdot, -, /)$ *it is true that any linear map* $f : V \to W$ *maps any linear combination of vectors* $v_1, \ldots, v_n$ *in* V *with coefficients* $a_1, \ldots, a_n$ *onto the linear combination of mapped vectors* $f(v_1), \ldots, f(v_n)$ *with the same coefficients, that is,*

$$\begin{aligned} &\forall n \, (n \in \mathbb{N}_+ \Rightarrow \forall a, v \, ([a : \{1, \ldots, n\} \to X \, \wedge \, v : \{1, \ldots, n\} \to V] \\ &\qquad \Rightarrow f\left(\sum_{i=1}^{n} a_i \cdot_V v_i\right) = \sum_{i=1}^{n} a_i \cdot_W f(v_i))). \end{aligned} \tag{10.63}$$

Proof. We let $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ be arbitrary vector spaces over any field $(X, +, \cdot, -, /)$ and $f : V \to W$ an arbitrary linear map. To prove the universal sentence with respect to n, we apply a proof by mathematical induction. In the base case ($n = 1$), we let $a : \{1, \ldots, n\} \to X$ be an arbitrary n-tuple of scalars and $v : \{1, \ldots, n\} \to V$ an arbitrary n-tuple of vectors in V. We obtain then the true equations

$$\begin{aligned} f\left(\sum_{i=1}^{n} a_i \cdot_V v_i\right) &= f\left(\sum_{i=1}^{1} a_i \cdot_V v_i\right) = f(a_1 \cdot_V v_1) = a_1 \cdot_W f(v_1) \\ &= \sum_{i=1}^{1} a_i \cdot_W f(v_i) = \sum_{i=1}^{n} a_i \cdot_W f(v_i) \end{aligned}$$

by applying substitution, (5.411), (10.49), again (5.411), and finally again substitution. To establish the induction step, we let $n \in \mathbb{N}_+$ be arbitrary, we make the induction assumption

$$\begin{aligned} &\forall a, v\,([a : \{1, \ldots, n\} \to X \,\wedge\, v : \{1, \ldots, n\} \to V] \\ &\quad \Rightarrow f\left(\sum_{i=1}^{n} a_i \cdot_V v_i\right) = \sum_{i=1}^{n} a_i \cdot_W f(v_i)), \end{aligned}$$

and we demonstrate that

$$\begin{aligned} &\forall a, v\,([a : \{1, \ldots, n+1\} \to X \,\wedge\, v : \{1, \ldots, n+1\} \to V] \\ &\quad \Rightarrow f\left(\sum_{i=1}^{n+1} a_i \cdot_V v_i\right) = \sum_{i=1}^{n+1} a_i \cdot_W f(v_i)) \end{aligned} \tag{10.64}$$

follows to be true as well. For this purpose, we let $a : \{1, \ldots, n+1\} \to X$ and $v : \{1, \ldots, n+1\} \to V$ be arbitrary. We obtain the true equations

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} a_i \cdot_V v_i\right) &= f\left(\left[\sum_{i=1}^{n} a_i \cdot_V v_i\right] +_V [a_{n+1} \cdot_V v_{n+1}]\right) \\ &= f\left(\sum_{i=1}^{n} a_i \cdot_V v_i\right) +_W f\left(a_{n+1} \cdot_V v_{n+1}\right) \\ &= \left[\sum_{i=1}^{n} a_i \cdot_W f(v_i)\right] +_W [a_{n+1} \cdot_W f(v_{n+1})] \\ &= \sum_{i=1}^{n+1} a_i \cdot_W f(v_i) \end{aligned}$$

using (5.417), (10.48), the induction assumption jointly with (10.49), and finally (5.417). This proves the implication in (10.64), in which a and v

are arbitrary, so that the universal sentence (10.64) follows to be true, too. As n was also arbitrary, we may therefore conclude that the induction step holds, besides the base case. We thus completed the proof of (10.63), and as $(V, +_V, \cdot_V)$, $(W, +_W, \cdot_W)$, $(X, +, \cdot, -, /)$ and $f : V \to W$ were initially arbitrary, we may now finally conclude that the proposed universal sentence is true. $\square$

Exercise 10.12. Show for any vector spaces $(U, +_U, \cdot_U)$, $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ over any field $(X, +, \cdot, -, /)$ that the composition $g \circ f$ of any linear maps $f : U \to V$ and $g : V \to W$ is itself a linear map.

(Hint: Apply Notation 3.6.)

Theorem 10.14 (Representation of composed linear maps by matrix products). *It is true for any n-by-p matrix-$\boldsymbol{A}$ and for any m-by-n-matrix $\boldsymbol{B}$ with values in any field Y that the product $\boldsymbol{BA}$ defines a linear map which is the composition of the linear maps defined by $\boldsymbol{A}$ and $\boldsymbol{B}$, in the sense that*

$$\forall \boldsymbol{x}\, (\boldsymbol{x} \in Y^p \Rightarrow \boldsymbol{B}(\boldsymbol{A}\boldsymbol{x}) = (\boldsymbol{BA})\boldsymbol{x}). \tag{10.65}$$

Proof. Letting $(Y, +, \cdot, -, /)$ be an arbitrary field, $\boldsymbol{A}$ an arbitrary n-by-p-matrix and $\boldsymbol{B}$ an arbitrary m-by-n-matrix each with values in Y, we may form the product $\boldsymbol{C} = \boldsymbol{BA}$, which is an m-by-p-matrix with values in Y. Setting up the vector spaces $(Y^p, +_{Y^p}, \boxdot_{Y^p})$, $(Y^n, +_{Y^n}, \boxdot_{Y^n})$ and $(Y^m, +_{Y^m}, \boxdot_{Y^m})$, the previous three matrices define the linear maps

$$\begin{aligned} f : Y^p \to Y^n, &\quad \boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}, \\ g : Y^n \to Y^m, &\quad \boldsymbol{y} \mapsto \boldsymbol{B}\boldsymbol{y}, \\ h : Y^p \to Y^m, &\quad \boldsymbol{x} \mapsto \boldsymbol{C}\boldsymbol{x} = (\boldsymbol{BA})\boldsymbol{x}. \end{aligned}$$

In view of the preceding Exercise 10.12, the composition $g \circ f : Y^p \to Y^m$ is a linear map, which we now show to be identical with h. For this purpose, we apply the Equality Criterion for functions, letting $\boldsymbol{x} \in Y^p$ be arbitrary. First, we observe the truth of the equations

$$(g \circ f)(\boldsymbol{x}) = g(f(\boldsymbol{x})) = g(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{A}\boldsymbol{x}) \tag{10.66}$$

in light of the notation for composed functions and the previous definitions of f and g. Let us use in the following the denotations $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{z} = \boldsymbol{B}(\boldsymbol{A}\boldsymbol{x})$ and $\boldsymbol{z}' = \boldsymbol{C}\boldsymbol{x}$. We may now apply the Equality Criterion for functions to prove that the latter two m-vectors are identical. Letting for this purpose

$i \in \{1, \ldots, m\}$ be arbitrary, we obtain

$$\begin{aligned}
z_i &= \sum_{k=1}^{n} b_{i,k} \cdot_Y y_k \\
&= \sum_{k=1}^{n} b_{i,k} \cdot_Y \left(\sum_{l=1}^{p} a_{k,l} \cdot_Y x_l \right) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{p} b_{i,k} \cdot_Y (a_{k,l} \cdot_Y x_l) \\
&= \sum_{l=1}^{p} \sum_{k=1}^{n} b_{i,k} \cdot_Y (a_{k,l} \cdot_Y x_l) \\
&= \sum_{l=1}^{p} x_l \cdot_Y \sum_{k=1}^{n} b_{i,k} \cdot_Y a_{k,l} \\
&= \sum_{l=1}^{p} x_l \cdot_Y c_{i,l} \\
&= \sum_{l=1}^{p} c_{i,l} \cdot_Y x_l \\
&= z_i'
\end{aligned}$$

by applying the Specification of the standard matrix-vector product first to $\boldsymbol{z} = \boldsymbol{By}$ (denoting $b_{i,k} = \boldsymbol{B}((i,k))$) and subsequently to $\boldsymbol{y} = \boldsymbol{Ax}$ (denoting $a_{k,l} = \boldsymbol{A}((k,l))$), the Generalized Distributive Law for semirings, the Interchange of nested n-fold sums, the associativity & commutativity of the field multiplication followed by the Generalized Distributive Law for semirings, the definition of the product of matrices (denoting $c_{i,l} = \boldsymbol{C}((i,l))$), again the commutativity of the field multiplication, and finally the Specification of the standard matrix-vector product with respect to $\boldsymbol{z}' = \boldsymbol{Cx}$. Since i was arbitrary, we may therefore conclude that $\boldsymbol{z}$ and $\boldsymbol{z}'$ are indeed identical. As this means that $\boldsymbol{B}(\boldsymbol{Ax})$ and $\boldsymbol{Cx}$ are identical, we obtain with the definition of the function h from (10.66)

$$(g \circ f)(\boldsymbol{x}) = \boldsymbol{Cx} = (\boldsymbol{BA})\boldsymbol{x} = h(\boldsymbol{x}).$$

As $\boldsymbol{x}$ was arbitrary, we may now further conclude that the linear map $g \circ f$ and the function h are identical, and that the universal sentence (10.65) holds. Thus, h constitutes a linear map defined by the matrix product $\boldsymbol{BA}$, which is the composition of the linear maps f and g defined by the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Since $(Y, +, \cdot, -, /)$, $\boldsymbol{A}$ and $\boldsymbol{B}$ were initially arbitrary, we may finally conclude that the theorem is true. $\square$

Definition 10.10 (Kernel, nullspace). We call for any vector spaces $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ over any field $(X, +, \cdot, -, /)$ and for any linear map $f : V \to W$ the inverse image of the zero vector 0_W the *kernel* or the *nullspace* of f, symbolically

$$\ker(f) = f^{-1}[\{0_W\}]. \tag{10.67}$$

Corollary 10.15. *For any vector spaces $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ over any field $(X, +, \cdot, -, /)$ and for any linear map $f : V \to W$, the elements of the kernel of f are characterized by*

$$\forall v\, (v \in \ker(f) \Leftrightarrow f(v) = 0_W). \tag{10.68}$$

Proof. For arbitrary vector spaces $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ over an arbitrary field $(X, +, \cdot, -, /)$ and for an arbitrary linear map $f : V \to W$, we see for an arbitrary constant v that the sentence $v \in \ker(f)$ is equivalent to $v \in f^{-1}[\{0_W\}]$ (by definition of a kernel), which in turn is equivalent to $f(v) \in \{0_W\}$ (by definition of an inverse image). The latter also is equivalent to $f(v) = 0_W$ in view of (2.169), so that the equivalence $v \in \ker(f) \Leftrightarrow f(v) = 0_W$ follows to be true. This equivalence is then evidently true for any v, $(V, +_V, \cdot_V)$, $(W, +_W, \cdot_W)$, $(X, +, \cdot, -, /)$ and $f : V \to W$. $\square$

Notation 10.3. Due to (10.68) the kernel/nullspace of a linear map f is commonly symbolized by

$$\ker(f) = \{v : f(v) = 0_W\}. \tag{10.69}$$

In case f is defined by means of an m-by-n-matrix $\boldsymbol{A}$, we symbolize the nullspace of f also by

$$N(\boldsymbol{A}) = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\}. \tag{10.70}$$

Note 10.6. In view of Corollary 10.15, the elements of the nullspace of the linear map $f : Y^n \to Y^m$ defined by any m-by-n-matrix $\boldsymbol{A}$ with values in any field Y (according to Theorem 10.11) are characterized by

$$\forall \boldsymbol{x}\, (\boldsymbol{x} \in N(\boldsymbol{A}) \Leftrightarrow \boldsymbol{A}\boldsymbol{x} = \mathbf{0}_m), \tag{10.71}$$

where $\mathbf{0}_m$ is the zero vector in Y^m.

Proposition 10.16. *For any vector spaces $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ over any field $(X, +, \cdot, -, /)$ and for any linear map $f : V \to W$ it is true that the kernel of f gives rise to the vector subspace*

$$(\ker(f), +_{\ker(f)}, \cdot_{\ker(f)}) \tag{10.72}$$

of $(V, +_V, \cdot_V)$, where $+_{\ker(f)}$ is the restriction of $+_V$ to $\ker(f) \times \ker(f)$ and $\cdot_{\ker(f)}$ the restriction of $\cdot_V$ to $X \times \ker(f)$.

Proof. Letting $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ be arbitrary vector spaces over any field $(X, +, \cdot, -, /)$ and $f : V \to W$ an arbitrary linear map, we first recall the truth of the equation (10.61), which implies

$$0_V \in \ker(f) \tag{10.73}$$

by virtue of (10.68), so that the set $\ker(f)$ is clearly nonempty. Furthermore, the inverse image $\ker(f)$ is a subset of the domain V of V (see Note 3.30). We may therefore form the mentioned restrictions $+_{\ker(f)}$ and $\cdot_{\ker(f)}$ according to Exercise 10.10, and we may apply the Vector Subspace Criterion to check that the ordered triple (10.72) constitutes a vector subspace of $(V, +_V, \cdot_V)$. To do this, we first take arbitrary elements $v, w \in \ker(f)$, for which we obtain $f(v) = 0_W$ and $f(w) = 0_W$ by virtue of (10.68). We may now infer the equations

$$f(v +_V w) = f(v) +_W f(w) = 0_W +_W 0_W = 0_W$$

by applying the additivity of the linear map f, substitutions, and the fact that 0_W is the neutral element of W with respect to the addition $+_W$. The resulting equation $f(v +_V w) = 0_W$ yields then $v +_V w \in \ker(f)$ with (10.68), and as v and w constitute arbitrary elements of $\ker(f)$, we may therefore conclude that the first requirement of a vector subspace concerning the addition is satisfied by (10.72).

Regarding the second requirement about the scalar multiplication, we let $a \in X$ and $v \in \ker(f)$ be arbitrary. The latter evidently implies $f(v) = 0_W$, with the consequence that

$$f(a \cdot_V v) = a \cdot_W f(v) = a \cdot_W 0_W = 0_W$$

(using the Cancellation Law for the zero vector). Thus, $f(a \cdot_V v) = 0_W$ gives us $a \cdot_V v \in \ker(f)$, as required. Since a and v were arbitrary, we may infer from these findings that (10.72) constitutes a vector subspace of $(V, +_V, \cdot_V)$. As $(V, +_V, \cdot_V)$, $(W, +_W, \cdot_W)$, $(X, +, \cdot, -, /)$ and $f : V \to W$ were initially all arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Notation 10.4. Applying Notation 10.3 to the vector subspace of $(Y^n, +_{Y^n}, \cdot_{Y^n})$ defined by the nullspace of an m-by-n-matrix $\boldsymbol{A}$ with values in a field Y, we write

$$(N(\boldsymbol{A}), +_{N(\boldsymbol{A})}, \cdot_{N(\boldsymbol{A})}). \tag{10.74}$$

Exercise 10.13. Show for any vector spaces $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ over any field $(X, +, \cdot, -, /)$ that the range of any linear map $f : V \to W$ induces the vector subspace

$$(\mathrm{ran}(f), +_{\mathrm{ran}(f)}, \cdot_{\mathrm{ran}(f)}) \tag{10.75}$$

of $(W, +_W, \cdot_W)$, where $+_{\mathrm{ran}(f)}$ is the restriction of $+_W$ to $\mathrm{ran}(f) \times \mathrm{ran}(f)$ and $\cdot_{\mathrm{ran}(f)}$ the restriction of $\cdot_W$ to $X \times \mathrm{ran}(f)$.
(Hint: Establish first

$$0_V \in \mathrm{ran}(f) \tag{10.76}$$

and apply then the Vector Subspace Criterion.)

Notation 10.5. When a linear map $f : Y^n \to Y^m$ is defined by an m-by-n-matrix $\boldsymbol{A}$ with values in Y, we write for the range of f also

$$R(\boldsymbol{A}), \tag{10.77}$$

and for the corresponding vector subspace of $(Y^m, +_{Y^m}, \cdot_{Y^m})$ accordingly

$$(R(\boldsymbol{A}), +_{R(\boldsymbol{A})}, \cdot_{R(\boldsymbol{A})}). \tag{10.78}$$

We now adjoin a partial ordering to a vector space.

Definition 10.11 ((Partially) ordered vector space). For any ordered field $(X, +, \cdot, -, /, <)$, any vector space $(V, +_V, \cdot_V)$ over X and

(1) for any irreflexive partial ordering $<_V$ of V we say that

$$(V, +_V, \cdot_V, <_V) \tag{10.79}$$

is a (*partially*) *ordered vector space* iff $<_V$ satisfies the monotony laws

$$\forall u, v, w\, (u, v, w \in V \Rightarrow [u <_V v \Rightarrow u +_V w <_V v +_V w]). \tag{10.80}$$

$$\forall a, v, w\, ([a \in X \wedge v, w \in V \wedge 0_X < a] \Rightarrow [u <_V v \Rightarrow a \cdot_V u <_V a \cdot_V v]). \tag{10.81}$$

(2) for any reflexive partial ordering $\leq_V$ of V we say that

$$(V, +_V, \cdot_V, \leq_V) \tag{10.82}$$

is (*partially*) *ordered vector space* iff $\leq_V$ satisfies the monotony laws

$$\forall u, v, w\, (u, v, w \in V \Rightarrow [u \leq_V v \Rightarrow u +_V w \leq_V v +_V w]). \tag{10.83}$$

$$\forall a, v, w\, ([a \in X \wedge v, w \in V \wedge 0_X < a] \Rightarrow [u \leq_V v \Rightarrow a \cdot_V u \leq_V a \cdot_V v]). \tag{10.84}$$

Note 10.7. In situations where both types of partial orderings, the irreflexive kind $<_V$ and the reflexive kind $\leq_V$, of a vector space V are used simultaneously, it is not necessarily the case that one is induced by the other.

Proposition 10.17. *For any set X and any ordered field $(Y,+,\cdot,-,/,<)$, it is true that the ordered quadruple $(Y^X,+_{Y^X},\cdot_{Y^X},\prec)$ containing the pointwise addition of functions (in Y^X), the scalar multiplication for functions (in Y^X) and the irreflexive partial ordering $\prec$ of Y^X established in Exercise 3.109 constitutes an ordered vector space over Y.*

Proof. Letting X be an arbitrary set and $(Y,+,\cdot,-,/,<)$ an arbitrary ordered field, we define the vector space $(Y^X,+_{Y^X},\boxdot_{Y^X})$ according to Theorem 10.6 and the irreflexive partial ordering $\prec$ of Y^X according to Exercise 3.109. Next, we show that $\prec$ is compatible with the monotony laws

$$\forall f,g,h\,(f,g,h\in Y^X \Rightarrow [f\prec g\Rightarrow f+_{Y^X}h\prec g+_{Y^X}h]), \tag{10.85}$$

$$\forall a,f,g\,([a\in Y\,\wedge\, f,g\in Y^X\,\wedge\, 0_Y<a]\Rightarrow[f\prec g\Rightarrow a\boxdot_{Y^X}f\prec a\boxdot_{Y^X}g]). \tag{10.86}$$

concerning an ordered vector space. Firstly, we let $f,g,h\in Y^X$ be arbitrary and assume $f\prec g$ to be true, so that the universal sentence

$$\forall x\,(x\in X\Rightarrow f(x)<g(x)) \tag{10.87}$$

is true by definition on $\prec$. To demonstrate that this implies $f+_{Y^X}h\prec g+_{Y^X}h$, we verify

$$\forall x\,(x\in X\Rightarrow(f+_{Y^X}h)(x)<(g+_{Y^X}h)(x)). \tag{10.88}$$

For this purpose, we let $x\in X$ be arbitrary, which implies $f(x)<g(x)$ with (10.87). It follows that

$$f(x)+h(x)<g(x)+h(x) \tag{10.89}$$

since $<$ satisfies the monotony law (7.98). By definition of the pointwise addition of functions, we find $f(x)+h(x)=(f+_{Y^X}h)(x)$ and $g(x)+h(x)=(g+_{Y^X}h)(x)$, so that substitutions into (10.89) yield the desired consequent $(f+h)(x)<(g+h)(x)$. Since x is arbitrary, we may infer from this the truth of the universal sentence (10.88), which means that $f+_{Y^X}h\prec g+_{Y^X}h$ is true. As f,g,h were arbitrary, it follows that the universal sentence (10.85) also holds.

Secondly, letting $a\in Y$ and $f,g\in Y^X$ be arbitrary and assuming both $0_Y<a$ and $f\prec g$ to be true, we show that this implies $a\boxdot_{Y^X}f\prec a\boxdot_{Y^X}g$. Letting $x\in X$ be arbitrary, we obtain with the assumption $f\prec g$ evidently $f(x)<g(x)$ and consequently $f(x)\cdot a<g(x)\cdot a$ because $<$ satisfies the monotony law (7.99). In view of the commutativity of $\cdot$, we obtain $a\cdot f(x)<a\cdot g(x)$, and then

$$(a\boxdot_{Y^X}f)(x)<(a\boxdot_{Y^X}g)(x)$$

with the definition of the scalar multiplication for functions. Since x is arbitrary, we may conclude that $a \boxdot_{Y^X} f \prec a \boxdot_{Y^X} g$ holds, and since a, f and g are also arbitrary, it follows now that (10.86) is true as well. Initially, the sets X and $(Y, +, \cdot, -, /, <)$ were arbitrary, so that $(Y^X, +_{Y^X}, \boxdot_{Y^X}, \prec)$ is indeed an ordered vector space. $\square$

Definition 10.12 (Ordered vector space of functions from X to Y). For any set X and any ordered field $(Y, +, \cdot, -, /, <)$, we call the ordered quadruple

$$(Y^X, +, \cdot, \prec) = (Y^X, +_{Y^X}, \boxdot_{Y^X}, \prec) \tag{10.90}$$

the *ordered vector space of functions from X to Y*.

10.2. Real Vector Spaces

We begin this section with a few basic definitions.

Definition 10.13 (Real vector space). We call any vector space $(V, +_V, \cdot_V)$ over the field $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, -_{\mathbb{R}}, /_{\mathbb{R}})$ of real numbers a *real vector space*.

Note 10.8. Taking the field of real numbers, the ordered triple $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}})$ constitutes a real vector space in view of Exercise 10.1. Furthermore, adding the standard linear ordering of $\mathbb{R}$ to this vector space yields by definition a (partially) ordered vector space.

Definition 10.14 (Real vector space of real numbers, ordered real vector space of real numbers). We call

$$(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}) \tag{10.91}$$

the *real vector space of real numbers* and

$$(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, <_{\mathbb{R}}) \tag{10.92}$$

the *ordered real vector space of real numbers*.

Note 10.9. We may form the real vector space $(\mathbb{R}^X, +_{\mathbb{R}^X}, \boxdot_{\mathbb{R}^X})$ of functions from (any set) X to $\mathbb{R}$, where the binary operation $+_{\mathbb{R}^X}$ (on $\mathbb{R}^X$) is the pointwise addition of functions (in $\mathbb{R}^X$) and where $\boxdot_{\mathbb{R}^X}$ is the scalar multiplication for functions (in $\mathbb{R}^X$). Proposition 5.19 shows that the constant function $g_0 = X \times \{0\}$ is the zero element of $\mathbb{R}^X$ with respect to $+_{\mathbb{R}^X}$. Thus, the zero vector $0_{\mathbb{R}^X} = g_0$ satisfies

$$\forall x\,(x \in X \Rightarrow 0_{\mathbb{R}^X}(x) = 0) \tag{10.93}$$

according to (3.534). The ordered vector space $(\mathbb{R}^X, +_{\mathbb{R}^X}, \boxdot_{\mathbb{R}^X}, \prec)$, where the irreflexive partial ordering $\prec$ of $\mathbb{R}^X$ is defined according to Exercise 3.109, can also be defined (see Proposition 10.17).

Definition 10.15 (Real vector space of real functions, ordered real vector space of real functions). We call

$$(\mathbb{R}^X, +_{\mathbb{R}^X}, \boxdot_{\mathbb{R}^X}) \tag{10.94}$$

the *real vector space of real functions* and

$$(\mathbb{R}^X, +_{\mathbb{R}^X}, \boxdot_{\mathbb{R}^X}, \prec) \tag{10.95}$$

the *ordered real vector space of real functions*.

Proposition 10.18. *In every real vector space of real functions, the product of any two real functions on X can be written in terms of a scalar multiplication and the First & Second Binomial Formula, that is,*

$$f \cdot_{\mathbb{R}^X} g = \frac{1}{4} \boxdot_{\mathbb{R}^X} \left[(f +_{\mathbb{R}^X} g)^2 -_{\mathbb{R}^X} (f -_{\mathbb{R}^X} g)^2 \right]. \tag{10.96}$$

Proof. We let X be an arbitrary set and f, g arbitrary real functions on X. The definition of a field implies for the field $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, -_{\mathbb{R}}, /_{\mathbb{R}})$ that $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, -_{\mathbb{R}})$ constitutes a commutative ring. $(\mathbb{R}^X, +_{\mathbb{R}^X}, \cdot_{\mathbb{R}^X}, -_{\mathbb{R}^X})$ is therefore also a commutative ring by virtue of Corollary 6.21, so that we may apply the Binomial Formulae to such functions. Note that $\cdot_{\mathbb{R}^X}$ represents the pointwise multiplication of functions in $\mathbb{R}^X$, so that $f \cdot_{\mathbb{R}^X} g$ is an element of and thus a vector in $\mathbb{R}^X$. We then obtain the equations (writing $f \pm g$ instead of $f \pm_{\mathbb{R}^X} g$ and $f \cdot g$ instead of $f \cdot_{\mathbb{R}^X} g$)

$$\begin{aligned}
&\frac{1}{4} \boxdot_{\mathbb{R}^X} \left[(f+g)^2 - (f-g)^2 \right] \\
&\quad = \frac{1}{4} \boxdot_{\mathbb{R}^X} \left[(f^2 + 2[f \cdot g] + g^2) - (f^2 - 2[f \cdot g] + g^2) \right] \\
&\quad = \frac{1}{4} \boxdot_{\mathbb{R}^X} \left[f^2 + 2(f \cdot g) + g^2 - g^2 - [f^2 - 2(f \cdot g)] \right] \\
&\quad = \frac{1}{4} \boxdot_{\mathbb{R}^X} \left[f^2 + 2(f \cdot g) + 0_{\mathbb{R}^X} + 2(f \cdot g) - f^2 \right] \\
&\quad = \frac{1}{4} \boxdot_{\mathbb{R}^X} \left[f^2 - f^2 + 2(f \cdot g) + 2(f \cdot g) \right] \\
&\quad = \frac{1}{4} \boxdot_{\mathbb{R}^X} [f \cdot g + f \cdot g + f \cdot g + f \cdot g] \\
&\quad = \frac{1}{4} \boxdot_{\mathbb{R}^X} (f \cdot g) + \frac{1}{4} \boxdot_{\mathbb{R}^X} (f \cdot g) + \frac{1}{4} \boxdot_{\mathbb{R}^X} (f \cdot g) + \frac{1}{4} \boxdot_{\mathbb{R}^X} (f \cdot g) \\
&\quad = \left(\frac{1}{4} +_{\mathbb{R}} \frac{1}{4} +_{\mathbb{R}} \frac{1}{4} +_{\mathbb{R}} \frac{1}{4} \right) \boxdot_{\mathbb{R}^X} (f \cdot g) = 1 \boxdot_{\mathbb{R}^X} (f \cdot g) \\
&\quad = f \cdot g
\end{aligned}$$

using the Binomial Formulae (6.387) and (6.388) with respect to the commutative ring $(\mathbb{R}^X, +_{\mathbb{R}^X}, \cdot_{\mathbb{R}^X}, -_{\mathbb{R}^X})$, Sign Law (6.52) with respect to the group $(\mathbb{R}^X, +_{\mathbb{R}^X})$ in connection with the commutativity and associativity of $+_{\mathbb{R}^X}$, the property of a negative together with Sign Law (6.53), the property of the zero element, the definition of a multiple, Property 2 of a vector space (twice), Property 3 of a vector space (twice), and finally Property 5 of a vector space.

As X, f and g are arbitrary, we conclude that the proposition holds, as claimed. □

Note 10.10. In the preceding definition, we may set in particular $X = \{1,\ldots,n\}$ for any $n \in \mathbb{N}_+$, so that $\mathbb{R}^X = \mathbb{R}^n$ (recalling Proposition 4.88). Here, the constant function $g_0 = \{1,\ldots,n\} \times \{0\}$ is the zero element of $\mathbb{R}^n$ with respect to $+_{\mathbb{R}^n}$. Thus, the zero vector $0_{\mathbb{R}^n} = g_0$, which we also symbolize by $\mathbf{0}$, satisfies

$$\forall i\,(i \in \{1,\ldots,n\} \Rightarrow \mathbf{0}_i = 0). \tag{10.97}$$

Definition 10.16 (Real vector space of real n-tuples, ordered real vector space of real n-tuples). For any $n \in \mathbb{N}_+$ we call

$$(\mathbb{R}^n, +_{\mathbb{R}^n}, \boxdot_{\mathbb{R}^n}) \tag{10.98}$$

the *real vector space of real n-tuples* and

$$(\mathbb{R}^n, +_{\mathbb{R}^n}, \boxdot_{\mathbb{R}^n}, \prec) \tag{10.99}$$

the *ordered real vector space of real n-tuples.*

Similarly, we may take the set ω of real sequences (with domain either $\mathbb{N}_+$ or $\mathbb{N}$) to define a vector space.

Definition 10.17 (Real vector space of real sequences, ordered real vector space of real sequences). We call

$$(\omega, +_\omega, \boxdot_\omega) \tag{10.100}$$

the *real vector space of real sequences* and

$$(\omega, +_\omega, \boxdot_\omega, \prec) \tag{10.101}$$

the *ordered real vector space of real sequences.*

Definition 10.18 (Convex set). For any real vector space $(V, +_V, \cdot_V)$ and any subset $E \subseteq V$, we say that E is *convex* iff the linear combination of any two vectors in E with coefficients $(1-t)$ and t is in E for all scalars t between 0 and 1, i.e. iff

$$\forall u, v\,(u, v \in E \Rightarrow \forall t\,(0 \leq t \leq 1 \Rightarrow t \cdot_V u +_V [1-t] \cdot_V v \in E)). \tag{10.102}$$

Definition 10.19 (Norm, normed (vector) space). For any real vector space $(V, +_V, \cdot_V)$ we say that a function

$$\|.\| : V \to \mathbb{R} \tag{10.103}$$

is a *norm* on $(V, +_V, \cdot_V)$ iff

1. $\|.\|$ has only nonnegative function values, that is,

$$\forall v\,(v \in V \Rightarrow \|v\| \geq_{\mathbb{R}} 0), \tag{10.104}$$

2. $\|.\|$ takes the value 0 precisely for the zero vector, that is,

$$\forall v\,(v \in V \Rightarrow [\|v\| = 0 \Leftrightarrow v = 0_V]), \tag{10.105}$$

3. $\|.\|$ is scalable in the sense that

$$\forall a, v\,([a \in \mathbb{R} \wedge v \in V] \Rightarrow [\|a \cdot_V v\| = |a| \cdot_{\mathbb{R}} \|v\|]), \tag{10.106}$$

4. $\|.\|$ satisfies the *triangle inequality*, that is,

$$\forall v, w\,(v, w \in V \Rightarrow \|v +_V w\| \leq_{\mathbb{R}} \|v\| +_{\mathbb{R}} \|w\|). \tag{10.107}$$

Then, for any $v \in V$, we call $\|v\|$ the *norm of* v. Furthermore, we call

$$(V, +_V, \cdot_V, \|.\|) \tag{10.108}$$

a *normed* (*vector*) *space*.

We now pick up the previously established idea of a metric.

Exercise 10.14. Define for any normed vector space $(V, +_V, \cdot_V, \|.\|)$ the function

$$d : V \times V \to \mathbb{R}, \quad (v, w) \mapsto d(v, w) = \|v -_V w\|\,. \tag{10.109}$$

(Hint: Proceed in analogy to Exercise 6.4.)

Theorem 10.19 (Definition of a metric via a norm). *For any normed vector space* $(V, +_V, \cdot_V, \|.\|)$*, it is true that the function* d *in (10.109) is a metric on* V*.*

Proof. We let $(V, +_V, \cdot_V, \|.\|)$ be an arbitrary normed vector space. Regarding Property 1 of a metric, we let $v, w \in V$ be arbitrary and notice that

$$d(v, w) = \|v -_V w\| \geq_{\mathbb{R}} 0$$

holds due to Property 1 of a norm; thus, since v and w are arbitrary, we conclude that d satisfies Property 1 of a metric.

Regarding Property 2, we let $v, w \in V$ be arbitrary and observe that the equivalences

$$\begin{aligned} d(v, w) = 0 &\Leftrightarrow \|v -_V w\| = 0_V \\ &\Leftrightarrow v -_V w = 0_V \\ &\Leftrightarrow v +_V (-w) = 0_V \end{aligned}$$

hold, using the specification of d, Property 2 of a norm and the definition of a subtraction. Here, we see that $-w$ is the negative of v (by definition of an inverse element), which means that $-w = -v$ holds. Thus, $v +_V (-w) = 0_V$ implies $w = v$ with Theorem 6.9b). Conversely, $v = w$ implies in conjunction with the true $v +_V (-v) = 0_V$ the equation $v +_V (-w) = 0_V$. This completes the proof of $d(v,w) = 0 \Leftrightarrow v = w$, and since v and w are arbitrary, we therefore conclude that Property 2 of a metric is also satisfied.

Regarding Property 3, we let $v, w \in V$ arbitrary and obtain the equations

$$\begin{aligned}
d(v,w) &= \|v -_V w\| \\
&= \|-(-(v -_V w))\| \\
&= \|-(w -_V v)\| \\
&= \|-(1 \cdot_V (w -_V v))\| \\
&= \|(-1) \cdot_V (w -_V v)\| \\
&= |-1| \cdot_{\mathbb{R}} \|w -_V v\| \\
&= |1| \cdot_{\mathbb{R}} d(w,v) \\
&= 1 \cdot_{\mathbb{R}} d(w,v) \\
&= d(w,v)
\end{aligned}$$

using the specification of d, the Sign Law (6.50) with respect to the (commutative) group $(V, +_V)$, Sign Law (6.53), Property 5 of a vector space, Sign Law (10.10) for vector spaces, Property 3 of a norm, the evenness of the absolute value function on $\mathbb{R}$ together with the specification of d, (8.409) with the evident fact that $1 \geq_{\mathbb{R}} 0$, and finally the property of the unity element 1. Since v and w are arbitrary, we thus conclude that Property 3 of a metric holds indeed.

Regarding Property 4 of a metric, we let $u, v, w \in V$ be arbitrary and observe first that

$$\begin{aligned}
d(u,v) &= \|u -_V v\| \\
&= \|(u -_V v) +_V 0_V\| \\
&= \|(u -_V v) +_V (w -_V w)\| \\
&= \|(u -_V w) +_V (w -_V v)\|
\end{aligned}$$

holds in light of the definition of d, the property of the zero element of the group $(V, +_V)$, the definition of an additive inverse and the commutativity of the vector addition. Now, since

$$\|(u -_V w) +_V (w -_V v)\| \leq_{\mathbb{R}} \|u -_V w\| +_{\mathbb{R}} \|w -_V v\|$$

holds with Property 4 of a norm, where

$$\|u -_V w\| +_{\mathbb{R}} \|w -_V v\| = d(u, w) +_{\mathbb{R}} d(w, v)$$

by definition of d, it follows that $d(u, v) \leq_{\mathbb{R}} d(u, w) +_{\mathbb{R}} d(w, v)$. Then, since u, v, w were arbitrary, we conclude that Property 4 of a metric holds as well. We thus showed that d is a metric on V, and since $(V, +_V, \cdot_V, \|.\|)$ is arbitrary, we conclude that the theorem is true. □

Definition 10.20 (Induced metric). We call for any normed vector space $(V, +_V, \cdot_V, \|.\|)$ the function

$$d : V \times V \to \mathbb{R}, \quad (v, w) \mapsto \|v -_V w\| \tag{10.110}$$

the *metric induced by* $\|.\|$.

Definition 10.21 (Dense subset (of a normed space), separable normed space). We say for any normed space $(V, +_V, \cdot_V, \|.\|)$ that a set A is a *dense subset* of V (alternatively, that a set A is *dense in* V) iff

1. A is a subset of V, that is,

$$A \subseteq V, \tag{10.111}$$

and

2. for any vector $v \in V$ and any positive real number r there exists an element $w \in A$ such that the distance from v to w is less than r, i.e.

$$\forall v, r\,([v \in V \;\wedge\; r >_{\mathbb{R}} 0] \Rightarrow \exists w\,(w \in A \;\wedge\; d(v, w) <_{\mathbb{R}} r)). \tag{10.112}$$

We then say that $(V, +_V, \cdot_V, \|.\|)$ is *separable* iff there exists a set A such that A is countable and such that A is a dense subset of V.

Definition 10.22 (Complete normed/Banach space). We say that a normed space $(V, +_V, \cdot_V, \|.\|)$ is *complete* or a *Banach space* iff any Cauchy sequence $f = (a_n)_{n \in \mathbb{N}_+}$ in V is convergent (with respect to the metric induced by $\|.\|$).

Definition 10.23 (Inner product, inner product/pre-Hilbert space). For any real vector space $(V, +_V, \cdot_V)$ we say that a function

$$\langle ., . \rangle : V \times V \to \mathbb{R} \tag{10.113}$$

is an *inner product* on $(V, +_V, \cdot_V)$ iff

1. $\langle .,. \rangle$ satisfies

$$\forall u, v, w\, (u, v, w \in V \Rightarrow \langle u +_V v, w\rangle = \langle u, w\rangle + \langle v, w\rangle), \tag{10.114}$$

2. $\langle .,. \rangle$ satisfies

$$\forall c, v, w\, ([c \in \mathbb{R} \wedge v, w \in V] \Rightarrow \langle c \cdot_V v, w\rangle = c \cdot \langle v, w\rangle), \tag{10.115}$$

3. $\langle .,. \rangle$ is symmetric in the sense that

$$\forall v, w\, (v, w \in V \Rightarrow \langle v, w\rangle = \langle w, v\rangle), \tag{10.116}$$

4. $\langle .,. \rangle$ is positive definite in the sense that

$$\forall v\, (v \in V \Rightarrow [\langle v, v\rangle \geq 0 \wedge (\langle v, v\rangle = 0 \Leftrightarrow v = 0_V)]). \tag{10.117}$$

Then, for any $v, w \in V$, we call $\langle v, w\rangle$ the *inner product of* v and w. In addition, we call

$$(V, +_V, \cdot_V, \langle .,. \rangle) \tag{10.118}$$

an *inner product space* (alternatively, a *Banach space*).

Note 10.11. Forming for an arbitrary $n \in \mathbb{N}$ the composition of the n-fold repeated addition $\sum_{i=1}^{n} : \mathbb{R}^n \to \mathbb{R}$ and the pointwise multiplication $\cdot_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ yields the function

$$\sum_{i=1}^{n} \circ \cdot_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{10.119}$$

according to Proposition 3.178. For arbitrary $v, w \in \mathbb{R}^n$, we have $v = (v_i \,|\, i \in \{1, \ldots, n\})$ and $w = (w_i \,|\, i \in \{1, \ldots, n\})$, and by definition of the pointwise product of two functions then $v \cdot_{\mathbb{R}^n} w = (v_i \cdot_{\mathbb{R}} w_i \,|\, i \in \{1, \ldots, n\})$. Then, the latter sequence is mapped to $\sum_{i=1}^{n} v_i \cdot_{\mathbb{R}} w_i$.

Definition 10.24 (Scalar product). For any $n \in \mathbb{N}$, we call the function (10.119) the *scalar product*, which also symbolize by

$$\bullet : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (v, w) \mapsto v \bullet w = \sum_{i=1}^{n} v_i \cdot_{\mathbb{R}} w_i. \tag{10.120}$$

Proposition 10.20. *The scalar product constitutes an inner product on* $(\mathbb{R}^n, +_{\mathbb{R}^n}, \boxdot_{\mathbb{R}^n})$ *for any* $n \in \mathbb{N}$.

Proof. To fix the scalar product, we take an arbitrary natural number n. Letting now $u, v, w \in \mathbb{R}^n$ be arbitrary, we find

$$\begin{aligned}(u +_{\mathbb{R}^n} v) \bullet w &= \sum_{i=1}^{n}(u_i + v_i) \cdot w_i \\ &= \sum_{i=1}^{n}(u_i \cdot w_i + v_i \cdot w_i) \\ &= \left(\sum_{i=1}^{n} u_i \cdot w_i\right) + \left(\sum_{i=1}^{n} v_i \cdot w_i\right) \\ &= (u \bullet w) + (v \bullet w)\end{aligned}$$

by using the definition of the pointwise addition on $\mathbb{R}^n$ with (10.120), the distributivity of $\cdot_{\mathbb{R}}$ over $+_{\mathbb{R}}$, the n-fold addition of the sum of two sequences, and finally again (10.120). Since u, v and w are arbitrary, we may therefore conclude that $\bullet$ satisfies Property 1 of an inner product on $(\mathbb{R}^n, +_{\mathbb{R}^n}, \boxdot_{\mathbb{R}^n})$. Similarly, letting $c \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$ be arbitrary, we obtain

$$(c \boxdot_{\mathbb{R}^n} v) \bullet w = \sum_{i=1}^{n}(c \cdot v_i) \cdot w_i = \sum_{i=1}^{n} c \cdot (v_i \cdot w_i) = c \cdot \sum_{i=1}^{n} v_i \cdot w_i = c \cdot (v \bullet w)$$

by means of (10.120), the definition of the scalar multiplication for functions from $\{1, \ldots, n\}$ to $\mathbb{R}$, the associativity of $\cdot_{\mathbb{R}}$, and the Generalized Distributive Law for semirings. Therefore, as c, v and w are arbitrary, Property 2 of an inner product turns out to hold as well. Furthermore, the commutativity of $\cdot_{\mathbb{R}}$ evidently yields for arbitrary $v, w \in \mathbb{R}^n$

$$v \bullet w = \sum_{i=1}^{n} v_i \cdot w_i = \sum_{i=1}^{n} w_i \cdot v_i = w \bullet v.$$

Because v and w are arbitrary, it follows that $\bullet$ is also symmetric. Finally, regarding Property 4, we take an arbitrary $v \in \mathbb{R}^n$, so that

$$v \bullet v = \sum_{i=1}^{n} v_i \cdot v_i = \sum_{i=1}^{n} v_i^2.$$

We now see in light of (6.244) that this sum is nonnegative, i.e., $v \bullet v \geq 0$. To show that $\bullet$ satisfies also the equivalence required by (10.117), we note that $v \bullet v = 0$ is equivalent to $\sum_{i=1}^{n} v_i^2 = 0$, where the latter is equivalent to

$$\forall i \, (i \in \{1, \ldots, n\} \Rightarrow v_i = 0)$$

in view of (6.245). Since this characterizes the terms of the zero vector $\mathbf{0}$ in $\mathbb{R}_n$ according to (10.97), the preceding universal sentence is equivalent to $v = \mathbf{0}$. Thus, $v \bullet v = 0$ is equivalent to $v = \mathbf{0}$ $[= 0_{\mathbb{R}^n}]$. Since v was arbitrary, we may infer from these findings that $\bullet$ is indeed positive definite. Because n was initially arbitrary, we may now finally conclude that the proposition holds, as claimed. □

Proposition 10.21. *For any inner product space $(V, +_V, \cdot_V, \langle ., . \rangle)$, the inner product of the negative of a vector $v \in V$ and a vector $w \in V$ equals the negative of the inner product of v and w, that is,*

$$\forall v, w\, (v, w \in V \Rightarrow \langle -v, w \rangle = -\langle v, w \rangle). \tag{10.121}$$

Proof. We let $(V, +_V, \cdot_V, \langle ., . \rangle)$ be an arbitrary inner product space and v, w arbitrary vectors in V. We obtain then the true equations

$$\begin{aligned} \langle -v, w \rangle &= \langle -(1 \cdot_V v), w \rangle \\ &= \langle (-1) \cdot_V v, w \rangle \\ &= (-1) \cdot \langle v, w \rangle \\ &= -(1 \cdot \langle v, w \rangle) \\ &= -\langle v, w \rangle \end{aligned}$$

by means of Property 5 of a vector space over $(\mathbb{R})$, the Sign Law (10.10), Property 2 of an inner product space, the Sign Law (6.64), and the fact that 1 is the neutral element of $\mathbb{R}$ with respect to the multiplication on $\mathbb{R}$. Since v and w are arbitrary, we may therefore conclude that the universal sentence (10.121) is true. Moreover, as $(V, +_V, \cdot_V, \langle ., . \rangle)$ was also arbitrary, we may further conclude that the proposition holds, as claimed. □

Corollary 10.22. *For any inner product space $(V, +_V, \cdot_V, \langle ., . \rangle)$,*

a) the inner product of a vector $v \in V$ and the negative of a vector $w \in V$ equals the negative of the inner product of v and w, that is,

$$\forall v, w\, (v, w \in V \Rightarrow \langle v, -w \rangle = -\langle v, w \rangle). \tag{10.122}$$

b) the inner product of the negative of a vector $v \in V$ and the negative of a vector $w \in V$ equals the inner product of v and w, that is,

$$\forall v, w\, (v, w \in V \Rightarrow \langle -v, -w \rangle = \langle v, w \rangle). \tag{10.123}$$

Proof. Letting $(V, +_V, \cdot_V, \langle ., . \rangle)$ be an arbitrary inner product space and v, w arbitrary vectors in V, we observe the truth of

$$\begin{aligned}\langle v, -w\rangle &= \langle -w, v\rangle \\ &= -\langle w, v\rangle \\ &= -\langle v, w\rangle\end{aligned}$$

in light of Property 3 of an inner product space and (10.121), and also the truth of

$$\begin{aligned}\langle -v, -w\rangle &= -\langle v, -w\rangle \\ &= -(-\langle v, w\rangle) \\ &= \langle v, w\rangle\end{aligned}$$

in light of (10.121), the previously established equation in (10.122), and the Sign Law (6.50). Since v, w and $(V, +_V, \cdot_V, \langle ., . \rangle)$ are arbitrary, we may infer from these findings the truth of the proposed universal sentences. □

Exercise 10.15. Show for any inner product space $(V, +_V, \cdot_V, \langle ., . \rangle)$ that the inner product involving the zero vector vanishes, that is,

$$\forall v\,(v \in V \Rightarrow [\langle v, 0_V\rangle = 0 \,\wedge\, \langle 0_V, v\rangle = 0]). \tag{10.124}$$

(Hint: Use (6.62) and (10.7).)

Exercise 10.16. Verify for any inner product space $(V, +_V, \cdot_V, \langle ., . \rangle)$ that

$$\forall c, v\,([c \in \mathbb{R} \,\wedge\, v, w \in V] \Rightarrow \langle c \cdot_V v, c \cdot_V w\rangle = c^2 \cdot \langle v, w\rangle). \tag{10.125}$$

Exercise 10.17. Verify for any inner product space $(V, +_V, \cdot_V, \langle ., . \rangle)$ that

$$\forall v, w\,(v, w \in V \Rightarrow \langle v \pm_V w, v \pm_V w\rangle = \langle v, v\rangle \pm 2 \cdot \langle v, w\rangle + \langle w, w\rangle). \tag{10.126}$$

Exercise 10.18. Verify for any inner product space $(V, +_V, \cdot_V, \langle ., . \rangle)$ that

$$\langle v -_V (c \cdot_V w), v -_V (c \cdot_V w)\rangle = \langle v, v\rangle - 2 \cdot c \cdot \langle v, w\rangle + c^2 \cdot \langle w, w\rangle \tag{10.127}$$

holds for any $c \in \mathbb{R}$ and any $v, w \in V$.
(Hint: Use (10.126), (10.10), (10.125), (6.65) and (6.63).)

Theorem 10.23 (Cauchy-Schwarz inequality). *For any inner product space $(V, +_V, \cdot_V, \langle ., . \rangle)$ the function $\|.\|$ in (10.133) satisfies*

$$\forall v, w\,(v, w \in V \Rightarrow \langle v, w\rangle^2 \leq \langle v, v\rangle \cdot \langle w, w\rangle). \tag{10.128}$$

Proof. We let $(V, +_V, \cdot_V, \langle .,. \rangle)$ be an arbitrary inner product space and v, w arbitrary vectors in V. We now consider the two cases $w = 0_V$ and $w \neq 0_V$. The first case $w = 0_V$ implies $\langle w, w \rangle = 0$ due to Property 4 of an inner product space. This equality evidently implies

$$\langle v, v \rangle \cdot \langle w, w \rangle = \langle v, v \rangle \cdot 0 = 0. \tag{10.129}$$

Furthermore, we obtain

$$\langle v, w \rangle^2 = \langle v, 0_V \rangle^2 = 0^2 = 0 \tag{10.130}$$

by applying substitution based on the current case assumption and (10.124). Since $0 \leq 0$ is true by the reflexivity of the total ordering $\leq_{\mathbb{R}}$, substitutions based on (10.129) and (10.130) give us the desired inequality in (10.128) in the first case.

The second case $w \neq 0_V$ clearly implies $\langle w, w \rangle \neq 0$ with Property 4 of an inner product, so that we may carry out the division $\langle w, v \rangle / \langle w, w \rangle$. Next, we observe in light of (10.127) the truth of

$$\begin{aligned}
&\left\langle v -_V \frac{\langle v, w \rangle}{\langle w, w \rangle} \cdot_V w, v -_V \frac{\langle v, w \rangle}{\langle w, w \rangle} \cdot_v w \right\rangle \\
&= \langle v, v \rangle - 2 \cdot \frac{\langle v, w \rangle}{\langle w, w \rangle} \cdot \langle v, w \rangle + \frac{\langle v, w \rangle}{\langle w, w \rangle} \cdot \frac{\langle v, w \rangle}{\langle w, w \rangle} \cdot \langle w, w \rangle \\
&= \langle v, v \rangle - 2 \cdot \frac{\langle v, w \rangle^2}{\langle w, w \rangle} + \frac{\langle v, w \rangle^2}{\langle w, w \rangle} \\
&= \langle v, v \rangle - \frac{\langle v, w \rangle^2}{\langle w, w \rangle}.
\end{aligned}$$

Since the inner product on the left-hand side of these equations is greater than or equal to zero by its positive definiteness, we find via substitution

$$0 \leq \langle v, v \rangle - \frac{\langle v, w \rangle^2}{\langle w, w \rangle}.$$

Applying now first the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$, and subsequently the Monotony Law for $\cdot_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ (noting that the evident fact $\langle w, w \rangle \geq 0$ implies with $\langle w, w \rangle \neq 0$ that $\langle w, w \rangle > 0$ holds), we find

$$\frac{\langle v, w \rangle^2}{\langle w, w \rangle} \cdot \langle w, w \rangle \leq \langle v, v \rangle \cdot \langle w, w \rangle$$

and therefore the desired inequality (10.128) also in the second case.

Since v, w and $(V, +_V, \cdot_V, \langle .,. \rangle)$ were all arbitrary, we therefore conclude that the theorem holds. □

Definition 10.25 (Orthogonal vectors). We say for any inner product space $(V, +_V, \cdot_V, \langle .,. \rangle)$ that two vectors v and w in V are *orthogonal* (alternatively, that v is *orthogonal to* w), symbolically

$$v \perp w, \tag{10.131}$$

iff the inner product of v and w vanishes, that is, iff

$$\langle v, w \rangle = 0. \tag{10.132}$$

Exercise 10.19. Define for any inner product space $(V, +_V, \cdot_V, \langle .,. \rangle)$ the function

$$\|.\| : V \to \mathbb{R}_+^0, \quad v \mapsto \|v\| = \sqrt{\langle v, v \rangle}. \tag{10.133}$$

(Hint: Apply Function definition by replacement.)

Proposition 10.24. *It is true for any inner product space* $(V, +_V, \cdot_V, \langle .,. \rangle)$ *that the function* $\|.\|$ *in (10.133) satisfies*

$$\forall v, w\, (v, w \in V \Rightarrow |\langle v, w \rangle| \leq \|v\| \cdot \|w\|). \tag{10.134}$$

Proof. Letting $(V, +_V, \cdot_V, \langle .,. \rangle)$ be an arbitrary inner product space and v, w arbitrary vectors in V, we observe first the truth of

$$|\langle v, w \rangle|^2 = \langle v, w \rangle^2$$

in light of Proposition 8.58. Secondly, we have

$$\begin{aligned} \|v\|^2 &= \sqrt{\langle v, v \rangle}^2 &= \langle v, v \rangle \\ \|w\|^2 &= \sqrt{\langle w, w \rangle}^2 &= \langle w, w \rangle \end{aligned}$$

by definition of the function $\|.\|$ in (10.133) and due to (8.425). Consequently, substitutions of the previous three equations into the Cauchy-Schwarz inequality yield

$$|\langle v, w \rangle|^2 \leq \|v\|^2 \cdot \|w\|^2 \quad [= (\|v\| \cdot \|w\|)^2].$$

Here, the real numbers $|\langle v, w \rangle|$ and $\|v\| \cdot \|w\|$ are evidently nonnegative. Now, since $\mathbb{R}_+^0$ defines an ordered elementary domain, we may apply the Monotony Law for the base and $\leq$ in order to simplify the preceding inequality to the one given in (10.134). As v and w were arbitrary vectors and $(V, +_V, \cdot_V, \langle .,. \rangle)$ an arbitrary inner product space, we may infer from this finding the truth of the proposition. $\square$

Exercise 10.20. Prove for any inner product space $(V, +_V, \cdot_V, \langle .,. \rangle)$ that

$$\forall a, v\,([a \in \mathbb{R} \wedge v \in V] \Rightarrow \|a \cdot_V v\|^2 = a^2 \cdot \|v\|^2). \tag{10.135}$$

(Hint: Use (10.133), (10.125), and (8.425).)

Exercise 10.21. Verify for any inner product space $(V, +_V, \cdot_V, \langle .,. \rangle)$ that

$$\forall v, w\,(v, w \in V \Rightarrow \|v \pm_V w\|^2 = \|v\|^2 \pm 2 \cdot \langle v, w \rangle + \|w\|^2). \tag{10.136}$$

(Hint: Recall (10.126).)

Theorem 10.25 (Definition of a norm via an inner product). *For any inner product space $(V, +_V, \cdot_V, \langle .,. \rangle)$, it is true that the function $\|.\|$ in (10.133) is a norm on $(V, +_V, \cdot_V)$.*

Proof. We let $(V, +_V, \cdot_V, \langle .,. \rangle)$ be an arbitrary inner product space.

Regarding Property 1 of a norm, we take an arbitrary $v \in V$ and observe that (10.133) yields $\|v\| \in \mathbb{R}_+^0$ (since $\mathbb{R}_+^0$ is a codomain) and thus $\|v\| \geq_{\mathbb{R}} 0$ (by definition of the set of nonnegative real numbers). As v is arbitrary, we therefore conclude that Property 1 of a norm is satisfied by (10.133).

Concerning Property 2, we let again $v \in V$ be arbitrary. We obtain then the true equivalences

$$\begin{aligned} \|v\| = 0 &\Leftrightarrow \sqrt{\langle v, v \rangle} = 0 \\ &\Leftrightarrow \langle v, v \rangle = 0 \\ &\Leftrightarrow v = 0_V \end{aligned}$$

with (10.133), (8.426) and Property 4 of an inner product. The resulting equivalence of $\|v\| = 0$ and $v = 0_V$ is then true for any $v \in V$, so that Property 2 of a norm is also satisfied by (10.133).

To establish the scalability of $\|.\|$, we now let $a \in \mathbb{R}$ and $v \in V$ be arbitrary. We now get the true equations

$$\begin{aligned} \|a \cdot_V v\|^2 &= a^2 \cdot \|v\|^2 \\ &= (a \cdot \|v\|)^2 \end{aligned} \tag{10.137}$$

using (10.135) and (5.484). Now, we obtain also

$$\begin{aligned} \|a \cdot_V v\| &= |\,\|a \cdot_V v\|\,| = \sqrt{\|a \cdot_V v\|^2} = \sqrt{(a \cdot \|v\|)^2} = |a \cdot \|v\|\,| = |a| \cdot |\,\|v\|\,| \\ &= |a| \cdot \|v\| \end{aligned}$$

using (8.409) in connection with the fact that the value $\|a \cdot_V v\|$ is an element of the codomain $\mathbb{R}_+^0$ of the function $\|.\|$ and thus greater than or equal to 0, (8.429), substitution based on (10.137), again (8.429), the Multiplicativity of the absolute value function on $\mathbb{R}$, and finally again (8.409) in connection with the fact that now the value $\|v\|$ is greater than or equal to 0. Because a and v are arbitrary, we therefore conclude that Property 3 of a norm is satisfied by (10.133), too.

Finally, we verify the triangle inequality, letting $v, w \in V$ be arbitrary. Let us observe first that $\langle v, w\rangle \leq |\langle v, w\rangle|$ holds by virtue of (8.410). Then, the Monotony Law for $\cdot_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ yields $2\cdot\langle v, w\rangle \leq 2\cdot|\langle v, w\rangle|$. This inequality in turn implies with the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$

$$2 \cdot \langle v, w\rangle + (\|v\|^2 + \|w\|^2) \leq 2 \cdot |\langle v, w\rangle| + (\|v\|^2 + \|w\|^2),$$

so that substitution based on (10.136) yields evidently

$$\|v +_V w\|^2 \leq \|v\|^2 + 2 \cdot |\langle v, w\rangle| + \|w\|^2 . \tag{10.138}$$

Using the previous two monotony laws, we also obtain from the inequality in (10.134)

$$2 \cdot |\langle v, w\rangle| + (\|v\|^2 + \|w\|^2) \leq 2 \cdot \|v\| \cdot \|w\| + (\|v\|^2 + \|w\|^2)$$

and therefore

$$\|v\|^2 + 2 \cdot |\langle v, w\rangle| + \|w\|^2 \leq \|v\|^2 + 2 \cdot \|v\| \cdot \|w\| + \|w\|^2 .$$

The conjunction of this inequality with the inequality (10.138) gives us then with the transitivity of the total ordering $\leq_{\mathbb{R}}$ the desired triangle inequality. As v and w are arbitrary, we therefore conclude that Property 4 of a norm is satisfied by (10.133) as well.

Consequently, since $(V, +_V, \cdot_V, \langle ., .\rangle)$ was initially arbitrary, we conclude that the theorem holds. □

Definition 10.26 (Induced norm). We call for any inner product space $(V, +_V, \cdot_V, \langle ., .\rangle)$ the function $\|.\|$ in (10.133) the *norm induced by the inner product* $\langle ., .\rangle$.

Corollary 10.26 (Pythagorean theorem). *For any inner product space* $(V, +_V, \cdot_V, \langle ., .\rangle)$*, it is true that the squared norm of the sum of any two orthogonal vectors* $v, w \in V$ *equals the sum of their squared norms, that is,*

$$\forall v, w\,([v, w \in V \,\wedge\, v\perp w] \Rightarrow \|v +_V w\|^2 = \|v\|^2 + \|w\|^2). \tag{10.139}$$

Proof. Letting $(V, +_V, \cdot_V, \langle ., .\rangle)$ be an arbitrary inner product space and v, w arbitrary vectors in V such that $v\perp w$ holds, we have by definition of orthogonal vectors $\langle v, w\rangle = 0$, so that (10.136) gives us

$$\|v +_V w\|^2 = \|v\|^2 + 2 \cdot 0 + \|w\|^2 = \|v\|^2 + \|w\|^2 .$$

As v and w were arbitrary vectors and $(V, +_V, \cdot_V, \langle ., . \rangle)$ an arbitrary inner product space, we may infer from this finding the truth of the corollary. $\square$

Corollary 10.27 (Parallelogram law). *It is true for any inner product space $(V, +_V, \cdot_V, \langle ., . \rangle)$ and any vectors $v, w \in V$ that*

$$\|v +_V w\|^2 + \|v -_V w\|^2 = 2 \cdot (\|v\|^2 + \|w\|^2). \tag{10.140}$$

Proof. Letting $(V, +_V, \cdot_V, \langle ., . \rangle)$ be an arbitrary inner product space and v, w arbitrary vectors in V, (10.136) yields evidently

$$\begin{aligned} \|v +_V w\|^2 + \|v -_V w\|^2 &= \|v\|^2 + 2 \cdot \langle v, w \rangle + \|w\|^2 \\ &\quad + \|v\|^2 - 2 \cdot \langle v, w \rangle + \|w\|^2 \\ &= 2 \cdot \|v\|^2 + 2 \cdot \|w\|^2 \\ &= 2 \cdot (\|v\|^2 + \|w\|^2), \end{aligned}$$

as desired. Because v, w and $(V, +_V, \cdot_V, \langle ., . \rangle)$ are arbitrary, the corollary follows to be true. $\square$

Definition 10.27 (Euclidean norm, L^2 norm, Euclidean metric, Euclidean distance). We call the norm induced by the scalar product, symbolically

$$\|.\|_2 : \mathbb{R}^n \to \mathbb{R}, \quad v \mapsto \|v\|_2 = \sqrt{v \bullet v} = \sqrt{\sum_{i=1}^n v_i^2}, \tag{10.141}$$

the *Euclidean norm* or the *L^2 norm.* Furthermore, we call the metric induced by the Euclidean norm, symbolically

$$\begin{aligned} d_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (v, w) \mapsto d_{\mathbb{R}^n}(v, w) &= \|v -_{\mathbb{R}^n} w\|_2 \\ &= \sqrt{\sum_{i=1}^n (v -_{\mathbb{R}^n} w)_i^2} \\ &= \sqrt{\sum_{i=1}^n (v_i - w_i)^2}, \end{aligned} \tag{10.142}$$

the *Euclidean metric* or the *Euclidean distance.*

Definition 10.28 (Hilbert space). We call

$$(V, +_V, \cdot_V, \langle ., . \rangle, \|.\|) \tag{10.143}$$

a *Hilbert space* iff

1. $(V, +_V, \cdot_V, \langle ., . \rangle)$ is a pre-Hilbert space,
2. $\|.\|$ is the norm induced by $\langle ., . \rangle$ and
3. $(V, +_V, \cdot_V, \|.\|)$ is complete.

10.3. Linear Equation Systems

In what follows, we recall that the standard matrix-vector product may be associated with any m-by-n matrix $\boldsymbol{A}$ and any n-vector $\boldsymbol{x}$, and that it constitutes an m-vector $\boldsymbol{y}$.

Definition 10.29 (Solution of a linear equation system). For any real m-by-n matrix $\boldsymbol{A}$, any real m-vector $\boldsymbol{b}$ and any real n-vector $\boldsymbol{x}$, we say that $\boldsymbol{x}$ is a *solution of the linear equation system*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{10.144}$$

iff this equation is true.

Notation 10.6. We abbreviate 'linear equation system' by 'LES'. We may write an LES component-wise as

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \tag{10.145}$$

or, in view of the definition of the standard matrix-vector product, as

$$\sum_{k=1}^{n} a_{1k} x_k = b_1 \tag{10.146}$$

$$\vdots$$

$$\sum_{k=1}^{n} a_{mk} x_k = b_m. \tag{10.147}$$

A more suggestive notation for the n-fold sums is given by

$$a_{1,1}x_1 + \cdots + a_{1,n}x_n = b_1 \tag{10.148}$$

$$\vdots$$

$$a_{m,1}x_1 + \cdots + a_{m,n}x_n = b_m. \tag{10.149}$$

Note 10.12. For any real m-by-n matrix $\boldsymbol{A}$ and any real m-vector $\boldsymbol{b}$ we may evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of a set $S(\boldsymbol{A})$ consisting of all the real n-vectors that are solutions of the linear equation system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, in the sense that

$$\forall \boldsymbol{x}\,(\boldsymbol{x} \in S(\boldsymbol{A}) \Leftrightarrow [\boldsymbol{x} \in \mathbb{R}^n \wedge \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}]). \tag{10.150}$$

Definition 10.30 (Solution set of an LES, particular solution of an LES, consistent/solvable & inconsistent/unsolvable LES, uniquely solvable LES). For any real m-by-n matrix $\boldsymbol{A}$ and any real m-vector $\boldsymbol{b}$ we call

(1) the set
$$S(\boldsymbol{A}) \tag{10.151}$$
the *solution set of the linear equation system* $\boldsymbol{Ax} = \boldsymbol{b}$ (alternatively, the *solution set of the linear equation system formed by* $\boldsymbol{A}$ *and* $\boldsymbol{b}$).

(2) every element of $S(\boldsymbol{A})$ a *particular solution* of $\boldsymbol{Ax} = \boldsymbol{b}$.

(3) the linear equation system $\boldsymbol{Ax} = \boldsymbol{b}$ *consistent* or *solvable* iff the solution set for $\boldsymbol{Ax} = \boldsymbol{b}$ has some element, i.e., iff
$$S(\boldsymbol{A}) \neq \emptyset. \tag{10.152}$$
Conversely, we call the linear equation system $\boldsymbol{Ax} = \boldsymbol{b}$ *inconsistent* or *unsolvable* iff $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution, i.e., iff
$$S(\boldsymbol{A}) = \emptyset. \tag{10.153}$$

(4) the linear equation system $\boldsymbol{Ax} = \boldsymbol{b}$ *uniquely solvable* iff it has a unique solution, that is, iff
$$\exists! \boldsymbol{x}\,(\boldsymbol{Ax} = \boldsymbol{b}). \tag{10.154}$$

We consider now the special case that the 'right-hand side' $\boldsymbol{b}$ of a linear equation system is given by the m-dimensional zero vector.

Definition 10.31 (Solution of a homogeneous LES, trivial solution of an LES). For any real m-by-n matrix $\boldsymbol{A}$ and any real n-vector $\boldsymbol{x}$, we say that $\boldsymbol{x}$ is a *solution of the homogeneous linear equation system*
$$\boldsymbol{Ax} = \mathbf{0}_m \tag{10.155}$$
iff this equation is true. Furthermore, we call $\mathbf{0}_n$ the *trivial solution* of $\boldsymbol{Ax} = \mathbf{0}_m$.

Note 10.13. For any real m-by-n-matrix $\boldsymbol{A}$, the set of all solutions of the homogeneous linear equation system $\boldsymbol{Ax} = \mathbf{0}_m$ is given by the nullspace $N(\boldsymbol{A})$ of $\boldsymbol{A}$ and therefore forms a vector subspace of $(\mathbb{R}^n, +_{\mathbb{R}^n}, \cdot_{\mathbb{R}^n})$, as shown by (10.74). In view of (10.62), the trivial solution $\mathbf{0}_n$ exists indeed.

Theorem 10.28 (Characterization of the solution set of a consistent LES). *For any real m-by-n-matrix $\boldsymbol{A}$ and any real m-vector $\boldsymbol{b}$ such that the linear equation system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent, and for any element $\bar{\boldsymbol{x}} \in S(\boldsymbol{A})$ of the corresponding solution set, it is true that the solution set $S(\boldsymbol{A})$ is characterized by*

$$\forall \boldsymbol{x}\,(\boldsymbol{x} \in S(\boldsymbol{A}) \Leftrightarrow \exists \boldsymbol{x}_0\,(\boldsymbol{x}_0 \in N(\boldsymbol{A}) \,\wedge\, \boldsymbol{x} = \bar{\boldsymbol{x}} +_{\mathbb{R}^n} \boldsymbol{x}_0)). \tag{10.156}$$

Proof. We take an arbitrary real m-by-n-matrix $\boldsymbol{A}$ and an arbitrary m-vector $\boldsymbol{b}$ such that the linear equation system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent. Thus, the corresponding solution set $S(\boldsymbol{A})$ is nonempty by definition. We now let $\bar{\boldsymbol{x}}$ be arbitrary, assuming $\bar{\boldsymbol{x}} \in S(\boldsymbol{A})$ to be true, which implies $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ and

$$\boldsymbol{A}\bar{\boldsymbol{x}} = \boldsymbol{b} \tag{10.157}$$

by virtue of (10.150). Next, we let $\boldsymbol{x}$ be arbitrary, assuming first $\boldsymbol{x} \in S(\boldsymbol{A})$ to be true. This assumption evidently implies $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{Ax} = \boldsymbol{b}$. Let us now take the n-vector

$$\bar{\boldsymbol{x}}_0 = \boldsymbol{x} - \bar{\boldsymbol{x}}, \tag{10.158}$$

for which we obtain the true equations

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x} + \boldsymbol{0} \\ &= \boldsymbol{x} + (\bar{\boldsymbol{x}} + [-\bar{\boldsymbol{x}}]) \\ &= (\boldsymbol{x} + [-\bar{\boldsymbol{x}}]) + \bar{\boldsymbol{x}} \\ &= \bar{\boldsymbol{x}}_0 +_{\mathbb{R}^n} \bar{\boldsymbol{x}} \\ &= \bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}_0. \end{aligned} \tag{10.159}$$

by applying the property of a zero element, the property of a negative, the associativity & commutativity of the addition on Y^n, substitution based on (10.158), and finally again the commutativity of the addition on Y^n. We obtain now also the true equations

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{Ax} - \boldsymbol{Ax} \\ &= \boldsymbol{A}(\bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}_0) - \boldsymbol{b} \\ &= (\boldsymbol{A}\bar{\boldsymbol{x}} + \boldsymbol{A}\bar{\boldsymbol{x}}_0) - \boldsymbol{b} \\ &= (\boldsymbol{b} + \boldsymbol{A}\bar{\boldsymbol{x}}_0) - \boldsymbol{b} \\ &= \boldsymbol{A}\bar{\boldsymbol{x}}_0 \end{aligned}$$

by applying the property of a negative, substitution based on the true equations (10.159) and $\boldsymbol{Ax} = \boldsymbol{b}$, the additivity property (10.59), substitution based on the true equation (10.157), and the commutativity & associativity of the addition on Y^n in conjunction with the property of a negative and with the property of a zero element. The truth of the resulting equation $\boldsymbol{A}\bar{\boldsymbol{x}}_0 = \mathbf{0}$ implies then that $\bar{\boldsymbol{x}}_0$ is an element of the nullspace of $\boldsymbol{A}$, that is, $\bar{\boldsymbol{x}}_0 \in N(\boldsymbol{A})$. In conjunction with (10.159), this demonstrates the truth of the existential sentence in (10.156), so that the first part ('⇒') of the equivalence in (10.156) holds.

To establish the second part ('⇐'), we conversely assume that there exists an element of $N(\boldsymbol{A})$, say $\bar{\boldsymbol{x}}_0'$, such that $\boldsymbol{x} = \bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}_0'$. Consequently, $\bar{\boldsymbol{x}}_0'$ is a solution of the homogeneous linear equation system $\boldsymbol{A}\bar{\boldsymbol{x}}_0' = \mathbf{0}$. We obtain then the true equations

$$\begin{aligned}\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{A}(\bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}_0') \\ &= \boldsymbol{A}\bar{\boldsymbol{x}} + \boldsymbol{A}\bar{\boldsymbol{x}}_0' \\ &= \boldsymbol{b} + \mathbf{0} \\ &= \boldsymbol{b}\end{aligned}$$

by applying substitution based on the true equation $\boldsymbol{x} = \bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}_0'$, the additivity property (10.59), two further substitutions based on (10.157) and $\boldsymbol{A}\bar{\boldsymbol{x}}_0' = \mathbf{0}$, and finally the property of a zero element. Thus, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is true, so that $\boldsymbol{x}$ is a solution of that linear equation system. Consequently, $\boldsymbol{x}$ is an element of the solution set $S(\boldsymbol{A})$, by definition. We thus completed the proof of the equivalence in (10.156), in which $\boldsymbol{x}$ is arbitrary, so that the universal sentence (10.156) follows now to be true. As $\boldsymbol{A}$, $\boldsymbol{b}$ and $\bar{\boldsymbol{x}}$ were initially also arbitrary, we may then further conclude that the stated theorem holds. □

Theorem 10.29 (Characterization of uniquely solvable consistent LES). *For any real m-by-n-matrix $\boldsymbol{A}$ and for any real m-vector $\boldsymbol{b}$ such that the linear equation system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent, it is true that the linear equation system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is uniquely solvable iff the nullspace of $\boldsymbol{A}$ is the singleton formed by the trivial solution $\mathbf{0}_n$ of $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}_m$, i.e.*

$$\exists! \boldsymbol{x}\,(\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}) \;\Leftrightarrow\; N(\boldsymbol{A}) = \{\mathbf{0}_n\}. \tag{10.160}$$

Proof. We let $\boldsymbol{A}$ be an arbitrary real m-by-n-matrix and $\boldsymbol{b}$ an arbitrary real m-vector such that the linear equation system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent. This means that the solution set $S(\boldsymbol{A})$ is nonempty, so that $S(\boldsymbol{A})$ has some element, say $\bar{\boldsymbol{x}}$. Consequently, the equation

$$\boldsymbol{A}\bar{\boldsymbol{x}} = \boldsymbol{b} \tag{10.161}$$

is true. To prove the first part ('⇒') of the equivalence (10.160), we assume the uniquely existential sentence to be true. To establish the desired consequent $N(\boldsymbol{A}) = \{\mathbf{0}_n\}$, we apply the Equality Criterion for sets, by proving accordingly

$$\forall \boldsymbol{x}\,(\boldsymbol{x} \in N(\boldsymbol{A}) \Leftrightarrow \boldsymbol{x} \in \{\mathbf{0}_n\}). \tag{10.162}$$

Letting $\boldsymbol{x}_0$ be arbitrary, we assume first $\boldsymbol{x}_0 \in N(\boldsymbol{A})$ to be true. Let us define now the n-vector

$$\bar{\boldsymbol{x}}' = \bar{\boldsymbol{x}} + \boldsymbol{x}_0. \tag{10.163}$$

According to the Characterization of the solution set of a linear equation system, this equation implies, in conjunction with $\boldsymbol{x}_0 \in N(\boldsymbol{A})$ and due to $\bar{\boldsymbol{x}} \in S(\boldsymbol{A})$, that $\bar{\boldsymbol{x}}' \in S(\boldsymbol{A})$ holds. Consequently, the equation $\boldsymbol{A}\bar{\boldsymbol{x}}' = \boldsymbol{b}$ is true, besides the previously established (10.161). The uniqueness part of the assumed uniquely existential sentence therefore yields $\bar{\boldsymbol{x}}' = \bar{\boldsymbol{x}}$. Applying now a substitution to (10.163) gives us $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}} + \boldsymbol{x}_0$, and subsequently

$$\boldsymbol{x}_0 = -\bar{\boldsymbol{x}} + (\bar{\boldsymbol{x}} + \boldsymbol{x}_0) = -\bar{\boldsymbol{x}} + \bar{\boldsymbol{x}} = \mathbf{0}_n$$

by using the properties of a zero element and of a negative, as well as the associativity of the addition on $\mathbb{R}^n$. The resulting equation $\boldsymbol{x}_0 = \mathbf{0}_n$ further implies $\boldsymbol{x}_0 \in \{\mathbf{0}_n\}$ due to (2.169), proving the first part of the equivalence in (10.162). Assuming conversely $\boldsymbol{x}_0 \in \{\mathbf{0}_n\}$ to hold, so that $\boldsymbol{x}_0 = \mathbf{0}_n$ is evidently true as well, we recall from Note 10.13 that $\mathbf{0}_n \in N(\boldsymbol{A})$ is true, so that substitution gives us already the desired $\boldsymbol{x}_0 \in N(\boldsymbol{A})$. Having proved the equivalence in (10.162), we may infer from this the truth of the equality $N(\boldsymbol{A}) = \{\mathbf{0}_n\}$ since $\boldsymbol{x}_0$ was arbitrary. We thus established the first part of the equivalence (10.160).

To establish the converse direction, we now assume $N(\boldsymbol{A}) = \{\mathbf{0}_n\}$ to be true. Then, since (10.161) holds, we see that the existential part of the uniquely existential sentence $\exists!\boldsymbol{x}\,(\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b})$ holds already. To prove the uniqueness part, we let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be arbitrary, assuming $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{A}\boldsymbol{x}' = \boldsymbol{b}$ to be true. These equations clearly shows that $\boldsymbol{x}, \boldsymbol{x}' \in S(\boldsymbol{A})$, so that there exists a particular element $\bar{\boldsymbol{x}}_0 \in N(\boldsymbol{A})$ satisfying $\boldsymbol{x} = \boldsymbol{x}' + \bar{\boldsymbol{x}}_0$ (according to the Characterization of the solution of a consistent linear equation system). Consequently, substitution based on the currently assumed antecedent yields $\bar{\boldsymbol{x}}_0 \in \{\mathbf{0}_n\}$, with the evident consequence that $\bar{\boldsymbol{x}}_0 = \mathbf{0}_n$. Combining the previous two equations evidently gives us

$$\boldsymbol{x} = \boldsymbol{x}' + \bar{\boldsymbol{x}}_0 = \boldsymbol{x}' + \mathbf{0}_n = \boldsymbol{x}',$$

so that the uniqueness part of the desired uniquely existential sentence turns out to be true as well (since $\boldsymbol{x}$ and $\boldsymbol{x}'$ were arbitrary). We thus completed the proof of the equivalence (10.160), and as $\boldsymbol{A}$ and $\boldsymbol{b}$ were initially arbitrary, we may infer from this the truth of the stated theorem. □

Part IV.

Stochastic Observation Data

Chapter 11.

Systems of Sets and Events

In this section, we introduce various types of set systems, leading us to the fundamental notion of an *event*.

11.1. π-Systems

We begin with a set system which is closed under pairwise intersections.

Definition 11.1 (π-system). For any set Ω we say that a set $\mathcal{K}$ is a π-*system* on Ω iff

1. $\mathcal{K}$ consists of subsets of Ω, that is,

$$\mathcal{K} \subseteq \mathcal{P}(\Omega), \tag{11.1}$$

2. $\mathcal{K}$ is nonempty, that is,

$$\mathcal{K} \neq \emptyset, \tag{11.2}$$

   and

3. $\mathcal{K}$ contains the intersection of any two sets in $\mathcal{K}$, that is,

$$\forall A, B\, (A, B \in \mathcal{K} \Rightarrow A \cap B \in \mathcal{K}), \tag{11.3}$$

Note 11.1. We may use the reflexive partial ordering of inclusion ($\subseteq$) to form the partially ordered set $(\mathcal{K}, \subseteq)$ for any π-system $\mathcal{K}$ on any set Ω.

Evidently, Property 3 allows us to establish a binary intersection operation on a π-system. The commutativity and the associativity of the intersection of two sets characterize then also the corresponding binary operation.

Exercise 11.1. Prove the following sentences for any set Ω and any π-system $\mathcal{K}$ on Ω.

a) There exists the unique binary operation

$$\cap_{\mathcal{K}} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}, \quad (A, B) \mapsto A \cap B. \tag{11.4}$$

(Hint: Proceed in analogy to the proof of Proposition 5.2.)

b) The binary operation $\cap_{\mathcal{K}}$ is idempotent.
(Hint: Apply Theorem 2.17.)

c) The ordered pair $(\mathcal{K}, \cap_{\mathcal{K}})$ is a commutative semigroup.
(Hint: Use Theorem 2.16 and Theorem 2.22.)

Theorem 11.1 (π-system of left-closed and right-open intervals). *It is true for any linearly ordered set $(\Omega, <)$ with $\Omega \neq \emptyset$ that the set $\mathcal{I} = \{[a, b) : a, b \in \Omega\}$ of left-closed and right-open intervals in Ω is a π-system on Ω.*

Proof. We let Ω and $<$ be arbitrary sets, assume $\Omega \neq \emptyset$, and assume that $<$ is a linear ordering of Ω. Property 1 of a π-system is satisfied by $\mathcal{I}$ because that set system is a specified subset of $\mathcal{P}(\Omega)$ according to Exercise 3.53d). Furthermore, the initial assumption $\Omega \neq \emptyset$ implies $\emptyset \in \mathcal{I}$ according to Corollary 3.121, so that $\mathcal{I} \neq \emptyset$ is clearly true; thus, $\mathcal{I}$ satisfies also Property 2 of a π-system. Regarding Property 3, we let $A, B \in \mathcal{I}$ be arbitrary and show that $A \cap B \in \mathcal{I}$ follows to be true. Since the set of left-closed and right-open intervals was specified to satisfy (3.387), it follows from $A \in \mathcal{I}$ that there are particular elements a_1, b_1 in Ω such that $(a_1, b_1] = A$. For the same reason, $B \in \mathcal{I}$ implies the existence of particular elements a_2, b_2 in Ω with $(a_2, b_2] = B$. Thus, our task is to establish $[a_1, b_1) \cap [a_2, b_2) \in \mathcal{I}$.

Let us observe now that the disjunction $\neg a_1 < b_1 \lor a_1 < b_1$ is true according to the Law of the Excluded Middle, so that we may prove the desired consequent by cases. The first case $\neg a_1 < b_1$ implies $(a_1, b_1] = \emptyset$ with (3.384) and therefore

$$[a_1, b_1) \cap [a_2, b_2) = \emptyset \cap [a_2, b_2) = \emptyset \quad [\in \mathcal{I}]$$

by applying substitution and (2.62) recalling the previous finding $\emptyset \in \mathcal{I}$. Thus, we find the desired $[a_1, b_1) \cap [a_2, b_2) \in \mathcal{I}$ in the first case.

We now assume in the second case that $a_1 < b_1$ is true. We notice that the disjunction $\neg a_2 < b_2 \lor a_2 < b_2$ is also true (in view of the Law of the Excluded Middle), allowing us to consider two further sub-cases. In analogy to the first case, the first sub-case $\neg a_2 < b_2$ implies $[a_2, b_2) = \emptyset$ and therefore

$$[a_1, b_1) \cap [a_2, b_2) = [a_1, b_1) \cap \emptyset = \emptyset \quad [\in \mathcal{I}],$$

as desired. In the second sub-case, we then assume $a_2 < b_2$ (alongside the current case assumption $a_1 < b_1$), that is,

$$a_2 < b_2 \wedge a_1 < b_1.$$

Evidently, the disjunction $\neg a_2 < b_1 \vee a_2 < b_1$ holds as well, on which basis we consider the following two sub-sub-cases. On the one hand, if $\neg a_2 < b_1$ is true, then we obtain

$$[a_1, b_1) \cap [a_2, b_2) = \emptyset \quad [\in \mathcal{I}] \tag{11.5}$$

due to (3.412). On the other hand, if $a_2 < b_1$ holds (alongside $a_2 < b_2$ and $a_1 < b_1$), that is,

$$a_2 < b_1 \wedge a_1 < b_1 \wedge a_2 < b_2$$

we may evidently consider the two further cases $\neg a_1 < b_2$ and $a_1 < b_2$. In case of $\neg a_1 < b_2$, we obtain again (11.5) because of (3.412). In the other case of $a_1 < b_2$ (holding besides $a_2 < b_1$, $a_1 < b_1$, and $a_2 < b_2$), that is,

$$a_1 < b_2 \wedge a_2 < b_1 \wedge a_1 < b_1 \wedge a_2 < b_2,$$

this means that the intervals are now neither empty nor disjoint. Let us observe that the linear ordering $<$ induces the total ordering $\leq$, so that the disjunction $a_1 \leq a_2 \vee a_2 \leq a_1$ is true. Based on this disjunction, we consider the next two possible sub-cases $a_1 \leq a_2$ versus $a_2 \leq a_1$. We first assume $a_1 \leq a_2$ (alongside $a_1 < b_2$, $a_2 < b_1$, $a_1 < b_1$, and $a_2 < b_2$), and we consider the final two sub-cases $b_1 \leq b_2$ versus $b_2 \leq b_1$, based on the fact that disjunction $b_1 \leq b_2 \vee b_2 \leq b_1$ is true because of the totality of the induced reflexive partial ordering $\leq$. Assuming first

$$b_1 \leq b_2 \quad (\wedge\, a_1 \leq a_2 \wedge a_1 < b_2 \wedge a_2 < b_1 \wedge a_1 < b_1 \wedge a_2 < b_2),$$

we apply (3.411) and the Commutative Law for the intersection of two sets to obtain the equations

$$[a_1, b_1) \cap [a_2, b_2) = [a_1, b_2) \cap [a_2, b_1) = [a_2, b_1) \cap [a_1, b_2).$$

Due to (3.410), the conjunction $a_1 \leq a_2 \wedge b_1 \leq b_2$ implies the inclusion $[a_2, b_1) \subseteq [a_1, b_2)$, with the consequence that

$$[a_2, b_1) \cap [a_1, b_2) = [a_2, b_1) \quad [\in \mathcal{I}]$$

holds according to (2.77), where we notice that $[a_2, b_1)$ is a left-closed and right-open interval. Thus, the previous equations give via substitution $[a_1, b_1) \cap [a_2, b_2) \in \mathcal{I}$, as desired. Considering the other case

$$b_2 \leq b_1 \quad (\wedge\, a_1 \leq a_2 \wedge a_1 < b_2 \wedge a_2 < b_1 \wedge a_1 < b_1 \wedge a_2 < b_2),$$

we have that the conjunction of $a_1 \leq a_2$ and $b_2 \leq b_1$ implies the inclusion $[a_2, b_2) \subseteq [a_1, b_1)$ with (3.410), so that

$$[a_1, b_1) \cap [a_2, b_2) = [a_2, b_2) \cap [a_1, b_1) = [a_2, b_2) \quad [\in \mathcal{I}]$$

follows with the Commutative Law for the intersection of two sets and (2.77).

We now switch to the sub-case $a_2 \leq a_1$ and consider again first $b_1 \leq b_2$, that is, we assume

$$b_1 \leq b_2 \ \ (\land\, a_2 \leq a_1 \,\land\, a_1 < b_2 \,\land\, a_2 < b_1 \,\land\, a_1 < b_1 \,\land\, a_2 < b_2).$$

Here, the conjunction $a_2 \leq a_1 \land b_1 \leq b_2$ implies $[a_1, b_1) \subseteq [a_2, b_2)$ with (3.410), so that we obtain

$$[a_1, b_1) \cap [a_2, b_2) = [a_1, b_1) \quad [\in \mathcal{I}]$$

with (2.77). Now, in the final sub-case of $b_2 \leq b_1$, which means that we assume

$$b_2 \leq b_1 \ \ (\land\, a_2 \leq a_1 \,\land\, a_1 < b_2 \,\land\, a_2 < b_1 \,\land\, a_1 < b_1 \,\land\, a_2 < b_2),$$

we have that $a_2 \leq a_1 \land b_2 \leq b_1$ implies $[a_1, b_2) \subseteq [a_2, b_1)$ with (3.410), which yields

$$[a_1, b_1) \cap [a_2, b_2) = [a_1, b_2) \cap [a_2, b_1) = [a_1, b_2) \quad [\in \mathcal{I}]$$

with (3.411) and (2.77), since $[a_1, b_2)$ is a left-closed and right-open interval.

We thus exhausted all of the (sub-)cases in proving that the intersection of two left-closed and right-open intervals $[a_1, b_1)$ and $[a_2, b_2)$ produces another left-closed and right-open interval. In any of these cases, we obtained for this intersection $\emptyset$, $[a_2, b_1)$, $[a_2, b_2)$, $[a_1, b_1)$, or $[a_1, b_2)$, thus invariably an element of $\mathcal{I}$. Since A and B were initially arbitrary, we therefore conclude that $\mathcal{I}$ satisfies indeed Property 3 of a π-system on Ω. Because Ω and $<$ were also arbitrary, we may infer from the previous findings that $\mathcal{I}$ constitutes a π-system for any such sets. □

Theorem 11.2 (π-system of open intervals). *It is true for any linearly ordered set $(\Omega, <)$ with $\Omega \neq \emptyset$ that the set $\mathcal{O} = \{(a, b) : a, b \in \Omega\}$ of open intervals in Ω is a π-system on Ω.*

Exercise 11.2. Establish the π-system of open intervals in analogy to the π-system of left-closed and right-open intervals.

11.2. Semirings of Sets

In a next step, we introduce a specific π-system that allows us to partition differences of sets (in that system) into certain types of sequences of sets (also in that system).

Corollary 11.3. *It is true for any set A that the singleton $C = \{(1, A)\}$ is a sequence $(C_i \mid i \in \{1, \ldots, 1\})$ of pairwise disjoint sets in $\{A\}$, and the union of the range of that sequence equals A, i.e.*

$$\bigcup \operatorname{ran}(C) = A. \tag{11.6}$$

Proof. Letting A be an arbitrary set, we have that $C = \{(1, A)\}$ is a function from $\{1\}$ [$= \{1, \ldots, 1\}$] to $\{A\}$ because of Corollary 3.156 (using also the notation of the initial segment of $\mathbb{N}_+$ up to $n = 1$). Because C is a surjection from $\{1\}$ to $\{A\}$ according to Corollary 3.194, the equation $\operatorname{ran}(C) = \{A\}$ holds (by definition of a surjection). Recalling now the fact that the union of the singleton formed by any set equals that set (see Proposition 2.66), it follows that

$$[\bigcup \operatorname{ran}(C) =] \quad \bigcup \{A\} = A.$$

Let us now verify that the family/sequence $C : \{1\} \to \{A\}$ has pairwise disjoint terms, i.e.

$$\forall i, j\, ([i, j \in \{1\} \wedge i \neq j] \Rightarrow C_i \cap C_j = \emptyset).$$

Letting $i, j \in \{1\}$ be arbitrary, we obtain $i = 1$ and $j = 1$ with (2.169), so that substitution yields $i = j$. Consequently, $i \neq j$ is false, and therefore also the antecedent $i, j \in \{1\} \wedge i \neq j$ of the implication to be proven. Thus, the implication itself is true, and since i and j are arbitrary, we may now infer from this that the terms of C are pairwise disjoint. Since A was arbitrary, we may therefore conclude that the corollary holds indeed. □

Definition 11.2 (Semiring of sets). For any set Ω we say that a set $\mathcal{S}$ is a *semiring of sets* on Ω iff

1. $\mathcal{S}$ consists of subsets of Ω, i.e.

$$\mathcal{S} \subseteq \mathcal{P}(\Omega), \tag{11.7}$$

2. $\mathcal{S}$ contains $\emptyset$, i.e.

$$\emptyset \in \mathcal{S}, \tag{11.8}$$

3. $\mathcal{S}$ contains the intersection of any two sets in $\mathcal{S}$, i.e.

$$\forall A, B\,(A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}), \tag{11.9}$$

   and

4. the difference of any two sets in $\mathcal{S}$ is the union of some sequence $C = (C_i \,|\, i \in \{1, \ldots, n\})$ of pairwise disjoint sets in $\mathcal{S}$, i.e.

$$\begin{aligned}\forall A, B\,(A, B \in \mathcal{S} \Rightarrow \exists n, C\,(&n \in \mathbb{N}_+ \,\wedge\, C : \{1, \ldots, n\} \to \mathcal{S} \\ &\wedge\, \forall i, j\,([i, j \in \{1, \ldots, n\} \,\wedge\, i \neq j] \Rightarrow C_i \cap C_j = \emptyset) \\ &\wedge\, A \setminus B = \bigcup \operatorname{ran}(C))).\end{aligned} \tag{11.10}$$

Note 11.2. Property 2 reveals that semirings of sets are not empty, so that the Properties 1 – 3 indeed establish a semiring of sets as a π-system (on the same set Ω). Thus, any semiring $\mathcal{S}$ gives rise to the commutative semigroup $(\mathcal{S}, \cap_{\mathcal{S}})$ (which need not necessarily contain the neutral element with respect to the binary intersection operation $\cap_{\mathcal{S}}$). Despite this connection of a semiring of sets to the concept of a binary operation, the former has no immediate relationship to a semiring.

To familiarize ourselves with the properties of a semiring, let us inspect a first simple example.

Proposition 11.4. *The natural number* $1 = \{\emptyset\}$ *is a semiring of sets on any set* Ω.

Proof. We let Ω be an arbitrary set. Since $\emptyset \in \mathcal{P}(\Omega)$ holds according to (3.15), we obtain the inclusion $\{\emptyset\} \subseteq \mathcal{P}(\Omega)$ with (2.184), which shows that $\{\emptyset\}$ satisfies Property 1 of a semiring of sets.

Because $\emptyset \in \{\emptyset\}$ also holds in view of (2.153), Property 2 of a semiring of sets is also satisfied.

To verify Property 3, we now take arbitrary sets $A, B \in \{\emptyset\}$, so that $A = \emptyset$ and $B = \emptyset$ follow to be true with (2.169). Then, we obtain for the intersection of these sets $A \cap B = \emptyset \cap \emptyset = \emptyset$ with (2.62), so that the previously established $\emptyset \in \{\emptyset\}$ implies $A \cap B \in \{\emptyset\}$, as desired. As A and B were arbitrary, we may therefore conclude that $\{\emptyset\}$ satisfies Property 3 of a semiring of sets; we thus showed that $\{\emptyset\}$ is a π-system.

Regarding Property 4, we let again $A, B \in \{\emptyset\}$ be arbitrary, so that we have $A = \emptyset$ and $B = \emptyset$ as before, which equations imply $A \setminus B = \emptyset \setminus \emptyset = \emptyset$ with (2.104). Let us now observe in light of Corollary 11.3 that $\bar{C} = \{(1, \emptyset)\}$ is a sequence $(\bar{C}_i \,|\, i \in \{1, \ldots, 1\})$ of pairwise disjoint sets in $\{\emptyset\}$ with

$$\bigcup \operatorname{ran}(\bar{C}) = \emptyset \quad [= A \setminus B],$$

so that $A \setminus B = \bigcup \mathrm{ran}(\bar{C})$ holds. Having found the positive natural number $\bar{n} = 1$ and the sequence $\bar{C} = (\bar{C}_i \,|\, i \in \{1, \ldots, \bar{n}\})$ of pairwise disjoint sets in $\{\emptyset\}$ satisfying $A \setminus B = \bigcup \mathrm{ran}(\bar{C})$, we thus see clearly that the existential sentence in (11.10) is satisfied by the sets A, B and $\mathcal{S} = \{\emptyset\}$. As A and B are arbitrary, we may therefore conclude that $\{\emptyset\}$ satisfies Property 4 of a semiring of sets.

Because Ω was arbitrary, we may now finally conclude that the proposition is true. □

Theorem 11.5 (Generation of a semiring of sets by means of a set of left-closed and right-open intervals). *It is true for any linearly ordered set $(\Omega, <)$ with $\Omega \neq \emptyset$ that the set $\mathcal{I} = \{[a, b) : a, b \in \Omega\}$ of left-closed and right-open intervals in Ω is a semiring of sets on Ω.*

Proof. We let Ω and $<$ be arbitrary sets, assume $\Omega \neq \emptyset$, and assume that $<$ is a linear ordering of Ω. Recalling that $\mathcal{I} = \{[a, b) : a, b \in \Omega\}$ is the π-system of left-closed and right-open intervals, Property 1 and Property 3 of a semiring of sets are naturally satisfied by that set system. Moreover, $\Omega \neq \emptyset$ implies $\emptyset \in \mathcal{I}$ with Corollary 3.121, so that Property 2 of a semiring of sets is also satisfied by $\mathcal{I}$. Concerning Property 4, we let $A, B \in \mathcal{I}$ be arbitrary. Because the set of left-closed and right-open intervals is specified by (3.387), it follows on the one hand from $A \in \mathcal{I}$ that there exist particular elements a_1, b_1 in Ω satisfying $(a_1, b_1] = A$. On the other hand, $B \in \mathcal{I}$ implies the existence of particular elements a_2, b_2 in Ω such that $(a_2, b_2] = B$. Thus, our task is to show that the set difference $A \setminus B$ can be written as the union of (the range) of some sequence $C = (C_i \,|\, i \in \{1, \ldots, n\})$ of pairwise disjoint sets in $\mathcal{I}$. We proceed by considering the cases and sub-cases from the proof of Theorem 11.5 based on the true disjunctions

$$\begin{aligned} &\neg a_1 < b_1 \ \vee \ a_1 < b_1, \\ &\neg a_2 < b_2 \ \vee \ a_2 < b_2, \\ &\neg a_2 < b_1 \ \vee \ a_2 < b_1, \\ &\neg a_1 < b_2 \ \vee \ a_1 < b_2. \end{aligned}$$

The first case $\neg a_1 < b_1$ implies $(a_1, b_1] = \emptyset$ with (3.384) and therefore

$$[a_1, b_1) \setminus [a_2, b_2) = \emptyset \setminus [a_2, b_2) = \emptyset = \bigcup \{\emptyset\} = \bigcup \mathrm{ran}(\{(1, \emptyset)\})$$

by applying substitution, (2.105), (2.199) and (3.640), where $\{(1, \emptyset)\}$ is a function/sequence with domain $\{1\}$ and codomain/range $\{\emptyset\}$. This sequence has pairwise disjoint sets in view of Corollary 11.3. Furthermore, the previous finding $\emptyset \in \mathcal{I}$ gives $\{\emptyset\} \subseteq \mathcal{I}$ with (2.184). We thus have that

$\{(1,\emptyset)\}$ is a sequence $\bar{C} : \{1,\ldots,\bar{n}\} \to \mathcal{I}$ of pairwise disjoint sets with $\bar{n} = 1$ $[\in \mathbb{N}_+]$ satisfying $[a_1, b_1) \setminus [a_2, b_2) = \bigcup \operatorname{ran}(\bar{C})$, as desired.

In the second case $a_1 < b_1$, we consider the first sub-case $\neg a_2 < b_2$, which yields $[a_2, b_2) = \emptyset$ again with (3.384) and therefore

$$[a_1, b_1) \setminus [a_2, b_2) = [a_1, b_1) \setminus \emptyset = \emptyset = \bigcup\{\emptyset\} = \bigcup \operatorname{ran}(\{(1,\emptyset)\}),$$

by means of substitution, (2.103), (2.199) and (3.640). We thus found the same sequence as in the first case having the required properties.

In the second sub-case $a_2 < b_2$, we consider first $\neg a_2 < b_1$, so that we obtain $[a_1, b_1) \cap [a_2, b_2) = \emptyset$ with (3.412) and consequently

$$[a_1, b_1) \setminus [a_2, b_2) = [a_1, b_1) = \bigcup\{[a_1, b_1)\} = \bigcup \operatorname{ran}(\{(1, [a_1, b_1))\}) \quad (11.11)$$

by using (2.107), (2.199), and (3.640). Here, the singleton $\{(1, [a_1, b_1))\}$ is a function/sequence with domain $\{1\}$ and codomain/range $\{[a_1, b_1)\}$, and this sequence has pairwise disjoint sets due to Corollary 11.3. Because the fact $[a_1, b_1) \in \mathcal{I}$ implies $\{[a_1, b_1)\} \subseteq \mathcal{I}$ according to (2.184), we may write $\{[a_1, b_1)\}$ as a sequence $\bar{D} : \{1,\ldots,\bar{n}\} \to \mathcal{I}$ with $\bar{n} = 1$ $[\in \mathbb{N}_+]$, having pairwise disjoint terms and satisfying the desired $[a_1, b_1) \setminus [a_2, b_2) = \bigcup \operatorname{ran}(\bar{D})$.

Considering now on the other hand $a_2 < b_1$, we assume first $\neg a_1 < b_2$, in which case we obtain again $[a_1, b_1) \cap [a_2, b_2) = \emptyset$ with (3.412) and therefore (11.11). Thus, we may use in case of $\neg a_1 < b_2$ the same sequence $\bar{D}$ of pairwise disjoint sets in $\mathcal{I}$ to form the difference of $[a_1, b_1)$ and $[a_2, b_2)$.

On the other hand, if $a_1 < b_2$ holds (besides $a_1 < b_1$, $a_2 < b_2$ and $a_2 < b_1$), we make the following observations. The preceding assumption $a_1 < b_2$ implies the truth of the disjunction $a_1 < b_2 \vee a_1 = b_2$ and therefore the truth of $a_1 \leq b_2$, using the definition of an induced reflexive partial ordering. Consequently, we obtain $[b_2, b_1) \subseteq [a_1, +\infty)$ with (3.454), and this inclusion gives us $[b_2, b_1) \cap [a_1, +\infty) = [b_2, b_1)$ with (2.77). Furthermore, the previously assumed $a_2 < b_1$ gives the true disjunction $a_2 < b_1 \vee a_2 = b_1$ and therefore $a_2 \leq b_1$. This inequality in turn implies $[a_1, a_2) \subseteq (-\infty, b_1)$ with (3.456), so that $[a_1, a_2) \cap (-\infty, b_1) = [a_1, a_2)$. Because of the previous two equations, we now obtain

$$\begin{aligned} [a_1, b_1) \setminus [a_2, b_2) &= (\,[a_1, a_2) \cap (-\infty, b_1)\,) \cup (\,[b_2, b_1) \cap [a_1, +\infty)\,) \\ &= [a_1, a_2) \cup [b_2, b_1) \end{aligned} \quad (11.12)$$

by applying (3.465) and then substitutions. Evidently, the two singletons $\{(1, [a_1, a_2))\}$ and $\{(2, [b_2, b_1))\}$ are functions with domain $\{1\}$ and $\{2\}$; since $1 \neq 2$ holds according to (4.167), we obtain $2 \notin \{1\}$ with (2.169). We may therefore apply Proposition 3.177 to construct the new function

$$\bar{F} = \{(1, [a_1, a_2))\} \cup \{(2, [b_2, b_1))\}$$

with domain $\{1\} \cup \{2\} = \{1, 2\}$, using (2.226). This domain is the initial segment of $\mathbb{N}_+$ up to $n = 2$, and we may now show by means of the Equality Criterion for sets that the range of $\bar{F}$ is given by $\{[a_1, a_2), [b_2, b_1)\}$. To do this, we prove the universal sentence

$$\forall Y\, (Y \in \operatorname{ran}(\bar{F}) \Leftrightarrow Y \in \{[a_1, a_2), [b_2, b_1)\}). \tag{11.13}$$

Letting Y be an arbitrary set and assuming first $Y \in \operatorname{ran}(\bar{F})$ to be true, there exists (by definition of a range) a constant, say $\bar{k}$, such that $(\bar{k}, Y) \in \bar{F}$ holds. By definition of the union of two sets, this implies that the disjunction of $(\bar{k}, Y) \in \{(1, [a_1, a_2))\}$ and $(\bar{k}, Y) \in \{(2, [b_2, b_1))\}$ is true, which we may write equivalently as the disjunction of $(\bar{k}, Y) = (1, [a_1, a_2))$ and $(\bar{k}, Y) = (2, [b_2, b_1))$ by applying (2.169). Let us prove the desired consequent $Y \in \{[a_1, a_2), [b_2, b_1)\}$ by cases, based on that disjunction. The first case $(\bar{k}, Y) = (1, [a_1, a_2))$ gives with the Equality Criterion for ordered pairs especially $Y = [a_1, a_2)$, so that the disjunction $Y = [a_1, a_2) \vee Y = [b_2, b_1)$ also holds. Thus, $Y \in \{[a_1, a_2), [b_2, b_1)\}$ follows to be true by definition of a pair. Using the same arguments as in the first case, the second case $(\bar{k}, Y) = (2, [b_2, b_1))$ yields especially $Y = [b_2, b_1)$ and then $Y = [a_1, a_2) \vee Y = [b_2, b_1)$, with the consequence that $Y \in \{[a_1, a_2), [b_2, b_1)\}$ holds again. Having completed the proof of the first part ('$\Rightarrow$') of the equivalence in (11.13), we now assume conversely $Y \in \{[a_1, a_2), [b_2, b_1)\}$ to be true, so that the definition of a pair gives us $Y = [a_1, a_2) \vee Y = [b_2, b_1)$. We use this true disjunction to prove the required consequent $Y \in \operatorname{ran}(\bar{F})$ by cases. In the first case $Y = [a_1, a_2)$, we obtain with the Equality Criterion for ordered pairs $(1, Y) = (1, [a_1, a_2))$ and therefore $(1, Y) \in \{(1, [a_1, a_2))\}$ with (2.169). Then, the disjunction

$$(1, Y) \in \{(1, [a_1, a_2))\} \vee (1, Y) \in \{(2, [b_2, b_1))\} \tag{11.14}$$

is also true, which evidently yields $(1, Y) \in \bar{F}$ by definition of $\bar{F}$ and of the union of two sets. This finding proves the existence of a constant k satisfying $(k, Y) \in \bar{F}$, so that $Y \in \operatorname{ran}(\bar{F})$ holds by definition of a range. Similarly, the second case $Y = [b_2, b_1)$ gives now $(2, Y) = (2, [b_2, b_1))$ and therefore $(2, Y) \in \{(2, [b_2, b_1))\}$. Then, the disjunction (11.14) is true again, and this gives $Y \in \operatorname{ran}(\bar{F})$ as before. Thus, the proof by cases is complete, so that the second part ('$\Leftarrow$') of the equivalence in (11.13) also holds. Since Y is arbitrary, we may now infer from the this findings the truth of the universal sentence (11.13), which in turn implies the desired equation

$\mathrm{ran}(\bar{F}) = \{[a_1, a_2), [b_2, b_1)\}$. Then, we obtain

$$\bigcup \mathrm{ran}(\bar{F}) = \bigcup \{[a_1, a_2), [b_2, b_1)\} = [a_1, a_2) \cup [b_2, b_1) = [a_1, b_1) \setminus [a_2, b_2), \tag{11.15}$$

using the notation for the union of two sets and recalling (11.12). Furthermore, $[a_1, a_2), [b_2, b_1) \in \mathcal{I}$ implies the inclusion $\{[a_1, a_2), [b_2, b_1)\} \subseteq \mathcal{I}$ with (2.164), which shows that $\mathcal{I}$ is a codomain of $\bar{F}$. It now remains for us to demonstrate that this sequence has pairwise disjoint terms. The current sub-case assumption $a_2 < b_2$ clearly implies $a_2 \leq b_2$ and then $\neg b_2 < a_2$ with the Negation Formula for $<$, so that we get

$$[a_1, a_2) \cap [b_2, b_1) = \emptyset$$

with (3.412). Letting now i and j be arbitrary and assuming $i, j \in \{1, 2\}$ as well as $i \neq j$ to be true, we obtain with the definition of a pair the true disjunctions $i = 1 \lor i = 2$ and $j = 1 \lor j = 2$. We now use the former disjunction to prove the required consequence $\bar{F}_i \cap \bar{F}_j = \emptyset$ by cases. In the first case $i = 1$, we see that the first part $j = 1$ of the latter disjunction is false in view of the previous assumption $i \neq j$, so that the second part $j = 2$ of that disjunction must be true; consequently,

$$\bar{F}_i \cap \bar{F}_j = \bar{F}_1 \cap \bar{F}_2 = [a_1, a_2) \cap [b_2, b_1) = \emptyset.$$

Similarly, the second case $i = 2$ shows that $j = 1$ must be true, so that

$$\bar{F}_i \cap \bar{F}_j = \bar{F}_2 \cap \bar{F}_1 = \bar{F}_1 \cap \bar{F}_2 = \emptyset,$$

where we applied the Commutative Law for the intersection of two sets. Thus, the proof by cases is complete, and since i and j are arbitrary, we may therefore conclude that $\bar{F} : \{1, 2\} \to \mathcal{I}$ is a sequence of pairwise disjoint sets, which satisfies $[a_1, b_1) \setminus [a_2, b_2) = \bigcup \mathrm{ran}(\bar{F})$ in view of (11.15).

We thus showed that the intersection of two left-closed and right-open intervals $[a_1, b_1)$ and $[a_2, b_2)$ can in any case be written as the union of (the range) of some sequence $C : \{1, \ldots, n\} \to \mathcal{I}$ of pairwise disjoint sets. Since A and B (forming these intervals) are arbitrary, we may now further conclude that $\mathcal{I}$ satisfies also Property 4 of a semiring of sets on Ω. Finally, as Ω and $<$ were initially arbitrary as well, we may infer from the previous findings that $\mathcal{I}$ constitutes a semiring of sets on Ω for any such sets. □

Definition 11.3 (Semiring of left-closed and right-open intervals). We call for any linearly ordered set $(\Omega, <)$ with $\Omega \neq \emptyset$ the set

$$\mathcal{I} = \{[a, b) : a, b \in \Omega\} \tag{11.16}$$

of all left-closed and right-open intervals in Ω the *semiring of left-closed and right-open intervals* (in Ω).

Note 11.3. Using the standard linear ordering $<_{\mathbb{R}}$ of the set of real numbers, we thus obtain the semiring

$$\mathcal{I} = \{[a, b) : a, b \in \mathbb{R}\} \tag{11.17}$$

of left-closed and right-open intervals in $\mathbb{R}$.

Theorem 11.6 (Stacking of a finite sequence of finite sequences). *The following sentences are true.*

a) For any natural number n and any sequence $N : \{1, \ldots, n\} \to \mathbb{N}$, there exists a unique function G with domain $\{1, \ldots, n\}$ such that

$$\forall i \, (i \in \{1, \ldots, n\} \Rightarrow G(i) = \left\{ \left(\sum_{j=1}^{i-1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{i-1} N_j \right) + N_i \right\}), \tag{11.18}$$

and this function is a sequence of pairwise disjoint sets.

b) Furthermore, the sequence G satisfies

$$\forall n \, (n \in \mathbb{N} \Rightarrow \forall N \, (N : \mathbb{N}^{\{1,\ldots,n\}} \Rightarrow \bigcup \operatorname{ran}(G) = \left\{ 1, \ldots, \sum_{i=1}^{n} N_i \right\})). \tag{11.19}$$

c) For any $n \in \mathbb{N}$, any sequence $N : \{1, \ldots, n\} \to \mathbb{N}$, any set Y and any sequence $F = (F_i \,|\, i \in \{1, \ldots, n\})$ with $F_i : \{1, \ldots, N_i\} \to Y$ for all $i \in \{1, \ldots, n\}$, it is true that there exists a unique sequence $f = (f_i \,|\, i \in \{1, \ldots, n\})$ such that

$$\forall i \, (i \in \{1, \ldots, n\} \Rightarrow f_i = F_i \circ t_i^{-1}) \tag{11.20}$$

with the translating sequence

$$t_i^{-1} : \left\{ \left(\sum_{j=1}^{i-1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{i-1} N_j \right) + N_i \right\} \to \{1, \ldots, N_i\} \tag{11.21}$$

being defined for any $i \in \{1, \ldots, n\}$ according to Proposition 5.115.

d) Furthermore, the range of the sequence $f = (f_i \,|\, i \in \{1, \ldots, n\})$ constitutes a compatible set of functions, and the union s of the range of f (i.e., $s = \bigcup \operatorname{ran}(f)$) is a sequence in Y with domain $\{1, \ldots, \sum_{i=1}^{n} N_i\}$

whose terms satisfy the two universal sentences

$$\forall k\,(k \in \{1,\dots,\sum_{i=1}^{n} N_i\} \tag{11.22}$$
$$\Rightarrow \exists i,j\,(i \in \{1,\dots,n\} \wedge j \in \{1,\dots,N_i\} \wedge s_k = F_i(j))),$$

$$\forall i,j\,([i \in \{1,\dots,n\} \wedge j \in \{1,\dots,N_i\}] \tag{11.23}$$
$$\Rightarrow \exists k\,(k \in \{1,\dots,\sum_{i=1}^{n} N_i\} \wedge F_i(j) = s_k)).$$

e) *Moreover, any two terms of s having different indexes are terms in different sequences in F or terms with different indexes within a single sequence in F, in the sense that*

$$\forall k,l\,([k,l \in \{1,\dots,\sum_{i=1}^{n} N_i\} \wedge k \neq l] \tag{11.24}$$
$$\Rightarrow \exists i,I,j,J\,(i,I \in \{1,\dots,n\} \wedge j \in \{1,\dots,N_i\} \wedge s_k = F_i(j)$$
$$\wedge\; J \in \{1,\dots,N_I\} \wedge s_l = F_I(J) \wedge [i \neq I \vee (i = I \wedge j \neq J)])).$$

Proof. Concerning a), we let n and N be arbitrary such that n is a natural number and such that N is a function from $\{1,\dots,n\}$ to $\mathbb{N}$, so that N is a sequence $(N_i \,|\, i \in \{1,\dots,n\})$ of natural numbers. We now apply Function definition by replacement and verify accordingly

$$\forall i\,(i \in \{1,\dots,n\} \Rightarrow \exists! y\,(y = \left\{\left(\sum_{j=1}^{i-1} N_j\right) + 1,\dots,\left(\sum_{j=1}^{i-1} N_j\right) + N_i\right\})), \tag{11.25}$$

letting i be arbitrary and assuming $i \in \{1,\dots,n\}$ to be true; clearly, we then have $1 \leq i$. Since $1 \leq 1$ is true because of the reflexivity of the standard total ordering $\leq_{\mathbb{N}}$, we may apply the Monotony Law for $-_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$ to obtain the inequality $1 - 1 \leq i - 1$. This shows that $i - 1$ is a natural number, so that the sum $\{\sum_{j=1}^{i-1} N_j$ is a specified natural number. Consequently, the two sums $(\sum_{j=1}^{i-1} N_j) + 1$ and $(\sum_{j=1}^{i-1} N_j) + N_i$ are also elements of $\mathbb{N}$, so that the intermediate segment of $\mathbb{N}_+$ from $(\sum_{j=1}^{i-1} N_j)^+$ to $(\sum_{j=1}^{i-1} N_j) + N_i$ is defined, which we may write also as stated in (11.25) by using (5.217). Then, the stated uniquely existential sentence follows to be true with (1.109). Since i is arbitrary, we may therefore conclude that the universal sentence (11.25) holds, which in turn implies the existence of a function G with domain $\{1,\ldots,n\}$ such that (11.18). Thus, G is a sequence $(G_i \,|\, i \in \{1,\ldots,n\})$ of intermediate segments of $\mathbb{N}_+$.

Next, we show that the terms of the sequence G are pairwise disjoint, that is,

$$\forall i,k\,([i,k \in \{1,\ldots,n\} \,\wedge\, i \neq k] \Rightarrow G_i \cap G_k = \emptyset). \tag{11.26}$$

To do this, we let i and k be arbitrary, and we assume $i,k \in \{1,\ldots,n\}$ as well as $i \neq k$ to be true. Because 1 is the minimum of $\{1,\ldots,n\}$, the former assumption implies $1 \leq i$ as well as $1 \leq k$. Let us now apply the definition of the empty set and prove the desired equation $G_i \cap G_k = \emptyset$ by demonstrating the truth of

$$\forall y\,(y \notin G_i \cap G_k). \tag{11.27}$$

For this purpose, we let y be arbitrary and prove the negation $y \notin G_i \cap G_k$ by contradiction, assuming $\neg y \notin G_i \cap G_k$ to be true and letting $y < y \wedge \neg y < y$ be the desired contradiction (in which $\neg y < y$ is true by virtue of the irreflexivity of the linear ordering $<$). Then, the Double Negation Law gives $y \in G_i \cap G_k$, so that $y \in G_i$ and $y \in G_k$ are both true by definition of the intersection of two sets, that is,

$$y \in \left\{ \left(\sum_{j=1}^{i-1} N_j\right) + 1, \ldots, \left(\sum_{j=1}^{i-1} N_j\right) + N_i \right\}, \tag{11.28}$$

$$y \in \left\{ \left(\sum_{j=1}^{k-1} N_j\right) + 1, \ldots, \left(\sum_{j=1}^{k-1} N_j\right) + N_k \right\}. \tag{11.29}$$

According to Proposition 4.62 and the recursive definition of the repeated sum, this means that the inequalities

$$\left(\sum_{j=1}^{i-1} N_j\right) + 1 \;\leq\; y \;\leq\; \sum_{j=1}^{i} N_j \tag{11.30}$$

$$\left(\sum_{j=1}^{k-1} N_j\right) + 1 \;\leq\; y \;\leq\; \sum_{j=1}^{k} N_j \tag{11.31}$$

are true. Furthermore, since the standard linear ordering $<_{\mathbb{N}}$ is connex, the initial assumption $i \neq k$ implies the truth of the disjunction $i < k \,\vee\, k < i$, which we now utilize to prove $y < y$ by cases.

In the first case $i < k$, we may in view of $1 \leq i$ apply the Monotony Law for $-_{\mathbb{N}}$ and $<_{\mathbb{N}}$ to obtain the inequality $i - 1 < k - 1$, which in turn implies $(i-1)^+ \leq k-1$ with (4.270). Consequently, we get $i \leq k-1$ using (5.217) in connection with (5.343). Applying now Corollary 5.119 in the context of the ordered elementary domain $(\mathbb{N}, +, \cdot, <)$ and the sequence $N : \{1, \ldots, n\} \to \mathbb{N}$, the inequality $i \leq k-1$ and the fact (4.153) imply the truth of

$$\sum_{j=1}^{i} N_j \;\leq\; \sum_{j=1}^{k-1} N_j \;<\; \left(\sum_{j=1}^{k-1} N_j\right) + 1.$$

Then, the Transitivity Formula for $\leq$ and $<$ yields

$$\sum_{j=1}^{i} N_j \;<\; \left(\sum_{j=1}^{k-1} N_j\right) + 1. \tag{11.32}$$

We now obtain from (11.30) – (11.32) the inequalities

$$y \;\leq\; \sum_{j=1}^{i} N_j \;<\; \left(\sum_{j=1}^{k-1} N_j\right) + 1 \;\leq\; y,$$

so that an application of the Transitivity Formula for $\leq$ and $<$ and then of the Transitivity Formula for $<$ and $\leq$ yields the desired $y < y$ (in the first case).

We may apply similar arguments in the second case $k < i$ to obtain $y < y$. To begin with, the previously found $1 \leq k$ allows us to infer from $k < i$ the truth of $k - 1 < i - 1$, with the consequence that $(k-1)^+ \leq i - 1$ and then $k \leq i - 1$ hold. This inequality gives now

$$\sum_{j=1}^{k} N_j \;\leq\; \sum_{j=1}^{i-1} N_j \;<\; \left(\sum_{j=1}^{i-1} N_j\right) + 1.$$

Consequently,

$$\sum_{j=1}^{k} N_j \;<\; \left(\sum_{j=1}^{i-1} N_j\right) + 1$$

holds, which leads in connection with (11.30) – (11.31) to

$$y \;\leq\; \sum_{j=1}^{k} N_j \;<\; \left(\sum_{j=1}^{i-1} N_j\right) + 1 \;\leq\; y.$$

We therefore arrive at $y < y$ also in the second case, which thus establishes the desired contradiction, completing the proof of $y \notin G_i \cap G_k$. Since y is arbitrary, we may infer from this the truth of the universal sentence (11.27), which in turn implies $G_i \cap G_k = \emptyset$ (by definition of the empty set). As i and j were also arbitrary, it follows from this that (11.26) holds, i.e. that the terms of the sequence $G = (G_i \,|\, i \in \{1, \ldots, n\})$ are indeed pairwise disjoint.

Concerning b), we will apply a proof by mathematical induction to establish (11.19). In the base case $n = 0$, letting N be arbitrary such that N is a function from $\{1, \ldots, n\}$ to $\mathbb{N}$, we have by definition of an initial segment of $\mathbb{N}_+$

$$\mathrm{dom}(N) = \{1, \ldots, n\} = \{1, \ldots, 0\} = \emptyset,$$

implying $\mathrm{ran}(N) = \emptyset$ with (3.119). We then obtain the equations

$$\bigcup \mathrm{ran}(G) = \bigcup \emptyset = \emptyset = \{1, \ldots, 0\} = \left\{1, \ldots, \sum_{i=1}^{0} N_i\right\} = \left\{1, \ldots, \sum_{i=1}^{n} N_i\right\}$$

by applying substitution, (2.205), the notation of initial segments of $\mathbb{N}_+$, (5.409) in the context of the addition on $\mathbb{N}$ and finally again substitution. Thus, the base case holds.

Regarding the induction step, we let $n \in \mathbb{N}$ be arbitrary, make the induction assumption

$$\forall N \,(N : \mathbb{N}^{\{1,\ldots,n\}} \Rightarrow \bigcup \mathrm{ran}(G^{(n)}) = \left\{1, \ldots, \sum_{i=1}^{n} N_i\right\}) \tag{11.33}$$

(writing $G^{(n)}$ instead of G), and show that

$$\forall N \,(N : \mathbb{N}^{\{1,\ldots,n+1\}} \Rightarrow \bigcup \mathrm{ran}(G^{(n+1)}) = \left\{1, \ldots, \sum_{i=1}^{n+1} N_i\right\}) \tag{11.34}$$

follows to be true. We take an arbitrary set N and assume N to be a sequence from $\{1, \ldots, n+1\}$ to $\mathbb{N}$. We now establish the equation in (11.28) by applying the Axiom of Extension, proving first the inclusion $\bigcup \mathrm{ran}(G^{(n+1)}) \subseteq \{1, \ldots, \sum_{i=1}^{n+1} N_i\}$, which we may write equivalently as (using the definition of a subset)

$$\forall y \,(y \in \bigcup \mathrm{ran}(G^{(n+1)}) \Rightarrow y \in \left\{1, \ldots, \sum_{i=1}^{n+1} N_i\right\}). \tag{11.35}$$

We let y be arbitrary and assume $y \in \bigcup \mathrm{ran}(G^{(n+1)})$ to be true. Since the sequence $G^{(n+1)}$ is defined to have the domain $\{1, \ldots, n+1\}$ according to (11.18), the preceding assumption implies the existence of a particular index $\bar{k} \in \{1, \ldots, n+1\}$ with $y \in G_{\bar{k}}^{(n+1)}$, so that

$$y \in \left\{ \left(\sum_{j=1}^{\bar{k}-1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{\bar{k}-1} N_j \right) + N_{\bar{k}} \right\}. \tag{11.36}$$

Consequently, we obtain the inequalities

$$1 \le \bar{k} \le n+1 \tag{11.37}$$

as well as the inequalities

$$\left(\sum_{j=1}^{\bar{k}-1} N_j \right) + 1 \le y \le \sum_{j=1}^{\bar{k}} N_j. \tag{11.38}$$

Since $\sum_{j=1}^{\bar{k}-1} N_j$ is a natural number satisfying $0 \le \sum_{j=1}^{\bar{k}-1} N_j$ because of (5.329), the Monotony Law for $+_{\mathbb{N}}$ and $\le_{\mathbb{N}}$ evidently yields the inequality $1 \le (\sum_{j=1}^{\bar{k}-1} N_j) + 1$. Together with the first inequality in (11.38), this implies now $1 \le y$ with the transitivity of the standard total ordering $\le_{\mathbb{N}}$. Furthermore, the second inequality in (11.37) gives $\sum_{j=1}^{\bar{k}} N_j \le \sum_{j=1}^{n+1} N_j$, so that the conjunction with the second inequality in (11.38) gives $y \le \sum_{j=1}^{n+1} N_j$, which we may write also as $y \le \sum_{i=1}^{n+1} N_i$. The truth of $1 \le y$ and the preceding inequality implies then $y \in \{1, \ldots, \sum_{i=1}^{n+1} N_i\}$, which finding proves the implication in (11.35). As y is arbitrary, we may therefore conclude that the universal sentence (11.35) holds, with the consequence that the inclusion $\bigcup \mathrm{ran}(G^{(n+1)}) \subseteq \{1, \ldots, \sum_{i=1}^{n+1} N_i\}$ is true.

We now show that the inclusion $\{1, \ldots, \sum_{i=1}^{n+1} N_i\} \subseteq \bigcup \mathrm{ran}(G^{(n+1)})$ also holds, by verifying equivalently

$$\forall y \, (y \in \left\{ 1, \ldots, \sum_{i=1}^{n+1} N_i \right\} \Rightarrow y \in \bigcup \mathrm{ran}(G^{(n+1)})). \tag{11.39}$$

We take an arbitrary y and assume that $y \in \{1, \ldots, \sum_{i=1}^{n+1} N_i\}$ holds, so that

$$1 \le y \le \sum_{i=1}^{n+1} N_i \quad \left[= \sum_{i=1}^{n} N_i + N_{n+1} \right] \tag{11.40}$$

is true. We prove the desired consequent $y \in \bigcup \mathrm{ran}(G^{(n+1)})$ by cases, based on the true disjunction $y \le \sum_{i=1}^{n} N_i \vee \neg y \le \sum_{i=1}^{n} N_i$ (given via the Law of the Excluded Middle). On the one hand, the first inequality $1 \le y$ implies with the first case $y \leq \sum_{i=1}^{n} N_i$ that $y \in \{1, \ldots, \sum_{i=1}^{n} N_i\}$ holds. Since the restriction of $N = (N_i \mid \{1, \ldots, n+1\})$ to $\{1, \ldots, n\}$ gives a sequence in $\mathbb{N}^{\{1,\ldots,n\}}$, we may use the induction assumption (11.33) to infer from the previous finding the truth of $y \in \bigcup \mathrm{ran}(G^{(n)})$. Thus, $y \in G_{\bar{k}}^{(n)}$ holds for a particular index $\bar{k} \in \{1, \ldots, n\}$, which we may write as (11.36) according to the definition of the function $G = G^{(n)}$ in a). Evidently, (11.36) is also true for the sequence $G = G^{(n+1)}$, so that we have $y \in G_{\bar{k}}^{(n+1)}$ and therefore $y \in \bigcup \mathrm{ran}(G^{(n+1)})$, as required. On the other hand, the second case $\neg y \leq \sum_{i=1}^{n} N_i$ implies $\sum_{i=1}^{n} N_i < y$ with the Negation Formula for $\leq$ (applied to the standard total ordering $\leq_{\mathbb{N}}$) and consequently

$$\left(\sum_{i=1}^{n} N_i\right) + 1 \;\leq\; y \quad \left[\leq \sum_{i=1}^{n} N_i + N_{n+1}\right],$$

recalling (11.40). Therefore, the definition of $G^{(n+1)}$ yields

$$y \in \left\{\left(\sum_{i=1}^{n} N_i\right) + 1, \ldots, \sum_{i=1}^{n} N_i + N_{n+1}\right\}$$
$$\left[= \left\{\left(\sum_{i=1}^{[n+1]-1} N_i\right) + 1, \ldots, \sum_{i=1}^{[n+1]-1} N_i + N_{n+1}\right\} = G_{n+1}^{(n+1)}\right],$$

which shows that there exists an index $i \in \{1, \ldots, n+1\}$ with $y \in G_i^{(n+1)}$, so that $y \in \bigcup \mathrm{ran}(G^{(n+1)})$ is true. We thus completed the proof by cases, and since y was arbitrary, we may now further conclude that (11.39) holds, which universal sentence in turn implies the truth of the second desired inclusion $\{1, \ldots, \sum_{i=1}^{n+1} N_i\} \subseteq \bigcup \mathrm{ran}(G^{(n+1)})$. Together with the already established converse inclusion, this implies the truth of the equation in (11.34), and because N was initially an arbitrary set, we may infer from this the truth of the universal sentence (11.34). As n was arbitrary, the induction step follows then to be true (besides the base case), so that the proof of b) by mathematical induction is complete.

Concerning c), we let n be an arbitrary natural number, N an arbitrary sequence from $\{1, \ldots, n\}$ to $\mathbb{N}$, Y and G arbitrary sets, and we assume F to be a sequence $(F_i \mid i \in \{1, \ldots, n\})$ such that any term F_i is a sequence in Y with domain $\{1, \ldots, n\}$. We may now apply Function definition by replacement to establish the function f with domain $\{1, \ldots, n\}$ and values satisfying (11.20). For this purpose, we verify

$$\forall i \, (i \in \{1, \ldots, n\} \Rightarrow \exists! y \, (y = F_i \circ t_i^{-1})), \tag{11.41}$$

letting i be arbitrary and assuming $i \in \{1, \ldots, n\}$ to be true. We then have that $F_i : \{1, \ldots, n\} \to Y$ is the term of the sequence F associated with the index i and moreover that the translating sequence (11.21) is defined according to Proposition 5.115. Consequently, we obtain the composed function/sequence

$$F_i \circ t_i^{-1} : \left\{ \left(\sum_{j=1}^{i-1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{i-1} N_j \right) + N_i \right\} \to Y \tag{11.42}$$

due to (3.604). Given that constant $F_i \circ t_i^{-1}$, the uniquely existential sentence in (11.41) follows to be true with (1.109). Because i is arbitrary, we may therefore conclude that the universal sentence (11.41) is true, so that there exists a unique function/sequence f with domain $\{1, \ldots, n\}$ such that (11.20).

Concerning d), we prove that $\mathrm{ran}(f)$ is a compatible set of functions by letting g and h be arbitrary elements of $\mathrm{ran}(f)$, so that there are (by definition of a range) particular constants $\bar{k}_g$ and $\bar{k}_h$ with $(\bar{k}_g, g), (\bar{k}_h, h) \in f$. Because f is a function/sequence with values (11.42) and with domain $\{1, \ldots, n\}$, we obtain

$$g = f_{\bar{k}_g} = F_{\bar{k}_g} \circ t_{\bar{k}_g}^{-1}, \tag{11.43}$$

$$h = f_{\bar{k}_h} = F_{\bar{k}_h} \circ t_{\bar{k}_h}^{-1}, \tag{11.44}$$

as well as $\bar{k}_g, \bar{k}_h \in \{1, \ldots, n\}$ (using the definition of a domain). In light of (11.42) and (11.18), we now see that the domains of $g = F_{\bar{k}_g} \circ t_{\bar{k}_g}^{-1}$ and of $h = F_{\bar{k}_h} \circ t_{\bar{k}_h}^{-1}$ are given by

$$\mathrm{dom}(g) = \left\{ \left(\sum_{j=1}^{\bar{k}_g - 1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{\bar{k}_g - 1} N_j \right) + N_{\bar{k}_g} \right\} = G_{\bar{k}_g},$$

$$\mathrm{dom}(h) = \left\{ \left(\sum_{j=1}^{\bar{k}_h - 1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{\bar{k}_h - 1} N_j \right) + N_{\bar{k}_h} \right\} = G_{\bar{k}_h}.$$

We now carry out a proof by cases, based on the true disjunction $\bar{k}_g = \bar{k}_h \lor \bar{k}_g \neq \bar{k}_h$, to show that g and h are compatible. In the first case $\bar{k}_g = \bar{k}_h$, we obtain from (11.43) – (11.44) via substitution $g = h$. Since the function g is compatible with itself (i.e., with g) in view of Corollary 3.173, it follows with the preceding equation (via substitution) that g and h are compatible. The second case $\bar{k}_g \neq \bar{k}_h$ implies with (11.26) that the domains $\mathrm{dom}(g) = G_{\bar{k}_g}$ and $\mathrm{dom}(h) = G_{\bar{k}_h}$ are disjoint, so that g and h turn out to be compatible also in the current second case because of Exercise 3.73. Thus, the proof by cases is complete, and as g and h are arbitrary, we may therefore conclude that $\mathrm{ran}(f)$ is a compatible set of functions. Consequently, we may apply Concatenation of functions to obtain the new function $\bigcup \mathrm{ran}(f)$; furthermore, there exists then a unique set (system) $\mathcal{D}$ consisting of the domains of all the functions in $\mathrm{ran}(f)$ in the sense that

$$\forall D\,(D \in \mathcal{D} \Leftrightarrow \exists g\,(g \in \mathrm{ran}(f) \;\wedge\; \mathrm{dom}(g) = D)), \tag{11.45}$$

and the union of this set system is the domain of the union of $\mathrm{ran}(f)$, i.e.

$$\mathrm{dom}(\bigcup \mathrm{ran}(f)) = \bigcup \mathcal{D}. \tag{11.46}$$

Here, we may readily prove that $\mathcal{D}$ is identical with the range of the sequence G, by verifying

$$\forall D\,(D \in \mathcal{D} \Leftrightarrow D \in \mathrm{ran}(G)). \tag{11.47}$$

Letting D be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $D \in \mathcal{D}$ to be true. This assumption implies the existence of a particular element $\bar{g} \in \mathrm{ran}(f)$ with $\mathrm{dom}(\bar{g}) = D$, using (11.45). By definition of a range, there is then also a particular constant $\bar{k}$ such that $(\bar{k}, \bar{g}) \in f$, which we may write in function/sequence notation equivalently as $\bar{g} = f_{\bar{k}}$. In light of the definition of a domain, we see in addition that $\bar{k} \in \{1, \ldots, n\}$ [$= \mathrm{dom}(f)$] holds, so that $f_{\bar{k}}$ is the composition $F_{\bar{k}} \circ t_{\bar{k}}^{-1}$. We therefore obtain with the previous findings, with (11.42) and with (11.18)

$$\begin{aligned} D &= \mathrm{dom}(\bar{g}) \\ &= \mathrm{dom}(f_{\bar{k}}) \\ &= \mathrm{dom}(F_{\bar{k}} \circ t_{\bar{k}}^{-1}) \\ &= \left\{ \left(\sum_{j=1}^{\bar{k}-1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{\bar{k}-1} N_j \right) + N_{\bar{k}} \right\} \\ &= G_{\bar{k}}, \end{aligned}$$

so that $(\bar{k}, D) \in G$ is true. This in turn implies $D \in \mathrm{ran}(G)$ (by definition of a range), completing the proof of the first part of the equivalence in (11.47). To prove the second part ('$\Leftarrow$') directly, we now assume conversely $D \in \mathrm{ran}(G)$ to be true, and we apply the arguments used in the proof of the first part of the equivalence in reversed order to establish the desired consequent $D \in \mathcal{D}$. First, the preceding assumption implies the existence of a particular constant $\bar{k}$ with $(\bar{k}, D) \in G$, which we may write as $D = G_{\bar{k}}$.

Moreover, $\bar{k}$ follows to be an element of the domain $\{1,\ldots,n\}$ of f, so that we obtain the equations

$$D = G_{\bar{k}} = \left\{ \left(\sum_{j=1}^{\bar{k}-1} N_j\right) + 1, \ldots, \left(\sum_{j=1}^{\bar{k}-1} N_j\right) + N_{\bar{k}} \right\} = \mathrm{dom}(F_{\bar{k}} \circ t_{\bar{k}}^{-1})$$
$$= \mathrm{dom}(f_{\bar{k}}),$$

resulting in $\mathrm{dom}(f_{\bar{k}}) = D$. Defining now $\bar{g} = f_{\bar{k}}$, so that $(\bar{k}, \bar{g}) \in f$ holds, we have $\bar{g} \in \mathrm{ran}(f)$. The conjunction of this and $\mathrm{dom}(f_{\bar{k}}) = D$ demonstrates the truth of the existential sentence in (11.45), which further implies $D \in \mathcal{D}$, as desired. Because D was arbitrary, we may therefore conclude that (11.45) holds, which universal sentence implies then the truth of the suggested equation $\mathcal{D} = \mathrm{ran}(G)$ by means of the Equality Criterion for sets. Consequently, we obtain the further equations

$$\mathrm{dom}(\bigcup \mathrm{ran}(f)) = \bigcup \mathcal{D} = \bigcup \mathrm{ran}(G) = \left\{1, \ldots, \sum_{i=1}^{n} N_i\right\}$$

recalling (11.46), applying substitution, and using (11.19).

We now introduce the convenient notation $s = \bigcup \mathrm{ran}(f)$ and establish (11.22), letting k be arbitrary and assuming $k \in \{1, \ldots, \sum_{i=1}^{n} N_i\}$ to be true, which means that k is in the domain of s. Therefore, $s_k = s(k)$ is the associated value/term of s, and we may write this equation as $(k, s_k) \in \bigcup \mathrm{ran}(f)$. Since f is a sequence with domain $\{1, \ldots, n\}$, we may apply the sequence notation in connection with the union of a family of sets and write $(k, s_k) \in \bigcup_{i=1}^{n} f_i$. Consequently, the Characterization of the union of a family of sets implies the existence of a particular index $\bar{i} \in \{1, \ldots, n\}$ such that $(k, s_k) \in f_{\bar{i}}$ holds. In view of (11.20), we may then write the preceding finding also as $(k, s_k) \in F_{\bar{i}} \circ t_{\bar{i}}^{-1}$. Using the fact that $t_{\bar{i}}^{-1}$ and the composition are functions and defining $\bar{j} = t_{\bar{i}}^{-1}(k)$, we obtain

$$s_k = (F_{\bar{i}} \circ t_{\bar{i}}^{-1})(k) = F_{\bar{i}}(t_{\bar{i}}^{-1}(k)) = F_{\bar{i}}(\bar{j})$$

and $(k, \bar{j}) \in t_{\bar{i}}^{-1}$. The latter shows that $\bar{j}$ is in the range of $t_{\bar{i}}^{-1}$, and since the this range is included in the codomain $\{1, \ldots, N_{\bar{i}}\}$ of $t_{\bar{i}}^{-1}$, it follows that $\bar{j} \in \{1, \ldots, N_{\bar{i}}\}$ is true. The previous findings prove the existence of constants i and j satisfying $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, N_i\}$ and $s_k = F_i(j)$, completing the proof of the implication in (11.22). Since k was arbitrary, we may therefore conclude that $s = \bigcup \mathrm{ran}(f)$ satisfies indeed the universal sentence (11.22).

Regarding (11.23), we now let i and j be arbitrary, assuming $i \in \{1, \ldots, n\}$ as well as $j \in \{1, \ldots, N_i\}$ to be true. Thus, i is in the domain of F and j in the domain of F_i, so that the value $F_i(j)$ is specified. Let us now observe that $f_i = F_i \circ t_i^{-1}$ implies via substitution

$$f_i \circ t_i = (F_i \circ t_i^{-1}) \circ t_i = F_i \circ (t_i^{-1} \circ t_i) = F_i \circ \mathrm{id}_{\{1,\ldots,N_i\}} = F_i$$

applying substitution, the Associative Law for function composition, (3.679), and the Neutrality of identity functions under composition in connection with the fact that $\{1, \ldots, N_i\}$ is the domain of the translating sequence t_i (which we used to define t_i^{-1} in c)). Consequently, substitution based on these equations yields

$$F_i(j) = (f_i \circ t_i)(j) = f_i(t_i(j))$$

Since the assumption $j\{1, \ldots, N_i\}$ implies that j is in the domain of t_i, the value $\bar{k} = t_i(j)$ is specified, so that another substitution gives us

$$F_i(j) = f_i(t_i(j)) = f_i(\bar{k});$$

thus, we have $(\bar{k}, F_i(j)) \in f_i$. Clearly, the term f_i is an element of the range of the sequence $f = (f_i \,|\, i \in \{1, \ldots, n\})$, with the consequence that $f_i \subseteq \bigcup \mathrm{ran}(f)$ $[= s]$ holds by virtue of (2.201). Then, the definition of a subset yields $(\bar{k}, F_i(j)) \in s$, which we may write as $F_i(j) = s_{\bar{k}}$, as desired. Since i and j were arbitrary, we may infer from this the truth of the universal sentence (11.23).

Next, we prove that s is a sequence in Y, that is, a sequence with codomain Y, which means by definition that the range of s is included in Y. For this task, we use the definition of a subset and prove the equivalent universal sentence

$$\forall y \, (y \in \mathrm{ran}(s) \Rightarrow y \in Y), \tag{11.48}$$

letting y be arbitrary in $\mathrm{ran}(s)$. Therefore, there exists by definition of a range a constant, say $\bar{k}$, such that $(\bar{k}, y) \in s$ holds. On the one hand, we may write this as $y = s_{\bar{k}}$, and on the other hand, $\bar{k} \in \{1, \ldots, \sum_{i=1}^{n} N_i\}$ $[= \mathrm{dom}(s)]$ follows to be true by definition of a domain. The latter finding further implies with (11.22) that there are constants, say I and J, such that $I \in \{1, \ldots, n\}$, $J \in \{1, \ldots, N_I\}$ and $[y =]$ $s_{\bar{k}} = F_I(J)$ are all true. The preceding equations give $y = F_I(J)$, which we may write also in the form $(J, y) \in F_I$, which shows that y is in the range of F_I. Because Y was in c) assumed to be a codomain of F_I, we have the inclusion $\mathrm{ran}(F_I) \subseteq Y$, so that the previously established $y \in \mathrm{ran}(F_I)$ implies the desired $y \in Y$ (by definition of a subset). Since y was arbitrary, we may therefore conclude that the universal sentence (11.48) holds, which in turn gives $\mathrm{ran}(s) \subseteq Y$, so that Y is indeed a codomain of s.

To prove e), we take arbitrary k and l in $\{1, \ldots, \sum_{i=1}^{n} N_i\}$ satisfying $k \neq l$.

We therefore obtain $k, l \in \bigcup \mathrm{ran}(G)$ with b), which we may write also as $k, l \in \bigcup_{i=1}^{n} G_i$, recalling from a) that G is a sequence with domain $\{1, \ldots, n\}$. According to the Characterization of the union of a family of sets, there is then a particular index $\bar{i} \in \{1, \ldots, n\}$ with $k \in G_{\bar{i}}$ as well as a particular index $\bar{I} \in \{1, \ldots, n\}$ with $l \in G_{\bar{I}}$. By definition of the function G, we therefore have

$$k \in \left\{ \left(\sum_{j=1}^{\bar{i}-1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{\bar{i}-1} N_j \right) + N_{\bar{i}} \right\} \quad [= \mathrm{dom}(t_{\bar{i}}^{-1})],$$
$$l \in \left\{ \left(\sum_{j=1}^{\bar{I}-1} N_j \right) + 1, \ldots, \left(\sum_{j=1}^{\bar{I}-1} N_j \right) + N_{\bar{I}} \right\} \quad [= \mathrm{dom}(t_{\bar{I}}^{-1})],$$

so that the value $\bar{j} = t_{\bar{i}}^{-1}(k)$ is evidently in the codomain $\{1, \ldots, N_{\bar{i}}\}$ of $t_{\bar{i}}^{-1}$ and the value $\bar{J} = t_{\bar{I}}^{-1}(l)$ in the codomain $\{1, \ldots, N_{\bar{I}}\}$ of $t_{\bar{I}}^{-1}$. Furthermore, the indexes $\bar{i}, \bar{I} \in \{1, \ldots, n\}$ give the terms $F_{\bar{i}}$ and $F_{\bar{I}}$ of the sequence F, where $F_{\bar{i}}$ is a sequence with domain $\{1, \ldots, N_{\bar{i}}\}$ and $F_{\bar{I}}$ a sequence with domain $\{1, \ldots, N_{\bar{I}}\}$. Thus, the element $\bar{j} \in \{1, \ldots, N_{\bar{i}}\}$ is associated with the value $F_{\bar{i}}(\bar{j})$, and the element $\bar{J} \in \{1, \ldots, N_{\bar{I}}\}$ is associated with the value $F_{\bar{I}}(\bar{J})$, for which two values we now obtain

$$F_{\bar{i}}(\bar{j}) = F_{\bar{i}}(t_{\bar{i}}^{-1}(k)) = (F_{\bar{i}} \circ t_{\bar{i}}^{-1})(k) = f_{\bar{i}}(k),$$
$$F_{\bar{I}}(\bar{J}) = F_{\bar{I}}(t_{\bar{I}}^{-1}(l)) = (F_{\bar{I}} \circ t_{\bar{I}}^{-1})(l) = f_{\bar{I}}(l),$$

applying substitutions, the notation for compositions, and c). Writing the resulting equations $F_{\bar{i}}(\bar{j}) = f_{\bar{i}}(k)$ and $F_{\bar{I}}(\bar{J}) = f_{\bar{I}}(l)$ in the form of

$$(k, F_{\bar{i}}(\bar{j})) \in f_{\bar{i}},$$
$$(l, F_{\bar{I}}(\bar{J})) \in f_{\bar{I}},$$

we see that the existential sentences

$$\exists i \, (i \in \{1, \ldots, n\} \wedge (k, F_{\bar{i}}(\bar{j})) \in f_i),$$
$$\exists i \, (i \in \{1, \ldots, n\} \wedge (k, F_{\bar{I}}(\bar{J})) \in f_i)$$

are both true. Consequently, the Characterization of the union of a family of sets and the definition of the sequence s in d) yields

$$(k, F_{\bar{i}}(\bar{j})) \in \bigcup_{i=1}^{n} f_i \quad \left[= \bigcup \mathrm{ran}(f) = s \right],$$
$$(l, F_{\bar{I}}(\bar{J})) \in \bigcup_{i=1}^{n} f_i \quad \left[= \bigcup \mathrm{ran}(f) = s \right],$$

so that we obtain $F_{\bar{i}}(\bar{j}) = s_k$ and $F_{\bar{I}}(\bar{J}) = s_l$ for the terms of the stacked sequence. To complete the proof of the desired existential sentence in (11.24), we now prove the disjunction

$$\bar{i} \neq \bar{I} \vee (\bar{i} = \bar{I} \wedge \bar{j} \neq \bar{J}) \tag{11.49}$$

by cases, based on the disjunction $\bar{i} = \bar{I} \vee \bar{i} \neq \bar{I}$ (which is true according to the Law of the Excluded Middle). The first case $\bar{i} = \bar{I}$ gives via substitution $t_{\bar{I}}^{-1}(l) = t_{\bar{i}}^{-1}(l)$. According to Proposition 5.115, the translating function $t_{\bar{i}}^{-1}$ is a bijection and thus an injection, so that the assumed $k \neq l$ implies

$$[\bar{j} =] \quad t_{\bar{i}}^{-1}(k) \neq t_{\bar{i}}^{-1}(l) \quad [= t_{\bar{I}}^{-1}(l) = \bar{J}]$$

with the Injection Criterion and therefore $\bar{j} \neq \bar{J}$. Thus, the conjunction $\bar{i} = \bar{I} \wedge \bar{j} \neq \bar{J}$ is true, and the disjunction (11.49) to be proven holds then as well. Since the second case $\bar{i} \neq \bar{I}$ immediately implies the truth of the disjunction (11.49), the proof by cases is complete. Having found particular constants $\bar{i}$, $\bar{I}$, $\bar{j}$ and $\bar{J}$ satisfying $\bar{i}, \bar{I} \in \{1, \ldots, n\}$, $\bar{j} \in \{1, \ldots, N_{\bar{i}}\}$, $s_k = F_{\bar{i}}(\bar{j})$, $\bar{J} \in \{1, \ldots, N_{\bar{I}}\}$, $s_l = F_{\bar{I}}(\bar{J})$ and (11.49), we thus proved the existential sentence in (11.24). Because k and l are arbitrary, we may infer from this the truth of the universal sentence (11.24).

Because n, N, Y and F were initially arbitrary, we may therefore conclude that the proposed sentences c) – e) are true, so that the proof of the theorem is complete. □

The next lemma generalizes Property 4 of a semiring of sets, making use of the preceding stacking theorem.

Lemma 11.7. *It is true for any set Ω, any semiring of sets $\mathcal{S}$ on Ω, any set $A \in \mathcal{S}$ and any sequence of sets $B : \{1, \ldots, m\} \to \mathcal{S}$ that the set difference of A and the union of B can be expressed as the union of some finite sequence of disjoint sets in $\mathcal{S}$, in the sense that*

$$\begin{aligned}
&\forall m\, (m \in \mathbb{N}_+ \Rightarrow \forall A, B\, ([A \in \mathcal{S} \wedge B : \{1, \ldots, m\} \to \mathcal{S}] \Rightarrow \exists n, C\, (n \in \mathbb{N}_+ \\
&\quad \wedge\ C : \{1, \ldots, n\} \to \mathcal{S} \wedge \forall i, j\, ([i, j \in \{1, \ldots, n\} \wedge i \neq j] \Rightarrow C_i \cap C_j = \emptyset) \\
&\quad \wedge\ A \setminus \bigcup \mathrm{ran}(B) = \bigcup \mathrm{ran}(C)). \qquad (11.50)
\end{aligned}$$

Proof. We take arbitrary sets Ω and $\mathcal{S}$ such that $\mathcal{S}$ is a semiring of sets on Ω and prove the universal sentence with respect to m by mathematical induction. Considering the base case ($m = 1$), we let A and B be arbitrary sets such that $A \in \mathcal{S}$ and $B : \{1\} \to \mathcal{S}$. Thus, the latter is the singleton $B = \{(1, B_1)\}$ according to Proposition 3.159, and this singleton is a surjection with $\mathrm{ran}(B) = \{B_1\}$ in view of Corollary 3.194. Since $B_1 \in \{B_1\}$

holds according to (2.153) and since the range $\{B_1\}$ of B is included in the codomain $\mathcal{S}$ of B, we obtain by definition of a subset $B_1 \in \mathcal{S}$. Furthermore, we obtain $\bigcup\{B_1\} = B_1$ with (2.199), so that substitution yields $\bigcup \mathrm{ran}(B) \in \mathcal{S}$. We thus have $A, \bigcup \mathrm{ran}(B) \in \mathcal{S}$, which implies with Property 4 of a semiring of sets that there are constants n and C such that n is a positive natural number and such that $C : \{1, \ldots, n\} \to \mathcal{S}$ is a sequence of pairwise disjoint sets with $A \setminus \bigcup \mathrm{ran}(B) = \bigcup \mathrm{ran}(C)$, proving the existential sentence in (11.50). As A and B are arbitrary sets, we may infer from this the truth of the base case.

Considering now the induction step, we let $m \in \mathbb{N}_+$ be arbitrary and make the induction assumption that, for all set $A \in \mathcal{S}$ and all sequences $B : \{1, \ldots, m\} \to \mathcal{S}$ there exists a sequence $C : \{1, \ldots, n\} \to \mathcal{S}$ of pairwise disjoint sets such that $A \setminus \bigcup \mathrm{ran}(B) = \bigcup \mathrm{ran}(C)$. We now prove that this assumption implies for any set $A \in \mathcal{S}$ and any sequence $B : \{1, \ldots, m+1\} \to \mathcal{S}$ the existence of a sequence $C : \{1, \ldots, n\} \to \mathcal{S}$ of pairwise disjoint sets with $A \setminus \bigcup \mathrm{ran}(B) = \bigcup \mathrm{ran}(C)$. For this purpose, we let A be an arbitrary set in the semiring of sets $\mathcal{S}$ and B an arbitrary set such that B is a sequence $(B_i \,|\, i \in \{1, \ldots, m+1\})$ of sets in $\mathcal{S}$. We then obtain the equations

$$\begin{aligned} A \setminus \bigcup \mathrm{ran}(B) &= A \setminus \bigcup_{i=1}^{m+1} B_i = A \setminus \left(\left[\bigcup_{i=1}^{m} B_i \right] \cup B_{m+1} \right) \\ &= \left(A \setminus \bigcup_{i=1}^{m} B_i \right) \setminus B_{m+1} \end{aligned}$$

by applying the notation for the union of a family of sets, then the Recursive evaluation of the union of a sequence of sets on an initial segment of $\mathbb{N}_+$, and then (2.225). In view of the induction assumption, there exists a particular sequence $\bar{C} : \{1, \ldots, \bar{n}\} \to \mathcal{S}$ of pairwise disjoint sets such that $A \setminus \bigcup_{i=1}^{m} B_i = \bigcup_{i=1}^{\bar{n}} \bar{C}_i$ holds, so that

$$A \setminus \bigcup \mathrm{ran}(B) = \left(\bigcup_{i=1}^{\bar{n}} \bar{C}_i \right) \setminus B_{m+1}.$$

An application of the Distributive Law for families of sets (3.823) yields then

$$A \setminus \bigcup \mathrm{ran}(B) = \bigcup_{i=1}^{\bar{n}} (\bar{C}_i \setminus B_{m+1}). \tag{11.51}$$

For every $i \in \{1, \ldots, \bar{n}\}$, the sets C_i and B_{m+1} are elements of $\mathcal{S}$, which is why there exists (for any $i \in \{1, \ldots, \bar{n}\}$) by virtue of Property 4 of a semiring of sets a positive natural number $N^{(i)}$ as well as a sequence

$D^{(i)} : \{1, \ldots, N^{(i)}\} \to \mathcal{S}$ of pairwise disjoint sets such that $\bar{C}_i \setminus B_{m+1} = \bigcup \operatorname{ran}(D^{(i)})$. We now intend to apply the Stacking of a finite sequence of finite sequences, which task we prepare by defining sequences $\bar{N} = (\bar{N}_i \,|\, i \in \{1, \ldots, \bar{n}\})$ and $\bar{D} = (\bar{D}_i \,|\, i \in \{1, \ldots, \bar{n}\})$ based on the preceding existential sentence. To do this, we need to apply the Axiom of Choice. We begin with the observation that the universal sentence

$$\begin{aligned} &\forall i\, (i \in \{1, \ldots, \bar{n}\} \Rightarrow \exists! \mathcal{D}^{(i)}\, \forall D\, (D \in \mathcal{D}^{(i)} \Leftrightarrow [D \in \mathcal{S}^{<\mathbb{N}_+} \\ &\quad \wedge\; \exists n^{(i)}, D^{(i)}\, (n^{(i)} \in \mathbb{N}_+ \;\wedge\; D^{(i)} : \{1, \ldots, n^{(i)}\} \to \mathcal{S} \\ &\quad \wedge\; \forall j, k\, ([j, k \in \{1, \ldots, n^{(i)}\} \;\wedge\; j \neq k] \Rightarrow D^{(i)}(j) \cap D^{(i)}(k) = \emptyset) \\ &\quad \wedge\; \bar{C}_i \setminus B_{m+1} = \bigcup \operatorname{ran}(D^{(i)}) \;\wedge\; D^{(i)} = D])) \end{aligned} \tag{11.52}$$

is true; indeed, letting $i \in \{1, \ldots, \bar{n}\}$ be arbitrary, the uniquely existential sentence in (11.52) follows to be true with the Axiom of Specification and with the Equality Criterion for sets, where the set $\mathcal{S}^{<\mathbb{N}_+}$ is defined according to Exercise 4.32. This universal sentence allows us to apply now Function definition of Replacement to establish a unique function/sequence $\mathcal{D}$ with domain $\{1, \ldots, \bar{n}\}$ satisfying

$$\forall i\, (i \in \{1, \ldots, \bar{n}\} \Rightarrow \mathcal{D}(i) = \mathcal{D}^{(i)}). \tag{11.53}$$

For this purpose, we prove the universal sentence

$$\forall i\, (i \in \{1, \ldots, \bar{n}\} \Rightarrow \exists! y\, (y = \mathcal{D}^{(i)})),$$

letting $i \in \{1, \ldots, \bar{n}\}$ be arbitrary. Consequently, the set $\mathcal{D}^{(i)}$ is uniquely specified according to (11.52). Consequently, the uniquely existential sentence $\exists! y\, (y = \mathcal{D}^{(i)})$ follows to be true with (1.109). Since i is arbitrary, we may therefore conclude that the universal sentence to be proven is indeed true, so that there is a unique function/sequence $\mathcal{D}$ with domain $\{1, \ldots, \bar{n}\}$ satisfying (11.53). Let us now check that $\emptyset \notin \operatorname{ran}(\mathcal{D})$ is true. The preceding negation is equivalent to

$$\forall Y\, (Y \in \operatorname{ran}(\mathcal{D}) \Rightarrow Y \neq \emptyset) \tag{11.54}$$

according to (2.5), which universal sentence we prove by letting Y be arbitrary and assuming $Y \in \operatorname{ran}(\mathcal{D})$ to be true. This assumption implies by definition of a range the existence of a particular constant $\bar{k}$ with $(\bar{k}, Y) \in \mathcal{D}$, which we may write as $Y = \mathcal{D}_{\bar{k}}$. Furthermore, $\bar{k}$ follows to be an element of the domain $\{1, \ldots, \bar{n}\}$ of $\mathcal{D}$, so that there exist (by virtue of Property 4 of a semiring of sets, as mentioned earlier) a particular constant $\bar{N}^{(\bar{k})} \in \mathbb{N}_+$ and a particular sequence $\bar{D}^{(\bar{k})} : \{1, \ldots, \bar{N}^{(\bar{k})}\} \to \mathcal{S}$ having pairwise disjoint terms and satisfying $\bar{C}_{\bar{k}} \setminus B_{m+1} = \bigcup \operatorname{ran}(\bar{D}^{(\bar{k})})$. Evidently then,

$\bar{D}^{(\bar{k})} \in \mathcal{S}^{<\mathbb{N}_+}$ and the equation $\bar{D}^{(\bar{k})} = \bar{D}^{(\bar{k})}$ hold. Thus, $\bar{D}^{(\bar{k})} \in \mathcal{S}^{<\mathbb{N}_+}$ and the existential sentence

$$\begin{aligned} &\exists n^{(i)}, D^{(i)}\,(n^{(i)} \in \mathbb{N}_+ \,\wedge\, D^{(i)} : \{1,\ldots,n^{(i)}\} \to \mathcal{S} \\ &\quad \wedge\, \forall j,k\,([j,k \in \{1,\ldots,n^{(i)}\} \,\wedge\, j \neq k] \Rightarrow D^{(i)}(j) \cap D^{(i)}(k) = \emptyset) \\ &\quad \wedge\, \bar{C}_{\bar{k}} \setminus B_{m+1} = \bigcup \mathrm{ran}(D^{(i)}) \,\wedge\, D^{(i)} = \bar{D}^{(\bar{k})}]) \end{aligned}$$

are both true, with the consequence that

$$\bar{D}^{(\bar{k})} \in \mathcal{D}^{(\bar{k})}\ [= \mathcal{D}_{\bar{k}} = Y]$$

in view of the equivalence in (11.52) and (11.53). The resulting $\bar{D}^{(\bar{k})} \in Y$ shows that there exists an element in Y, so that Y is clearly nonempty. Thus, the proof of the implication in (11.54) is complete, and as Y was arbitrary, we may infer from this the truth of the universal sentence (11.54) and then the truth of the equivalent negation $\emptyset \notin \mathrm{ran}(\mathcal{D})$. This negation in turn implies with the Axiom of Choice the existence of a particular function $\bar{h} : \mathrm{ran}(\mathcal{D}) \to \bigcup \mathrm{ran}(\mathcal{D})$ with

$$\forall K\,(K \in \mathrm{ran}(\mathcal{D}) \Rightarrow \bar{h}(K) \in K). \tag{11.55}$$

Consequently, the composition $\bar{D} = \bar{h} \circ \mathcal{D}$ turns out to be a function from $\{1,\ldots,\bar{n}\}$ to $\bigcup \mathrm{ran}(\mathcal{D})$ because of Proposition 3.178. To bring out more clearly the structure of this composition, we establish now

$$\begin{aligned} &\forall i\,(i \in \{1,\ldots,\bar{n}\} \Rightarrow \exists! n^{(i)}\,(n^{(i)} \in \mathbb{N}_+ \,\wedge\, \bar{D}_i : \{1,\ldots,n^{(i)}\} \to \mathcal{S} \\ &\quad \wedge\, \forall j,k\,([j,k \in \{1,\ldots,n^{(i)}\} \,\wedge\, j \neq k] \Rightarrow \bar{D}_i(j) \cap \bar{D}_i(k) = \emptyset) \\ &\quad \wedge\, \bar{C}_i \setminus B_{m+1} = \bigcup \mathrm{ran}(\bar{D}_i))), \end{aligned} \tag{11.56}$$

letting $i \in \{1,\ldots,\bar{n}\}$ be arbitrary. Thus, the associated value of $\bar{D}$ is

$$\bar{D}_i = (\bar{h} \circ \mathcal{D})_i = \bar{h}(\mathcal{D}_i).$$

Since $\mathcal{D}_i = \mathcal{D}(i)$ can be written as $(i, \mathcal{D}_i) \in \mathcal{D}$, which demonstrates that $\mathcal{D}_i \in \mathrm{ran}(\mathcal{D})$ is true, we obtain $\bar{h}(\mathcal{D}_i) \in \mathcal{D}_i$ with (11.55), and therefore after substitution $\bar{D}_i \in \mathcal{D}_i\ [= \mathcal{D}^{(i)}]$. Because of the equivalence in (11.52), the resulting $\bar{D}_i \in \mathcal{D}^{(i)}$ implies especially that there exist particular constants $\bar{n}^{(i)} \in \mathbb{N}_+$ and $\bar{D}^{(i)}$ such that $\bar{D}^{(i)} : \{1,\ldots,\bar{n}^{(i)}\} \to \mathcal{S}$, the universal sentence

$$\forall j,k\,([j,k \in \{1,\ldots,\bar{n}^{(i)}\} \,\wedge\, j \neq k] \Rightarrow \bar{D}^{(i)}(j) \cap \bar{D}^{(i)}(k) = \emptyset),$$

$\bar{C}_i \setminus B_{m+1} = \bigcup \mathrm{ran}(\bar{D}^{(i)})$, and $\bar{D}^{(i)} = \bar{D}_i$ hold. Applying now substitutions based on this equation to the previous findings gives the true conjunction of $\bar{n}^{(i)} \in \mathbb{N}_+$, $\bar{D}_i : \{1,\ldots,\bar{n}^{(i)}\} \to \mathcal{S}$,

$$\forall j,k\,([j,k \in \{1,\ldots,\bar{n}^{(i)}\} \,\wedge\, j \neq k] \Rightarrow \bar{D}_i(j) \cap \bar{D}_i(k) = \emptyset),$$

and $\bar{C}_i \setminus B_{m+1} = \bigcup \operatorname{ran}(\bar{D}_i)$. Having found a particular constant $\bar{n}^{(i)}$ satisfying this multiple conjunction, we thus established the existential part of the uniquely existential sentence in (11.56). Regarding the uniqueness part, we now take arbitrary constants $n_1^{(i)}$ and $n_2^{(i)}$ such that $n_1^{(i)}, n_2^{(i)} \in \mathbb{N}_+$,

$$\bar{D}_i : \{1, \ldots, n_1^{(i)}\} \to \mathcal{S},$$
$$\bar{D}_i : \{1, \ldots, n_2^{(i)}\} \to \mathcal{S},$$

the universal sentences

$$\forall j, k\,([j, k \in \{1, \ldots, n_1^{(i)}\} \wedge j \neq k] \Rightarrow \bar{D}_i(j) \cap \bar{D}_i(k) = \emptyset),$$
$$\forall j, k\,([j, k \in \{1, \ldots, n_2^{(i)}\} \wedge j \neq k] \Rightarrow \bar{D}_i(j) \cap \bar{D}_i(k) = \emptyset),$$

and $\bar{C}_i \setminus B_{m+1} = \bigcup \operatorname{ran}(\bar{D}_i)$ are satisfied by these constants. Here, we see that the domain of $\bar{D}_i$ is given by

$$\operatorname{dom}(\bar{D}_i) = \{1, \ldots, n_1^{(i)}\} = \{1, \ldots, n_2^{(i)}\}.$$

Then, due to $n_1^{(i)}, n_2^{(i)} \in \mathbb{N}_+$, the latter equation implies $n_1^{(i)} = n_2^{(i)}$ with (4.252), completing the proof of the uniqueness part and thus the proof of the uniquely existential sentence (11.56). Since i is arbitrary, we may therefore conclude that the universal sentence (11.56) is true. Consequently, we may apply Function definition by replacement to establish the unique function $\bar{N}$ with domain $\{1, \ldots, \bar{n}\}$, i.e. the sequence $\bar{N} = (\bar{N}_i \,|\, i \in \{1, \ldots, \bar{n}\})$, such that

$$\begin{aligned} &\forall i\,(i \in \{1, \ldots, \bar{n}\} \Rightarrow [\bar{N}_i \in \mathbb{N}_+ \wedge \bar{D}_i : \{1, \ldots, \bar{N}_i\} \to \mathcal{S} \\ &\quad \wedge \forall j, k\,([j, k \in \{1, \ldots, \bar{N}_i\} \wedge j \neq k] \Rightarrow \bar{D}_i(j) \cap \bar{D}_i(k) = \emptyset) \\ &\quad \wedge \bar{C}_i \setminus B_{m+1} = \bigcup \operatorname{ran}(\bar{D}_i)]). \end{aligned} \tag{11.57}$$

We now verify that $\mathbb{N}$ is a codomain of $\bar{N}$, i.e. that the inclusion $\operatorname{ran}(\bar{N}) \subseteq \mathbb{N}$ holds. Applying the definition of a subset and letting $y \in \operatorname{ran}(\bar{N})$ be true, there is then (by definition of a range) a particular constant $\bar{k}$ satisfying $(\bar{k}, y) \in \bar{N}$, so that we have $y = \bar{N}_{\bar{k}}$ and moreover (by definition of a domain) $\bar{k} \in \{1, \ldots, \bar{n}\}$ $[= \operatorname{dom}(\bar{N})]$. The latter further implies especially

$$[y =] \quad \bar{N}_{\bar{k}} \in \mathbb{N}_+ \quad [\subseteq \mathbb{N}]$$

with (11.57) and (2.308). Consequently, we obtain $y \in \mathbb{N}$ by using the definition of a subset, and as y was arbitrary, we may infer from this finding that $y \in \operatorname{ran}(\bar{N})$ implies $y \in \mathbb{N}$ for any y, so that $\operatorname{ran}(\bar{N}) \subseteq \mathbb{N}$ follows indeed to be true. Thus, the sequence $\bar{N} = (\bar{N}_i \,|\, i \in \{1, \ldots, \bar{n}\})$ constitutes a function $\bar{N} : \{1, \ldots, \bar{n}\} \to \mathbb{N}$. Furthermore, we see in light of (11.57) that the sequence $\bar{D} = (\bar{D}_i \,|\, i \in \{1, \ldots, \bar{n}\})$ satisfies $\bar{D}_i : \{1, \ldots, \bar{N}_i\} \to \mathcal{S}$ for all $i \in \{1, \ldots, \bar{n}\}$. We are now in a position to apply Stacking of a finite sequence of finite sequences to $\bar{D}$ to obtain the new stacked sequence $\bar{E} = (\bar{E}_k \,|\, k \in \{1, \ldots, \sum_{i=1}^{\bar{n}} \bar{N}_i\})$ in $\mathcal{S}$ whose terms satisfy

$$\begin{aligned} &\forall k \, (k \in \{1, \ldots, \sum_{i=1}^{\bar{n}} \bar{N}_i\} && (11.58)\\ &\quad \Rightarrow \exists i, j \, (i \in \{1, \ldots, \bar{n}\} \wedge j \in \{1, \ldots, \bar{N}_i\} \wedge \bar{E}_k = \bar{D}_i(j))), \end{aligned}$$

$$\begin{aligned} &\forall i, j \, ([i \in \{1, \ldots, \bar{n}\} \wedge j \in \{1, \ldots, \bar{N}_i\}] && (11.59)\\ &\quad \Rightarrow \exists k \, (k \in \{1, \ldots, \sum_{i=1}^{\bar{n}} \bar{N}_i\} \wedge \bar{D}_i(j) = \bar{E}_k)). \end{aligned}$$

Recalling the equation (11.51), we may now use (11.57) in connection with (11.58) to establish also the equation

$$\bigcup_{i=1}^{\bar{n}} (\bar{C}_i \setminus B_{m+1}) = \bigcup_{k=1}^{\sum_{i=1}^{\bar{n}} \bar{N}_i} \bar{E}_k. \tag{11.60}$$

To do this, we apply the Equality Criterion for sets and prove the equivalent universal sentence

$$\forall y \, (y \in \bigcup_{i=1}^{\bar{n}} (\bar{C}_i \setminus B_{m+1}) \Leftrightarrow y \in \bigcup_{k=1}^{\sum_{i=1}^{\bar{n}} \bar{N}_i} \bar{E}_k), \tag{11.61}$$

letting y be arbitrary. To prove the first part ('$\Rightarrow$') of the equivalence directly, we assume

$$y \in \bigcup_{i=1}^{\bar{n}} (\bar{C}_i \setminus B_{m+1}), \tag{11.62}$$

so that

$$y \in \bar{C}_I \setminus B_{m+1} \tag{11.63}$$

holds for a particular $I \in \{1, \ldots, \bar{n}\}$ (according to the Characterization of the union of a family of sets). Then, (11.57) yields in particular $\bar{C}_I \setminus B_{m+1} = \bigcup \mathrm{ran}(\bar{D}_I)$, so that substitution gives us

$$y \in \bigcup \mathrm{ran}(\bar{D}_I). \tag{11.64}$$

Because $\bar{D}_I$ is a sequence with domain $\{1,\ldots,\bar{N}_I\}$, we may write

$$y \in \bigcup_{j=1}^{\bar{N}_I} \bar{D}_I(j), \tag{11.65}$$

with the consequence that

$$y \in \bar{D}_I(J) \tag{11.66}$$

holds for a particular $J \in \{1,\ldots,\bar{N}_I\}$ (using again the Characterization of the union of a family of sets). Next, the conjunction of $I \in \{1,\ldots,\bar{n}\}$ and $J \in \{1,\ldots,\bar{N}_I\}$ implies with (11.59) the existence of a particular constant $K \in \{1,\ldots,\sum_{i=1}^{\bar{n}} \bar{N}_i\}$ with $\bar{D}_I(J) = \bar{E}_K$, so that substitution gives us

$$y \in \bar{E}_K. \tag{11.67}$$

Thus, there is a constant $k \in \{1,\ldots,\sum_{i=1}^{\bar{n}} \bar{N}_i\}$ such that $y \in \bar{E}_k$, and therefore we obtain (once again by means of the Characterization of the union of a family of sets) the desired

$$y \in \bigcup_{k=1}^{\sum_{i=1}^{\bar{n}} \bar{N}_i} \bar{E}_k. \tag{11.68}$$

We may prove the second part ('$\Leftarrow$') of the equivalence in (11.61) directly by applying analogous arguments as in the proof of the first part. Assuming (11.68) to be true, we evidently obtain (11.67) for some particular $K \in \{1,\ldots,\sum_{i=1}^{\bar{n}} \bar{N}_i\}$. The latter implies with (11.58) the existence of particular $I \in \{1,\ldots,\bar{n}\}$ and $J \in \{1,\ldots,\bar{N}_I\}$ satisfying $\bar{E}_K = \bar{D}_I(J)$, so that (11.67) yields (11.66). Thus, there is some $j \in \{1,\ldots,\bar{N}_I\}$ with $y \in \bar{D}_I(j)$, which existential sentence implies now (11.65), which we may write also as (11.64). Because $I \in \{1,\ldots,\bar{n}\}$ implies $\bar{C}_I \setminus B_{m+1} = \bigcup \mathrm{ran}(\bar{D}_I)$, we furthermore obtain (11.63). We thus see that there is some $i \in \{1,\ldots,\bar{n}\}$ with $y \in \bar{C}_i \setminus B_{m+1}$, and this existential sentence implies (11.62), as desired. Because y was arbitrary, we may infer from these findings the truth of (11.61), which universal sentence in turn implies the suggested equation (11.60). Combining now this equation with (11.51), we arrive at

$$A \setminus \bigcup \mathrm{ran}(B) = \bigcup_{k=1}^{\sum_{i=1}^{\bar{n}} \bar{N}_i} \bar{E}_k. \tag{11.69}$$

It remains for us to prove that the terms of the sequence $\bar{E} = (\bar{E}_k \,|\, k \in \{1,\ldots,\sum_{i=1}^{\bar{n}} \bar{N}_i\})$ are pairwise disjoint, that is,

$$\forall k,l\,([k,l \in \left\{1,\ldots,\sum_{i=1}^{\bar{n}} \bar{N}_i\right\} \wedge\ k \neq l] \Rightarrow \bar{E}_k \cap \bar{E}_l = \emptyset). \tag{11.70}$$

Letting k and l be arbitrary and assuming $k, l \in \{1, \ldots, \sum_{i=1}^{\bar{n}} \bar{N}_i\}$ as well as $k \neq l$ to be true, we prove $\bar{E}_k \cap \bar{E}_l = \emptyset$ via the definition of the empty set, by verifying

$$\forall y \, (y \notin \bar{E}_k \cap \bar{E}_l). \tag{11.71}$$

We take an arbitrary y and we prove the sentence $y \notin \bar{E}_k \cap \bar{E}_l$ by contradiction, assuming its negation to be true, so that the Double Negation Law yields $y \in \bar{E}_k \cap \bar{E}_l$, which in turn implies the truth of $y \in \bar{E}_k$ and of $y \in \bar{E}_l$ by definition of the intersection of two sets. Because $\bar{E}$ was defined by stacking the sequence $\bar{D}$, we may apply Theorem 11.6e) to infer from the assumption $k, l \in \{1, \ldots, \sum_{i=1}^{\bar{n}} \bar{N}_i\}$ the existence of particular indexes $\bar{i}, \bar{I} \in \{1, \ldots, \bar{n}\}$, $\bar{j} \in \{1, \ldots, \bar{N}_{\bar{i}}$ and $\bar{J} \in \{1, \ldots, \bar{N}_{\bar{I}}$ such that $\bar{E}_k = \bar{D}_{\bar{i}}(\bar{j})$, $\bar{E}_l = \bar{D}_{\bar{I}}(\bar{J})$ and the disjunction

$$\bar{i} \neq \bar{I} \,\vee\, (\bar{i} = \bar{I} \,\wedge\, \bar{j} \neq \bar{J}) \tag{11.72}$$

are true. Consequently, $y \in \bar{D}_{\bar{i}}(\bar{j})$ and $y \in \bar{D}_{\bar{I}}(\bar{J})$ hold, and we therefore see that the two existential sentences

$$\exists j \, (j \in \{1, \ldots, \bar{N}_{\bar{i}}\} \,\wedge\, y \in \bar{D}_{\bar{i}}(j))$$
$$\exists j \, (j \in \{1, \ldots, \bar{N}_{\bar{I}}\} \,\wedge\, y \in \bar{D}_{\bar{I}}(j))$$

are both true. We then obtain

$$y \in \bigcup_{j=1}^{\bar{N}_{\bar{i}}} \bar{D}_{\bar{i}}(j)) \quad \left[= \bigcup \operatorname{ran}(\bar{D}_{\bar{i}}) = \bar{C}_{\bar{i}} \setminus B_{m+1}\right],$$
$$y \in \bigcup_{j=1}^{\bar{N}_{\bar{I}}} \bar{D}_{\bar{I}}(j)) \quad \left[= \bigcup \operatorname{ran}(\bar{D}_{\bar{I}}) = \bar{C}_{\bar{I}} \setminus B_{m+1}\right],$$

and thus the true conjunction

$$y \in \bar{C}_{\bar{i}} \setminus B_{m+1} \,\wedge\, y \in \bar{C}_{\bar{I}} \setminus B_{m+1}.$$

In view of the definition of a set difference, this conjunction implies

$$(y \in \bar{C}_{\bar{i}} \,\wedge\, y \notin B_{m+1}) \,\wedge\, (y \in \bar{C}_{\bar{I}} \,\wedge\, y \notin B_{m+1}),$$

which in turn gives

$$(y \in \bar{C}_{\bar{i}} \,\wedge\, y \in \bar{C}_{\bar{I}}) \,\wedge\, (y \notin B_{m+1} \,\wedge\, y \notin B_{m+1}),$$

with the Associative and the Commutative Law for the conjunction. This conjunction implies in particular the truth of

$$y \in \bar{C}_{\bar{i}} \,\wedge\, y \in \bar{C}_{\bar{I}}$$

so that the definition of the intersection yields

$$y \in \bar{C}_{\bar{i}} \cap \bar{C}_{\bar{I}}. \tag{11.73}$$

We may now establish the contradiction $y \in \emptyset \wedge y \notin \emptyset$ by cases, based on the true disjunction (11.72). Here, $y \notin \emptyset$ is true by definition of the empty set, and we demonstrate in the following that $y \in \emptyset$ also holds in both cases. The first case $\bar{i} \neq \bar{I}$ implies $\bar{C}_{\bar{i}} \cap \bar{C}_{\bar{I}} = \emptyset$, recalling that we previously established $\bar{C} : \{1, \ldots, \bar{n}\} \to \mathcal{S}$ as a sequence of pairwise disjoint sets. Then, (11.73) gives the desired consequent $y \in \emptyset$ via substitution based on the preceding equation. The second case $\bar{i} = \bar{I} \wedge \bar{j} \neq \bar{J}$ implies especially $\bar{i} = \bar{I}$, and with this equation the previously found $y \in \bar{D}_{\bar{I}}(\bar{J})$ gives us $y \in \bar{D}_{\bar{i}}(\bar{J})$, besides $y \in \bar{D}_{\bar{i}}(\bar{j})$. By definition of the intersection of two sets, $y \in \bar{D}_{\bar{i}}(\bar{j}) \cap \bar{D}_{\bar{i}}(\bar{J})$ follows to be true. Let us now observe in light of (11.56) that $\bar{D}_{\bar{i}}$ is a sequence having pairwise disjoint terms, so that the current case assumption $\bar{j} \neq \bar{J}$ implies $\bar{D}_{\bar{i}}(\bar{j}) \cap \bar{D}_{\bar{i}}(\bar{J}) = \emptyset$. We therefore obtain via substitution $y \in \emptyset$ also in the second case, so that the previously mentioned contradiction holds in any case. This completes the proof of the negation in (11.71), and as y is arbitrary, we may infer from the truth of that negation the truth of the universal sentence (11.71). Consequently, the equation $\bar{E}_k \cap \bar{E}_l = \emptyset$ is indeed true; because k and l are arbitrary as well, we may now further conclude that (11.70) also holds. We thus established a particular sequence $\bar{E} : \{1, \ldots, \sum_{i=1}^{\bar{n}} \bar{N}_i\} \to \mathcal{S}$ satisfying (11.69) and (11.70), where $\sum_{i=1}^{\bar{n}} \bar{N}_i \in \mathbb{N}_+$ holds. This proves the existential sentence in (11.50), and since A and B were arbitrary, the universal sentence with respect to A and B follows to be also true.

Because m was arbitrary as well, we may further conclude that the induction step holds (besides the base case), so that the universal sentence (11.50) is true. Finally, the sets Ω and $\mathcal{S}$ were initially also arbitrary, so that we may infer from this finding the truth of the stated lemma. □

Collecting the Cartesian products formed by elements of two semirings of sets yields a new one.

Lemma 11.8. *For any sets Ω_1 and Ω_2, any semiring of sets $\mathcal{S}_1$ on Ω_1 and any semiring of sets $\mathcal{S}_2$ on Ω_2, it is true that there exists a unique set system $\mathcal{S}_1 \otimes \mathcal{S}_2$ consisting of all Cartesian products $A \times B$ with $A \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$, in the sense that*

$$\forall Y \, (Y \in \mathcal{S}_1 \otimes \mathcal{S}_2 \Leftrightarrow \exists A, B \, (A \in \mathcal{S}_1 \wedge B \in \mathcal{S}_2 \wedge Y = A \times B)). \tag{11.74}$$

This set system $\mathcal{S}_1 \otimes \mathcal{S}_2$ is a semiring of sets on $\Omega_1 \times \Omega_2$.

Proof. We let Ω_1, Ω_2, $\mathcal{S}_1$ and $\mathcal{S}_1$ be arbitrary sets, assuming $\mathcal{S}_1$ to be a semiring of sets on Ω_1 and $\mathcal{S}_2$ to be a semiring of sets on Ω_2. We may then use the Axiom of Specification in connection with the Equality Criterion for sets to establish the unique existence of a set $\mathcal{S}_1 \otimes \mathcal{S}_2$ such that

$$\begin{aligned}&\forall Y\,(Y \in \mathcal{S}_1 \otimes \mathcal{S}_2 \Leftrightarrow [Y \in \mathcal{P}(\Omega_1 \times \Omega_2)\\ &\quad \wedge\ \exists A, B\,(A \in \mathcal{S}_1\ \wedge\ B \in \mathcal{S}_2\ \wedge\ Y = A \times B)]),\end{aligned} \tag{11.75}$$

We now demonstrate that the set system $\mathcal{S}_1 \otimes \mathcal{S}_2$ satisfies also (11.74). Letting Y be arbitrary and assuming first $Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$ to be true, the desired consequent

$$\exists A, B\,(A \in \mathcal{S}_1\ \wedge\ B \in \mathcal{S}_2\ \wedge\ Y = A \times B) \tag{11.76}$$

of the first part ('$\Rightarrow$') of the equivalence in (11.74) follows in particular to be true with (11.75). Assuming now conversely that existential sentence to be true, there are then particular sets $\bar{A} \in \mathcal{S}_1$ and $\bar{B} \in \mathcal{S}_2$ with $Y = \bar{A} \times \bar{B}$. According to Property 1 of a semiring of sets, $\bar{A} \in \mathcal{S}_1$ and $\bar{B} \in \mathcal{S}_2$ imply $\bar{A} \in \mathcal{P}(\Omega_1)$ as well as $\bar{B} \in \mathcal{P}(\Omega_2)$ with the definition of a subset, consequently $\bar{A} \subseteq \Omega_1$ as well as $\bar{B} \subseteq \Omega_2$ with the definition of a power set. Then, the conjunction of these two inclusions implies $\bar{A} \times \bar{B} \subseteq \Omega_1 \times \Omega_2$ with (3.40), and therefore $\bar{A} \times \bar{B} \in \mathcal{P}(\Omega_1 \times \Omega_2)$ (using again the definition of a power set). The conjunction of this finding and the assumed existential sentence (11.76) in turn implies $Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$ with (11.75), so that the second part ('$\Leftarrow$') of the equivalence to be proven also holds. Since Y is arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (11.74).

Let us now observe in light of (11.75) that $Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$ implies especially $Y \in \mathcal{P}(\Omega_1 \times \Omega_2)$ for any Y, so that

$$\mathcal{S}_1 \otimes \mathcal{S}_2 \subseteq \mathcal{P}(\Omega_1 \times \Omega_2)$$

follows to be true by definition of a subset. Thus, $\mathcal{S}_1 \otimes \mathcal{S}_2$ satisfies Property 1 of a semiring of sets on $\Omega_1 \times \Omega_2$.

Next, we notice on the one hand that $\emptyset \in \mathcal{S}_1$ and $\emptyset \in \mathcal{S}_2$ are both due to Property 2 of a semiring of sets with respect to $\mathcal{S}_1$ and $\mathcal{S}_2$. On the other hand, $\emptyset = \emptyset \times \emptyset$ holds according to (3.27). Thus, there exist sets A and B satisfying $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$ and $\emptyset = A \times B$, so that $\emptyset \in \mathcal{S}_1 \otimes \mathcal{S}_2$ follows to be true with (11.74). This shows that Property 2 of a semiring of sets is also satisfied by $\mathcal{S}_1 \otimes \mathcal{S}_2$.

Regarding Property 3, we establish now the truth of

$$\forall X, Y\,(X, Y \in \mathcal{S}_1 \otimes \mathcal{S}_2 \Rightarrow X \cap Y \in \mathcal{S}_1 \otimes \mathcal{S}_2), \tag{11.77}$$

letting $X, Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$ be arbitrary. Here, $X \in \mathcal{S}_1 \otimes \mathcal{S}_2$ implies with (11.74) the existence of particular sets $A_1 \in \mathcal{S}_1$ and $B_1 \in \mathcal{S}_2$ with $X = A_1 \times B_1$, and for the same reason $Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$ implies the existence of particular sets $A_2 \in \mathcal{S}_1$ and $B_2 \in \mathcal{S}_2$ with $Y = A_2 \times B_2$. We now use the fact that

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2) \tag{11.78}$$

due to (3.63), where the previously found $A_1, A_2 \in \mathcal{S}_1$ and $B_1, B_2 \in \mathcal{S}_2$ imply $A_1 \cap A_2 \in \mathcal{S}_1$ as well as $B_1 \cap B_2 \in \mathcal{S}_2$ with Property 3 of a semiring of sets (with respect to $\mathcal{S}_1$ and $\mathcal{S}_2$). These findings and the equation (11.78) demonstrate that there are sets A, B such that $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$ and

$$(A_1 \times B_1) \cap (A_2 \times B_2) = A \times B$$

hold, so that (11.74) yields

$$(A_1 \times B_1) \cap (A_2 \times B_2) \in \mathcal{S}_1 \otimes \mathcal{S}_2.$$

Then, substitutions based on the previously obtained equations $X = A_1 \times B_1$ and $Y = A_2 \times B_2$ give $X \cap Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$, proving the implication in (11.77). Since X and Y were arbitrary, we may infer from the truth of this implication the truth of the universal sentence (11.77), which means that the set system $\mathcal{S}_1 \otimes \mathcal{S}_2$ has Property 3 of a semiring of sets.

Finally, concerning Property 4 of a semiring of sets, we verify

$$\begin{aligned}
&\forall X, Y\, (X, Y \in \mathcal{S}_1 \otimes \mathcal{S}_2 \Rightarrow \exists n, C\, (n \in \mathbb{N}_+ \wedge C : \{1, \ldots, n\} \to \mathcal{S}_1 \otimes \mathcal{S}_2 \\
&\quad \wedge \forall i, j\, ([i, j \in \{1, \ldots, n\} \wedge i \neq j] \Rightarrow C_i \cap C_j = \emptyset) \\
&\quad \wedge X \setminus Y = \bigcup \operatorname{ran}(C))).
\end{aligned} \tag{11.79}$$

To do this, we take arbitrary sets $X, Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$, so that there are in view of (11.74) particular sets $A_1, A_2 \in \mathcal{S}_1$ and $B_1, B_2 \in \mathcal{S}_2$ with $X = A_1 \times B_1$ and $Y = A_2 \times B_2$. Since Property 4 of a semiring of sets applies to both $\mathcal{S}_1$ and $\mathcal{S}_2$, there are then particular positive natural natural numbers n_1 and n_2 as well as particular sequences $C_1 : \{1, \ldots, n_1\} \to \mathcal{S}_1$ and $C_2 : \{1, \ldots, n_2\} \to \mathcal{S}_2$ satisfying the universal sentences

$$\forall i, j\, ([i, j \in \{1, \ldots, n_1\} \wedge i \neq j] \Rightarrow C_1(i) \cap C_1(j) = \emptyset) \tag{11.80}$$

$$\forall i, j\, ([i, j \in \{1, \ldots, n_2\} \wedge i \neq j] \Rightarrow C_2(i) \cap C_2(j) = \emptyset) \tag{11.81}$$

and the equations

$$A_1 \setminus A_2 = \bigcup \operatorname{ran}(C_1) \quad \left[= \bigcup_{j=1}^{n_1} C_1(j)\right] \tag{11.82}$$

$$B_1 \setminus B_2 = \bigcup \operatorname{ran}(C_2) \quad \left[= \bigcup_{j=1}^{n_2} C_2(j)\right]. \tag{11.83}$$

Because $A_1, A_2 \in \mathcal{S}_1$ and $B_1, B_2 \in \mathcal{S}_2$ imply $A_1, A_2 \in \mathcal{P}(\Omega)$ and $B_1, B_2 \in \mathcal{P}(\Omega)$ with Property 1 of a semiring of sets (applied to $\mathcal{S}_1$ and $\mathcal{S}_2$) in connection with the definition of a subset, we have that A_1, A_2, B_1 and B_2 are all subsets of Ω (by definition of a power set). We may therefore utilize (3.69) to infer from this the truth of the equation

$$(A_1 \times B_1) \setminus (A_2 \times B_2) = [A_1 \times (B_1 \setminus B_2)] \cup [(A_1 \setminus A_2) \times (B_1 \cap B_2)], \tag{11.84}$$

where

$$[A_1 \times (B_1 \setminus B_2)] \cap [(A_1 \setminus A_2) \times (B_1 \cap B_2)] = \emptyset \tag{11.85}$$

holds due to (3.68). Substitutions based on the equations $X = A_1 \times B_1$ and $Y = A_2 \times B_2$, (11.82) – (11.83) and the Commutative Law for the union & intersection of two sets allow us to write (11.84) – (11.84) in the form

$$X \setminus Y = \left[\left(\bigcup_{j=1}^{n_1} C_1(j)\right) \times (B_1 \cap B_2)\right] \cup \left[A_1 \times \left(\bigcup_{j=1}^{n_2} C_2(j)\right)\right],$$

$$\emptyset = \left[\left(\bigcup_{j=1}^{n_1} C_1(j)\right) \times (B_1 \cap B_2)\right] \cap \left[A_1 \times \left(\bigcup_{j=1}^{n_2} C_2(j)\right)\right],$$

and then also as

$$X \setminus Y = \left(\bigcup_{j=1}^{n_1} [C_1(j) \times (B_1 \cap B_2)]\right) \cup \left(\bigcup_{j=1}^{n_2} [A_1 \times C_2(j)]\right), \tag{11.86}$$

$$\emptyset = \left(\bigcup_{j=1}^{n_1} [C_1(j) \times (B_1 \cap B_2)]\right) \cap \left(\bigcup_{j=1}^{n_2} [A_1 \times C_2(j)]\right), \tag{11.87}$$

by using (3.824) – (3.825). We thus have the two sequences

$$D_1 = (C_1(j) \times [B_1 \cap B_2] \,|\, j \in \{1, \ldots, n_1\})$$
$$D_2 = (A_1 \times C_2(j) \,|\, j \in \{1, \ldots, n_2\})$$

with $n_1, n_2 \in \mathbb{N}_+$, which allow us to rewrite (11.86) – (11.87) shorter as

$$X \setminus Y = \left(\bigcup_{j=1}^{n_1} D_1(j)\right) \cup \left(\bigcup_{j=1}^{n_2} D_2(j)\right), \tag{11.88}$$

$$\emptyset = \left(\bigcup_{j=1}^{n_1} D_1(j)\right) \cap \left(\bigcup_{j=1}^{n_2} D_2(j)\right). \tag{11.89}$$

Let us now apply Proposition 4.85 to define on the one hand the sequence $D = (D_i \,|\, i \in \{1,2\})$ having the terms D_1 and D_2, and on the other hand the sequence $n = (n_i \,|\, i \in \{1,2\})$ in $\mathbb{N}_+$ given by the terms $n_1, n_2 \in \mathbb{N}_+$. Because $\mathbb{N}_+$ is codomain of n, the inclusion $\mathrm{ran}(n) \subseteq \mathbb{N}_+$ holds. Since the inclusion $\mathbb{N}_+ \subseteq \mathbb{N}$ is true as well according to (2.308), we obtain also the inclusion $\mathrm{ran}(n) \subseteq \mathbb{N}$ by means of (2.13). We thus have $n : \{1,2\} \to \mathbb{N}$. Moreover, we can show that the sequence $D = (D_i \,|\, i \in \{1,2\})$ satisfies $D_i : \{1, \ldots, n_i\} \to \mathcal{S}_1 \otimes \mathcal{S}_2$ for all $i \in \{1,2\}$, which we do by demonstrating that $\mathcal{S}_1 \otimes \mathcal{S}_2$ is a codomain of D_i for all $i \in \{1,2\}$. To do this, we prove the universal sentence

$$\forall i \, (i \in \{1,2\} \Rightarrow \mathrm{ran}(D_i) \subseteq \mathcal{S}_1 \otimes \mathcal{S}_2). \tag{11.90}$$

We let i be arbitrary, assume $i \in \{1,2\}$ to be true, so that the definition of a pair gives the true disjunction $i = 1 \vee i = 2$, and we establish the consequent $\mathrm{ran}(D_i) \subseteq \mathcal{S}_1 \otimes \mathcal{S}_2$ by means of the definition of a subset. For this purpose, we prove the universal sentence

$$\forall Y \, (Y \in \mathrm{ran}(D_i) \Rightarrow Y \in \mathcal{S}_1 \otimes \mathcal{S}_2), \tag{11.91}$$

letting Y be arbitrary and assuming $Y \in \mathrm{ran}(D_i)$ to be true. Then, the definition of a range yields a particular constant $\bar{j}$ such that $(\bar{j}, Y) \in D_i$ holds, which we may write as $Y = D_i(\bar{j})$. Using now the preceding disjunction for a proof of $Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$ by cases, the first case $i = 1$ gives via substitution and by definition of D_1

$$Y = D_1(\bar{j}) = C_1(\bar{j}) \times [B_1 \cap B_2]. \tag{11.92}$$

Here, we see that the term $C_1(\bar{j})$, for which we may write $(\bar{j}, C_1(\bar{j})) \in C_1$, is evidently contained in the range of C_1, which range is in turn included in the codomain $\mathcal{S}_1$ of C_1. Consequently, $C_1(\bar{j}) \in \mathcal{S}_1$ holds according to (2.13), and we see in light of Property 3 of a semiring of sets (applied in connection with $\mathcal{S}_2$) that $B_1 \cap B_2 \in \mathcal{S}_2$ is implied by $B_1, B_2 \in \mathcal{S}_2$. Thus, in view of (11.92), there exist sets A and B with $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$ and $Y = A \times B$, so that $Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$ follows to be true with (11.74), as desired. On the other hand, in the second case $i = 2$, the previously established $Y = D_i(\bar{j})$ yields via substitution and by definition of D_2

$$Y = D_2(\bar{j}) = A_1 \times C_2(\bar{j}), \tag{11.93}$$

where $A_1 \in \mathcal{S}_1$ holds. Furthermore, the term $C_2(\bar{j})$, for which we may write $(\bar{j}, C_2(\bar{j})) \in C_2$, is thus element of the range of C_2, which range is itself included in the codomain $\mathcal{S}_2$ of C_2. Therefore, $C_2(\bar{j}) \in \mathcal{S}_2$ is true again due to (2.13). In light of these findings and (11.93), we notice that there are again sets A and B with $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$ and $Y = A \times B$, with the evident consequence that the desired $Y \in \mathcal{S}_1 \otimes \mathcal{S}_2$ holds also in the current second case. Because Y was arbitrary, we may infer from this finding the truth of the universal sentence (11.91) and thus the truth of the inclusion $\mathrm{ran}(D_i) \subseteq \mathcal{S}_1 \otimes \mathcal{S}_2$. Since i was also arbitrary, we may now further conclude that the universal sentence (11.90) is true, so that the sequence $D = (D_i \,|\, i \in \{1,2\})$ satisfies indeed $D_i : \{1,\dots,n_i\} \to \mathcal{S}_1 \otimes \mathcal{S}_2$ for all $i \in \{1,2\}$.

We are therefore in a position to apply Stacking of a finite sequence of finite sequences to establish the sequence E in $\mathcal{S}_1 \otimes \mathcal{S}_2$ with domain $\{1,\dots,\sum_{i=1}^{n} n_i\}$ whose terms satisfy the two universal sentences

$$\begin{aligned}&\forall k\,(k \in \{1,\dots,\sum_{i=1}^{2} n_i\}\\ &\quad\Rightarrow \exists i,j\,(i \in \{1,\dots,2\} \wedge j \in \{1,\dots,n_i\} \wedge E_k = D_i(j)))\end{aligned} \tag{11.94}$$

and

$$\begin{aligned}&\forall i,j\,([i \in \{1,\dots,2\} \wedge j \in \{1,\dots,n_i\}]\\ &\quad\Rightarrow \exists k\,(k \in \{1,\dots,\sum_{i=1}^{2} n_i\} \wedge D_i(j) = E_k)).\end{aligned} \tag{11.95}$$

Here, we may also write $\{1,\dots,2\} = \{1,2\}$ according to (4.242) and $\sum_{i=1}^{2} n_i = n_1 + n_2$ with (5.413). Now, we may use these equations in connection with the Equality Criterion for sets to prove the equation

$$\left(\bigcup_{j=1}^{n_1} D_1(j)\right) \cup \left(\bigcup_{j=1}^{n_2} D_2(j)\right) = \bigcup_{k=1}^{n_1+n_2} E_k. \tag{11.96}$$

For this purpose, we establish the equivalent universal sentence

$$\forall y \,(y \in \left(\bigcup_{j=1}^{n_1} D_1(j)\right) \cup \left(\bigcup_{j=1}^{n_2} D_2(j)\right) \Leftrightarrow y \in \bigcup_{k=1}^{\bar{n}_1+\bar{n}_2} E_k), \tag{11.97}$$

letting y be arbitrary. We prove the first part ('$\Rightarrow$') of the equivalence directly, assuming

$$y \in \left(\bigcup_{j=1}^{n_1} D_1(j)\right) \cup \left(\bigcup_{j=1}^{n_2} D_2(j)\right), \tag{11.98}$$

to be true, so that the disjunction

$$y \in \bigcup_{j=1}^{n_1} D_1(j) \vee y \in \bigcup_{j=1}^{n_2} D_2(j) \tag{11.99}$$

holds by definition of the union of two sets. In the following, we utilize this true disjunction to prove the desired consequent $y \in \bigcup_{k=1}^{\bar{n}_1+\bar{n}_2} E_k$ by cases. In the first case $y \in \bigcup_{j=1}^{n_1} D_1(j)$, the Characterization of the union of a family of sets shows that there exists a particular constant $J \in \{1, \ldots, n_1\}$ such that $y \in D_1(J)$ holds. Since $1 \in \{1, 2\}$ is evidently also true, we obtain by means of (11.95) a particular constant $K \in \{1, \ldots, n_1 + n_2\}$ satisfying $D_1(J) = E_K$. With this equation, $y \in D_1(J)$ implies through substitution $y \in E_K$, which finding shows that there is some constant $k \in \{1, \ldots, n_1 + n_2\}$ such that $y \in E_k$. Therefore, the Characterization of the union of a family of sets yields the desired

$$y \in \bigcup_{k=1}^{n_1+n_2} E_k. \tag{11.100}$$

In analogy to the first case, we have in the second case $y \in \bigcup_{j=1}^{n_2} D_2(j)$ that there is a particular index $\bar{J} \in \{1, \ldots, n_2\}$ such that $y \in D_2(\bar{J})$ is true. Furthermore, $2 \in \{1, 2\}$ holds evidently, so that (11.95) gives a particular index $\bar{K} \in \{1, \ldots, n_1 + n_2\}$ with $D_2(\bar{J}) = E_{\bar{K}}$. Therefore, substitution results in $y \in E_{\bar{K}}$, so that there exists some index $k \in \{1, \ldots, n_1 + n_2\}$ satisfying $y \in E_k$. We thus arrive again at the desired consequent (11.100) of the first part of the equivalence in (11.97).

We may also prove the second part ('$\Leftarrow$') of the equivalence in (11.97) directly, assuming now (11.100) to be true and essentially reversing the arguments used within the proof of the first part. Thus, the preceding assumption implies $y \in E_K$ for a particular constant $K \in \{1, \ldots, n_1 + n_2\}$, which in turn implies according to (11.94) that there are further particular constants $I \in \{1,2\}$ and $J \in \{1,\ldots,n_I\}$ such that $E_K = D_I(J)$ holds. With this equation, $y \in E_K$ gives us now $y \in D_I(J)$. Furthermore, the definition of a pair yields the true disjunction $I = 1 \vee I = 2$, based on which we prove the disjunction (11.99) by cases. Substitution in $y \in D_I(J)$ and $J \in \{1,\ldots,n_I\}$ based on the first case $I = 1$ gives $y \in \bar{D}_1(J)$ as well as $J \in \{1,\ldots,n_1\}$, with the consequence that $y \in \bigcup_{j=1}^{n_1} D_1(j)$ is true; then, the desired disjunction (11.99) holds as well. Similarly, substitution based on the second case $I = 2$ yields $y \in D_2(J)$ as well as $J \in \{1,\ldots,n_2\}$, so that $y \in \bigcup_{j=1}^{n_2} D_2(j)$; therefore, the disjunction (11.99) turns out to be true again, completing the proof by cases. Having established that disjunction, we now obtain (11.98) with the definition of the union of two sets, completing also the proof of the second part of the equivalence in (11.97). Because y is arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (11.97), which in turn implies the truth of the proposed equation (11.96). Combining this equation with (11.88) yields then

$$X \setminus Y = \bigcup_{k=1}^{n_1+n_2} E_k \quad [= \mathrm{ran}(E)]. \tag{11.101}$$

Our next task is to prove that the terms of $E = (E_k \,|\, k \in \{1,\ldots,n_1+n_2\})$ are pairwise disjoint, i.e. that the sequence E has the definite property

$$\forall k,l\,([k,l \in \{1,\ldots,n_1+n_2\} \wedge k \neq l] \Rightarrow E_k \cap E_l = \emptyset). \tag{11.102}$$

Letting k and l be arbitrary such that $k,l \in \{1,\ldots,n_1+n_2\}$ and $k \neq l$ are true, we apply now the definition of the empty set to establish the consequent $E_k \cap E_l = \emptyset$. For this purpose, we prove the universal sentence

$$\forall x\,(x \notin E_k \cap E_l), \tag{11.103}$$

letting x be arbitrary. Next, we prove the negation $x \notin E_k \cap E_l$ by contradiction, by demonstrating the truth of

$$x \in \emptyset \wedge x \notin \emptyset. \tag{11.104}$$

Here, $x \notin \emptyset$ is true by definition of the empty set, so that we are required to establish merely the truth of the first part $x \in \emptyset$. To begin with the proof by contradiction, we assume the negation of the negation to be proven to be true, so that the Double negation Law gives us the true sentence $x \in E_k \cap E_l$. Therefore, $x \in E_k$ and $x \in E_l$ follow to be both true with the definition of the intersection of two sets. Then, due to Part e) of the Stacking of a finite sequence of finite sequences, the assumed $k, l \in \{1, \ldots, n_1 + n_2\}$ implies that there exist particular constants i_1, i_2, j_1, j_2 that satisfy simultaneously $i_1, i_2 \in \{1, 2\}$, $j_1 \in \{1, \ldots, n_{i_1}\}$, $E_k = D_{i_1}(j_1)$, $j_2 \in \{1, \ldots, n_{i_2}\}$, $E_l = D_{i_2}(j_2)$ and

$$i_1 \neq i_2 \vee (i_1 = i_2 \wedge j_1 \neq j_2). \tag{11.105}$$

Applying substitutions based on $E_k = D_{i_1}(j_1)$ and $E_l = D_{i_2}(j_2)$ to, respectively, $x \in E_k$ and $x \in E_l$, we get $x \in D_{i_1}(j_1)$ as well as $x \in D_{i_2}(j_2)$. These findings demonstrate in view of the previously established $j_1 \in \{1, \ldots, n_{i_1}\}$ and $j_2 \in \{1, \ldots, n_{i_2}\}$ firstly that $x \in D_{i_1}(j)$ holds for some $j \in \{1, \ldots, n_{i_1}\}$, and secondly that $x \in D_{i_2}(j)$ is true for some $j \in \{1, \ldots, n_{i_2}\}$. These two existential sentences in turn imply with the Characterization of the union of a family of sets $x \in \bigcup_{j=1}^{n_{i_1}} D_{i_1}(j)$ as well as $x \in \bigcup_{j=1}^{n_{i_2}} D_{i_2}(j)$, so that we obtain by means of the definition of the intersection of two sets

$$x \in \left(\bigcup_{j=1}^{n_{i_1}} D_{i_1}(j) \right) \cap \left(\bigcup_{j=1}^{n_{i_2}} D_{i_2}(j) \right). \tag{11.106}$$

After this preparatory work, we now apply a proof by cases based on the true disjunction (11.105) in order to establish the lacking first part $x \in \emptyset$ of the contradiction (11.104). In the first case $i_1 \neq i_2$, we notice in light of the previously found $i_1, i_2 \in \{1, 2\}$ and the definition of a pair that the disjunctions

$$i_1 = 1 \vee i_1 = 2 \tag{11.107}$$

$$i_2 = 1 \vee i_2 = 2 \tag{11.108}$$

are both true. We apply now (within the current first case) a proof by (sub-)cases based on the first of these disjunctions to establish $x \in \emptyset$. In the first subcase $i_1 = 1$, we first prove $i_2 = 2$ by contradiction, by establishing the truth of the conjunction $i_1 = i_2 \wedge i_1 \neq i_2$, whose second part is true according to the current case assumption $i_1 \neq i_2$; assuming the negation $\neg i_2 = 2$ to be true, the first part $i_2 = 1$ of the true disjunction (11.108) must then be true, so that substitution based on the current sub-case assumption $i_1 = 1$ yields $i_1 = i_2$, completing the proof of the contradiction $i_1 = i_2 \wedge i_1 \neq i_2$ and thus the proof of $i_2 = 2$. Then, the truth of $i_1 = 1$ and $i_2 = 2$ gives after substitutions $n_{i_1} = n_1$ as well as $n_{i_2} = n_2$, which equations in turn imply via substitutions in (11.106)

$$x \in \left(\bigcup_{j=1}^{n_1} D_1(j) \right) \cap \left(\bigcup_{j=1}^{n_2} D_2(j) \right), \tag{11.109}$$

Recalling the disjointness property (11.89), we finally arrive through another substitution at the desired $x \in \emptyset$. Applying a similar proof by contradiction as in the first subcase, the second subcase $i_1 = 2$ leads to $i_2 = 1$, since assuming $\neg i_2 = 1$ to be true implies now the truth of the second part $i_2 = 2$ of the disjunction (11.108) and therefore the truth of $i_1 = i_2$ (as in the first subcase), which equation contradicts the current case assumption $i_1 \neq i_2$. Thus, the equations $i_1 = 2$ and $i_2 = 1$ are both true, and substitutions based on them give the equations $n_{i_1} = n_2$ and $n_{i_2} = n_1$, and (11.106) implies therefore

$$x \in \left(\bigcup_{j=1}^{n_2} D_2(j) \right) \cap \left(\bigcup_{j=1}^{n_1} D_1(j) \right).$$

Because of the Commutative Law of the intersection of two sets, we can write the preceding findings equivalently as (11.109), so that a substitution based on (11.89) yields $x \in \emptyset$ also for the second subcase, completing the proof of $x \in \emptyset$ for the first case.

In the second case, we assume $i_1 = i_2 \wedge j_1 \neq j_2$ to be true, so that the previously established $x \in D_{i_1}(j_1)$ and $x \in D_{i_2}(j_2)$ imply first $x \in D_{i_1}(j_1) \wedge x \in D_{i_1}(j_2)$ and then (by definition of the intersection of two sets) $x \in D_{i_1}(j_1) \cap D_{i_1}(j_2)$. Let us consider again the true disjunction (11.107) and prove $D_{i_1}(j_1) \cap D_{i_1}(j_2) = \emptyset$ by corresponding (sub)-cases. On the one hand, if $i_1 = 1$ holds (besides the assumed $i_1 = i_2$), then $j_1 \in \{1, \ldots, n_{i_1}\}$ and $j_2 \in \{1, \ldots, n_{i_2}\}$ give $j_1 \in \{1, \ldots, n_1\}$ as well as $j_2 \in \{1, \ldots, n_1\}$, and we obtain the equations

$$\begin{aligned}
D_{i_1}(j_1) \cap D_{i_1}(j_2) &= D_1(j_1) \cap D_1(j_2) \\
&= (C_1(j_1) \times [B_1 \cap B_2]) \cap (C_1(j_2) \times [B_1 \cap B_2]) \\
&= (C_1(j_1) \cap C_1(j_2)) \times (B_1 \cap B_2) \\
&= \emptyset \times (B_1 \cap B_2) \\
&= \emptyset
\end{aligned}$$

by applying substitution, the definition of the sequence D_1, (3.66), then (11.80) based on the previous finding $j_1, j_2 \in \{1, \ldots, n_1\}$ alongside the assumption $j_1 \neq j_2$, and finally (3.27). On the other hand, if $i_1 = 2$ is true, then $j_1 \in \{1, \ldots, n_{i_1}\}$ and $j_2 \in \{1, \ldots, n_{i_2}\}$ imply $j_1 \in \{1, \ldots, n_2\}$ as well as $j_2 \in \{1, \ldots, n_2\}$, and then the truth of equations

$$\begin{aligned} D_{i_1}(j_1) \cap D_{i_1}(j_2) &= D_2(j_1) \cap D_2(j_2) \\ &= (A_1 \times C_2(j_1)) \cap (A_1 \times C_2(j_2)) \\ &= (A_1 \cap A_2) \times (C_2(j_1) \cap C_2(j_2)) \\ &= (A_1 \cap A_2) \times \emptyset \\ &= \emptyset \end{aligned}$$

by means of substitution, the definition of the sequence D_2, (3.66), (11.81) based on the previous finding $j_1, j_2 \in \{1, \ldots, n_2\}$ in connection with the assumed $j_1 \neq j_2$, and (3.27). We thus find $[x \in]\ D_{i_1}(j_1) \cap D_{i_1}(j_2) = \emptyset$ for both sub-cases, so that $x \in \emptyset$ holds also in the second case, completing the proof of the contradiction (11.104) and thus the proof of the negation $x \notin E_k \cap E_l$. As x was arbitrary, we may therefore conclude that the universal sentence (11.103) holds, which now further implies $E_k \cap E_l = \emptyset$. Here, k and l were also arbitrary, so that the universal sentence (11.102) follows to be also true.

In summary, $E : \{1, \ldots, n_1 + n_2\} \to \mathcal{S}_1 \otimes \mathcal{S}_2$ is a sequence of pairwise disjoint sets satisfying $X \setminus Y = \mathrm{ran}(E)$, where $n_1, n_2 \in \mathbb{N}_+$ evidently implies $n_1 + n_2 \in \mathbb{N}_+$. This means that $X \setminus Y$ satisfies the existential sentence in (11.79), and because X and Y were arbitrary, we may infer from the truth of that the difference of any two sets in $\mathcal{S}_1 \otimes \mathcal{S}_2$ can be written as the union of some sequence of pairwise disjoint sets in $\mathcal{S}_1 \otimes \mathcal{S}_2$ having some finite domain. Thus, the set system $\mathcal{S}_1 \otimes \mathcal{S}_2$ possesses the definite Property 4 of a semiring of sets on $\Omega_1 \times \Omega_2$.

Since Ω_1, Ω_2, $\mathcal{S}_1$ and $\mathcal{S}_1$ were initially arbitrary sets, we may now finally infer from the previous findings that the stated lemma is true. □

11.3. Rings of Sets

Next, we consider a type of set system that is closed under pairwise unions and set differences (which will turn out to be a special kind of π-system).

Definition 11.4 (Ring of sets). For any set Ω we say that a set $\mathcal{R}$ is a *ring of sets* on Ω iff

1. $\mathcal{R}$ consists of subsets of Ω, that is,

$$\mathcal{R} \subseteq \mathcal{P}(\Omega), \tag{11.110}$$

2. $\mathcal{R}$ contains the empty set, that is,

$$\emptyset \in \mathcal{R}, \tag{11.111}$$

3. $\mathcal{R}$ contains the union of any two sets in $\mathcal{R}$, that is,

$$\forall A, B\,(A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}), \tag{11.112}$$

and

4. $\mathcal{R}$ contains the set difference of any two sets in $\mathcal{R}$, that is,

$$\forall A, B\,(A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}). \tag{11.113}$$

Proposition 11.9. *The power set of any set Ω is a ring of sets on Ω.*

Proof. Letting Ω be an arbitrary, we immediately see that $\mathcal{P}(\Omega) \subseteq \mathcal{P}(\Omega)$ holds in light of (2.10). Furthermore, $\emptyset \in \mathcal{P}(\Omega)$ is true according to (3.15). Thus, $\mathcal{P}(\Omega)$ satisfies Property 1 and Property 2 of a ring of sets on Ω.

To establish Property 3, we take arbitrary sets A, B and assume $A, B \in \mathcal{P}(\Omega)$ to be true. We then obtain the inclusions $A \subseteq \Omega$ and $B \subseteq \Omega$ with the definition of a power set, and therefore the inclusion $A \cup B \subseteq \Omega$ with (2.252), which in turn implies the desired $A \cup B \in \mathcal{P}(\Omega)$ (by definition of a power set). Since A and B are arbitrary, we may infer from this finding that $\mathcal{P}(\Omega)$ satisfies Property 3 of a ring of sets on Ω.

Finally, regarding Property 4, we again take arbitrary sets A, B such that $A, B \in \mathcal{P}(\Omega)$. Consequently, we obtain (as before) the inclusion $A \subseteq \Omega$, and furthermore the inclusion $A \setminus B \subseteq A$ because of (2.125). These two inclusions give us then $A \setminus B \subseteq \Omega$ with (2.13), thus $A \setminus B \in \mathcal{P}(\Omega)$ as desired. Because A and B are arbitrary, we may therefore conclude that $\mathcal{P}(\Omega)$ satisfies also Property 4 of a ring of sets on Ω.

As Ω was initially an arbitrary set, the proposed sentence follows true. □

Exercise 11.3. Verify that the natural number $1 = \{\emptyset\}$ is a ring of sets on any set Ω.

(Hint: Use some of the arguments in the proof of Proposition 11.4 and in addition (2.216).)

The fact that rings of sets are closed under pairwise unions allows us to immediately define the corresponding binary operation, which inherits some of the set-theoretical characteristics of the union.

Exercise 11.4. Establish the following sentences for any set Ω and any ring of sets $\mathcal{R}$ on Ω.

a) There exists the unique binary operation

$$\cup_{\mathcal{R}} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}, \quad (A, B) \mapsto A \cup B. \tag{11.114}$$

(Hint: Proceed in analogy to the proof of Exercise 11.1a).)

b) $\cup_{\mathcal{R}}$ is idempotent. (Hint: Use Theorem 2.71.)

c) $\emptyset$ is the neutral element of $\mathcal{R}$ with respect to $\cup_{\mathcal{R}}$.
(Hint: Use (2.216).)

d) The ordered pair $(\mathcal{R}, \cup_{\mathcal{R}})$ is a commutative semigroup.
(Hint: Apply Theorem 2.70 and Theorem 2.74.)

Note 11.4. Because $\cup_{\mathcal{R}}$ is a binary operation on $\mathcal{R}$ and since the neutral element ($\emptyset$) with respect to $\cup_{\mathcal{R}}$ exists in $\mathcal{R}$, we may in addition define the n-fold binary operation

$$\bigcup_{i=1}^{n} : \mathcal{R}^{\{1,\ldots,n\}} \to \mathcal{R}, \quad (A_i \,|\, i \in \{1,\ldots,n\}) \mapsto \bigcup_{i=1}^{n} A_i \tag{11.115}$$

(for any natural number and any ring of sets $\mathcal{R}$ on any set Ω), so that rings of sets are also closed under n-fold unions. We introduced the function symbol $\bigcup_{i=1}^{n}$ already in Notation 4.7 to denote the union of the range of the mapped sequence; the following proposition shows that both kinds of union are equivalent representations of the same set. Thus, we may apply for instance Theorem 3.233 also in the context of n-fold unions on rings of sets.

Proposition 11.10. *For any set Ω and any ring of sets $\mathcal{R}$ on Ω, it is true that the n-fold binary union operation (11.115) satisfies for any $n \in \mathbb{N}$ and any sequence of sets $f = (A_i \,|\, i \in \{1,\ldots,n\})$ in $\mathcal{R}$ the equation*

$$\bigcup_{i=1}^{n} A_i = \bigcup \operatorname{ran}(f). \tag{11.116}$$

Proof. Letting Ω be an arbitrary set and $\mathcal{R}$ an arbitrary ring of sets on Ω, we may apply a proof by mathematical induction. In the base case $n = 0$, we see that the domain $\{1,\ldots,0\}$ of an arbitrary sequence $f = (A_i\, i \in \{1,\ldots,0\})$ in $\mathcal{R}$ is empty, so that its range is also empty according to (3.118). We therefore obtain $\bigcup \operatorname{ran}(f) = \emptyset$ with (2.205). On the other hand, we have $\bigcup_{i=1}^{0} A_i = \emptyset$ according to (5.389), because $\emptyset$ is the neutral element with respect to $\cup_{\mathcal{R}}$ (see Exercise 11.4c)). Thus, the equation (11.116) holds for $n = 0$.

Regarding the induction step, we now take an arbitrary natural number n and make the induction assumption that (11.116) holds for any sequence of sets $(A_i \,|\, i \in \{1,\ldots,n\})$ in $\mathcal{R}$. Next, we take an arbitrary sequence of sets $f = (A_i \,|\, i \in \{1,\ldots,n+1\})$ and establish the equation

$$\bigcup_{i=1}^{n+1} A_i = \bigcup \operatorname{ran}(f). \tag{11.117}$$

For this purpose, we apply the Equality Criterion for sets and take an arbitrary y, assuming first $y \in \bigcup_{i=1}^{n+1} A_i$ to be true. Because of (5.394), we therefore obtain $y \in (\bigcup_{i=1}^{n} A_i) \cup A_{n+1}$, so that the disjunction of $y \in \bigcup_{i=1}^{n} A_i$ and $y \in A_{n+1}$ holds by definition of the union of two sets. In case of $y \in \bigcup_{i=1}^{n} A_i$, it follows with the induction step that y is in the union of the range of the sequence $(A_i \mid i \in \{1, \ldots, n\})$. According to the Characterization of the union of a family of sets, there exists then an index in $\{1, \ldots, n\}$, say $\bar{k}$, such that $y \in A_{\bar{k}}$ holds. Clearly, $\{1, \ldots, n\}$ is included in $\{1, \ldots, n+1\}$, so that $\bar{k}$ follows to be in $\{1, \ldots, n+1\}$ (by definition of a subset). In the other case $y \in A_{n+1}$, we see that the index $n+1$ is in $\{1, \ldots, n+1\}$. Thus, there exists in any case an index i in $\{1, \ldots, n+1\}$ with $y \in A_i$, so that y is in the union of the range of the sequence $(A_i \mid i \in \{1, \ldots, n+1\})$.

We now conversely assume $y \in \bigcup \mathrm{ran}(f)$ to be true. According to the Characterization of the union of a family of sets, there exists then an element in the index set $\{1, \ldots, n+1\}$, say $\bar{k}$, such that $y \in A_{\bar{k}}$ holds. Clearly, we have that the preceding initial segment is the union of $\{1, \ldots, n\}$ and $\{n+1\}$, so that $\bar{k} \in \{1, \ldots, n\}$ or $\bar{k} \in \{n+1\}$ follows to be true by definition of the union of two sets. In the first case $\bar{k} \in \{1, \ldots, n\}$, we see that there is an index i in $\{1, \ldots, n\}$ with $y \in A_i$, so that y follows to be in the union of the (range of) sequence $(A_i \mid i \in \{1, \ldots, n\})$, and therefore to be in $\bigcup_{i=1}^{n} A_i$ (because of the induction assumption). Consequently, y is also in the union $(\bigcup_{i=1}^{n} A_i) \cup A_{n+1}$, using the definition of a subset in connection with (2.245). In the other case $\bar{k} \in \{n+1\}$, we obtain $\bar{k} = n+1$, so that $y \in A_{\bar{k}}$ yields $y \in A_{n+1}$. Therefore, y follows again to be an element in the union $(\bigcup_{i=1}^{n} A_i) \cup A_{n+1}$, applying again the definition of a subset in connection with (2.245). In light of (5.394), we thus see that $y \in \bigcup_{i=1}^{n+1} A_i$ holds in any case.

Because y is arbitrary, we may now infer from these findings equality of the sets $\bigcup_{i=1}^{n+1} A_i$ and $\bigcup \mathrm{ran}(f)$. As the sequence f and also n were arbitrary, we may therefore conclude that the induction step holds (besides the base case), completing the proof by mathematical induction. As Ω and $\mathcal{R}$ were arbitrary as well, the proposed universal sentences follows to be true. □

Exercise 11.5. Prove the following sentences for any set Ω, any ring of sets $\mathcal{R}$ on Ω and any sequence of sets $f = (A_n)_{n \in \mathbb{N}_+}$ in $\mathcal{R}$.

a) There exists a unique sequence of sets $u = (U_n)_{n \in \mathbb{N}_+}$ in $\mathcal{R}$ (and thus in $\mathcal{P}(\Omega)$) which satisfies

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow U_n = \bigcup_{i=1}^{n} A_i). \tag{11.118}$$

(Hint: Proceed as in Exercise 5.54 and apply Note 11.4.)

b) Furthermore, the sequence of sets $u = (U_n)_{n\in\mathbb{N}_+}$ is isotone with respect to the reflexive partial ordering of inclusions $\subseteq_{\mathcal{R}}$ and $\subseteq_{\mathcal{P}(\Omega)}$.

(Hint: Apply Corollary 4.71 with Exercise 4.29, (2.245) and (5.394).)

c) Moreover, the sequence $u = (U_n)_{n\in\mathbb{N}_+}$ converges isotonely with respect to $\subseteq_{\mathcal{P}(\Omega)}$ both to its union and the union of the sequence $f = (A_n)_{n\in\mathbb{N}_+}$, i.e.

$$\overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} U_n = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} A_n. \tag{11.119}$$

(Hint: Use Corollary 4.72a), Theorem 2.6, Theorem 3.233a), (11.118), (11.116), and (4.247).)

Proposition 11.11. *For any set Ω, any ring of sets $\mathcal{R}$ on Ω and any sequence of pairwise disjoint sets $f = (A_i \,|\, i \in \{1, \dots, n+1\})$, i.e. for which $A_j \cap A_k = \emptyset$ holds for all $j, k \in \{1, \dots, n\}$ with $j \neq k$, it is true that the n-fold union of $(A_i \,|\, i \in \{1, \dots, n\})$ and the term A_{n+1} are disjoint sets, i.e.*

$$\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1} = \emptyset. \tag{11.120}$$

Proof. Letting Ω, $\mathcal{R}$ and f be arbitrary sets such that $\mathcal{R}$ is a ring of sets and such that $f = (A_i \,|\, i \in \{1, \dots, n+1\})$ is a sequence of sets satisfying

$$\forall j, k\,([j, k \in \{1, \dots, n+1\} \wedge j \neq k] \Rightarrow A_j \cap A_k = \emptyset), \tag{11.121}$$

we prove (11.120) by verifying the equivalent universal sentence (applying the definition of the empty set)

$$\forall y\,(y \notin \left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}). \tag{11.122}$$

We take an arbitrary y and apply a proof by contradiction to prove the negation. Thus, we assume $\neg y \notin (\bigcup_{i=1}^{n} A_i) \cap A_{n+1}$ to be true, which yields $y \in (\bigcup_{i=1}^{n} A_i) \cap A_{n+1}$ with the Double Negation Law. Consequently, $y \in \bigcup_{i=1}^{n} A_i$ and $y \in A_{n+1}$ follow to be both true by definition of the intersection of two sets. Here, the first part of the conjunction implies with the Characterization of the union of a sequence (which we may apply in view of the previous Proposition 11.10) that there exists an index in $\{1, \dots, n\}$, say $\bar{k}$, such that $y \in A_{\bar{k}}$ holds. Together with $y \in A_{n+1}$, this yields $y \in$

$A_{\bar{k}} \cap A_{n+1}$ (again with the definition of the intersection of two sets),Â´so that the preceding intersection is clearly nonempty, i.e. $A_{\bar{k}} \cap A_{n+1} \neq \emptyset$.

Now, since the initial segment $\{1, \ldots, n\}$ is evidently a subset of the initial segment $\{1, \ldots, n+1\}$, we have that $\bar{k} \in \{1, \ldots, n\}$ implies $\bar{k} \in \{1, \ldots, n+1\}$. Furthermore, $\bar{k} \in \{1, \ldots, n\}$ implies clearly $\bar{k} \neq n+1$, where $n+1 \in \{1, \ldots, n+1\}$. We thus have that $\bar{k}, n+1 \in \{1, \ldots, n+1\}$ and $\bar{k} \neq n+1$ are both true, so that (11.121) gives $A_{\bar{k}} \cap A_{n+1} = \emptyset$.

We thus obtained a contradiction, which means that the negation in (11.122) is true. Because y is arbitrary, we may therefore conclude that the universal sentence (11.122) holds, which in turn implies (11.120). Then, as the sets Ω, $\mathcal{R}$ and f were also arbitrary, we may further conclude that the proposition is true. □

Lemma 11.12. *Show that every isotone sequence $A : \mathbb{N}_+ \to \mathcal{R}$ of sets in any ring of sets $\mathcal{R}$ can be written as the union of a sequence $B : \mathbb{N}_+ \to \mathcal{R}$ of pairwise disjoint sets in the form*

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n = A_1 \cup \bigcup_{n=2}^{\infty} (A_n \setminus A_{n-1}), \tag{11.123}$$

(where $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ for all $n \neq 1$). Show also that the following equation holds for any $n \in \mathbb{N}_+$:

$$A_n = \bigcup_{i=1}^{n} B_i \tag{11.124}$$

Proof. We let $\mathcal{R}$ be an arbitrary ring of sets (on an arbitrary set Ω) and $A : \mathbb{N}_+ \to \mathcal{R}$ an arbitrary sequence of sets. We now use Function definition by replacement to establish the desired sequence of sets $B : \mathbb{N}_+ \to \mathcal{K}$. For this purpose, we prove

$$\forall n\, (n \in \mathbb{N}_+ \Rightarrow \exists! Y\, ([n = 1 \Rightarrow Y = A_n] \wedge [n \neq 1 \Rightarrow Y = A_n \setminus A_{n-1}])), \tag{11.125}$$

letting $n \in \mathbb{N}_+$ be arbitrary. Thus, n is in the domain of the sequence A, so that the corresponding term A_n is specified. Concerning the existential part, we first assume $n = 1$ to be true. Thus, replacing Y by the constant A_n in $Y = A_n$ yields a true sentence, so that the first implication holds. Assuming now $n \neq 1$ to be true, it evidently follows from $n \in \mathbb{N}_+$ that $n - 1 \geq 1$ is true, so that $n - 1$ is in the domain of the sequence A. Then, the set $A_n \setminus A_{n-1}$ is uniquely determined, so that the second implication also holds. Regarding the uniqueness part, we let Y and Y' be arbitrary such that the conjunctions

$$[n = 1 \Rightarrow Y = A_n] \wedge [n \neq 1 \Rightarrow Y = A_n \setminus A_{n-1}]$$
$$[n = 1 \Rightarrow Y' = A_n] \wedge [n \neq 1 \Rightarrow Y' = A_n \setminus A_{n-1}]$$

hold. In case of $n = 1$, we obtain therefore $Y = A_n$ as well as $Y' = A_n$, with the consequence that $Y = Y'$. In the other case $n \neq 1$, we obtain $Y = A_n \setminus A_{n-1}$ as well as $Y' = A_n \setminus A_{n-1}$, so that $Y = Y'$ follows to be true again. Since Y and Y' were arbitrary, we may therefore conclude that the uniqueness part also holds. We thus proved the uniquely existential sentence, and since n was arbitrary, we may infer from this the truth of the uniquely existential sentence (11.125), and therefore the existence of a unique function B with domain $\mathbb{N}_+$ such that

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow ([n = 1 \Rightarrow B(n) = A_n] \wedge [n \neq 1 \Rightarrow B(n) = A_n \setminus A_{n-1}])). \tag{11.126}$$

We now show that the range of the sequence B is included in $\mathcal{R}$. To do this, we let $Y \in \operatorname{ran}(B)$ be arbitrary. By definition of a range, there exists then a particular constant n with $(n, Y) \in B$. Using the notations for functions, we may write this also as $Y = B(n)$. By definition of a domain, we also find $n \in \mathbb{N}_+$ $[= \operatorname{dom}(B)]$. Now, if $n = 1$ is true, then (11.126) yields $Y = A_n$, which is clearly an element of the codomain $\mathcal{R}$ of the sequence A. If $n \neq 1$ is true, then we find $Y = A_n \setminus A_{n-1}$, which set difference is again in $\mathcal{R}$ by virtue of Property 4 of a ring of sets. As Y was arbitrary, we may therefore conclude that the range of the sequence B is indeed included in $\mathcal{R}$. This means that $\mathcal{R}$ is a codomain of B.

Next, we show that all terms of the sequence B are pairwise disjoint. Letting $m, n \in \mathbb{N}_+$ be arbitrary such that $m \neq n$, we note that $m < n$ or $n < m$ holds according to the Characterization of comparability with respect to the linear ordering $<_{\mathbb{N}}$. If $m < n$ is true, we consider the two subcases $m = 1$ and $m \neq 1$. In the first subcase, we have $B_m = B_1 = A_1$ due to (11.126). Note that $m \in \mathbb{N}_+$ implies $l \leq m$ with (4.278); in conjunction with the current case assumption $m < n$, this gives us $1 < n$ with the Transitivity Formula for $\leq$ and $<$. Thus, $n \neq 1$ is evidently true, so that $B_n = A_n \setminus A_{n-1}$. Furthermore, the preceding inequality $1 < n$ evidently yields $0 < n - 1$ with (4.272) and therefore $1 \leq n - 1$ with (4.270). This inequality in turn implies $A_1 \subseteq A_{n-1}$ because of the isotony of the sequence A in connection with Proposition 3.257. Now, since $A_{n-1} \cap (A_n \setminus A_{n-1}) = \emptyset$ is also true according to (2.111), we obtain

$$[B_m \cap B_n =] \quad A_1 \cap (A_n \setminus A_{n-1}) = \emptyset$$

with (2.119), thus $B_m \cap B_n = \emptyset$ as desired. In the second subcase $m \neq 1$, we now have $B_m = A_m \setminus A_{m-1}$ besides $B_n = A_n \setminus A_{n-1}$. Observe that

$m < n$ evidently implies $m - 1 < n - 1$ and therefore $m \leq n - 1$, so that $A_m \subseteq A_{n-1}$. In connection with the evident inclusion $A_m \setminus A_{m-1} \subseteq A_m$, this implies $A_m \setminus A_{m-1} \subseteq A_{n-1}$ with the transitivity property of $\subseteq$ in (2.13). This inclusion and the previously found equation $A_{n-1} \cap (A_n \setminus A_{n-1}) = \emptyset$ further imply

$$[B_m \cap B_n =] \quad (A_m \setminus A_{m-1}) \cap (A_n \setminus A_{n-1}) = \emptyset,$$

again by means of (2.119). Thus, the desired disjointness of B_m and B_n holds in the first case $m < n$. We may establish the second case $n < m$ similarly to the first case, simply by exchanging m and n throughout the derivation for the first case. Thus, $B : \mathbb{N}_+ \to \mathcal{R}$ constitutes a sequence of pairwise disjoint sets.

Finally, we use mathematical induction to prove (11.124). In the base case ($n = 1$), we find

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{1} B_i = B_1 = B_n = A_1 = A_n$$

by applying substitutions, (5.391) in connection with the binary operation $\cup$ on $\mathcal{R}$, and (11.126). Within the induction step, we let $n \in \mathbb{N}_+$ be arbitrary, and we make the induction assumption that (11.124) holds. We then obtain

$$\bigcup_{i=1}^{n+1} B_i = \bigcup_{i=1}^{n} B_i \cup B_{n+1} = A_n \cup B_{n+1} = A_n \cup (A_{n+1} \setminus A_n) = A_{n+1}$$

using (5.394), the induction assumption, (11.126) in connection with the finding that $n \in \mathbb{N}_+$ evidently implies $n \geq 1$ and therefore $n+1 \geq 2$ so that $n + 1 \neq 1$, and finally (2.263) in connection with the assumption that A is isotone so that $A_n \subseteq A_{n+1}$. These equations give us $\bigcup_{i=1}^{n+1} B_i = A_{n+1}$, as desired. As n was arbitrary, we may therefore conclude that the induction step holds, besides the base case. We thus completed the proof of (11.124).

Since $\mathcal{R}$ and A were initially arbitrary, we may infer from the previous findings the truth of the stated lemma. □

Exercise 11.6. Write down a detailed proof for the second case within the demonstration that the terms of $B : \mathbb{N}_+ \to \mathcal{R}$, as defined in Lemma 11.12, are pairwise disjoint.

We now see that rings of sets are also closed under symmetric differences and under pairwise intersections.

Corollary 11.13. *Show for any set Ω and any ring of sets $\mathcal{R}$ on Ω that $\mathcal{R}$ contains the symmetric difference of any two sets in $\mathcal{R}$, that is,*

$$\forall A, B\,(A, B \in \mathcal{R} \Rightarrow A\Delta B \in \mathcal{R}). \tag{11.127}$$

Proof. We let Ω and $\mathcal{R}$ be arbitrary sets and assume that $\mathcal{R}$ is a ring of sets on Ω. Next, we take arbitrary sets $A, B \in \mathcal{R}$ (so that $B, A \in \mathcal{R}$ is also true) and observe in light of Property 4 of a ring of sets that $(A \setminus B), (B \setminus A) \in \mathcal{R}$ follows to be true, which in turn implies

$$[A\Delta B =] \quad (A \setminus B) \cup (B \setminus A) \in \mathcal{R}$$

with the definition of a symmetric difference and Property 3 of a ring of sets. Thus, the implication in (11.127) is true, and since A and B are arbitrary, we may therefore conclude that $\mathcal{R}$ is closed under symmetric differences. As Ω and $\mathcal{R}$ were initially also arbitrary, the proposed universal sentence follows to be true. □

Due to this result, we may now establish a binary operation on a ring of sets based on the symmetric difference in exactly the same way as we defined the binary intersection operation on a π-systems.

Exercise 11.7. Prove the following sentences for any set Ω and any ring of sets $\mathcal{R}$ on Ω.

a) There exists the unique binary operation

$$\Delta_{\mathcal{R}} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}, \quad (A, B) \mapsto A\Delta B. \tag{11.128}$$

(Hint: Proceed in analogy to the proof of Exercise 11.1a).)

b) The ordered pair $(\mathcal{R}, \Delta_{\mathcal{R}})$ is a commutative semigroup.
(Hint: Use Theorem 2.93 and Theorem 2.101.)

Proposition 11.14. *For any set Ω and any ring of sets $\mathcal{R}$ on Ω, it is true that*

a) $\emptyset$ is the neutral element of $\mathcal{R}$ with respect to $\Delta_{\mathcal{R}}$.

b) $(\mathcal{R}, \Delta_{\mathcal{R}})$ is a commutative group.

Proof. Letting Ω and $\mathcal{R}$ be arbitrary such that $\mathcal{R}$ is a ring of sets on Ω, we observe that $\emptyset \in \mathcal{R}$ holds according to Property 2 of a ring of sets, and we verify that $\emptyset$ satisfies the definition of a neutral element, that is,

$$\forall A\,(A \in \mathcal{R} \Rightarrow [\emptyset\Delta A = A \,\wedge\, A\Delta\emptyset = A]). \tag{11.129}$$

For this purpose, we take an arbitrary set $A \in \mathcal{R}$, so that (2.269) yields already the desired equations $\emptyset \Delta A = A$ and $A \Delta \emptyset = A$. Since A is arbitrary, we may therefore conclude that $\emptyset$ is indeed the neutral element of $\mathcal{R}$ with respect to $\Delta_{\mathcal{R}}$. Thus, $(\mathcal{R}, \Delta_{\mathcal{R}})$ satisfies Property 1 of a group.

Concerning b), it then remains only for us to verify Property 2 of a group. Letting $A \in \mathcal{R}$ be arbitrary, we notice that the equation $A \Delta A = \emptyset$ holds according to (2.270). Then, the conjunction

$$A \Delta A = \emptyset \wedge A \Delta A = \emptyset \tag{11.130}$$

is also true because of the Idempotent Law for the conjunction. Thus, there exists an element $A^{-1} \in \mathcal{R}$ with $A \Delta A^{-1} = \emptyset \wedge A^{-1} \Delta A = \emptyset$. Since A is arbitrary, we may infer from the truth of this existential sentence that the inverse element with respect to $\Delta_{\mathcal{R}}$ exists for every element of $\mathcal{R}$. Thus, $(\mathcal{R}, \Delta_{\mathcal{R}})$ satisfies also Property 2 of a group; because $\Delta_{\mathcal{R}}$ is a commutative, $(\mathcal{R}, \Delta_{\mathcal{R}})$ constitutes then a commutative group.

As Ω and $\mathcal{R}$ were arbitrary, we may conclude that the proposition is true. □

Note 11.5. Since $\emptyset = 0$ turned out to be the neutral element of a ring of sets with respect to $\Delta_{\mathcal{R}}$, which binary operation is based on the Boolean sum/symmetric difference, it seems justified to view $\Delta_{\mathcal{R}}$ as an addition with zero element $0 = \emptyset$. Following this interpretation, we write for the inverse element A^{-1} of any element $A \in \mathcal{R}$ accordingly $-A$, for which (11.130) showed that

$$A = A^{-1} \; [= -A]. \tag{11.131}$$

Thus, the inverse element of any $A \in \mathcal{R}$ is identical with A itself.

Definition 11.5 (Boolean addition). For any set Ω and any ring of sets $\mathcal{R}$ on Ω, we call the binary operation $\Delta_{\mathcal{R}}$ in (11.128) the *Boolean addition* on $\mathcal{R}$.

Corollary 11.15. *For any set Ω and any ring of sets $\mathcal{R}$ on Ω, it is true that the Boolean addition $\Delta_{\mathcal{R}}$ is a subtraction on $\mathcal{R}$.*

Proof. We let Ω and $\mathcal{R}$ be arbitrary, assume that $\mathcal{R}$ is a ring of sets on Ω, and verify that the supposed subtraction $\Delta_{\mathcal{R}}$ maps indeed any ordered pair $(A, B) \in \mathcal{R} \times \mathcal{R}$ to the difference of B and A, i.e. to the (Boolean) sum of B and the negative $-A$ of A. For this purpose, we let $(A, B) \in \mathcal{R} \times \mathcal{R}$ be arbitrary, so that $A \in \mathcal{R}$ and $B \in \mathcal{R}$ hold by definition of the Cartesian product of two sets. We then obtain the equation $B \Delta A = B \Delta -A$ via substitution based on (11.131), which shows that the symmetric difference Δ yields indeed the genuine difference of B and A. Since A and

B are arbitrary, we may therefore conclude that $\Delta_{\mathcal{R}}$ is a subtraction on $\mathcal{R}$. Because Ω and $\mathcal{R}$ were initially also arbitrary, we may further conclude that proposed universal sentence holds. $\square$

The fact that rings of sets are closed under symmetric differences allows us then to establish the useful fact that rings of sets are also closed under pairwise intersections.

Exercise 11.8. Show that any ring of sets $\mathcal{R}$ on any set Ω contains the intersection of any two sets in $\mathcal{R}$, that is,

$$\forall A, B\,(A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}). \tag{11.132}$$

(Hint: Proceed similarly as in the proof of Corollary 11.13, using now (11.112), (11.127), (11.113), and (2.280).)

Proposition 11.16. *Any ring of sets $\mathcal{R}$ on any set Ω is*

a) a semiring of sets on Ω.

b) a π-system on Ω.

Proof. Letting Ω and $\mathcal{R}$ be arbitrary such that $\mathcal{R}$ is a ring of sets on Ω, we see that Property 1 and Property 2 of a ring of sets are the same as Property 1 and Property 2 of a semiring of sets. Because $\mathcal{R}$ is closed under pairwise intersections according to (11.132), $\mathcal{R}$ satisfies also Property 3 of a semiring of sets. To establish Property 4 of a semiring of sets, we take arbitrary sets $A, B \in \mathcal{R}$ and observe that $A \setminus B \in \mathcal{R}$ follows to be true with Property 4 of a ring of sets. In view of Corollary 11.3, we then have that $\bar{C} = \{(1, A \setminus B)\}$ is a sequence of pairwise disjoint sets from $\{1, \ldots, 1\}$ to $\{A \setminus B\}$, for which $\bigcup \mathrm{ran}(\bar{C}) = A \setminus B$ holds. Because the previously found $A \setminus B \in \mathcal{R}$ implies $\{A \setminus B\} \subseteq \mathcal{R}$ with (2.184), we thus see that $\bar{C}$ is a sequence in $\mathcal{R}$, so that we showed in summary that there exist sets n and C such that $n \in \mathbb{N}_+$ and such that C is a sequence of pairwise disjoint sets from $\{1, \ldots, n\}$ to $\mathcal{R}$ with $A \setminus B = \bigcup \mathrm{ran}(C)$. Since A and B are arbitrary sets, we may infer from this that the ring of sets $\mathcal{R}$ satisfies Property 4 of a semiring of sets, completing the proof of a). Consequently, $\mathcal{R}$ is also a π-system, as mentioned in Note 11.2. Finally, because Ω and $\mathcal{R}$ were arbitrary, we may conclude that the proposition is true. $\square$

Note 11.6. According to the findings of Exercise 11.1, we may thus form for any ring of sets $\mathcal{R}$ on any set Ω the binary intersection operation

$$\cap_{\mathcal{R}} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}, \quad (A, B) \mapsto A \cap B, \tag{11.133}$$

which is idempotent, commutative and associative. The latter two properties show that $(\mathcal{R}, \cap_{\mathcal{R}})$ is a commutative semigroup.

Proposition 11.17. *For any set Ω and any ring of sets $\mathcal{R}$ on Ω, it is true that the ordered quadruple $(\mathcal{R}, \Delta_{\mathcal{R}}, \cap_{\mathcal{R}}, \Delta_{\mathcal{R}})$ is*

a) a commutative ring with zero element $\emptyset$.

b) a Boolean ring.

Proof. Letting Ω and $\mathcal{R}$ be arbitrary such that $\mathcal{R}$ is a ring of sets on Ω, we may view $\Delta_{\mathcal{R}}$ as an addition (the Boolean addition), $\cap_{\mathcal{R}}$ as a multiplication, and $\Delta_{\mathcal{R}}$ as a subtraction, according to Corollary 11.15. Concerning a), we observe in light of Proposition 11.14b) and Note 11.6 that $(\mathcal{R}, \Delta_{\mathcal{R}}, \cap_{\mathcal{R}}, \Delta_{\mathcal{R}})$ satisfies, respectively, Property 1 and Property 2 of a ring. Moreover, $\emptyset$ is the neutral element of $\mathcal{R}$ with respect to the (Boolean) addition $\Delta_{\mathcal{R}}$ according to Proposition 11.14a), thus $\emptyset$ is the zero element of $\mathcal{R}$. Now, it only remains for us to verify that the 'multiplication' $\cap_{\mathcal{R}}$ is distributive over the (Boolean) addition $\Delta_{\mathcal{R}}$, which we do by proving

$$\forall A, B, C\,(A, B, C \in \mathcal{R} \Rightarrow A \cap [B \Delta C] = [A \cap B]\,\Delta\,[A \cap C]). \tag{11.134}$$

Letting $A, B, C \in \mathcal{R}$ be arbitrary, we obtain the desired equation immediately with (2.272). Consequently, we may infer from this finding that $(\mathcal{R}, \Delta_{\mathcal{R}}, \cap_{\mathcal{R}}, \Delta_{\mathcal{R}})$ satisfies also the remaining Property 3 of a ring. Because the 'multiplication' $\cap_{\mathcal{R}}$ is also commutative according to Note 11.6, we thus have that the ordered quadruple $(\mathcal{R}, \Delta_{\mathcal{R}}, \cap_{\mathcal{R}}, \Delta_{\mathcal{R}})$ is indeed a commutative ring (with zero element $\emptyset$).

Concerning b), we simply need to recall that the 'multiplication' $\cap_{\mathcal{R}}$ is idempotent (see Note 11.6).

As Ω and $\mathcal{R}$ were initially arbitrary, we may therefore conclude that the proposition is true. □

Definition 11.6 (Boolean ring of sets). For any set Ω and any ring of sets $\mathcal{R}$ on Ω, we call

$$(\mathcal{R}, \Delta_{\mathcal{R}}, \cap_{\mathcal{R}}, \Delta_{\mathcal{R}}) \tag{11.135}$$

a *Boolean ring of sets* (on Ω).

We encounter now a mechanism for constructing a ring of sets in a unique manner from a given set system.

Theorem 11.18 (Generation of rings of sets). *The following sentences are true for any set Ω and for any set system $\mathcal{K} \subseteq \mathcal{P}(\Omega)$.*

a) There exists a unique set $\mathcal{U}$ consisting of all rings of sets on Ω (in $\mathcal{P}(\mathcal{P}(\Omega))$) which include $\mathcal{K}$, i.e.

$$\forall \mathcal{R}\,(\mathcal{R} \in \mathcal{U} \Leftrightarrow [\mathcal{R} \in \mathcal{P}(\mathcal{P}(\Omega)) \wedge (\mathcal{R} \text{ is a ring of sets on } \Omega \wedge \mathcal{K} \subseteq \mathcal{R})]). \tag{11.136}$$

This set $\mathcal{U}$ *is nonempty and satisfies also*

$$\forall \mathcal{R}\,(\mathcal{R} \in \mathcal{U} \Leftrightarrow [\mathcal{R} \text{ is a ring of sets on } \Omega \wedge \mathcal{K} \subseteq \mathcal{R}]). \tag{11.137}$$

b) *Then, the intersection* $\mathcal{R}(\mathcal{K}) = \bigcap \mathcal{U}$ *is itself a ring of sets on* Ω *that includes* $\mathcal{K}$.

c) *Furthermore, this ring of sets* $\mathcal{R}(\mathcal{K})$ *on* Ω *is the smallest ring of sets on* Ω *that includes* $\mathcal{K}$ *in the sense that*

$$\forall \mathcal{R}\,([\mathcal{R} \text{ is a ring of sets on } \Omega \wedge \mathcal{K} \subseteq \mathcal{R}] \Rightarrow \mathcal{R}(\mathcal{K}) \subseteq \mathcal{R}). \tag{11.138}$$

Proof. We take arbitrary sets Ω and $\mathcal{K}$, and we assume the latter to be a subset of the power set of the former. Concerning a), we may evidently apply the Axiom of Specification alongside the Equality Criterion for sets to establish the unique existence of a set $\mathcal{U}$ with the property (11.136). To show that the set $\mathcal{U}$ satisfies also (11.137), we take an arbitrary set $\mathcal{R}$ and assume first $\mathcal{R} \in \mathcal{U}$ to be true. Then, this assumption implies with (11.136) especially that $\mathcal{R}$ is a ring of sets on Ω with $\mathcal{K} \subseteq \mathcal{R}$, which is the desired consequent of the first part ('$\Rightarrow$') of the equivalence in (11.137). We now conversely assume $\mathcal{R}$ to be a ring of sets on Ω with $\mathcal{K} \subseteq \mathcal{R}$. Thus, $\mathcal{R} \subseteq \mathcal{P}(\Omega)$ holds according to Property 1 of a ring of sets on Ω, and this inclusion implies $\mathcal{R} \in \mathcal{P}(\mathcal{P}(\Omega))$ by definition of a power set. Together with the preceding assumption, this finding implies now $\mathcal{R} \in \mathcal{U}$ with (11.136), completing the proof of the equivalence in (11.137). As $\mathcal{R}$ was arbitrary, we may therefore conclude that $\mathcal{U}$ satisfies indeed the universal sentence (11.137). Let us now observe in light of Proposition 11.9 that $\mathcal{P}(\Omega)$ is a ring of sets on Ω; since $\mathcal{K} \subseteq \mathcal{P}(\Omega)$ is also true by assumption, it follows from this with (11.137) that $\mathcal{P}(\Omega) \in \mathcal{U}$ holds. Thus, the set $\mathcal{U}$ is clearly nonempty, so that we may form the intersection $\bigcap \mathcal{U}$, which we denote in the following also by $\mathcal{R}(\mathcal{K})$.

Concerning b), we now verify that the set $\mathcal{R}(\mathcal{K}) = \bigcap \mathcal{U}$ satisfies the Properties 1 – 4 of a ring of sets on Ω. To establish Property 1, i.e. to establish $\mathcal{R}(\mathcal{K}) \subseteq \mathcal{P}(\Omega)$, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall A\,(A \in \bigcap \mathcal{U} \Rightarrow A \in \mathcal{P}(\Omega)). \tag{11.139}$$

We take an arbitrary set A such that $A \in \bigcap \mathcal{U}$ holds. Then, by definition of the intersection of a set system, the universal sentence

$$\forall \mathcal{R}\,(\mathcal{R} \in \mathcal{U} \Rightarrow A \in \mathcal{R}) \tag{11.140}$$

holds. Because we established $\mathcal{U} \neq \emptyset$, there exists an element in that set, say $\bar{\mathcal{R}} \in \mathcal{U}$, which follows to be a ring of sets on Ω in view of (11.137).

Consequently, we have $\bar{\mathcal{R}} \subseteq \mathcal{P}(\Omega)$ according to Property 1 of a ring of sets on Ω. Furthermore, $\bar{\mathcal{R}} \in \mathcal{U}$ implies $A \in \bar{\mathcal{R}}$ with (11.140) and therefore $A \in \mathcal{P}(\Omega)$ with the preceding inclusion (applying the definition of a subset). This finding completes the proof of the implication in (11.139), and as A was arbitrary, the universal sentence (11.139) follows to be true. Therefore, the inclusion $\mathcal{R}(\mathcal{K}) \subseteq \mathcal{P}(\Omega)$ is also true, so that $\mathcal{R}(\mathcal{K})$ satisfies indeed Property 1 of a ring of sets on Ω.

We continue with Property 2 and verify $\emptyset \in \mathcal{R}(\mathcal{K})$ by proving the equivalent universal sentence (applying the definition of the intersection of a set system)

$$\forall \mathcal{R} \, (\mathcal{R} \in \mathcal{U} \Rightarrow \emptyset \in \mathcal{R}). \tag{11.141}$$

Letting $\mathcal{R}$ be arbitrary and assuming $\mathcal{R} \in \mathcal{U}$ to be true, it follows with (11.137) that $\mathcal{R}$ is a ring of sets on Ω (with $\mathcal{K} \subseteq \mathcal{R}$). Consequently, $\emptyset \in \mathcal{R}$ is true because $\mathcal{R}$ satisfies Property 2 of a ring of sets on Ω. Thus, the implication in (11.141) is true, and since $\mathcal{R}$ was arbitrary, we may therefore conclude that the universal sentence (11.141) holds, which in turn implies $\emptyset \in \mathcal{R}(\mathcal{K})$, as desired.

Regarding Property 3 and Property 4, we let A and B be arbitrary and assume $A, B \in \mathcal{R}(\mathcal{K})$ (i.e., $A, B \in \bigcap \mathcal{U}$) to be true, so that the universal sentences

$$\forall \mathcal{R} \, (\mathcal{R} \in \mathcal{U} \Rightarrow A \in \mathcal{R}) \tag{11.142}$$

$$\forall \mathcal{R} \, (\mathcal{R} \in \mathcal{U} \Rightarrow B \in \mathcal{R}) \tag{11.143}$$

hold by definition of the intersection of a set system. To establish the desired consequents $A \cup B \in \bigcap \mathcal{U}$ and $A \setminus B \in \bigcap \mathcal{U}$, we need to prove

$$\forall \mathcal{R} \, (\mathcal{R} \in \mathcal{U} \Rightarrow A \cup B \in \mathcal{R}), \tag{11.144}$$

$$\forall \mathcal{R} \, (\mathcal{R} \in \mathcal{U} \Rightarrow A \setminus B \in \mathcal{R}). \tag{11.145}$$

Letting $\mathcal{R} \in \mathcal{U}$ be arbitrary, we obtain on the one hand $A, B \in \mathcal{R}$ with (11.142) – (11.143). On the other hand, $\mathcal{R} \in \mathcal{U}$ implies that $\mathcal{R}$ is a ring of sets on Ω, so that the preceding finding implies $A \cup B \in \mathcal{R}$ as well as $A \setminus B \in \mathcal{R}$, according to Property 3 and Property 4 of a ring of sets on Ω. As $\mathcal{R}$ was arbitrary, we may therefore conclude that the universal sentences (11.144) – (11.145) both hold, so that $A \cup B \in \bigcap \mathcal{U}$ and $A \setminus B \in \bigcap \mathcal{U}$ follow to be both true by definition of the intersection of a set system. We thus showed that $A, B \in \mathcal{R}(\mathcal{K})$ implies $A \cup B \in \mathcal{R}(\mathcal{K})$ as well as $A \setminus B \in \mathcal{R}(\mathcal{K})$, and since A and B are arbitrary, we may infer from these findings that Property 3 and Property 4 of a ring of set are both satisfied by $\mathcal{R}(\mathcal{K})$.

Let us now verify that $\bigcap \mathcal{U}$ includes $\mathcal{K}$, i.e. that $\mathcal{K} \subseteq \bigcap \mathcal{U}$. For this purpose, we prove the universal sentence

$$\forall A \, (A \in \mathcal{K} \Rightarrow A \in \bigcap \mathcal{U}), \tag{11.146}$$

letting A be an arbitrary set and assuming $A \in \mathcal{K}$ to be true. To establish the desired consequent $A \in \bigcap \mathcal{U}$, we apply the definition of the intersection of a set system and prove the equivalent universal sentence

$$\forall \mathcal{R}\, (\mathcal{R} \in \mathcal{U} \Rightarrow A \in \mathcal{R}). \tag{11.147}$$

We take an arbitrary set $\mathcal{R}$, and we assume $\mathcal{R} \in \mathcal{U}$, so that (11.137) yields especially $\mathcal{K} \subseteq \mathcal{R}$. Because of this inclusion, the assumed $A \in \mathcal{K}$ implies $A \in \mathcal{R}$, which is the desired consequent of the implication in (11.147). Because $\mathcal{R}$ is arbitrary, we may now infer from the truth of that implication the truth of the universal sentence (11.147), which in turn gives $A \in \bigcap \mathcal{U}$. Thus, the proof of the implication in (11.146) is complete, and as A was arbitrary, we may further conclude that (11.146) holds. The truth of this universal sentence implies then the truth of the desired inclusion $\mathcal{K} \subseteq \bigcap \mathcal{U}$, so that $\mathcal{R}(\mathcal{K})$ is indeed a ring of sets on Ω that includes $\mathcal{K}$.

It now remains for us to establish c). To do this, we take an arbitrary set $\mathcal{R}$ and assume $\mathcal{R}$ to be a ring of sets on Ω such that $\mathcal{K} \subseteq \mathcal{R}$ also holds. These assumptions imply $\mathcal{R} \in \mathcal{U}$ by definition of the latter set, which then further implies $\bigcap \mathcal{U} \subseteq \mathcal{R}$ with (2.92), and thus the desired inclusion $\mathcal{R}(\mathcal{K}) \subseteq \mathcal{R}$. Since $\mathcal{R}$ was arbitrary, we may therefore conclude that c) also holds.

Because Ω and $\mathcal{K}$ were initially arbitrary sets, we may now finally conclude that the theorem is true. □

Definition 11.7 (Generated ring of sets, generating system for a ring of sets). For any set Ω and any set system $\mathcal{K} \subseteq \mathcal{P}(\Omega)$, we call the set system

$$\mathcal{R}(\mathcal{K}) \tag{11.148}$$

satisfying (11.137) the *ring of sets* (*on* Ω) *generated by* $\mathcal{K}$. We then say that $\mathcal{K}$ is the *generating system* for $\mathcal{R}(\mathcal{K})$.

Theorem 11.19 (Generation of rings of sets by means of semirings of sets). *It is true for any set Ω and any semiring of sets $\mathcal{S}$ on Ω that there exists a unique set $\mathcal{X}$ consisting of all the subsets of Ω that are unions of finite sequences of disjoint sets in $\mathcal{S}$, in the sense that*

$$\begin{aligned} &\forall A\, (A \in \mathcal{X} \Leftrightarrow [A \in \mathcal{P}(\Omega) \wedge \exists n, C\, (n \in \mathbb{N}_+ \wedge C : \{1,\ldots,n\} \to \mathcal{S} \\ &\quad \wedge \forall i,j\, ([i,j \in \{1,\ldots,n\} \wedge i \neq j] \Rightarrow C_i \cap C_j = \emptyset) \wedge A = \bigcup \operatorname{ran}(C))]). \end{aligned} \tag{11.149}$$

Then, this set is identical with the ring of sets $\mathcal{R}(\mathcal{S})$ generated by the semiring of sets $\mathcal{S}$, i.e.

$$\mathcal{X} = \mathcal{R}(\mathcal{S}). \tag{11.150}$$

Proof. Letting Ω and $\mathcal{S}$ be arbitrary sets and assuming $\mathcal{S}$ to be a semiring of sets on Ω, we may evidently utilize the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of a set $\mathcal{X}$ such that (11.149) holds. We now prove that the two inclusions $\mathcal{X} \subseteq \mathcal{R}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S}) \subseteq \mathcal{X}$ are true, which will then imply the truth of the equation $\mathcal{X} = \mathcal{R}(\mathcal{S})$ with the Axiom of Extension.

To establish $\mathcal{X} \subseteq \mathcal{R}(\mathcal{S})$, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall A\,(A \in \mathcal{X} \Rightarrow A \in \mathcal{R}(\mathcal{S})). \tag{11.151}$$

For this purpose, we take an arbitrary set A, and we assume $A \in \mathcal{X}$ to be true. In view of (11.149) there is then a particular positive natural number $\bar{n}$ and a particular sequence $\bar{C} : \{1, \ldots, \bar{n}\} \to \mathcal{S}$ of pairwise disjoint sets (in $\mathcal{S}$) such that $A = \bigcup \operatorname{ran}(\bar{C})$ holds. Thus, $\mathcal{S}$ is a codomain $\bar{C}$, so that the inclusion $\operatorname{ran}(\bar{C}) \subseteq \mathcal{S}$ is true. Furthermore, since the generated ring of sets $\mathcal{R}(\mathcal{S})$ includes its generating system $\mathcal{S}$, which means that the inclusion $\mathcal{S} \subseteq \mathcal{R}(\mathcal{S})$ also holds, we obtain $\operatorname{ran}(\bar{C}) \subseteq \mathcal{R}(\mathcal{S})$ with (2.13). Thus, $\mathcal{R}(\mathcal{S})$ is also a codomain of $\bar{C}$, so that we have the sequence $\bar{C} : \{1, \ldots, \bar{n}\} \to \mathcal{R}(\mathcal{S})$ in the (generated) ring of sets $\mathcal{R}(\mathcal{S})$. In view of Note 11.4 and Proposition 11.10, we therefore obtain

$$[A = \bigcup \operatorname{ran}(\bar{C}) =] \bigcup_{i=1}^{n} \bar{C}_i \in \mathcal{R}(\mathcal{S}),$$

proving the implication in (11.151). As A was arbitrary, we may now conclude that the universal sentence (11.151) is true, which then yields $\mathcal{X} \subseteq \mathcal{R}(\mathcal{S})$, as desired.

Next, we verify that $\mathcal{X}$ is a ring of sets on Ω that includes $\mathcal{S}$, which will then imply $\mathcal{R}(\mathcal{S}) \subseteq \mathcal{X}$ with Theorem 11.18c). We first establish the inclusion $\mathcal{S} \subseteq \mathcal{X}$, by proving

$$\forall A\,(A \in \mathcal{S} \Rightarrow A \in \mathcal{X}). \tag{11.152}$$

To do this, we let $A \in \mathcal{S}$ be arbitrary and use the equivalence in (11.149) to obtain the desired consequent $A \in \mathcal{X}$. On the one hand, $A \in \mathcal{S}$ implies $A \in \mathcal{P}(\Omega)$ with Property 1 of a semiring of sets on Ω (using the definition of a subset). On the other hand, Corollary 11.3 shows that the set A defines the sequence $\bar{C} = \{(1, A)\} = (\bar{C}_i \,|\, i \in \{1, \ldots, 1\})$ of pairwise disjoint sets in $\{A\}$ with $\bigcup \operatorname{ran}(\bar{C}) = A$. Since $A \in \mathcal{S}$ gives $\{A\} \subseteq \mathcal{S}$ with (2.184), it is true that $\bar{C}$ is a sequence in $\mathcal{S}$. Thus, there are sets n and C such that $n \in \mathbb{N}_+$, $C : \{1, \ldots, n\} \to \mathcal{S}$, the disjointness of the terms of C and the equation $A = \bigcup \operatorname{ran}(C)$ hold. Together with $A \in \mathcal{P}(\Omega)$, this existential sentence yields the desired $A \in \mathcal{X}$ with (11.149). As A was arbitrary, we may therefore conclude that (11.152) is true, which universal sentence implies then $\mathcal{S} \subseteq \mathcal{X}$ by definition of a subset.

Next, we verify that $\mathcal{X}$ satisfies all four properties of a ring of sets. Regarding Property 1, we simply observe in (11.149) that $A \in \mathcal{X}$ implies $A \in \mathcal{P}(\Omega)$ for any set A, so that the inclusion $\mathcal{X} \subseteq \mathcal{P}(\Omega)$ follows to be true by definition of a subset. Furthermore, because $\emptyset \in \mathcal{S}$ holds in view of Property 2 of a semiring of sets, it follows with the previously established inclusion $\mathcal{S} \subseteq \mathcal{X}$ (again with the definition of a subset) that $\emptyset \in \mathcal{X}$ also holds. Thus, $\mathcal{X}$ satisfies also Property 2 of a ring of sets.

To establish Property 3 and 4, we demonstrate first that $\mathcal{X}$ is closed under pairwise intersections, that is,

$$\forall A, B\,(A, B \in \mathcal{X} \Rightarrow A \cap B \in \mathcal{X}). \tag{11.153}$$

We take arbitrary sets $\bar{A}, \bar{B} \in \mathcal{X}$, so that (by definition of the set $\mathcal{X}$) we have on the one hand $\bar{A}, \bar{B} \in \mathcal{P}(\Omega)$. On the other hand, there exist particular positive natural numbers $\bar{m}$ and $\bar{n}$ as well as particular sequences $\bar{C} : \{1, \ldots, \bar{m}\} \to \mathcal{S}$ and $\bar{D} : \{1, \ldots, \bar{n}\} \to \mathcal{S}$ having pairwise disjoint terms, satisfying

$$\bar{A} = \bigcup \operatorname{ran}(\bar{C}), \tag{11.154}$$

$$\bar{B} = \bigcup \operatorname{ran}(\bar{D}). \tag{11.155}$$

Thus, $\mathcal{S}$ is a codomain of the sequences $\bar{C}$ and $\bar{D}$, so that their ranges are included in that semiring of sets, that is, $\operatorname{ran}(\bar{C}) \subseteq \mathcal{S}$ and $\operatorname{ran}(\bar{D}) \subseteq \mathcal{S}$ are both true. Recalling now that the generated ring of sets $\mathcal{R}(\mathcal{S})$ includes the generating system $\mathcal{S}$, i.e. $\mathcal{S} \subseteq \mathcal{R}(\mathcal{S})$, we obtain the inclusions $\operatorname{ran}(\bar{C}) \subseteq \mathcal{R}(\mathcal{S})$ and $\operatorname{ran}(\bar{D}) \subseteq \mathcal{R}(\mathcal{S})$ by applying (2.13). Thus, $\mathcal{R}(\mathcal{S})$ is also a codomain of $\bar{C}$ and $\bar{D}$, which are therefore sequences $\bar{C} : \{1, \ldots, \bar{m}\} \to \mathcal{R}(\mathcal{S})$ and $\bar{D} : \{1, \ldots, \bar{n}\} \to \mathcal{R}(\mathcal{S})$. We now prove that $\mathcal{X}$ contains the intersection of the two sets $\bar{A}$ and $\bar{B}$, using again the definition of the set $\mathcal{X}$ in (11.149). On the one hand, the previously found $\bar{A}, \bar{B} \in \mathcal{P}(\Omega)$ implies $\bar{A} \cap \bar{B} \in \mathcal{P}(\Omega)$ since $\mathcal{P}(\Omega)$ is a ring of sets on Ω (see Proposition 11.9), which is thus closed under pairwise intersections (see Exercise 11.8). On the other hand, we may prove the existence of a positive natural number N and of a sequence $E : \{1, \ldots, N\} \to \mathcal{S}$ of pairwise disjoint sets such that $\bar{A} \cap \bar{B} = \bigcup \operatorname{ran}(E)$ holds. Let us begin with the observation that $\{1, \ldots, \bar{m}\}$ and $\{1, \ldots, \bar{n}\}$ are finite sets with cardinalities $|\{1, \ldots, \bar{m}\}| = \bar{m}$ and $|\{1, \ldots, \bar{n}\}| = \bar{n}$, according to (4.506). Due to the Finiteness of the Cartesian product of two finite sets, we therefore have that $\{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\}$ is a finite set, so that there exist (by definition) a particular natural number $\bar{N}$ and a particular bijection $\bar{c} : \{1, \ldots, \bar{N}\} \rightleftarrows \{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\}$. Thus, the sets $\{1, \ldots, \bar{N}\}$ and $\{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\}$ are equinumerous because of (4.514), which means that their cardinalities are identical, that is,

$$|\{1, \ldots, \bar{N}\}| = |\{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\}|. \tag{11.156}$$

Here, we have on the one hand $|\{1, \ldots, \bar{N}\}| = \bar{N}$ in view of (4.506), and on the other hand $|\{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\}| = \bar{m} \cdot_{\mathbb{N}} \bar{n}$ by definition of the multiplication $\cdot_{\mathbb{N}}$. Consequently, we may apply substitutions to (11.156) to obtain $\bar{N} = \bar{m} \cdot_{\mathbb{N}} \bar{n}$. Thus, $\bar{c}$ is the bijection

$$\bar{c} : \{1, \ldots, \bar{m} \cdot \bar{n}\} \rightleftarrows \{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\}, \tag{11.157}$$

which we now use to establish the unique existence of a sequence $E = (E_k \,|\, k \in \{1, \ldots, \bar{m} \cdot \bar{n}\})$ whose terms are defined by

$$\forall k\, (k \in \{1, \ldots, \bar{m} \cdot \bar{n}\} \Rightarrow \exists i, j\, (\bar{c}(k) = (i, j) \,\wedge\, E_k = \bar{C}_i \cap \bar{D}_j)). \tag{11.158}$$

For this purpose, we apply Function definition by replacement and verify accordingly

$$\forall k\, (k \in \{1, \ldots, \bar{m} \cdot \bar{n}\} \Rightarrow \exists! Y\, (\exists i, j\, (\bar{c}(k) = (i, j) \wedge Y = \bar{C}_i \cap \bar{D}_j))), \tag{11.159}$$

letting $\bar{k} \in \{1, \ldots, \bar{m} \cdot \bar{n}\}$ be arbitrary. Regarding the existential part, we notice that $\bar{k}$ is in the domain of $\bar{c}$, so that the corresponding value $y = \bar{c}(\bar{k})$ is evidently in the codomain/range $\{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\}$ of $\bar{c}$. It then follows with the definition of the Cartesian product of two sets that there are particular constants $\bar{i} \in \{1, \ldots, \bar{m}\}$ and $\bar{j} \in \{1, \ldots, \bar{n}\}$ which satisfy $(\bar{i}, \bar{j}) = \bar{c}(\bar{k})$. Thus, $\bar{i}$ is an index of the sequence $\bar{C}$ and $\bar{j}$ an index of the sequence $\bar{D}$, with associated terms $\bar{C}_{\bar{i}}$ and $\bar{D}_{\bar{j}}$. We may therefore form the intersection $\bar{Y} = \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}}$, so that we found constants i and j that satisfy $\bar{c}(\bar{k}) = (i, j)$ and $\bar{Y} = \bar{C}_i \cap \bar{D}_j$. Since $\bar{Y}$ satisfies the preceding existential sentence, the existential part of the uniquely existential sentence in (11.159) therefore holds.

Regarding the uniqueness part, we let Y and Y' be arbitrary and assume these to satisfy the existential sentence with respect to i and j. Thus, there are particular elements $\bar{i}, \bar{j}$ with $\bar{c}(\bar{k}) = (\bar{i}, \bar{j})$ and $Y = \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}}$ as well as particular elements $\bar{i}', \bar{j}'$ with $\bar{c}(\bar{k}) = (\bar{i}', \bar{j}')$ and $Y' = \bar{C}_{\bar{i}'} \cap \bar{D}_{\bar{j}'}$. Combining the two equations for $\bar{c}(\bar{k})$ yields then $(\bar{i}, \bar{j}) = (\bar{i}', \bar{j}')$, which equation implies with the Equality Criterion for ordered pairs $\bar{i} = \bar{i}'$ and $\bar{j} = \bar{j}'$. With these equations and the previous equations for Y and Y', we obtain by means of substitutions

$$Y = \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}} = \bar{C}_{\bar{i}'} \cap \bar{D}_{\bar{j}'} = Y',$$

and therefore $Y = Y'$. Because Y and Y' are arbitrary, we may infer from this finding that the uniqueness part also holds, so that the proof of the uniquely existential sentence in (11.159) is complete. Since $\bar{k}$ was also arbitrary, we may now further conclude that the universal sentence (11.159) is true, which in turn implies the unique existence of a function E with domain $\{1, \ldots, \bar{m} \cdot \bar{n}\}$ such that (11.158). We may thus write E in sequence notation as $(E_k \,|\, k \in \{1, \ldots, \bar{m} \cdot \bar{n}\})$. Let us check that $\mathcal{S}$ is a codomain of this sequence, i.e. that $\mathrm{ran}(E) \subseteq \mathcal{S}$ holds. Applying the definition of a subset, we take an arbitrary set Y and assume $Y \in \mathrm{ran}(E)$ to be true. By definition of a range, there is then a constant $\bar{k}$ such that $(\bar{k}, Y) \in E$ holds. Since E is a function/sequence, we may write this also as $Y = E(\bar{k}) = E_{\bar{k}}$. Furthermore, $(\bar{k}, Y) \in E$ implies by definition of a domain that $\bar{k} \in \{1, \ldots, \bar{m} \cdot \bar{n}\}$ $[= \mathrm{dom}(E)]$ is true, which in turn implies with (11.158) that there are constants, say $\bar{i}$ and $\bar{j}$, satisfying $\bar{c}(\bar{k}) = (\bar{i}, \bar{j})$ and $[Y =]$ $E_{\bar{k}} = \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}}$. Here, $\bar{C}_{\bar{i}}$ and $\bar{D}_{\bar{j}}$ are terms of the sequences $\bar{C}$ and $\bar{D}$ in $\mathcal{S}$, so that $\bar{C}_{\bar{i}}, \bar{D}_{\bar{j}}$ holds. Recalling that $\mathcal{S}$ is a semiring of sets, we therefore obtain (with Property 3 of a semiring of sets)

$$[Y = E_{\bar{k}} =] \quad \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}} \in \mathcal{S}.$$

We thus showed that $Y \in \mathrm{ran}(E)$ implies $Y \in \mathcal{S}$, and as Y was arbitrary, we may therefore infer from this implication the truth of the inclusion $\mathrm{ran}(E) \subseteq \mathcal{S}$. Consequently, we have that $E = (E_k \,|\, k \in \{1, \ldots, \bar{m} \cdot \bar{n}\})$ is a sequence of sets in $\mathcal{S}$. Recalling that $\bar{m}, \bar{n} \in \mathbb{N}_+$ holds, so that the $\bar{m}, \bar{n} \in \mathbb{N}$, $\bar{m} \neq 0$ and $\bar{n} \neq 0$ are true by definition of the set of positive natural numbers, we obtain then $\bar{m} \cdot \bar{n} \in \mathbb{N}$ by definition of the multiplication on $\mathbb{N}$ and furthermore $\bar{m} \cdot \bar{n} \neq 0$ with the Criterion for zero-divisor freeness (applied to the semiring of natural numbers). Thus, we have $\bar{m} \cdot \bar{n} \in \mathbb{N}_+$.

Let us now prove that the terms of this sequence are pairwise disjoint, that is,

$$\forall i, j \, ([i, j \in \{1, \ldots, \bar{m} \cdot \bar{n}\} \,\wedge\, i \neq j] \Rightarrow E_i \cap E_j = \emptyset). \tag{11.160}$$

To do this, we take arbitrary indexes $i, j \in \{1, \ldots, \bar{m} \cdot \bar{n}\}$ with $i \neq j$. According to (11.158), there are then particular constants $\bar{k}, \bar{l}$ satisfying $\bar{c}(i) = (\bar{k}, \bar{l})$ and $E_i = \bar{C}_{\bar{k}} \cap \bar{D}_{\bar{l}}$, as well as particular constants $\bar{\bar{k}}, \bar{\bar{l}}$ such that $\bar{c}(j) = (\bar{\bar{k}}, \bar{\bar{l}})$ and $E_j = \bar{C}_{\bar{\bar{k}}} \cap \bar{D}_{\bar{\bar{l}}}$. Recalling that $\bar{c}$ is a bijection, which is by definition an injection, we may now apply the Injection Criterion to infer from the assumed $i, j \in \{1, \ldots, \bar{m} \cdot \bar{n}\}$ and $i \neq j$ the truth of

$$[(\bar{k}, \bar{l}) =] \quad \bar{c}(i) \neq \bar{c}(j) \quad [= (\bar{\bar{k}}, \bar{\bar{l}})].$$

The resulting inequality $(\bar{k}, \bar{l}) \neq (\bar{\bar{k}}, \bar{\bar{l}})$ implies then with the Equality Criterion for ordered pairs that the negation $\neg(\bar{k} = \bar{\bar{k}} \wedge \bar{l} = \bar{\bar{l}})$ holds. Applying

De Morgan's Law for the conjunction gives now the true disjunction

$$\bar{k} \neq \bar{\bar{k}} \vee \bar{l} \neq \bar{\bar{l}}. \tag{11.161}$$

To establish the desired consequent $E_i \cap E_j = \emptyset$, we apply the definition of the empty set and prove the equivalent universal sentence

$$\forall y \, (y \notin E_i \cap E_j). \tag{11.162}$$

For this purpose, we take an arbitrary y and prove $y \notin E_i \cap E_j$ by cases, based on (11.161). In the first case $\bar{k} \neq \bar{\bar{k}}$, we may prove $y \notin E_i \cap E_j$ by contradiction, by establishing the contradiction

$$\bar{C}_{\bar{k}} \cap \bar{C}_{\bar{\bar{k}}} = \emptyset \wedge \bar{C}_{\bar{k}} \cap \bar{C}_{\bar{\bar{k}}} \neq \emptyset. \tag{11.163}$$

Assuming the negation $\neg y \notin E_i \cap E_j$ to be true, so that the Double Negation Law gives the true sentence $y \in E_i \cap E_j$, we obtain with the definition of the intersection of two sets first the true conjunction of

$$y \in E_i \quad [= \bar{C}_{\bar{k}} \cap \bar{D}_{\bar{l}}], \tag{11.164}$$
$$y \in E_j \quad [= \bar{C}_{\bar{\bar{k}}} \cap \bar{D}_{\bar{\bar{l}}}], \tag{11.165}$$

and then in particular the truth of $y \in \bar{C}_{\bar{k}}$ and $y \in \bar{C}_{\bar{\bar{k}}}$. We thus have the true conjunction $y \in \bar{C}_{\bar{k}} \wedge y \in \bar{C}_{\bar{\bar{k}}}$, so that another application of the definition of the intersection of two sets yields then $y \in \bar{C}_{\bar{k}} \cap \bar{C}_{\bar{\bar{k}}}$. This findings clearly shows that $\bar{C}_{\bar{k}} \cap \bar{C}_{\bar{\bar{k}}} \neq \emptyset$ is true. However, the current case assumption $\bar{k} \neq \bar{\bar{k}}$ implies also the truth of $\bar{C}_{\bar{k}} \cap \bar{C}_{\bar{\bar{k}}} = \emptyset$, because we initially assumed $\bar{C}$ to be a sequence of pairwise disjoint sets. We therefore obtained the contradiction (11.163), so that $y \notin E_i \cap E_j$ holds in the first case.

In the second case $\bar{l} \neq \bar{\bar{l}}$, we may prove $y \notin E_i \cap E_j$ again by contradiction, using this time the contradiction

$$\bar{D}_{\bar{l}} \cap \bar{D}_{\bar{\bar{l}}} = \emptyset \wedge \bar{D}_{\bar{l}} \cap \bar{D}_{\bar{\bar{l}}} \neq \emptyset. \tag{11.166}$$

Assuming for this purpose $\neg y \notin E_i \cap E_j$ to hold, we obtain as in the first case $y \in E_i \cap E_j$ and therefore the conjunction of (11.164) and (11.165). This yields especially (by definition of the intersection of two sets) $y \in \bar{D}_{\bar{l}}$ and $y \in \bar{D}_{\bar{\bar{l}}}$ and then $y \in \bar{D}_{\bar{l}} \cap \bar{D}_{\bar{\bar{l}}}$, which intersection is thus evidently not empty. However, the current case assumption $\bar{l} \neq \bar{\bar{l}}$ implies $\bar{D}_{\bar{l}} \cap \bar{D}_{\bar{\bar{l}}} = \emptyset$ since we initially assumed the sequence $\bar{D}$ to have pairwise disjoint sets. Consequently, we arrived at the contradiction (11.166), proving $y \notin E_i \cap E_j$ also for the second case.

As y was arbitrary, we may therefore conclude that the universal sentence (11.162) holds, so that $E_i \cap E_j = \emptyset$ holds indeed. Because i and j were also arbitrary, we may infer from this that $E = (E_k \mid k \in \{1, \ldots, \bar{m} \cdot \bar{n}\})$ is a sequence of pairwise disjoint sets (in $\mathcal{S}$).

It now remains for us to prove that $\bar{A} \cap \bar{B} = \bigcup \mathrm{ran}(E)$ holds, which we do by applying the Equality Criterion for sets, i.e. by verifying

$$\forall y\,(y \in \bar{A} \cap \bar{B} \Leftrightarrow y \in \bigcup \mathrm{ran}(E)). \tag{11.167}$$

We take an arbitrary y and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming

$$y \in \bar{A} \cap \bar{B} \tag{11.168}$$

to be true. Then, substitutions based on (11.154) and (11.155) give

$$y \in \bigcup \mathrm{ran}(\bar{C}) \cap \bigcup \mathrm{ran}(\bar{D}), \tag{11.169}$$

which we may write also as (using the notation for the union of a sequence of sets on an initial segment in connection with the definition of the intersection of two sets)

$$y \in \bigcup_{i=1}^{\bar{m}} \bar{C}_i \,\wedge\, y \in \bigcup_{j=1}^{\bar{n}} \bar{D}_j. \tag{11.170}$$

Next, we use the Characterization of the union of a family of sets to obtain

$$\exists i\,(i \in \{1, \ldots, \bar{m}\} \,\wedge\, y \in \bar{C}_i) \,\wedge\, \exists j\,(j \in \{1, \ldots, \bar{n}\} \,\wedge\, y \in \bar{D}_j). \tag{11.171}$$

Thus, there are particular indexes $\bar{i} \in \{1, \ldots, \bar{m}\}$ and $\bar{j} \in \{1, \ldots, \bar{n}\}$ such that $y \in \bar{C}_{\bar{i}}$ and $y \in \bar{D}_{\bar{j}}$ are both true. We then obtain $(\bar{i}, \bar{j}) \in \{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\}$ with the definition of the Cartesian product of two sets, and also $y \in \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}}$ with the definition of the intersection of two sets. Let us now observe that the inverse of the bijection $\bar{c}$ in (11.157) is given by the bijection

$$\bar{c}^{-1} : \{1, \ldots, \bar{m}\} \times \{1, \ldots, \bar{n}\} \rightleftarrows \{1, \ldots, \bar{m} \cdot \bar{n}\},$$

according to (3.683). This shows that $(\bar{i}, \bar{j})$ is in the domain of $\bar{c}^{-1}$, so that the associated function value $\bar{k} = \bar{c}^{-1}((\bar{i}, \bar{j}))$ is in the codomain/range of that inverse, i.e. $\bar{k} \in \{1, \ldots, \bar{m} \cdot \bar{n}\}$. On the one hand, this implies with (11.158) the existence of particular indexes $\bar{i}', \bar{j}'$ with $\bar{c}(\bar{k}) = (\bar{i}', \bar{j}')$ and $E_{\bar{k}} = \bar{C}_{\bar{i}'} \cap \bar{D}_{\bar{j}'}$. On the other hand, $\bar{k} = \bar{c}^{-1}((\bar{i}, \bar{j}))$ implies $(\bar{i}, \bar{j}) = \bar{c}(\bar{k})$ with the Characterization of the function values of an inverse function. Combining the two equations for $\bar{c}(\bar{k})$ gives than $(\bar{i}', \bar{j}') = (\bar{i}, \bar{j})$ and therefore $\bar{i}' = \bar{i}$ as well as $\bar{j}' = \bar{j}$ with the Equality Criterion for ordered pairs. Consequently, we may write the term $E_{\bar{k}}$ also as the intersection $E_{\bar{k}} = \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}}$. Recalling now that $y \in \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}}$ holds, $y \in E_{\bar{k}}$ follows to be true (besides the previously obtained $\bar{k} \in \{1, \ldots, \bar{m} \cdot \bar{n}\}$). We thus see that the existential sentence

$$\exists k \, (k \in \{1, \ldots, \bar{m} \cdot \bar{n}\} \, \wedge \, y \in E_k) \tag{11.172}$$

is true, which then further implies (with the Characterization of the union of a family of sets)

$$y \in \bigcup_{k=1}^{m \cdot n} E_k, \tag{11.173}$$

and therefore (using again the notation for the union of a sequence of sets)

$$y \in \bigcup \mathrm{ran}(E). \tag{11.174}$$

This completes the proof of the first part of the equivalence in (11.167). regarding the second part ('$\Leftarrow$'), we now assume (11.174) to be true, which gives then evidently (11.173) and subsequently (11.172). Thus, there is a particular index $\bar{k} \in \{1, \ldots, \bar{m} \cdot \bar{n}\}$ such that $y \in E_{\bar{k}}$ holds. Because of (11.158), there are then particular $\bar{i}$ and $\bar{j}$ satisfying $\bar{c}(\bar{k}) = (\bar{i}, \bar{j})$ as well as $E_{\bar{k}} = \bar{C}_{\bar{i}} \cap \bar{D}_{\bar{j}}$). Thus, y follows to be element of the preceding intersection, so that $y \in \bar{C}_{\bar{i}}$ and $y \in \bar{D}_{\bar{j}}$ both hold. Obviously, $\bar{i}$ is an index of the sequence $\bar{C}$ and $\bar{j}$ is an index of the sequence $\bar{D}$; in other words, $\bar{i}$ is in the domain $\{1, \ldots, \bar{m}\}$ of $\bar{C}$ and $\bar{j}$ in the domain $\{1, \ldots, \bar{n}\}$ of $\bar{D}$. The previous findings clearly show that the existential sentences in (11.171) are both true, so that the equivalent sentences (11.170), (11.169) and finally (11.168) also hold.

Thus, the proof of the equivalence in (11.167) is complete, and as y was arbitrary, we may infer from this now the truth of the universal sentence (11.167). This gives us in turn the desired equation $\bar{A} \cap \bar{B} = \bigcup \mathrm{ran}(E)$. Together with the already established fact that $E = (E_k \,|\, k \in \{1, \ldots, \bar{m} \cdot \bar{n}\})$ is a sequence of pairwise disjoint sets in $\mathcal{S}$ with $\bar{m} \cdot \bar{n} \in \mathbb{N}_+$, this shows us that the intersection $\bar{A} \cap \bar{B}$ satisfies the existential sentence in (11.149), alongside the already established $\bar{A} \cap \bar{B} \in \mathcal{P}(\Omega)$. Consequently, we obtain indeed $\bar{A} \cap \bar{B} \in \mathcal{X}$, and since $\bar{A}, \bar{B}$ are arbitrary, we may therefore conclude that the universal sentence (11.153) is true, which shows that $\mathcal{X}$ is closed under pairwise intersections.

Next, we verify that $\mathcal{X}$ is closed also under set differences, that is,

$$\forall A, B \, (A, B \in \mathcal{X} \Rightarrow A \setminus B \in \mathcal{X}). \tag{11.175}$$

To do this, we let again $\bar{A}$ and $\bar{B}$ be arbitrary in $\mathcal{X}$, so that there are by definition of $\mathcal{X}$ particular numbers $\bar{m}, \bar{n} \in \mathbb{N}_+$ as well as particular sequences $\bar{C} : \{1, \ldots, \bar{m}\} \to \mathcal{S}$ and $\bar{D} : \{1, \ldots, \bar{n}\} \to \mathcal{S}$ of pairwise disjoint sets with

$\bar{A} = \bigcup \operatorname{ran}(\bar{C})$ and $\bar{B} = \bigcup \operatorname{ran}(\bar{D})$. Thus, we may write

$$\begin{aligned}
\bar{A} \setminus \bar{B} &= \left[\bigcup \operatorname{ran}(\bar{C})\right] \setminus \left[\bigcup \operatorname{ran}(\bar{D})\right] \\
&= \left(\bigcup_{i=1}^{\bar{m}} \bar{C}_i\right) \setminus \left(\bigcup_{j=1}^{\bar{n}} \bar{D}_j\right) \\
&= \bigcup_{i=1}^{\bar{m}} \left(\bar{C}_i \setminus \bigcup_{j=1}^{\bar{n}} \bar{D}_j\right) \\
&= \bigcup_{i=1}^{\bar{m}} \left[\bar{C}_i \setminus \bigcup \operatorname{ran}(\bar{D})\right] \qquad (11.176)
\end{aligned}$$

using the Distributive Law for families of sets (3.823). As before, $\bar{A}, \bar{B} \in \mathcal{X}$ yields $\bar{A}, \bar{B} \in \mathcal{P}(\Omega)$ according to the specification of the set $\mathcal{X}$, and therefore $\bar{A} \setminus \bar{B} \in \mathcal{P}(\Omega)$ (recalling that this power set is a ring of sets on Ω). To establish $\bar{A} \setminus \bar{B} \in \mathcal{X}$, the task is now to prove the existence of a natural number K and of a sequence $F : \{1, \ldots, K\} \to \mathcal{S}$ having pairwise disjoint terms and satisfying $\bar{A} \setminus \bar{B} = \bigcup \operatorname{ran}(F)$. Now, due to Lemma 11.7, we may express every set $\bar{C}_i \setminus \bigcup_{j=1}^{\bar{n}} \bar{D}_j$ as the union of some finite sequence of disjoint sets in $\mathcal{S}$, and we may stack all these unions within some sequence $F : \{1, \ldots, K\} \to \mathcal{S}$.

To begin with, let us observe the truth of the universal sentence

$$\begin{aligned}
&\forall i\, (i \in \{1, \ldots, \bar{m}\} \Rightarrow \exists! \mathcal{E}^{(i)}\, \forall Y\, (Y \in \mathcal{E}^{(i)} \Leftrightarrow [Y \in \mathcal{S}^{<\mathbb{N}_+} \qquad (11.177) \\
&\quad \wedge\ \exists n^{(i)}, E^{(i)}\, (n^{(i)} \in \mathbb{N}_+ \ \wedge\ E^{(i)} : \{1, \ldots, n^{(i)}\} \to \mathcal{S} \\
&\quad \wedge\ \forall j, k\, ([j, k \in \{1, \ldots, n^{(i)}\} \ \wedge\ j \neq k] \Rightarrow E^{(i)}(j) \cap E^{(i)}(k) = \emptyset) \\
&\quad \wedge\ \bar{C}_i \setminus \operatorname{ran}(\bar{D}) = \bigcup \operatorname{ran}(E^{(i)}) \ \wedge\ E^{(i)} = Y]))
\end{aligned}$$

in light of the Axiom of Specification and the Equality Criterion for sets (recalling that $\mathcal{S}^{<\mathbb{N}_+}$ is defined according to Exercise 4.32). We may now apply Function definition of Replacement based on this universal sentence to establish the unique existence of a function/sequence $\mathcal{E}$ with domain $\{1, \ldots, \bar{m}\}$ such that

$$\forall i\, (i \in \{1, \ldots, \bar{m}\} \Rightarrow \mathcal{E}(i) = \mathcal{E}^{(i)}). \qquad (11.178)$$

To prove the required universal sentence

$$\forall i\, (i \in \{1, \ldots, \bar{m}\} \Rightarrow \exists! y\, (y = \mathcal{E}^{(i)})),$$

we let $i \in \{1, \ldots, \bar{m}\}$ be arbitrary, so that the set $\mathcal{E}^{(i)}$ exists uniquely because of (11.177). Therefore, the uniquely existential sentence $\exists! y\,(y = \mathcal{E}^{(i)})$ turns out to be true with (1.109), and as i was arbitrary, we may now conclude that the required universal sentence holds indeed. Consequently, there is a unique function/sequence $\mathcal{E}$ with domain $\{1, \ldots, \bar{m}\}$ such that (11.178). In the next step, we verify that $\emptyset \notin \operatorname{ran}(\mathcal{E})$ holds as well, which negation is equivalent to

$$\forall Y\,(Y \in \operatorname{ran}(\mathcal{E}) \Rightarrow Y \neq \emptyset) \tag{11.179}$$

due to (2.5). To prove this universal sentence, we let Y be arbitrary, and we assume $Y \in \operatorname{ran}(\mathcal{E})$, which assumption implies with the definition of a range that there exists a constant, say $\bar{k}$, such that $(\bar{k}, Y) \in \mathcal{E}$ holds. On the one hand, we may write this finding as $Y = \mathcal{E}_{\bar{k}}$; on the other hand, it follows with the definition of a domain that $\bar{k} \in \{1, \ldots, \bar{m}\}$ $[= \operatorname{dom}(\mathcal{E})$ is true. Thus, $\bar{k}$ is also in the domain of $\bar{C}$, so that the corresponding term is given by $\bar{C}_{\bar{k}} = \bar{C}(\bar{k})$. Writing this equation in the form $(\bar{k}, \bar{C}_{\bar{k}}) \in \bar{C}$, we evidently obtain with the definition of a range and of a codomain $\bar{C}_{\bar{k}} \in \operatorname{ran}(\bar{C})$ $[\subseteq \mathcal{S}]$, so that the definition of a subset yields $\bar{C}_{\bar{k}} \in \mathcal{S}$. Recalling now $\bar{n} \in \mathbb{N}_+$ and of $\bar{D} : \{1, \ldots, \bar{n}\} \to \mathcal{S}$, it follows with Lemma 11.7 that there are sets, say $\bar{N}^{(\bar{k})}$ and $\bar{E}^{(\bar{k})}$, such that $\bar{N}^{(\bar{k})} \in \mathbb{N}_+$, $\bar{E}^{(\bar{k})} : \{1, \ldots, \bar{N}^{(\bar{k})}\} \to \mathcal{S}$, the universal sentence

$$\forall i, j\,([i, j \in \{1, \ldots, \bar{N}^{(\bar{k})}\} \wedge i \neq j] \Rightarrow \bar{E}_i^{(\bar{k})} \cap \bar{E}_j^{(\bar{k})} = \emptyset),$$

and the equation $\bar{C}_{\bar{k}} \setminus \bigcup \operatorname{ran}(\bar{D}) = \bigcup \operatorname{ran}(\bar{E}^{(\bar{k})})$ are all true. Here, $\bar{N}^{(\bar{k})} \in \mathbb{N}_+$ gives $\bar{E}^{(\bar{k})} \in \mathcal{S}^{<\mathbb{N}_+}$, and the equation $\bar{E}^{(\bar{k})} = \bar{E}^{(\bar{k})}$ clearly holds as well. Then, the conjunction of $\bar{E}^{(\bar{k})} \in \mathcal{S}^{<\mathbb{N}_+}$ and the existential sentence

$$\begin{aligned}
&\exists n^{(i)}, E^{(i)}\,(n^{(i)} \in \mathbb{N}_+ \wedge E^{(i)} : \{1, \ldots, n^{(i)}\} \to \mathcal{S} \\
&\quad \wedge \forall j, k\,([j, k \in \{1, \ldots, n^{(i)}\} \wedge j \neq k] \Rightarrow E^{(i)}(j) \cap E^{(i)}(k) = \emptyset) \\
&\quad \wedge \bar{C}_{\bar{k}} \setminus \bigcup \operatorname{ran}(\bar{D}) = \bigcup \operatorname{ran}(\bar{E}^{(\bar{k})}) \wedge E^{(i)} = \bar{E}^{(\bar{k})}])
\end{aligned}$$

holds, which conjunction further implies

$$\bar{E}^{(\bar{k})} \in \mathcal{E}^{(\bar{k})} \; [= \mathcal{E}_{\bar{k}} = Y]$$

due to the equivalence in (11.177) and (11.178). The resulting $\bar{E}^{(\bar{k})} \in Y$ demonstrates the existence of an element in Y, which evidently means that $Y \neq \emptyset$ holds, as desired. As Y was arbitrary, we may therefore conclude that (11.179) is true, which universal sentence implies then the truth of the negation $\emptyset \notin \operatorname{ran}(\mathcal{E})$. Consequently, we see in light of the Axiom of Choice that there is a particular function $\bar{H} : \operatorname{ran}(\mathcal{E}) \to \bigcup \operatorname{ran}(\mathcal{E})$ such that

$$\forall K\,(K \in \operatorname{ran}(\mathcal{E}) \Rightarrow \bar{H}(K) \in K). \tag{11.180}$$

We then obtain the composition $\bar{E} = \bar{H} \circ \mathcal{E} : \{1, \ldots, \bar{m}\}$ to $\bigcup \operatorname{ran}(\mathcal{E})$ with Proposition 3.178, which we may now show to satisfy

$$\begin{aligned} &\forall i\,(i \in \{1, \ldots, \bar{m}\} \Rightarrow \exists! n^{(i)}\,(n^{(i)} \in \mathbb{N}_+ \,\wedge\, \bar{E}_i : \{1, \ldots, n^{(i)}\} \to \mathcal{S} \\ &\quad \wedge\, \forall j, k\,([j, k \in \{1, \ldots, n^{(i)}\} \,\wedge\, j \neq k] \Rightarrow \bar{E}_i(j) \cap \bar{E}_i(k) = \emptyset) \\ &\quad \wedge\, \bar{C}_i \setminus \bigcup \operatorname{ran}(\bar{D}) = \bigcup \operatorname{ran}(\bar{E}_i))). \end{aligned} \tag{11.181}$$

We take an arbitrary $i \in \{1, \ldots, \bar{m}\}$, which gives for the associated value of $\bar{E}$

$$\bar{E}_i = (\bar{H} \circ \mathcal{E})_i = \bar{H}(\mathcal{E}_i).$$

Writing $\mathcal{E}_i = \mathcal{E}(i)$ as $(i, \mathcal{E}_i) \in \mathcal{E}$, we now see that $\mathcal{E}_i \in \operatorname{ran}(\mathcal{E})$ is true, so that $\bar{H}(\mathcal{E}_i) \in \mathcal{E}_i$ follows to be true with (11.180), and substitution yields then $\bar{E}_i \in \mathcal{E}_i$ $[= \mathcal{E}^{(i)}]$. Due to the equivalence in (11.177), the resulting $\bar{E}_i \in \mathcal{E}^{(i)}$ implies in particular that there are sets, say $\bar{n}^{(i)}$ and $\bar{E}^{(i)}$, such that $\bar{n}^{(i)} \in \mathbb{N}_+$, $\bar{E}^{(i)} : \{1, \ldots, \bar{n}^{(i)}\} \to \mathcal{S}$, the universal sentence

$$\forall j, k\,([j, k \in \{1, \ldots, \bar{n}^{(i)}\} \,\wedge\, j \neq k] \Rightarrow \bar{E}^{(i)}(j) \cap \bar{E}^{(i)}(k) = \emptyset),$$

$\bar{C}_i \setminus \bigcup \operatorname{ran}(\bar{D}) = \bigcup \operatorname{ran}(\bar{E}^{(i)})$, and $\bar{E}^{(i)} = \bar{E}_i$ are all true. This equation allows us to carry out substitutions to the previous findings to obtain the conjunction of $\bar{n}^{(i)} \in \mathbb{N}_+$, $\bar{E}_i : \{1, \ldots, \bar{n}^{(i)}\} \to \mathcal{S}$,

$$\forall j, k\,([j, k \in \{1, \ldots, \bar{n}^{(i)}\} \,\wedge\, j \neq k] \Rightarrow \bar{E}_i(j) \cap \bar{E}_i(k) = \emptyset),$$

and $\bar{C}_i \setminus \bigcup \operatorname{ran}(\bar{D}) = \bigcup \operatorname{ran}(\bar{E}_i)$. This multiple conjunction clearly shows that the existential part of the uniquely existential sentence in (11.181). To establish the uniqueness part, we let $n_1^{(i)}$ and $n_2^{(i)}$ be arbitrary in $\mathbb{N}_+$ such that

$$\bar{E}_i : \{1, \ldots, n_1^{(i)}\} \to \mathcal{S},$$
$$\bar{E}_i : \{1, \ldots, n_2^{(i)}\} \to \mathcal{S},$$

the universal sentences

$$\forall j, k\,([j, k \in \{1, \ldots, n_1^{(i)}\} \,\wedge\, j \neq k] \Rightarrow \bar{E}_i(j) \cap \bar{E}_i(k) = \emptyset),$$
$$\forall j, k\,([j, k \in \{1, \ldots, n_2^{(i)}\} \,\wedge\, j \neq k] \Rightarrow \bar{E}_i(j) \cap \bar{E}_i(k) = \emptyset),$$

and $\bar{C}_i \setminus \bigcup \operatorname{ran}(\bar{D}) = \bigcup \operatorname{ran}(\bar{E}_i)$ are true. Here, we have for the the domain of $\bar{E}_i$

$$\operatorname{dom}(\bar{E}_i) = \{1, \ldots, n_1^{(i)}\} = \{1, \ldots, n_2^{(i)}\},$$

where the second equation implies $n_1^{(i)} = n_2^{(i)}$ with (4.252) because of $n_1^{(i)}, n_2^{(i)} \in \mathbb{N}_+$. Having found this equation, we completed the proof of the uniqueness part and thus the proof of the uniquely existential sentence (11.181). As i was arbitrary, we may therefore conclude that (11.181) is true, which universal sentence allows us to establish via Function definition by replacement the unique function $\bar{N}$ with domain $\{1,\ldots,\bar{m}\}$, i.e. the unique sequence $\bar{N} = (\bar{N}_i \,|\, i \in \{1,\ldots,\bar{m}\})$, such that

$$\forall i\,(i \in \{1,\ldots,\bar{m}\} \Rightarrow [\bar{N}_i \in \mathbb{N}_+ \,\wedge\, \bar{E}_i : \{1,\ldots,\bar{N}_i\} \to \mathcal{S}$$
$$\wedge\, \forall j,k\,([j,k \in \{1,\ldots,\bar{N}_i\} \,\wedge\, j \neq k] \Rightarrow \bar{E}_i(j) \cap \bar{E}_i(k) = \emptyset)$$
$$\wedge\, \bar{C}_i \setminus \bigcup \mathrm{ran}(\bar{D}) = \bigcup \mathrm{ran}(\bar{E}_i)]). \tag{11.182}$$

Let us now check that $\mathbb{N}$ is a codomain of $\bar{N}$, i.e. that the inclusion $\mathrm{ran}(\bar{N}) \subseteq \mathbb{N}$ is true. We apply for this purpose the definition of a subset and let $y \in \mathrm{ran}(\bar{N})$ be arbitrary, so that there is (by definition of a range) a constant, say $\bar{k}$, that satisfies $(\bar{k}, y) \in \bar{N}$. On the one hand, we may write this in function/sequence notation as $y = \bar{N}_{\bar{k}}$, and on the other hand we obtain (by definition of a domain) $\bar{k} \in \{1,\ldots,\bar{m}\}$ $[= \mathrm{dom}(\bar{N})]$. The latter finding implies in particular

$$[y =] \quad \bar{N}_{\bar{k}} \in \mathbb{N}_+ \quad [\subseteq \mathbb{N}]$$

according to (11.182) and (2.308), resulting in $y \in \mathbb{N}$ (via the definition of a subset). Because y is arbitrary, we may therefore conclude that $y \in \mathrm{ran}(\bar{N})$ implies $y \in \mathbb{N}$ for all y, and this means that the inclusion $\mathrm{ran}(\bar{N}) \subseteq \mathbb{N}$ holds indeed. We thus have $\bar{N} : \{1,\ldots,\bar{m}\} \to \mathbb{N}$, and in addition the terms of the sequence $\bar{E} = (\bar{E}_i \,|\, i \in \{1,\ldots,\bar{m}\})$ satisfy $\bar{E}_i : \{1,\ldots,\bar{N}_i\} \to \mathcal{S}$ for any $i \in \{1,\ldots,\bar{n}\}$ in view of (11.182). Having established these two sequences, we may subsequently apply Stacking of a finite sequence of finite sequences in order to define the stacked sequence $\bar{F} = (\bar{F}_k \,|\, k \in \{1,\ldots,\sum_{i=1}^{\bar{m}} \bar{N}_i\})$ in $\mathcal{S}$ whose terms are characterized by

$$\forall k\,(k \in \{1,\ldots,\sum_{i=1}^{\bar{m}} \bar{N}_i\} \tag{11.183}$$
$$\Rightarrow \exists i,j\,(i \in \{1,\ldots,\bar{m}\} \,\wedge\, j \in \{1,\ldots,\bar{N}_i\} \,\wedge\, \bar{F}_k = \bar{E}_i(j))),$$

$$\forall i,j\,([i \in \{1,\ldots,\bar{m}\} \,\wedge\, j \in \{1,\ldots,\bar{N}_i\}] \tag{11.184}$$
$$\Rightarrow \exists k\,(k \in \{1,\ldots,\sum_{i=1}^{\bar{m}} \bar{N}_i\} \,\wedge\, \bar{E}_i(j) = \bar{F}_k)).$$

We now utilize (11.182) – (11.183) to prove

$$\bigcup_{i=1}^{\bar{n}} \left[\bar{C}_i \setminus \bigcup \mathrm{ran}(\bar{D})\right] = \bigcup_{k=1}^{\sum_{i=1}^{\bar{m}} \bar{N}_i} \bar{F}_k \tag{11.185}$$

by means of the Equality Criterion for sets, by verifying accordingly

$$\forall y\,(y \in \bigcup_{i=1}^{\bar{m}} \left[\bar{C}_i \setminus \bigcup \operatorname{ran}(\bar{D})\right] \Leftrightarrow y \in \bigcup_{k=1}^{\sum_{i=1}^{\bar{m}} \bar{N}_i} \bar{F}_k). \tag{11.186}$$

We let y be arbitrary, and we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming

$$y \in \bigcup_{i=1}^{\bar{m}} \left[\bar{C}_i \setminus \bigcup \operatorname{ran}(\bar{D})\right] \tag{11.187}$$

to be true. The Characterization of the union of a family of sets shows then that there is an index, say $I \in \{1, \ldots, \bar{m}\}$, such that

$$y \in \bar{C}_I \setminus \bigcup \operatorname{ran}(\bar{D}) \tag{11.188}$$

holds. Next, $I \in \{1, \ldots, \bar{m}\}$ implies with (11.182) especially the equation $\bar{C}_I \setminus \bigcup \operatorname{ran}(\bar{D}) = \bigcup \operatorname{ran}(\bar{E}_I)$, which allows us to apply substitution to (11.188) in order to obtain

$$y \in \bigcup \operatorname{ran}(\bar{E}_I). \tag{11.189}$$

Here, $\bar{E}_I$ is a sequence with domain $\{1, \ldots, \bar{N}_I\}$, so that we can write for (11.189) also

$$y \in \bigcup_{j=1}^{\bar{N}_I} \bar{E}_I(j). \tag{11.190}$$

The Characterization of the union of a family of sets shows us now also that there is an index, say $J \in \{1, \ldots, \bar{N}_I\}$, with

$$y \in \bar{E}_I(J). \tag{11.191}$$

Moreover, the conjunction of $I \in \{1, \ldots, \bar{m}\}$ and $J \in \{1, \ldots, \bar{N}_I\}$ implies with (11.184) that there exists an index, say $K \in \{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\}$, satisfying with $\bar{E}_I(J) = \bar{F}_K$, which equation we use for a substitution in (11.191) to obtain

$$y \in \bar{F}_K. \tag{11.192}$$

This finding demonstrates the existence of a constant $k \in \{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\}$ with $y \in \bar{F}_k$, so that the Characterization of the union of a family of sets yields

$$y \in \bigcup_{k=1}^{\sum_{i=1}^{\bar{m}} \bar{N}_i} \bar{F}_k, \tag{11.193}$$

which is the desired consequent of the first part of the equivalence in (11.186).

We establish the second part ('$\Leftarrow$') of the equivalence by applying similar arguments as before, assuming (11.193) to hold. Therefore, there is an index, say $K \in \{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\}$, such that (11.192) holds. Here, $K \in \{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\}$ implies with (11.183) that there are indexes, say I and J, satisfying $I \in \{1, \ldots, \bar{m}\}$, $J \in \{1, \ldots, \bar{N}_I\}$ and $\bar{F}_K = \bar{E}_I(J)$. Therefore, (11.192) implies (11.191), which demonstrates that there exists a constant $j \in \{1, \ldots, \bar{N}_I\}$ with $y \in \bar{E}_I(j)$. This existential sentence in turn implies (11.190), which can be written in the form of (11.189). Since $I \in \{1, \ldots, \bar{m}\}$ implies the truth of the equation $\bar{C}_I \setminus \bigcup \mathrm{ran}(\bar{D}) = \bigcup \mathrm{ran}(\bar{E}_I)$, we obtain then (11.188). We thus see that there exists a constant $i \in \{1, \ldots, \bar{m}\}$ with $y \in \bar{C}_i \setminus \bigcup \mathrm{ran}(\bar{D})$, which existential sentence further implies the desired consequent (11.187) of the second part of the equivalence. We thus completed the proof of that equivalence, and as y was arbitrary, we may therefore conclude that (11.186) holds. This universal sentence implies now the proposed equation (11.185), which yields in connection with (11.176)

$$\bar{A} \setminus \bar{B} = \bigcup_{k=1}^{\sum_{i=1}^{\bar{m}} \bar{N}_i} \bar{F}_k. \tag{11.194}$$

We may readily prove next that $\bar{F} = (\bar{F}_k \,|\, k \in \{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\})$ is a sequence of pairwise disjoint sets, by verifying

$$\forall k, l\, ([k, l \in \left\{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\right\} \wedge\, k \neq l] \Rightarrow \bar{F}_k \cap \bar{F}_l = \emptyset). \tag{11.195}$$

We take arbitrary $k, l \in \{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\}$ satisfying $k \neq l$, and we establish the desired consequent $\bar{F}_k \cap \bar{F}_l = \emptyset$ by means of the definition of the empty set, demonstrating the truth of the universal sentence

$$\forall y\, (y \notin \bar{F}_k \cap \bar{F}_l). \tag{11.196}$$

Letting y be arbitrary, we prove the desired negation $y \notin \bar{F}_k \cap \bar{F}_l$ by contradiction, assuming that the negation of that negation is true. We therefore obtain with the Double Negation Law the true sentence $y \in \bar{F}_k \cap \bar{F}_l$ and then with the definition of the intersection of two sets $y \in \bar{F}_k$ as well as $y \in \bar{F}_l$. In light of Theorem 11.6e), we notice that the assumed $k, l \in \{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\}$ implies that there are indexes, say $\bar{i}$ and $\bar{I}$, contained in $\{1, \ldots, \bar{m}\}$ and satisfying $\bar{j} \in \{1, \ldots, \bar{N}_{\bar{i}}$, $\bar{J} \in \{1, \ldots, \bar{N}_{\bar{I}}$, $\bar{F}_k = \bar{E}_{\bar{i}}(\bar{j})$, $\bar{F}_l = \bar{E}_{\bar{I}}(\bar{J})$ and moreover the disjunction

$$\bar{i} \neq \bar{I} \,\vee\, (\bar{i} = \bar{I} \,\wedge\, \bar{j} \neq \bar{J}). \tag{11.197}$$

Therefore, the previously found $y \in \bar{F}_k$ and $y \in \bar{F}_l$ give via substitutions $y \in \bar{E}_{\bar{i}}(\bar{j})$ as well as $y \in \bar{E}_{\bar{I}}(\bar{J})$, which sentences demonstrate the truth of the existential sentences

$$\exists j\,(j \in \{1,\ldots,\bar{N}_{\bar{i}}\} \,\wedge\, y \in \bar{E}_{\bar{i}}(j)),$$
$$\exists j\,(j \in \{1,\ldots,\bar{N}_{\bar{I}}\} \,\wedge\, y \in \bar{E}_{\bar{I}}(j)).$$

Consequently, we obtain with the Characterization of the union of a family of sets

$$y \in \bigcup_{j=1}^{\bar{N}_{\bar{i}}} \bar{E}_{\bar{i}}(j) \quad \left[= \bigcup \operatorname{ran}(\bar{E}_{\bar{i}}) = \bar{C}_{\bar{i}} \setminus \bigcup \operatorname{ran}(\bar{D})\right],$$
$$y \in \bigcup_{j=1}^{\bar{N}_{\bar{I}}} \bar{E}_{\bar{I}}(j) \quad \left[= \bigcup \operatorname{ran}(\bar{E}_{\bar{I}}) = \bar{C}_{\bar{I}} \setminus \bigcup \operatorname{ran}(\bar{D})\right],$$

and then the true conjunction

$$y \in \bar{C}_{\bar{i}} \setminus \bigcup \operatorname{ran}(\bar{D}) \,\wedge\, y \in \bar{C}_{\bar{I}} \setminus \bigcup \operatorname{ran}(\bar{D}),$$

which implies with the definition of a set difference

$$[y \in \bar{C}_{\bar{i}} \,\wedge\, y \notin \bigcup \operatorname{ran}(\bar{D})] \,\wedge\, [y \in \bar{C}_{\bar{I}} \,\wedge\, y \notin \bigcup \operatorname{ran}(\bar{D})].$$

Therefore,

$$[y \in \bar{C}_{\bar{i}} \,\wedge\, y \in \bar{C}_{\bar{I}}] \,\wedge\, [y \notin \bigcup \operatorname{ran}(\bar{D}) \,\wedge\, y \notin \bigcup \operatorname{ran}(\bar{D})],$$

follows to be true with the Associative and the Commutative Law for the conjunction, so that

$$y \in \bar{C}_{\bar{i}} \,\wedge\, y \in \bar{C}_{\bar{I}}$$

is true in particular. An application of the definition of the intersection gives us

$$y \in \bar{C}_{\bar{i}} \cap \bar{C}_{\bar{I}}, \tag{11.198}$$

which allows us now to prove the contradiction $y \in \emptyset \wedge y \notin \emptyset$ by cases using the true disjunction (11.197). Let us first observe the truth of $y \notin \emptyset$ in light of the definition of the empty set. The first case $\bar{i} \neq \bar{I}$ implies $\bar{C}_{\bar{i}} \cap \bar{C}_{\bar{I}} = \emptyset$ with the fact that the previously established sequence of sets $\bar{C} : \{1,\ldots,\bar{m}\} \to \mathcal{S}$ has pairwise disjoint terms. Applying then a substitution based on the preceding equation to (11.198) yields $y \in \emptyset$, as desired. The second case $\bar{i} = \bar{I} \wedge \bar{j} \neq \bar{J}$ implies in particular the truth of the equation

$\bar{i} = \bar{I}$, which we may use to infer from the previously obtained $y \in \bar{E}_{\bar{i}}(\bar{j})$ and $y \in \bar{E}_{\bar{I}}(\bar{J})$ the truth of $y \in \bar{E}_{\bar{i}}(\bar{j})$ and $y \in \bar{E}_{\bar{i}}(\bar{J})$ via a substitution. The definition of the intersection of two sets then yields $y \in \bar{E}_{\bar{i}}(\bar{j}) \cap \bar{E}_{\bar{i}}(\bar{J})$. Because of (11.182), the sequence of sets $\bar{E}_{\bar{i}}$ has pairwise disjoint terms, which is why the current case assumption $\bar{j} \neq \bar{J}$ implies $\bar{E}_{\bar{i}}(\bar{j}) \cap \bar{E}_{\bar{i}}(\bar{J}) = \emptyset$. Consequently, we obtain $y \in \emptyset$ via substitution, which finding thus establishes the proposed contradiction in any case, completing the proof of the negation in (11.196). Since y is arbitrary, we may therefore conclude that the universal sentence (11.196) holds, so that the equation $\bar{E}_k \cap \bar{E}_l = \emptyset$ follows to be true as well. Because k and l were arbitrary as well, we may infer from the truth of that equation the truth of the universal sentence (11.195), which means that the terms of the sequence $\bar{F} : \{1, \ldots, \sum_{i=1}^{\bar{m}} \bar{N}_i\} \to \mathcal{S}$ are disjoint; here, $\sum_{i=1}^{\bar{m}} \bar{N}_i$ is a positive natural number, and the sequence $\bar{F}$ satisfies (11.194).

Thus, the existential sentence in (11.149) is satisfied by $\bar{A} \setminus \bar{B}$, and since we already found $\bar{A} \setminus \bar{B} \in \mathcal{P}(\Omega)$ to be true as well, $\bar{A} \setminus \bar{B} \in \mathcal{X}$ follows then to be true with (11.149). Since $\bar{A}$ and $\bar{B}$ were arbitrary, we may infer from the preceding the finding the truth of (11.175), which universal sentence shows us that the set system $\mathcal{X}$ is indeed closed under set differences, satisfying thus Property 4 of a ring of sets.

It will now be useful to show that $\mathcal{X}$ is closed under symmetric differences, i.e.

$$\forall A, B\, (A, B \in \mathcal{X} \Rightarrow A \Delta B \in \mathcal{X}). \tag{11.199}$$

We take arbitrary sets A and B in $\mathcal{X}$, so that the set differences $A \setminus B$ and $B \setminus A$ follow to be both in $\mathcal{X}$ as well, according to (11.175). Because of (11.149), $A \setminus B \in \mathcal{X}$ and $B \setminus A \in \mathcal{X}$ imply the existence of particular constants $\bar{n}_1, \bar{n}_2 \in \mathbb{N}_+$ as well as of particular sequences $\bar{C}_1 : \{1, \ldots, \bar{n}_1\} \to \mathcal{S}$ and $\bar{C}_2 : \{1, \ldots, \bar{n}_2\} \to \mathcal{S}$ satisfying the universal sentences

$$\forall i, j\, ([i, j \in \{1, \ldots, \bar{n}_1\} \wedge i \neq j] \Rightarrow \bar{C}_1(i) \cap \bar{C}_1(j) = \emptyset) \tag{11.200}$$

$$\forall i, j\, ([i, j \in \{1, \ldots, \bar{n}_2\} \wedge i \neq j] \Rightarrow \bar{C}_2(i) \cap \bar{C}_2(j) = \emptyset) \tag{11.201}$$

and the equations

$$A \setminus B = \bigcup \operatorname{ran}(\bar{C}_1) \quad \left[= \bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j)\right], \tag{11.202}$$

$$B \setminus A = \bigcup \operatorname{ran}(\bar{C}_2) \quad \left[= \bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j)\right]. \tag{11.203}$$

According to the definition of the symmetric difference, we may therefore write

$$A\Delta B = (A \setminus B) \cup (B \setminus A) = \left(\bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j)\right) \cup \left(\bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j)\right). \tag{11.204}$$

We then obtain the true equations

$$\emptyset = (A \setminus B) \cap (B \setminus A) = \left(\bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j)\right) \cap \left(\bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j)\right) \tag{11.205}$$

with (2.106) and (11.204).

Next, we use Proposition 4.85 to define on the one hand the sequence $\bar{F} = (\bar{C}_i \,|\, i \in \{1,2\})$ having the terms $\bar{F}_1 = \bar{C}_1$ and $\bar{F}_2 = \bar{C}_2$, and on the other hand the sequence $\bar{N} = (\bar{n}_i \,|\, i \in \{1,2\})$ in $\mathbb{N}_+$ with terms $\bar{N}_1 = \bar{n}_1$ and $\bar{N}_2 = \bar{n}_2$. Since $\mathrm{ran}(\bar{N}) \subseteq \mathbb{N}_+$ is thus true by definition of a codomain and as the inclusion $\mathbb{N}_+ \subseteq \mathbb{N}$ holds according to (2.308), we obtain $\mathrm{ran}(\bar{N}) \subseteq \mathbb{N}$ with (2.13). This inclusion shows that $\bar{N}$ is a sequence in $\mathbb{N}$, i.e. $\bar{N} : \{1,2\} \to \mathbb{N}$. Furthermore, we see in light of the definition of a pair that $\bar{C}_i : \{1,\ldots,\bar{n}_i\} \to \mathcal{S}$ holds for all $i \in \{1,2\}$. We may therefore apply Stacking of a finite sequence of finite sequences to obtain the stacked sequence $\bar{G}$ in $\mathcal{S}$ with domain $\{1,\ldots,\sum_{i=1}^{2} \bar{n}_i\}$ whose terms satisfy the two universal sentences

$$\forall k\,(k \in \{1,\ldots,\sum_{i=1}^{2} \bar{n}_i\} \tag{11.206}$$
$$\Rightarrow \exists i,j\,(i \in \{1,2\} \wedge j \in \{1,\ldots,\bar{n}_i\} \wedge \bar{G}_k = \bar{C}_i(j))),$$

$$\forall i,j\,([i \in \{1,2\} \wedge j \in \{1,\ldots,\bar{n}_i\}] \tag{11.207}$$
$$\Rightarrow \exists k\,(k \in \{1,\ldots,\sum_{i=1}^{2} \bar{n}_i\} \wedge \bar{C}_i(j) = \bar{G}_k)).$$

Here, we may also write $\sum_{i=1}^{2} \bar{n}_i = \bar{n}_1 + \bar{n}_2$ in view of (5.413). Based on these findings, we apply now the Equality Criterion for sets to prove the equation

$$\left(\bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j)\right) \cup \left(\bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j)\right) = \bigcup_{k=1}^{\bar{n}_1+\bar{n}_2} \bar{G}_k. \tag{11.208}$$

To do this, we verify the equivalent universal sentence

$$\forall y \, (y \in \left(\bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j) \right) \cup \left(\bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j) \right) \Leftrightarrow y \in \bigcup_{k=1}^{\bar{n}_1+\bar{n}_2} \bar{G}_k), \tag{11.209}$$

letting y be arbitrary. Addressing the first part ('$\Rightarrow$') of the equivalence, we assume

$$y \in \left(\bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j) \right) \cup \left(\bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j) \right), \tag{11.210}$$

so that the disjunction

$$y \in \bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j) \vee y \in \bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j) \tag{11.211}$$

follows to be true by definition of the union of two sets. We use this disjunction to prove the desired consequent $y \in \bigcup_{k=1}^{\bar{n}_1+\bar{n}_2} \bar{G}_k$ by cases. The first case $y \in \bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j)$ implies with the Characterization of the union of a family of sets that there is a particular index $\bar{J} \in \{1, \ldots, \bar{n}_1\}$ with $y \in \bar{C}_1(\bar{J})$. Since $1 \in \{1, 2\}$ is clearly true, it follows now with (11.207) that there is a particular index $\bar{K} \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ such that $\bar{C}_1(\bar{J}) = \bar{G}_{\bar{K}}$ holds. Therefore, $y \in \bar{C}_1(\bar{J})$ yields via substitution $y \in \bar{G}_{\bar{K}}$, which demonstrates the existence of some $k \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ satisfying $y \in \bar{G}_k$, so that the Characterization of the union of a family of sets gives

$$y \in \bigcup_{k=1}^{\bar{n}_1+\bar{n}_2} \bar{G}_k, \tag{11.212}$$

as desired. Similarly, the second case $y \in \bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j)$ implies the existence of a particular index $\bar{\bar{J}} \in \{1, \ldots, \bar{n}_2\}$ with $y \in \bar{C}_2(\bar{\bar{J}})$. Because of the evident $2 \in \{1, 2\}$, we obtain with (11.207) a particular index $\bar{\bar{K}} \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ such that $\bar{C}_2(\bar{\bar{J}}) = \bar{G}_{\bar{\bar{K}}}$ is true. Consequently, substitution yields $y \in \bar{G}_{\bar{\bar{K}}}$, which show that there is a $k \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ with $y \in \bar{G}_k$. Therefore, we arrive again at the desired (11.212).

Regarding the second part ('$\Leftarrow$') of the equivalence in (11.209), we conversely assume (11.212) to be true, so that $y \in \bar{G}_K$ is true for a particular index $K \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$. This in turn implies with (11.206) the existence of particular indexes $I \in \{1, 2\}$ and $J \in \{1, \ldots, \bar{n}_I\}$ satisfying $\bar{G}_K = \bar{C}_I(J)$; we thus obtain $y \in \bar{C}_I(J)$ via substitution. By definition of a pair, $I \in \{1, 2\}$ implies $I = 1 \vee I = 2$, which disjunction we use now to prove (11.211) by cases. On the one hand, $I = 1$ yields $y \in \bar{C}_1(J)$ and $J \in \{1, \ldots, \bar{n}_1\}$ through substitutions, so that $y \in \bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j)$ follows to be true; then, the disjunction (11.211) to be proven also holds. On the other, $I = 2$ gives similarly to the first case $y \in \bar{C}_2(J)$ and $J \in \{1, \ldots, \bar{n}_2\}$, with the consequence that $y \in \bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j)$; thus, the disjunction (11.211) is true again, so that the proof by cases is complete. That disjunction implies now the truth of (11.210) by definition of the union of two sets. This completes the proof of the second part of the equivalence in (11.209), and since y was arbitrary, we may therefore conclude that the universal sentence (11.209) is true.

Consequently, the stated equation (11.208) holds indeed, which we may combine with the equations (11.204) to obtain

$$A \Delta B = \bigcup_{k=1}^{\bar{n}_1+\bar{n}_2} \bar{G}_k. \tag{11.213}$$

Next, we check that the terms of $\bar{G} = (\bar{G}_k \,|\, k \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\})$ are pairwise disjoint, i.e. that the sequence $\bar{G}$ satisfies

$$\forall k, l\, ([k, l \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\} \,\wedge\, k \neq l] \Rightarrow \bar{G}_k \cap \bar{G}_l = \emptyset). \tag{11.214}$$

We take arbitrary indexes $k, l \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ such that $k \neq l$ holds, and we establish $\bar{G}_k \cap \bar{G}_l = \emptyset$ by means of the definition of the empty set, i.e. by verifying

$$\forall x\, (x \notin \bar{G}_k \cap \bar{G}_l). \tag{11.215}$$

We let x be arbitrary, and we prove the sentence $x \notin \bar{G}_k \cap \bar{G}_l$ by contradiction, by establishing the contradiction

$$x \in \emptyset \,\wedge\, x \notin \emptyset. \tag{11.216}$$

For this purpose, we assume that the negation of the sentence to be proven holds. This assumption implies the truth of $x \in \bar{G}_k \cap \bar{G}_l$ with the Double Negation Law, and subsequently also of $x \in \bar{G}_k \,\wedge\, x \in \bar{G}_l$ (using the definition of the intersection of two sets). Now, Part e) of the Stacking of a finite sequence of finite sequences shows us that $k, l \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ implies the existence of particular constants i_1, i_2, j_1, j_2 satisfying simultaneously $i_1, i_2 \in \{1, 2\}$, $j_1 \in \{1, \ldots, \bar{N}_{i_1}\}$, $\bar{G}_k = \bar{C}_{i_1}(j_1)$, $j_2 \in \{1, \ldots, \bar{N}_{i_2}\}$, $\bar{G}_l = \bar{C}_{i_2}(j_2)$ and

$$i_1 \neq i_2 \,\vee\, (i_1 = i_2 \,\wedge\, j_1 \neq j_2). \tag{11.217}$$

Because of the two implied equations, the truth of $x \in \bar{G}_k$ and $x \in \bar{G}_l$ implies via substitutions the truth of $x \in \bar{C}_{i_1}(j_1)$ and $x \in \bar{C}_{i_2}(j_2)$, respectively. These findings show on the one hand that $x \in \bar{C}_{i_1}(j)$ is true for some $j \in \{1, \ldots, \bar{N}_{i_1}\}$, and on the other hand that $x \in \bar{C}_{i_2}(j)$ holds for some $j \in \{1, \ldots, \bar{N}_{i_2}\}$. These two existential sentences further imply $x \in \bigcup_{j=1}^{\bar{N}_{i_1}} \bar{C}_{i_1}(j)$ and $x \in \bigcup_{j=1}^{\bar{N}_{i_2}} \bar{C}_{i_2}(j)$ with the Characterization of the union of a family of sets, so that the definition of the intersection of two sets yields

$$x \in \left(\bigcup_{j=1}^{\bar{N}_{i_1}} \bar{C}_{i_1}(j) \right) \cap \left(\bigcup_{j=1}^{\bar{N}_{i_2}} \bar{C}_{i_2}(j) \right). \tag{11.218}$$

Let us now use the true disjunction (11.217) to establish the contradiction (11.216), in which the second part $x \notin \emptyset$ is already true by definition of the empty set). Regarding the first case $i_1 \neq i_2$, let us observe that the previously established $i_1, i_2 \in \{1, 2\}$ implies (by definition of a pair) the truth of the disjunctions

$$i_1 = 1 \lor i_1 = 2, \tag{11.219}$$

$$i_2 = 1 \lor i_2 = 2. \tag{11.220}$$

We use the first of these disjunctions to consider the two subcases $i_1 = 1$ and $i_1 = 2$ within the current first case $i_1 \neq i_2$. The first subcase $i_1 = 1$ allows us to prove $i_2 = 2$ by contradiction. Indeed, assuming the negation $\neg i_2 = 2$ implies the truth of the first part $i_2 = 1$ of the disjunction (11.220) and thus the truth of $i_1 = i_2$, which contradicts the case assumption $i_1 \neq i_2$. Thus, $i_1 = 1$ and $i_2 = 2$ are both true, so that we obtain $\bar{N}_{i_1} = \bar{N}_1 = \bar{n}_1$ as well as $\bar{N}_{i_2} = \bar{N}_2 = \bar{n}_2$. Substitutions in (11.218) based on these equations give then

$$x \in \left(\bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j) \right) \cap \left(\bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j) \right), \tag{11.221}$$

and another substitution based on (11.205) yields $x \in \emptyset$, as required by the proposed contradiction (11.216). In analogy to the first subcase, the second subcase $i_1 = 2$ leads to $i_2 = 1$, because assuming $\neg i_2 = 1$ to be true implies the truth of the second part $i_2 = 2$ of the disjunction (11.220) and therefore the truth of $i_1 = i_2$, in contradiction to the current case assumption $i_1 \neq i_2$. Having established the equations $i_1 = 2$ and $i_2 = 1$, we may carry out substitutions to obtain first the equations $\bar{N}_{i_1} = \bar{N}_2 = \bar{n}_2$ and $\bar{N}_{i_2} = \bar{N}_1 = \bar{n}_1$. Therefore, (11.218) yields

$$x \in \left(\bigcup_{j=1}^{\bar{n}_2} \bar{C}_2(j) \right) \cap \left(\bigcup_{j=1}^{\bar{n}_1} \bar{C}_1(j) \right),$$

which we may write also as (11.221) due to the Commutative Law of the intersection of two sets. In view of (11.205), we thus arrive at $x \in \emptyset$ also in the second subcase, so that the proof of the first case is complete.

In the second case, we assume that $i_1 = i_2 \wedge j_1 \neq j_2$ is true, so that the previously obtained $x \in \bar{C}_{i_1}(j_1)$ and $x \in \bar{C}_{i_2}(j_2)$ imply first $x \in \bar{C}_{i_1}(j_1) \wedge x \in \bar{C}_{i_1}(j_2)$ and then $x \in \bar{C}_{i_1}(j_1) \cap \bar{C}_{i_1}(j_2)$ with the definition of the intersection of two sets. We now apply a proof by (sub-)cases based on the true disjunction (11.219) to prove $\bar{C}_{i_1}(j_1) \cap \bar{C}_{i_1}(j_2) = \emptyset$. If $i_1 = 1$ holds (besides the assumed $i_1 = i_2$), then these equations yields $\bar{N}_{i_1} = \bar{N}_1 = \bar{n}_1$ as well as $\bar{N}_{i_2} = \bar{N}_{i_1} = \bar{N}_1 = \bar{n}_1$. Consequently, $j_1 \in \{1, \ldots, \bar{N}_{i_1}\}$ and $j_2 \in \{1, \ldots, \bar{N}_{i_2}\}$ imply $j_1, j_2 \in \{1, \ldots, \bar{n}_1\}$ and $j_1 \neq j_2$ are both true, so that (11.200) gives us $\bar{C}_1(j_1) \cap \bar{C}_1(j_2) = \emptyset$ and therefore (after back-substitutions) indeed $\bar{C}_{i_1}(j_1) \cap \bar{C}_{i_1}(j_2) = \emptyset$. Similarly, in the other sub-case of $i_1 = 2$, we obtain $\bar{N}_{i_1} = \bar{N}_2 = \bar{n}_2$ and $\bar{N}_{i_2} = \bar{N}_{i_1} = \bar{N}_2 = \bar{n}_2$. With these equations, $j_1 \in \{1, \ldots, \bar{N}_{i_1}\}$ and $j_2 \in \{1, \ldots, \bar{N}_{i_2}\}$ imply the truth of $j_1, j_2 \in \{1, \ldots, \bar{n}_2\}$, where $j_1 \neq j_2$ also holds by assumption. It then follows with (11.201) that $\bar{C}_2(j_1) \cap \bar{C}_2(j_2) = \emptyset$ holds, which evidently gives $\bar{C}_{i_1}(j_1) \cap \bar{C}_{i_1}(j_2) = \emptyset$ again. We thus find $x \in \emptyset$ in the second case as well, so that the proof of the contradiction (11.216) is complete.

Therefore, the sentence $x \notin \bar{G}_k \cap \bar{G}_l$ is true, and since x was arbitrary, (11.215) follows to be true as well. This universal sentence in turn implies $\bar{G}_k \cap \bar{G}_l = \emptyset$, and as k and l were arbitrary, we may further conclude that (11.214) holds. Thus, $\bar{G} : \{1, \ldots, \bar{n}_1 + \bar{n}_2\} \to \mathcal{S}$ is a sequence having pairwise disjoint terms, which we already showed to satisfy (11.213) and where $\bar{n}_1, \bar{n}_2 \in \mathbb{N}_+$ evidently implies $n_1 + n_2 \in \mathbb{N}_+$. Thus, the symmetric difference $A \Delta B$ satisfies the existential sentence in (11.149). Moreover, the initial assumption $A, B in \mathcal{X}$ implies $A, B \in \mathcal{P}(\Omega)$ with (11.149), which evidently gives $A \setminus B, B \setminus A \in \mathcal{P}(\Omega)$ and then

$$[A \Delta B =] \quad (A \setminus B) \cup (B \setminus A) \in \mathcal{P}(\Omega)$$

with the previously mentioned fact that $\mathcal{P}(\Omega)$ constitutes a ring of sets. Then, the conjunction of the resulting $A \Delta B \in \mathcal{P}(\Omega)$ and the preceding existential sentence implies $A \Delta B \in \mathcal{X}$ again with (11.149), proving the implication in (11.199). Since A and B were arbitrary, we may infer from the truth of that implication the truth of the universal sentence (11.199), so that the set system $\mathcal{X}$ is indeed closed under symmetric differences.

Recalling from (11.153) that $\mathcal{X}$ is closed under pairwise intersection, we are ready to show that $\mathcal{X}$ is closed also under pairwise unions, as required by Property 3 of a ring of sets. To do this, we prove the universal sentence

$$\forall A, B\, (A, B \in \mathcal{X} \Rightarrow A \cup B \in \mathcal{X}), \tag{11.222}$$

letting $A, B \in \mathcal{X}$ be arbitrary. We then obtain $A \Delta B \in \mathcal{X}$ with 11.199 and $A \cap B \in \mathcal{X}$ with (11.153), i.e. $A \Delta B, A \cap B \in \mathcal{X}$. This finding implies

$$[A \cup B =] \quad (A \Delta B) \, \Delta \, (A \cap B) \in \mathcal{X}$$

by means of (2.277) and 11.199, which shows that $A \cup B \in \mathcal{X}$ is true. As A and B are arbitrary, we may therefore conclude that the universal sentence (11.222) holds. Thus, $\mathcal{X}$ satisfies Property 3 of a ring of sets.

Having established $\mathcal{X}$ as a ring of sets on Ω (that includes the semiring of sets $\mathcal{S}$ on Ω), it now follows with Theorem 11.18c) that the ring of sets $\mathcal{R}(\mathcal{S})$ is included in $\mathcal{X}$. The conjunction of the already established inclusion $\mathcal{X} \subseteq \mathcal{R}(\mathcal{S})$ and the preceding inclusion $\mathcal{R}(\mathcal{S}) \subseteq \mathcal{X}$ gives then the equation (11.150) via the Axiom of Extension. Because Ω and $\mathcal{S}$ were initially arbitrary sets, the stated theorem finally follows to be true. □

Note 11.7. The preceding theorem shows for any set Ω and for any semiring of sets $\mathcal{S}$ on Ω that the generated ring of sets $\mathcal{R}(\mathcal{S})$ contains precisely every subset of Ω which may be written as the union of some sequence of disjoint sets in $\mathcal{S}$ with finite domain.

Clearly, we may then use a semiring $\mathcal{I} = \{[a, b) : a, b \in \Omega\}$ as a generating system to produce a ring of sets based on left-closed and right-open intervals.

Definition 11.8 (Ring of one-dimensional figures, one-dimensional figure). We call for any linearly ordered set $(\Omega, <)$ with $\Omega \neq \emptyset$ the generated ring of sets

$$\mathcal{F} = \mathcal{R}(\mathcal{I}) \tag{11.223}$$

(generated by the semiring of left-closed and right-open intervals in Ω) the *ring of one-dimensional figures* in Ω. Here, we call each element of $\mathcal{F}$ a *one-dimensional figure* (in Ω).

Note 11.8. According to the Generation of rings of sets by means of semirings of sets, every one-dimensional figure is the union of a finite number of disjoint left-closed and right-open intervals.

11.4. Algebras of Sets

The following type of set system, which is closed under pairwise unions and complementation, further specializes a ring of sets and may also be viewed as a particular kind of Boolean algebra.

Definition 11.9 (Algebra of sets). For any set Ω we say that a set $\mathcal{A}$ is an *algebra of sets* on Ω iff

1. $\mathcal{A}$ consists of subsets of Ω, i.e.
$$\mathcal{A} \subseteq \mathcal{P}(\Omega), \tag{11.224}$$

2. $\mathcal{A}$ contains the empty set, i.e.
$$\emptyset \in \mathcal{A}, \tag{11.225}$$

3. $\mathcal{A}$ contains the union of any two sets in $\mathcal{A}$, i.e.
$$\forall A, B\, (A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}), \tag{11.226}$$
and

4. $\mathcal{A}$ contains the complement (with respect to Ω) of any set in $\mathcal{A}$, i.e.
$$\forall A\, (A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}). \tag{11.227}$$

Because (11.225) implies $[\Omega =]\ \emptyset^c \in \mathcal{A}$ with (2.133) and (11.227) for an arbitrary algebra of sets $\mathcal{A}$ on an arbitrary set Ω, we obtain the following characteristic property of algebras of sets.

Corollary 11.20. *Any algebra of sets $\mathcal{A}$ on any set Ω contains Ω, i.e.*
$$\Omega \in \mathcal{A}. \tag{11.228}$$

Proposition 11.21. *For any set Ω, it is true that the ring of sets $\mathcal{R} = \{\emptyset\}$ on Ω is an algebra of sets on Ω iff $\Omega = \emptyset$.*

Proof. We let Ω be an arbitrary set, so that $\mathcal{R} = \{\emptyset\}$ is a ring of sets on Ω according to Exercise 11.3. To establish the first part ('$\Rightarrow$') of the stated equivalence, we assume that $\{\emptyset\}$ is an algebra of sets, so that $\Omega \in \{\emptyset\}$ holds in view of the preceding Corollary 11.20. This in turn implies $\Omega = \emptyset$ with (2.169), as desired.

Regarding the second part ('$\Leftarrow$'), we now assume that $\Omega = \emptyset$ holds. As a ring of sets, $\{\emptyset\}$ satisfies evidently the first three properties of an algebra of sets. Concerning Property 4, we take an arbitrary $A \in \{\emptyset\}$, so that $A = \emptyset$ follows to be true with (2.169). Since $\emptyset \subseteq \emptyset\ [= \Omega]$ holds according to (2.45), we may form the complement of $A = \emptyset$ with respect to Ω, which is $A^c = \emptyset^c = \Omega = \emptyset$. Because $\emptyset \in \{\emptyset\}$ is evidently true, we obtain via substitution based on the previous equations the desired consequent $A^c \in \{\emptyset\}$. As A was arbitrary, we may therefore conclude that $\{\emptyset\}$ satisfies Property 4 of an algebra of sets. Thus, $\{\emptyset\}$ is indeed an algebra of sets on $\Omega = \emptyset$, completing the proof of the equivalence.

Since Ω was initially arbitrary, we may now further conclude that the proposed universal sentence is true. □

Note 11.9. The equivalence in Proposition 11.21 shows that the ring of sets $\mathcal{R} = \{\emptyset\}$ on a set $\Omega \neq \emptyset$ is not an algebra of sets on Ω.

To turn $\{\emptyset\}$ into an algebra of sets on an arbitrary (empty or nonempty) set Ω, we apparently need to incorporate Ω into the set.

Proposition 11.22. *It is true for any set Ω that the pair $\{\emptyset, \Omega\}$ constitutes an algebra of sets on Ω.*

Proof. We let Ω be an arbitrary set. We establish Property 1 of an algebra of sets for $\{\emptyset, \Omega\}$ by verifying

$$\forall A\,(A \in \{\emptyset, \Omega\} \Rightarrow A \in \mathcal{P}(\Omega)). \tag{11.229}$$

Letting $A \in \{\emptyset, \Omega\}$ be arbitrary, we have by definition of a pair that $A = \emptyset$ or $A = \Omega$ is true. Let us observe in light of (3.15) $\emptyset, \Omega \in \mathcal{P}(\Omega)$ holds. Thus, each of the cases $A = \emptyset$ and $A = \Omega$ gives the desired consequent $A \in \mathcal{P}(\Omega)$. Since A is arbitrary, we may therefore conclude that the universal sentence (11.229) is true, so that the definition of a subset yields the true inclusion $\{\emptyset, \Omega\} \subseteq \mathcal{P}(\Omega)$, as required by Property 1 of an algebra of sets.

Because $\emptyset \in \{\emptyset, \Omega\}$ holds according to (2.151), Property 2 is also satisfied by $\{\emptyset, \Omega\}$.

Regarding Property 3, we take arbitrary sets $A, B \in \{\emptyset, \Omega\}$, so that the definition of a pair gives the true disjunctions $A = \emptyset \vee A = \Omega$ and $B = \emptyset \vee B = \Omega$. In the first case $A = \emptyset$ and the first subcase $B = \emptyset$, we obtain

$$A \cup B = \emptyset \cup \emptyset = \emptyset \quad [\in \{\emptyset, \Omega\}],$$

and in the second subcase $B = \Omega$

$$A \cup B = \emptyset \cup \Omega = \Omega \quad [\in \{\emptyset, \Omega\}],$$

applying substitutions based on the (sub)case assumption, (2.216) and again (2.151). Similarly, the second case $A = \Omega$ and the first subcase $B = \emptyset$ yields

$$A \cup B = \Omega \cup \emptyset = \emptyset \quad [\in \{\emptyset, \Omega\}],$$

and with the second subcase $B = \Omega$ (using now the Idempotent Law for the union of two sets)

$$A \cup B = \Omega \cup \Omega = \Omega \quad [\in \{\emptyset, \Omega\}].$$

Thus, $A \cup B \in \{\emptyset, \Omega\}$ holds in any case, and since A and B were initially arbitrary sets, we may infer from this finding that $\{\emptyset, \Omega\}$ satisfies also Property 3 of an algebra of sets.

Finally, we may establish also Property 4 by taking an arbitrary $A \in \{\emptyset, \Omega\}$, so that $A = \emptyset$ or $A = \Omega$ holds. In case $A = \emptyset$, we obtain

$$A^c = \emptyset^c = \Omega \quad [\in \{\emptyset, \Omega\}]$$

due to (2.133), and in case of $A = \Omega$

$$A^c = \Omega^c = \emptyset \quad [\in \{\emptyset, \Omega\}]$$

because of (2.134). This completes the proof by cases of $A^c \in \{\emptyset, \Omega\}$, and as A was arbitrary, we may further conclude that Property 4 is satisfied as well by $\{\emptyset, \Omega\}$. This set system is thus an algebra of sets on Ω, and because Ω was initially arbitrary, we may finally conclude that the proposed universal sentence is true. □

Property 4 of an algebra of sets allows us to view such set systems as being closed under *complementations*.

Exercise 11.9. Show for any set Ω and any algebra of sets $\mathcal{A}$ on Ω that there exists the unique function

$$^c_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}, \quad A \mapsto A^c. \tag{11.230}$$

(Hint: Apply Method 3.3 in connection with (1.109).)

The other three properties have been considered already in the context of rings of sets.

Proposition 11.23. *Any algebra of sets $\mathcal{A}$ on any set Ω is*

a) a ring of sets on Ω.

b) a semiring of sets on Ω.

c) a π-system on Ω.

Proof. Letting Ω and $\mathcal{A}$ be arbitrary sets and assuming $\mathcal{A}$ to be an algebra of sets on Ω, we notice that $\mathcal{A}$ satisfies the first three properties of a ring of sets by virtue of (11.224) – (11.226). Regarding Property 4 of a ring of sets, we take arbitrary sets $A, B \in \mathcal{A}$ and show that $A \setminus B \in \mathcal{A}$ follows to be true. Because $A, B \in \mathcal{A}$ implies $A, B \in \mathcal{P}(\Omega)$ with (11.224) by definition of a subset, we may form the complements A^c,B^c and derive the equations

$$\begin{aligned} A \setminus B &= A \cap B^c \\ &= (A^c)^c \cap B^c \\ &= (A^c \cup B)^c \end{aligned}$$

by means of (2.138), (2.136) and De Morgan's Law for the union of two sets. The assumed $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$ with (11.227), and this implies together with the assumed $B \in \mathcal{A}$ that $A^c \cup B \in \mathcal{A}$ holds, according to (11.226). Consequently, $(A^c \cup B)^c \in \mathcal{A}$ is also true in view of (11.227), so that substitution based on the previous equations yields the desired $A \setminus B \in \mathcal{A}$. Since A and B are arbitrary, we may therefore conclude that $\mathcal{A}$ is closed under set difference and satisfies thus Property 4 of a ring of sets. As we established $\mathcal{A}$ as a ring of sets on Ω, we have that $\mathcal{A}$ as also a semiring of sets on Ω and a π-system on Ω due to Proposition 11.16. Because Ω and $\mathcal{A}$ were initially arbitrary, we may infer from these findings that the proposition is indeed true. □

Note 11.10. We have for any algebra of sets $\mathcal{A}$ on any set Ω the binary operations

$$\cup_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (A, B) \mapsto A \cup B, \tag{11.231}$$
$$\Delta_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (A, B) \mapsto A \Delta B, \tag{11.232}$$
$$\cap_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (A, B) \mapsto A \cap B, \tag{11.233}$$

according to Exercise 11.4, Exercise 11.7 and Note 11.6.

Exercise 11.10. Show for any set Ω and any algebra of sets $\mathcal{A}$ on Ω that Ω is the neutral element of $\mathcal{A}$ with respect to $\cap_{\mathcal{A}}$.
(Hint: Apply (2.77).)

Note 11.11. We may summarize the previous findings as follows: For any algebra of sets $\mathcal{A}$ on any set Ω,

a) $(\mathcal{A}, \cup_{\mathcal{A}})$ is a commutative semigroup with neutral element $\emptyset$ and idempotent $\cup_{\mathcal{A}}$,

b) $(\mathcal{A}, \Delta_{\mathcal{A}})$ is a commutative group with neutral element $\emptyset$,

c) $(\mathcal{A}, \cap_{\mathcal{A}})$ is a commutative semigroup with neutral element Ω and idempotent $\cap_{\mathcal{A}}$.

d) $(\mathcal{A}, \Delta_{\mathcal{A}}, \cap_{\mathcal{A}}, \Delta_{\mathcal{A}})$ is a commutative ring with zero element $\emptyset$ and unity element Ω.

e) $(\mathcal{A}, \Delta_{\mathcal{A}}, \cap_{\mathcal{A}}, \Delta_{\mathcal{A}})$ is a Boolean ring with unity element Ω.

Proposition 11.24. *For any set Ω and any algebra of sets $\mathcal{A}$ on Ω, it is true that the ordered quadruple $(\mathcal{A}, \cup_{\mathcal{A}}, \cap_{\mathcal{A}}, {}^c_{\mathcal{A}})$ is a Boolean algebra.*

Proof. Letting Ω and $\mathcal{A}$ be arbitrary sets and assuming $\mathcal{A}$ to be an algebra of sets on Ω, we view $\Delta_{\mathcal{A}}$ as an addition and $\cap_{\mathcal{A}}$ as the multiplication operation, according to Note 11.11d). We now verify first that the binary operation $\cup_{\mathcal{A}}$ maps every ordered pair (A, B) to $(A\Delta B)\Delta(A \cap B)$, as required by (6.59). For this purpose, we take arbitrary sets A, B and assume $(A, B) \in \mathcal{A}\times\mathcal{A}$ to be true, so that $\cup_{\mathcal{A}}(A, B) = A\cup B$. Because (2.277) yields also the equation $A \cup B = (A\Delta B)\Delta(A \cap B)$, we obtain via substitution

$$\cup_{\mathcal{A}}(A, B) = (A\Delta B)\Delta(A \cap B).$$

Because A and B are arbitrary, we may therefore conclude that $\cup_{\mathcal{A}}$ corresponds to the binary operation $\vee$ of a Boolean algebra. Then, we may take the other binary operation $\wedge$ of a Boolean algebra to be the 'multiplication' $\cap_{\mathcal{A}}$. It now remains for us to prove that the function ${}^c_{\mathcal{A}}$ maps every element A of $\mathcal{A}$ to $A\Delta\Omega$, as required by (6.60). Letting $A \in \mathcal{A}$ be arbitrary, we see that ${}^c_{\mathcal{A}}(A) = A^c$ holds; since $A \in \mathcal{A}$ implies $A \subseteq \mathcal{P}(\Omega)$ with Property 1 of an algebra of sets, so that $A \subseteq \Omega$ is true by definition of a power set, we also obtain $A^c = A\Delta\Omega$ with (2.271), because $A \in \mathcal{A}$, so that substitution yields

$${}^c_{\mathcal{A}}(A) = A\Delta\Omega.$$

As the set A was arbitrary, we may infer from this finding that ${}^c_{\mathcal{A}}$ corresponds to the function $'$ in a Boolean algebra. In summary, we thus have that the Boolean ring $(\mathcal{A}, \cup_{\mathcal{A}}, \cap_{\mathcal{A}}, {}^c_{\mathcal{A}})$ with unity element Ω constitutes a Boolean algebra. Since Ω and $\mathcal{A}$ were arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Definition 11.10 (Boolean algebra of sets). For any set Ω and any algebra of sets $\mathcal{A}$ on Ω, we call

$$(\mathcal{A}, \cup_{\mathcal{A}}, \cap_{\mathcal{A}}, {}^c_{\mathcal{A}}) \tag{11.234}$$

a *Boolean algebra of sets* (on Ω).

Note 11.12. Having established the binary operation $\cup_{\mathcal{A}}$ with neutral element $\emptyset \in \mathcal{A}$ and the binary operation $\cap_{\mathcal{A}}$ with neutral element $\Omega \in \mathcal{A}$, we are in a position to define the corresponding n-fold binary operations

$$\bigcup_{i=1}^{n} : \mathcal{A}^{\{1,\ldots,n\}} \to \mathcal{A}, \quad (A_i \,|\, i \in \{1, \ldots, n\}) \mapsto \bigcup_{i=1}^{n} A_i, \tag{11.235}$$

$$\bigcap_{i=1}^{n} : \mathcal{A}^{\{1,\ldots,n\}} \to \mathcal{A}, \quad (A_i \,|\, i \in \{1, \ldots, n\}) \mapsto \bigcap_{i=1}^{n} A_i \tag{11.236}$$

(for any natural number n and any algebra of sets $\mathcal{A}$ on any set Ω), so that algebras of sets are also closed under n-fold unions and intersections. Because algebras of sets are ring of sets, we observe in light of Proposition 11.10 that the n-fold union $\bigcup_{i=1}^{n}$ yields the union of the range of a sequence in $\mathcal{A}$ with domain $\{1,\ldots,n\}$ in the sense of Notation 4.7.

The following proposition establishes the preceding observation also for n-fold intersections.

Proposition 11.25. *It is true for any set Ω and any algebra of sets $\mathcal{A}$ on Ω that the n-fold binary intersection operation (11.236) satisfies for any $n \in \mathbb{N}_+$ and any sequence of sets $f = (A_i \,|\, i \in \{1,\ldots,n\})$ in $\mathcal{A}$ the equation*

$$\bigcap_{i=1}^{n} A_i = \bigcap \mathrm{ran}(f). \tag{11.237}$$

Proof. We let Ω and $\mathcal{A}$ be arbitrary sets, we assume that $\mathcal{A}$ is an algebra of sets on Ω, and we apply a proof by mathematical induction to establish the stated universal sentence with respect to n. In the base case $n = 1$, we take an arbitrary set f and assume that $f = (A_i\, i \in \{1,\ldots,1\})$ is a sequence of sets in $\mathcal{A}$. According to the notation for initial segments of $\mathbb{N}_+$, we have that the domain $\{1,\ldots,1\}$ of this sequence equals $\{1\}$. Therefore, this sequence is the singleton $f = \{(1, A_1)\}$ according to Proposition 3.159, which singleton is a surjection from $\{1\}$ to $\{A_1\}$ because of Corollary 3.194, so that $\mathrm{ran}(f) = \{A_1\}$ holds. Consequently, we obtain $\bigcap \mathrm{ran}(f) = A_1$ with (2.170). On the other hand, we have $\bigcap_{i=1}^{1} A_i = A_1$ $[= \bigcap \mathrm{ran}(f)]$ according to (5.391), so that the equation (11.237) holds for $n = 1$.

Regarding the induction step, we now take an arbitrary positive natural number n and make the induction assumption that (11.237) holds for any sequence of sets $(A_i \,|\, i \in \{1,\ldots,n\})$ in $\mathcal{A}$. Letting now f be arbitrary and assuming $f = (A_i \,|\, i \in \{1,\ldots,n+1\})$ to be a sequence of sets in $\mathcal{A}$, we need to demonstrate the truth of the equation

$$\bigcap_{i=1}^{n+1} A_i = \bigcap \mathrm{ran}(f). \tag{11.238}$$

To do this, we apply the Equality Criterion for sets and take an arbitrary y, assuming first $y \in \bigcap_{i=1}^{n+1} A_i$ to hold. In view of (5.394), we therefore obtain $y \in (\bigcap_{i=1}^{n} A_i) \cap A_{n+1}$, and the conjunction of $y \in \bigcap_{i=1}^{n} A_i$ and $y \in A_{n+1}$ is then also true by definition of the intersection of two sets. The first part of this conjunction implies with the induction assumption that y is in the intersection of the range of the (restricted) sequence of sets $(A_i \,|\, i \in \{1,\ldots,n\})$; consequently, the universal sentence

$$\forall i\,(i \in \{1,\ldots,n\} \Rightarrow y \in A_i) \tag{11.239}$$

holds according to the Characterization of the intersection of a sequence of sets. We use the latter argument now also to establish $y \in \bigcap \operatorname{ran}(f)$, which we do by proving accordingly

$$\forall i\,(i \in \{1,\ldots,n+1\} \Rightarrow y \in A_i), \tag{11.240}$$

letting $i \in \{1,\ldots,n+1\}$ $[= \{1,\ldots,n\} \cup \{n+1\}]$ be arbitrary, so that $i \in \{1,\ldots,n\}$ or $i \in \{n+1\}$ holds by definition of the union of two sets. On the one hand, if $i \in \{1,\ldots,n\}$ is true, then (11.239) gives $y \in A_i$, as desired. On the other hand, if $i \in \{n+1\}$ holds, then (2.169) yields $i = n+1$, so that the previously established $y \in A_{n+1}$ gives the desired consequent $y \in A_i$ via substitution. As i was arbitrary, we may therefore conclude that the universal sentence (11.240) is true, with the mentioned consequence that $y \in \bigcap \operatorname{ran}(f)$.

We now conversely assume $y \in \bigcap \operatorname{ran}(f)$ to be true, so that (11.240) follows to be true with the Characterization of the intersection of a sequence of sets. Let us now establish the truth of (11.239). We take an arbitrary $i \in \{1,\ldots,n\}$, so that the disjunction $i \in \{1,\ldots,n\} \vee i \in \{n+1\}$ is also true, so that we evidently obtain $i \in \{1,\ldots,n\} \cup \{n+1\}$ $[= \{1,\ldots,n+1\}]$. Therefore, (11.240) yields $y \in A_i$, proving the implication in (11.239). Since i is arbitrary, we may therefore conclude that the universal sentence (11.239) holds, which in turn implies that y is in the intersection of (the range) of the sequence of sets $(A_i \,|\, i \in \{1,\ldots,n\})$. Consequently, $y \in \bigcap_{i=1}^{n} A_i$ follows to be true with the induction assumption. Next, we observe the evident truth of $n+1 \in \{1,\ldots,n+1\}$, which implies $y \in A_{n+1}$ with (11.240). Thus, $y \in \bigcap_{i=1}^{n} A_i$ and $y \in A_{n+1}$ are both true, so that $y \in (\bigcap_{i=1}^{n} A_i) \cap A_{n+1}$ $[= \bigcap_{i=1}^{n+1} A_i]$ also holds.

We may infer from the previous findings that $y \in \bigcap_{i=1}^{n+1} A_i$ implies $y \in \bigcap \operatorname{ran}(f)$ and vice versa for any y, so that the equation (11.238) follows to be true. Since f and n were arbitrary, we may therefore conclude that the induction step holds (besides the base case). As Ω and $\mathcal{A}$ were initially arbitrary, we may finally conclude that proposed universal sentences are true. □

11.5. λ-Systems

Definition 11.11 (λ-/Dynkin system). For any set Ω we say that a set $\mathcal{D}$ is a *λ-system* (alternatively, a *Dynkin system*) on Ω iff

1. $\mathcal{D}$ consists of subsets of Ω, that is,

$$\mathcal{D} \subseteq \mathcal{P}(\Omega), \tag{11.241}$$

2. $\mathcal{D}$ contains Ω, that is,
$$\Omega \in \mathcal{D}, \tag{11.242}$$

3. $\mathcal{D}$ contains the union of any sequence $A = (A_n)_{n\in\mathbb{N}_+}$ of pairwise disjoint sets in $\mathcal{D}$, that is,
$$\begin{aligned} &\forall A\,([A : \mathbb{N}_+ \to \mathcal{D} \,\wedge\, \forall m, n\,([m, n \in \mathbb{N}_+ \,\wedge\, m \neq n] \Rightarrow A_m \cap A_n = \emptyset)] \\ &\quad \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}), \end{aligned} \tag{11.243}$$
and

4. $\mathcal{D}$ contains the complement of any set in $\mathcal{D}$, that is,
$$\forall A\,(A \in \mathcal{D} \Rightarrow A^c \in \mathcal{D}). \tag{11.244}$$

Corollary 11.26. *Any λ-system on any set Ω contains $\emptyset$.*

Proof. Letting Ω be an arbitrary set and $\mathcal{D}$ an arbitrary λ-system on Ω, it follows from (11.242) with (11.244) that $\Omega^c \in \mathcal{D}$ holds. Since the complement is take with respect to Ω, we obtain $\Omega^c = \emptyset$ with (2.134), so that substitution yields indeed $\emptyset \in \mathcal{D}$, which then follows to be true for any Ω and any $\mathcal{D}$. □

Proposition 11.27. *For any set Ω, it is true that any λ-system $\mathcal{D}$ on Ω contains the difference of any set B in $\mathcal{D}$ and any set A in $\mathcal{D}$ if A is included in B, that is,*
$$\forall A, B\,([A, B \in \mathcal{D} \,\wedge\, A \subseteq B] \Rightarrow B \setminus A \in \mathcal{D}). \tag{11.245}$$

Proof. We let Ω and $\mathcal{D}$ be arbitrary sets and assume $\mathcal{D}$ to be a λ-system on Ω. Next, we let A and B be arbitrary sets and assume in $A, B \in \mathcal{D}$ to be true. Then, $B^c \in \mathcal{D}$ follows to be also true because of Property 4 of a λ-system. Since $\mathcal{D}$ contains also $\emptyset$, as shown by the preceding corollary, we may now apply Exercise 5.40b) to define the sequence of sets $(A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{D}$ with $A_1 = A$, $A_2 = B^c$ and $A_n = \emptyset$ for any $n > 2$, and we then obtain for the union of this sequence $\bigcup_{n=1}^{\infty} A_n = A \cup B^c$ according to (5.363). We now verify that the terms of the sequence of sets $(A_n)_{n\in\mathbb{N}_+}$ are pairwise disjoint, i.e. that the universal sentence
$$\forall m, n\,([m, n \in \mathbb{N}_+ \,\wedge\, m \neq n] \Rightarrow A_m \cap A_n = \emptyset). \tag{11.246}$$
Letting $m, n \in \mathbb{N}_+$ be arbitrary, we now consider the three cases $m = 1$, $m = 2$ and $m > 2$ (as used within the definition of the sequence of sets).

In the first case $m = 1$, we see that the assumption $m \neq n$ yields $n \neq 1$, so that only the two subcases $n = 2$ and $n > 2$ are possible concerning n. In the first subcase $n = 2$, we thus have $A_m = A_1 = A$ and $A_n = A_2 = B^c$. Since we previously assumed $A \subseteq B$, we obtain $A \setminus B = \emptyset$ with (2.118) and therefore $A \cap B^c = \emptyset$ by means of (2.138), which in turn gives the desired $A_m \cap A_n = \emptyset$. The second subcase $n > 2$ gives $A_n = \emptyset$ and therefore $A_m \cap A_n = \emptyset$ according to (2.62). In the second case $m = 2$, we see now that $n \neq 2$ holds, so that the two subcases $n = 1$ and $n > 2$ are possible. If on the one hand $n = 1$ is true, so that $A_m = A_2 = B^c$ and $A_n = A_1 = A$, we obtain with the Commutative Law for the intersection of two sets and the finding $A \cap B^c = \emptyset$ of the first case the equations $A_m \cap A_n = B^c \cap A = A \cap B^c = \emptyset$, as desired. On the other hand, if $n > 2$ is true, we obtain $A_n = \emptyset$ and therefore $A_m \cap A_n = \emptyset$ as in the first case. Similarly, the third case $m > 2$ yields $A_m = \emptyset$ and consequently $A_m \cap A_n = \emptyset \cap A_n = \emptyset$. Thus, the disjointness of A_m and A_n holds in any case, and since m and n were initially arbitrary, we may infer from this the truth of (11.246). This shows that $\bigcup_{n=1}^{\infty} A_n = A \cup B^c$ is the union of a sequence of pairwise disjoint sets in $\mathcal{D}$, which union therefore follows to be an element of $\mathcal{D}$ due Property 3 of a λ-system. In view of Property 4, the complement of that union, i.e. $(A \cup B^c)^c$ is then also in $\mathcal{D}$; observing the truth of the equations

$$(A \cup B^c)^c = A^c \cap (B^c)^c = A^c \cap B = B \cap A^c = B \setminus A$$

in light of De Morgan's Law for the union of two sets, (2.136), the Commutative Law for the intersection of two sets and (2.138), we now see that the previously established $(A \cup B^c)^c \in \mathcal{D}$ implies $B \setminus A \in \mathcal{D}$. This proves the implication in (11.245), and as A and B are arbitrary, we may therefore conclude that the universal sentence (11.245) holds. Moreover, because the sets Ω and $\mathcal{D}$ were arbitrary as well, we may further conclude that the stated proposition is true. $\square$

11.6. σ-Algebras

Definition 11.12 (σ-algebra, measurable space, event/measurable set). For any set Ω we say that a nonempty set $\mathcal{A}$ is a *σ-algebra* (of sets) on Ω iff

1. $\mathcal{A}$ consists of subsets of Ω, that is,
$$\mathcal{A} \subseteq \mathcal{P}(\Omega), \tag{11.247}$$
2. $\mathcal{A}$ contains Ω, that is,
$$\Omega \in \mathcal{A}, \tag{11.248}$$

3. $\mathcal{A}$ contains the union of any sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in $\mathcal{A}$, that is,

$$\forall A\,(A:\mathbb{N}_+\to\mathcal{A}\;\Rightarrow\;\bigcup_{n=1}^{\infty}A_n\in\mathcal{A}),\tag{11.249}$$

   and

4. $\mathcal{A}$ contains the complement of any set in $\mathcal{A}$, that is,

$$\forall A\,(A\in\mathcal{A}\Rightarrow A^c\in\mathcal{A}).\tag{11.250}$$

We then call $(\Omega,\mathcal{A})$ a *measurable space* and the elements of $\mathcal{A}$ *events* or alternatively *measurable sets* (in $\mathcal{A}$).

Unlike an algebra of sets, a σ-algebra contains the limit of any monotone sequence $A=(A_n)_{n\in\mathbb{N}_+}$ of its sets essentially by virtue of Property 3. To establish this fact, we use in the following the partially ordered set $(\mathcal{A},\subseteq_{\mathcal{A}})$.

Corollary 11.28. *For any set Ω and any σ-algebra $\mathcal{A}$ on Ω, it is true that any sequence of sets $A=(A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{A}$ which is isotone with respect to the reflexive partial ordering of inclusion $\subseteq_{\mathcal{A}}$ converges isotonely to the limit*

$$\overset{\subseteq_{\mathcal{A}}}{\lim_{n\to\infty}}A_n=\bigcup_{n=1}^{\infty}A_n.\tag{11.251}$$

Proof. Letting Ω, $\mathcal{A}$ and A be arbitrary sets such that $\mathcal{A}$ is a σ-algebra on Ω and such that A is a sequence $(A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{A}$ which is isotone with respect to $\subseteq_{\mathcal{A}}$, we see in particular that $A:\mathbb{N}_+\to\mathcal{A}$, which implies with Property 3 of a σ-algebra on Ω that $\bigcup_{n=1}^{\infty}A_n\in\mathcal{A}$. Since $\mathcal{A}\subseteq\mathcal{P}(\Omega)$ holds according to Property 1 of a σ-algebra on Ω, we may therefore apply Corollary 4.74 to obtain the equation (11.251). Since Ω, $\mathcal{A}$ and A were initially arbitrary sets, we may then infer from the truth of this equation the truth of the stated corollary. $\square$

Let us inspect two examples of a σ-algebra.

Proposition 11.29. *For any set Ω, it is true that the pair $\{\emptyset,\Omega\}$ is a σ-algebra on Ω.*

Proof. Letting Ω be an arbitrary and recalling that $\{\emptyset,\Omega\}$ is an algebra of sets on Ω (see Proposition 11.22), which clearly contains Ω, we immediately see that this pair satisfies Property 1, Property 2 and Property 4 of a σ-algebra on Ω. To establish Property 3, we take an arbitrary set A and assume $A:\mathbb{N}_+\to\{\emptyset,\Omega\}$, so that A is a sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in $\{\emptyset,\Omega\}$.

We may write the desired consequent $\bigcup_{n=1}^{\infty} A_n \in \{\emptyset, \Omega\}$ equivalently as the disjunction

$$\bigcup_{n=1}^{\infty} A_n = \emptyset \vee \bigcup_{n=1}^{\infty} A_n = \Omega \tag{11.252}$$

by means of the definition of a pair. Let us now prove this disjunction by considering the two cases $\Omega \in \operatorname{ran}(A)$ and $\Omega \notin \operatorname{ran}(A)$.

In the first case $\Omega \in \operatorname{ran}(A)$, we may apply the Equality Criterion for sets to establish the equation $\bigcup_{n=1}^{\infty} A_n = \Omega$. To do this, we prove the universal sentence

$$\forall \omega\, (\omega \in \bigcup_{n=1}^{\infty} A_n \Leftrightarrow \omega \in \Omega), \tag{11.253}$$

letting ω be arbitrary. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $\omega \in \bigcup_{n=1}^{\infty} A_n$ to hold. It then follows with the Characterization of the union of a family of sets that there is an element of $\mathbb{N}_+$, say $\bar{n}$, with $\omega \in A_{\bar{n}}$. As A was assumed to be a function/sequence, we may write $A_{\bar{n}} = A(\bar{n})$ and then $(\bar{n}, A_{\bar{n}}) \in A$, which shows that $A_{\bar{n}} \in \operatorname{ran}(A)$ holds by in view of the definition of a range. Moreover, since A was assumed to be a sequence in $\{\emptyset, \Omega\}$, we have the inclusion $\operatorname{ran}(A) \subseteq \{\emptyset, \Omega\}$ by definition of a codomain. Consequently, $A_{\bar{n}} \in \operatorname{ran}(A)$ implies $A_{\bar{n}} \in \{\emptyset, \Omega\}$ with the definition of a subset, so that the disjunction $A_{\bar{n}} = \emptyset \vee A_{\bar{n}} = \Omega$ holds by definition of a pair. Here, we may prove by contradiction that the first part of the disjunction is false, i.e. that $A_{\bar{n}} \neq \emptyset$ is true. Indeed, assuming $\neg A_{\bar{n}} \neq \emptyset$ to be true, so that the Double Negation Law yields $A_{\bar{n}} = \emptyset$, the previously established $\omega \in A_{\bar{n}}$ gives $\omega \in \emptyset$ via substitution, which evidently contradicts the fact $\omega \notin \emptyset$ holds by definition of the empty set. We thus proved that $A_{\bar{n}} = \emptyset$ is false, so that the second part $A_{\bar{n}} = \Omega$ of the preceding disjunction is true. Consequently, $\omega \in A_{\bar{n}}$ gives now $\omega \in \Omega$ via substitution, as desired.

Regarding the second part ('$\Leftarrow$') of the equivalence in (11.253), we conversely assume $\omega \in \Omega$ to be true. Let us now observe that the current case assumption $\Omega \in \operatorname{ran}(A)$ implies the existence of a particular constant $\bar{m}$ such that $(\bar{m}, \Omega) \in A$, which we may write also as $\Omega = A_{\bar{m}}$. With this equation, the assumed $\omega \in \Omega$ gives $\omega \in A_{\bar{m}}$. Furthermore, recalling that $\mathbb{N}_+$ is the domain of the sequence A, it follows from $(\bar{m}, \Omega) \in A$ that $\bar{m} \in \mathbb{N}_+$ holds. We thus showed that there exists an m for which the conjunction $m \in \mathbb{N}_+ \wedge \omega \in A_{\bar{m}}$ is true, so that we obtain $\omega \in \bigcup_{n=1}^{\infty} A_n$ with the Characterization of the union of a family of sets. This completes the proof of the equivalence, and as ω was arbitrary, we may therefore conclude that the universal sentence (11.253) is true, which in turn implies the truth of the equation $\bigcup_{n=1}^{\infty} A_n = \Omega$. Consequently, the disjunction (11.252) holds in the first case.

In the second case $\Omega \notin \mathrm{ran}(A)$, we may establish $\bigcup_{n=1}^{\infty} A_n = \emptyset$ by using the definition of the empty set. For this purpose, we prove

$$\forall \omega \, (\omega \notin \bigcup_{n=1}^{\infty} A_n). \tag{11.254}$$

We let ω be arbitrary and prove $\omega \notin \bigcup_{n=1}^{\infty} A_n$ by contradiction, assuming $\neg \omega \notin \bigcup_{n=1}^{\infty} A_n$ to be true, so that $\omega \in \bigcup_{n=1}^{\infty} A_n$ follows to be true (with the Double Negation Law). Thus, there evidently exists a particular $\bar{n} \in \mathbb{N}_+$ for which $\omega \in A_{\bar{n}}$ holds. We may now apply exactly the same arguments as in the first case to infer from these findings the truth of the disjunction $A_{\bar{n}} = \emptyset \vee A_{\bar{n}} = \Omega$, whose second part we may establish as false. To do this, we notice that the case assumption $\Omega \notin \mathrm{ran}(A)$ implies the truth of the negation $\neg \exists n \, ((n, \Omega) \in A)$, applying the Law of Contraposition to the definition of a range, and this negation gives the universal sentence $\forall n \, (\neg (n, \Omega) \in A)$ with the Negation Law for existential sentences. Consequently, the negation $\neg (\bar{n}, \Omega) \in A$ holds, which we may write also as $\neg \Omega = A_{\bar{n}}$ in sequence notation. Thus, the second part of the preceding disjunction is false, so that first part $A_{\bar{n}} = \emptyset$ is true. But then the true $\omega \in A_{\bar{n}}$ yields $\omega \in \emptyset$, which clearly contradicts the fact $\omega \notin \emptyset$. This finding completes the proof of the negation in (11.254), and since ω was arbitrary, we may now conclude that the universal sentence (11.254) holds, so that $\bigcup_{n=1}^{\infty} A_n = \emptyset$ holds indeed. Then, the disjunction (11.252) is also true, which thus holds in any case.

With this disjunction, the definition of a pair gives the desired consequent $\bigcup_{n=1}^{\infty} A_n \in \{\emptyset, \Omega\}$, and as A was arbitrary, we may infer from this finding that $\{\emptyset, \Omega\}$ satisfies Property 3 (and thus all of the properties) of a σ-algebra on Ω.

Because Ω was initially an arbitrary set, we may finally conclude that the proposition is true. □

Exercise 11.11. Show that the power set of any set Ω is a σ-algebra on Ω. (Hint: Use (2.10), (3.15), (3.12), (3.784), Definition 2.2, and (2.137).)

Proposition 11.30. *Any σ-algebra $\mathcal{A}$ on any set Ω is*

a) a λ-system on Ω.

b) an algebra of sets on Ω.

c) a ring of sets on Ω.

d) a semiring of sets on Ω.

e) a π-system on Ω.

Proof. We take arbitrary sets Ω and $\mathcal{A}$, and we assume that $\mathcal{A}$ is a σ-algebra on Ω.

Concerning a), the properties of a σ-algebra on Ω clearly show that $\mathcal{A}$ satisfies Property 1, Property 2 and Property 4 of a λ-system on Ω. Regarding Property 3, we take an arbitrary set A and assume $A : \mathbb{N}_+ \to \mathcal{A}$ as well as the universal sentence

$$\forall m, n\,([m, n \in \mathbb{N}_+ \;\wedge\; m \neq n] \Rightarrow A_m \cap A_n = \emptyset)$$

to be true. Then, the former assumption $A : \mathbb{N}_+ \to \mathcal{A}$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ with (11.249), and since A is arbitrary, we may therefore conclude that A satisfies Property 3 of a λ-system. Thus, $\mathcal{A}$ is a λ-system on Ω.

Concerning b), we observe first that $\mathcal{A}$ satisfies Property 1 and Property 4 of an algebra of sets on Ω by virtue of Property 1 and Property 4 of a σ-algebra on Ω. Let us now apply Corollary 11.26 with the finding a) to obtain

$$\emptyset \in \mathcal{A}; \tag{11.255}$$

thus, the σ-algebra $\mathcal{A}$ on Ω satisfies Property 2 of an algebra of sets on Ω. To prove that $\mathcal{A}$ satisfies also Property 3 of an algebra of sets on Ω, we show that $\mathcal{A}$ contains the union of any two sets A and B in $\mathcal{A}$, that is,

$$\forall A, B\,(A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}). \tag{11.256}$$

To do this, we assume $\bar{A}, \bar{B} \in \mathcal{A}$ to be true, and we define the sequence of sets $(A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{A}$ with terms $A_1 = \bar{A}$, $A_2 = \bar{B}$ and $A_n = \emptyset$ for all $n > 2$, whose union is given by $\bigcup_{n=1}^{\infty} A_n = \bar{A} \cup \bar{B}$ according to Exercise 5.40. Then, Property 3 of a σ-algebra gives first $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ and subsequently $\bar{A} \cup \bar{B} \in \mathcal{A}$ via substitution based on the preceding equation. Because $\bar{A}$ and $\bar{B}$ are arbitrary, we may therefore conclude that $\mathcal{A}$ is indeed closed under pairwise unions, and satisfies thus Property 3 of an algebra of sets on Ω as well.

Because $\mathcal{A}$ is then an algebra of sets on Ω, the sentences c) – e) follow immediately to be true with Proposition 11.23. As the sets Ω and $\mathcal{A}$ were initially arbitrary, we may infer from the previous findings the truth of the proposition. □

The preceding proposition allows us to immediately define the following important set-theoretical functions on a σ-algebra.

Note 11.13. We have for any σ-algebra on any set Ω the function

$$^{c}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}, \quad A \mapsto A^c. \tag{11.257}$$

in view of Exercise 11.9 as well as the binary operations

$$\cup_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (A, B) \mapsto A \cup B, \tag{11.258}$$

$$\Delta_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (A, B) \mapsto A \Delta B, \tag{11.259}$$

$$\cap_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (A, B) \mapsto A \cap B, \tag{11.260}$$

according to Note 11.10.

The function for forming the complements of the sets in a σ-algebra is a convenient tool for showing that any σ-algebra is closed under intersections of sequences $(A_n)_{n \in \mathbb{N}_+}$.

Proposition 11.31. *For any set Ω and any σ-algebra on Ω, it is true that $\mathcal{A}$ contains the intersection of any sequence $(A_n)_{n \in \mathbb{N}_+}$ of sets in $\mathcal{A}$, that is,*

$$\forall A \, (A : \mathbb{N}_+ \to \mathcal{A} \;\Rightarrow\; \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}), \tag{11.261}$$

Proof. Letting Ω and $\mathcal{A}$ be arbitrary sets and assuming $\mathcal{A}$ to be a σ-algebra on Ω, we now take an arbitrary set A and assume moreover $A : \mathbb{N}_+ \to \mathcal{A}$, which means that $A = (A_n)_{n \in \mathbb{N}_+}$ is a sequence of sets in $\mathcal{A}$. Then, we obtain the composition ${}^c_{\mathcal{A}} \circ A : \mathbb{N}_+ \to \mathcal{A}$ with Proposition 3.178, which is evidently a sequence of sets in $\mathcal{A}$ whose terms are defined by the mapping

$$n \mapsto ({}^c_{\mathcal{A}} \circ A)(n) = {}^c_{\mathcal{A}} \, (A(n)) = {}^c_{\mathcal{A}} \, (A_n) = A_n^c,$$

so that we may write this sequence as $A' = (A_n^c)_{n \in \mathbb{N}_+}$. Furthermore, $A' : \mathbb{N}_+ \to \mathcal{A}$ implies with Property 3 of a σ-algebra on Ω that $\bigcup_{n=1}^{\infty} A_n^c \in \mathcal{A}$ holds, and this in turn implies

$$\left(\bigcup_{n=1}^{\infty} A_n^c \right)^c \in \mathcal{A} \tag{11.262}$$

with Property 4 of a σ-algebra. Now, the inclusions $\mathrm{ran}(A') \subseteq \mathcal{A} \subseteq \mathcal{P}(\Omega)$ are true because $\mathcal{A}$ is a codomain of A' and because $\mathcal{A}$ is a σ-algebra on Ω, so that $\mathrm{ran}(A') \subseteq \mathcal{P}(\Omega)$ follows to be true with (2.13). This inclusion shows us that $A' = (A_n^c)_{n \in \mathbb{N}_+}$ is a sequence of sets in $\mathcal{P}(\Omega)$ for which the index set $\mathbb{N}_+$ is clearly nonempty. We may therefore apply De Morgan's Law for the union of a family of sets to obtain the equation

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c \right)^c,$$

so that the desired consequent in (11.261) follows from (11.262) via substitution. Because A, Ω and $\mathcal{A}$ were arbitrary, we may therefore conclude that the proposed sentence is true. □

Exercise 11.12. Verify for any set Ω and any σ-algebra $\mathcal{A}$ on Ω that any sequence of sets $A = (A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{A}$ which is antitone with respect to the reflexive partial ordering of inclusion $\subseteq_{\mathcal{A}}$ converges antitonely to the limit

$$\overset{\subseteq_{\mathcal{A}}}{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} A_n. \tag{11.263}$$

(Hint: Proceed in analogy to the proof of Corollary 11.28 and apply now Proposition 11.31 in connection with Exercise 4.22.)

Proposition 11.32. *It is true for any set Ω, any subset $\Omega_1 \subseteq \Omega$ and any σ-algebra $\mathcal{A}_{\Omega_1}$ on Ω_1 that there is a unique set (system) $\mathcal{A}_\Omega$ consisting of all subsets of Ω whose intersections with Ω_1 are in $\mathcal{A}_{\Omega_1}$, in the sense that*

$$\forall X\,(X \in \mathcal{A}_\Omega \Leftrightarrow [X \in \mathcal{P}(\Omega) \wedge \Omega_1 \cap X \in \mathcal{A}_{\Omega_1}]). \tag{11.264}$$

Then, this set system $\mathcal{A}_\Omega$ is a σ-algebra on Ω.

Proof. We take arbitrary sets Ω, Ω_1 and $\mathcal{A}_{\Omega_1}$, assuming that Ω_1 is included in Ω and assuming that $\mathcal{A}_{\Omega_1}$ is a σ-algebra on Ω_1. We then see in light of the Axiom of Specification and the Equality Criterion for sets that the proposed uniquely existential sentence is true. Next, we demonstrate that the set system $\mathcal{A}_{\Omega_1}$ satisfies all four properties of a σ-algebra on Ω.

Regarding Property 1, we notice in (11.264) that $X \in \mathcal{A}_\Omega$ implies especially $X \in \mathcal{P}(\Omega)$ for any X, so that the required inclusion

$$\mathcal{A}_\Omega \subseteq \mathcal{P}(\Omega) \tag{11.265}$$

follows to be true by definition of a subset.

Regarding Property 2, we firstly have $\Omega \in \mathcal{P}(\Omega)$ due to (3.15). Secondly, we obtain $\Omega_1 \cap \Omega = \Omega_1$ with (2.77), where $\Omega_1 \in \mathcal{A}_{\Omega_1}$ holds with Property 2 of a σ-algebra, so that substitution yields $\Omega_1 \cap \Omega \in \mathcal{A}_{\Omega_1}$. The conjunction of these two findings implies now $\Omega \in \mathcal{A}_\Omega$ with (11.264), as required.

Regarding Property 3, we take an arbitrary set A, assume A to be a function/sequence $(A_n)_{n\in\mathbb{N}_+}$ from $\mathbb{N}_+$ to $\mathcal{A}_\Omega$, and we verify that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_\Omega$ holds. To begin with, we notice that the inclusion $\mathrm{ran}(A) \subseteq \mathcal{A}_\Omega$ holds with the preceding assumption and the definition of a codomain; in conjunction with the inclusion (11.265), this gives us $\mathrm{ran}(A) \subseteq \mathcal{P}(\Omega)$ with (2.13), which shows that $A : \mathbb{N}_+ \to \mathcal{P}(\Omega)$. Since $\mathcal{P}(\Omega)$ is a σ-algebra on Ω according to Exercise 11.11, we therefore obtain with Property 3 of a σ-algebra

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{P}(\Omega). \tag{11.266}$$

We now establish, as a useful preparation, the universal sentence

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \Omega_1 \cap A_n \in \mathcal{A}_{\Omega_1}), \tag{11.267}$$

letting $n \in \mathbb{N}_+$ [$= \mathrm{dom}(A)$] be arbitrary, so that the associated term is given by $A_n \in \mathcal{A}_\Omega$. Consequently, we obtain with (11.264) especially $\Omega_1 \cap A_n \in \mathcal{A}_{\Omega_1}$, as desired. As n was arbitrary, we may therefore conclude that the universal sentence (11.267) holds indeed. Let us in addition notice the truth of the equation

$$\Omega_1 \cap \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (\Omega_1 \cap A_n) \tag{11.268}$$

in light of the Distributive Law for families of sets (3.822). We demonstrate next that the corresponding sequence of sets $B = (\Omega_1 \cap A_n)_{n \in \mathbb{N}_+}$ has codomain $\mathcal{A}_{\Omega_1}$, i.e. that the inclusion $\mathrm{ran}(B) \subseteq \mathcal{A}_{\Omega_1}$ is true. For this purpose, we apply the definition of a subset and let $Y \in \mathrm{ran}(B)$ be arbitrary, so that the definition of a range yields a particular constant $\bar{n}$ with $(\bar{n}, Y) \in B$. Then, the definition of a domain gives us $\bar{n} \in \mathbb{N}_+$ [$= \mathrm{dom}(B)$], and we may write in function/sequence notation $Y = B_{\bar{n}} = \Omega_1 \cap A_{\bar{n}}$. Consequently, we obtain $\Omega_1 \cap A_{\bar{n}} \in \mathcal{A}_{\Omega_1}$ with (11.267) and then $Y \in \mathcal{A}_{\Omega_1}$ by means of substitution. Here, Y is arbitrary, so that the inclusion $\mathrm{ran}(B) \subseteq \mathcal{A}_{\Omega_1}$ follows now to be true, which shows that $B : \mathbb{N}_+ \to \mathcal{A}_{\Omega_1}$. By virtue of Property 3 of a σ-algebra, the preceding finding implies then $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}_{\Omega_1}$, so that the definition of the sequence B yields via substitution $\bigcup_{n=1}^{\infty}(\Omega_1 \cap A_n) \in \mathcal{A}_{\Omega_1}$. In view of (11.268), we therefore obtain the

$$\Omega_1 \cap \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_{\Omega_1}. \tag{11.269}$$

The conjunction of (11.266) and (11.269) in turn implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_\Omega$ with (11.264), as desired. Since A was initially arbitrary, we may infer from this finding that the set system $\mathcal{A}_\Omega$ possesses Property 3 of a σ-algebra.

Regarding Property 4, we let $A \in \mathcal{A}_\Omega$ be arbitrary and show that $A^c \in \mathcal{A}_\Omega$ follows to be true. On the one hand, the preceding assumption implies $A \in \mathcal{P}(\Omega)$ with (11.264), with the consequence that $A^c \in \mathcal{P}(\Omega)$ (recalling that $\mathcal{P}(\Omega)$ is a σ-algebra on Ω, which power set satisfies thus Property 4 of a σ-algebra on Ω). On the other hand, we obtain $\Omega_1 \cap A \in \mathcal{A}_{\Omega_1}$, and therefore $(\Omega_1 \cap A)^c \in \mathcal{A}_{\Omega_1}$ (using the initial assumption that $\mathcal{A}_{\Omega_1}$ is a σ-algebra on Ω_1); here, the complement is take with respect to Ω_1. Observing now the truth of the equations

$$(\Omega_1 \cap A)^c = \Omega_1^c \cup A^c = \emptyset \cup A^c = A^c$$

in view of De Morgan's Law for the intersection of two sets, (2.134) and (2.216), the previous finding $(\Omega_1 \cap A)^c \in \mathcal{A}_{\Omega_1}$ gives us $A^c \in \mathcal{A}_{\Omega_1}$ via substitution. Because $\Omega_1 \in \mathcal{A}_{\Omega_1}$ is also true, as mentioned earlier, we also have that $\Omega_1 \cap A^c \in \mathcal{A}_{\Omega_1}$ (recalling from Note 11.13 that any σ-algebra is closed under pairwise intersections). The conjunction of $A^c \in \mathcal{P}(\Omega)$ and the preceding finding implies then $A^c \in \mathcal{A}_\Omega$ with (11.264). Since A was arbitrary, we may therefore conclude that $\mathcal{A}_\Omega$ is indeed closed under the formation of complements.

Having thus established Property 1 – Property 4 of a σ-algebra on Ω for the set system $\mathcal{A}_\Omega$, we may infer from these findings the truth of the proposition, because the sets Ω, Ω_1 and $\mathcal{A}_{\Omega_1}$ were initially arbitrary. □

As a preparation for the next theorem, the idea of Proposition 2.91 is now generalized to sequences of sets.

Lemma 11.33. *The following sentences are true for any sequence of sets* $A = (A_n)_{n \in \mathbb{N}_+}$.

a) There exists a unique sequence $B = (B_n)_{n \in \mathbb{N}_+}$ *such that*

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i). \tag{11.270}$$

b) The unions of the sequences A *and* B *are identical, that is,*

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n. \tag{11.271}$$

c) The terms of the sequence $B = (B_n)_{n \in \mathbb{N}_+}$ *are pairwise disjoint.*

Proof. We let A be an arbitrary set and assume A to be a sequence $(A_n)_{n \in \mathbb{N}_+}$ of sets. Concerning a), we apply Function definition by replacement and verify for this purpose

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow \exists! y \, (y = A_n \setminus \bigcup_{i=1}^{n-1} A_i)), \tag{11.272}$$

letting $n \in \mathbb{N}_+$ be arbitrary. This assumption implies $1 \leq_{\mathbb{N}} n$ with (4.278) and therefore evidently $0 \leq_{\mathbb{N}} n -_{\mathbb{N}} 1$ with the Monotony Law for $-_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$, which shows that $n -_{\mathbb{N}} 1 \in \mathbb{N}$. Thus, the initial segment $\{1, \ldots, n - 1\}$ of $\mathbb{N}_+$ is defined, and we may restrict the sequence $(A_n)_{n \in \mathbb{N}_+}$ to that initial segment to obtain the sequence of sets $s = (A_i \,|\, i \in \{1, \ldots, n-1\})$, whose union is given by the set $\bigcup_{i=1}^{n-1} A_i = \bigcup \mathrm{ran}(s)$. We now see that the uniquely existential sentence in (11.272) is true in light of (1.109). Since n is arbitrary, we may therefore conclude that the universal sentence (11.272) holds, which implies then the unique existence of a function B with domain $\mathbb{N}_+$ such that $B(n) = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ holds for any $n \in \mathbb{N}_+$. Thus, B is a sequence $(B_n)_{n\in\mathbb{N}_+}$ with terms defined by (11.270).

Concerning b), we apply the Equality Criterion for sets to prove the stated equation, by verifying

$$\forall y\, (y \in \bigcup_{n=1}^{\infty} A_n \Leftrightarrow y \in \bigcup_{n=1}^{\infty} B_n). \tag{11.273}$$

We take an arbitrary y and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $y \in \bigcup_{n=1}^{\infty} A_n$ to be true. According to the Characterization of the union of a family of sets, there exists then an element of $\mathbb{N}_+$, say $\bar{n}$, with $y \in A_{\bar{n}}$. Let us now apply the Axiom of Specification in connection with the Equality Criterion for sets to establish the unique existence of a set M satisfying

$$\forall m\, (m \in M \Leftrightarrow [m \in \mathbb{N}_+ \,\wedge\, y \in A_m]). \tag{11.274}$$

Because $\bar{n} \in \mathbb{N}_+$ and $y \in A_{\bar{n}}$ are both true, we therefore obtain $\bar{n} \in M$, so that the set M is clearly nonempty. The preceding universal sentence also shows that $m \in M$ implies $m \in \mathbb{N}_+$ for any m, so that M is a (nonempty) subset of $\mathbb{N}_+$. We may the apply (4.284) to infer from this finding that $k = \min M$ exists. Since $k \in M$ is true by definition of a minimum, the inclusion $M \subseteq \mathbb{N}_+$ gives $k \in \mathbb{N}_+$ by definition of a subset, and therefore $1 \leq_{\mathbb{N}} k$ with (4.278). By definition of an induced irreflexive partial ordering, the preceding inequality means that the disjunction $1 = k \vee 1 <_{\mathbb{N}} k$ is true. We now use this disjunction to prove the desired consequent $y \in \bigcup_{n=1}^{\infty} B_n$ by cases.

In the first case that $k = 1$ is true, it follows from $k \in M$ with (11.274) in particular the truth of $y \in A_k$ $[= A_1]$. Observing now that the sequence $s = (A_i \,|\, i \in \{1, \ldots, 0\})$ has an empty domain according to the notation for initial segment of $\mathbb{N}_+$ and consequently an empty range due to (3.118), we obtain

$$\begin{aligned} B_1 &= A_1 \setminus \bigcup_{i=1}^{1-1} A_i = A_1 \setminus \bigcup_{i=1}^{0} A_i = A_1 \setminus \bigcup \mathrm{ran}(s) = A_1 \setminus \bigcup \emptyset = A_1 \setminus \emptyset \\ &= A_1 \end{aligned}$$

by applying the definition of the sequence B, (5.345), substitution based on the previous observation $\mathrm{ran}(s) = \emptyset$, (2.205), and (2.102). With these equations, the previously established $y \in A_1$ yields $y \in B_1$, which shows that there exists an $n \in \mathbb{N}_+$ with $y \in B_n$, so that $y \in \bigcup_{n=1}^{\infty} B_n$ follows to be true with the Characterization of the union of a family of sets.

The second case $1 <_{\mathbb{N}} k$ implies with the Monotony Law for $-_{\mathbb{N}}$ and $<_{\mathbb{N}}$ the inequality $0 <_{\mathbb{N}} k -_{\mathbb{N}} 1$, which shows on the one hand that $k -_{\mathbb{N}} 1 \in \mathbb{N}$ holds, and which gives on the other hand $k -_{\mathbb{N}} 1 \neq 0$ with the Characterization of comparability of the standard linear ordering $<_{\mathbb{N}}$; thus, $k -_{\mathbb{N}} 1 \in \mathbb{N}_+$ holds by definition of the set of positive natural numbers. We are now in a position to prove the universal sentence

$$\forall j\,(j \in \{1, \ldots, k-1\} \Rightarrow y \notin A_j). \tag{11.275}$$

To do this, we let j be arbitrary and prove the implication by contradiction, assuming both $j \in \{1, \ldots, k-1\}$ and $\neg y \notin A_j$ to be true, where the latter evidently gives $y \in A_j$ with the Double Negation Law. Because the initial segment $\{1, \ldots, k-1\}$ consists of positive natural numbers, $j \in \mathbb{N}_+$ also holds, so that $j \in M$ follows to be true with (11.274). Since $k = \min M$ is a lower bound for M by definition of a minimum, we have that $k \leq_{\mathbb{N}} j$ holds. However the assumed $j \in \{1, \ldots, k-1\}$ implies $j \leq_{\mathbb{N}} k-1$ with (4.275), where evidently $k - 1 <_{\mathbb{N}} [k-1] + 1\ [= k]$, so that $j <_{\mathbb{N}} k$ follows to be true with the Transitivity Formula for $\leq$ and $<$, and this inequality further implies $\neg k \leq_{\mathbb{N}} j$ with the Negation Formula for $\leq$. This contradicts the previously established $k \leq_{\mathbb{N}} j$, so that the proof of the implication in (11.275) is complete. Since j was arbitrary, we may therefore conclude that the universal sentence (11.275) is true, which in turn implies the truth of

$$\neg\exists j\,(j \in \{1, \ldots, k-1\} \wedge y \in A_j)$$

with the Negation Law for existential conjunctions. An application of the Characterization of the union of a sequence of sets yields then the equivalent negation $\neg y \in \bigcup_{i=1}^{k-1} A_i$, which implies together with the previous finding $y \in A_k$ in view of the definition of a set difference $y \in A_k \setminus \bigcup_{i=1}^{k-1} A_i$, so that $k \in \mathbb{N}_+$ implies $y \in B_k$ with (11.270). Thus, there is an $n \in \mathbb{N}_+$ with $y \in B_n$, which means according to the Characterization of the union of a sequence of sets that $y \in \bigcup_{n=1}^{\infty} B_n$. Since we found this result to be true in both cases, the first part of the equivalence in (11.273) holds.

To establish the second part ('$\Leftarrow$'), we now assume conversely $y \in \bigcup_{n=1}^{\infty} B_n$ to be true. Clearly, this means that there is a positive natural number, say $\bar{n}$, such that $y \in B_{\bar{n}}$ holds. Consequently, (11.270) yields $y \in A_{\bar{n}} \setminus \bigcup_{i=1}^{\bar{n}-1} A_i$, so that $y \in A_{\bar{n}}$ is especially true (by definition of a set difference). This shows that there exists a positive natural number n such that $y \in A_n$ holds, which evidently means that $y \in \bigcup_{n=1}^{\infty} A_n$ is true. Thus, the proof of the equivalence in (11.273) is complete, and since y was arbitrary, we may therefore conclude that the universal sentence (11.273) is true. This universal sentence in turn implies the equation (11.271) with the Equality Criterion for sets, completing the proof of b).

Concerning c), we establish the pairwise disjointness of the terms of B by proving the universal sentence

$$\forall m, n\,([m, n \in \mathbb{N}_+ \;\wedge\; m \neq n] \Rightarrow B_m \cap B_n = \emptyset). \tag{11.276}$$

We take arbitrary positive natural numbers m and n such that $m \neq n$ holds, and show that $B_m \cap B_n = \emptyset$ follows to be true. For this purpose, we use the definition of the empty set and prove the universal sentence

$$\forall y\,(y \notin B_m \cap B_n), \tag{11.277}$$

letting y be arbitrary. We may now prove the desired negation $y \notin B_m \cap B_n$ by establishing the contradiction

$$(y \in A_m \;\wedge\; y \notin A_m) \;\vee\; (y \in A_n \;\wedge\; y \notin A_n), \tag{11.278}$$

assuming $\neg y \notin B_m \cap B_n$ to be true, so that the Double Negation Law gives the true sentence $y \in B_m \cap B_n$. By definition of the intersection of two sets, the conjunction $y \in B_m \;\wedge\; y \in B_n$ holds then, which we may write as

$$y \in A_m \setminus \bigcup_{i=1}^{m-1} A_i \;\wedge\; y \in A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

by using (11.270), and then also as

$$\left(y \in A_m \;\wedge\; \neg y \in \bigcup_{i=1}^{m-1} A_i\right) \;\wedge\; \left(y \in A_n \;\wedge\; \neg y \in \bigcup_{i=1}^{n-1} A_i\right)$$

by means of the definition of a set difference. Furthermore, the Characterization of the union of a family of sets yields the true sentence

$$\begin{aligned}&[y \in A_m \;\wedge\; \neg\exists i\,(i \in \{1, \ldots, m-1\} \;\wedge\; y \in A_i)]\\ &\quad\wedge\; [y \in A_n \;\wedge\; \neg\exists i\,(i \in \{1, \ldots, n-1\} \;\wedge\; y \in A_i)]\,,\end{aligned}$$

and then after applying the Negation Law for existential conjunctions

$$[y \in A_m \;\wedge\; \forall i\,(i \in \{1, \ldots, m-1\} \Rightarrow y \notin A_i)] \tag{11.279}$$
$$\quad\wedge\; [y \in A_n \;\wedge\; \forall i\,(i \in \{1, \ldots, n-1\} \Rightarrow y \notin A_i)]\,. \tag{11.280}$$

Because the standard linear ordering $<_{\mathbb{N}}$ is connex, the disjunction of $m <_{\mathbb{N}} n$, $n <_{\mathbb{N}} m$ and $m = n$ holds, where $m = n$ was assumed to be false in (11.276), so that the disjunction $m <_{\mathbb{N}} n \vee n <_{\mathbb{N}} m$ is true.

In the first case that $m <_{\mathbb{N}} n$ is true, we evidently obtain $m + 1 \leq_{\mathbb{N}} n$ with (4.270) and then $[1 \leq_{\mathbb{N}}]\ m \leq_{\mathbb{N}} n - 1$ with the initial assumption $m \in \mathbb{N}_+$ and the Monotony Law of $-_{\mathbb{N}}$ and $\leq_{\mathbb{N}}$. Here, the transitivity of $\leq_{\mathbb{N}}$ yields $1 \leq_{\mathbb{N}} n - 1$, so that $n - 1 \in \mathbb{N}_+$ is evidently true. With this finding, the previous inequality $m \leq_{\mathbb{N}} n - 1$ implies $m \in \{1, \ldots, n - 1\}$ with (4.275), which further implies $y \notin A_m$ with the universal sentence in (11.280). Recalling the truth of the first part of the conjunction in (11.279), we thus established the true conjunction $y \in A_m \wedge y \notin A_m$, which is clearly a contradiction, and the disjunction (11.278) is then also false, as desired.

In the first case $n <_{\mathbb{N}} m$, we obtain in analogy to the first case $n+1 \leq_{\mathbb{N}} m$ and then $[1 \leq_{\mathbb{N}}]\ n \leq_{\mathbb{N}} m - 1$ (now with the initially assumed $n \in \mathbb{N}_+$). Consequently, $1 \leq_{\mathbb{N}} m - 1$ holds, which shows that $m - 1 \in \mathbb{N}_+$. With this, $n \leq_{\mathbb{N}} m - 1$ implies $n \in \{1, \ldots, m - 1\}$ and furthermore $y \notin A_n$ with the universal sentence in (11.279). In view of the first part of the conjunction in (11.280), we thus obtained the contradiction $y \in A_n \wedge y \notin A_n$, and the disjunction (11.278) follows therefore to be false as well.

This completes the proof by contradiction of $y \notin B_m \cap B_n$, and since y was arbitrary, we may infer from the truth of this negation the truth of the universal sentence (11.277), which implies $B_m \cap B_n = \emptyset$ (by definition of the empty set). Because m and n were also arbitrary, we may now further conclude that the universal sentence (11.276) is true, which shows that the terms of the sequence $B = (B_n)_{n \in \mathbb{N}_+}$ are indeed pairwise disjoint. Thus, the proof of c) is complete.

As the set A was initially arbitrary, we may finally conclude that the stated lemma holds. □

Theorem 11.34 (π-λ Characterization of a σ-algebra). *The following sentences are true for any set Ω and any λ-system $\mathcal{D}$ on Ω.*

a) If $\mathcal{D}$ is a π-system on Ω, then $\mathcal{D}$ contains the difference of any sets B and A in $\mathcal{D}$, that is,

$$\forall A, B\,(A, B \in \mathcal{D} \Rightarrow A \setminus B \in \mathcal{D}). \tag{11.281}$$

b) If $\mathcal{D}$ is a π-system on Ω, it is then also true for any $A, B \in \mathcal{D}$ that there exists a unique sequence of sets $(A_n)_{n \in \mathbb{N}_+}$ in $\mathcal{D}$ with terms $A_1 = A$, $A_2 = B \setminus A$ as well as $A_n = \emptyset$ for all $n > 2$, and these terms are pairwise disjoint.

c) *Moreover, if $\mathcal{D}$ is a π-system on Ω, then $\mathcal{D}$ contains the union of any two sets A and B in $\mathcal{D}$, that is,*

$$\forall A, B\,(A, B \in \mathcal{D} \Rightarrow A \cup B \in \mathcal{D}). \tag{11.282}$$

d) *Furthermore, if $\mathcal{D}$ is a π-system on Ω, then there exists the unique binary operation*

$$\cup_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}, \quad (A, B) \mapsto A \cup B, \tag{11.283}$$

and $\emptyset$ is the neutral element of $\mathcal{D}$ with respect to $\cup_{\mathcal{D}}$. Then, there also exists for any $n \in \mathbb{N}$ the n-fold binary operation

$$\bigcup_{i=1}^{n} : \mathcal{D}^{\{1,\ldots,n\}} \to \mathcal{D}, \quad (A_i \,|\, i \in \{1, \ldots, n\}) \mapsto \bigcup_{i=1}^{n} A_i, \tag{11.284}$$

which gives the union of the range of any mapped sequence of sets $s = (A_i \,|\, i \in \{1, \ldots, n\})$ in the sense of

$$\bigcup_{i=1}^{n} A_i = \bigcup \mathrm{ran}(s). \tag{11.285}$$

e) *$\mathcal{D}$ is a σ-algebra on Ω iff $\mathcal{D}$ is a π-system on Ω.*

Proof. We take an arbitrary sets Ω and $\mathcal{D}$, assuming that $\mathcal{D}$ is a λ-system on Ω.

Concerning a), we assume that $\mathcal{D}$ is a π-system on Ω and show that $\mathcal{D}$ follows then to be closed under set differences. Letting A and B be arbitrary sets such that $A, B \in \mathcal{D}$ holds, we notice first that this implies $A \cap B \in \mathcal{D}$ because of Property 3 of a π-system. Then, since $A \cap B \subseteq A$ is true because of (2.74), we obtain $A \setminus (A \cap B) \in \mathcal{D}$ with (11.245). As the equation $A \setminus B = A \setminus (A \cap B)$ also holds according to (2.98), we may apply substitution and write the preceding finding as $A \setminus B \in \mathcal{D}$, which proves the implication in (11.281). Because A and B were arbitrary, we may therefore conclude that the a) is true.

Concerning b), we take arbitrary sets A and B, assuming $A, B \in \mathcal{D}$ to be true. Then, $B \setminus A \in \mathcal{D}$ holds according to (11.281), and $\emptyset \in \mathcal{D}$ because of Corollary 11.26). We may therefore define the sequence of sets $(A_n)_{n \in \mathbb{N}_+}$ in $\mathcal{D}$ having the terms $A_1 = A$, $A_2 = B \setminus A$ and $A_n = \emptyset$ for all $n > 2$, as done in Exercise 5.40b). We now verify that the terms of the sequence of sets $(A_n)_{n \in \mathbb{N}_+}$ are pairwise disjoint, i.e. that the universal sentence

$$\forall m, n\,([m, n \in \mathbb{N}_+ \,\wedge\, m \neq n] \Rightarrow A_m \cap A_n = \emptyset) \tag{11.286}$$

holds. Letting $m, n \in \mathbb{N}_+$ be arbitrary, we now consider the same three cases $m = 1$, $m = 2$, $m > 2$ and corresponding subcases for n as in the proof of Proposition 11.27. In the first case $m = 1$ and the first subcase $n = 2$, we thus have $A_m = A_1 = A$ and $A_n = A_2 = B \setminus A$. Since $A \cap (B \setminus A) = \emptyset$ holds according to (2.111), we obtain the desired $A_m \cap A_n = \emptyset$ via substitutions. The second subcase $n > 2$ gives $A_n = \emptyset$ and therefore $A_m \cap A_n = \emptyset$ according to (2.62). In the second case $m = 2$ and the first subcase $n = 1$, it follows that $A_m = A_2 = B \setminus A$ and $A_n = A_1 = A$ are true, so that we obtain with the Commutative Law for the intersection of two sets and the finding $A \cap (B \setminus A) = \emptyset$ of the first case the equations

$$A_m \cap A_n = (B \setminus A) \cap A = A \cap (B \setminus A) = \emptyset,$$

as desired. On the other hand, if $n > 2$ is true, we obtain $A_n = \emptyset$ and therefore $A_m \cap A_n = \emptyset$ as in the first case. Similarly, the third case $m > 2$ yields $A_m = \emptyset$ and consequently $A_m \cap A_n = \emptyset \cap A_n = \emptyset$. Thus, the disjointness of A_m and A_n holds in any case, and since m and n were initially arbitrary, we may infer from this the truth of (11.286). A and B were also arbitrary, we may therefore conclude that b) holds, as claimed.

Concerning c), we take arbitrary sets A and B in $\mathcal{D}$ and define according to b) the sequence of sets $(A_n)_{n \in \mathbb{N}_+}$ in $\mathcal{D}$ with pairwise disjoint terms $A_1 = A$, $A_2 = B \setminus A$ as well as $A_n = \emptyset$ for all $n > 2$. Consequently, the union $\bigcup_{n=1}^{\infty} A_n$ of that sequence turns out to be an element of $\mathcal{D}$ by Property 3 of a λ-system. We then obtain the equations

$$\bigcup_{n=1}^{\infty} A_n = A \cup (B \setminus A) = A \cup B$$

by applying Exercise 5.40 and (2.265), so that the previously established $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$ gives the desired $A \cup B \in \mathcal{D}$ via substitution. As A and B were arbitrary, we may therefore conclude that c) also holds.

Concerning d), it is then a straightforward task to define the binary operation (11.283) and to establish $\emptyset$ as the corresponding neutral element of $\mathcal{D}$ (see Exercise 11.13). Based on these results, we may define now for any $n \in \mathbb{N}$ the n-fold binary union operation (11.284), which can be shown to satisfy (11.285).

Concerning e), we prove the first part ('$\Rightarrow$') of the equivalence by assuming that $\mathcal{D}$ is a σ-algebra on Ω; thus, $\mathcal{D}$ is a π-system on Ω according to Proposition 11.30e).

To establish the second part ('$\Rightarrow$'), we now conversely assume that $\mathcal{D}$ is a π-system on Ω, so that $\mathcal{D}$ satisfies Property 1, Property 2 and Property

4 of a σ-algebra by virtue of Property 1, Property 2 and Property 4 of a λ-system. To show that $\mathcal{D}$ satisfies also Property 3 of a σ-algebra, we take an arbitrary set A, for which we assume that $A = (A_n)_{n\in\mathbb{N}_+}$ is a sequence of sets in $\mathcal{D}$, and we establish the truth of $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$. Because of Lemma 11.33a), there exists the sequence of sets $B = (B_n)_{n\in\mathbb{N}_+}$ with terms $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$. Let us verify that B is a sequence with codomain $\mathcal{D}$, i.e. that $\operatorname{ran}(B) \subseteq \mathcal{D}$ holds. We apply for this purpose the definition of a subset and verify the equivalent universal sentence

$$\forall Y\,(Y \in \operatorname{ran}(B) \Rightarrow Y \in \mathcal{D}). \tag{11.287}$$

We take an arbitrary set Y and assume $Y \in \operatorname{ran}(B)$, so that there is by definition of a range a constant, say $\bar{n}$, with $(\bar{n}, Y) \in B$. Since B is a function/sequence, we may write this also as

$$Y = B(\bar{n}) = B_{\bar{n}} = A_{\bar{n}} \setminus \bigcup_{i=1}^{\bar{n}-1} A_i.$$

In view of (11.284), $\bigcup_{i=1}^{\bar{n}-1} A_i \in \mathcal{D}$ holds, which n-fold union represents also the union of the (range) of the sequence $(A_i \,|\, i \in \{1, \ldots, \bar{n}-1\})$ used to define the sequence B, as shown in d). Evidently, the initial assumption $A : \mathbb{N}_+ \to \mathcal{D}$ yields then also $A_{\bar{n}} \in \mathcal{D}$. It then follows with (11.281) that

$$[Y = B_{\bar{n}} =] \quad A_{\bar{n}} \setminus \bigcup_{i=1}^{\bar{n}-1} A_i \in \mathcal{D}.$$

These equations give the desired consequent $Y \in \mathcal{D}$, and since Y is arbitrary, we may therefore conclude that (11.287) is true, so that the inclusion $\operatorname{ran}(B) \subseteq \mathcal{D}$ holds indeed. Thus, B is a sequence of sets in $\mathcal{D}$, whose terms are pairwise disjoint due to Lemma 11.33c). Because of Property 3 of a λ-system, we then obtain $\bigcup_{n=1}^{\infty} B_n \in \mathcal{D}$. Furthermore, Lemma 11.33b) gives the true equation $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$, so that substitution yields $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$. Because A was initially an arbitrary set, we may infer from this that the $\lambda-$ and π-system $\mathcal{D}$ on Ω satisfies also Property 3 of a σ-algebra on Ω, which therefore constitutes a σ-algebra (on Ω). Thus, the proof of the equivalence e) is complete.

Since Ω and $\mathcal{D}$ were initially arbitrary sets, we may therefore infer from the truth of a) – e) the truth of the stated theorem. □

Exercise 11.13. Establish Part d) of Theorem 11.34.

(Hint: Proceed as in Exercise 11.4 and use Corollary 11.26.)

Note 11.11 and the definition of a Boolean algebra of sets apply also directly to σ-algebras.

Note 11.14. For any σ-algebra $\mathcal{A}$ on any set Ω, we have that

a) $(\mathcal{A}, \cup_{\mathcal{A}})$ is a commutative semigroup with neutral element $\emptyset$ and idempotent $\cup_{\mathcal{A}}$,

b) $(\mathcal{A}, \Delta_{\mathcal{A}})$ is a commutative group with neutral element $\emptyset$,

c) $(\mathcal{A}, \cap_{\mathcal{A}})$ is a commutative semigroup with neutral element Ω and idempotent $\cap_{\mathcal{A}}$.

d) $(\mathcal{A}, \Delta_{\mathcal{A}}, \cap_{\mathcal{A}}, \Delta_{\mathcal{A}})$ is a commutative ring with zero element $\emptyset$ and unity element Ω.

e) $(\mathcal{A}, \Delta_{\mathcal{A}}, \cap_{\mathcal{A}}, \Delta_{\mathcal{A}})$ is a Boolean ring with unity element Ω.

f) $(\mathcal{A}, \cup_{\mathcal{A}}, \cap_{\mathcal{A}}, {}^{c}_{\mathcal{A}})$ is a Boolean algebra of sets on Ω.

Furthermore, Note 11.12 shows that σ-algebras are closed under n-fold unions and intersections.

Note 11.15. For any σ-algebra $\mathcal{A}$ on any set Ω, we may define the n-fold binary operations

$$\bigcup_{i=1}^{n} : \mathcal{A}^{\{1,\ldots,n\}} \to \mathcal{A}, \quad (A_i \,|\, i \in \{1, \ldots, n\}) \mapsto \bigcup_{i=1}^{n} A_i, \tag{11.288}$$

$$\bigcap_{i=1}^{n} : \mathcal{A}^{\{1,\ldots,n\}} \to \mathcal{A}, \quad (A_i \,|\, i \in \{1, \ldots, n\}) \mapsto \bigcap_{i=1}^{n} A_i, \tag{11.289}$$

where $\bigcup_{i=1}^{n} A_i$ yields the union and $\bigcap_{i=1}^{n} A_i$ the intersection of the (range of) a sequence $(A_i \,|\, i \in \{1, \ldots, n\})$ in $\mathcal{A}$, according to Proposition 11.10 and Proposition 11.25.

We now establish a principle for constructing σ-algebras in close analogy to the Generation of rings of sets.

Theorem 11.35 (Generation of σ-algebras). *The following sentences are true for any set Ω and for any set system $\mathcal{K} \subseteq \mathcal{P}(\Omega)$.*

a) *There exists a unique set $\mathcal{U}$ consisting of all σ-algebras on Ω (in $\mathcal{P}(\mathcal{P}(\Omega))$) which include $\mathcal{K}$, i.e.*

$$\forall \mathcal{A} \,(\mathcal{A} \in \mathcal{U} \Leftrightarrow [\mathcal{A} \in \mathcal{P}(\mathcal{P}(\Omega)) \,\wedge\, (\mathcal{A} \text{ is a } \sigma\text{-algebra on } \Omega \,\wedge\, \mathcal{K} \subseteq \mathcal{A})]). \tag{11.290}$$

This set $\mathcal{U}$ is nonempty and satisfies also

$$\forall \mathcal{A} \,(\mathcal{A} \in \mathcal{U} \Leftrightarrow [\mathcal{A} \text{ is a } \sigma\text{-algebra on } \Omega \,\wedge\, \mathcal{K} \subseteq \mathcal{A}]). \tag{11.291}$$

b) Then, the intersection $\mathcal{A}(\mathcal{K}) = \bigcap \mathcal{U}$ is itself a σ-algebra on Ω that includes $\mathcal{K}$.

c) Furthermore, this σ-algebra $\mathcal{A}(\mathcal{K})$ on Ω is the smallest σ-algebra on Ω that includes $\mathcal{K}$ in the sense that

$$\forall \mathcal{A}\,([\mathcal{A} \text{ is a } \sigma\text{−algebra on } \Omega \,\wedge\, \mathcal{K} \subseteq \mathcal{A}] \Rightarrow \mathcal{A}(\mathcal{K}) \subseteq \mathcal{A}). \tag{11.292}$$

Proof. We take arbitrary sets Ω and $\mathcal{K}$ such that $\mathcal{K} \subseteq \mathcal{P}(\Omega)$ holds. We may establish a) in analogy to Part a) of the Generation of rings of sets.

Regarding b), we may demonstrate that the set $\mathcal{A}(\mathcal{K}) = \bigcap \mathcal{U}$ satisfies Property 1, Property 2 and Property 4 of a σ-algebra on Ω and moreover that $\mathcal{A}(\mathcal{K})$ includes $\mathcal{K}$, similarly to the corresponding proofs for generated rings of sets.

Regarding Property 3, we let A be arbitrary and assume $A : \mathbb{N}_+ \to \mathcal{A}(\mathcal{K})$ (i.e., $A : \mathbb{N}_+ \to \bigcap \mathcal{U}$) to be true. We thus have $\mathrm{ran}(A) \subseteq \bigcap \mathcal{U}$ by definition of a codomain. To establish the desired consequent $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathcal{U}$, we need to prove the universal sentence

$$\forall \mathcal{A}\,(\mathcal{A} \in \mathcal{U} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}). \tag{11.293}$$

Letting $\mathcal{A} \in \mathcal{U}$ be arbitrary, we obtain first the inclusion $\bigcap \mathcal{U} \subseteq \mathcal{A}$ with (2.92); together with the previously established inclusion $\mathrm{ran}(A) \subseteq \bigcap \mathcal{U}$, this implies $\mathrm{ran}(A) \subseteq \mathcal{A}$ with (2.13), which shows that $\mathcal{A}$ is also a codomain of the sequence A, so that $A : \mathbb{N}_+ \to \mathcal{A}$. Furthermore, $\mathcal{A} \in \mathcal{U}$ implies that $\mathcal{A}$ is a σ-algebra on Ω. Because of Property 3 of a σ-algebra on Ω, the union of the preceding sequence follows then to be in $\mathcal{A}$, that is, we obtain $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ as desired. As $\mathcal{A}$ was arbitrary, we may therefore conclude that the universal sentence (11.293) holds, so that $\bigcup_{n=1}^{\infty} A_n \in \bigcap \mathcal{U}$ follows to be true by definition of the intersection of a set system. We thus showed that $A : \mathbb{N}_+ \to \mathcal{A}(\mathcal{K})$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}(\mathcal{K})$, and since A is arbitrary, we may infer from this finding that Property 3 of a σ-algebra on Ω is satisfied by $\mathcal{A}(\mathcal{K})$.

The remaining Part c) is established exactly like Part c) of the Generation of rings of sets. Because Ω and $\mathcal{K}$ were initially arbitrary, we may conclude that the stated theorem holds indeed. □

Exercise 11.14. Establish the missing parts in the proof of Theorem 11.35.

Definition 11.13 (Generated σ-algebra, generating system for a σ-algebra). For any set Ω and any set system $\mathcal{K} \subseteq \mathcal{P}(\Omega)$, we call the set

$$\mathcal{A}(\mathcal{K}) \tag{11.294}$$

that satisfies Theorem 11.35a,b) the *σ-algebra (on Ω) generated by $\mathcal{K}$*. We then say that $\mathcal{K}$ is the *generating system for $\mathcal{A}(\mathcal{K})$*.

Note 11.16. Any σ-algebra $\mathcal{A}(\mathcal{K})$ generated by any system $\mathcal{K}$ of subsets of any set Ω satisfies by virtue of (11.255) and Property 2 of a σ-algebra

$$\emptyset \in \mathcal{A}(\mathcal{K}), \tag{11.295}$$
$$\Omega \in \mathcal{A}(\mathcal{K}). \tag{11.296}$$

Note 11.17. For any linearly ordered set $(\Omega, <)$ with $\Omega \neq \emptyset$, the semiring $\mathcal{I}$ of left-closed and right-open intervals in Ω generates the σ-algebra

$$\mathcal{A}(\mathcal{I}), \tag{11.297}$$

where

$$\mathcal{I} \subseteq \mathcal{A}(\mathcal{I}). \tag{11.298}$$

Corollary 11.36. *It is true for any linearly ordered set $(\Omega, <)$ with $\Omega \neq \emptyset$ and any elements $a, b \in \Omega$ that the left-closed and right-open interval from a to b is an element of the σ-algebra generated by the semiring $\mathcal{I}$ of left-closed and right-open intervals in Ω, i.e.*

$$[a, b) \in \mathcal{A}(\mathcal{I}). \tag{11.299}$$

Proof. Letting Ω, $<$, a and b be arbitrary such that $\Omega \neq \emptyset$ holds, such that $(\Omega, <)$ is linearly ordered, and such that $a, b \in \Omega$ is true, we clearly have $[a, b) \in \{[a, b) : a, b \in \Omega\}$ $[= \mathcal{I}]$ by definition of the set of left-closed and right-open intervals as well as by definition of the semiring of left-closed and right-open intervals in Ω. In view of (11.298), we therefore obtain (11.299) with the definition of a subset. Since Ω, $<$, a and b were initially arbitrary, we may now infer from this finding the truth of the stated universal sentence. $\square$

Note 11.18. The σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$ satisfies accordingly (11.299) and

$$\emptyset \in \mathcal{A}(\mathcal{I}), \tag{11.300}$$
$$\mathbb{R} \in \mathcal{A}(\mathcal{I}). \tag{11.301}$$

We end this section with three useful facts about generated σ-algebras.

Proposition 11.37. *It is true that any σ-algebra on any set Ω is generated by itself, that is,*

$$\forall \Omega, \overline{\mathcal{A}}\, (\overline{\mathcal{A}} \text{ is a } \sigma\text{−algebra on } \Omega \;\Rightarrow\; \overline{\mathcal{A}} = \mathcal{A}(\overline{\mathcal{A}}). \tag{11.302}$$

Proof. We let Ω and $\overline{\mathcal{A}}$ be arbitrary sets, assume that $\overline{\mathcal{A}}$ is a σ-algebra on Ω, and show that the conjunction

$$\overline{\mathcal{A}} \subseteq \mathcal{A}(\overline{\mathcal{A}}) \ \wedge \ \mathcal{A}(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{A}} \tag{11.303}$$

holds, which will imply the desired equation $\overline{\mathcal{A}} = \mathcal{A}(\overline{\mathcal{A}})$ with the Axiom of Extension. On the one hand, the σ-algebra $\mathcal{A}(\overline{\mathcal{A}})$ generated by $\overline{\mathcal{A}}$ includes the generating system (see Part b) of the Generation of σ-algebras), that is, $\overline{\mathcal{A}} \subseteq \mathcal{A}(\overline{\mathcal{A}})$. On the other hand, since $\overline{\mathcal{A}}$ is a σ-algebra on Ω and since $\overline{\mathcal{A}} \subseteq \overline{\mathcal{A}}$ is also true according to (2.10), it follows with Part c) of the Generation of σ-algebras that $\mathcal{A}(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{A}}$ holds as well. We thus proved the conjunction (11.303), which in turn implies $\overline{\mathcal{A}} = \mathcal{A}(\overline{\mathcal{A}})$. As Ω and $\overline{\mathcal{A}}$ were arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Lemma 11.38 (Inclusion Criterion for generated σ-algebras). *It is true for any set Ω that, if a subset $\mathcal{K}_1$ of the power set $\mathcal{P}(\Omega)$ is included in a subset $\mathcal{K}_2$ of $\mathcal{P}(\Omega)$, then the σ-algebra generated by $\mathcal{K}_1$ is included in the σ-algebra generated by $\mathcal{K}_2$, that is,*

$$\forall \Omega, \mathcal{K}_1, \mathcal{K}_2\, ([\mathcal{K}_1 \subseteq \mathcal{P}(\Omega) \ \wedge \ \mathcal{K}_2 \subseteq \mathcal{P}(\Omega)] \Rightarrow [\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \mathcal{A}(\mathcal{K}_1) \subseteq \mathcal{A}(\mathcal{K}_2)]). \tag{11.304}$$

Proof. We let in the following Ω, $\mathcal{K}_1$ and $\mathcal{K}_2$ be arbitrary sets such that $\mathcal{K}_1$ and $\mathcal{K}_2$ are both included in $\mathcal{P}(\Omega)$, so that we may use these set systems to generate the σ-algebras $\mathcal{A}(\mathcal{K}_1)$ and $\mathcal{A}(\mathcal{K}_2)$. Next, we assume furthermore that $\mathcal{K}_1 \subseteq \mathcal{K}_2$ is true. Let us now observe that the σ-algebra $\mathcal{A}(\mathcal{K}_2)$ generated by $\mathcal{K}_2$ includes the generating system (see Part b) of the Generation of σ-algebras), that is, $\mathcal{K}_2 \subseteq \mathcal{A}(\mathcal{K}_2)$. Together with the previously assumed inclusion, this implies $\mathcal{K}_1 \subseteq \mathcal{A}(\mathcal{K}_2)$ with (2.13). Thus, $\mathcal{A}(\mathcal{K}_2)$ is a σ-algebra on Ω which includes $\mathcal{K}_1$. It follows then with Part c) of the Generation of σ-algebras that $\mathcal{A}(\mathcal{K}_1) \subseteq \mathcal{A}(\mathcal{K}_2)$ is true, as desired. Since Ω, $\mathcal{K}_1$ and $\mathcal{K}_2$ were initially arbitrary sets, we may therefore conclude that the proposed universal sentence holds. □

Theorem 11.39 (Equality Criterion for generated σ-algebras). *It is true for any set Ω and any set systems $\mathcal{K}_1 \subseteq \mathcal{P}(\Omega)$ and $\mathcal{K}_2 \subseteq \mathcal{P}(\Omega)$ that the σ-algebra generated by $\mathcal{K}_1$ is identical with the σ-algebra generated by $\mathcal{K}_2$ if $\mathcal{K}_1$ is included in the σ-algebra generated by $\mathcal{K}_2$ and if $\mathcal{K}_2$ is included in the σ-algebra generated by $\mathcal{K}_1$, i.e.*

$$[\mathcal{K}_1 \subseteq \mathcal{A}(\mathcal{K}_2) \ \wedge \ \mathcal{K}_2 \subseteq \mathcal{A}(\mathcal{K}_1)] \ \Rightarrow \ \mathcal{A}(\mathcal{K}_1) = \mathcal{A}(\mathcal{K}_2). \tag{11.305}$$

Proof. We take arbitrary sets Ω, $\mathcal{K}_1$ and $\mathcal{K}_2$, assume $\mathcal{K}_1 \subseteq \mathcal{P}(\Omega)$ and $\mathcal{K}_2 \subseteq \mathcal{P}(\Omega)$, and assume moreover $\mathcal{K}_1 \subseteq \mathcal{A}(\mathcal{K}_2)$ as well as $\mathcal{K}_2 \subseteq \mathcal{A}(\mathcal{K}_1)$ to be true.

Next, we prove the conjunction

$$\mathcal{A}(\mathcal{K}_1) \subseteq \mathcal{A}(\mathcal{K}_2) \ \wedge \ \mathcal{A}(\mathcal{K}_2) \subseteq \mathcal{A}(\mathcal{K}_1), \tag{11.306}$$

which will imply the desired equation $\mathcal{A}(\mathcal{K}_1) = \mathcal{A}(\mathcal{K}_2)$ with the Axiom of Extension. Since $\mathcal{A}(\mathcal{K}_1)$ and $\mathcal{A}(\mathcal{K}_2)$ are both σ-algebras on Ω by definition of a generated σ-algebra, we obtain with (11.302) the two equations

$$\mathcal{A}(\mathcal{A}(\mathcal{K}_1)) = \mathcal{A}(\mathcal{K}_1),$$
$$\mathcal{A}(\mathcal{A}(\mathcal{K}_2)) = \mathcal{A}(\mathcal{K}_2).$$

Furthermore, $\mathcal{A}(\mathcal{K}_2)$ satisfies, by Property 1 of a σ-algebra on Ω, the inclusion $\mathcal{A}(\mathcal{K}_2) \subseteq \mathcal{P}(\Omega)$. Together with the assumed inclusion $\mathcal{K}_1 \subseteq \mathcal{P}(\Omega)$, this implies with the Inclusion Criterion for generated σ-algebras the truth of the implication

$$\mathcal{K}_1 \subseteq \mathcal{A}(\mathcal{K}_2) \ \Rightarrow \ \mathcal{A}(\mathcal{K}_1) \subseteq \mathcal{A}(\mathcal{A}(\mathcal{K}_2)).$$

As we previously assumed the antecedent to be true, the consequent

$$\mathcal{A}(\mathcal{K}_1) \subseteq \mathcal{A}(\mathcal{A}(\mathcal{K}_2)) \quad [= \mathcal{A}(\mathcal{K}_2)]$$

follows to be also true, so that $\mathcal{A}(\mathcal{K}_1) \subseteq \mathcal{A}(\mathcal{K}_2))$ holds. This finding completes the proof of the first part of the conjunction (11.306).

Similarly, $\mathcal{A}(\mathcal{K}_1)$ satisfies $\mathcal{A}(\mathcal{K}_1) \subseteq \mathcal{P}(\Omega)$ (due to Property 1 of a σ-algebra on Ω). This inclusion, together with the assumed $\mathcal{K}_2 \subseteq \mathcal{P}(\Omega)$, gives then with the Inclusion Criterion for generated σ-algebras

$$\mathcal{K}_2 \subseteq \mathcal{A}(\mathcal{K}_1) \ \Rightarrow \ \mathcal{A}(\mathcal{K}_2) \subseteq \mathcal{A}(\mathcal{A}(\mathcal{K}_1)),$$

the antecedent of which implication is true by assumption. Consequently,

$$\mathcal{A}(\mathcal{K}_2) \subseteq \mathcal{A}(\mathcal{A}(\mathcal{K}_1)) \quad [= \mathcal{A}(\mathcal{K}_1)]$$

is true, which shows that the second part $\mathcal{A}(\mathcal{K}_2) \subseteq \mathcal{A}(\mathcal{K}_1))$ of the conjunction (11.306) holds as well. Having thus established the truth of that conjunction, the equation in (11.305) follows to be also true, proving the implication (11.305). Because the sets Ω, $\mathcal{K}_1$ and $\mathcal{K}_2$ were initially arbitrary, we may then infer from the truth of this implication the truth of the theorem. □

As a first useful application, the Equality Criterion for generated σ-algebras allows us to establish the following result for the σ-algebra $\mathcal{A}(\mathcal{F})$ generated by the ring of one-dimensional figures.

Proposition 11.40. *It is true for any linearly ordered set* $(\Omega, <)$ *with* $\Omega \neq \emptyset$ *that the* σ*-algebra generated by the semiring of left-closed and right-open intervals in* Ω *is identical with the* σ*-algebra generated by the ring of one-dimensional figures in* Ω*, that is,*

$$\mathcal{A}(\mathcal{I}) = \mathcal{A}(\mathcal{F}). \tag{11.307}$$

Proof. Letting Ω and $<$ be arbitrary sets such that $\Omega \neq \emptyset$ holds and such that $(\Omega, <)$ is linearly ordered, we establish the two inclusions $\mathcal{I} \subseteq \mathcal{A}(\mathcal{F})$ and $\mathcal{F} \subseteq \mathcal{A}(\mathcal{I})$, the conjunction of which will then imply the desired equation (11.307) by means of the Equality Criterion for generated σ-algebras. On the one hand, the fact that the inclusion

$$\mathcal{I} \subseteq \mathcal{R}(\mathcal{I}) \; [= \mathcal{F}]$$

holds by definition of a generated ring (recalling also the notation for the ring of one-dimensional figures in Ω) and that in addition the inclusion

$$\mathcal{F} \subseteq \mathcal{A}(\mathcal{F})$$

is true by definition of a generated σ-algebra implies the inclusion $\mathcal{I} \subseteq \mathcal{A}(\mathcal{F})$ with (2.13).

On the other hand, we may prove the inclusion $\mathcal{F} \subseteq \mathcal{A}(\mathcal{I})$ via the definition of a subset, i.e. by demonstrating the truth of

$$\forall A \, (A \in \mathcal{F} \Rightarrow A \in \mathcal{A}(\mathcal{I})). \tag{11.308}$$

To do this, we take an arbitrary set A, and we assume the antecedent $A \in \mathcal{F}$ $[= \mathcal{R}(\mathcal{I})]$ to be true. According to the Generation of rings of sets by means of semirings of sets, there are then a particular positive natural number $\bar{n}$ and a particular sequence $\bar{C} : \{1, \ldots, \bar{n}\} \to \mathcal{I}$ of pairwise disjoint sets whose union equals A, i.e. such that $A = \bigcup \mathrm{ran}(\bar{C})$. Being a codomain of $\bar{C}$, the range of $\bar{C}$ is thus included in the semiring of sets $\mathcal{I}$. Furthermore, $\mathcal{I}$ is included in the σ-algebra $\mathcal{A}(\mathcal{I})$ generated by that semiring of sets, so that the range of $\bar{C}$ is included also in $\mathcal{A}(\mathcal{I})$, which shows that this σ-algebra is another codomain of the sequence $\bar{C}$, i.e. we have $\bar{C} : \{1, \ldots, \bar{n}\} \to \mathcal{A}(\mathcal{I})$. In view of Note 11.15, the σ-algebra $\mathcal{A}(\mathcal{I})$ is closed under n-fold unions, which represent unions of ranges of sequences of sets. Consequently, the union

$$A = \bigcup \mathrm{ran}(\bar{C}) = \bigcup_{i=1}^{\bar{n}} \bar{C}_i$$

turns out to be an element of $\mathcal{A}(\mathcal{I})$, which finding proves the implication in (11.308). Since A is arbitrary, we may now infer from the truth of that implication the truth of the universal sentence (11.308), and therefore the truth of the inclusion $\mathcal{F} \subseteq \mathcal{A}(\mathcal{I})$.

Thus, we obtain the desired equation (11.307), as mentioned before. Here, the sets Ω and $<$ were arbitrary, so that the proposed universal sentence follows to be true. $\square$

We now state a method for constructing a σ-algebra from a given σ-algebra on a set Ω, such that the new σ-algebra is on a subset Ω_1 of Ω.

Theorem 11.41. *The following sentences are true for any measurable space $(\Omega, \mathcal{A})$ and any subset $\Omega_1 \subseteq \Omega$.*

a) There exists a unique set (system) $\mathcal{A}|\Omega_1$ containing precisely every set in $\mathcal{P}(\Omega_1)$ that is the intersection of Ω_1 with some measurable set A in $\mathcal{A}$.

b) This set $\mathcal{A}|\Omega_1$ satisfies then

$$\forall X\,(X \in \mathcal{A}|\Omega_1 \Leftrightarrow \exists A\,(A \in \mathcal{A} \wedge \Omega_1 \cap A = X)). \tag{11.309}$$

c) Furthermore, $\mathcal{A}|\Omega_1$ is a σ-algebra on Ω_1.

d) If Ω_1 is a measurable set in $\mathcal{A}$, then the σ-algebra $\mathcal{A}|\Omega_1$ on Ω_1 consists of all subsets of Ω_1 that are measurable sets in $\mathcal{A}$, i.e.

$$\forall A\,(A \in \mathcal{A}|\Omega_1 \Leftrightarrow [A \in \mathcal{A} \wedge A \subseteq \Omega_1]). \tag{11.310}$$

Proof. We let Ω, $\mathcal{A}$ and Ω_1 be arbitrary sets such that $\mathcal{A}$ is a σ-algebra on Ω and such that $\Omega_1 \subseteq \Omega$ holds. We may then concerning a) evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of a set $\mathcal{A}|\Omega_1$ satisfying

$$\forall X\,(X \in \mathcal{A}|\Omega_1 \Leftrightarrow [X \in \mathcal{P}(\Omega_1) \wedge \exists A\,(A \in \mathcal{A} \wedge \Omega_1 \cap A = X)]). \tag{11.311}$$

Concerning b), we take an arbitrary set X and assume first $X \in \mathcal{A}|\Omega_1$. Then, (11.311) gives in particular the existential sentence

$$\exists A\,(A \in \mathcal{A} \wedge \Omega_1 \cap A = X), \tag{11.312}$$

which is the desired consequent of the first part ('$\Rightarrow$') of the equivalence in (11.309) to be proven. To establish the second part ('$\Leftarrow$'), we now assume the preceding existential sentence to be true, so that there is a particular set $\bar{A} \in \mathcal{A}$ with $\Omega_1 \cap \bar{A} = X$. We observe now that $\Omega_1 \cap \bar{A} \subseteq \Omega_1$ is true according to (2.74), so that the resulting inclusion $X \subseteq \Omega_1$ yields $X \in \mathcal{P}(\Omega_1)$ with the definition of a power set. The conjunction of this finding and the assumed existential sentence (11.312) implies then with (11.311) $X \in \mathcal{A}|\Omega_1$, which is the desired consequent of the second part of the equivalence in (11.309), which is thus true. As X was arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (11.309).

Concerning c), we begin with the verification that $\mathcal{A}|\Omega_1$ satisfies Property 1 of a σ-algebra on Ω_1. For this purpose, we observe in (11.311) that $X \in \mathcal{A}|\Omega_1$ implies especially $X \in \mathcal{P}(\Omega_1)$ for any X, so that the required inclusion $\mathcal{A}|\Omega_1 \subseteq \mathcal{P}(\Omega_1)$ follows to be true by definition of a subset.

Regarding Property 2, we notice that the initial assumption $\Omega_1 \subseteq \Omega$ implies $\Omega_1 \cap \Omega = \Omega_1$. As $\Omega \in \mathcal{A}$ is also true according to Property 2 of a σ-algebra on Ω, we thus see that there exists an element of $\mathcal{A}$ such that the intersection of Ω_1 and that element equals Ω_1 (i.e., the set $X = \Omega_1$ satisfies the existential sentence in (11.309)). Therefore, $\Omega_1 \in \mathcal{A}|\Omega_1$ follows to be true, as required by Property 2 of a σ-algebra on Ω_1.

Regarding Property 3, we take an arbitrary set A, assume $A : \mathbb{N}_+ \to \mathcal{A}|\Omega_1$ to hold, and we show that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}|\Omega_1$ is implied. To do this, we will show that there is a particular set $\overline{A} \in \mathcal{A}$ satisfying $\Omega_1 \cap \overline{A} = \bigcup_{n=1}^{\infty} A_n$, which will imply the desired consequent with (11.309). More specifically, we may show that there exists a particular sequence $\overline{B} = (\overline{B}_n)_{n \in \mathbb{N}_+}$ in $\mathcal{A}$ such that $\Omega_1 \cap \bigcup_{n=1}^{\infty} \overline{B}_n = \bigcup_{n=1}^{\infty} A_n$ holds, where $\bigcup_{n=1}^{\infty} \overline{B}_n \in \mathcal{A}$ will be true in view of Property 3 of the σ-algebra $\mathcal{A}$. To prepare the construction of $\overline{B}$, we establish the universal sentence

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow \exists C \, (C \in \mathcal{A} \, \wedge \, \Omega_1 \cap C = A_n)). \tag{11.313}$$

Letting $n \in \mathbb{N}_+$ be arbitrary, the term $\bar{C} = A_n$ of the sequence A in $\mathcal{A}|\Omega_1$ is clearly an element of the codomain, i.e. $A_n \in \mathcal{A}|\Omega_1$, which implies the truth of the existential sentence in (11.313) with (11.309). As n is arbitrary, we may therefore conclude that the universal sentence (11.313) is true indeed. Choosing now in Lemma 3.237 $I = \mathbb{N}_+$, $X = \Omega_1$ and $\mathcal{K} = \mathcal{A}$, we obtain now a particular sequence $\overline{B} = (\overline{B}_n)_{n \in \mathbb{N}_+}$ in $\mathcal{A}$ such that

$$\forall n \, (n \in \mathbb{N}_+ \Rightarrow \Omega_1 \cap \bar{B}_n = A_n).$$

Here, $\bigcup_{n=1}^{\infty} \bar{B}_n \in \mathcal{A}$ follows to be true for that sequence, recalling that $\mathcal{A}$ is a σ-algebra. Furthermore, the sequence $(\overline{B}_n)_{n \in \mathbb{N}_+}$ defines according to Corollary 3.238 the unique sequence $S = (\Omega_1 \cap B_n)_{n \in \mathbb{N}_+}$ such that

$$(A_n)_{n \in \mathbb{N}_+} = (\Omega_1 \cap B_n)_{n \in \mathbb{N}_+},$$

i.e. such that $A = S$. We then obtain by means of the Distributive Law

(3.822) for families of sets and substitution

$$\Omega_1 \cap \bigcup_{n=1}^{\infty} \bar{B}_n = \bigcup_{n=1}^{\infty} (\Omega_1 \cap \bar{B}_n) = \bigcup \mathrm{ran}(S) = \bigcup \mathrm{ran}(A) = \bigcup_{n=1}^{\infty} A_n.$$

Thus, the set $\bigcup_{n=1}^{\infty} \bar{B}_n$ in $\mathcal{A}$ satisfies the equation $\Omega_1 \cap \bigcup_{n=1}^{\infty} \bar{B}_n = \bigcup_{n=1}^{\infty} A_n$. The existence of a set $A \in \mathcal{A}$ with $\Omega_1 \cap A = \bigcup_{n=1}^{\infty} A_n$ implies then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}|\Omega_1$ with (11.309), as required. As A was arbitrary, we may now finally conclude that $\mathcal{A}|\Omega_1$ satisfies Property 3 of a σ-algebra on Ω_1.

Regarding Property 4, we let $A \in \mathcal{A}|\Omega_1$ be arbitrary. In view of (11.309), there exists then an element in $\mathcal{A}$, say $\bar{C}$, with $\Omega_1 \cap \bar{C} = A$. Let us recall that the inclusions $\mathcal{A}|\Omega_1 \subseteq \mathcal{P}(\Omega_1)$ and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ hold, so that the definition of a subset yields $A \in \mathcal{P}(\Omega_1)$ as well as $\bar{C} \in \mathcal{P}(\Omega)$, and therefore (using the definition of a power set) $A \subseteq \Omega_1$ as well as $\bar{C} \subseteq \Omega$. We then obtain

$$\begin{aligned} A^c &= \Omega_1 \setminus A = \Omega_1 \setminus (\Omega_1 \cap \bar{C}) = \Omega_1 \setminus \bar{C} = (\Omega_1 \cap \Omega) \setminus \bar{C} = \Omega_1 \cap (\Omega \setminus \bar{C}) \\ &= \Omega_1 \cap \bar{C}^c \end{aligned}$$

by applying the definition of a complement (with respect to Ω_1), substitution, (2.98), (2.77) based on the initial assumption $\Omega_1 \subseteq \Omega$, (2.100), and finally again the definition of a complement (now with respect to Ω). Since $\bar{C} \in \mathcal{A}$ implies $\bar{C}^c \in \mathcal{A}$ with Property 4 of a σ-algebra, we see that there exists a set C satisfying both $C \in \mathcal{A}$ and $\Omega_1 \cap C = A^c$. Consequently, A^c evidently turns out to be an element in $\mathcal{A}|\Omega_1$. As A was arbitrary, we may therefore conclude that Property 4 of a σ-algebra applies also to $\mathcal{A}|\Omega_1$. We thus showed that $\mathcal{A}|\Omega_1$ is a σ-algebra on Ω_1.

Concerning d), we further assume $\Omega_1 \in \mathcal{A}$ to be true and let A be an arbitrary set. To prove the first part ('$\Rightarrow$') of the equivalence in (11.310), we assume moreover $A \in \mathcal{A}|\Omega_1$. According to b), there exists then a set, $\bar{C}$, such that $\bar{C} \in \mathcal{A}$ and $\Omega_1 \cap \bar{C} = A$ hold. On the one hand, the assumption $\Omega_1 \in \mathcal{A}$ and the previous finding $\bar{C} \in \mathcal{A}$ imply $[A =]\ \Omega_1 \cap \bar{C} \in \mathcal{A}$, because the σ-algebra $\mathcal{A}$ is closed under pairwise intersections according to Note 11.13. On the other hand, $[A =]\ \Omega_1 \cap \bar{C} \subseteq \Omega_1$ holds because of (2.74). We thus found $A \in \mathcal{A}$ and $A \subseteq \Omega_1$ to be both true, so that the first part ('$\Rightarrow$') of the equivalence holds.

Regarding the second part ('$\Leftarrow$'), we conversely assume now that $A \in \mathcal{A}$ and $A \subseteq \Omega_1$ are both true. The latter assumption implies then $A \cap \Omega_1 = A$ with (2.77), and the Commutative Law for the intersection of two sets yields $\Omega_1 \cap A = A \cap \Omega_1$ $[= A]$. We thus have a particular set A that satisfies both $A \in \mathcal{A}$ and $\Omega_1 \cap A = A$, so that A follows to be in $\mathcal{A}|\Omega_1$, according to b). The proof of the equivalence is now complete, and since A is arbitrary, we may therefore conclude that d) is also true.

Since the sets Ω, $\mathcal{A}$ and Ω_1 were initially arbitrary, we may then infer from the previous findings the truth of the stated theorem. □

Definition 11.14 (Trace (of a) σ-algebra). For any measurable space $(\Omega, \mathcal{A})$ and any subset Ω_1 of Ω we call

$$\mathcal{A}|\Omega_1 \tag{11.314}$$

the *trace* (σ*-algebra*) of $\mathcal{A}$ in Ω_1. We symbolize this set also by

$$\{\Omega_1 \cap A : A \in \mathcal{A}\} \tag{11.315}$$

Corollary 11.42. *It is true for any measurable space* $(\Omega, \mathcal{A})$ *and any subset* Ω_1 *of* Ω *that the trace of* $\mathcal{A}$ *in* Ω *is included in* $\mathcal{A}$ *if* Ω_1 *is a measurable set of* $\mathcal{A}$*, i.e.*

$$\Omega_1 \in \mathcal{A} \ \Rightarrow \ \mathcal{A}|\Omega_1 \subseteq \mathcal{A}. \tag{11.316}$$

Proof. Letting Ω, $\mathcal{A}$ and Ω_1 be arbitrary sets such that $(\Omega, \mathcal{A})$ is a measurable space and such that Ω_1 is a subset of Ω in $\mathcal{A}$, we see in light of Theorem 11.41d) that $A \in \mathcal{A}|\Omega_1$ implies especially $A \in \mathcal{A}$ for any A, so that the proposed inclusion (11.316) follows to be true with the definition of a subset. Since Ω, $\mathcal{A}$ and Ω_1 are arbitrary, the corollary follows therefore to be true. □

11.7. Topologies

Definition 11.15 (Topology, topological space, open set). For any set Ω, we say that a set (system) $\mathcal{O}$ is a *topology* on Ω iff

1. $\mathcal{O}$ consists of subsets of Ω, that is,
$$\mathcal{O} \subseteq \mathcal{P}(\Omega), \tag{11.317}$$
2. $\mathcal{O}$ contains Ω, that is,
$$\Omega \in \mathcal{O}, \tag{11.318}$$
3. $\mathcal{O}$ contains the union of any system of sets in $\mathcal{O}$, that is,
$$\forall \mathcal{K}\, (\mathcal{K} \subseteq \mathcal{O} \Rightarrow \bigcup \mathcal{K} \in \mathcal{O}), \tag{11.319}$$
and
4. $\mathcal{O}$ contains the intersection of any two sets in $\mathcal{O}$, that is,
$$\forall U, V\, (U, V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}), \tag{11.320}$$

where we call $(\Omega, \mathcal{O})$ a *topological space* and any element of $\mathcal{O}$ an *open set* in Ω (with respect to $\mathcal{O}$).

Having now the general idea of a topological space at our disposal, further related concepts are now introduced.

Definition 11.16 (Closed set, limit point, Hausdorff space). For any topological space $(\Omega, \mathcal{O})$,

(1) we say that a subset A of Ω is a *closed set* in Ω (with respect to $\mathcal{O}$) iff its complement (with respect to Ω) is open in Ω (with respect to $\mathcal{O}$), i.e. iff
$$A^c \in \mathcal{O}. \tag{11.321}$$

(2) we say for any subset $A \subseteq \Omega$ that an element $\omega \in \Omega$ is a *limit point* of A in Ω (with respect to $\mathcal{O}$) iff any open set U of $\mathcal{O}$ containing ω also contains some element $\nu \in A$ different from ω, i.e. iff
$$\forall U\, ([U \in \mathcal{O} \,\wedge\, \omega \in U] \Rightarrow \exists \nu\, (\nu \in A \,\wedge\, \nu \neq \omega)). \tag{11.322}$$

(3) we say that $(\Omega, \mathcal{O})$ is a *Hausdorff space* iff there exist, for any distinct elements ω and ν in Ω, disjoint open sets U and V in Ω containing ω and ν, respectively, i.e. iff
$$\begin{aligned} &\forall \omega, \nu\, ([\omega, \nu \in \Omega \,\wedge\, \omega \neq \nu] \\ &\quad \Rightarrow \exists U, V\, (U, V \in \mathcal{O} \,\wedge\, U \cap V = \emptyset \,\wedge\, \omega \in U \,\wedge\, \nu \in V)). \end{aligned} \tag{11.323}$$

Corollary 11.43. *Any topology $\mathcal{O}$ on any set Ω contains the empty set, i.e.*

$$\emptyset \in \mathcal{O}. \tag{11.324}$$

Furthermore, $\emptyset$ and Ω are closed sets in Ω.

Proof. Letting Ω and $\mathcal{O}$ be arbitrary sets such that $(\Omega, \mathcal{O})$ is a topological space, we see that $\emptyset \subseteq \mathcal{O}$ is true due to (2.43). Then, Property 3 of a topology on Ω gives $\bigcup \emptyset \in \mathcal{O}$, where $\bigcup \emptyset = \emptyset$ holds according to (2.205). Consequently, substitution yields on the one hand (11.324), as desired.

Furthermore, we have on the one hand $\emptyset^c = \Omega$ according to (2.134), so that Property 2 of a topology (on Ω) yields via substitution $\emptyset^c \in \mathcal{O}$; on the other hand, we have $\Omega^c = \emptyset$ according to (2.133), so that (11.324) gives $\Omega^c \in \mathcal{O}$ (again through substitution). These two findings imply that $\emptyset$ and Ω are closed sets in Ω, by definition.

Since Ω and $\mathcal{O}$ were initially arbitrary sets, we may therefore conclude that the proposed universal sentence holds. □

Proposition 11.44. *It is true for any set Ω that the pair $\{\emptyset, \Omega\}$ is a topology on Ω.*

Proof. Letting Ω be an arbitrary, we see in light of (3.15) that $\emptyset, \Omega \in \mathcal{P}(\Omega)$ holds, so that the inclusion

$$\{\emptyset, \Omega\} \subseteq \mathcal{P}(\Omega),$$

required by Property 1 of a topology on Ω, follows to be true with (2.164). Since $\Omega \in \{\emptyset, \Omega\}$ holds according to (2.151), Property 2 of a topology on Ω is also satisfied by $\{\emptyset, \Omega\}$. To show that this pair satisfies Property 3 of a topology on Ω, we can proceed similarly as in the proof that $\{\emptyset, \Omega\}$ satisfies Property 3 of a σ-algebra on Ω. Regarding Property 4, let us now recall from Proposition 11.29 and Proposition 11.30e) that the pair $\{\emptyset, \Omega\}$ is a σ-algebra on Ω and therefore a π-system on Ω, which thus satisfies the required

$$\forall U, V\, (U, V \in \{\emptyset, \Omega\} \Rightarrow U \cap V \in \{\emptyset, \Omega\}).$$

Because Ω was initially an arbitrary set, we may infer from the previous findings that the proposition holds, as claimed. □

Exercise 11.15. Prove for any Ω that the pair $\{\emptyset, \Omega\}$ satisfies indeed Property 3 of a topology on Ω.

(Hint: Proceed in analogy to the proof of Proposition 11.29, using now the definition of the union of a set system instead of the Characterization of the union of a family of sets, and using (2.4).)

We now use the previous findings to establish the unique existence of a topology on the empty set.

Proposition 11.45. *The singleton $\mathcal{O} = \{\emptyset\}$ is the only topology on $\emptyset$.*

Proof. We see in light of Proposition 11.44 and the definition of a singleton that

$$\mathcal{O} = \{\emptyset\} = \{\emptyset, \emptyset\}$$

is a topology on $\Omega = \emptyset$. We thus completed the first step within the Proof of a uniquely existential sentence according to Method 1.18. To complete the proof, we verify now the universal sentence

$$\forall \mathcal{O}' \, (\mathcal{O}' \text{ is a topology on } \emptyset \Rightarrow \{\emptyset\} = \mathcal{O}'), \tag{11.325}$$

letting $\mathcal{O}'$ be an arbitrary topology on $\Omega = \emptyset$. We prove the equation via the Axiom of Extension by verifying the two inclusions $\{\emptyset\} \subseteq \mathcal{O}'$ and $\mathcal{O}' \subseteq \{\emptyset\}$. On the one hand, we have $\emptyset \in \mathcal{O}'$ in view of Corollary 11.43, so that the first inclusion $\{\emptyset\} \subseteq \mathcal{O}'$ follows to be true with (2.184). On the other hand, we have $\mathcal{O}' \subseteq \mathcal{P}(\emptyset)$ $[= \{\emptyset\}]$ by Property 1 of a topology and with (3.17), so that the second inclusion $\mathcal{O}' \subseteq \{\emptyset\}$ also hold. Consequently, the equation $\{\emptyset\} = \mathcal{O}'$ is also true, and since $\mathcal{O}'$ is arbitrary, we may therefore conclude that the universal sentence (11.325) holds. Thus, the proof that the topology $\mathcal{O} = \{\emptyset\}$ on $\emptyset$ exists uniquely is complete. □

Exercise 11.16. Show that the power set of any set Ω is a topology on Ω.

(Hint: Apply a proof similarly to the proof that the power set of any set Ω is a σ-algebra on Ω, using Exercise 11.11 in connection with Proposition 11.30e).)

Definition 11.17 (Discrete & indiscrete topology). For any set Ω, we call

(1) the power set $\mathcal{P}(\Omega)$ the *discrete topology* on Ω and

(2) the pair $\{\emptyset, \Omega\}$ the *indiscrete topology* on Ω.

Note 11.19. Any topology $\mathcal{O}$ on any set Ω, containing $\emptyset, \Omega$ and being thus nonempty, constitutes evidently a π-system on Ω. Therefore, there exists the unique binary operation

$$\cap_{\mathcal{O}} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}, \quad (U, V) \mapsto U \cap V, \tag{11.326}$$

where $\cap_{\mathcal{O}}$ is idempotent and where $(\mathcal{O}, \cap_{\mathcal{O}})$ is a commutative semigroup.

Exercise 11.17. Show for any set Ω and any topology $\mathcal{O}$ on Ω that Ω is the neutral element of $\mathcal{O}$ with respect to $\cap_{\mathcal{O}}$.

(Hint: Proceed similarly as in Exercise 11.10.)

Note 11.20. Since $\cap_{\mathcal{O}}$ is a binary operation on $\mathcal{O}$ concerning which the neutral element Ω exists in $\mathcal{O}$, the corresponding n-fold binary operation

$$\bigcap_{i=1}^{n} : \mathcal{O}^{\{1,\ldots,n\}} \to \mathcal{O}, \quad (U_i \,|\, i \in \{1,\ldots,n\}) \mapsto \bigcap_{i=1}^{n} U_i \tag{11.327}$$

is also defined for any natural number and any topology $\mathcal{O}$ on any set Ω. Thus, topologies are closed under n-fold intersections. ; the following proposition shows that both kinds of union are equivalent representations of the same set. Thus, we may apply for instance Theorem 3.233 also in the context of n-fold unions on rings of sets.

Any recursively defined intersection in the sense of (11.327) can be shown to be identical with the intersection of the range of the sequence of sets considered.

Exercise 11.18. Show for any set Ω and any topology $\mathcal{O}$ on Ω that the n-fold binary intersection operation (11.327) satisfies for any $n \in \mathbb{N}$ and any sequence of sets $f = (U_i \,|\, i \in \{1,\ldots,n\})$ in $\mathcal{O}$ the equation

$$\bigcap_{i=1}^{n} U_i = \bigcap \operatorname{ran}(f). \tag{11.328}$$

(Proceed as in the proof of Proposition 11.25.)

Theorem 11.46 (Characterization of open sets). *It is true for any topological space* $(\Omega, \mathcal{O})$ *that a set* U *is open in* Ω *iff there is, for any element* ω *of* U*, an open set in* Ω *that contains* ω *and that is included in* U*, i.e.*

$$\forall U\,(U \in \mathcal{O} \Leftrightarrow \forall \omega\,(\omega \in U \Rightarrow \exists V\,(V \in \mathcal{O} \wedge \omega \in V \wedge V \subseteq U))). \tag{11.329}$$

Proof. We take arbitrary sets Ω, $\mathcal{O}$ and U, and we assume that $(\Omega, \mathcal{O})$ constitutes a topological space. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $U \in \mathcal{O}$ to be true and let then ω be arbitrary, assuming in addition that $\omega \in U$ holds. Since the inclusion $U \subseteq U$ is also true in view of (2.10), we have the true multiple conjunction $U \in \mathcal{O} \wedge \omega \in U \wedge U \subseteq U$, which shows that the existential sentence

$$\exists V\,(V \in \mathcal{O} \wedge \omega \in V \wedge V \subseteq U) \tag{11.330}$$

is true. As ω was arbitrary, we may therefore conclude that the universal sentence

$$\forall\omega\,(\omega \in U \Rightarrow \exists V\,(V \in \mathcal{O} \wedge \omega \in V \wedge V \subseteq U)) \tag{11.331}$$

holds, so that the proof of the implication '$\Rightarrow$' is complete.

To prove the second part ('$\Leftarrow$') of the equivalence, we assume now (11.331) to be true. Next, we specify for every $\omega \in U$ a unique set $\mathcal{Y}$ consisting of all open sets of $\mathcal{O}$ which contain ω and which are included in U, that is,

$$\forall\omega\,(\omega \in U \Rightarrow \exists!\mathcal{Y}\,(\forall V\,(V \in \mathcal{Y} \Leftrightarrow [V \in \mathcal{O} \wedge \omega \in V \wedge V \subseteq U]))). \tag{11.332}$$

Indeed, letting ω be arbitrary and assuming $\omega \in U$ to be true, the uniquely existential sentence follows to be true with the Axiom of Specification and the Equality Criterion for sets. Since ω is arbitrary, (11.332) follows therefore to be true, and the truth of this universal sentence implies then – according to Function definition by replacement – the existence of a particular function $\bar{f}$ with domain U such that

$$\forall\omega\,(\omega \in U \Rightarrow \forall V\,(V \in \bar{f}(\omega) \Leftrightarrow [V \in \mathcal{O} \wedge \omega \in V \wedge V \subseteq U])). \tag{11.333}$$

Here, we can prove that the empty set is not an element of the range of $\bar{f}$, by verifying

$$\forall X\,(X \in \mathrm{ran}(\bar{f}) \Rightarrow X \neq \emptyset). \tag{11.334}$$

letting X be arbitrary and assuming $X \in \mathrm{ran}(\bar{f})$ to be true, it follows by definition of a range that there exists a constant, say $\bar{\omega}$, such that $(\bar{\omega}, X) \in \bar{f}$. We thus have according to the definition of a domain $\bar{\omega} \in U$ [$= \mathrm{dom}(\bar{f})$], and this element $\bar{\omega}$ is associated with the value $X = \bar{f}(\bar{\omega})$, which satisfies

$$\forall V\,(V \in \bar{f}(\bar{\omega}) \Leftrightarrow [V \in \mathcal{O} \wedge \omega \in V \wedge V \subseteq U]). \tag{11.335}$$

Now, $\bar{\omega} \in U$ implies with (11.331) that there exists a set, say $\bar{V}$, such that the multiple conjunction $\bar{V} \in \mathcal{O} \wedge \omega \in \bar{V} \wedge \bar{V} \subseteq U$ is satisfied. Consequently, $\bar{V} \in \bar{f}(\bar{\omega})$ [$= X$] turns out to be true because of (11.335), where the resulting $\bar{V} \in X$ demonstrates that X is a nonempty set. As X was arbitrary, we therefore conclude that (11.334) holds, and this universal implies $\mathrm{ran}(\bar{f}) \neq \emptyset$ due to (2.5). This negation allows us to apply the Axiom of Choice in order to infer the existence of a particular function $\bar{g} : \mathrm{ran}(\bar{f}) \to \bigcup \mathrm{ran}(\bar{f})$ such that

$$\forall X\,(X \in \mathrm{ran}(\bar{f}) \Rightarrow \bar{g}(X) \in X). \tag{11.336}$$

The composition $\bar{g} \circ \bar{f}$ is then a function from U to $\bigcup \mathrm{ran}(\bar{f})$, according to Proposition 3.178. Here, we can write this composition also in the surjective form $\bar{g} \circ \bar{f} : U \twoheadrightarrow \mathrm{ran}(\bar{g} \circ \bar{f})$. We prove now that the union of the range of this composition is identical with U, by applying the Equality Criterion for sets and by proving accordingly the universal sentence

$$\forall \omega \, (\omega \in \bigcup \mathrm{ran}(\bar{g} \circ \bar{f}) \Leftrightarrow \omega \in U). \tag{11.337}$$

We take an arbitrary ω, and we assume first $\omega \in \bigcup \mathrm{ran}(\bar{g} \circ \bar{f})$ to be true. The definition of the union of a set system gives then a particular set $\bar{Y} \in \mathrm{ran}(\bar{g} \circ \bar{f})$ for which $\omega \in \bar{Y}$ holds. By definition of a range and by definition of a domain, there exists also a particular element $\bar{\nu} \in U$ $[= \mathrm{dom}(\bar{g} \circ \bar{f})]$ such that $(\bar{\nu}, \bar{Y}) \in \bar{g} \circ \bar{f}$. Because the preceding composition is a function, we can also write $\bar{Y} = \bar{g}(\bar{f}(\bar{\nu}))$, where $\bar{f}(\bar{\nu})$ is evidently an element of the range of $\bar{f}$, so that (11.336) yields $\bar{Y} \in \bar{f}(\bar{\nu})$. Consequently, (11.335) gives us especially the inclusion $\bar{Y} \subseteq U$, so that the previously found $\omega \in \bar{Y}$ implies $\omega \in U$ (by definition of a subset). We thus proved the first part ('$\Rightarrow$') of the equivalence in (11.337).

Regarding the second part ('$\Leftarrow$'), we assume conversely $\omega \in U$ to be true. Thus, ω is an element of the domain of $\bar{g} \circ \bar{f}$, which is associated with the value $\bar{g} \circ \bar{f}(\omega) \in \mathrm{ran}(\bar{g} \circ \bar{f})$. Here, (11.336) yields $\bar{g}(\bar{f}(\omega)) \in \bar{f}(\omega)$, and this implies now especially $\omega \in \bar{g}(\bar{f}(\omega))$ with (11.335). Writing this in the form $\omega \in \bar{g} \circ \bar{f}(\omega)$, we see in light of $\bar{g} \circ \bar{f}(\omega) \in \mathrm{ran}(\bar{g} \circ \bar{f})$ and the definition of the union of a set system that $\omega \in \bigcup \mathrm{ran}(\bar{g} \circ \bar{f})$ holds. This completes the proof of the equivalence, in which ω is arbitrary, so that (11.337) is true.

Consequently, the equivalent equality

$$\bigcup \mathrm{ran}(\bar{g} \circ \bar{f}) = U \tag{11.338}$$

is indeed true. Let us establish now also the truth of the inclusion

$$\mathrm{ran}(\bar{g} \circ \bar{f}) \subseteq \mathcal{O}. \tag{11.339}$$

According to the definition of a subset, we let X be arbitrary, and we assume $X \in \mathrm{ran}(\bar{g} \circ \bar{f})$ to be true. Since U is the domain of the composition domain, we have then $(\bar{\omega}, X) \in \bar{g} \circ \bar{f}$ for some particular element $\bar{\omega} \in U$. In view of (11.336), the value $X = \bar{g} \circ \bar{f}(\bar{\omega}) = \bar{g}(\bar{f}(\bar{\omega}))$ is an element of $\bar{f}(\bar{\omega})$ and, due to (11.335), also an element of $\mathcal{O}$. Having shown that $X \in \mathrm{ran}(\bar{g} \circ \bar{f})$ implies $X \in \mathcal{O}$, we can infer from the truth of this implication the truth of the inclusion (11.339) since X was arbitrary. Because of Property 3 of a topology, that inclusion implies now

$$\bigcup \mathrm{ran}(\bar{g} \circ \bar{f}) \in \mathcal{O},$$

so that substitution based on (11.338) yields $U \in \mathcal{O}$. Thus, the second part ('$\Leftarrow$') of the equivalence in (11.329) holds, too. Since U, Ω and $\mathcal{O}$ were initially arbitrary, we therefore conclude that that the theorem is true. □

Proposition 11.47. *It is true for any topological space* $(\Omega, \mathcal{O})$ *that the intersection of any nonempty system* $\mathcal{K}$ *of closed sets in* Ω *with respect to* $\mathcal{O}$ *is itself a closed set in* Ω *(with respect to* $\mathcal{O}$*).*

Proof. We let Ω, $\mathcal{O}$ and $\mathcal{K}$ be arbitrary sets such that $\mathcal{O}$ is a topology on Ω, such that $\mathcal{K} \neq \emptyset$ holds, and such that

$$\forall C\,(C \in \mathcal{K} \Rightarrow C \text{ is a closed set in } \Omega) \tag{11.340}$$

is true. According to Exercise 3.100, there exist then particular sets $\bar{I}$ and $\bar{A}$ such that $\bar{A} : \bar{I} \to \mathcal{K}$ and $\bigcap \mathcal{K} = \bigcap_{i \in \bar{I}} \bar{A}_i$ hold. Based on these findings, we may now demonstrate that $\bar{A}$ is a family of sets in the power set of Ω, i.e. that the inclusion $\mathrm{ran}(\bar{A}) \subseteq \mathcal{P}(\Omega)$ holds. Applying for this purpose the definition of a subset, we let $C \in \mathrm{ran}(\bar{A})$ be arbitrary and recall from the proof of Exercise 3.100 that $\mathrm{ran}(\bar{A}) = \mathcal{K}$ is true, so that substitution based on this equation yields $C \in \mathcal{K}$. This in turn implies with (11.340) that C is a closed set in Ω, which is thus a subset of Ω by definition. Consequently, C is an element of $\mathcal{P}(\Omega)$ by definition of a power set, and because C is arbitrary, we may now infer from this the truth of the inclusion $\mathrm{ran}(\bar{A}) \subseteq \mathcal{P}(\Omega)$. We may therefore view $\bar{A}$ as a family $(\bar{A}_i)_{i \in \bar{I}}$ in $\mathcal{P}(\Omega)$. According to Exercise 3.101, we may then define the family $s = (\bar{A}_i^c)_{i \in \bar{I}}$, whose terms we now show to satisfy

$$\forall i\,(i \in \bar{I} \Rightarrow \bar{A}_i^c \in \mathcal{O}). \tag{11.341}$$

Letting i be an arbitrary index in $\bar{I}$, the corresponding term $\bar{A}_i$ is then element of the codomain $\mathcal{K}$ of $\bar{A}$, and therefore a closed set in Ω according to (11.340). By definition of a closed set, we thus have $\bar{A}_i^c \in \mathcal{O}$, and as i was arbitrary, we may now further conclude that (11.341) is indeed true.

Let us now observe the truth of the equation

$$\Big(\bigcap_{i \in \bar{I}} \bar{A}_i\Big)^c = \bigcup_{i \in \bar{I}} \bar{A}_i^c = \bigcup \mathrm{ran}(s) \tag{11.342}$$

in light of De Morgan's Law for the intersection of a family of sets and the definition of the union of a family of sets (in connection with the definition of the family s). Furthermore, let us establish here the truth of the inclusion $\mathrm{ran}(s) \subseteq \mathcal{O}$ via the definition of a subset, letting U be an arbitrary set and assuming $U \in \mathrm{ran}(s)$. By definition of a range, there exists then a particular constant $\bar{k}$ such that $(\bar{k}, U) \in s$ holds, which we may write also as $U = s_{\bar{k}} = \bar{A}_{\bar{k}}^c$ with the definition of the family s. Moreover, the definition of a domain shows that $\bar{k} \in I$ $[= \mathrm{dom}(s)]$ is true, which gives us $\bar{A}_{\bar{k}}^c \in \mathcal{O}$ because of (11.341). Since U was arbitrary, we may therefore conclude that $\mathrm{ran}(s) \subseteq \mathcal{O}$ holds indeed. In view of Property 3 of a topology, we obtain then $\bigcup \mathrm{ran}(s) \in \mathcal{O}$, which shows in light of the equations (11.342) that the set $(\bigcap_{i\in\bar{I}}\bar{A}_i)^c$ is open in Ω with respect to $\mathcal{O}$, and therefore that the intersection $\bigcap_{i\in\bar{I}}\bar{A}_i$ is closed in Ω (with respect to $\mathcal{O}$). As the sets Ω, $\mathcal{O}$ and $\mathcal{K}$ were initially arbitrary, we may infer from this finding now the truth of the proposed universal sentence. $\square$

Exercise 11.19. Establish the following sentences for any topological space $(\Omega,\mathcal{O})$ and any subset $A\subseteq\Omega$.

a) There exists a unique set $\mathcal{X}$ consisting of all closed sets in Ω which include A, in the sense that

$$\forall C\,(C\in\mathcal{X}\Leftrightarrow[C\text{ is a closed set in }\Omega\ \wedge\ A\subseteq C]). \tag{11.343}$$

b) Then, the intersection $\mathrm{cl}(A)=\bigcap\mathcal{X}$ is itself a closed set in Ω that includes A.

c) Moreover, $\mathrm{cl}(A)$ is the smallest closed set in Ω that includes A in the sense that

$$\forall C\,([C\text{ is a closed set in }\Omega\ \wedge\ A\subseteq C]\Rightarrow\mathrm{cl}(A)\subseteq C). \tag{11.344}$$

(Hint: Proceed similarly as in the proof of Theorem 11.35.)

The previously specified intersection of all closed sets including a a given subset gives rise to the following definitions.

Definition 11.18 (Closure, dense subset, separable topological space). We say for any topological space $(\Omega,\mathcal{O})$

(1) and any subset $A\subseteq\Omega$ that the intersection

$$\mathrm{cl}(A) \tag{11.345}$$

of all closed sets in Ω (with respect to Ω) that include A is the *closure* of A in Ω (with respect to $\mathcal{O}$).

(2) and any subset $A\subseteq\Omega$ that A is a *dense subset* of Ω (alternatively, that A is *dense* in Ω) with respect to $\mathcal{O}$ iff the closure of A is identical with Ω, i.e. iff

$$\mathrm{cl}(A)=\Omega. \tag{11.346}$$

(3) that $(\Omega,\mathcal{O})$ is *separable* iff there exists a set A such that A is countable and such that A is a dense subset of Ω.

Corollary 11.48. *It is true for any topological space* $(\Omega, \mathcal{O})$ *that the closure of any closed set* A *in* Ω *is identical with* A*, that is,*

$$\forall A\,(A \text{ is a closed set in } \Omega \Rightarrow \mathrm{cl}(A) = A). \tag{11.347}$$

Proof. We let Ω, $\mathcal{O}$ and A be arbitrary sets such that $\mathcal{O}$ is a topology on Ω and such that A is a closed set in Ω with respect to $\mathcal{O}$. Since $A \subseteq A$ is also true according to (2.10), we obtain $\mathrm{cl}(A) \subseteq A$ with (11.344). On the other hand, the inclusion $A \subseteq \mathrm{cl}(A)$ holds according to Exercise 11.19b). The previous two inclusions give us then $\mathrm{cl}(A) = A$ with the Axiom of Extension, and since A, Ω and $\mathcal{O}$ were arbitrary, we may therefore conclude that the corollary is true. □

Note 11.21. Since $\emptyset$ is a closed set in any set Ω with respect to any topology $\mathcal{O}$ on Ω according to Corollary 11.43, we see in light of (11.347) that

$$\mathrm{cl}(\emptyset) = \emptyset. \tag{11.348}$$

Furthermore, the choice $\Omega = \emptyset$ gives rise to the topology $\{\emptyset, \Omega\} = \{\emptyset\}$ in view of Proposition 11.44 and the definition of a singleton, so that (11.348) shows also that $\emptyset$ is a dense subset of $\emptyset$ with respect to that topology. Then, recalling that the zero $0 = \emptyset$ is a natural number, which is a finite set according to Corollary 4.98 and thus a countable set by definition, we see that the topological space $(\emptyset, \{\emptyset\})$ is separable.

Theorem 11.49 (Characterization of the elements of a closure). *It is true for any topological space* $(\Omega, \mathcal{O})$ *and any subset* $A \subseteq \Omega$ *that an element* ω *of* Ω *is an element of the closure of* A *in* Ω *(with respect to* $\mathcal{O}$*) iff* A *and any open set of* $\mathcal{O}$ *containing* ω *are not disjoint, i.e.*

$$\forall \omega\,(\omega \in \Omega \Rightarrow [\omega \in \mathrm{cl}(A) \Leftrightarrow \forall U\,([U \in \mathcal{O} \wedge \omega \in U] \Rightarrow A \cap U \neq \emptyset)]). \tag{11.349}$$

Proof. Letting Ω, $\mathcal{O}$, A and ω be arbitrary and assuming $(\Omega, \mathcal{O})$ to be a topological space, A to be a subset of Ω as well as ω to be an element of Ω, we prove the first part ('$\Rightarrow$') of the equivalence in (11.349) by contraposition. For this purpose, we assume that the negation of the universal sentence with respect to U is true, and we show that the negation $\omega \notin \mathrm{cl}(A)$ follows to be true. Because of the Negation Law for universal implications, we obtain the true existential sentence

$$\exists U\,(U \in \mathcal{O} \wedge \omega \in U \wedge \neg A \cap U \neq \emptyset), \tag{11.350}$$

which means that there exists set, say $\bar{U}$, such that $U \in \mathcal{O}$, $\omega \in \bar{U}$ and $\neg A \cap U \neq \emptyset$ hold. Since $\bar{U}$ is open in Ω, we evidently have $\bar{U} \subseteq \Omega$, and its complement $\bar{U}^c$ is a closed set in Ω, by definition. Furthermore, the preceding negation yields the equation $A \cap U = \emptyset$ with the Double Negation Law; in conjunction with inclusions $A \subseteq \Omega$ and $\bar{U} \subseteq \Omega$, this implies $A \subseteq \bar{U}^c$ with (2.141). Together with the fact that $\bar{U}^c$ is a closed set in Ω, this further implies $\text{cl}(A) \subseteq \bar{U}^c$ with (11.344), i.e. with the fact that the closure of A is the smallest closed set of which A is a subset. Let us now observe that the previously established $\omega \in \bar{U}$ implies on the one hand $\omega \in \Omega$ with the definition of a subset in connection with the inclusion $\bar{U} \subseteq \Omega$, and on the other hand $\omega \in (\bar{U}^c)^c$ with (2.136), so that we obtain $\omega \notin \bar{U}^c$ by means of (2.132). Together with the inclusion $\text{cl}(A) \subseteq \bar{U}^c$, this gives us now $\omega \notin \text{cl}(A)$ with ω (2.9), which finding proves the first part of the equivalence in (11.349).

We prove the second part ('$\Leftarrow$') of that equivalence also by contraposition, assuming now $\omega \notin \text{cl}(A)$ to be true. Recalling that the closure $\text{cl}(A)$ is itself a closed set in Ω, according to Theorem 11.19b), we have then on the one hand that $\text{cl}(A)$ is a subset of Ω. On the other hand, its complement $[\text{cl}(A)]^c$ is an open set in Ω, i.e. $\Omega \setminus \text{cl}(A) \in \mathcal{O}$. Here, the initially assumed $\omega \in \Omega$ and $\omega \notin \text{cl}(A)$ imply $\omega \in [\text{cl}(A)]^c$ by definition of a set difference. Finally, recalling the truth of the inclusion $A \subseteq \text{cl}(A)$ (mentioned in the proof of the first part of the equivalence) and of the inclusion $\text{cl}(A) \subseteq \Omega$, we obtain $A \cap [\text{cl}(A)]^c = \emptyset$ with (2.140). We thus found an element of $\mathcal{O}$ that contains ω and whose intersection with A is empty, so that the existential sentence (11.350) is true. Then, another application of the Negation Law for universal implications shows that the negation of the universal sentence with respect to U in (11.349) is also true, completing the proof by contraposition, and thus the proof of the equivalence in (11.349).

Since Ω, $\mathcal{O}$, A and ω were initially arbitrary, we may therefore conclude that the theorem holds indeed. □

Theorem 11.50 (Denseness Criterion for topological spaces). *It is true for any topological space* $(\Omega, \mathcal{O})$ *and any subset* $A \subseteq \Omega$ *that* A *is dense in* Ω *(with respect to* $\mathcal{O}$*) iff* A *and any nonempty open set in* Ω *are not disjoint, i.e.*

$$\text{cl}(A) = \Omega \;\Leftrightarrow\; \forall U\,([U \in \mathcal{O} \wedge U \neq \emptyset] \Rightarrow A \cap U \neq \emptyset). \tag{11.351}$$

Proof. We take arbitrary sets Ω, $\mathcal{O}$ and A, for which we assume that $\mathcal{O}$ is a topology on Ω and that A is included in Ω. We prove the first part ('$\Rightarrow$') of the stated equivalence directly, assuming $\text{cl}(A) = \Omega$, letting then A be an arbitrary set, and assuming furthermore $U \in \mathcal{O}$ and $U \neq \emptyset$. The latter clearly implies that there exists an element in U, say $\bar{\omega}$. As an open set of the topology $\mathcal{O}$ on Ω, U is evidently a subset of Ω, so that $\bar{\omega} \in U$ implies $\bar{\omega} \in \Omega$ with the definition of a subset. Therefore, substitution based on the assumed equation yields $\bar{\omega} \in \mathrm{cl}(A)$. The previous findings imply now with the Characterization of the elements of a closure that $A \cap U \neq \emptyset$ is true, as desired. Since U is arbitrary, we may therefore conclude that the first implication '$\Rightarrow$' holds.

To establish the second implication '$\Leftarrow$' in (11.351), we now assume the universal sentence with respect to U to be true, and we prove the equation $\mathrm{cl}(A) = \Omega$ by means of the Equality Criterion for sets. For this purpose, we verify the universal sentence

$$\forall \omega \, (\omega \in \mathrm{cl}(A) \Leftrightarrow \omega \in \Omega), \tag{11.352}$$

letting ω be arbitrary. Since the closure of A is a closed set in Ω according to Exercise 11.19, it is by definition a subset of Ω. Therefore, assuming $\omega \in \mathrm{cl}(A)$ implies $\omega \in \Omega$ by definition of a subset. Assuming conversely $\omega \in \Omega$ to be true, we may now establish the universal sentence

$$\forall U \, ([U \in \mathcal{O} \wedge \omega \in U] \Rightarrow A \cap U \neq \emptyset). \tag{11.353}$$

To do this, we take an arbitrary open set U of $\mathcal{O}$ containing ω. We thus see that $U \neq \emptyset$ holds, which implies together with the assumed $U \in \mathcal{O}$ the desired consequent $A \cap U \neq \emptyset$ by means of the assumed universal sentence with respect to U in (11.351). Because U is arbitrary, we may now infer from this finding the truth of the universal sentence (11.353), which in turn implies – in view of $\omega \in \Omega$ and by virtue of the Characterization of the elements of a closure – that $\omega \in \mathrm{cl}(A)$ holds. We thus proved the equivalence in (11.352), so that the universal sentence (11.352) follows to be true (since ω was arbitrary). This completes the proof of the equation $\mathrm{cl}(A)$, and thus the proof of the equivalence in (11.351).

As the sets Ω, $\mathcal{O}$ and A were initially arbitrary, we may finally infer from the truth of that equivalence the truth of the stated theorem. □

Exercise 11.20. Show for any topological space $(\Omega, \mathcal{O})$ that the closure of a subset A of Ω is included in the closure of a subset B of Ω if A is included in B, i.e.

$$\forall A, B \, ([A \subseteq \Omega \wedge B \subseteq \Omega] \Rightarrow [A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)]). \tag{11.354}$$

(Hint: Apply the Characterization of the elements of a closure in connection with Exercise 11.19, the definition of the intersection of two sets and Proposition 2.11.)

The following theorem and definition have some noteworthy analogies to Theorem 11.41 and the concept of a trace σ-algebras.

Theorem 11.51. *The following sentences are true for any topological space* $(\Omega, \mathcal{O})$ *and any subset* $\Omega_1 \subseteq \Omega$.

a) There exists a unique set (system) $\mathcal{O}|\Omega_1$ *containing precisely every set in* $\mathcal{P}(\Omega_1)$ *which is the intersection of* Ω_1 *and some open set* U *in* $\mathcal{O}$.

b) This set $\mathcal{O}|\Omega_1$ *satisfies then*

$$\forall X \, (X \in \mathcal{O}|\Omega_1 \Leftrightarrow \exists U \, (U \in \mathcal{O} \wedge \Omega_1 \cap U = X)). \tag{11.355}$$

c) Moreover, $\mathcal{O}|\Omega_1$ *is a topology on* Ω_1.

d) If Ω_1 *is an open set of* $\mathcal{O}$*, then the topology* $\mathcal{O}|\Omega_1$ *on* Ω_1 *consists of all subsets of* Ω_1 *that are open sets of* $\mathcal{O}$*, i.e.*

$$\forall U \, (U \in \mathcal{O}|\Omega_1 \Leftrightarrow [U \in \mathcal{O} \wedge U \subseteq \Omega_1]). \tag{11.356}$$

Exercise 11.21. Prove Theorem 11.51.

(Hint: Proceed in analogy to the proof of Theorem 11.41.)

Definition 11.19 (Subspace topology, topological subspace). For any topological space $(\Omega, \mathcal{O})$ and any subset $\Omega_1 \subseteq \Omega$ we call the topology

$$\mathcal{O}|\Omega_1 \tag{11.357}$$

(specified according to Theorem 11.51) the *subspace topology* of $\mathcal{O}$ in Ω_1 and

$$(\Omega_1, \mathcal{O}|\Omega_1) \tag{11.358}$$

a *topological subspace* of $(\Omega, \mathcal{O})$.

Exercise 11.22. Prove for any topological space $(\Omega, \mathcal{O})$ and any subset Ω_1 of Ω that the subspace topology of $\mathcal{O}$ in Ω_1 is included in $\mathcal{O}$ if Ω_1 is an open set of $\mathcal{O}$, i.e.

$$\Omega_1 \in \mathcal{O} \;\Rightarrow\; \mathcal{O}|\Omega_1 \subseteq \mathcal{O}. \tag{11.359}$$

(Hint: Carry out a proof in analogy to the proof of Corollary 11.42.)

Proposition 11.52. *It is true for any topological space* $(\Omega, \mathcal{O})$ *that the subspace topology of* $\mathcal{O}$ *in* Ω *is identical with the topology* $\mathcal{O}$*, i.e.*

$$\mathcal{O}|\Omega = \mathcal{O}. \tag{11.360}$$

Proof. Letting Ω and $\mathcal{O}$ be arbitrary and assuming $(\Omega, \mathcal{O})$ to be a topological space, we have the true inclusion $\Omega \subseteq \Omega$ according to (2.10), so that the subspace topology $\mathcal{O}|\Omega$ is indeed defined. Furthermore, since $\Omega \in \mathcal{O}$ is true by Property 2 of a topology on Ω, we obtain with (11.359) the inclusion $\mathcal{O}|\Omega \subseteq \mathcal{O}$. We apply now the definition of a subset to establish the converse inclusion $\mathcal{O} \subseteq \mathcal{O}|\Omega$, letting X be an arbitrary open set of $\mathcal{O}$. Due to Property 1 of a topology on Ω, we have that $X \in \mathcal{O}$ implies $X \in \mathcal{P}(\Omega)$ and therefore $X \subseteq \Omega$ (using the definition of a power set). This inclusion in turn implies $X \cap \Omega = X$ with (2.77), which equation we may write also as $\Omega \cap X = X$. In view of $X \in \mathcal{O}$, the existential sentence $\exists U\, (U \in \mathcal{O} \wedge \Omega \cap U = X)$ is thus true, so that (11.355) yields $X \in \mathcal{O}|\Omega$, as desired. Because X is arbitrary in $\mathcal{O}$, we may therefore conclude that the inclusion $\mathcal{O} \subseteq \mathcal{O}|\Omega$ holds indeed. In conjunction with the inclusion $\mathcal{O}|\Omega \subseteq \mathcal{O}$, this gives us now the desired equation (11.360) by means of the Axiom of Extension. As Ω and $\mathcal{O}$ were initially arbitrary, the proposition follows therefore to be true. $\square$

11.7.1. Topological bases

Next, we inspect a mechanism for obtaining topologies from relatively simple-structured systems of sets, whose required properties are fixed by the following definition.

Definition 11.20 (Basis for a topology). For any set Ω, we say that a set (system) $\mathcal{K}_\Omega$ is a *basis* (*for a topology*) on Ω iff

1. $\mathcal{K}_\Omega$ consists of subsets of Ω, i.e.

$$\mathcal{K}_\Omega \subseteq \mathcal{P}(\Omega), \tag{11.361}$$

2. any element ω of Ω is contained in some set in $\mathcal{K}_\Omega$, i.e.

$$\forall \omega\, (\omega \in \Omega \Rightarrow \exists A\, (A \in \mathcal{K}_\Omega \;\wedge\; \omega \in A)), \tag{11.362}$$

   and

3. for any ω and any two sets A_1, A_2 in $\mathcal{K}_\Omega$ containing ω there exists a set A_3 in $\mathcal{K}_\Omega$ which contains ω and which is included in the intersection of A_1 and A_2, i.e.

$$\begin{aligned}&\forall A_1, A_2, \omega\, ([A_1, A_2 \in \mathcal{K}_\Omega \;\wedge\; \omega \in A_1 \;\wedge\; \omega \in A_2] \\ &\quad \Rightarrow \exists A_3\, (A_3 \in \mathcal{K}_\Omega \;\wedge\; \omega \in A_3 \;\wedge\; A_3 \subseteq A_1 \cap A_2)).\end{aligned} \tag{11.363}$$

Corollary 11.53. *Any basis $\mathcal{K}_\Omega$ for a topology on any set Ω is nonempty, that is,*

$$\forall \Omega, \mathcal{K}_\Omega\, (\mathcal{K}_\Omega \text{ is a basis for a topology on } \Omega \Rightarrow \mathcal{K}_\Omega \neq \emptyset). \tag{11.364}$$

Proof. Letting Ω and $\mathcal{K}_\Omega$ be arbitrary sets such that the latter is a basis for a topology on the former, and letting ω also be arbitrary, we see that the implication in (11.362) is true. Thus, its consequent is also true (irrespective of the truth value of its antecedent), so that there is a set, say $\bar{A}$, satisfying $\bar{A} \in \mathcal{K}_\Omega$ as well as $\omega \in \bar{A}$. Clearly, the set $\mathcal{K}_\Omega$ is nonempty, which fact then follows to be true for any Ω and any $\mathcal{K}_\Omega$. $\square$

Exercise 11.23. Prove for any set Ω and any basis $\mathcal{K}_\Omega$ for a topology on Ω that $\mathcal{K}_\Omega \setminus \{\emptyset\}$ constitutes again a basis for a topology on Ω.

(Hint: Show that $\mathcal{K}_\Omega \setminus \{\emptyset\}$ satisfies all three defining properties of a basis for a topology on Ω, using in particular (2.42) and (2.169).)

Theorem 11.54 (Generation of a topology by means of a basis). *It is true for any set Ω and any basis $\mathcal{K}_\Omega$ for a topology on Ω that there is a unique set (system) $\mathcal{O}(\mathcal{K}_\Omega)$ consisting of all sets B in the power set of Ω such that every element ω in B is contained in some set in $\mathcal{K}_\Omega$ which is included in B, i.e.*

$$\forall B\,(B \in \mathcal{O}(\mathcal{K}_\Omega) \Leftrightarrow \tag{11.365}$$
$$[B \in \mathcal{P}(\Omega) \wedge \forall \omega\,(\omega \in B \Rightarrow \exists A\,(A \in \mathcal{K}_\Omega \wedge \omega \in A \wedge A \subseteq B))]).$$

Then, the set $\mathcal{O}(\mathcal{K}_\Omega)$ is a topology on Ω.

Proof. We let Ω be an arbitrary set and $\mathcal{K}_\Omega$ an arbitrary basis for a topology on Ω. The unique existence of a set $\mathcal{O}(\mathcal{K}_\Omega)$ satisfying (11.365) evidently follows with the Axiom of Specification and the Equality Criterion for sets. We now verify the Properties 1 - 4 of a topology on Ω with respect to the set $\mathcal{O}(\mathcal{K}_\Omega)$.

Property 1 is evident since $B \in \mathcal{O}(\mathcal{K}_\Omega)$ implies in view of (11.365) especially $B \in \mathcal{P}(\Omega)$ for any B, so that $\mathcal{O}(\mathcal{K}_\Omega) \subseteq \mathcal{P}(\Omega)$ follows to be true by definition of a subset.

Regarding Property 2, we first notice the truth of $\Omega \in \mathcal{P}(\Omega)$ in view of (3.15). Letting now ω be arbitrary and assuming $\omega \in \Omega$ to be true, it follows with Property 2 of a basis for a topology that there exists a set, say $\bar{A}$, such that $\bar{A} \in \mathcal{K}_\Omega$ and $\omega \in \bar{A}$ both hold. The former implies now $\bar{A} \in \mathcal{P}(\Omega)$ with Property 1 of a basis for a topology, using the definition of a subset. Therefore, $\bar{A} \subseteq \Omega$ holds by definition of a power set. We thus showed that there is a set A satisfying the conjunction of $A \in \mathcal{K}_\Omega$, $\omega \in A$ and $A \subseteq \Omega$, so that the implication in (11.365) holds for the constant $B = \Omega$. As ω is arbitrary, we conclude that the universal sentence in (11.365) with respect to ω is true, which then implies – together with the initially noted $\Omega \in \mathcal{P}(\Omega)$ – the truth of $\Omega \in \mathcal{O}(\mathcal{K}_\Omega)$. Thus, $\mathcal{O}(\mathcal{K}_\Omega)$ satisfies indeed Property 2 of a topology on Ω.

Regarding Property 3, we let $\mathcal{K}$ be an arbitrary set, we assume $\mathcal{K} \subseteq \mathcal{O}(\mathcal{K}_\Omega)$ to be true, and we show that this assumption implies $\bigcup \mathcal{K} \in \mathcal{O}(\mathcal{K}_\Omega)$. For this purpose, we will apply again (11.365), and we begin with the verification of $\bigcup \mathcal{K} \in \mathcal{P}(\Omega)$. Because of the definition of a power set, we may write the latter equivalently as $\bigcup \mathcal{K} \subseteq \Omega$, which inclusion we prove by applying the definition of subset. To do this, we take an arbitrary ω such that $\omega \in \bigcup \mathcal{K}$ is true. According to the definition of the union of a set system, there exists then a set, say $\bar{B}$, satisfying both $\bar{B} \in \mathcal{K}$ and $\omega \in \bar{B}$. Let us recall the truth of the inclusions $\mathcal{K} \subseteq \mathcal{O}(\mathcal{K}_\Omega)\mathcal{P}(\Omega)$, which give $\mathcal{K} \subseteq \mathcal{P}(\Omega)$ with (2.13) and therefore $\forall B\,(B \in \mathcal{K} \Rightarrow B \in \mathcal{P}(\Omega))$ by definition of a subset. Consequently, the previously found $\bar{B} \in \mathcal{K}$ yields $\bar{B} \in \mathcal{P}(\Omega)$ and then

$\bar{B} \subseteq \Omega$ by definition of a power set. Therefore, the previously established $\omega \in \bar{B}$ evidently implies $\omega \in \Omega$ by definition of a subset. We thus showed that $\omega \in \bigcup \mathcal{K}$ implies $\omega \in \Omega$, and since ω was arbitrary, we may infer from the truth of this implication the truth of the inclusion $\bigcup \mathcal{K} \subseteq \Omega$. This finding establishes $\bigcup \mathcal{K} \in \mathcal{P}(\Omega)$, and we may now verify that the set $\bigcup \mathcal{K}$ satisfies also the second part of the conjunction in (11.365).

We take an arbitrary ω, assume $\omega \in \bigcup \mathcal{K}$ to be true, and demonstrate the truth of the existential sentence

$$\exists A \, (A \in \mathcal{K}_\Omega \, \wedge \, \omega \in A \, \wedge \, A \subseteq \bigcup \mathcal{K}). \tag{11.366}$$

Due to the preceding assumption there exists, by definition of the union of a set system, a particular set $\bar{B} \in \mathcal{K}$ with $\omega \in \bar{B}$. On the one hand, $\bar{B} \in \mathcal{K}$ implies the inclusion $\bar{B} \subseteq \bigcup \mathcal{K}$ with (2.201), and on the other hand $\bar{B} \in \mathcal{O}(\mathcal{K}_\Omega)$ with assumed $\mathcal{K} \subseteq \mathcal{O}(\mathcal{K}_\Omega)$ (using the definition of a subset). The latter implies with the specification of the set $\mathcal{O}(\mathcal{K}_\Omega)$ in (11.365) especially the truth of the universal sentence

$$\forall \omega \, (\omega \in \bar{B} \Rightarrow \exists A \, (A \in \mathcal{K}_\Omega \, \wedge \, \omega \in A \, \wedge \, A \subseteq \bar{B})),$$

so that the previously found $\omega \in \bar{B}$ implies the existence of a particular set $\bar{A} \in \mathcal{K}_\Omega$ satisfying also $\omega \in \bar{A}$ and the inclusion $\bar{A} \subseteq \bar{B}$. Recalling the inclusion $\bar{B} \subseteq \bigcup \mathcal{K}$, we therefore obtain $\bar{A} \subseteq \bigcup \mathcal{K}$ with (2.13). These properties of the set $\bar{A}$ thus demonstrate the truth of the existential sentence (11.366), and as ω was arbitrary, we may therefore conclude that the second part of the conjunction (11.365) is also satisfies by the union $\bigcup \mathcal{K}$. Therefore, this union turns out to be an element of $\mathcal{O}(\mathcal{K}_\Omega)$, which was to be proven. As $\mathcal{K}$ was arbitrary, we then conclude that $\mathcal{O}(\mathcal{K}_\Omega)$ satisfies Property 3 of a topology on Ω.

Regarding Property 4, we let $U_1, U_2 \in \mathcal{O}(\mathcal{K}_\Omega)$ be arbitrary, and we show that $U_1 \cap U_2 \in \mathcal{O}(\mathcal{K}_\Omega)$ follows to be true. The assumptions $U_1, U_2 \in \mathcal{O}(\mathcal{K}_\Omega)$ imply with (11.365) especially $U_1, U_2 \in \mathcal{P}(\Omega)$, consequently $U_1 \subseteq \Omega$ as well as $U_2 \subseteq \Omega$ by definition of a power sets, therefore $U_1 \cap U_2 \subseteq \Omega$ with (2.85), and then also $U_1 \cap U_2 \in \mathcal{P}(\Omega)$ (using again the definition of a power set). Next, we show that the union $U_1 \cap U_2$ satisfies also the second part of the conjunction in (11.365), i.e.

$$\forall \omega \, (\omega \in U_1 \cap U_2 \Rightarrow \exists A \, (A \in \mathcal{K}_\Omega \, \wedge \, \omega \in A \, \wedge \, A \subseteq U_1 \cap U_2)). \tag{11.367}$$

For this purpose, we let ω be arbitrary and assume $\omega \in U_1 \cap U_2$ to be true, so that $\omega \in U_1$ and $\omega \in U_2$ follow to be both true by definition of the intersection of two sets. Now, the assumptions $U_1, U_2 \in \mathcal{P}(\Omega)$ also imply with (11.365) the two universal sentences

$$\forall\omega\,(\omega \in U_1 \Rightarrow \exists A\,(A \in \mathcal{K}_\Omega \,\wedge\, \omega \in A \,\wedge\, A \subseteq U_1)),$$
$$\forall\omega\,(\omega \in U_2 \Rightarrow \exists A\,(A \in \mathcal{K}_\Omega \,\wedge\, \omega \in A \,\wedge\, A \subseteq U_2)),$$

so that the previously found $\omega \in U_1$ and $\omega \in U_2$ imply the existence of particular constants $\bar{A}_1$ and $\bar{A}_2$ satisfying the conjunctions

$$\bar{A}_1 \in \mathcal{K}_\Omega \,\wedge\, \omega \in \bar{A}_1 \wedge \bar{A}_1 \subseteq U_1,$$
$$\bar{A}_2 \in \mathcal{K}_\Omega \,\wedge\, \omega \in \bar{A}_2 \wedge \bar{A}_2 \subseteq U_2.$$

Since $\bar{A}_1, \bar{A}_2 \in \mathcal{K}_\Omega$, $\omega \in \bar{A}_1$ and $\omega \in \bar{A}_2$ are thus all true, we may utilize Property 3 of a basis for a topology to infer from these findings that there is a set, say $\bar{A}_3$, such that $\bar{A}_3 \in \mathcal{K}_\Omega$, $\omega \in \bar{A}_3$ and $\bar{A}_3 \subseteq \bar{A}_1 \cap \bar{A}_2$ hold. Let us observe that the previous multiple conjunctions give $\bar{A}_1 \subseteq U_1 \wedge \bar{A}_2 \subseteq U_2$, which conjunction implies $\bar{A}_1 \cap \bar{A}_2 \subseteq U_1 \cap U_2$ with (2.80). Together with the previously obtained inclusion $\bar{A}_3 \subseteq \bar{A}_1 \cap \bar{A}_2$, this further implies $\bar{A}_3 \subseteq U_1 \cap U_2$ with (2.13). Given the properties of $\bar{A}_3$, we thus showed that there exists a set A which satisfies $A \in \mathcal{K}_\Omega$, $\omega \in A$ and $A \subseteq U_1 \cap U_2$, so that the proof of the implication in (11.367) is complete. As ω was arbitrary, we therefore conclude that the universal sentence (11.367) holds, which in turn implies – together with $U_1 \cap U_2 \in \mathcal{P}(\Omega)$ – that $U_1 \cap U_2 \in \mathcal{O}(\mathcal{K}_\Omega)$ holds, according to the specification of the set $\mathcal{O}(\mathcal{K}_\Omega)$ in (11.365). Because U_1 and U_2 were arbitrary, we may conclude that $\mathcal{O}(\mathcal{K}_\Omega)$ satisfies Property 4 of a topology on Ω. Thus, $\mathcal{O}(\mathcal{K}_\Omega)$ constitutes a topology on Ω.

Since Ω and $\mathcal{K}_\Omega$ were initially arbitrary sets, the proposed universal sentence follows finally to be true. □

Definition 11.21 (Topology generated by a basis). For any set Ω and any basis $\mathcal{K}_\Omega$ for a topology on Ω, we call the set

$$\mathcal{O}(\mathcal{K}_\Omega), \tag{11.368}$$

that satisfies (11.365) the *topology generated by* $\mathcal{K}_\Omega$. We then say also that the basis $\mathcal{K}_\Omega$ generates $\mathcal{O}(\mathcal{K}_\Omega)$.

Proposition 11.55. *It is true for any set Ω that the topology $\mathcal{O}(\mathcal{K}_\Omega)$ generated by any basis $\mathcal{K}_\Omega$ includes this basis, i.e.*

$$\mathcal{K}_\Omega \subseteq \mathcal{O}(\mathcal{K}_\Omega). \tag{11.369}$$

Proof. We let Ω and $\mathcal{K}_\Omega$ be arbitrary sets such that $\mathcal{K}_\Omega$ is a basis for a topology on Ω, generating $\mathcal{O}(\mathcal{K}_\Omega)$. We now prove (11.369) by verifying

$$\forall B\,(B \in \mathcal{K}_\Omega \Rightarrow B \in \mathcal{O}(\mathcal{K}_\Omega)). \tag{11.370}$$

We let $B \in \mathcal{K}_\Omega$ be arbitrary and observe first that $B \in \mathcal{P}(\Omega)$ follows to be true with Property 1 of a basis for a topology on Ω and with the definition of a subset. Next, we prove

$$\forall \omega\, (\omega \in B \Rightarrow \exists A\, (A \in \mathcal{K}_\Omega \,\wedge\, \omega \in A \,\wedge\, A \subseteq B)), \tag{11.371}$$

letting ω be arbitrary and assuming $\omega \in B$ to be true. Thus, the set B satisfies the conjunctions $B \in \mathcal{K}_\Omega \wedge \omega \in B \wedge B \subseteq B$, where the inclusion holds according to (2.10), so that the existential sentence in (11.371) is true. As ω was arbitrary, we may therefore conclude that the universal sentence (11.371) holds, which implies now together with the previously established $B \in \mathcal{P}(\Omega)$ that $B \in \mathcal{O}(\mathcal{K}_\Omega)$ is true, according to the definition of a topology generated by a basis. Since B was also arbitrary, we may further conclude that the universal sentence (11.370) holds, which yields the desired inclusion (11.369) by definition of a subset. Finally, because the sets Ω and $\mathcal{K}_\Omega$ were initially arbitrary, we may infer from the truth of that inclusion the truth of the proposed universal sentence. □

Theorem 11.56 (Inclusion Criterion for topologies generated by bases). *It is true for any set Ω and any bases $\mathcal{K}_\Omega, \mathcal{K}'_\Omega$ for a topology on Ω that*

$$\begin{aligned} &\mathcal{O}(\mathcal{K}_\Omega) \subseteq \mathcal{O}(\mathcal{K}'_\Omega) \\ &\quad \Leftrightarrow\ \forall \omega, A\, ([A \in \mathcal{K}_\Omega \,\wedge\, \omega \in A] \Rightarrow \exists B\, (B \in \mathcal{K}'_\Omega \,\wedge\, \omega \in B \,\wedge\, B \subseteq A)). \end{aligned} \tag{11.372}$$

Proof. We let Ω, $\mathcal{K}_\Omega$ and $\mathcal{K}'_\Omega$ be arbitrary sets such that $\mathcal{K}_\Omega$ and $\mathcal{K}'_\Omega$ both constitute a basis for a topology on Ω. Thus, $\mathcal{O}(\mathcal{K}_\Omega)$ and $\mathcal{O}(\mathcal{K}'_\Omega)$ constitute the corresponding generated topologies on Ω.

To establish the first part ('$\Rightarrow$') of the equivalence, we assume the inclusion $\mathcal{O}(\mathcal{K}_\Omega) \subseteq \mathcal{O}(\mathcal{K}'_\Omega)$ to be true, which means by definition of subset that

$$\forall U\, (U \in \mathcal{O}(\mathcal{K}_\Omega) \Rightarrow U \in \mathcal{O}(\mathcal{K}'_\Omega)). \tag{11.373}$$

We now let ω and A be arbitrary such that $A \in \mathcal{K}_\Omega$ and $\omega \in A$ hold. Since the basis $\mathcal{K}_\Omega$ is included in the corresponding generated topology $\mathcal{O}(\mathcal{K}_\Omega)$ according to (11.369), we clearly see that $A \in \mathcal{K}_\Omega$ implies $A \in \mathcal{O}(\mathcal{K}_\Omega)$. This in turn yields $A \in \mathcal{O}(\mathcal{K}'_\Omega)$ with (11.373), and this finding implies the universal sentence

$$\forall \omega\, (\omega \in A \Rightarrow \exists B\, (B \in \mathcal{K}'_\Omega \,\wedge\, \omega \in B \,\wedge\, B \subseteq A)) \tag{11.374}$$

according to the Generation of a topology by means of a basis. Thus, the previously made assumption $\omega \in A$ implies precise the existential sentence to be shown; since ω and A were arbitrary, we may therefore conclude that the first part of the equivalence (11.372) holds.

Regarding the second part ('$\Leftarrow$') of that equivalence, we now assume the right-hand side to be true and verify (11.373). Letting U be an arbitrary open set of $\mathcal{O}(\mathcal{K}_\Omega)$, the Generation of a topology by means of a basis shows us that

$$\forall \omega\,(\omega \in U \Rightarrow \exists T\,(T \in \mathcal{K}_\Omega \,\wedge\, \omega \in T \,\wedge\, T \subseteq U)) \tag{11.375}$$

is true. Similarly, to prove that U is also an open set of $\mathcal{O}(\mathcal{K}'_\Omega)$, we may establish the truth of

$$\forall \omega\,(\omega \in U \Rightarrow \exists B\,(B \in \mathcal{K}'_\Omega \,\wedge\, \omega \in B \,\wedge\, B \subseteq U)). \tag{11.376}$$

Letting $\omega \in U$ be arbitrary, it follows with (11.375) that there is a particular basis element $\bar{T} \in \mathcal{K}_\Omega$ with $\omega \in \bar{T}$ and $\bar{T} \subseteq U$. Due to the assumed right-hand side of the equivalence to be proven, it follows from $\bar{T} \in \mathcal{K}_\Omega$ and $\omega \in \bar{T}$ that there is a particular basis element $\bar{B} \in \mathcal{K}'_\Omega$ satisfying $\omega \in \bar{B}$ and $\bar{B} \subseteq \bar{T}$. The conjunction of the latter and the previously established inclusion $\bar{T} \subseteq U$ yields then $\bar{B} \subseteq U$ with the transitivity of $\subseteq$. The truth of $\bar{B} \in \mathcal{K}'_\Omega$, $\omega \in \bar{B}$ and $\bar{B} \subseteq U$ in turn demonstrates the truth of the existential sentence in (11.376). As ω was arbitrary, we therefore conclude that the universal sentence (11.376) holds. According to the Generation of a topology by means of a basis, $U \in \mathcal{O}(\mathcal{K}'_\Omega)$ follows now be true (observing that the open set U of $\mathcal{O}(\mathcal{K}_\Omega)$ is an element of the power set $\mathcal{P}(\Omega)$ by Property 1 of a topology on Ω). Thus, the proof of the implication in (11.373) is complete, in which U is arbitrary, so that the universal sentence (11.373) follows to be true as well. This completes the proof of the inclusion and thus the proof of the equivalence (11.372).

Initially, the sets Ω, $\mathcal{K}_\Omega$ and $\mathcal{K}'_\Omega$ were arbitrary, so that we may conclude that the theorem is indeed true. □

Theorem 11.57 (Characterization of the elements of a topology generated by a basis). *For any set Ω and any basis $\mathcal{K}_\Omega$ for a topology on Ω, the generated topology $\mathcal{O}(\mathcal{K}_\Omega)$ consists of all unions of systems of basis sets, i.e.*

$$\forall B\,(B \in \mathcal{O}(\mathcal{K}_\Omega) \Leftrightarrow \exists \mathcal{G}\,(\mathcal{G} \subseteq \mathcal{K}_\Omega \,\wedge\, B = \bigcup \mathcal{G})). \tag{11.377}$$

Proof. We let Ω and $\mathcal{K}_\Omega$ be arbitrary sets, and we assume that $\mathcal{K}_\Omega$ is a basis for a topology on Ω, generating the topology $\mathcal{O}(\mathcal{K}_\Omega)$. Now, we also let B be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence in (11.377) directly, assuming $B \in \mathcal{O}(\mathcal{K}_\Omega)$. This implies with Theorem 11.54 especially the truth of the universal sentence

$$\forall \omega\, (\omega \in B \Rightarrow \exists A\, (A \in \mathcal{K}_\Omega \,\wedge\, \omega \in A \,\wedge\, A \subseteq B)). \tag{11.378}$$

Using the Axiom of Specification and the Equality Criterion for sets, we then have that there exists a unique set (system) $\mathcal{G}'$ which consists of all the sets A in the basis that contain an $\omega \in B$ and that are included in B, i.e. which satisfies

$$\forall A\, (A \in \mathcal{G}' \Leftrightarrow [A \in \mathcal{K}_\Omega \,\wedge\, \exists \omega\, (\omega \in A) \,\wedge\, A \subseteq B]). \tag{11.379}$$

Because $A \in \mathcal{G}'$ implies for any A in particular $A \in \mathcal{K}_\Omega$, it follows by definition of a subset that $\mathcal{G}' \subseteq \mathcal{K}_\Omega$. We now apply the Equality Criterion for sets to verify $B = \bigcup \mathcal{G}'$, by demonstrating the truth of the universal sentence

$$\forall \omega\, (\omega \in B \Leftrightarrow \omega \in \bigcup \mathcal{G}'). \tag{11.380}$$

Letting $\bar{\omega}$ be arbitrary, we may write the equivalence equivalently as (using the definition of the union of a set system)

$$\bar{\omega} \in B \Leftrightarrow \exists A\, (A \in \mathcal{G}' \,\wedge\, \bar{\omega} \in A). \tag{11.381}$$

Assuming first $\bar{\omega} \in B$ to be true, it follows with (11.378) that there is a particular set $\bar{A}$ such that $\bar{A} \in \mathcal{K}_\Omega$, $\bar{\omega} \in \bar{A}$ and $\bar{A} \subseteq B$ hold. These findings clearly show that the conjunctions

$$\bar{A} \in \mathcal{K}_\Omega \,\wedge\, \exists \omega\, (\omega \in \bar{A}) \,\wedge\, \bar{A} \subseteq B$$

are true, so that $\bar{A} \in \mathcal{G}'$ follows to be true with (11.379). Alongside $\bar{\omega} \in \bar{A}$, this in turn demonstrates the truth of the existential sentence in (11.381). Assuming conversely that there exists an element of $\mathcal{G}'$, say $\bar{A}$, with $\bar{\omega} \in \bar{A}$ and $\bar{A} \subseteq B$, it follows from $\bar{\omega} \in \bar{A}$ that $\bar{\omega} \in B$ holds (by definition of a subset), completing the proof of the equivalence. Thus, the equivalence in (11.380) also holds, and since $\bar{\omega}$ is arbitrary, we may now conclude that $B = \bigcup \mathcal{G}'$ is indeed true. We thus proved the first part of the equivalence in (11.377).

We now prove the second part ('$\Leftarrow$') of that equivalence, assuming that there exists a particular set $\bar{\mathcal{G}}$ satisfying $\bar{\mathcal{G}} \subseteq \mathcal{K}_\Omega$ and $B = \bigcup \bar{\mathcal{G}}$. Because the inclusion (11.369) is also true, we obtain the inclusion $\bar{\mathcal{G}} \subseteq \mathcal{O}(\mathcal{K}_\Omega)$ with (2.13). By Property 3 of a topology, the preceding inclusion in turn implies $[B =]\ \bigcup \bar{\mathcal{G}} \in \mathcal{O}(\mathcal{K}_\Omega)$, so that $B \in \mathcal{O}(\mathcal{K}_\Omega)$ holds. Thus, the proof of the equivalence in (11.377) is complete, and as B, Ω and $\mathcal{K}_\Omega$ were arbitrary, we therefore conclude that the theorem is true. $\square$

Note 11.22. For any set Ω and any basis $\mathcal{K}_\Omega$ for a topology on Ω, the generated topology $\mathcal{O}(\mathcal{K}_\Omega)$ contains $B = \emptyset$ due to Corollary 11.43. Correspondingly, the empty set $\bar{\mathcal{G}} = \emptyset$ satisfies the required conjunction on the right-hand side of the equivalence in (11.377), i.e. $\emptyset \subseteq \mathcal{K}_\Omega \wedge \emptyset = \bigcup \emptyset$, because of (2.43) and (2.205).

Theorem 11.58 (Characterization of the basis generating a given topology). *The following implication holds for any topological space* $(\Omega, \mathcal{O})$ *and any subset* $\mathcal{K} \subseteq \mathcal{O}$. *If any element* ω *of any open set* B *in* $\mathcal{O}$ *is contained in some set* A *in* $\mathcal{K}$ *such that* A *is included in* B, *i.e. if* $(\Omega, \mathcal{O})$ *and* $\mathcal{K}$ *satisfy*

$$\forall B, \omega\,([B \in \mathcal{O} \wedge \omega \in B] \Rightarrow \exists A\,(A \in \mathcal{K} \wedge \omega \in A \wedge A \subseteq B)), \tag{11.382}$$

then

a) $\mathcal{K}$ *is a basis for a topology on* Ω, *and*

b) $\mathcal{O}$ *is the topology generated by the basis* $\mathcal{K}_\Omega = \mathcal{K}$.

Proof. We let Ω, $\mathcal{O}$ and $\mathcal{K}$ be arbitrary sets such that $(\Omega, \mathcal{O})$ is a topological space and such that $\mathcal{K} \subseteq \mathcal{O}$ holds. We now prove the stated implication directly, assuming that (11.382) holds. Regarding the first part a) of the desired conjunction, we observe in light of the assumed $\mathcal{K} \subseteq \mathcal{O}$ and in light of the true inclusion $\mathcal{O} \subseteq \mathcal{P}(\Omega)$ (which holds by virtue of Property 1 of a topology on Ω) that $\mathcal{K} \subseteq \mathcal{P}(\Omega)$ follows to be true with (2.13), which finding shows that $\mathcal{K}$ satisfies Property 1 of a basis for a topology on Ω.

To verify Property 2 of a basis, i.e. to verify

$$\forall \omega\,(\omega \in \Omega \Rightarrow \exists A\,(A \in \mathcal{K} \wedge \omega \in A)), \tag{11.383}$$

we let $\omega \in \Omega$ be arbitrary and notice that $\Omega \in \mathcal{O}$ holds by Property 2 of a topology; then, the conjunction of $\Omega \in \mathcal{O}$ and $\omega \in \Omega$ implies with (11.382) that there exists a set, say $\bar{A}$, satisfying $\bar{A} \in \mathcal{K}$, $\omega \in \bar{A}$ (and $\bar{A} \subseteq \Omega$). This proves the existential sentence in (11.383), and since ω is arbitrary, we therefore conclude that $\mathcal{K}$ satisfies indeed Property 2 of a basis for a topology on Ω.

Regarding Property 3 of a basis, i.e.

$$\begin{aligned} &\forall A_1, A_2, \omega\,([A_1, A_2 \in \mathcal{K} \wedge \omega \in A_1 \wedge \omega A_2] \\ &\quad \Rightarrow \exists A_3\,(A_3 \in \mathcal{K} \wedge \omega \in A_3 \wedge A_3 \subseteq A_1 \cap A_2)). \end{aligned} \tag{11.384}$$

we let A_1, A_2 and ω be arbitrary such that $A_1, A_2 \in \mathcal{K}$, $\omega \in A_1$ and $\omega \in A_2$ hold. The latter two assumptions imply then $\omega \in A_1 \cap A_2$ by definition of the intersection of two sets. As $\mathcal{K}$ is a subset of $\mathcal{O}$, it follows that A_1 and

A_2 are elements also of $\mathcal{O}$, so that $A_1 \cap A_2 \in \mathcal{O}$ is true due to Property 4 of a topology. Then, the conjunction of $A_1 \cap A_2 \in \mathcal{O}$ and $\omega \in A_1 \cap A_2$ implies with (11.382) that there is a set, say A_3, which satisfies $A_3 \in \mathcal{K}$, $\omega \in A_3$ and $A_3 \subseteq A_1 \cap A_2$. These findings demonstrate the truth of the existential sentence in (11.384). Therefore, because A_1, A_2 and ω are arbitrary, we now conclude that the universal sentence (11.384) holds, so that $\mathcal{K}$ satisfies also Property 3 of a basis. In summary, we thus proved that $\mathcal{K}_\Omega = \mathcal{K}$ is a basis for a topology on Ω.

We now prove b), i.e. $\mathcal{O} = \mathcal{O}(\mathcal{K})$, by verifying

$$\forall B\,(B \in \mathcal{O} \Leftrightarrow B \in \mathcal{O}(\mathcal{K})). \tag{11.385}$$

Letting B be arbitrary, we first assume $B \in \mathcal{O}$ and show that this implies $B \in \mathcal{O}(\mathcal{K})$. To do this, we verify the conjunction

$$B \in \mathcal{P}(\Omega) \,\wedge\, \forall\omega\,(\omega \in B \Rightarrow \exists A\,(A \in \mathcal{K} \,\wedge\, \omega \in A \,\wedge\, A \subseteq B)), \tag{11.386}$$

which will imply the desired $B \in \mathcal{O}(\mathcal{K})$ with Theorem 11.54. Recalling that $\mathcal{O} \subseteq \mathcal{P}(\Omega)$ holds according to Property 1 of a topology on Ω, it follows from $B \in \mathcal{O}$ with the definition of a subset that the first part $B \in \mathcal{P}(\Omega)$ of the conjunction is true. To establish the second part, we let ω be arbitrary, assume $\omega \in B$ to be true, and we notice that the conjunction of $B \in \mathcal{O}$ and $\omega \in B$ already implies with (11.382) the truth of the existential sentence in (11.386). As ω is arbitrary, we may therefore conclude that the universal sentence (11.386) is true, so that the proof of the conjunction (11.386) is complete. As mentioned before, this gives now $B \in \mathcal{O}(\mathcal{K})$, completing the proof of the first part ('$\Rightarrow$') of the equivalence in (11.385).

Regarding the second part, we conversely assume $B \in \mathcal{O}(\mathcal{K})$ to be true. Due to (11.377), there exists then a system of sets, say $\bar{\mathcal{G}}$, such that $\bar{\mathcal{G}} \subseteq \mathcal{K}$ and $B = \bigcup \bar{\mathcal{G}}$ hold. The preceding inclusion implies now together with the initially assumed inclusion $\mathcal{K} \subseteq \mathcal{O}$ that $\bar{\mathcal{G}} \subseteq \mathcal{O}$ holds, using once again (2.13). Because of Property 3 of a topology, we obtain then $[B =]\ \bigcup \bar{\mathcal{G}} \in \mathcal{O}$, which shows that the second part ('$\Leftarrow$') of the equivalence (11.385) also holds.

Since B is arbitrary, we may therefore conclude that (11.385) holds, which universal sentence implies now the truth of the equation $\mathcal{O} = \mathcal{O}(\mathcal{K})$ with the Equality Criterion for sets. Thus, $\mathcal{O}$ is indeed the topology generated by the basis $\mathcal{K}$, so that the proof of b) is complete as well.

As Ω, $\mathcal{O}$ and $\mathcal{K}$ were initially arbitrary sets, we may now conclude that the theorem holds, as claimed. □

Proposition 11.59. *The following sentences are true for any set Ω and for any basis $\mathcal{K}_\Omega$ for a topology on Ω.*

a) There exists a unique set $\mathcal{U}$ consisting of all topologies on Ω which include $\mathcal{K}_\Omega$, in the sense that

$$\forall \mathcal{O}\,(\mathcal{O} \in \mathcal{U} \Leftrightarrow [\mathcal{O} \text{ is a topology on } \Omega \,\wedge\, \mathcal{K}_\Omega \subseteq \mathcal{O}]). \tag{11.387}$$

b) Then, the intersection $\bigcap \mathcal{U}$ is itself a topology on Ω that includes $\mathcal{K}_\Omega$.

c) Furthermore, the intersection $\bigcap \mathcal{U}$ is the topology generated by $\mathcal{K}_\Omega$, i.e.

$$\bigcap \mathcal{U} = \mathcal{O}(\mathcal{K}_\Omega). \tag{11.388}$$

d) Moreover, the generated topology $\mathcal{O}(\mathcal{K}_\Omega)$ is the smallest topology on Ω that includes $\mathcal{K}_\Omega$ in the sense that

$$\forall \mathcal{O}\,([\mathcal{O} \text{ is a topology on } \Omega \,\wedge\, \mathcal{K}_\Omega \subseteq \mathcal{O}] \Rightarrow \mathcal{O}(\mathcal{K}_\Omega) \subseteq \mathcal{O}). \tag{11.389}$$

Proof. We take arbitrary sets Ω and $\mathcal{K}_\Omega$ such that $\mathcal{K}_\Omega$ is a basis for a topology on Ω. Then, a) – b) are proved similarly as for the Generation of σ-algebras.

Concerning c), we establish the truth of the two inclusions $\bigcap \mathcal{U} \subseteq \mathcal{O}(\mathcal{K}_\Omega)$ and $\mathcal{O}(\mathcal{K}_\Omega) \subseteq \bigcap \mathcal{U}$, which will imply the stated equation (11.388) with the Axiom of Extension. On the one hand, we have that the generated topology $\mathcal{O}(\mathcal{K}_\Omega)$ is a topology on Ω which includes its basis $\mathcal{K}_\Omega$ because of Proposition 11.55; therefore, we obtain $\mathcal{O}(\mathcal{K}_\Omega) \in \mathcal{U}$ with (11.387) and then $\bigcap \mathcal{U} \subseteq \mathcal{O}(\mathcal{K}_\Omega)$ due to (2.92).

On the other hand, we may prove the universal sentence

$$\forall V\,(V \in \mathcal{O}(\mathcal{K}_\Omega) \Rightarrow V \in \bigcap \mathcal{U}), \tag{11.390}$$

letting V be an arbitrary open set in $\mathcal{O}(\mathcal{K}_\Omega)$. It then follows with the Characterization of the elements of a topology generated by a basis that there exists a set, say $\bar{\mathcal{G}}$, such that $\bar{\mathcal{G}} \subseteq \mathcal{K}_\Omega$ and $V = \bigcup \bar{\mathcal{G}}$ are both true. We already showed in b) that $\bigcap \mathcal{U}$ is a topology on Ω satisfying $\mathcal{K}_\Omega \subseteq \bigcap \mathcal{U}$, which inclusion implies with the previously established inclusion $\bar{\mathcal{G}} \subseteq \mathcal{K}_\Omega$ the truth of $\bar{\mathcal{G}} \subseteq \bigcap \mathcal{U}$. Furthermore, $\bigcap \mathcal{U}$ satisfies especially Property 3 of a topology on Ω, so that the preceding inclusion yields $\bigcup \bar{\mathcal{G}} \in \bigcap \mathcal{U}$ and then via substitution $V \in \bigcap \mathcal{U}$, as desired. As V was arbitrary, we may therefore conclude that (11.390) is true, which universal sentence in turn implies $\mathcal{O}(\mathcal{K}_\Omega) \subseteq \bigcap \mathcal{U}$ by definition of a subset. Thus, both of the proposed inclusions hold, so that the stated equation (11.388) follows to be true.

Concerning d), we take an arbitrary set $\mathcal{O}$ and assume $\mathcal{O}$ to be a topology on Ω such that $\mathcal{K}_\Omega \subseteq \mathcal{O}$ also holds. These assumptions imply $\mathcal{O} \in \mathcal{U}$ by definition of the latter set, which then further implies $\bigcap \mathcal{U} \subseteq \mathcal{O}$ with (2.92), as desired. Since $\mathcal{O}$ was arbitrary, we may therefore conclude in view of c) that d) also holds.

Because Ω and $\mathcal{K}_\Omega$ were arbitrary, we may now conclude that the stated proposition is true. $\square$

Exercise 11.24. Prove Proposition 11.59a,b).

(Hint: One can follow along similar lines of arguments as in the Generation of σ-algebras.)

Theorem 11.60 (Characterization of the elements of a closure by means of a basis). *It is true for any set Ω, any basis $\mathcal{K}_\Omega$ for a topology on Ω and any subset $A \subseteq \Omega$ that an element ω of Ω is an element of the closure of A in Ω (with respect to the generated topology $\mathcal{O}(\mathcal{K}_\Omega)$) iff A and any basis element of $\mathcal{K}_\Omega$ are not disjoint, i.e.*

$$\forall \omega\, (\omega \in \Omega \Rightarrow [\omega \in \mathrm{cl}(A) \Leftrightarrow \forall B\, ([B \in \mathcal{K}_\Omega \,\wedge\, \omega \in B] \Rightarrow A \cap B \neq \emptyset)]). \tag{11.391}$$

Proof. We take arbitrary sets Ω, $\mathcal{K}_\Omega$ and A such that $\mathcal{K}_\Omega$ is a basis for a topology on Ω and such that A is a subset of Ω. Thus, $\mathcal{K}_\Omega$ generates the topology $\mathcal{O}(\mathcal{K}_\Omega)$ on Ω, and the closure $\mathrm{cl}(A)$ of A in Ω with respect to $\mathcal{O}(\mathcal{K}_\Omega)$ is defined. Next, we take an arbitrary ω and assume $\omega \in \Omega$ to be true.

We prove the first part ('$\Rightarrow$') of the equivalence in (11.391) directly, assuming $\omega \in \mathrm{cl}(A)$ to hold. Consequently, recalling the truth of $\omega \in \Omega$, the Characterization of the elements of a closure gives us the true universal sentence

$$\forall U\, ([U \in \mathcal{O}(\mathcal{K}_\Omega) \,\wedge\, \omega \in U] \Rightarrow A \cap U \neq \emptyset). \tag{11.392}$$

To prove the desired consequent

$$\forall B\, ([B \in \mathcal{K}_\Omega \,\wedge\, \omega \in B] \Rightarrow A \cap B \neq \emptyset) \tag{11.393}$$

of the implication '$\Rightarrow$', we let B be an arbitrary set and assume that $B \in \mathcal{K}_\Omega$ and $\omega \in B$ are both true. Since the basis $\mathcal{K}_\Omega$ is included in the corresponding generated topology $\mathcal{O}(\mathcal{K}_\Omega)$ according to Proposition 11.55, the previously made assumption $B \in \mathcal{K}_\Omega$ implies with the definition of a subset $B \in \mathcal{O}(\mathcal{K}_\Omega)$. In conjunction with the other assumption $\omega \in B$, this further implies $A \cap B \neq \emptyset$ with (11.392), which finding proves the implication in (11.393). Here, B is arbitrary, so that the desired universal sentence (11.393) follows to be true.

We prove the second part ('$\Leftarrow$') of the equivalence in (11.391) also directly, so that we assume now (11.393) to be true. We establish the desired consequent $\omega \in \mathrm{cl}(A)$ by means of the Characterization of the elements of a closure, by proving the universal sentence (11.392). Letting $U \in \mathcal{O}(\mathcal{K}_\Omega)$ be arbitrary such that $\omega \in U$ holds, it follows with the Generation of a topology by means of a basis that there exists a particular basis element $\bar{B}$ in $\mathcal{K}_\Omega$ which contains ω and which is included in U. Here, $\bar{B} \in \mathcal{K}_\Omega$ and $\omega \in \bar{B}$ imply with the assumed universal sentence (11.393) that $A \cap \bar{B} \neq \emptyset$ holds. Thus, there evidently exists an element in $A \cap \bar{B}$, say $\bar{\omega}$, and this element is then both in A and in $\bar{B}$ according to the definition of the intersection of two sets. For the same reason, the truth of $\bar{\omega} \in A$ and of the assumed $\omega \in U$ implies then $\bar{\omega} \in A \cap U$, which clearly shows that $A \cap U$ is nonempty. We thus completed the proof of the implication in (11.392), and since U is arbitrary, we may infer from the truth of this implication the truth of the universal sentence (11.392), and consequently the truth of the desired consequent $\omega \in \mathrm{cl}(A)$ (using the Characterization of the elements of a closure in connection with the initial assumption $\omega \in \Omega$).

Having thus completed the proof of the equivalence in (11.391), we may now conclude that the stated theorem is indeed true, because ω and also the sets Ω, $\mathcal{K}_\Omega$, A were arbitrary. □

Theorem 11.61 (Basis for a subspace topology). *The following sentences are true for any set Ω, any basis $\mathcal{K}_\Omega$ for a topology on Ω and any subset $\Omega_1 \subseteq \Omega$.*

a) There exists a unique set (system) $\mathcal{K}_{\Omega_1}$ containing precisely every set in $\mathcal{P}(\Omega_1)$ which is the intersection of Ω_1 and some basis element B in $\mathcal{K}_\Omega$.

b) This set $\mathcal{K}_{\Omega_1}$ satisfies then

$$\forall X\,(X \in \mathcal{K}_{\Omega_1} \Leftrightarrow \exists B\,(B \in \mathcal{K}_\Omega \wedge \Omega_1 \cap B = X)). \tag{11.394}$$

c) Moreover, $\mathcal{K}_{\Omega_1}$ is a basis for a topology on Ω_1 that generates the subspace topology of $\mathcal{O}(\mathcal{K}_\Omega)$ in Ω_1, i.e.

$$\mathcal{O}(\mathcal{K}_{\Omega_1}) = \mathcal{O}(\mathcal{K}_\Omega)|\Omega_1. \tag{11.395}$$

Proof. We let Ω, $\mathcal{K}_\Omega$ and Ω_1 be arbitrary sets assuming that $\mathcal{K}_\Omega$ is a basis for a topology on Ω, generating thus the topology $\mathcal{O}(\mathcal{K}_\Omega)$, and assuming furthermore that $\Omega_1 \subseteq \Omega$ holds.

We may then concerning a) evidently apply the Axiom of Specification and the Equality Criterion for sets to prove the unique existence of a set $\mathcal{K}_{\Omega_1}$ satisfying

$$\forall X\,(X \in \mathcal{K}_{\Omega_1} \Leftrightarrow [X \in \mathcal{P}(\Omega_1) \wedge \exists B\,(B \in \mathcal{K}_\Omega \wedge \Omega_1 \cap B = X)]). \tag{11.396}$$

Concerning b), we take an arbitrary set X and assume first $X \in \mathcal{K}_{\Omega_1}$. Then, (11.396) gives in particular the existential sentence

$$\exists B\,(B \in \mathcal{K}_\Omega \,\wedge\, \Omega_1 \cap B = X), \tag{11.397}$$

which is the desired consequent of the first part ('$\Rightarrow$') of the equivalence in (11.394) to be proven. To establish the second part ('$\Leftarrow$'), we now assume the preceding existential sentence to be true, so that there is a particular basis element $\bar{B} \in \mathcal{K}_\Omega$ with $\Omega_1 \cap \bar{B} = X$. We observe now that $\Omega_1 \cap \bar{B} \subseteq \Omega_1$ is true according to (2.74), so that the resulting inclusion $X \subseteq \Omega_1$ yields $X \in \mathcal{P}(\Omega_1)$ with the definition of a power set. The conjunction of this finding and the assumed existential sentence (11.397) implies then with (11.396) $X \in \mathcal{K}_{\Omega_1}$, which is the desired consequent of the second part of the equivalence in (11.394), which is thus true. As X was arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (11.394).

To prove c), we first prove that the set $\mathcal{K}_{\Omega_1}$ is included in the subspace topology $\mathcal{O}(\mathcal{K}_\Omega)|\Omega_1$. For this purpose, we use the definition of a subset and establish accordingly the universal sentence

$$\forall X\,(X \in \mathcal{K}_{\Omega_1} \Rightarrow X \in \mathcal{O}(\mathcal{K}_\Omega)|\Omega_1), \tag{11.398}$$

letting $\bar{X}$ be arbitrary and assuming $\bar{X} \in \mathcal{K}_{\Omega_1}$ to hold. Consequently, (11.394) gives us a particular set $\bar{B} \in \mathcal{K}_\Omega$ with $\Omega_1 \cap \bar{B} = \bar{X}$. Since the topology $\mathcal{O}(\mathcal{K}_\Omega)$ includes its basis $\mathcal{K}_\Omega$ according to Proposition 11.55, we therefore obtain $\bar{B} \in \mathcal{O}(\mathcal{K}_\Omega)$ with the definition of a subset. In conjunction with the preceding, this demonstrates the truth of the existential sentence

$$\exists U\,(U \in \mathcal{O}(\mathcal{K}_\Omega) \,\wedge\, \Omega_1 \cap U = \bar{X}),$$

so that the definition of a subspace topology yields $\bar{X} \in \mathcal{O}(\mathcal{K}_\Omega)|\Omega_1$, proving the implication in (11.398). Here, $\bar{X}$ is arbitrary, so that the universal sentence (11.398) follows to be true, and this further implies $\mathcal{K}_{\Omega_1} \subseteq \mathcal{O}(\mathcal{K}_\Omega)|\Omega_1$ with the definition of a subset.

We are now in a position to apply the Characterization of the basis generating a given topology, and we verify accordingly

$$\forall B,\omega\,([B \in \mathcal{O}(\mathcal{K}_\Omega)|\Omega_1 \,\wedge\, \omega \in B] \Rightarrow \exists A\,(A \in \mathcal{K}_{\Omega_1} \,\wedge\, \omega \in A \,\wedge\, A \subseteq B)). \tag{11.399}$$

Letting $\bar{B}$ and $\bar{\omega}$ be arbitrary such that $\bar{B} \in \mathcal{O}(\mathcal{K}_\Omega)|\Omega_1$ and $\bar{\omega} \in \bar{B}$ are both true, it follows with the definition of a subspace topology that there exists a set, say $\bar{U}$, such that $\bar{U} \in \mathcal{O}(\mathcal{K}_\Omega)$ and $\Omega_1 \cap \bar{U} = \bar{B}$ hold. Because of this equation, $\bar{\omega} \in \bar{B}$ gives us $\bar{\omega} \in \Omega_1 \cap \bar{U}$ by means of substitution, and subsequently $\bar{\omega} \in \Omega_1$ as well as $\bar{\omega} \in \bar{U}$ by definition of the intersection of two sets. Here, $\bar{U} \in \mathcal{O}(\mathcal{K}_\Omega)$ and $\bar{\omega} \in \bar{U}$ imply the existence of a particular set $\bar{A} \in \mathcal{K}_\Omega$ with $\bar{\omega} \in \bar{A}$ and $\bar{A} \subseteq \bar{U}$, according to the Generation of a topology by means of a basis. Then, $\bar{A} \in \mathcal{K}_\Omega$ and the true equation $\Omega_1 \cap \bar{A} = \Omega_1 \cap \bar{A}$ show that there exists a set B satisfying both $B \in \mathcal{K}_\Omega$ and $\Omega_1 \cap B = \Omega_1 \cap \bar{A}$, so that

$$\Omega_1 \cap \bar{A} \in \mathcal{K}_{\Omega_1} \tag{11.400}$$

follows to be true with (11.394). Furthermore, the truth of the previously established $\bar{\omega} \in \Omega_1$ and $\bar{\omega} \in \bar{A}$ implies with the definition of the intersection of two sets

$$\bar{\omega} \in \Omega_1 \cap \bar{A}. \tag{11.401}$$

Moreover, we may apply (2.76) to the previously established $\bar{A} \subseteq \bar{U}$ to obtain

$$\Omega_1 \cap \bar{A} \subseteq \Omega_1 \cap \bar{U} \quad [= B]. \tag{11.402}$$

Having thus found the set $\Omega_1 \cap \bar{A}$ satisfying (11.400) – (11.402), we see now that the existential sentence in (11.399) is true. Because $\bar{B}$ and $\bar{\omega}$ are arbitrary, we may therefore conclude that the universal sentence (11.399) holds, so that the set system $\mathcal{K}_{\Omega_1}$ follows to be a basis for a topology on Ω_1, and the topology $\mathcal{O}(\mathcal{K}_\Omega)|\Omega_1$ is generated by that basis $\mathcal{K}_{\Omega_1}$. We thus proved the equation (11.395), and since Ω, $\mathcal{K}_\Omega$ and Ω_1 were initially arbitrary sets, we may now finally conclude that the stated theorem is indeed true. □

Definition 11.22 (Second-countable topological space, Second Countability Axiom). We say that a topological space $(\Omega, \mathcal{O})$ is *second-countable* (alternatively, that a topological space $(\Omega, \mathcal{O})$ *satisfies the Second Countability Axiom*) iff there exists a basis for a topology on Ω which generates $\mathcal{O}$ and which is a countable set.

We now establish the useful fact that any basis generating a given second-countable topology includes a countable basis generating the same topology.

Lemma 11.62. *It is true for any second-countable topological space* $(\Omega, \mathcal{O})$ *and for any basis* $\mathcal{K}'_\Omega$ *(for a topology on* Ω*) generating* $\mathcal{O}$ *that there exists a subset* $\mathcal{K}^{(c)}_\Omega$ *of* $\mathcal{K}'_\Omega$ *such that* $\mathcal{K}^{(c)}_\Omega$ *is a countable basis (for a topology on* Ω*) generating* $\mathcal{O}$*.*

Proof. We take arbitrary sets Ω and $\mathcal{O}$, assume that $(\Omega, \mathcal{O})$ is a topological space, and assume also that there exists a set, say $\overline{\mathcal{K}}_\Omega$, such that $\overline{\mathcal{K}}_\Omega$ is a countable set and at the same time a basis for a topology on Ω generating $\mathcal{O}$. We then take an arbitrary set $\mathcal{K}'_\Omega$ and assume furthermore that this set is a basis for a topology on Ω generating $\mathcal{O}$ as well. Since the set $\overline{\mathcal{K}}_\Omega$ is nonempty according to Corollary 11.53 and moreover countable by assumption, there exists in view of the Countability Criterion (4.653) a surjection from $\mathbb{N}$ to $\overline{\mathcal{K}}_\Omega$, say $\bar{f}$. We may therefore write this function as the sequence of sets $\bar{f} = (\bar{A}_n)_{n\in\mathbb{N}}$, whose terms are precisely the distinct basis elements in $\overline{\mathcal{K}}_\Omega$. Next, we show for any $m, n \in \mathbb{N}$ that there is a unique set $\mathcal{K}_{(m,n)}$ consisting of all the basis elements A in $\mathcal{K}'_\Omega$ such that $\bar{A}_m \subseteq A \subseteq \bar{A}_n$ holds. Indeed, letting m and n be arbitrary natural numbers, we may evidently apply the Axiom of Specification and the Equality Criterion for sets to establish the unique existence of a set $\mathcal{K}_{(m,n)}$ such that

$$\forall A\,(A \in \mathcal{K}_{(m,n)} \Leftrightarrow [A \in \mathcal{K}'_\Omega \,\wedge\, A_m \subseteq A \,\wedge\, A \subseteq A_n]). \tag{11.403}$$

Here, we notice that $\mathcal{K}_{(m,n)}$ is empty in case there is no basis element $A \in \mathcal{K}'_\Omega$ satisfying $A_m \subseteq A \subseteq A_n$. Let us observe here that $A \in \mathcal{K}_{(m,n)}$ implies $A \in \mathcal{K}'_\Omega$ for any A, so that the definition of a subset yields the inclusion $\mathcal{K}_{(m,n)} \subseteq \mathcal{K}'_\Omega$. Consequently, $\mathcal{K}_{(m,n)} \in \mathcal{P}(\mathcal{K}'_\Omega)$ is true by definition of a power set. Since m and n are arbitrary, we may therefore conclude that the set $\mathcal{K}_{(m,n)}$ is a uniquely specified element of $\mathcal{P}(\mathcal{K}'_\Omega)$ for any $m, n \in \mathbb{N}$.

In a next step, we apply again the Axiom of Specification in connection with the Equality Criterion for sets to uniquely specify the set $\mathcal{X}$ consisting of all the nonempty sets $\mathcal{K}_{(m,n)}$ in $\mathcal{P}(\mathcal{K}'_\Omega)$ with $m, n \in \mathbb{N}$, i.e. we can show that there exists a unique set $\mathcal{X}$ such that

$$\begin{aligned}&\forall K\,(K \in \mathcal{X} \\ &\quad \Leftrightarrow [K \in \mathcal{P}(\mathcal{K}'_\Omega) \,\wedge\, \exists m, n\,(m, n \in \mathbb{N} \,\wedge\, \mathcal{K}_{(m,n)} \neq \emptyset \,\wedge\, \mathcal{K}_{(m,n)} = K)]).\end{aligned} \tag{11.404}$$

Let us check that the set system $\mathcal{X}$ does indeed not contain the empty set, by verifying

$$\forall K\,(K \in \mathcal{X} \Rightarrow K \neq \emptyset).$$

Letting K be an arbitrary set and assuming $K \in \mathcal{X}$ to be true, it follows with (11.404) in particular that there exist constants, say $\bar{m}$ and $\bar{n}$, satisfying $\bar{m}, \bar{n} \in \mathbb{N}$, $\mathcal{K}_{(\bar{m},\bar{n})} \neq \emptyset$, and $\mathcal{K}_{(\bar{m},\bar{n})} = K$. Consequently, substitution yields the desired consequent $K \neq \emptyset$, and as K was arbitrary, we may therefore conclude that the preceding universal sentence to be proven holds indeed. We therefore obtain the suggested negation $\emptyset \notin \mathcal{X}$ with (2.5).

After this preparation, we may now apply the Axiom of Choice to establish the existence of a particular function $\bar{g} : \mathcal{X} \to \bigcup \mathcal{X}$ which picks out a specific basis element A from each of the nonempty sets $\mathcal{K}_{(m,n)}$ in $\mathcal{X}$, i.e. which has the property

$$\forall K\,(K \in \mathcal{X} \Rightarrow \bar{g}(K) \in K)). \tag{11.405}$$

We now prove by means of the Characterization of the basis generating a given topology that $\mathcal{K} = \operatorname{ran}(\bar{g})$ is a basis for a topology on Ω generating

$\mathcal{O}$. For this purpose, we let B and ω be arbitrary, assume that $B \in \mathcal{O}$ and $\omega \in B$ are both true, and show that there is a set A in $\mathcal{K}$ which contains ω and which is included in B.

Since $\overline{\mathcal{K}}_\Omega$ generates $\mathcal{O}$, it follows from $B \in \mathcal{O}$ and $\omega \in B$ – according to the Generation of a topology by means of a basis – that there exists a particular set $\bar{X} \in \overline{\mathcal{K}}_\Omega$ containing ω with $\bar{X} \subseteq B$. The previously established surjection $\bar{f} : \mathbb{N} \twoheadrightarrow \overline{\mathcal{K}}_\Omega$ has the range $\mathrm{ran}(\bar{f}) = \overline{\mathcal{K}}_\Omega$, so that substitution gives $\bar{X} \in \mathrm{ran}(\bar{f})$. By definition of a range, there is then a constant, say $\bar{N}$, such that $(\bar{N}, \bar{X}) \in \bar{f}$ holds. We may write the latter also in function/sequence notation as $\bar{X} = \bar{f}(\bar{N}) = \bar{A}_{\bar{M}}$. Thus, $\bar{A}_{\bar{N}}$ is a set containing ω (i.e., $\omega \in \bar{A}_{\bar{N}}$) with $\bar{A}_{\bar{N}} \subseteq B$. Since $\mathcal{O}$ includes the basis $\overline{\mathcal{K}}_\Omega$ it is generated by (see Proposition 11.55), we have that $\bar{X} \in \overline{\mathcal{K}}_\Omega$ implies by definition of a subset $\bar{X} \in \mathcal{O}$, thus $\bar{A}_{\bar{N}} \in \mathcal{O}$ is true. Because $\mathcal{K}'_\Omega$ also generates $\mathcal{O}$, it follows from $\bar{A}_{\bar{N}} \in \mathcal{O}$ and $\omega \in \bar{A}_{\bar{N}}$ – again according to the Generation of a topology by means of a basis – that there is a particular set $\bar{Y}$ in $\mathcal{K}'_\Omega$ containing ω such that $\bar{Y} \subseteq \bar{A}_{\bar{N}}$. Then, as $\mathcal{O}$ includes its generating system $\mathcal{K}'_\Omega$, it follows from $\bar{Y} \in \mathcal{K}'_\Omega$ that $\bar{Y} \in \mathcal{O}$ is true. Together with $\omega \in \bar{Y}$, the latter implies – once again according to the Generation of a topology by means of a basis (now again with respect to $\overline{\mathcal{K}}_\Omega$) – the existence of a particular set $\bar{Z} \in \overline{\mathcal{K}}_\Omega$ [$= \mathrm{ran}(\bar{f})$] containing ω and with $\bar{Z} \subseteq \bar{Y}$. Here, $\bar{Z} \in \mathrm{ran}(\bar{f})$ implies (by definition of a range) the existence of a particular constant $\bar{M}$ satisfying $(\bar{M}, \bar{Z}) \in \bar{f}$, which we may write also as $\bar{Z} = \bar{f}(\bar{M}) = \bar{A}_{\bar{M}}$. Therefore, we obtain after substitutions $\omega \in \bar{A}_{\bar{M}}$ as well as $\bar{A}_{\bar{M}} \subseteq \bar{Y}$. Recalling the truth of the inclusion $\bar{Y} \subseteq \bar{A}_{\bar{N}}$, we thus found a particular constant $\bar{Y}$ that satisfies $\bar{Y} \in \mathcal{K}'_\Omega$, $\bar{A}_{\bar{M}} \subseteq \bar{Y}$ and $\bar{Y} \subseteq \bar{A}_{\bar{N}}$, where $\bar{M}$ and $\bar{N}$ are natural numbers. Consequently, $\bar{Y} \in \mathcal{K}_{(\bar{M},\bar{N})}$ follows to be true with (11.403), and this finding clearly shows that the set $\bar{K} = \mathcal{K}_{(\bar{M},\bar{N})}$ is nonempty. We thus established the existence of constants m and n such that $m, n \in \mathbb{N}$, $\mathcal{K}_{(m,n)} \neq \emptyset$ and $\mathcal{K}_{(m,n)} = \bar{K}$ hold. Together with the previously mentioned fact that any such set is a specified element of $\mathcal{P}(\mathcal{K}'_\Omega)$, so that $\bar{K} \in \mathcal{P}(\mathcal{K}'_\Omega)$ is in particular true, the preceding existential sentence implies $\bar{K} \in \mathcal{X}$ with (11.404). We therefore obtain with (11.405) $\bar{g}(\bar{K}) \in \bar{K}$ [$= \mathcal{K}_{(\bar{M},\bar{N})}$]. The resulting $\bar{g}(\bar{K}) \in \mathcal{K}_{(\bar{M},\bar{N})}$ then implies with (11.403) the inclusions

$$[\omega \in] \quad A_{\bar{M}} \subseteq \bar{g}(\bar{K}) \wedge \bar{g}(\bar{K}) \subseteq A_{\bar{N}} \quad [\subseteq B],$$

where $\omega \in A_{\bar{M}}$ and $A_{\bar{N}} \subseteq B$ were established previously. Consequently, the definition of a subset yields $\omega \in \bar{g}(\bar{K})$, and an application of (2.13) yields $\bar{g}(\bar{K}) \subseteq B$. Furthermore, the value $\bar{g}(\bar{K})$ is clearly an element of the range of the function $\bar{g}$, that is, $\bar{g}(\bar{K}) \in \mathcal{K}$ [$= \mathrm{ran}(\bar{g})$]. These findings show that $\bar{g}(\bar{K})$ is a set in $\mathcal{K}$ included in B and containing ω. The existence of such a set implies now with the Characterization of the basis generating a given topology that $\mathcal{K}$ is a basis for a topology on Ω generating $\mathcal{O}$.

It remains for us to show that $\mathcal{K}$ is a countable set. For this purpose, we apply Function definition by replacement to prove the unique existence of a function h with domain $\mathbb{N} \times \mathbb{N}$ such that

$$\forall z\,(z \in \mathbb{N} \times \mathbb{N} \Rightarrow \exists m, n\,(h((m,n)) = \mathcal{K}_{(m,n)} \,\wedge\, (m,n) = z)). \tag{11.406}$$

To do this, we establish accordingly the truth of the universal sentence

$$\forall z\,(z \in \mathbb{N} \times \mathbb{N} \Rightarrow \exists! Y\,(\exists m, n\,(Y = \mathcal{K}_{(m,n)} \,\wedge\, (m,n) = z))), \tag{11.407}$$

letting $z \in \mathbb{N} \times \mathbb{N}$ be arbitrary. By definition of the Cartesian product of two sets, there are then particular numbers $\bar{m}, \bar{n} \in \mathbb{N}$ with $(\bar{m}, \bar{n}) = z$. As shown before, the set $\mathcal{K}_{(m,n)}$ is uniquely specified for any natural numbers m and n, so that we obtain the particular set $\bar{Y} = \mathcal{K}_{(\bar{m},\bar{n})}$. On the one hand, this demonstrates the existence of constants m, n satisfying $\bar{Y} = \mathcal{K}_{(m,n)}) \wedge (m,n) = z$. On the other hand, the fact that $\bar{Y}$ satisfies that existential sentence shows us that the existential part of the uniquely existential sentence in (11.407) holds. Evidently, we may carry out the variant of the Proof of a uniquely existential sentence (according to Method 1.18), letting Y' be an arbitrary set satisfying the existential sentence with respect to m and n. Thus, there are particular constants $\bar{m}', \bar{n}'$ such that $Y' = \mathcal{K}_{(\bar{m}',\bar{n}')}$ and $(\bar{m}', \bar{n}') = z$. Combining the two previous equations for z yields $(\bar{m}', \bar{n}') = (\bar{m}, \bar{n})$, and these equations give in turn $\bar{m}' = \bar{m}$ as well as $\bar{n}' = \bar{n}$ with the Equality Criterion for ordered pairs. We therefore obtain

$$Y' = \mathcal{K}_{(\bar{m}',\bar{n}')} = \mathcal{K}_{(\bar{m},\bar{n})} = \bar{Y},$$

and we may infer from the truth of the resulting equation $\bar{Y} = Y'$ that the uniquely existential sentence in (11.407) is true (since Y' was arbitrary). Because z was also arbitrary, we may now further conclude that the universal sentence (11.407) holds, and this sentence in turn implies the unique existence of a function h with domain $\mathbb{N} \times \mathbb{N}$ and property (11.406). Then, $h : \mathbb{N} \times \mathbb{N} \to \mathrm{ran}(h)$ is a surjection by definition. Due to the Equivalence of $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ in connection with the symmetry of the Equivalence relation of equinumerosity, there exists a bijection and thus a surjection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, say $\bar{c}$. Then, it follows with the Surjectivity of the composition of two surjections that $h \circ \bar{c}$ is a surjection from $\mathbb{N}$ to $\mathrm{ran}(h)$. Due to the Countability Criterion (4.653), the existence of such a surjection implies the countability of the set $\mathrm{ran}(h)$. Next, we verify $\mathcal{X} \subseteq \mathrm{ran}(h)$, that is,

$$\forall K\,(K \in \mathcal{X} \Rightarrow \mathrm{ran}(h)).$$

Letting K be arbitrary and assuming $K \in \mathcal{X}$ to be true, we see in light of (11.404) that there exist particular numbers $\bar{m}, \bar{n} \in \mathbb{N}$ such that $\mathcal{K}_{(\bar{m},\bar{n})} \neq \emptyset$ and $\mathcal{K}_{(\bar{m},\bar{n})} = K$) hold. By definition of the Cartesian product of two sets, we therefore have $(\bar{m}, \bar{n}) \in \mathbb{N} \times \mathbb{N}$, so that (11.406) yields two particular constants $\bar{m}', \bar{n}'$ satisfying $h((\bar{m}', \bar{n}')) = \mathcal{K}_{(\bar{m}',\bar{n}')}$ and $(\bar{m}', \bar{n}') = (\bar{m}, \bar{n})$. The latter equation gives with the Equality Criterion for ordered pairs $\bar{m}' = \bar{m}$ and $\bar{n}' = \bar{n}$, so that substitutions yield

$$K = \mathcal{K}_{(\bar{m},\bar{n})} = \mathcal{K}_{(\bar{m}',\bar{n}')} = h((\bar{m}', \bar{n}')) = h((\bar{m}, \bar{n})).$$

We may write the resulting equation $K = h((\bar{m}, \bar{n}))$ also as $((\bar{m}, \bar{n}), K) \in h$, which shows in light of the definition of a range that $K \in \operatorname{ran}(h)$ is true, as desired. Since K was arbitrary, we may therefore infer from this finding the truth of the inclusion $\mathcal{X} \subseteq \operatorname{ran}(h)$ (using the definition of a subset). It then follows from this that the subset $\mathcal{X}$ of the countable set $\operatorname{ran}(h)$ is itself countable, according to (4.655). Let us bring out more clearly now that the set $\mathcal{X}$ is nonempty. To begin with, the previously established basis $\mathcal{K} = \operatorname{ran}(\bar{g})$ for a topology (generating $\mathcal{O}$) is a nonempty set in view of Corollary 11.53. Therefore, there evidently exists a set in $\operatorname{ran}(\bar{g})$, say $\bar{K}$. Then, there exists (by definition of a range) also a set, say $\bar{X}$, such that $(\bar{X}, \bar{K}) \in \bar{g}$ holds. This in turn gives (by definition of a domain) $\bar{X} \in \mathcal{X}$ [$= \operatorname{dom}(\bar{g})$], which clearly shows that $\mathcal{X} \neq \emptyset$. Thus, $\mathcal{X}$ is a nonempty countable set, so that the previously mentioned Countability Criterion implies the existence of a particular surjection $\bar{s}$ from $\mathbb{N}$ to $\mathcal{X}$. Because we may write the function $\bar{g} : \mathcal{X} \to \bigcup \mathcal{X}$ also in the form of a surjection $\bar{g} : \mathcal{X} \to \operatorname{ran}(\bar{g})$, we may apply once again the Surjectivity of the composition of two surjections in order to obtain the new surjection $\bar{g} \circ \bar{s}$ from $\mathbb{N}$ to $\operatorname{ran}(\bar{g})$ [$= \mathcal{K}$]. The existence of such a surjection implies then that the basis $\mathcal{K}$ for a topology on Ω (generating $\mathcal{O}$) is countable. Having found this particular basis $\mathcal{K}$, we thus established the existence of a countable basis $\mathcal{K}_{\Omega}^{(c)}$ (for a topology on Ω) generating $\mathcal{O}$.

Since Ω and $\mathcal{O}$ were initially arbitrary sets, we may now finally conclude that the stated lemma is indeed true. □

Order topologies

We prepare now the definition of a topology consisting of certain kinds intervals.

Exercise 11.25. Establish for any partially ordered set $(\Omega, <_{\Omega})$ such that

a) the minimum of Ω exists the unique set $\{[\min \Omega, b) : b \in \Omega\}$ consisting of all the left-closed and right-open intervals from the minimum of Ω

to an element b in Ω, in the sense that

$$\forall Z\,(Z \in \{[\min\Omega, b) : b \in \Omega\} \Leftrightarrow [Z \in \{[a,b) : a, b \in \Omega\} \wedge \exists b\,(\,Z = [\min\Omega, b)\,)]). \tag{11.408}$$

b) the maximum of Ω exists the unique set $\{(a, \max\Omega] : a \in \Omega\}$ consisting of all the left-open and right-closed intervals from an element a in Ω to the maximum of Ω, in the sense that

$$\forall Z\,(Z \in \{(a, \max\Omega] : a \in \Omega\} \Leftrightarrow [Z \in \{(a,b] : a, b \in \Omega\} \wedge \exists a\,(\,Z = (a, \max\Omega]\,)]). \tag{11.409}$$

Lemma 11.63. *The following sentences are true for any linearly ordered set* $(\Omega, <_\Omega)$.

a) The intersection of any open interval in Ω and any left-closed and right-open interval beginning in the minimum of Ω is itself an open interval in Ω, i.e.

$$\forall A, B\,([A \in \{(a,b) : a, b \in \Omega\} \wedge B \in \{[\min\Omega, b) : b \in \Omega\}] \Rightarrow A \cap B \in \{(a,b) : a, b \in \Omega\}). \tag{11.410}$$

b) The intersection of any open interval in Ω and any left-open and right-closed interval ending in the maximum of Ω is itself an open interval in Ω, i.e.

$$\forall A, B\,([A \in \{(a,b) : a, b \in \Omega\} \wedge B \in \{(a, \max\Omega] : a \in \Omega\}] \Rightarrow A \cap B \in \{(a,b) : a, b \in \Omega\}). \tag{11.411}$$

c) The intersection of two left-closed and right-open intervals beginning in the minimum of Ω is itself such an interval, i.e.

$$\forall A, B\,(A, B \in \{[\min\Omega, b) : b \in \Omega\} \Rightarrow A \cap B \in \{[\min\Omega, b) : b \in \Omega\}). \tag{11.412}$$

d) The intersection of two left-open and right-closed intervals ending in the maximum of Ω is itself such an interval, i.e.

$$\forall A, B\,(A, B \in \{(a, \max\Omega] : a \in \Omega\} \Rightarrow A \cap B \in \{(a, \max\Omega] : a \in \Omega\}). \tag{11.413}$$

e) *The intersection of a left-closed and right-open intervals beginning in the minimum of* Ω *and a left-open and right-closed intervals ending in the maximum of* Ω *is an open interval, i.e.*

$$\forall A, B\,([A \in \{[\min\Omega, b) : b \in \Omega\} \,\wedge\, B \in \{(a, \max\Omega] : a \in \Omega\}] \\ \Rightarrow A \cap B \in \{(a, b) : a, b \in \Omega\}). \tag{11.414}$$

Proof. We let Ω and $<_\Omega$ be arbitrary sets, and we assume $(\Omega, <_\Omega)$ to be linearly ordered. Concerning a), we take arbitrary sets A and B such that $A \in \{(a, b) : a, b \in \Omega\}$ and $B \in \{[\min\Omega, b) : b \in \Omega\}$ are true. By definition of the set of open intervals in Ω, there are then particular elements $\bar{a}$ and $\bar{b}$ in Ω with $(\bar{a}, \bar{b}) = A$. Furthermore, according to the specification of the set $\{[\min\Omega, b) : b \in \Omega\}$ in (11.408), there is a particular element $\bar{c} \in \Omega$ with $B = [\min\Omega, \bar{c})$. We now prove the desired consequent by cases, based on the true disjunction $\bar{b} \leq_\Omega \bar{c} \,\vee\, \bar{c} \leq_\Omega \bar{b}$ (which holds due to the totality of the partial ordering $\leq_\Omega$ induced by the linear ordering $<_\Omega$).

In the first case $\bar{b} \leq_\Omega \bar{c}$, we may apply the Equality Criterion for sets to establish the equation

$$(\bar{a}, \bar{b}) \cap [\min\Omega, \bar{c}) = (\bar{a}, \bar{b}), \tag{11.415}$$

by verifying the universal sentence

$$\forall\omega\,(\,\omega \in (\bar{a}, \bar{b}) \cap [\min\Omega, \bar{c}) \Leftrightarrow \omega \in (\bar{a}, \bar{b})\,). \tag{11.416}$$

Letting ω be arbitrary and assuming first $\omega \in (\bar{a}, \bar{b}) \cap [\min\Omega, \bar{c})$ to be true, we obtain with the definition of the intersection of two sets in particular the desired consequent $\omega \in (\bar{a}, \bar{b})$ of the first part ('$\Rightarrow$') of the equivalence to be proven. Assuming now conversely $\omega \in (\bar{a}, \bar{b})$ to be true, we obtain by definition of an open interval $\omega \in \Omega$ as well as the inequalities $\bar{a} <_\Omega \omega$ and $\omega <_\Omega \bar{b}$. On the one hand, the latter inequality implies together with the current case assumption $\bar{b} \leq_\Omega \bar{c}$ by means of the Transitivity Formula for $<$ and $\leq$ that $\omega <_\Omega \bar{c}$ is true. On the other hand, $\omega \in \Omega$ implies $\min\Omega \leq_\Omega \omega$ since the minimum $\min\Omega$ is a lower bound for Ω. Thus, the conjunctions

$$(\bar{a} <_\Omega \omega \,\wedge\, \omega <_\Omega \bar{b}) \,\wedge\, (\min\Omega \leq_\Omega \omega \,\wedge\, \omega <_\Omega \bar{c}) \tag{11.417}$$

evidently hold, so that

$$\omega \in (\bar{a}, \bar{b}) \,\wedge\, \omega \in [\min\Omega, \bar{c})$$

follows to be true with the definition of an open interval and with the definition of a left-closed and right-open interval, which conjunction now gives the desired consequent of the second part ('$\Leftarrow$') of the equivalence in (11.416). As ω was arbitrary, we may therefore conclude that (11.416) holds, which universal sentence in turn gives the equation (11.415). This equation shows that the intersection of $A = (\bar{a}, \bar{b})$ and $B = [\min \Omega, \bar{c})$ is an open interval in Ω, so that $A \cap B \in \{(a,b) : a, b \in \Omega\}$ follows to be true.

In the second case $\bar{c} \leq_\Omega \bar{b}$, we proceed similarly to the first part, proving now the equation

$$(\bar{a}, \bar{b}) \cap [\min \Omega, \bar{c}) = (\bar{a}, \bar{c}) \tag{11.418}$$

via the Equality Criterion for sets, i.e. via a proof of the universal sentence

$$\forall \omega \, (\, \omega \in (\bar{a}, \bar{b}) \cap [\min \Omega, \bar{c}) \Leftrightarrow \omega \in (\bar{a}, \bar{c}) \,). \tag{11.419}$$

We let ω be arbitrary and observe that the assumption $\omega \in (\bar{a}, \bar{b}) \cap [\min \Omega, \bar{c})$ implies the truth of the conjunctions (11.417), with the consequence that $\bar{a} <_\Omega \omega$ and $\omega <_\Omega \bar{c}$ are in particular true. We thus obtain $\omega \in (\bar{a}, \bar{c})$, as desired. Regarding the second part ('$\Leftarrow$') of the equivalence in (11.419), we now assume that $\omega \in (\bar{a}, \bar{c})$ holds, so that $\omega \in \Omega$, $\bar{a} <_\Omega \omega$ and $\omega <_\Omega \bar{c}$ are all true. The conjunction of the latter inequality and the current case assumption $\bar{c} \leq_\Omega \bar{b}$ evidently yields $\omega <_\Omega \bar{b}$, and we clearly also have $\min \Omega \leq_\Omega \omega$ (as in the first case). Thus, we may form again the conjunctions in (11.417), which then give the desired consequent of the second part of the equivalence in (11.419). Because ω is arbitrary, we may infer from the truth of that equivalence the truth of the equation (11.418). As in the first case, we now see in light of this equation that the intersection of $A = (\bar{a}, \bar{b})$ and $B = [\min \Omega, \bar{c})$ is an open interval in Ω, so that $A \cap B \in \{(a,b) : a, b \in \Omega\}$ holds again.

Thus, the proof by cases in complete, and since A and B were arbitrary, we may therefore conclude that Part a) of the lemma is true. The parts b) – d) can be proved analogously.

Concerning e), we let A and B be arbitrary and assume $A \in \{[\min \Omega, b) : b \in \Omega\}$ as well as $B \in \{(a, \max \Omega] : a \in \Omega\}$ are true. By definition of the sets $\{[\min \Omega, b) : b \in \Omega\}$ and $\{(a, \max \Omega] : a \in \Omega\}$, there are then particular elements $\bar{b}$ and $\bar{a}$ in Ω with $A = [\min \Omega, \bar{b})$ and $B = (\bar{a}, \max \Omega]$. Let us observe here that $\min \Omega$ is a lower bound and that $\max \Omega$ is an upper bound for Ω, so that $\bar{a} \in \Omega$ implies $\min \Omega \leq_\Omega \bar{a}$ and $\bar{b} \in \Omega$ implies $\bar{b} \leq_\Omega \max \Omega$. The conjunction of these two inequalities implies with (3.415) the equation

$$[A \cap B =] \quad [\min \Omega, \bar{b}) \cap (\bar{a}, \max \Omega] = (\bar{a}, \bar{b})$$

This shows that there exist elements a and b in Ω such that $(a,b) = A \cap B$ holds, so that this conjunction turns out to be an element in the set of open intervals, by definition of that set system. Here, the sets A and B are arbitrary, so that the proposed universal sentence (11.414) follows now to be true.

Finally, as the sets Ω and $<_\Omega$ were initially arbitrary, we may infer from the truth of a) – e) the truth of the stated lemma. $\square$

Exercise 11.26. Prove Lemma 11.63b) – d).

Theorem 11.64 (Basis for a topology determined by a linear ordering). *The following sentences are true for any linearly ordered set* $(\Omega, <_\Omega)$ *such that* Ω *is neither empty nor a singleton.*

a) If the minimum and the maximum of Ω *(with respect to* $<_\Omega$*) do not exist, then the set*

$$\{(a,b) : a,b \in \Omega\} \tag{11.420}$$

of open intervals in Ω *constitutes a basis for a topology on* Ω*.*

b) If the minimum of Ω *does exist and the maximum of* Ω *does not exist, then the union*

$$\{(a,b) : a,b \in \Omega\} \cup \{[\min \Omega, b) : b \in \Omega\} \tag{11.421}$$

of all open intervals and of all left-closed and right-open intervals beginning in the minimum of Ω *(in* Ω*) is a basis for a topology on* Ω*.*

c) If the minimum of Ω *does not exist and the maximum of* Ω *does exist, then the union*

$$\{(a,b) : a,b \in \Omega\} \cup \{(a, \max \Omega] : a \in \Omega\} \tag{11.422}$$

of all open intervals and of all left-open and right-closed intervals ending in the maximum of Ω *(in* Ω*) is a basis for a topology on* Ω*.*

d) If the minimum and the maximum of Ω *do both exist, then the union*

$$\{(a,b) : a,b \in \Omega\} \cup \{[\min \Omega, b) : b \in \Omega\} \cup \{(a, \max \Omega] : a \in \Omega\} \tag{11.423}$$

of all open intervals, of all left-closed and right-open intervals beginning in the minimum of Ω *and of all left-open and right-closed intervals ending in the maximum of* Ω *is a basis for a topology on* Ω*.*

Proof. We let Ω and $<_\Omega$ be arbitrary sets, assuming $\Omega \neq \emptyset$ as well as $\forall a\, (\Omega \neq \{a\})$ to be true, and assuming moreover $<_\Omega$ to be a linear ordering of Ω.

Concerning a), we assume also that the two negations

$$\neg\exists m\,(m \in \Omega \wedge \forall\omega\,(\omega \in \Omega \Rightarrow m \leq_\Omega \omega)) \tag{11.424}$$

$$\neg\exists m\,(m \in \Omega \wedge \forall\omega\,(\omega \in \Omega \Rightarrow \omega \leq_\Omega m)) \tag{11.425}$$

are true. Since $\{(a,b) : a,b \in \Omega\}$ is the π-system of open intervals in Ω, this set system naturally satisfies Property 1 of a basis for a topology on Ω. To establish Property 2, we take an arbitrary $\bar{\omega} \in \Omega$ and show that there is some open interval (a,b) containing $\bar{\omega}$. For this purpose, we apply the Negation Law for existential conjunctions and write the two assumed negations equivalently as

$$\forall m\,(m \in \Omega \Rightarrow \neg\forall\omega\,(\omega \in \Omega \Rightarrow m \leq_\Omega \omega)), \tag{11.426}$$

$$\forall m\,(m \in \Omega \Rightarrow \neg\forall\omega\,(\omega \in \Omega \Rightarrow \omega \leq_\Omega m)). \tag{11.427}$$

Therefore, $\bar{\omega} \in \Omega$ implies the truth of the two negations

$$\neg\forall\omega\,(\omega \in \Omega \Rightarrow \bar{\omega} \leq_\Omega \omega), \tag{11.428}$$

$$\neg\forall\omega\,(\omega \in \Omega \Rightarrow \omega \leq_\Omega \bar{\omega}), \tag{11.429}$$

which we may write also as (applying now the Negation Formula for universal implications)

$$\exists\omega\,(\omega \in \Omega \wedge \neg\bar{\omega} \leq_\Omega \omega), \tag{11.430}$$

$$\exists\omega\,(\omega \in \Omega \wedge \neg\omega \leq_\Omega \bar{\omega}). \tag{11.431}$$

Thus, there exists on the one hand a particular constant, say $\bar{a}$, such that $\bar{a} \in \Omega$ and $\neg\bar{\omega} \leq_\Omega \bar{a}$ hold. Here, the latter negation implies $\bar{a} <_\Omega \bar{\omega}$ with the Negation Formula for $\leq$. On the other hand, there is a particular constant $\bar{b} \in \Omega$ with $\neg\bar{b} \leq_\Omega \bar{\omega}$, so that this constant evidently satisfies $\bar{\omega} <_\Omega \bar{b}$. We thus obtained the true inequalities $\bar{a} <_\Omega \bar{\omega} <_\Omega \bar{b}$, which imply because of $\bar{a}, \bar{b} \in \Omega$ by definition of an open interval $\bar{\omega} \in (\bar{a}, \bar{b})$. This finding demonstrates the existence of an element in $\{(a,b) : a,b \in \Omega\}$ containing $\bar{\omega}$, and since $\bar{\omega}$ is arbitrary, we may therefore conclude that Property 2 of a basis for a topology on Ω is indeed satisfied by $\{(a,b) : a,b \in \Omega\}$.

Regarding Property 3, we now let A_1, A_2 and ω be arbitrary, and we assume $A_1, A_2 \in \{(a,b) : a,b \in \Omega\}$, $\omega \in A_1$ as well as $\omega \in A_2$ to be true. Because the π-system $\{(a,b) : a,b \in \Omega\}$ is closed under pairwise intersections, the former assumption implies $A_1 \cap A_2 \in \{(a,b) : a,b \in \Omega\}$. Moreover, the other two assumptions imply $\omega \in A_1 \cap A_2$ by definition of the intersection of two sets. Observing now that $A_1 \cap A_2 \subseteq A_1 \cap A_2$ is also true according to (2.10), we found out that there exists a set A_3 that satisfies the conjunction of $A_3 \in \{(a,b) : a,b \in \Omega\}$, $\omega \in A_3$ and $A_3 \subseteq A_1 \cap A_2$. Since

A_1, A_2 and ω are arbitrary, we may therefore infer from these findings that $\{(a,b) : a, b \in \Omega\}$ satisfies also Property 3 of a basis for a topology on Ω.

Concerning b), we now assume on the one hand that the minimum $\min \Omega$ exists and on the other hand that the maximum of Ω does not exist, i.e.

$$\neg \exists m \, (m \in \Omega \wedge \forall \omega \, (\omega \in \Omega \Rightarrow \omega \leq_\Omega m)). \tag{11.432}$$

We see in light of (11.408) $Z \in \{[\min \Omega, b) : b \in \Omega\}$ implies $\{[a,b) : a, b \in \Omega\}$ for any Z, so that the inclusions

$$\{[\min \Omega, b) : b \in \Omega\} \subseteq \{[a,b) : a, b \in \Omega\} \quad [\subseteq \mathcal{P}(\Omega)]$$

hold, recalling also (3.390); we thus obtain $\{[\min \Omega, b) : b \in \Omega\} \subseteq \mathcal{P}(\Omega)$ with (2.13). The conjunction of this inclusion and the already established inclusion $\{(a,b) : a, b \in \Omega\} \subseteq \mathcal{P}(\Omega)$ implies now that the union (11.421) is also included in $\mathcal{P}(\Omega)$, according to (2.252). This finding shows us that this union satisfies Property 1 of a basis for a topology.

To verify Property 2, we let $\bar{\omega} \in \Omega$ be arbitrary and notice that the assumed existence of the minimum $\min \Omega$ means that $\min \Omega$ is a lower bound for Ω contained in Ω. Thus, we have the true universal sentence

$$\forall \omega \, (\omega \in \Omega \Rightarrow \min \Omega \leq_\Omega \omega) \tag{11.433}$$

and in addition $\min \Omega \in \Omega$. Thus, $\bar{\omega} \in \Omega$ yields $\min \Omega \leq_\Omega \bar{\omega}$. Furthermore, we observe in light of the proof of a) that the assumption (11.432) implies the existence of a particular constant $\bar{b} \in \Omega$ such that $\bar{\omega} <_\Omega \bar{b}$ turns out to be true. Combining the previous two findings to $\min \Omega \leq_\Omega \bar{\omega} <_\Omega \bar{b}$ and recalling the truth of $\min \Omega, \bar{b} \in \Omega$, we now obtain $\bar{\omega} \in [\min \Omega, \bar{b})$ with the definition of a left-closed and right-open interval (beginning in $\min \Omega$). Consequently, we find that $\bar{\omega} \in \{[\min \Omega, b) : b \in \Omega\}$ holds, and the disjunction of $\bar{\omega} \in \{(a,b) : a, b \in \Omega\}$ and the preceding finding is then also true, so that $\bar{\omega}$ turns out to be an element of the union (11.421) by definition of the union of two sets. This shows that $\bar{\omega}$ is in some element of (11.421), and as $\bar{\omega}$ was arbitrary, we may infer from this that the set (11.421) satisfies indeed Property 2 of a basis for a topology on Ω.

Next, we establish Property 3 by letting A_1, A_2 and ω be arbitrary such that A_1 and A_2 are elements of the union (11.421) and such that ω is contained both in A_1 and A_2. Thus, $\omega \in A_1 \cap A_2$ and $A_1 \cap A_2 \subseteq A_1 \cap A_2$ follow to be true with the definition of the intersection of two sets and with (2.13). We now prove that $A_1 \cap A_2$ is an element of the union (11.421), which we do by considering two cases and two subcases based on the two disjunctions

$$A_1 \in \{(a,b) : a, b \in \Omega\} \vee A_1 \in \{[\min \Omega, b) : b \in \Omega\},$$
$$A_2 \in \{(a,b) : a, b \in \Omega\} \vee A_2 \in \{[\min \Omega, b) : b \in \Omega\},$$

which follow to be true from the previous assumption by definition of the union of two sets. In the first case $A_1 \in \{(a,b) : a,b \in \Omega\}$ and the first subcase $A_2 \in \{(a,b) : a,b \in \Omega\}$, we obtain $A_1 \cap A_2 \in \{(a,b) : a,b \in \Omega\}$, as mentioned already in a). Then, the disjunction

$$A_1 \cap A_2 \in \{(a,b) : a,b \in \Omega\} \vee A_1 \cap A_2 \in \{[\min \Omega, b) : b \in \Omega\} \quad (11.434)$$

is also true, so that $A_1 \cap A_2$ follows to be in the union (11.421), as desired. In the second subcase $A_2 \in \{[\min \Omega, b) : b \in \Omega\}$, we obtain for the intersection $A_1 \cap A_2 \in \{(a,b) : a,b \in \Omega\}$ now because of (11.410), so that the disjunction (11.434) holds as in the first subcase. Similarly, the second case $A_1 \in \{[\min \Omega, b) : b \in \Omega\}$ and the first subcase $A_2 \in \{(a,b) : a,b \in \Omega\}$ give $[A_1 \cap A_2 =]\ A_2 \cap A_1 \in \{(a,b) : a,b \in \Omega\}$ with the Commutative Law for the intersection of two sets and again with (11.410), and therefore again (11.434). Finally, the second subcase $A_2 \in \{[\min \Omega, b) : b \in \Omega\}$ gives in view of the current assumption $A_1 \in \{[\min \Omega, b) : b \in \Omega\}$ for the intersection $A_1 \cap A_2 \in \{[\min \Omega, b) : b \in \Omega\}$ by virtue of (11.412) and consequently the disjunction (11.434), so that $A_1 \cap A_2$ is element of the union (11.421) in any case.

We thus proved the existence of a set A_3 such that A_3 is in (11.421), such that $\omega \in A_3$ holds, and such that $A_3 \subseteq A_1 \cap A_2$ is also true. Since A_1, A_2 and ω are arbitrary, we may therefore conclude that Property 3 of a basis for a topology on Ω is also satisfied by the set system $\{[\min \Omega, b) : b \in \Omega\}$, which thus constitutes such a basis (if Ω has a minimum).

We can prove Part c) in analogy to Part b).

Concerning d), we now assume that the minimum $\min \Omega$ and the maximum $\max \Omega$ of Ω with respect to $\leq_\Omega$ both exist. Let us recall from the proof of b) that the union (11.421) is a subset of $\mathcal{P}(\Omega)$. Furthermore, (11.409) shows that $Z \in \{(a, \max \Omega] : a \in \Omega\}$ implies $\{(a,b] : a,b \in \Omega\}$ for any Z, which yields by definition of a subset the inclusions

$$\{(a, \max \Omega] : a \in \Omega\} \subseteq \{(a,b] : a,b \in \Omega\} \quad [\subseteq \mathcal{P}(\Omega)],$$

where the second inclusion holds according to (3.389). These inclusions imply now $\{(a, \max \Omega] : a \in \Omega\} \subseteq \mathcal{P}(\Omega)$ because of (2.13). Thus, the union (11.421) and $\{(a, \max \Omega] : a \in \Omega\}$ are both subsets of $\mathcal{P}(\Omega)$, so that the union (11.423) is also as subset of $\mathcal{P}(\Omega)$ due to (2.252). This means that the set (11.423) satisfies Property 1 of a basis for a topology.

We continue with the verification of Property 2 of a basis, letting $\bar{\omega} \in \Omega$ be arbitrary. Since the minimum $\min \Omega$ is by definition a lower bound and the maximum $\max \Omega$ by definition an upper bound for Ω, which are both in Ω, we have on the one hand the true universal sentences

$$\forall\omega\,(\omega \in \Omega \Rightarrow \min\Omega \leq_\Omega \omega), \tag{11.435}$$

$$\forall\omega\,(\omega \in \Omega \Rightarrow \omega \leq_\Omega \max\Omega), \tag{11.436}$$

and on the other hand $\min\Omega, \max\Omega \in \Omega$. Consequently, $\bar{\omega} \in \Omega$ yields $\min\Omega \leq_\Omega \bar{\omega}$ as well as $\bar{\omega} \leq_\Omega \max\Omega$, which inequalities we may write also as the disjunctions

$$\min\Omega <_\Omega \bar{\omega} \;\vee\; \min\Omega = \bar{\omega},$$

$$\bar{\omega} <_\Omega \max\Omega \;\vee\; \bar{\omega} = \max\Omega,$$

using the definition of an induced reflexive partial ordering. We now prove by cases (based on the first disjunction) and sub-cases (based on the second disjunction) that there is an interval in one of the three systems in (11.423) containing $\bar{\omega}$. Beginning with the first case $\min\Omega <_\Omega \bar{\omega}$ and the first subcase $\bar{\omega} <_\Omega \max\Omega$, we immediately see that $\bar{\omega}$ is element of the open interval $(\min\Omega, \max\Omega)$. Furthermore, $\min\Omega <_\Omega \bar{\omega}$ and the second subcase $\bar{\omega} = \max\Omega$ (which evidently implies $\bar{\omega} \leq_\Omega \max\Omega$) reveals that $\bar{\omega}$ is in the left-open and right-closed interval $(\min\Omega, \max\Omega]$ ending in $\max\Omega$. On the other hand, the second case $\min\Omega = \bar{\omega}$, implying $\min\Omega \leq_\Omega \bar{\omega}$, and the first subcase $\bar{\omega} <_\Omega \max\Omega$ show that the left-closed and right-open interval $[\min\Omega, \max\Omega)$ contains $\bar{\omega}$. Let us finally consider $\min\Omega = \bar{\omega}$ jointly with the second subcase $\bar{\omega} = \max\Omega$, where $\min\Omega$, $\bar{\omega}$ and $\max\Omega$ represent a single element of Ω. Now, as we assumed Ω to be neither empty nor a singleton, there exists an element in Ω, say $\bar{y}$, such that $\bar{\omega} \neq \bar{y}$ holds, according to (2.21). Then, we obtain with the connexity of the linear ordering $<_\Omega$ the true disjunction $\bar{\omega} <_\Omega \bar{y} \vee \bar{y} <_\Omega \bar{\omega}$, which we now use for another proof by cases. If $\bar{\omega} <_\Omega \bar{y}$ is true, we recall that $\min\Omega = \bar{\omega}$ implies $\min\Omega \leq_\Omega \bar{\omega}$, so that $\bar{\omega}$ is evidently an element of the left-closed and right-open interval $[\min\Omega, \bar{y})$ beginning in $\min\Omega$. On the other hand, if $\bar{y} <_\Omega \bar{\omega}$ holds, then we recall that $\bar{\omega} = \max\Omega$ implies $\bar{\omega} \leq_\Omega \max\Omega$, so that $\bar{\omega}$ is now in the left-open and right-closed interval $(\bar{y}, \max\Omega]$ ending in $\max\Omega$. In summary, the element $\bar{\omega}$ is in any case element of an interval in $\{(a,b) : a, b \in \Omega\}$, in $\{[\min\Omega, b) : b \in \Omega\}$, or in $\{(a, \max\Omega] : a \in \Omega\}$, which means that there is (in any case) an interval in the union (11.423) containing $\bar{\omega}$. Since $\bar{\omega}$ was arbitrary, we may therefore conclude that the set (11.423) satisfies Property 2 of a basis for a topology on Ω.

Next, we seek to establish Property 3 for the system (11.423), by taking arbitrary A_1, A_2 and ω, assuming A_1 and A_2 to be elements of the system (11.423), and assuming $\omega \in A_1$ as well as $\omega \in A_2$ to hold. Clearly, we then have for the choice $\bar{A}_3 = A_1 \cap A_2$ already the required $\omega \in \bar{A}_3$ and the required $\bar{A}_3 \subseteq A_1 \cap A_2$, so that it only remains for us to verify that $\bar{A}_3$ is also element of the interval system (11.423). Since the sets A_1 and A_2 are elements of the threefold union (11.423), we obtain the disjunctions

$$\begin{aligned} A_1 \in \{(a,b) : a,b \in \Omega\} &\vee A_1 \in \{[\min \Omega, b) : b \in \Omega\} \\ &\vee A_1 \in \{(a, \max \Omega] : a \in \Omega\}, \\ A_2 \in \{(a,b) : a,b \in \Omega\} &\vee A_2 \in \{[\min \Omega, b) : b \in \Omega\} \\ &\vee A_2 \in \{(a, \max \Omega] : a \in \Omega\}, \end{aligned}$$

where we omit the brackets in view of the Associative Law for the disjunction, so that we can carry out the proof by considering three cases and then three sub-cases. The first case $A_1 \in \{(a,b) : a,b \in \Omega\}$ and the first sub-case $A_2 \in \{(a,b) : a,b \in \Omega\}$ give $[A_1 \cap A_2 =]\ \bar{A}_3 \in \{(a,b) : a,b \in \Omega\}$, as discussed already in the proof of a) and b); therefore, the disjunction

$$\bar{A}_3 \in \{(a,b) : a,b \in \Omega\} \vee \bar{A}_3 \in \{[\min \Omega, b) : b \in \Omega\} \tag{11.437}$$

$$\vee\ \bar{A}_3 \in \{(a, \max \Omega] : a \in \Omega\} \tag{11.438}$$

holds. We noted also in the proof of b) that $A_1 \in \{(a,b) : a,b \in \Omega\}$ and the second sub-case $A_2 \in \{[\min \Omega, b) : b \in \Omega\}$ imply $[A_1 \cap A_2 =]\ \bar{A}_3 \in \{(a,b) : a,b \in \Omega\}$, with the consequence that the disjunction (11.437) is true again. Next, the first case $A_1 \in \{(a,b) : a,b \in \Omega\}$ in connection with the third sub-case $A_2 \in \{(a, \max \Omega] : a \in \Omega\}$ produces for the intersection again $\bar{A}_3 \in \{(a,b) : a,b \in \Omega\}$ according to (11.411), as for the first and second sub-case. Next, as explained in the proof of b), the second case $A_1 \in \{[\min \Omega, b) : b \in \Omega\}$ and the first sub-case $A_2 \in \{(a,b) : a,b \in \Omega\}$ give $\bar{A}_3 \in \{(a,b) : a,b \in \Omega\}$, and the second case in combination with the second sub-case $A_2 \in \{[\min \Omega, b) : b \in \Omega\}$ yields $\bar{A}_3 \in \{[\min \Omega, b) : b \in \Omega\}$. Furthermore, the second case $A_1 \in \{[\min \Omega, b) : b \in \Omega\}$ and the third sub-case $A_2 \in \{(a, \max \Omega] : a \in \Omega\}$ imply $[A_1 \cap A_2 =]\ \bar{A}_3 \in \{(a,b) : a,b \in \Omega\}$ because of (11.414), so that all three sub-cases of the second case lead to the true disjunction (11.437). Now, the third case $A_1 \in \{(a, \max \Omega] : a \in \Omega\}$ in connection with the first sub-case $A_2 \in \{(a,b) : a,b \in \Omega\}$ as well as in connection with the second sub-case $A_2 \in \{[\min \Omega, b) : b \in \Omega\}$ imply

$$[\bar{A}_3 = A_1 \cap A_2 =]\ A_2 \cap A_1 \in \{(a,b) : a,b \in \Omega\}$$

with (11.411) and (11.414), respectively. Moreover, the third case $A_1 \in \{(a, \max \Omega] : a \in \Omega\}$ and the third sub-case $A_2 \in \{(a, \max \Omega] : a \in \Omega\}$ lead to $[A_1 \cap A_2 =]\ \bar{A}_3 \in \{(a, \max \Omega] : a \in \Omega\}$ in view of (11.413). We therefore obtain the true disjunction (11.437) also for all three sub-cases within the third case, completing the proof that $\bar{A}_3$ is element of the union (11.423).

We thus showed that there exists a set A_3 in the interval system (11.423) with $\omega \in A_3$ and $A_3 \subseteq A_1 \cap A_2$. Here, A_1, A_2 and ω were arbitrary, so that this interval system satisfies indeed Property 3 of a basis for a topology on Ω (besides the already established Property 1 and Property 2).

Having completed the proof of d) and recalling that the sets Ω and $<_\Omega$ were initially arbitrary, we may now finally conclude that the stated theorem is true. □

Exercise 11.27. Establish Part c) of the preceding theorem in analogy to Part b).

Definition 11.23 (Order topology). For any linearly ordered set $(\Omega, <_\Omega)$ such that Ω is neither empty nor a singleton,

(1) if the minimum and the maximum of Ω (with respect to $<_\Omega$) do not exist, then we call the topology

$$\mathcal{O}(\{(a,b) : a,b \in \Omega\}) \tag{11.439}$$

generated by the set of open intervals in Ω the *order topology* on Ω (with respect to $<_\Omega$).

(2) if the minimum of Ω does exist and the maximum of Ω does not exist, then we call the topology

$$\mathcal{O}(\{(a,b) : a,b \in \Omega\} \cup \{[\min \Omega, b) : b \in \Omega\}) \tag{11.440}$$

generated by the set of all open intervals and all left-closed and right-open intervals beginning in the minimum of Ω the *order topology* on Ω (with respect to $<_\Omega$).

(3) if the minimum of Ω does not exist and the maximum of Ω does exist, then we call the topology

$$\mathcal{O}(\{(a,b) : a,b \in \Omega\} \cup \{(a, \max \Omega] : a \in \Omega\}) \tag{11.441}$$

generated by the set of all open intervals and all left-open and right-closed intervals ending in the maximum of Ω the *order topology* on Ω (with respect to $<_\Omega$).

(4) if the minimum and the maximum of Ω do both exist, then we call the topology

$$\mathcal{O}(\{(a,b) : a,b \in \Omega\} \cup \{[\min \Omega, b) : b \in \Omega\} \cup \{(a, \max \Omega] : a \in \Omega\}) \tag{11.442}$$

generated by the set of all open intervals, of all left-closed and right-open intervals beginning in the minimum of Ω and of all left-open and right-closed intervals ending in the maximum of Ω the *order topology* on Ω (with respect to $<_\Omega$).

In any case, we symbolize the order topology on a linearly ordered set $(\Omega, <_\Omega)$ also by

$$\mathcal{O}_{<_\Omega}. \tag{11.443}$$

Proposition 11.65. *The order topology $\mathcal{O}_{<_\Omega}$ exists for any linearly ordered set $(\Omega, <_\Omega)$ such that Ω is neither empty nor a singleton, and its basis includes in any case the set of open intervals in Ω*

Proof. Letting Ω and $<_\Omega$ be arbitrary such that Ω is neither empty nor a singleton and such that $(\Omega, <_\Omega)$ is linearly ordered, we consider first the two cases

$$\neg\exists m\,(m \in \Omega \wedge \forall\omega\,(\omega \in \Omega \Rightarrow \omega \leq_\Omega m)) \tag{11.444}$$

$$\exists m\,(m \in \Omega \wedge \forall\omega\,(\omega \in \Omega \Rightarrow \omega \leq_\Omega m)) \tag{11.445}$$

and within each of these the further two sub-cases

$$\neg\exists m\,(m \in \Omega \wedge \forall\omega\,(\omega \in \Omega \Rightarrow m \leq_\Omega \omega)) \tag{11.446}$$

$$\exists m\,(m \in \Omega \wedge \forall\omega\,(\omega \in \Omega \Rightarrow m \leq_\Omega \omega)). \tag{11.447}$$

Then, the first case (11.444) in connection with the first sub-case (11.446) means that the minimum and the maximum of Ω (with respect to $<_\Omega$) do not exist, so that the set (11.420) of open intervals in Ω constitutes the basis generating the order topology $\mathcal{O}_{<_\Omega}$ according to (11.439), which basis includes the set of open intervals in Ω in view of (2.10).

In connection with the second sub-case (11.447), we then have that the maximum of Ω still does not exist whereas $\min\Omega$ does exist, so that the union (11.421) is the basis generating $\mathcal{O}_{<_\Omega}$ according to (11.440), and this basis includes again the set of open intervals in Ω because of (2.245).

In the second case (11.445), $\max\Omega$ exists, and the minimum of Ω does not exist in the first sub-case. Thus, the order topology is generated by the set (11.422), of which union the set of open intervals is a subset, again because of (2.245).

Finally, $\max\Omega$ and $\min\Omega$ both exist in the second sub-case of the second case, where $\mathcal{O}_{<_\Omega}$ is generated by the basis (11.423); this threefold union evidently includes again the set of open intervals in Ω, using the Associative Law for the union of two sets alongside (2.245).

Since Ω and $<_\Omega$ are arbitrary, we may therefore conclude that the proposition corollary is indeed true. □

Using the standard linear orderings of $\mathbb{R}$ and of $\overline{\mathbb{R}}$, we obtain two important instances of an order topology.

Note 11.23. Since the minimum and the maximum of $\mathbb{R}$ do not exist (see Corollary 8.15), we see in light of the definition of an order topology that the set of open intervals in $\mathbb{R}$ generates as a basis the order topology on the set of real numbers, that is,

$$\mathcal{O}_{<_{\mathbb{R}}} = \mathcal{O}(\{(a,b) : a,b \in \mathbb{R}\}); \tag{11.448}$$

here, the inclusion

$$\{(a,b) : a,b \in \mathbb{R}\} \subseteq \mathcal{O}_{<_{\mathbb{R}}} \tag{11.449}$$

holds according to Proposition 11.55.

Furthermore, since $-\infty$ is the minimum and $+\infty$ the maximum of $\overline{\mathbb{R}}$ (see Corollary 9.13), the order topology on the set of extended real numbers is given by

$$\mathcal{O}_{<_{\overline{\mathbb{R}}}} = \mathcal{O}(\{(a,b) : a,b \in \overline{\mathbb{R}}\} \cup \{[-\infty,b) : b \in \overline{\mathbb{R}}\} \cup \{(a,+\infty] : a \in \overline{\mathbb{R}}\}), \tag{11.450}$$

and the generating basis is evidently included in $\mathcal{O}_{<_{\overline{\mathbb{R}}}}$.

Definition 11.24 (Standard/order topology on $\mathbb{R}$ & $\overline{\mathbb{R}}$). We call

(1) the order topology

$$\mathcal{O}_{<_{\mathbb{R}}} \tag{11.451}$$

on the set of real numbers also the *standard topology* on $\mathbb{R}$.

(2) the order topology

$$\mathcal{O}_{<_{\overline{\mathbb{R}}}} \tag{11.452}$$

on the set of extended real numbers also the *standard topology* on $\overline{\mathbb{R}}$.

Note 11.24. We have with Corollary 11.43 and Property 2 of a topology

$$\emptyset \in \mathcal{O}_{<_{\mathbb{R}}}, \tag{11.453}$$

$$\mathbb{R} \in \mathcal{O}_{<_{\mathbb{R}}}, \tag{11.454}$$

as well as

$$\emptyset \in \mathcal{O}_{<_{\overline{\mathbb{R}}}}, \tag{11.455}$$

$$\overline{\mathbb{R}} \in \mathcal{O}_{<_{\overline{\mathbb{R}}}}. \tag{11.456}$$

Moreover we have for any $a,b \in \mathbb{R}$ in view of the inclusion (11.449)

$$(a,b)_{\mathbb{R}} \in \mathcal{O}_{<_{\mathbb{R}}} \tag{11.457}$$

and for any $a, b \in \overline{\mathbb{R}}$ because of the inclusion mentioned in Note 11.23

$$(a, b)_{\overline{\mathbb{R}}} \in \mathcal{O}_{<_{\overline{\mathbb{R}}}}, \tag{11.458}$$

Thus, open intervals are open sets in the order topology with respect to the same linear ordering. We thus have in particular

$$(-\infty, +\infty)_{\overline{\mathbb{R}}} \in \mathcal{O}_{<_{\overline{\mathbb{R}}}}, \tag{11.459}$$

and then, recalling the equation $\mathbb{R} = (-\infty, +\infty)_{\overline{\mathbb{R}}}$ from Proposition 9.11

$$\mathbb{R} \in \mathcal{O}_{<_{\overline{\mathbb{R}}}}. \tag{11.460}$$

Definition 11.25 (ε-neighborhood). We call for any real numbers a and $\varepsilon >_{\mathbb{R}} 0$ the open interval

$$V_\varepsilon(a) = (a -_{\mathbb{R}} \varepsilon, a +_{\mathbb{R}} \varepsilon) \tag{11.461}$$

in $\mathbb{R}$ the ε-*neighborhood* of a.

Exercise 11.28. Show that every ε-neighborhood of any real number contains that number, that is,

$$\forall a, \varepsilon\,([a, \varepsilon \in \mathbb{R} \wedge \varepsilon >_{\mathbb{R}} 0] \Rightarrow a \in V_\varepsilon(a)). \tag{11.462}$$

(Hint: Apply the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$.)

Note 11.25. Exercise 11.28 shows that all ε-neighborhoods are nonempty.

The result of the following exercise follows immediately from (3.407).

Exercise 11.29. Prove the following implication for any real number a and any positive real numbers $\varepsilon_1, \varepsilon_2$.

$$\varepsilon_1 \leq_{\mathbb{R}} \varepsilon_2 \;\Rightarrow\; V_{\varepsilon_1}(a) \subseteq V_{\varepsilon_2}(a) \tag{11.463}$$

Lemma 11.66. *It is true for any $a, b \in \mathbb{R}$ that the open interval $(a, b)_{\mathbb{R}}$ includes an ε-neighborhood for each of its elements, i.e.*

$$\forall y\,(y \in (a, b)_{\mathbb{R}} \Rightarrow \exists \varepsilon\,(\varepsilon >_{\mathbb{R}} 0 \wedge V_\varepsilon(y) \subseteq (a, b)_{\mathbb{R}})). \tag{11.464}$$

Proof. We let a and b be arbitrary real numbers, let then y also be arbitrary, and assume $y \in (a, b)_{\mathbb{R}}$ to be true. Thus, the inequalities $a <_{\mathbb{R}} y <_{\mathbb{R}} b$ hold by definition of an open interval in $\mathbb{R}$. The Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$ gives us therefore evidently $0 <_{\mathbb{R}} y - a$ and $0 <_{\mathbb{R}} b - y$ (where we write $-$ instead of $-_{\mathbb{R}}$ for greater clarity). We then observe in light of the totality of $\leq_{\mathbb{R}}$ that $y - a \leq_{\mathbb{R}} b - y$ or $b - y \leq_{\mathbb{R}} y - a$ holds. We use this disjunction to prove the desired existential sentence by cases.

In the first case $y-a \leq_{\mathbb{R}} b-y$, we denote the difference $y-a$ by $\bar{\varepsilon}$, so that the previously found $0 <_{\mathbb{R}} y - a$ yields $\bar{\varepsilon} >_{\mathbb{R}} 0$ via substitution. Thus, the ε-neighborhood $V_{\bar{\varepsilon}}(y) = (y-\bar{\varepsilon}, y+\bar{\varepsilon})$ is defined, and we can show in addition that it is included in the open interval $(a, b)_{\mathbb{R}}$. For this purpose, we apply the definition of a subset and let accordingly $x \in V_{\bar{\varepsilon}}(y)$ be arbitrary. We thus have $x \in (y - \bar{\varepsilon}, y + \bar{\varepsilon})$, which implies $y - \bar{\varepsilon} <_{\mathbb{R}} x <_{\mathbb{R}} y + \bar{\varepsilon}$ by definition of an open interval. Since $\bar{\varepsilon} = y - a$ evidently implies $a = y - \bar{\varepsilon}$, it follows from the preceding inequalities through substitution that $a <_{\mathbb{R}} x <_{\mathbb{R}} y + \bar{\varepsilon}$. Let us observe next on the one hand that the current case assumption $y - a \leq_{\mathbb{R}} b - y$ implies $y + y - a \leq_{\mathbb{R}} b$ with the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$, and that $\bar{\varepsilon} = y - a$ yields $y + \bar{\varepsilon} = y + y - a$, so that $y + \bar{\varepsilon} \leq_{\mathbb{R}} b$ follows to be true. In conjunction with the previously found inequality $x <_{\mathbb{R}} y + \bar{\varepsilon}$, this implies $x <_{\mathbb{R}} b$ with the Transitivity Formula for $<$ and $\leq$. Having thus established $a <_{\mathbb{R}} x$ and $x <_{\mathbb{R}} b$, we may evidently infer from these inequalities $x \in (a, b)_{\mathbb{R}}$. Because x was arbitrary in $V_{\bar{\varepsilon}}(y)$, we can infer from this finding the truth of the inclusion $V_{\bar{\varepsilon}}(y) \subseteq (a, b)_{\mathbb{R}}$. Recalling the truth of $\bar{\varepsilon} >_{\mathbb{R}} 0$, we see now that the existential sentence in (11.464) is true for the first case.

In the second case $b - y \leq_{\mathbb{R}} y - a$, we define $\bar{\varepsilon} = b - y$, with the consequence that $\bar{\varepsilon} >_{\mathbb{R}} 0$ holds by virtue of the previously established $0 <_{\mathbb{R}} b-y$. We demonstrate now in analogy to the first case that the resulting ε-neighborhood $V_{\bar{\varepsilon}}(y) = (y - \bar{\varepsilon}, y + \bar{\varepsilon})$ constitutes a subset of $(a, b)_{\mathbb{R}}$. Letting again $x \in V_{\bar{\varepsilon}}(y)$ be arbitrary, which means that $x \in (y - \bar{\varepsilon}, y + \bar{\varepsilon})$ is true, we obtain $y - \bar{\varepsilon} <_{\mathbb{R}} x <_{\mathbb{R}} y + \bar{\varepsilon}$ again. Now, the equality $\bar{\varepsilon} = b - y$ gives us $b = y + \bar{\varepsilon}$, so that we can write the preceding inequalities also as $y - \bar{\varepsilon} <_{\mathbb{R}} x <_{\mathbb{R}} b$. Furthermore, the current case assumption yields $a \leq_{\mathbb{R}} y + y - b$, and $\bar{\varepsilon} = b - y$ gives $y - \bar{\varepsilon} = y - (b - y) = y + y - b$. Combining these findings, we get $a \leq_{\mathbb{R}} y-\bar{\varepsilon}$, and this implies in connection with $y - \bar{\varepsilon} <_{\mathbb{R}} x$ that $a <_{\mathbb{R}} x$ is true (using the Transitivity Formula for $\leq$ and $<$). Because $x <_{\mathbb{R}} b$ also holds, we obtain then $x \in (a, b)_{\mathbb{R}}$, where x is arbitrary in $V_{\bar{\varepsilon}}(y)$, so that the inclusion $V_{\bar{\varepsilon}}(y) \subseteq (a, b)_{\mathbb{R}}$ turns out to be true. Due to $\bar{\varepsilon} >_{\mathbb{R}} 0$, the existential sentence in (11.464) holds therefore also for the second case.

The proof of the implication in (11.464) is now complete, and since y, a and b are arbitrary, we may therefore conclude that the stated lemma holds. □

Exercise 11.30. Establish the following sentences.

a) $\emptyset$ includes an ε-neighborhood for each of its elements, i.e.

$$\forall y\,(y \in \emptyset \Rightarrow \exists \varepsilon\,(\varepsilon >_{\mathbb{R}} 0 \,\wedge\, V_\varepsilon(y) \subseteq \emptyset)), \tag{11.465}$$

b) $\mathbb{R}$ includes an ε-neighborhood for each real number, i.e.

$$\forall y\,(y \in \mathbb{R} \Rightarrow \exists \varepsilon\,(\varepsilon >_{\mathbb{R}} 0 \,\wedge\, V_\varepsilon(y) \subseteq \mathbb{R})). \tag{11.466}$$

c) The interval $(-\infty, b)$ includes for any $b \in \mathbb{R}$ an ε-neighborhood for each of its elements, i.e.

$$\forall y\,(y \in (-\infty, b)_{\mathbb{R}} \Rightarrow \exists \varepsilon\,(\varepsilon >_{\mathbb{R}} 0 \,\wedge\, V_\varepsilon(y) \subseteq (-\infty, b)_{\mathbb{R}})). \tag{11.467}$$

(Hint: Consider ε-neighborhoods with $\bar{\epsilon} = b - y$.)

d) The interval $(a, +\infty)$ includes for any $a \in \mathbb{R}$ an ε-neighborhood for each of its elements, i.e.

$$\forall y\,(y \in (a, +\infty, b)_{\mathbb{R}} \Rightarrow \exists \varepsilon\,(\varepsilon >_{\mathbb{R}} 0 \,\wedge\, V_\varepsilon(y) \subseteq (a, +\infty)_{\mathbb{R}})). \tag{11.468}$$

Lemma 11.67. *It is true for every open set U of the standard topology on $\mathbb{R}$ and for every real number y that U includes some ε-neighborhood of y, that is,*

$$\forall U, y\,([U \in \mathcal{O}_{<_{\mathbb{R}}} \,\wedge\, y \in U] \Rightarrow \exists \varepsilon\,(\varepsilon >_{\mathbb{R}} 0 \,\wedge\, V_\varepsilon(y) \subseteq U))). \tag{11.469}$$

Proof. We let U and y be arbitrary, assuming $U \in \mathcal{O}_{<_{\mathbb{R}}}$ and $y \in U$ to be true. Let us recall from Note 11.23 that the standard/order topology $\mathcal{O}_{<_{\mathbb{R}}}$ is generated by the basis $\{(a, b) : a, b \in \mathbb{R}\}$. According to the Generation of a topology by means of a basis, there exists then a particular set $\bar{A} \in \{(a, b) : a, b \in \mathbb{R}\}$ such that $y \in \bar{A}$ and $\bar{A} \subseteq U$ hold. By definition of that basis, there are therefore particular real numbers $\bar{a}$ and $\bar{b}$ with $\bar{A} = (\bar{a}, \bar{b})$, so that we obtain after substitutions $y \in (\bar{a}, \bar{b})$ and $(\bar{a}, \bar{b}) \subseteq U$. In view of Lemma 11.66, the former finding implies the existence of a particular real number $\bar{\varepsilon} >_{\mathbb{R}} 0$ for which $V_{\bar{\varepsilon}}(y) \subseteq (\bar{a}, \bar{b})$ is satisfied. This inclusion implies now in conjunction with the previously established inclusion $(\bar{a}, \bar{b}) \subseteq U$ the truth of $V_{\bar{\varepsilon}}(y) \subseteq U$ by virtue of (2.13). This inclusion demonstrates in connection with $\bar{\varepsilon} >_{\mathbb{R}} 0$ that the existential sentence in (11.469) holds. Thus, the proof of the implication in (11.469) is complete, and since U and y are arbitrary, we may therefore conclude that the lemma is indeed true. □

Theorem 11.68 (Characterization of closed sets in $\mathbb{R}$). *It is true for every subset $A \subseteq \mathbb{R}$ that A is closed in $\mathbb{R}$ (with respect to the standard* *topology $\mathcal{O}_{<_{\mathbb{R}}}$) iff every element $y \in A^c$ has some ε-neighborhood disjoint from A, that is,*

$$\forall A\,(A \subseteq \mathbb{R} \\ \Rightarrow [A \text{ is closed in } \mathbb{R} \Leftrightarrow \forall y\,(y \in A^c \Rightarrow \exists \varepsilon\,(\varepsilon >_{\mathbb{R}} 0 \,\wedge\, V_\varepsilon(y) \cap A = \emptyset))]). \tag{11.470}$$

Proof. We take an arbitrary set A, and we assume A to be a subset of $\mathbb{R}$. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume in addition that A is a closed set in $\mathbb{R}$ with respect to the order topology $\mathcal{O}_{<_{\mathbb{R}}}$. Then, we let y be arbitrary, and we assume moreover $y \in A^c$ to be true. Here, the complement $A^c = \mathbb{R} \setminus A$ is an open set in $\mathbb{R}$ (with respect to $\mathcal{O}_{<_{\mathbb{R}}}$), by definition. The conjunction of $A^c \in \mathcal{O}_{<_{\mathbb{R}}}$ and $y \in A^c$ implies now with Lemma 11.67 that there exists a particular real number $\bar{\varepsilon} >_{\mathbb{R}} 0$ such that the $\bar{\varepsilon}$-neighborhood $V_{\bar{\varepsilon}}(y)$ is included in the open set A^c. Let us observe here that the initial assumption $A \subseteq \mathbb{R}$ implies $A^c \subseteq \mathbb{R}$ in view of (2.137). Then, the conjunction of the preceding inclusions $V_{\bar{\varepsilon}}(y) \subseteq A^c$ and $A^c \subseteq \mathbb{R}$ implies the truth of $V_{\bar{\varepsilon}}(y) \cap (A^c)^c = \emptyset$ because of (2.140), so that the equation $V_{\bar{\varepsilon}}(y) \cap A = \emptyset$ follows to be true with (2.136). Alongside $\bar{\varepsilon} >_{\mathbb{R}} 0$, this finding shows us that the existential sentence in (11.470) is true. As y was arbitrary, we may therefore conclude that the first part of the equivalence to be proven holds.

To establish the second part ('$\Leftarrow$') of the equivalence, we assume the universal sentence

$$\forall y\,(y \in A^c \Rightarrow \exists \varepsilon\,(\varepsilon >_{\mathbb{R}} 0 \,\wedge\, V_\varepsilon(y) \cap A = \emptyset)) \tag{11.471}$$

to be true, and we demonstrate that A follows to be a closed set in $\mathbb{R}$. This means by definition that A^c is an open set of $\mathcal{O}_{<_{\mathbb{R}}}$, which assertion in turn is equivalent to the universal sentence

$$\forall y\,(y \in A^c \Rightarrow \exists V\,(V \in \mathcal{O}_{<_{\mathbb{R}}} \,\wedge\, y \in V \,\wedge\, V \subseteq A^c)), \tag{11.472}$$

according to the Characterization of open sets. To prove the preceding universal sentence, we let y be arbitrary, and we assume $y \in A^c$ to be true. This assumption implies with (11.471) that there exists a constant, say $\bar{\varepsilon}$, such that $\varepsilon >_{\mathbb{R}} 0$ and $V_{\bar{\varepsilon}}(y) \cap A = \emptyset$ are satisfied. The latter equation further implies $V_{\bar{\varepsilon}}(y) \subseteq A^c$ with (2.141). Moreover, $V_{\bar{\varepsilon}}(y)$ constitutes the open interval $(y - \bar{\varepsilon}, y + \bar{\varepsilon})$ by definition of an ε-neighborhood, with the consequence that $V_{\bar{\varepsilon}}(y) \in \mathcal{O}_{<_{\mathbb{R}}}$ holds, according to (11.457). In addition, we find $y \in V_{\bar{\varepsilon}}(y)$ to be true by virtue of (11.462). These findings demonstrate the truth of the existential sentence (11.472), and since y was arbitrary, we may therefore conclude that the universal sentence (11.472) holds. Then, the equivalent sentence that A^c is an open set of $\mathcal{O}_{<_{\mathbb{R}}}$ is also true, which means that A is indeed a closed set in $\mathbb{R}$.

Thus, the proof of the equivalence in (11.470) is complete. Since A was initially arbitrary, we may now infer from the truth of this equivalence the truth of the universal sentence (11.470). $\square$

The order topology on $\mathbb{R}$ is not only generated by the basis set of open intervals in $\mathbb{R}$, but also by the set of open intervals in $\mathbb{Q}$, which is defined by means of the standard ordering of $\mathbb{Q}$.

Theorem 11.69 (Second-countability of $(\mathbb{R}, \mathcal{O}_{<_\mathbb{R}})$). *The following sentences are true.*

a) The set of open intervals in $\mathbb{Q}$ is a basis for a topology on $\mathbb{R}$.

b) Furthermore, the standard/order topology on the set of real numbers is generated by that basis, i.e.

$$\mathcal{O}_{<_\mathbb{R}} = \mathcal{O}(\{(a,b) : a,b \in \mathbb{Q}\}). \tag{11.473}$$

c) Moreover, that basis is a countable set, and the topological space $(\mathbb{R}, \mathcal{O}_{<_\mathbb{R}})$ is second-countable.

Proof. Based on the topological space $(\mathbb{R}, \mathcal{O}_{<_\mathbb{R}})$, we establish a) and b) using the Characterization of the basis generating a given topology. For this purpose, we first prove the inclusion

$$\{(a,b) : a,b \in \mathbb{Q}\} \subseteq \{(a,b) : a,b \in \mathbb{R}\} \tag{11.474}$$

via the definition of a subset. Letting B be arbitrary in $\{(a,b) : a,b \in \mathbb{Q}\}$, there exist then, by definition of the set of open intervals in $\mathbb{Q}$, elements of $\mathbb{Q}$, say $\bar{a}$ and $\bar{b}$, with $(\bar{a}, \bar{b}) = B$. Here, $\bar{a}, \bar{b} \in \mathbb{Q}$ evidently implies $\bar{a}, \bar{b} \in \mathbb{R}$ so that $[B =]\ (\bar{a}, \bar{b}) \in \{(a,b) : a,b \in \mathbb{R}\}$ is clearly true. Since B is arbitrary, we may therefore infer from this finding the truth of the desired inclusion (11.474). Recalling from (11.448) that the order topology on $\mathbb{R}$ is generated by the set of open intervals in $\mathbb{R}$, which basis is included in the order topology according to Proposition 11.55, we have

$$\{(a,b) : a,b \in \mathbb{R}\} \subseteq \mathcal{O}_{<_\mathbb{R}}.$$

Together with (11.474), this inclusion implies now with (2.13)

$$\{(a,b) : a,b \in \mathbb{Q}\} \subseteq \mathcal{O}_{<_\mathbb{R}}.$$

We may therefore proceed with application of the Characterization of the basis generating a given topology. We let ω and B be arbitrary, assuming $B \in \mathcal{O}_{<_\mathbb{R}}$ and $\omega \in B$ to hold. Since $\{(a,b) : a,b \in \mathbb{R}\}$ is a basis generating

$\mathcal{O}_{<_{\mathbb{R}}}$, it follows from these assumptions – according to the Generation of a topology by means of a basis – that there exists a set, say $\bar{A}$, which satisfies $\bar{A} \in \{(a,b) : a,b \in \mathbb{R}\}$, $\omega \in \bar{A}$ and $\bar{A} \subseteq B$. By definition of the set of open intervals (in $\mathbb{R}$), there are then particular numbers $\bar{a}, \bar{b} \in \mathbb{R}$ such that $\bar{A} = (\bar{a}, \bar{b})$. Based on this equation, substitutions give us $\omega \in (\bar{a}, \bar{b})$ and $(\bar{a}, \bar{b}) \subseteq B$, where the former yields $\bar{a} <_{\mathbb{R}} \omega$ as well as $\omega <_{\mathbb{R}} \bar{b}$, by definition of an open interval (in $\mathbb{R}$). Recalling now that $\mathbb{R}$ is a separably ordered set with respect to $\mathbb{Q}$ (see Corollary 8.12), so that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, it follows on the one hand from the previously established $\bar{a}, \omega \in \mathbb{R}$ and $\bar{a} <_{\mathbb{R}} \omega$ that there exists a particular number $\bar{p} \in \mathbb{Q}$ satisfying

$$\bar{a} <_{\mathbb{R}} \bar{p} <_{\mathbb{R}} \omega.$$

On the other hand, it follows from $\omega, \bar{b} \in \mathbb{R}$ and $\omega <_{\mathbb{R}} \bar{b}$ that there is a particular number $\bar{q} \in \mathbb{Q}$ with

$$\omega <_{\mathbb{R}} \bar{q} <_{\mathbb{R}} \bar{b}.$$

We thus found in particular the inequalities $\bar{p} <_{\mathbb{R}} \omega <_{\mathbb{R}} \bar{q}$, which we may write also as $\bar{p} <_{\mathbb{Q}} \omega <_{\mathbb{Q}} \bar{q}$ because of our previous findings $\bar{p}, \bar{q} \in \mathbb{Q}$. Consequently, the definition of an open interval (in $\mathbb{Q}$) gives

$$\omega \in (\bar{p}, \bar{q}), \tag{11.475}$$

and

$$(\bar{p}, \bar{q}) \in \{(a,b) : a,b \in \mathbb{Q}\} \tag{11.476}$$

holds by definition of the set of open intervals (in $\mathbb{Q}$). Furthermore, the previously obtained inequalities $\bar{a} <_{\mathbb{R}} \bar{p}$ and $\bar{q} <_{\mathbb{R}} \bar{b}$ imply the truth of the disjunctions $\bar{a} <_{\mathbb{R}} \bar{p} \vee \bar{a} = \bar{p}$ and $\bar{q} <_{\mathbb{R}} \bar{b} \vee \bar{q} = \bar{b}$ and then, by definition of an induced reflexive partial ordering, the truth of $\bar{a} \leq_{\mathbb{R}} \bar{p}$ and $\bar{q} \leq_{\mathbb{R}} \bar{b}$. The conjunction of these two inequalities in turn imply the inclusion $(\bar{p}, \bar{q}) \subseteq (\bar{a}, \bar{b})$ with Proposition 3.125a). In connection with the previously established inclusion $(\bar{a}, \bar{b}) \subseteq B$, this yields

$$(\bar{p}, \bar{q}) \subseteq B \tag{11.477}$$

with (2.13). In view of (11.475) – (11.477), we thus showed that there exists a set $A \in \{(a,b) : a,b \in \mathbb{Q}\}$ with $\omega \in A$ and $A \subseteq B$, which existential sentence implies now with Theorem 11.58 firstly that $\{(a,b) : a,b \in \mathbb{Q}\}$ is a basis for a topology on $\mathbb{R}$ and secondly that $\mathcal{O}_{<_{\mathbb{R}}}$ is the topology generated by that basis.

It now remains for us to prove that

$$A \in \{(a,b) : a,b \in \mathbb{Q}\}$$

is a countable set. For this purpose, we first apply Function definition by replacement to demonstrate the unique existence of a function f with domain $\mathbb{Q} \times \mathbb{Q}$ such that, for any $z \in \mathbb{Q} \times \mathbb{Q}$, the value $f(z)$ is the open interval $(p,q)_{<_\mathbb{Q}}$ with respect to $<_\mathbb{Q}$ for some p,q with z being the ordered pair (p,q). To do this, we need to prove the universal sentence

$$\forall z\,(z \in \mathbb{Q} \times \mathbb{Q} \Rightarrow \exists! Y\,(\exists p,q\,(Y = (p,q)_{<_\mathbb{Q}} \wedge (p,q) = z))). \qquad (11.478)$$

We let $\bar{z} \in \mathbb{Q} \times \mathbb{Q}$ be arbitrary, so that there are, by definition of the Cartesian product of two sets, particular numbers $\bar{p} \in \mathbb{Q}$ and $\bar{q} \in \mathbb{Q}$ for which $(\bar{p},\bar{q}) = \bar{z}$. Thus, the open interval $\bar{Y} = (\bar{p},\bar{q})$ is defined, and the previous two equations show us now that the existential sentence

$$\exists p,q\,(\bar{Y} = (p,q)_{<_\mathbb{Q}} \wedge (p,q) = \bar{z})$$

holds. This in turn demonstrates the truth of the existential sentence

$$\exists Y\,(\exists p,q\,(Y = (p,q)_{<_\mathbb{Q}} \wedge (p,q) = \bar{z})),$$

proving the existential part of the uniquely existential sentence in (11.478). To establish the uniqueness part, we let Y and Y' be arbitrary such that

$$\exists p,q\,(Y = (p,q)_{<_\mathbb{Q}} \wedge (p,q) = z) \wedge \exists p,q\,(Y' = (p,q)_{<_\mathbb{Q}} \wedge (p,q) = z)$$

holds. This means that there exist on the one hand constants, say $\bar{p}$ and $\bar{q}$, satisfying $Y = (\bar{p},\bar{q})_{<_\mathbb{Q}}$ and $(\bar{p},\bar{q}) = \bar{z}$. On the other hand, there are constants, say $\bar{p}'$ and $\bar{q}'$, such that $Y' = (\bar{p},\bar{q})_{<_\mathbb{Q}}$ and $(\bar{p},\bar{q}) = \bar{z}$. Combining now the two equations for $\bar{z}$, we obtain $(\bar{p}',\bar{q}') = (\bar{p},\bar{q})$, which equation further implies with the Equality Criterion for ordered pairs $\bar{p}' = \bar{p}$ as well as $\bar{q}' = \bar{q}$. Applying now substitutions based on these two equations, we obtain for the open intervals Y and Y' the relationships

$$Y' = (\bar{p}',\bar{q}')_{<_\mathbb{Q}} = (\bar{p},\bar{q})_{<_\mathbb{Q}} = Y,$$

so that the desired $Y' = Y$ follows to be true. Because Y and Y' are arbitrary, we may therefore conclude that the uniqueness part also holds, which finding completes the proof of the uniquely existential sentence in (11.478). As $\bar{z}$ was also arbitrary, we may now infer from this consequent the truth of the universal sentence (11.478). Consequently, there exists a unique function f with domain $\mathbb{Q} \times \mathbb{Q}$ such that

$$\forall z\,(z \in \mathbb{Q} \times \mathbb{Q} \Rightarrow \exists p,q\,(f(z) = (p,q)_{<_\mathbb{Q}} \wedge (p,q) = z)). \qquad (11.479)$$

Next, we show that f is a surjection from $\mathbb{Q} \times \mathbb{Q}$ to $\{(a,b) : a,b \in \mathbb{Q}\}$, by applying (3.631). We now prove accordingly the universal sentence

$$\forall Y\,(Y \in \{(a,b) : a,b \in \mathbb{Q}\} \Leftrightarrow \exists z\,(f(z) = Y)), \qquad (11.480)$$

letting Y be arbitrary. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $Y \in \{(a,b) : a,b \in \mathbb{Q}\}$ to be true, so that there exist (by definition of the set of open intervals in $\mathbb{Q}$) particular rational numbers $\bar{p}$ and $\bar{q}$ for which $(\bar{p},\bar{q})_{<_{\mathbb{Q}}} = Y$. Then, $(\bar{p},\bar{q}) \in \mathbb{Q} \times \mathbb{Q}$ [$= \mathrm{dom}(f)$] follows to be true by definition of the Cartesian product of two sets. In view of (11.479), there are then also particular constants $\bar{p}', \bar{q}'$ such that $f((\bar{p},\bar{q})) = (\bar{p}',\bar{q}')_{<_{\mathbb{Q}}}$ and $(\bar{p}',\bar{q}') = (\bar{p},\bar{q})$. The last equation implies with the Equality Criterion for ordered pairs the two equations $\bar{p}' = \bar{p}$ and $\bar{q}' = \bar{q}$, which allow us now to write the other equation via substitutions equivalently as

$$f((\bar{p},\bar{q})) = (\bar{p},\bar{q})_{<_{\mathbb{Q}}} \; [= Y].$$

The resulting equation $f((\bar{p},\bar{q})) = Y$ shows us that there exists a set z for which $f(z) = Y$ holds, proving the first part of the equivalence in (11.480). To establish the second part ('$\Leftarrow$'), we now conversely assume that existential sentence to be true, so that there is a particular set $\bar{z}$ that satisfies $f(\bar{z}) = Y$. This equation shows us that $\bar{z}$ is in the domain $\mathbb{Q} \times \mathbb{Q}$ of the function f, so that (11.479) gives us particular constants $\bar{\bar{p}}, \bar{\bar{q}}$ satisfying $f((\bar{\bar{p}},\bar{\bar{q}})) = (\bar{\bar{p}},\bar{\bar{q}})_{<_{\mathbb{Q}}}$ as well as $(\bar{\bar{p}},\bar{\bar{q}}) = \bar{z}$. We therefore obtain through substitution

$$Y = f(\bar{z}) = f((\bar{\bar{p}},\bar{\bar{q}})) = (\bar{\bar{p}},\bar{\bar{q}})_{<_{\mathbb{Q}}},$$

and the previously found $\bar{z} \in \mathbb{Q} \times \mathbb{Q}$ gives $(\bar{\bar{p}},\bar{\bar{q}}) \in \mathbb{Q} \times \mathbb{Q}$; the latter furthermore implies (by definition of the Cartesian product of two sets) $\bar{\bar{p}} \in \mathbb{Q}$ and $\bar{\bar{q}} \in \mathbb{Q}$. These findings clearly imply the truth of the desired consequent $Y \in \{(a,b) : a,b \in \mathbb{Q}\}$ of the second part of the equivalence in (11.480). Because the set Y is arbitrary, we may therefore infer from the truth of that equivalence the truth of the universal sentence (11.480), which then further implies that f is a surjection with domain $\mathbb{Q} \times \mathbb{Q}$ and codomain/range $\{(a,b) : a,b \in \mathbb{Q}\}$.

Recalling the Countability of $\mathbb{Q}$ and the Countability of the Cartesian product of two countable sets, we have that $\mathbb{Q} \times \mathbb{Q}$ is countable; here, $\mathbb{Q}$ is clearly a nonempty set, so that $\mathbb{Q} \times \mathbb{Q} \neq \emptyset$ follows evidently to be true with (3.27). According to the Countability Criterion (4.653), there exists then a surjection from $\mathbb{N}$ to $\mathbb{Q} \times \mathbb{Q}$, say $\bar{g}$, whose composition with f gives us the surjection

$$f \circ \bar{g} : \mathbb{N} \twoheadrightarrow \{(a,b) : a,b \in \mathbb{Q}\}$$

by means of the Surjectivity of the composition of two surjections. The existence of such a surjection implies then with the Countability Criterion that the set $\{(a,b) : a,b \in \mathbb{Q}\}$ is countable. The sentences a) - c) imply that the topological space $(\mathbb{R}, \mathcal{O}_{<_{\mathbb{R}}})$ is second-countable, by definition. $\square$

One benefit of generating the order topology on $\mathbb{R}$ by means of the countable basis of open intervals in $\mathbb{Q}$ lies in the following result that any open set in that topology may then be expressed as a 'countable union' of basis elements.

Proposition 11.70. *Any open set in the order topology on* $\mathbb{R}$ *can be written as the union of a sequence of open intervals in* $\mathbb{Q}$*, in the sense that*

$$\forall U\,(U \in \mathcal{O}_{<_{\mathbb{R}}} \Rightarrow \exists B\,(B : \mathbb{N}_+ \to \{(a,b) : a,b \in \mathbb{Q}\} \wedge U = \bigcup_{n=1}^{\infty} B_n)). \tag{11.481}$$

Proof. We let U be an arbitrary set in $\mathcal{O}_{<_{\mathbb{R}}}$, which topology is generated by the basis $\{(a,b) : a,b \in \mathbb{Q}\}$ according to the preceding theorem. According to the Characterization of the elements of a topology generated by a basis, the assumption $U \in \mathcal{O}_{<_{\mathbb{R}}}$ implies that there exists a set (system), say $\bar{\mathcal{G}}$, such that $\bar{\mathcal{G}} \subseteq \{(a,b) : a,b \in \mathbb{Q}\}$ and $U = \bigcup \bar{\mathcal{G}}$ hold. Next, we consider the two cases $\bar{\mathcal{G}} = \emptyset$ and $\bar{\mathcal{G}} \neq \emptyset$ (given by the Law of the Excluded Middle) to establish the desired existential sentence.

In the first case $\bar{\mathcal{G}} = \emptyset$, which implies $[U =]\ \bigcup \bar{\mathcal{G}} = \emptyset$ with (2.205) and thus $U = \emptyset$, we take the constant function $\bar{C} : \mathbb{N}_+ \to \{\emptyset\}$, whose value $\emptyset$ is element of $\{(a,b) : a,b \in \mathbb{Q}\}$ according to (3.397) as a consequence of the evident fact that $\mathbb{Q} \neq \emptyset$. We therefore obtain the inclusion $\{\emptyset\} \subseteq \{(a,b) : a,b \in \mathbb{Q}\}$ with (2.184), which shows in light of the definition of a codomain that

$$\bar{C} : \mathbb{N}_+ \to \{(a,b) : a,b \in \mathbb{Q}\}. \tag{11.482}$$

Since the domain $\mathbb{N}_+$ of the constant function $\bar{C}$ is clearly nonempty, it follows with Proposition 3.193 that $\bar{C} : \mathbb{N}_+ \to \{\emptyset\}$ is a surjection, so that $\{\emptyset\}$ is its range. Viewing $\bar{C}$ as the sequence $\bar{C} = (\bar{C}_n)_{n \in \mathbb{N}_+}$, we obtain for the empty open set U

$$U = \emptyset = \bigcup \{\emptyset\} = \bigcup \operatorname{ran}(\bar{C}) = \bigcup_{n=1}^{\infty} \bar{C}_n. \tag{11.483}$$

using the current case assumption, (2.199), the previous finding that $\{\emptyset\}$ is the range of $\bar{C}$, and the notation for the union of a sequence of sets. Then, (11.482) and (11.483) show us that the existential sentence in (11.481) holds for the current first case.

In the second case $\bar{\mathcal{G}} \neq \emptyset$, we first recall from the preceding theorem that $\{(a,b) : a,b \in \mathbb{Q}\}$ is a countable set, so that the subset $\bar{\mathcal{G}}$ of that basis is also countable due to Corollary 4.139. Then, $\bar{\mathcal{G}} \neq \emptyset$ implies with the Countability Criterion that there exists a particular surjection $\bar{f} : \mathbb{N} \twoheadrightarrow \bar{\mathcal{G}}$.

Because the sets $\mathbb{N}_+$ and $\mathbb{N}$ are equinumerous according to (4.667), there is a particular bijection $\bar{g} : \mathbb{N}_+ \rightleftarrows \mathbb{N}$, which is by definition a surjection. Exploiting now the Surjectivity of the composition of two surjections, we obtain the surjection $\bar{B} = \bar{f} \circ \bar{g}$ from $\mathbb{N}_+$ to $\bar{\mathcal{G}}$, whose range $\bar{\mathcal{G}}$ we found earlier to be included in $\{(a,b) : a, b \in \mathbb{Q}\}$. According to the definition of a codomain, we thus have

$$\bar{B} : \mathbb{N}_+ \to \{(a,b) : a, b \in \mathbb{Q}\}. \tag{11.484}$$

We may write this function also in sequence notation as $\bar{B} = (\bar{B}_n)_{n \in \mathbb{N}_+}$ and apply the notation for the union of a sequence of sets to obtain for the open set U

$$U = \bigcup \bar{\mathcal{G}} = \bigcup \mathrm{ran}(\bar{B}) = \bigcup_{n=1}^{\infty} \bar{B}_n. \tag{11.485}$$

In view of (11.484) and (11.485), the existential sentence in (11.481) is true also in the second case.

Since U was initially arbitrary, we may therefore conclude that the proposition holds. □

Metric topologies

Theorem 11.71 (Basis for a topology determined by a metric). *It is true for any metric space (Ω, d) that the set $\mathcal{B}_\Omega^{(d)}$ of open balls with respect to d is a basis for a topology on Ω.*

Proof. We let Ω and d be arbitrary sets, and we assume that (Ω, d) is a metric space, and we prove that $\mathcal{B}_\Omega^{(d)}$ satisfies the Properties $1-3$ of a basis for a topology on Ω. Observing in light of the specification of the set of open balls that $A \in \mathcal{B}_\Omega^{(d)}$ implies especially $A \in \mathcal{P}(\Omega)$ for any set A, we see then also that the inclusion $\mathcal{B}_\Omega^{(d)} \subseteq \mathcal{P}(\Omega)$ follows (by definition of a subset) to be true, as required by Property 1 of a basis for a topology on Ω.

Regarding Property 2, we let $\bar{\omega}$ be arbitrary, assume $\bar{\omega} \in \Omega$ to be true, and we show that there is a set A in $\mathcal{B}_\Omega^{(d)}$ such that $\bar{\omega} \in A$ holds. In view of the basic fact 1 is a positive real number, we see that the open ball $\bar{A} = B_d(\bar{\omega}, 1)$ is defined, which contains its center $\bar{\omega}$ according to Corollary 8.66. The previous two sentences clearly show that there exist constants ω_0, ϵ satisfying $\omega_0 \in \Omega$, $\epsilon \in \mathbb{R}_+$ and $B_d(\omega_0, \epsilon) = \bar{A}$, which existential sentence implies $\bar{A} \in \mathcal{B}_\Omega^{(d)}$ with (8.458). This finding, alongside the previously established $\bar{\omega} \in \bar{A}$ $[= B_d(\bar{\omega}, 1)]$, moreover shows that the desired existential sentence $\exists A\, (A \in \mathcal{B}_\Omega^{(d)} \wedge \bar{\omega} \in A)$ is true. Since $\bar{\omega}$ is arbitrary, we may therefore conclude that $\mathcal{B}_\Omega^{(d)}$ satisfies indeed Property 2 of a basis for a topology.

Finally, regarding Property 3, we let A_1, A_2 and ω be arbitrary such that $A_1, A_2 \in \mathcal{B}_\Omega^{(d)}$, $\omega \in A_1$ and $\omega \in A_2$ hold. Now, since A_1 and A_2 are thus (by definition of the set $\mathcal{B}_\Omega^{(d)}$) open balls, which contain ω according to the preceding assumptions, we see in light of Proposition 8.68 that $\omega \in A_1$ implies that there exists an open ball with center ω, say $B_d(\omega, \bar{r}_2)$, such that it is included in A_1. For the same reason, $\omega \in A_2$ implies the existence of a particular open ball $B_d(\omega, \bar{r}_2)$ again with center ω such that it is included in A_2. To establish the desired consequent

$$\exists A_3 \, (A_3 \in \mathcal{B}_\Omega^{(d)} \, \wedge \, \omega \in A_3 \, \wedge \, A_3 \subseteq A_1 \cap A_2), \tag{11.486}$$

we carry out a proof by cases based on the disjunction $\bar{r}_1 \leq_\mathbb{R} \bar{r}_2 \vee \bar{r}_2 \leq_\mathbb{R} \bar{r}_1$ (which is true because of the totality of the reflexive partial ordering $\leq_\mathbb{R}$). The first case $\bar{r}_1 \leq_\mathbb{R} \bar{r}_2$ implies with Exercise 8.37

$$B_d(\omega, \bar{r}_1) \subseteq B_d(\omega, \bar{r}_2) \quad [\subseteq A_2]$$

and therefore $B_d(\omega, \bar{r}_1) \subseteq A_2$ with (2.13). The conjunction of this inclusion and the previously established inclusion $B_d(\omega, \bar{r}_1) \subseteq A_1$ implies then $B_d(\omega, \bar{r}_1) \subseteq A_1 \cap A_2$ with (2.84), which shows that there exists an element A_3 of $\mathcal{B}_\Omega^{(d)}$ which contains ω and which is included in $A_1 \cap A_2$.

Similarly, the second case $\bar{r}_2 \leq \mathbb{R}\bar{r}_1$ yields

$$B_d(\omega, \bar{r}_2) \subseteq B_d(\omega, \bar{r}_1) \quad [\subseteq A_1]$$

and consequently $B_d(\omega, \bar{r}_2) \subseteq A_1$. The conjunction of this result and the previously obtained $B_d(\omega, \bar{r}_2) \subseteq A_2$ implies now $B_d(\omega, \bar{r}_2) \subseteq A_1 \cap A_2$, proving the existence of an element A_3 in $\mathcal{B}_\Omega^{(d)}$ that contains ω and that is included in $A_1 \cap A_2$. Thus, the existential sentence (11.486) is true in any case, and since A_1, A_2 and ω are arbitrary, we may therefore conclude that Property 3 of a basis for a topology is satisfied by $\mathcal{B}_\Omega^{(d)}$ as well.

This completes the verification that $\mathcal{B}_\Omega^{(d)}$ is a basis for a topology on Ω. Finally, as the sets Ω and d were initially arbitrary, we may infer from this the truth of the proposed universal sentence. □

Definition 11.26 (Metric topology, metrizable topological space, separable metric space). We call

(1) for any metric space (Ω, d) the topology

$$\mathcal{O}_d = \mathcal{O}(\mathcal{B}_\Omega^{(d)}) \tag{11.487}$$

generated by the basis $\mathcal{B}_{\Omega}^{(d)}$ (as defined in Theorem 11.71) the *metric topology* on Ω *induced by* d.

(2) a topological space $(\Omega, \mathcal{O})$ *metrizable* iff $\mathcal{O}$ is induced by some metric.

(3) a metric space (Ω, d) a *separable metric space* iff the topological space $(\Omega, \mathcal{O}_d)$ is separable.

Exercise 11.31. Show for any metric space (Ω, d) and for any subset V of Ω that V is open in Ω with respect to the metric topology $\mathcal{O}_d$ on Ω induced by d iff, for any element ω in V, there is an element ω_0 in Ω as well as a positive real number ϵ such that the open ball with center ω_0 and radius ϵ (with respect to d) is included in V, i.e.

$$V \in \mathcal{O}_d \;\Leftrightarrow\; \forall \omega \,(\omega \in V \Rightarrow \exists \omega_0, \epsilon \,(\omega_0 \in \Omega \;\wedge\; \epsilon >_{\mathbb{R}} 0 \;\wedge\; B_d(\omega_0, \epsilon) \subseteq V)). \tag{11.488}$$

(Hint: Apply Theorem 11.54 and Theorem 11.71.)

Before proceeding with general metric spaces, we define two instances of such cases for real numbers and real n-tuples.

Definition 11.27 (Euclidean topology on $\mathbb{R}$ & on $\mathbb{R}^n$). We call

(1) the metric topology

$$\mathcal{O}_{d_{\mathbb{R}}} = \mathcal{O}(\mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}) \tag{11.489}$$

on $\mathbb{R}$ induced by the absolute difference function on $\mathbb{R}$ the *Euclidean topology on* $\mathbb{R}$.

(2) for any $n \in \mathbb{N}$ the metric topology

$$\mathcal{O}_{d_{\mathbb{R}^n}} = \mathcal{O}(\mathcal{B}_{\mathbb{R}^n}^{(d_{\mathbb{R}^n})}) \tag{11.490}$$

on $\mathbb{R}^n$ induced by the Euclidean metric the *Euclidean topology on* $\mathbb{R}^n$.

Theorem 11.72 (Equality of the order topology and the Euclidean topology on $\mathbb{R}$). *It is true that the standard/order topology and the Euclidean topology on* $\mathbb{R}$ *are identical, that is,*

$$\mathcal{O}_{<_{\mathbb{R}}} = \mathcal{O}_{d_{\mathbb{R}}}. \tag{11.491}$$

Proof. We first establish the inclusion

$$\mathcal{O}(\{(a,b) : a, b \in \mathbb{R}\}) \;\subseteq\; \mathcal{O}(\mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}) \tag{11.492}$$

by means of the Inclusion Criterion for topologies generated by bases, i.e., by verifying

$$\forall \omega, A\,([A \in \{(a,b) : a,b \in \mathbb{R}\} \,\wedge\, \omega \in A] \\ \Rightarrow \exists B\,(B \in \mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}) \,\wedge\, \omega \in B \,\wedge\, B \subseteq A). \tag{11.493}$$

Letting $A \in \{(a,b) : a,b \in \mathbb{R}\}$ and $\omega \in A$ be arbitrary, we clearly see from the latter that $A \neq \emptyset$, so that $A \notin \{\emptyset\}$ follows to be true with (2.169); together with the former, that means

$$A \in \{(a,b) : a,b \in \mathbb{R}\} \setminus \{\emptyset\} \quad \left[= \mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}\right] \tag{11.494}$$

by definition of a set difference and due to (8.461). We thus found a particular set which satisfies $A \in \mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}$ and $\omega \in A$; since $A \subseteq A$ also holds according to (2.10), we see that the existential sentence in (11.493) is satisfied by ω and A. As these constants were arbitrary, we may therefore conclude that the universal sentence (11.493) is true, with the consequence that the inclusion (11.492) also holds.

Similarly, we may prove the inclusion

$$\mathcal{O}(\mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}) \;\subseteq\; \mathcal{O}(\{(a,b) : a,b \in \mathbb{R}\}) \tag{11.495}$$

via the verification of

$$\forall \omega, A\,([A \in \mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})} \,\wedge\, \omega \in A] \\ \Rightarrow \exists B\,(B \in \{(a,b) : a,b \in \mathbb{R}\} \,\wedge\, \omega \in B \,\wedge\, B \subseteq A). \tag{11.496}$$

Taking arbitrary $A \in \mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})}$ and $\omega \in A$, the former evidently implies $A \in \{(a,b) : a,b \in \mathbb{R}\} \setminus \{\emptyset\}$ and therefore $A \in \{(a,b) : a,b \in \mathbb{R}\}$; as $\omega \in A$ and $A \subseteq A$ also hold, the existential sentence in (11.496) is true. As ω and A were arbitrary, we may infer from this finding the truth of the universal sentence (11.496) and consequently also the truth of the inclusion (11.495) – applying again the Inclusion Criterion for topologies generated by bases.

The truth of the two inclusions (11.492) and (11.495) implies now the equality $\mathcal{O}(\{(a,b) : a,b \in \mathbb{R}\}) = \mathcal{O}(\mathcal{B}_{\mathbb{R}}^{(d_{\mathbb{R}})})$ by means of the Axiom of Extension, so that proposed equation holds in view of (11.448) and (11.489). □

Note 11.26. The second-countability of $(\mathbb{R}, \mathcal{O}_{<_{\mathbb{R}}})$ implies with the equality of the order topology and the Euclidean topology on $\mathbb{R}$ that the topological space $(\mathbb{R}, \mathcal{O}_{d_{\mathbb{R}}})$ is second-countable.

Theorem 11.73 (Hausdorff Property of topological spaces involving the metric topology). *It is true for any metric space (Ω, d) that the topological space $(\Omega, \mathcal{O}_d)$ is a Hausdorff space.*

Proof. Letting Ω and d be arbitrary sets such that (Ω, d) constitutes a metric space, we can define the corresponding metric topology $\mathcal{O}_d = \mathcal{O}(\mathcal{B}_\Omega^{(d)})$. To show that the resulting topological space $(\Omega, \mathcal{O}_d)$ is a Hausdorff space, we verify its defining property

$$\forall \omega, \nu\,([\omega, \nu \in \Omega \,\wedge\, \omega \neq \nu] \\ \Rightarrow \exists U, V\,(U, V \in \mathcal{O}_d \,\wedge\, U \cap V = \emptyset \,\wedge\, \omega \in U \,\wedge\, \nu \in V)). \tag{11.497}$$

We take arbitrary ω and ν, assuming these sets to be distinct element of Ω. Since the given metric is a function $d : \Omega \times \Omega \to \mathbb{R}$, the ordered pair (ω, ν) is associated with the value $d(\omega, \nu)$ in $\mathbb{R}$, according to the Function Criterion. This value satisfies $d(\omega, \nu) \geq_{\mathbb{R}} 0$ because of Property 1 of a metric, and the initial assumption $\omega \neq \nu$ implies $\neg d(\omega, \nu) = 0$ with Property 2 of a metric and the Law of Contraposition. As the preceding inequality gives the disjunction $d(\omega, \nu) >_{\mathbb{R}} 0 \vee d(\omega, \nu) = 0$ because of the definition of an induced irreflexive partial ordering, we see in light of the preceding negation that its second part is false, so that its first part $d(\omega, \nu) >_{\mathbb{R}} 0$ must be true. We consider now the real number $\epsilon = \frac{1}{2} \cdot d(\omega, \nu)$, where the evident fact $\frac{1}{2} >_{\mathbb{R}} 0$ implies in connection with the preceding inequality and Monotony Law for $\cdot$ and $<$ that $\epsilon >_{\mathbb{R}} 0$ is true. Consequently, we may form the open balls $B_d(\omega, \epsilon)$ and $B_d(\nu, \epsilon)$, which are elements of the basis $\mathcal{B}_\Omega^{(d)}$ according to Theorem 11.71. Since this basis is included in the metric topology $\mathcal{O}_d$ according to Proposition 11.55, it follows by definition of a subset that the two open balls ore open sets in Ω, that is,

$$B_d(\omega, \epsilon), B_d(\nu, \epsilon) \in \mathcal{O}_d. \tag{11.498}$$

Next, we prove that these two open balls are disjoint, that is,

$$B_d(\omega, \epsilon) \cap B_d(\nu, \epsilon) = \emptyset. \tag{11.499}$$

To establish this equation, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall x\,(x \notin B_d(\omega, \epsilon) \cap B_d(\nu, \epsilon)\,), \tag{11.500}$$

letting x be arbitrary. We prove $x \notin B_d(\omega, \epsilon) \cap B_d(\nu, \epsilon)$ by establishing the contradiction

$$2\epsilon <_{\mathbb{R}} 2\epsilon \,\wedge\, \neg 2\epsilon <_{\mathbb{R}} 2\epsilon, \tag{11.501}$$

assuming the negation $\neg x \notin B_d(\omega, \epsilon) \cap B_d(\nu, \epsilon)$ to be true, so that the Double Negation Law gives us $x \in B_d(\omega, \epsilon) \cap B_d(\nu, \epsilon)$. Then, $x \in B_d(\omega, \epsilon)$ and $x \in B_d(\nu, \epsilon)$ are true by definition of the intersection of two sets, and the definition of an open ball yields therefore the inequalities $d(x, \omega) <_{\mathbb{R}} \epsilon$

as well as $d(x,\nu) <_{\mathbb{R}} \epsilon$. As $d(x,\omega) = d(\omega,x)$ holds by Property 3 of a metric, the former inequality implies $d(\omega,x) <_{\mathbb{R}} \epsilon$ via substitution. In view of the Additivity of $<$-inequalities for ordered integral domains, we obtain now from the two inequalities $d(\omega,x) <_{\mathbb{R}} \epsilon$ and $d(x,\nu) <_{\mathbb{R}} \epsilon$

$$d(\omega,x) +_{\mathbb{R}} d(x,\nu) <_{\mathbb{R}} \epsilon +_{\mathbb{R}} \epsilon \quad [= 2\epsilon].$$

Because $d(\omega,\nu) \leq_{\mathbb{R}} d(\omega,x) +_{\mathbb{R}} d(x,\nu)$ also holds due to the Triangle Inequality (i.e., Property 4 of a metric), we get $d(\omega,\nu) <_{\mathbb{R}} 2\epsilon$ by applying the Transitivity Formula for $\leq$ and $<$. Recalling now our choice $\epsilon = \frac{1}{2} \cdot d(\omega,\nu)$, which we may evidently write also as $2\epsilon = d(\omega,\nu)$, it follows via substitution that $2\epsilon <_{\mathbb{R}} 2\epsilon$ is true. Because $\neg 2\epsilon <_{\mathbb{R}} 2\epsilon$ is also true by virtue of the irreflexivity of $<_{\mathbb{R}}$, we arrived at the desired contradiction (11.501). We thus proved $x \notin B_d(\omega,\epsilon) \cap B_d(\nu,\epsilon)$, and since x is arbitrary, we therefore conclude that the universal sentence (11.500) holds. Consequently, the equivalent equation (11.499) is also true.

Let us now recall that every open ball contains its center (see Corollary 8.66), so that we find

$$\omega \in B_d(\omega,\epsilon) \ \wedge \ \nu \in B_d(\nu,\epsilon). \tag{11.502}$$

Then, the conjunction of (11.498), (11.499) and (11.502) shows us that the existential sentence in (11.497) is true. As ω and ν were arbitrary, we may therefore conclude that the universal sentence (11.497) holds, which means that $(\Omega, \mathcal{O}_d)$ is a Hausdorff space. Initially, the sets Ω and $\mathcal{O}$ were also arbitrary, so that the theorem follows finally to be true. □

Note 11.27. The topological spaces $(\mathbb{R}, \mathcal{O}_{d_{\mathbb{R}}})$ and $(\mathbb{R}, \mathcal{O}_{d_{\mathbb{R}^n}})$ for any $n \in \mathbb{N}$ are Hausdorff spaces.

Theorem 11.74 (Equivalence of separable metric spaces and second-countable topological spaces). *The following sentences are true.*

a) For any separable metric space (Ω, d), where d induces the metric topology $\mathcal{O}_d$ on Ω, and for any countable, dense subset C of Ω, there exists a unique set (system)

$$\mathcal{B}_{\Omega}^{(d,C)} = \{B_d(c_0,q) : c_0 \in C \wedge q \in \mathbb{Q}_+\} \tag{11.503}$$

consisting of all open balls $B_d(c_0,q)$ in $\{B_d(\omega_0,\epsilon) \,|\, \omega_0 \in \Omega \wedge \epsilon \in \mathbb{R}_+\}$ with center $c_0 \in C$ and radius $q \in \mathbb{Q}_+$, and this set is a countable basis for a topology on Ω generating $\mathcal{O}_d$.

b) For any set Ω and any metric d on Ω inducing the metric topology $\mathcal{O}_d$ on Ω, the metric space (Ω, d) is separable iff the topological space $(\Omega, \mathcal{O}_d)$ is second-countable.

Proof. Concerning a), we take arbitrary sets Ω, d and C, we assume that (Ω, d) is a separable metric space, so that there exists (by definition) a subset of Ω which is countable and dense (with respect to the induced metric topology $\mathcal{O}_d$). We now let C be an arbitrary countable, dense subset of Ω. According to Theorem 11.71, the metric d determines then the unique basis $\mathcal{B}_\Omega^{(d)}$ that generates $\mathcal{O}_d$. This basis consists of all the open balls $B_d(\omega_0, \epsilon)$ with $\omega_0 \in \Omega$ and $\epsilon \in \mathbb{R}_+$. Evidently, we can utilize now the Axiom of Specification in connection with the Equality Criterion for sets to prove the unique existence of a set $\mathcal{B}_\Omega^{(d,C)}$ satisfying

$$\forall A\,(A \in \mathcal{B}_\Omega^{(d,C)} \Leftrightarrow [A \in \mathcal{B}_\Omega^{(d)} \wedge \exists c_0, q\,(c_0 \in C \wedge q \in \mathbb{Q}_+ \wedge B_d(c_0, q) = A)]). \tag{11.504}$$

Let us observe here for later reference that $A \in \mathcal{B}_\Omega^{(d,C)}$ implies in particular $A \in \mathcal{B}_\Omega^{(d)}$ for any A, so that the inclusion

$$\mathcal{B}_\Omega^{(d,C)} \subseteq \mathcal{B}_\Omega^{(d)} \tag{11.505}$$

follows to be true by definition of a subset. To establish the countability of the set $\mathcal{B}_\Omega^{(d,C)}$, we now apply Function definition by replacement to establish the mapping

$$b : C \times \mathbb{Q}_+ \to \mathcal{B}_\Omega^{(d,C)}, \qquad (c_0, q) \mapsto B_d(c_0, q). \tag{11.506}$$

For this purpose, we prove the universal sentence

$$\forall z\,(z \in C \times \mathbb{Q}_+ \Rightarrow \exists! Y\,(\exists c_0, q\,(Y = B_d(c_0, q) \wedge (c_0, q) = z))), \tag{11.507}$$

letting $\bar{z}$ be arbitrary and assuming $\bar{z} \in C \times \mathbb{Q}_+$ to be true. By definition of the Cartesian product of two sets, there are then particular elements $\bar{c}_0 \in C$ and $\bar{q} \in \mathbb{Q}_+$ with $(\bar{c}_0, \bar{q}) = \bar{z}$. Since the inclusion $C \subseteq \Omega$ is true by assumption and since the inclusion $\mathbb{Q}_+ \subseteq \mathbb{R}_+$ evidently holds as well, it follows from $\bar{c}_0 \in C$ and $\bar{q} \in \mathbb{Q}_+$ (by definition of a subset) that $\bar{c}_0 \in \Omega$ and $\bar{q} \in \mathbb{R}_+$. Consequently, the open ball $\bar{Y} = B_d(\bar{c}_0, \bar{q})$ is defined, and the existential sentence

$$\exists c_0, q\,(\bar{Y} = B_d(c_0, q) \wedge (c_0, q) = \bar{z})$$

is thus true. Since $\bar{Y}$ satisfies the preceding existential sentence, this proves the existential part of the uniquely existential sentence in (11.507). Regarding the uniqueness part, we now take arbitrary set Y and Y', and we assume that these sets both satisfy the existential sentence with respect to c_0 and q. Thus, there exist on the one hand particular constants $\bar{c}_0, \bar{q}$ satisfying $Y = B_d(\bar{c}_0, \bar{q})$ and $(\bar{c}_0, \bar{q}) = \bar{z}$. On the other hand, there are particular constants $\bar{c}_0', \bar{q}'$ such that $Y' = B_d(\bar{c}_0', \bar{q}')$ and $(\bar{c}_0', \bar{q}') = \bar{z}$. Substitution based on the two equations for $\bar{z}$ yields then first $(\bar{c}_0', \bar{q}') = (\bar{c}_0, \bar{q})$, and subsequently with the Equality Criterion for ordered pairs $\bar{c}_0' = \bar{c}_0$ as well as $\bar{q}' = \bar{q}$. Then, further substitutions based on these two equations give

$$Y' = B_d(\bar{c}_0', \bar{q}') = B_d(\bar{c}_0, \bar{q}) = Y,$$

and therefore the desired $Y' = Y$. As Y and Y' are arbitrary, we may infer from the truth of this equation that the uniqueness part also holds, so that the proof of the uniquely existential sentence in (11.507) is complete. Because $\bar{z}$ is also arbitrary, we may now further conclude that the universal sentence (11.507) is true, which implies then in turn the unique existence of a function b with domain $C \times \mathbb{Q}_+$ such that

$$\forall z\,(z \in C \times \mathbb{Q}_+ \Rightarrow \exists c_0, q\,(b((c_0, q)) = B_d(c_0, q) \,\wedge\, (c_0, q) = z)). \quad (11.508)$$

We now utilize (3.631) to prove that this function b is a surjection (from $C \times \mathbb{Q}_+$) to $\mathcal{B}_\Omega^{(d,C)}$, for which purpose we verify the universal sentence

$$\forall Y\,(Y \in \mathcal{B}_\Omega^{(d,C)} \Leftrightarrow \exists z\,(b(z) = Y)). \quad (11.509)$$

We take an arbitrary set Y and assume first $Y \in \mathcal{B}_\Omega^{(d,C)}$. In view of (11.504), there are then constants, say $\bar{c}_0$ and $\bar{q}$, such that $\bar{c}_0 \in C$, $\bar{q} \in \mathbb{Q}_+$ and $B_d(\bar{c}_0, \bar{q}) = Y$ hold. We therefore obtain $(\bar{c}_0, \bar{q}) \in C \times \mathbb{Q}_+$ $[= \mathrm{dom}(b)]$ by definition of the Cartesian product of two sets. Because of (11.508), there exist then particular constants $\bar{c}_0', \bar{q}'$ satisfying $b((\bar{c}_0, \bar{q})) = B_d(\bar{c}_0', \bar{q}')$ and $(\bar{c}_0', \bar{q}') = (\bar{c}_0, \bar{q})$. Here, the latter equation yields (with the Equality Criterion for ordered pairs) the two equations $\bar{c}_0' = \bar{c}_0$ and $\bar{q}' = \bar{q}$, so that we may write the other equation after substitutions also as

$$b((\bar{c}_0, \bar{q})) = B_d(\bar{c}_0, \bar{q})\ [= Y].$$

The resulting equation $b((\bar{c}_0, \bar{q})) = Y$ clearly demonstrates the existence of a set z with $b(z) = Y$, so that the first part ('$\Rightarrow$') of the equivalence in (11.509) holds. Regarding the second part ('$\Leftarrow$'), we now conversely assume that there exists a set, say $\bar{z}$, such that $b(\bar{z}) = Y$ holds. Thus, $\bar{z}$ is evidently element of the domain $C \times \mathbb{Q}_+$ of the function b, implying with (11.508) the existence of particular constants $\bar{\bar{c}}_0, \bar{\bar{q}}$ with $b((\bar{\bar{c}}_0, \bar{\bar{q}})) = B_d(\bar{\bar{c}}_0, \bar{\bar{q}})$ and $(\bar{\bar{c}}_0, \bar{\bar{q}}) = \bar{z}$. Consequently, we obtain by means of substitution

$$Y = b(\bar{z}) = b((\bar{\bar{c}}_0, \bar{\bar{q}})) = B_d(\bar{\bar{c}}_0, \bar{\bar{q}}).$$

Furthermore, the previously obtained $\bar{z} \in C \times \mathbb{Q}_+$ yields $(\bar{\bar{c}}_0, \bar{\bar{q}}) \in C \times \mathbb{Q}_+$, and therefore (with the definition of the Cartesian product of two sets)

$\bar{\bar{c}}_0 \in C$ as well as $\bar{\bar{q}} \in \mathbb{Q}_+$. The previous two findings clearly show that the existential sentence

$$\exists c_0, q\,(c_0 \in C \,\wedge\, q \in \mathbb{Q}_+ \,\wedge\, B_d(c_0, q) = Y)$$

is true. Moreover, $\bar{\bar{c}}_0 \in C$ as well as $\bar{\bar{q}} \in \mathbb{Q}_+$ evidently imply $\bar{\bar{c}}_0 \in \Omega$ and $\bar{\bar{q}} \in \mathbb{R}_+$, so that the previously established set $Y = B_d(\bar{\bar{c}}_0, \bar{\bar{q}})$ turns out to be an open ball in the set $\mathcal{B}_\Omega^{(d)}$ of open balls with center in Ω and radius in $\mathbb{R}_+$. Then, the conjunction of this finding $Y \in \mathcal{B}_\Omega^{(d)}$ and the preceding existential sentence implies with (11.504) $Y \in \mathcal{B}_\Omega^{(d,C)}$, as desired. Since Y was arbitrary, we may therefore conclude that the universal sentence (11.509) holds, which in turn implies (as mentioned earlier) that b is a surjection from (the domain) $C \times \mathbb{Q}_+$ to (the range) $\mathcal{B}_\Omega^{(d,C)}$.

Let us now observe in light of the Countability of $\mathbb{Q}$ that the subset $\mathbb{Q}_+$ of $\mathbb{Q}$ is also countable, according to Corollary 4.139. Together with the assumed countability of C, this implies with the Countability of the Cartesian product of two countable sets that $C \times \mathbb{Q}_+$ is countable. Because the disjunction of $C \times \mathbb{Q}_+ = \emptyset$ and $C \times \mathbb{Q}_+ \neq \emptyset$ is true according to the Law of the Excluded Middle, we may now prove by cases that the set $\mathcal{B}_\Omega^{(d,C)}$ is also countable.

The first case $[\mathrm{dom}(b) =]\ C \times \mathbb{Q}_+ = \emptyset$ implies $[\mathcal{B}_\Omega^{(d,C)} =]\ \mathrm{ran}(b) = \emptyset$ with (3.118), with the consequence that $\mathcal{B}_\Omega^{(d,C)} = \emptyset$. This finding in turn implies with the Countability Criterion (4.653) the desired result that $\mathcal{B}_\Omega^{(d,C)}$ is countable.

In the second case $C \times \mathbb{Q}_+ \neq \emptyset$, the countability of $C \times \mathbb{Q}_+$ implies with the Countability Criterion (4.653) that there exists a surjection from $\mathbb{N}$ to $C \times \mathbb{Q}_+$, say $\bar{f}$. Consequently, the composition of the previously established surjection $b : C \times \mathbb{Q}_+ \twoheadrightarrow \mathcal{B}_\Omega^{(d,C)}$ and $\bar{f} : \mathbb{N} \twoheadrightarrow C \times \mathbb{Q}_+$ yields the surjection $b \circ \bar{f} : \mathbb{N} \twoheadrightarrow \mathcal{B}_\Omega^{(d,C)}$ with the Surjectivity of the composition of two surjections. Thus, there exists a surjection from $\mathbb{N}$ to $\mathcal{B}_\Omega^{(d,C)}$, which fact implies (again with the aforementioned Countability Criterion) that $\mathcal{B}_\Omega^{(d,C)}$ is a countable set, completing the proof by cases.

Our next task is to prove that $\mathcal{B}_\Omega^{(d,C)}$ is a basis for a topology on Ω that generates $\mathcal{O}_d$. To do this, we use the Characterization of the basis generating a given topology and verify accordingly

$$\forall B, \omega\,([B \in \mathcal{O}_d \wedge \omega \in B] \Rightarrow \exists A\,(A \in \mathcal{B}_\Omega^{(d,C)} \wedge \omega \in A \wedge A \subseteq B)). \quad (11.510)$$

We let B and ω be arbitrary, assume $B \in \mathcal{O}_d$ as well as $\omega \in B$ to be true, and demonstrate the existence of a set A in $\mathcal{B}_\Omega^{(d,C)}$ such that this set contains ω and is included in B. As the basis

$$\mathcal{B}_{\Omega}^{(d)} = \{B_d(\omega_0, \epsilon) : \omega_0 \in \Omega \wedge \epsilon \in \mathbb{R}_+\}$$

generates $\mathcal{O}_d$, it follows from $B \in \mathcal{O}_d$ and then from $\omega \in B$ that there exists a particular set $\bar{A} \in \mathcal{B}_{\Omega}^{(d)}$ with $\omega \in \bar{A}$ and $\bar{A} \subseteq B$, according to (11.365). Thus, $\bar{A}$ is some open ball, i.e. there exists a particular element $\bar{\omega}_0 \in \Omega$ as well as a particular number $\bar{\epsilon} \in \mathbb{R}_+$ such that $\bar{A} = B_d(\bar{\omega}_0, \bar{\epsilon})$, according to Theorem 11.71b). We therefore have

$$B_d(\bar{\omega}_0, \bar{\epsilon}) \in \mathcal{B}_{\Omega}^{(d)} \ \wedge \ \omega \in B_d(\bar{\omega}_0, \bar{\epsilon}) \ \wedge \ B_d(\bar{\omega}_0, \bar{\epsilon}) \subseteq B. \tag{11.511}$$

Next, we show that there is a set $A \in \mathcal{B}_{\Omega}^{(d,C)}$ with $\omega \in A$ and $A \subseteq B_d(\bar{\omega}_0, \bar{\epsilon})$. By definition of an open ball, the second part $\omega \in B_d(\bar{\omega}_0, \bar{\epsilon})$ of the conjunction in (11.511) implies $d(\omega, \bar{\omega}_0) <_{\mathbb{R}} \bar{\epsilon}$, which evidently gives

$$0 <_{\mathbb{R}} \bar{\epsilon} -_{\mathbb{R}} d(\omega, \bar{\omega}_0)$$

with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$ then

$$0 <_{\mathbb{R}} \frac{\bar{\epsilon} -_{\mathbb{R}} d(\omega, \bar{\omega}_0)}{2}$$

with the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$. Since $(\mathbb{R}, <_{\mathbb{R}})$ is separably ordered with respect to $\mathbb{Q}$, the latter inequality implies that there exists an element of $\mathbb{Q}$, say $\bar{q}$, such that

$$0 <_{\mathbb{R}} \bar{q} <_{\mathbb{R}} \frac{\bar{\epsilon} -_{\mathbb{R}} d(\omega, \bar{\omega}_0)}{2}. \tag{11.512}$$

Here, we notice on the one hand that $\bar{q}$ is a positive rational number, i.e. $\bar{q} \in \mathbb{Q}_+$, so that $\bar{q} \in \mathbb{R}_+$ is evidently also true. Moreover, we observe in light of the previous assumptions (and Property 1 of a topology on Ω) that

$$[\omega \in] \ B \in \mathcal{O}_d \ [\subseteq \mathcal{P}(\Omega)]$$

gives $[\omega \in] \, B \in \mathcal{P}(\Omega)$ by definition of a subset, then $[\omega \in] \, B \subseteq \Omega$ by definition of a power set, and therefore $\omega \in \Omega$ (using again by definition of a subset). The previous findings $\omega \in \Omega$ and $\bar{q} \in \mathbb{R}_+$ give rise to the open ball $B_d(\omega, \bar{q})$ in the basis $\mathcal{B}_{\Omega}^{(d)}$, which is included in the topology $\mathcal{O}_d$ generated by that basis, according to Proposition 11.55. We therefore obtain $B_d(\omega, \bar{q}) \in \mathcal{O}_d$ with the definition of a subset. Since any open ball contains its center according to Corollary 8.66, we have in addition $B_d(\omega, \bar{q}) \neq \emptyset$. Recalling now the assumption that C is a dense subset of

Ω with respect to $\mathcal{O}_d$, the previous two findings imply with the Denseness Criterion of topological spaces that $C \cap B_d(\omega, \bar{q}) \neq \emptyset$ is true. Thus, there exists an element in $C \cap B_d(\omega, \bar{q})$, say $\bar{z}$, and the definition of the intersection of two sets gives us then $\bar{z} \in C$ and $\bar{z} \in B_d(\omega, \bar{q})$. By definition of an open ball, this yields first $d(\bar{z}, \omega) <_{\mathbb{R}} \bar{q}$ and subsequently with Property 3 of a metric

$$d(\omega, \bar{z}) <_{\mathbb{R}} \bar{q}. \tag{11.513}$$

The latter implies $\omega \in B_d(\bar{z}, \bar{q})$ with the definition of an open ball, where $B_d(\bar{z}, \bar{q})$ evidently is an element of $\mathcal{B}_\Omega^{(d,C)}$ because of $\bar{z} \in C$ and $\bar{q} \in \mathbb{Q}_+$. We now show that the inclusion $B_d(\bar{z}, \bar{q}) \subseteq B_d(\bar{\omega}_0, \bar{\epsilon})$ holds. Applying the definition of a subset, we let x be arbitrary and assume $x \in B_d(\bar{z}, \bar{q})$ to be true, which means (by definition of an open ball)

$$d(x, \bar{z}) <_{\mathbb{R}} \bar{q}. \tag{11.514}$$

The desired consequent to be established is then $x \in B_d(\bar{\omega}_0, \bar{\epsilon})$, or equivalently $d(x, \bar{\omega}_0) <_{\mathbb{R}} \bar{\epsilon}$. Applying twice the Triangle Inequality (i.e., Property 4 of a metric), we obtain

$$d(x, \bar{\omega}_0) \leq_{\mathbb{R}} d(x, \omega) +_{\mathbb{R}} d(\omega, \bar{\omega}_0) \tag{11.515}$$

as well as

$$d(x, \omega) \leq_{\mathbb{R}} d(x, \bar{z}) +_{\mathbb{R}} d(\bar{z}, \omega). \tag{11.516}$$

Then, the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ to (11.516) gives

$$d(x, \omega) +_{\mathbb{R}} d(\omega, \bar{\omega}_0) \leq_{\mathbb{R}} d(x, \bar{z}) +_{\mathbb{R}} d(\bar{z}, \omega) +_{\mathbb{R}} d(\omega, \bar{\omega}_0),$$

and the conjunction of this and (11.515) leads with the transitivity of the standard total ordering $\leq_{\mathbb{R}}$ to

$$d(x, \bar{\omega}_0) \leq_{\mathbb{R}} d(x, \bar{z}) +_{\mathbb{R}} d(\bar{z}, \omega) +_{\mathbb{R}} d(\omega, \bar{\omega}_0). \tag{11.517}$$

The inequality (11.514) yields with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$

$$d(x, \bar{z}) +_{\mathbb{R}} d(\bar{z}, \omega) <_{\mathbb{R}} \bar{q} +_{\mathbb{R}} d(\bar{z}, \omega), \tag{11.518}$$

and inequality (11.513) implies $d(\bar{z}, \omega) <_{\mathbb{R}} \bar{q}$ with the symmetry (i.e., Property 3) of a metric and therefore (using again the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$)

$$d(\bar{z}, \omega) +_{\mathbb{R}} \bar{q} <_{\mathbb{R}} \bar{q} +_{\mathbb{R}} \bar{q} \quad (= 2\bar{q}). \tag{11.519}$$

The conjunction of the two inequalities (11.518) and (11.519) in turn implies with the commutativity of $+_{\mathbb{R}}$ and the transitivity of the standard linear ordering $<_{\mathbb{R}}$

$$d(x, \bar{z}) +_{\mathbb{R}} d(\bar{z}, \omega) <_{\mathbb{R}} 2\bar{q}$$

Applying now the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$ again results in

$$d(x,\bar{z}) +_{\mathbb{R}} d(\bar{z},\omega) +_{\mathbb{R}} d(\omega,\bar{\omega}_0) <_{\mathbb{R}} 2\bar{q} +_{\mathbb{R}} d(\omega,\bar{\omega}_0).$$

Combining this with (11.517) leads, with the Transitivity Formula for $\leq$ and $<$, to

$$d(x,\bar{\omega}_0) <_{\mathbb{R}} 2\bar{q} +_{\mathbb{R}} d(\omega,\bar{\omega}_0). \tag{11.520}$$

Now, as (11.512) evidently implies with the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$

$$2\bar{q} <_{\mathbb{R}} \bar{\epsilon} -_{\mathbb{R}} d(\omega,\bar{\omega}_0),$$

which finding in turn implies (with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$)

$$2\bar{q} +_{\mathbb{R}} d(\omega,\bar{\omega}_0) <_{\mathbb{R}} \bar{\epsilon} -_{\mathbb{R}} d(\omega,\bar{\omega}_0) +_{\mathbb{R}} d(\omega,\bar{\omega}_0). \quad [=\bar{\epsilon}],$$

The conjunction of this and (11.520) gives with the transitivity of $<_{\mathbb{R}}$ the inequality $d(x,\bar{\omega}_0) <_{\mathbb{R}} \bar{\epsilon}$, which then yields the desired $x \in B_d(\bar{\omega}_0,\bar{\epsilon})$ with the definition of an open ball. We thus showed that $x \in B_d(\bar{z},\bar{q})$ implies $x \in B_d(\bar{\omega}_0,\bar{\epsilon})$, and because x is arbitrary, we may infer from this implication the truth of the inclusion $B_d(\bar{z},\bar{q}) \subseteq B_d(\bar{\omega}_0,\bar{\epsilon})$. The conjunction of this inclusion and the inclusion $B_d(\bar{\omega}_0,\bar{\epsilon}) \subseteq B$ in (11.510) implies furthermore $B_d(\bar{z},\bar{q}) \subseteq B$ with (2.13). Together with the already established $B_d(\bar{z},\bar{q}) \in \mathcal{B}_\Omega^{(d,C)}$ and $\omega \in B_d(\bar{z},\bar{q})$, this proves the existence of a set A satisfying $A \in \mathcal{B}_\Omega^{(d,C)}$, $\omega \in A$ and $A \subseteq B$, so that the implication in (11.510) is true. Since B and ω are arbitrary, we therefore conclude that the universal sentence (11.510) holds, which implies now the desired results that $\mathcal{B}_\Omega^{(d,C)}$ is a basis for a topology on Ω generating $\mathcal{O}_d$.

Initially, Ω ,d and C were arbitrary sets, so that the universal sentence a) follows to be true.

Concerning b), we let Ω and d be arbitrary sets such that d is a metric on Ω. We then denote by $\mathcal{O}_d$ the topology on Ω induced by that metric. To establish the first part ('$\Rightarrow$') of the stated equivalence, we assume the metric space (Ω, d) to be separable. Consequently, there exists a subset of Ω that is both countable and dense (with respect to $\mathcal{O}_d$), say $\bar{C}$. According to a), there exists then a basis $\mathcal{B}_\Omega^{(d,\bar{C})}$ for a topology on Ω generating $\mathcal{O}_d$, so that the topological space $(\Omega, \mathcal{O}_d)$ is second-countable by definition. Thus, the implication '$\Rightarrow$' is true.

To prove the converse implication '$\Leftarrow$', we assume now the topological space $(\Omega, \mathcal{O}_d)$ to be second-countable, so that there exists a countable basis generating $\mathcal{O}_d$, say $\bar{\mathcal{K}}_\Omega^{(c)}$. Consequently, the set difference $\bar{\mathcal{K}}_\Omega^{(c)} \setminus \{\emptyset\}$ is also a basis for a topology on Ω according to Exercise 11.23. Furthermore, this set difference constitutes a subset of the countable basis $\bar{\mathcal{K}}_{\Omega}^{(c)}$ because of (2.125), and is therefore itself countable in view of (4.655). Since this countable basis $\bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\}$ is nonempty due to (11.364), we see in light of the Countability Criterion (4.653) that there exists a particular surjection

$$\bar{f} : \mathbb{N} \twoheadrightarrow \bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\}.$$

Moreover, we note that $\emptyset \notin \bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\}$ is true because of (2.179), so that the Axiom of Choice gives us also a particular function

$$\bar{g} : \bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\} \to \bigcup(\bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\}) \tag{11.521}$$

such that

$$\forall B\,(B \in \bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\} \Rightarrow \bar{g}(B) \in B). \tag{11.522}$$

Then, we can write $\bar{g}$ in its surjective form as

$$\bar{g} : \bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\} \twoheadrightarrow \operatorname{ran}(\bar{g}), \tag{11.523}$$

so that the Surjectivity of the composition of two surjections gives

$$\bar{g} \circ \bar{f} : \mathbb{N} \twoheadrightarrow \operatorname{ran}(\bar{g}).$$

Thus, the range of $\bar{g}$ turns out to be a countable set according to the Countability Criterion (4.653), and we show that $\operatorname{ran}(\bar{g})$ is a (countable) subset of Ω. For this purpose, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall y\,(y \in \operatorname{ran}(\bar{g}) \Rightarrow y \in \Omega). \tag{11.524}$$

Letting y be arbitrary and assuming that $y \in \operatorname{ran}(\bar{g})$ is true, we notice in (11.521) that the union $\bigcup(\bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\})$ is a codomain of $\bar{g}$, so that the inclusion

$$\operatorname{ran}(\bar{g} \subseteq \bigcup(\bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\})$$

holds. Therefore, the definition of a subset yields $y \in \bigcup(\bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\})$, which in turn implies with the definition of the union of a set system that there is a particular set $\bar{B} \in \bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\}$ for which $y \in \bar{B}$. By Property 1 of a basis for a topology, we have in addition the inclusion

$$\bar{\mathcal{K}}_{\Omega}^{(c)} \setminus \{\emptyset\} \subseteq \mathcal{P}(\Omega),$$

so that another application of the definition of a subset results in $\bar{B} \in \mathcal{P}(\Omega)$. Consequently, the inclusion

$$\bar{B} \subseteq \Omega$$

is true by definition of a power set, which allows us now to infer from $y \in \bar{B}$ the truth of $y \in \Omega$ (using once again the definition of a subset). This finding proves the implication in (11.524), in which y is arbitrary, so that the universal sentence (11.524) follows to be true. We thus proved that $\mathrm{ran}(\bar{g})$ is a countable subset of Ω.

It now remains for us to prove that $\mathrm{ran}(\bar{g})$ is dense with respect to $\mathcal{O}_d$, i.e. that the closure of $\mathrm{ran}(\bar{g})$ equals Ω. We accomplish this task by means of the Denseness Criterion for topological spaces, through the verification of the universal sentence

$$\forall U\,([U \in \mathcal{O}_d \wedge U \neq \emptyset] \Rightarrow \mathrm{ran}(\bar{g}) \cap U \neq \emptyset). \tag{11.525}$$

We take an arbitrary nonempty open set U of the metric topology $\mathcal{O}_d$. Clearly, there exists then an element in U, say $\bar{\omega}$. According to the Generation of a topology by means of a basis, the truth of $U \in \mathcal{O}_d$ and the truth of $\bar{\omega} \in U$ implies the existence of a particular basis element $\bar{A} \in \bar{\mathcal{K}}_\Omega^{(c)}$ such that $\bar{\omega} \in \bar{A}$ and $\bar{A} \subseteq U$ hold. Thus, the set $\bar{A}$ is clearly nonempty, which fact implies $\bar{A} \notin \{\emptyset\}$ with (2.169), so that $\bar{A} \in \bar{\mathcal{K}}_\Omega^{(c)} \setminus \{\emptyset\}$ follows to be true by definition of a set difference. This means on the one hand that $\bar{A}$ is in the domain of the function $\bar{g}$, so that the associated value $\bar{g}(\bar{A})$ is in the codomain $\mathrm{ran}(\bar{g})$ of $\bar{g}$, as indicated by (11.523). On the other hand, $\bar{A} \in \bar{\mathcal{K}}_\Omega^{(c)} \setminus \{\emptyset\}$ implies $\bar{g}(\bar{A}) \in \bar{A}$ with (11.522), so that the previously found inclusion $\bar{A} \subseteq U$ yields $\bar{g}(\bar{A}) \in U$ by definition of a subset. We thus found $\bar{g}(\bar{A}) \in \mathrm{ran}(\bar{g})$ and $\bar{g}(\bar{A}) \in U$ to be both true, so that $\bar{g}(\bar{A}) \in \mathrm{ran}(\bar{g}) \cap U$ holds by definition of the intersection of two sets. This shows that the intersection $\mathrm{ran}(\bar{g}) \cap U$ is nonempty, so that the proof of the implication in (11.525) is complete. Here, U was arbitrary, which allows us now to infer from this the truth of the universal sentence (11.525) and therefore the truth of the equation

$$\mathrm{cl}(\mathrm{ran}(\bar{g})) = \Omega.$$

This means that $\mathrm{ran}(\bar{g})$ is dense in Ω with respect to $\mathcal{O}_d$. In summary, $\mathrm{ran}(\bar{g})$ constitutes therefore a countable and dense subset of Ω. The existence of such a set in turn demonstrates that the topological space $(\Omega, \mathcal{O}_d)$ is separable, so that the metric space (Ω, d) is separable by definition.

We thus completed the proof of the equivalence, and as Ω and d were arbitrary, we can now conclude that Part b) of the theorem holds, too. $\square$

Note 11.28. In other words, every metrizable topological space $(\Omega, \mathcal{O})$ has a countable basis if there exists a countable, dense subset of Ω.

Note 11.29. Since the topological space $(\mathbb{R}, \mathcal{O}_{d_\mathbb{R}})$ is second-countable (see Note 11.26), we now see in light of the equivalence of separable metric spaces and second-countable topological spaces that the metric space $(\mathbb{R}, d_{\mathbb{R}})$ is separable. By definition, the topological space $(\mathbb{R}, \mathcal{O}_{d_{\mathbb{R}}})$ is therefore separable.

Proposition 11.75. *It is true for any separable metric space (Ω, d) that every nonempty open set U in the induced metric topology $\mathcal{O}_d$ on Ω can be written as the union of a sequence of basis elements in $\mathcal{B}_\Omega^{(d)}$, in the sense*

$$\forall U\,([U \in \mathcal{O}_d \,\wedge\, U \neq \emptyset] \Rightarrow \exists B\,(B : \mathbb{N}_+ \to \mathcal{B}_\Omega^{(d)} \,\wedge\, U = \bigcup_{n=1}^{\infty} B_n)). \quad (11.526)$$

Proof. We let Ω, d and U be arbitrary sets, assume that (Ω, d) is a separable metric space, and assume moreover that U is a nonempty element of $\mathcal{O}_d$. The former assumption implies then (by definition of a separable metric space) the existence of a particular countable, dense subset $\bar{C}$ of Ω. We then see in light of Theorem 11.74 that there exists the countable basis $\mathcal{B}_\Omega^{(d,\bar{C})}$ for a topology on Ω generating $\mathcal{O}_d$. This countable basis is by the way a subset of $\mathcal{B}_\Omega^{(d)}$ according to (11.505). In view of the Characterization of the elements of a topology generated by a basis, it follows from $U \in \mathcal{O}_d$ that there exists a particular subset $\bar{\mathcal{G}} \subseteq \mathcal{B}_\Omega^{(d,\bar{C})}$ with $U = \bigcup \bar{\mathcal{G}}$. The initial assumption $U \neq \emptyset$ gives us then via substitution $\bigcup \bar{\mathcal{G}} \neq \emptyset$, which negation implies $\bar{\mathcal{G}} \neq \emptyset$ with (2.207). Here, since $\mathcal{B}_\Omega^{(d,C)}$ is countable, the subset $\bar{\mathcal{G}}$ is also countable because of Corollary 4.139. Furthermore, because $\bar{\mathcal{G}}$ is nonempty, it follows with the Countability Criterion (4.653) that there exists a surjection from $\mathbb{N}$ to $\bar{\mathcal{G}}$, say $\bar{f} : \mathbb{N} \twoheadrightarrow \bar{\mathcal{G}}$. Recalling that $\mathbb{N}_+$ and $\mathbb{N}$ are equinumerous according to (4.667), so that there is a bijection from $\mathbb{N}_+$ to $\mathbb{N}$, say $\bar{g}$, we have in particular that this function is a surjection, i.e. $\bar{g} : \mathbb{N}_+ \twoheadrightarrow \mathbb{N}$. Due to the Surjectivity of the composition of two surjections, we obtain now the surjection $\bar{f} \circ \bar{g} : \mathbb{N}_+ \twoheadrightarrow \bar{\mathcal{G}}$, which thus constitutes a sequence with range $\bar{\mathcal{G}}$ $[\subseteq \mathcal{B}_\Omega^{(d,\bar{C})} \subseteq \mathcal{B}_\Omega^{(d)}]$, and thus a sequence in $\mathcal{B}_\Omega^{(d)}$, using the definition of a codomain and (2.13). Denoting this sequence $\bar{f} \circ \bar{g}$ by $\bar{B} = (\bar{B}_n)_{n \in \mathbb{N}_+}$, we obtain

$$U = \bigcup \bar{\mathcal{G}} = \bigcup \operatorname{ran}(\bar{B}) = \bigcup_{n=1}^{\infty} \bar{B}_n,$$

using the notation for the union of a sequence of sets. Clearly, this finding demonstrates the proof of the existential sentence in (11.526). Since Ω, d and U were initially arbitrary sets, we may therefore conclude that the proposition is true. □

Exercise 11.32. Show for any second-countable topological space $(\Omega, \mathcal{O})$ that every nonempty open set U can be written as the union $\bigcup_{n=1}^{\infty} B_n$ of some sequence $B = (B_n)_{n\in\mathbb{N}_+}$ of basis elements.

(Hint: Devise a proof in analogy to Proposition 11.75.)

11.7.2. Topological subbases

Besides generating a topology by means of a basis, we may alternatively use the following kind of set system for this purpose.

Definition 11.28 (Subbasis for a topology). For any set Ω we say that a set system $\mathcal{C}_\Omega$ is a *subbasis* (*for a topology*) on Ω iff $\mathcal{C}_\Omega$ is a covering of Ω, i.e. iff

1. $\mathcal{C}_\Omega$ consists of subsets of Ω, that is,
$$\mathcal{C}_\Omega \subseteq \mathcal{P}(\Omega) \tag{11.527}$$
and
2. the union of $\mathcal{C}_\Omega$ equals Ω, that is,
$$\bigcup \mathcal{C}_\Omega = \Omega. \tag{11.528}$$

Theorem 11.76 (Generation of a basis for a topology by means of a subbasis). *For any set Ω and any subbasis $\mathcal{C}_\Omega$ for a topology on Ω, it is true that there exists a unique set (system) $\mathcal{K}_{\mathcal{C}_\Omega}$ consisting of all intersections of finite sequences of subbasis elements, in the sense that*

$$\begin{aligned}&\forall C\,(C \in \mathcal{K}_{\mathcal{C}_\Omega} \\ &\quad\Leftrightarrow \exists m, A\,(m \in \mathbb{N}_+ \,\wedge\, A : \{1,\ldots,m\} \to \mathcal{C}_\Omega \,\wedge\, C = \bigcap \operatorname{ran}(A))]),\end{aligned} \tag{11.529}$$

and this set system $\mathcal{K}_{\mathcal{C}_\Omega}$ is a basis for a topology on Ω.

Proof. We let Ω and $\mathcal{C}_\Omega$ be arbitrary sets and assume $\mathcal{C}_\Omega$ to be a subbasis for a topology on Ω. We can evidently use the Axiom of Specification and the Equality Criterion for sets in the usual way to prove the unique existence of a set $\mathcal{K}_{\mathcal{C}_\Omega}$ satisfying

$$\begin{aligned}&\forall C\,(C \in \mathcal{K}_{\mathcal{C}_\Omega} \Leftrightarrow [C \in \mathcal{P}(\Omega) \,\wedge\, \exists m, A\,(n \in \mathbb{N}_+ \,\wedge\, A : \{1,\ldots,m\} \to \mathcal{C}_\Omega \\ &\quad\wedge\, C = \bigcap \operatorname{ran}(A))]).\end{aligned} \tag{11.530}$$

Letting now C be arbitrary, we see that the assumption $C \in \mathcal{K}_{\mathcal{C}_\Omega}$ implies in particular the existential sentence

$$\exists m, A\,(m \in \mathbb{N}_+ \,\wedge\, A : \{1,\ldots,m\} \to \mathcal{C}_\Omega \,\wedge\, C = \bigcap \operatorname{ran}(A)),$$

so that the first part ('$\Rightarrow$') of the equivalence in (11.529) holds for the set $\mathcal{K}_{\mathcal{C}_\Omega}$. Assuming now conversely the truth of the preceding existential sentence, so that there exist a particular positive natural number $\bar{m}$ and a particular sequence $\bar{A} : \{1, \ldots, \bar{m}\} \to \mathcal{C}_\Omega$ satisfying $C = \bigcap \mathrm{ran}(\bar{A})$, we observe on the one hand that the inclusion $\mathrm{ran}(\bar{A}) \subseteq \mathcal{C}_\Omega$ holds by definition of a codomain. Let us on the other hand recall that the inclusion $\mathcal{C}_\Omega \subseteq \mathcal{P}(\Omega)$ holds according to Property 1 of a subbasis for a topology on Ω. We therefore obtain the inclusion $\mathrm{ran}(\bar{A}) \subseteq \mathcal{P}(\Omega)$ with (2.13). Because the assumed $\bar{m} \in \mathbb{N}_+$ implies according to the definition of an initial segment that $\{1, \ldots, \bar{m}\}$ is nonempty, which set is the domain of $\bar{A}$, we obtain $\mathrm{ran}(\bar{A}) \neq \emptyset$ with (3.119). This finding implies now together with $\mathrm{ran}(\bar{A}) \subseteq \mathcal{P}(\Omega)$ that $[C =] \bigcap \mathrm{ran}(A) \in \mathcal{P}(\Omega)$ is true, according to (3.18). Then, the conjunction of the resulting $C \in \mathcal{P}(\Omega)$ and the assumed existential sentence implies $C \in \mathcal{K}_{\mathcal{C}_\Omega}$ with (11.530), so that the second part ('$\Leftarrow$') of the equivalence in (11.530) also holds, completing the proof of that equivalence. Since C was arbitrary, we may therefore conclude that the set $\mathcal{K}_{\mathcal{C}_\Omega}$ satisfies indeed the universal sentence (11.530).

Next, we verify that $\mathcal{K}_{\mathcal{C}_\Omega}$ is a basis for a topology on Ω. Firstly, we see in light of (11.530) that $C \in \mathcal{K}_{\mathcal{C}_\Omega}$ implies especially $C \in \mathcal{P}(\Omega)$ for any C, so that $\mathcal{K}_{\mathcal{C}_\Omega}$ is included in $\mathcal{P}(\Omega)$ by definition of a subset, as required by Property 1 of a basis for a topology on Ω.

Regarding Property 2, we verify

$$\forall \omega\, (\omega \in \Omega \Rightarrow \exists A\, (A \in \mathcal{K}_{\mathcal{C}_\Omega} \wedge \omega \in A)), \tag{11.531}$$

letting ω be arbitrary and assuming $\omega \in \Omega$ to be true. This assumption implies via substitution $\omega \in \bigcup \mathcal{C}_\Omega$ in view of Property 2 of a subbasis, so that there exists by definition of the union of a set system a particular subbasis element $\bar{A}_1 \in \mathcal{C}_\Omega$ such that $\omega \in \bar{A}_1$ holds. Consequently, the singleton $\bar{A} = \{(1, \bar{A}_1)\}$ is a surjection from (the domain) $\{1\} = \{1, \ldots, 1\}$ to (the range) $\{\bar{A}_1\}$, according to (3.640). We therefore obtain

$$\bigcap \mathrm{ran}(\bar{A}) = \bigcap \{\bar{A}_1\} = \bar{A}_1$$

by applying substitution and (2.170). Furthermore, the previously established $\bar{A}_1 \in \mathcal{C}_\Omega$ implies $[\mathrm{ran}(\bar{A}) =] \{\bar{A}_1\} \subseteq \mathcal{C}_\Omega$ with (2.184), which shows that $\mathcal{C}_\Omega$ is a codomain of $\bar{A}$. We thus established the existence of a positive natural number m and of a sequence $A : \{1, \ldots, m\} \to \mathcal{C}_\Omega$ satisfying $\bar{A}_1 = \bigcap \mathrm{ran}(\bar{A})$, so that $\bar{A}_1 \in \mathcal{K}_{\mathcal{C}_\Omega}$ follows to be true with (11.529). Recalling $\omega \in \bar{A}_1$, the preceding finding in turn demonstrates the truth of the existential sentence in (11.531). Since ω was arbitrary, we may therefore conclude that the universal sentence (11.531) is true, which means that

Property 2 of a basis for a topology on Ω is satisfied by $\mathcal{K}_{\mathcal{C}_\Omega}$.

Next, we establish Property 3 by proving

$$\forall A_1, A_2, \omega\,([A_1, A_2 \in \mathcal{K}_{\mathcal{C}_\Omega} \wedge \omega \in A_1 \wedge \omega \in A_2] \\ \Rightarrow \exists A_3\,(A_3 \in \mathcal{K}_{\mathcal{C}_\Omega} \wedge \omega \in A_3 \wedge A_3 \subseteq A_1 \cap A_2)). \tag{11.532}$$

We take arbitrary A_1, A_2 and ω such that $A_1 \in \mathcal{K}_{\mathcal{C}_\Omega}$, $A_2 \in \mathcal{K}_{\mathcal{C}_\Omega}$, $\omega \in A_1$ and $\omega \in A_2$ are all true. In view of (11.529), the first two assumptions imply the existence of two particular positive natural numbers $\bar{n}_1$ and $\bar{n}_2$ as well as the existence of two particular sequences $\bar{B}_1 : \{1, \ldots, \bar{n}_1\} \to \mathcal{C}_\Omega$ and $\bar{B}_2 : \{1, \ldots, \bar{n}_2\} \to \mathcal{C}_\Omega$ such that $A_1 = \bigcap \mathrm{ran}(\bar{B}_1)$ and $A_2 = \bigcap \mathrm{ran}(\bar{B}_2)$ hold. Thus, we may write for the intersection of these two sets

$$A_1 \cap A_2 = \left[\bigcap \mathrm{ran}(\bar{B}_1)\right] \cap \left[\bigcap \mathrm{ran}(\bar{B}_2)\right] = \left(\bigcap_{j=1}^{\bar{n}_1} \bar{B}_1(j)\right) \cap \left(\bigcap_{j=1}^{\bar{n}_2} \bar{B}_2(j)\right). \tag{11.533}$$

Here, we can utilize Proposition 4.85 to define sequence $\bar{B} = (\bar{B}_i \,|\, i \in \{1,2\})$ having the terms $\bar{B}_1$ and $\bar{B}_2$ and in addition the sequence $\bar{n} = (\bar{n}_i \,|\, i \in \{1,2\})$ in $\mathbb{N}_+$ with terms $\bar{n}_1, \bar{n}_2 \in \mathbb{N}_+$. Thus, $\mathbb{N}_+$ is codomain of $\bar{n}$, satisfying by definition $\mathrm{ran}(\bar{n}) \subseteq \mathbb{N}_+$, for which the inclusion $\mathbb{N}_+ \subseteq \mathbb{N}$ also holds in view of (2.308). Consequently, we obtain the further inclusion $\mathrm{ran}(\bar{n}) \subseteq \mathbb{N}$ by using (2.13), so that $\bar{n}$ is a sequence also in $\mathbb{N}$, i.e. $\bar{n} : \{1,2\} \to \mathbb{N}$. As the sequence $\bar{B} = (\bar{B}_i \,|\, i \in \{1,2\})$ clearly satisfies $\bar{B}_i : \{1, \ldots, \bar{n}_i\} \to \mathcal{C}_\Omega$ for all $i \in \{1,2\}$, we can now apply Stacking of a finite sequence of finite sequences to define the new, stacked sequence $\bar{C}$ in $\mathcal{C}_\Omega$ with domain $\{1, \ldots, \sum_{i=1}^{2} \bar{n}_i\} = \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ and terms satisfying

$$\forall k\,(k \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\} \\ \Rightarrow \exists i, j\,(i \in \{1,2\} \wedge j \in \{1, \ldots, \bar{n}_i\} \wedge \bar{C}_k = \bar{B}_i(j))), \tag{11.534}$$

$$\forall i, j\,([i \in \{1,2\} \wedge j \in \{1, \ldots, \bar{n}_i\}] \\ \Rightarrow \exists k\,(k \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\} \wedge \bar{B}_i(j) = \bar{C}_k)), \tag{11.535}$$

where we applied (4.242) and (5.413) to simplify the notation. Next, we apply the Equality Criterion for sets to establish the equation

$$\left(\bigcap_{j=1}^{\bar{n}_1} \bar{B}_1(j)\right) \cap \left(\bigcap_{j=1}^{\bar{n}_2} \bar{B}_2(j)\right) = \bigcap_{k=1}^{\bar{n}_1+\bar{n}_2} \bar{C}_k, \tag{11.536}$$

via the verification of the equivalent universal sentence

$$\forall y \left(y \in \left(\bigcap_{j=1}^{\bar{n}_1} \bar{B}_1(j)\right) \cap \left(\bigcap_{j=1}^{\bar{n}_2} \bar{B}_2(j)\right) \Leftrightarrow y \in \bigcap_{k=1}^{\bar{n}_1+\bar{n}_2} \bar{C}_k\right). \tag{11.537}$$

Letting y be arbitrary and assuming first

$$y \in \left(\bigcap_{j=1}^{\bar{n}_1} \bar{B}_1(j)\right) \cap \left(\bigcap_{j=1}^{\bar{n}_2} \bar{B}_2(j)\right) \tag{11.538}$$

to be true, we now obtain with the definition of the intersection of two sets $y \in \bigcap_{j=1}^{\bar{n}_1} \bar{B}_1(j)$ and $y \in \bigcap_{j=1}^{\bar{n}_2} \bar{B}_2(j)$. We then see in light of the Characterization of the intersection of a family of sets that the two universal sentences

$$\forall j \left(j \in \{1, \ldots, \bar{n}_1\} \Rightarrow y \in \bar{B}_1(j)\right) \tag{11.539}$$

$$\forall j \left(j \in \{1, \ldots, \bar{n}_2\} \Rightarrow y \in \bar{B}_2(j)\right) \tag{11.540}$$

are true. Based on these sentences, we may prove subsequently that the universal sentence

$$\forall k \left(k \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\} \Rightarrow y \in \bar{C}_k\right) \tag{11.541}$$

also holds. Letting $k \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ be arbitrary, it follows from this with (11.534) that there exist particular indexes $I \in \{1, 2\}$ and $J \in \{1, \ldots, \bar{n}_I\}$ such that $\bar{C}_k = \bar{B}_I(J)$. Here, the definition of a pair gives us the true disjunction $I = 1 \vee I = 2$, which we use to prove $y \in \bar{C}_k$ by cases. The first case $I = 1$ yields via substitution $J \in \{1, \ldots, \bar{n}_1\}$ and then with (11.539)

$$y \in \bar{B}_1(J) \quad [= \bar{B}_I(J) = \bar{C}_k].$$

Similarly, the second $I = 2$ gives $J \in \{1, \ldots, \bar{n}_2\}$ and now with (11.540)

$$y \in \bar{B}_2(J) \quad [= \bar{B}_I(J) = \bar{C}_k].$$

We thus find $y \in \bar{C}_k$ to be true in both cases, so that the implication in (11.541) holds. Since k is arbitrary, we may then infer from the truth of this implication the truth of the universal sentence (11.541), which in turn implies (due to the Characterization of the intersection of a family of sets)

$$y \in \bigcap_{k=1}^{\bar{n}_1+\bar{n}_2} \bar{C}_k. \tag{11.542}$$

This is the desired consequent of the first part ('$\Rightarrow$') of the equivalence in (11.537), which implication thus holds.

Concerning the second implication ('$\Leftarrow$'), we assume (11.542), so that the universal sentence (11.541) evidently follows to be true. This sentence allows us to establish also the two universal sentences (11.539) and (11.540). Letting $j_1 \in \{1, \ldots, \bar{n}_1\}$ and $j_2 \in \{1, \ldots, \bar{n}_2\}$ be arbitrary and observing the truth of $1, 2 \in \{1, 2\}$, we obtain with (11.535) particular indexes $k_1, k_2 \in \{1, \ldots, \bar{n}_1 + \bar{n}_2\}$ satisfying $\bar{B}_1(j_1) = \bar{C}_{k_1}$ as well as $\bar{B}_2(j_2) = \bar{C}_{k_2}$. Therefore,

$$
\begin{aligned}
y &\in \bar{C}_{k_1} \quad [= \bar{B}_1(j_1)] \\
y &\in \bar{C}_{k_2} \quad [= \bar{B}_2(j_2)]
\end{aligned}
$$

follow to be true with (11.541). Here, the resulting $y \in \bar{B}_1(j_1)$ proves the implication in (11.539) and the other finding $y \in \bar{B}_2(j_2)$ proves the implication in (11.540). Since j_1 and j_2 were initially arbitrary, we may infer from the truth of these implications the truth of the corresponding universal sentences (11.539) and (11.540), which then further imply, respectively, $y \in \bigcap_{j=1}^{\bar{n}_1} \bar{B}_1(j)$ and $y \in \bigcap_{j=1}^{\bar{n}_2} \bar{B}_2(j)$. Finally, the simultaneous truth of the latter two findings in turn implies (11.538), so that the second part of the equivalence in (11.537) also holds. As y was arbitrary, we may therefore conclude that the universal sentence (11.537) is true, and this sentence yields then the desired equation (11.536).

This equation can now be combined with the equation (11.533), with the consequence that

$$
\begin{aligned}
A_1 \cap A_2 &= \bigcap_{k=1}^{\bar{n}_1 + \bar{n}_2} \bar{C}_k \\
&= \bigcap \operatorname{ran}(\bar{C}).
\end{aligned}
$$

Because the latter equation is based on the evidently positive natural number $\bar{n}_1 + \bar{n}_2$ and on the function $\bar{C} : \{1, \ldots, \bar{n}_1 + \bar{n}_2\} \to \mathcal{C}_\Omega$, we see that the set $\bar{A}_3 = A_1 \cap A_2$ satisfies the existential sentence in (11.529), so that $\bar{A}_3 \in \mathcal{K}_{\mathcal{C}_\Omega}$ follows to be true. Let us recall the assumptions $\omega \in A_1$ and $\omega \in A_2$, so that the definition of the intersection of two sets yields $\omega \in A_1 \cap A_2$ $[= \bar{A}_3]$. Because $\bar{A}_3 \subseteq \bar{A}_3$ holds according to (2.10), substitution based on the preceding equation gives us the inclusion $\bar{A}_3 \subseteq A_1 \cap A_2$. In light of the previous findings, we now see that there exists a set A_3 for which $A_3 \in \mathcal{K}_{\mathcal{C}_\Omega}$, $\omega \in A_3$ and $A_3 \subseteq A_1 \cap A_2$ are all true. This existential sentence proves the implication in (11.532), and as A_1, A_2 and ω were arbitrary, we may infer from the truth of that implication the truth of the universal sentence (11.532). Thus, the set system $\mathcal{K}_{\mathcal{C}_\Omega}$ satisfies Property 3 of a basis for a topology on Ω.

Since Ω and $\mathcal{C}_\Omega$ are arbitrary, we may therefore conclude that the stated theorem is true. $\square$

Definition 11.29 (Topology generated by a subbasis). For any set Ω and any subbasis $\mathcal{C}_\Omega$ for a topology on Ω, we say that the topology

$$\mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega}), \tag{11.543}$$

generated by the basis $\mathcal{K}_{\mathcal{C}_\Omega}$ defined in Theorem 11.76, is the *topology generated by $\mathcal{C}_\Omega$*.

Theorem 11.77 (Characterization of the elements of a topology generated by a subbasis). *The topology generated by any subbasis $\mathcal{C}_\Omega$ on any set Ω contains precisely every set B which is the union of finite intersections of subbasis elements, in the sense that*

$$\begin{aligned}&\forall B\,(B \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega}) \Leftrightarrow \exists \mathcal{G}\,(B = \bigcup \mathcal{G} \,\wedge\, \forall C\,(C \in \mathcal{G} \\ &\quad\Rightarrow \exists m, A\,(m \in \mathbb{N}_+ \,\wedge\, A : \{1,\ldots,m\} \to \mathcal{C}_\Omega \,\wedge\, C = \bigcap \operatorname{ran}(A)))))).\end{aligned} \tag{11.544}$$

Proof. We let Ω and $\mathcal{C}_\Omega$ be arbitrary such that $\mathcal{C}_\Omega$ is a subbasis for a topology on Ω, generating $\mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})$. Next, we also let $\bar{B}$ be arbitrary, and we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\bar{B} \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})$ to be true. According to the Characterization of the elements of a topology generated by a basis, there exists then a set, say $\bar{\mathcal{G}}$, such that the inclusion $\bar{\mathcal{G}} \subseteq \mathcal{K}_{\mathcal{C}_\Omega}$ and the equation

$$\bar{B} = \bigcup \bar{\mathcal{G}} \tag{11.545}$$

are both true. By definition of a subset, we therefore obtain the true universal sentence

$$\forall C\,(C \in \bar{\mathcal{G}} \Rightarrow C \in \mathcal{K}_{\mathcal{C}_\Omega}), \tag{11.546}$$

which we now use to establish

$$\forall C\,(C \in \bar{\mathcal{G}} \Rightarrow \exists m, A\,(m \in \mathbb{N}_+ \,\wedge\, A : \{1,\ldots,m\} \to \mathcal{C}_\Omega \,\wedge\, C = \bigcap \operatorname{ran}(A))). \tag{11.547}$$

We let C be arbitrary and assume $C \in \bar{\mathcal{G}}$ to be true, so that (11.546) yields $C \in \mathcal{K}_{\mathcal{C}_\Omega}$. Then, according to the Generation of a basis for a topology by means of a subbasis, this finding implies the truth of the existential sentence in (11.547). Since C was arbitrary, we may therefore conclude that the universal sentence (11.547) holds, as suggested. Alongside the previously obtained equation (11.545), this demonstrates the truth of the desired existential sentence in (11.544), proving the first part of the equivalence.

To establish the second part ('$\Leftarrow$'), we now conversely assume that existential sentence to be true, so that there exists a particular set $\bar{\mathcal{G}}$ satisfying the equation (11.545) as well as the universal sentence (11.547). Based on the latter sentence, we may now prove the universal sentence (11.546), letting $C \in \bar{\mathcal{G}}$ be arbitrary, so that the existential sentence in (11.547) follows to be true. We therefore obtain (with the Generation of a basis for a topology by means of a subbasis) $C \in \mathcal{K}_{\mathcal{C}_\Omega}$, proving indeed (11.546) since C was arbitrary. That universal sentence in turn implies (with the definition of a subset) $\bar{\mathcal{G}} \subseteq \mathcal{K}_{\mathcal{C}_\Omega}$. The conjunction of that inclusion and the already established equation (11.545) shows us (in light of the Characterization of the elements of a topology generated by a basis) that $\bar{B} \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})$ is true. Thus, the proof of the equivalence in (11.544) is complete.

Because B, Ω and $\mathcal{C}_\Omega$ were arbitrary, we may now infer from the truth of that equivalence the truth of the stated theorem. □

Proposition 11.78. *It is true for any set Ω that the topology $\mathcal{O}(\mathcal{K}_{C_\Omega})$ generated by any subbasis $\mathcal{C}_\Omega$ includes this subbasis, i.e.*

$$\mathcal{C}_\Omega \subseteq \mathcal{O}(\mathcal{K}_{C_\Omega}). \tag{11.548}$$

Proof. We take arbitrary sets Ω and $\mathcal{C}_\Omega$ where we assume that $\mathcal{C}_\Omega$ is a subbasis for a topology on Ω, generating thus $\mathcal{O}(\mathcal{K}_{C_\Omega})$. To establish the proposed inclusion, we apply the definition of a subset and prove accordingly the equivalent universal sentence

$$\forall B\,(B \in \mathcal{C}_\Omega \Rightarrow B \in \mathcal{O}(\mathcal{K}_{C_\Omega})), \tag{11.549}$$

letting $B \in \mathcal{C}_\Omega$ be arbitrary, and making then use of the Characterization of the elements of a topology generated by a subbasis. Let us define for this purpose the singleton $\bar{\mathcal{G}} = \{B\}$, for which we have

$$B = \bigcup \bar{\mathcal{G}} \tag{11.550}$$

because of (2.199). Then, the singleton $\bar{A} = \{(1, B)\}$ is a function with domain $\{1\}$ and range $\{B\}$ according to Corollary 3.194; here, we may write $\{1\} = \{1, \ldots, \bar{m}\}$ using the notation for initial segments of $\mathbb{N}_+$ and defining $\bar{m} = 1$. Since $B \in \mathcal{C}_\Omega$ implies $\{B\} \subseteq \mathcal{C}_\Omega$ with (2.184), it follows with the definition of a codomain that $\bar{A} : \{1, \ldots, \bar{m}\} \to \mathcal{C}_\Omega$, where $\bar{m} \in \mathbb{N}_+$ is evidently true. In addition, we obtain the equation $B = \bigcap\{B\}$ with (2.170), so that substitution yields $B = \bigcap \operatorname{ran}(\bar{A})$. We are now in a position to establish the universal sentence

$$\forall C\,(C \in \bar{\mathcal{G}} \Rightarrow \exists m, A\,(m \in \mathbb{N}_+ \wedge A : \{1, \ldots, m\} \to \mathcal{C}_\Omega \wedge C = \bigcap \operatorname{ran}(A))). \tag{11.551}$$

We let C be arbitrary and assume $C \in \bar{\mathcal{G}}$ $[= \{B\}]$, so that $C = B$ follows to be true with (2.169). Consequently, substitution yields $C = \bigcap \operatorname{ran}(\bar{A})$; the conjunction of the previously found $\bar{m} \in \mathbb{N}_+$, $\bar{A} : \{1, \ldots, \bar{m}\} \to \mathcal{C}_\Omega$ and the preceding equation demonstrates then the truth of the desired existential sentence in (11.551). Since C was arbitrary, we may therefore conclude that the universal sentence (11.551) holds. Since the conjunction of (11.550) and (11.551) shows us now that the set B satisfies the existential sentence

$$\begin{aligned}&\exists \mathcal{G}\,(B = \bigcup \mathcal{G} \ \wedge\ \forall C\,(C \in \mathcal{G}\\&\quad \Rightarrow \exists m, A\,(m \in \mathbb{N}_+ \ \wedge\ A : \{1, \ldots, m\} \to \mathcal{C}_\Omega \ \wedge\ C = \bigcap \operatorname{ran}(A)))),\end{aligned}$$

we finally obtain the desired $B \in \mathcal{O}(\mathcal{K}_{C_\Omega})$ with the Characterization of the elements of a topology generated by a subbasis. Because B was arbitrary, we may infer from this the truth of (11.549), which universal sentence in turn yields the inclusion (11.548). As Ω and $\mathcal{C}_\Omega$ were initially also arbitrary, the proposed universal sentence follows then to be true. □

Proposition 11.79. *The following sentences hold for any set Ω and any subbasis $\mathcal{C}_\Omega$ for a topology on Ω.*

a) There exists a unique set $\mathcal{V}$ consisting of all topologies on Ω which include $\mathcal{C}_\Omega$, in the sense that

$$\forall \mathcal{O}\,(\mathcal{O} \in \mathcal{V} \Leftrightarrow [\mathcal{O} \text{ is a topology on } \Omega \ \wedge\ \mathcal{C}_\Omega \subseteq \mathcal{O}]). \tag{11.552}$$

b) Then, the intersection $\bigcap \mathcal{V}$ is itself a topology on Ω that includes $\mathcal{C}_\Omega$.

c) Furthermore, the intersection $\bigcap \mathcal{V}$ is the topology generated by $\mathcal{C}_\Omega$, i.e.

$$\bigcap \mathcal{V} = \mathcal{O}(\mathcal{K}_{C_\Omega}). \tag{11.553}$$

d) Moreover, the generated topology $\mathcal{O}(\mathcal{K}_{C_\Omega})$ is the smallest topology on Ω that includes $\mathcal{C}_\Omega$ in the sense that

$$\forall \mathcal{O}\,([\mathcal{O} \text{ is a topology on } \Omega \ \wedge\ \mathcal{C}_\Omega \subseteq \mathcal{O}] \Rightarrow \mathcal{O}(\mathcal{K}_{C_\Omega}) \subseteq \mathcal{O}). \tag{11.554}$$

Proof. We take arbitrary sets Ω and $\mathcal{C}_\Omega$ such that $\mathcal{C}_\Omega$ is a subbasis for a topology on Ω. Then, Part a) and Part b) are proved in analogy to the corresponding parts of Proposition 11.59.

Concerning c), we establish the truth of the two inclusions $\bigcap \mathcal{V} \subseteq \mathcal{O}(\mathcal{K}_{C_\Omega})$ and $\mathcal{O}(\mathcal{K}_{C_\Omega}) \subseteq \bigcap \mathcal{V}$, which will imply the stated equation (11.553) with the Axiom of Extension. On the one hand, we have that the generated topology $\mathcal{O}(\mathcal{K}_{C_\Omega})$ is a topology on Ω which includes its subbasis C_Ω because of Proposition 11.78; therefore, we obtain $\mathcal{O}(\mathcal{K}_{C_\Omega}) \in \mathcal{V}$ with (11.552) and then $\bigcap \mathcal{V} \subseteq \mathcal{O}(\mathcal{K}_{C_\Omega})$ due to (2.92).

On the other hand, we may prove the universal sentence

$$\forall V\,(V \in \mathcal{O}(\mathcal{K}_{C_\Omega}) \Rightarrow V \in \bigcap \mathcal{V}), \tag{11.555}$$

letting V be an arbitrary open set in $\mathcal{O}(\mathcal{K}_{C_\Omega})$. It then follows with the Characterization of the elements of a topology generated by a subbasis that there exists a set, say $\bar{\mathcal{G}}$, such that $V = \bigcup \bar{\mathcal{G}}$ and the universal sentence

$$\forall C\,(C \in \bar{\mathcal{G}} \Rightarrow \exists m, A\,(m \in \mathbb{N}_+ \,\wedge\, A : \{1, \ldots, m\} \to \mathcal{C}_\Omega \,\wedge\, C = \bigcap \mathrm{ran}(A))) \tag{11.556}$$

are both true. We can now prove the universal sentence

$$\forall C\,(C \in \bar{\mathcal{G}} \Rightarrow C \in \bigcap \mathcal{V}), \tag{11.557}$$

letting C be arbitrary and assuming $C \in \bar{\mathcal{G}}$ to hold. In view of (11.556), this assumption implies the existence of a particular positive natural number $\bar{m}$ and of a particular finite sequence $\bar{A} : \{1, \ldots, \bar{m}\} \to \mathcal{C}_\Omega$ with $C = \bigcap \mathrm{ran}(\bar{A})$. Let us observe here that the subbasis $\mathcal{C}_\Omega$ is on the one hand a codomain of $\bar{A}$, so that the range of $\bar{A}$ is included in $\mathcal{C}_\Omega$; on the other hand, the subbasis is included in the topology $\bigcap \mathcal{V}$, as shown in b). Therefore, the range of $\bar{A}$ is also included in the topology $\bigcap \mathcal{V}$, according to (2.13), so that $\bar{A} = (\bar{A}_i \,|\, i \in \{1, \ldots, \bar{m}\})$ is a sequence in the topology $\bigcap \mathcal{V}$. Because of this, we can utilize Exercise 11.18 (based on Property 4 of a topology) to write the intersection

$$C = \bigcap \mathrm{ran}(\bar{A}) = \bigcap_{i=1}^{\bar{m}} \bar{A}_i$$

in terms of the $\bar{m}$-fold repeated binary operation $\bigcap_{i=1}^{\bar{m}} : \bigcap \mathcal{V}^{\{1,\ldots,\bar{m}\}} \to \bigcap \mathcal{V}$. We thus have $C \in \bigcap \mathcal{V}$, proving the implication in (11.557). Because C is arbitrary, we may infer from this implication the truth of the universal sentence (11.557), which now further implies $\bar{\mathcal{G}} \subseteq \bigcap \mathcal{V}$ with the definition of a subset. Because $\bigcap \mathcal{V}$ satisfies especially Property 3 of a topology on Ω, the preceding inclusion yields $\bigcup \bar{\mathcal{G}} \in \bigcap \mathcal{V}$ and then via substitution $V \in \bigcap \mathcal{V}$, as desired. As V was arbitrary, we may therefore conclude that (11.555) is true, which universal sentence in turn implies $\mathcal{O}(\mathcal{K}_{C_\Omega}) \subseteq \bigcap \mathcal{V}$ by definition of a subset. Thus, both of the proposed inclusions hold, so that the stated equation (11.553) follows to be true.

Part d) is proved exactly like Proposition 11.59d). Because Ω and $\mathcal{C}_\Omega$ were initially arbitrary, we may now conclude that the proposed universal sentence is true. □

Exercise 11.33. Prove Proposition 11.79a,b,d).

Theorem 11.80 (Subbasis for a subspace topology). *The following sentences are true for any set Ω, any subbasis $\mathcal{C}_\Omega$ for a topology on Ω and any subset $\Omega_1 \subseteq \Omega$.*

a) There exists a unique set (system) $\mathcal{C}_{\Omega_1}$ containing precisely every set in $\mathcal{P}(\Omega_1)$ which is the intersection of Ω_1 and some subbasis element C in $\mathcal{C}_\Omega$.

b) This set $\mathcal{C}_{\Omega_1}$ satisfies then

$$\forall X\,(X \in \mathcal{C}_{\Omega_1} \Leftrightarrow \exists C\,(C \in \mathcal{C}_\Omega \,\wedge\, \Omega_1 \cap C = X)). \tag{11.558}$$

c) Moreover, $\mathcal{C}_{\Omega_1}$ is a subbasis for a topology on Ω_1 that generates the subspace topology of $\mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})$ in Ω_1, i.e.

$$\mathcal{O}(\mathcal{K}_{\mathcal{C}_{\Omega_1}}) = \mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})|\Omega_1. \tag{11.559}$$

Proof. We take arbitrary Ω, $\mathcal{C}_\Omega$ and Ω_1 such that $\mathcal{C}_\Omega$ is a subbasis for a topology on Ω – generating thus the topology $\mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})$ – and such that $\Omega_1 \subseteq \Omega$ is true.

To prove a), we can use the Axiom of Specification in connection with the Equality Criterion for sets (in the usual way) to establish the unique existence of a set $\mathcal{C}_{\Omega_1}$ for which

$$\forall X\,(X \in \mathcal{C}_{\Omega_1} \Leftrightarrow [X \in \mathcal{P}(\Omega_1) \,\wedge\, \exists C\,(C \in \mathcal{C}_\Omega \,\wedge\, \Omega_1 \cap C = X)]). \tag{11.560}$$

To prove then b), we let X be arbitrary, assuming first $X \in \mathcal{C}_{\Omega_1}$ to be true. Thus, (11.560) yields especially the existential sentence

$$\exists C\,(C \in \mathcal{C}_\Omega \,\wedge\, \Omega_1 \cap C = X), \tag{11.561}$$

so that the first part ('$\Rightarrow$') of the equivalence in (11.558) holds. To prove the second part ('$\Leftarrow$') of the equivalence, we conversely assume the preceding existential sentence to be true, which means that there is a subbasis element, say $\bar{C} \in \mathcal{C}_\Omega$, such that $\Omega_1 \cap \bar{C} = X$ is satisfied. Because $\Omega_1 \cap \bar{C} \subseteq \Omega_1$ also holds in view of (2.74), we obtain via substitution $X \subseteq \Omega_1$ and therefore $X \in \mathcal{P}(\Omega_1)$ by definition of a power set. In conjunction with the assumed existential sentence (11.561), this implies $X \in \mathcal{C}_{\Omega_1}$ due to (11.560), so that the second part of the equivalence in (11.558) also holds. As the set X was arbitrary, we may now conclude that the universal sentence (11.558) is indeed true.

To prove c), we observe first in light of Theorem 11.61 that the basis $\mathcal{K}_{\mathcal{C}_\Omega}$ (generated by the given subbasis $\mathcal{C}_\Omega$) defines the basis $\mathcal{K}_{\Omega_1}$ satisfying

$$\forall X\,(X \in \mathcal{K}_{\Omega_1} \Leftrightarrow \exists C\,(C \in \mathcal{K}_{\mathcal{C}_\Omega} \,\wedge\, \Omega_1 \cap C = X)) \tag{11.562}$$

and generating the subspace topology of $\mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})$ in Ω_1, i.e.

$$\mathcal{O}(\mathcal{K}_{\Omega_1}) = \mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})|\Omega_1. \tag{11.563}$$

To obtain from this the desired equation (11.559), we now establish the equation

$$\mathcal{K}_{\mathcal{C}_{\Omega_1}} = \mathcal{K}_{\Omega_1}, \tag{11.564}$$

making use of the Equality Criterion for sets. To do this, we prove the universal sentence

$$\forall Y\, (Y \in \mathcal{K}_{\mathcal{C}_{\Omega_1}} \Leftrightarrow Y \in \mathcal{K}_{\Omega_1}), \tag{11.565}$$

letting Y be an arbitrary set. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume $Y \in \mathcal{K}_{\mathcal{C}_{\Omega_1}}$ to be true. According to the Generation of a basis for a topology by means of a subbasis, there exists then a particular positive natural number $\bar{m}$ and a particular function/sequence $\bar{A}$ from $\{1, \ldots, \bar{m}\}$ to $\mathcal{C}_{\Omega_1}$ such that $Y = \bigcap \operatorname{ran}(\bar{A})$. Based on this sequence $\bar{A} = (\bar{A}_i \,|\, i \in \{1, \ldots, \bar{m}\})$, let us now verify the universal sentence

$$\forall i\, (i \in \{1, \ldots, \bar{m}\} \Leftrightarrow \exists C\, (C \in \mathcal{C}_\Omega \,\wedge\, \Omega_1 \cap C = \bar{A}_i)). \tag{11.566}$$

We take an arbitrary index $i \in \{1, \ldots, \bar{m}\}$, so that the corresponding term $\bar{A}_i$ is evidently in the codomain $\mathcal{C}_{\Omega_1}$ of $\bar{A}$. In view of (11.560), $\bar{A}_i \in \mathcal{C}_{\Omega_1}$ implies then in particular the existential sentence in (11.566), so that the universal sentence (11.566) follows to be true, because i was arbitrary. We may therefore apply Lemma 3.237, using the denotations $I = \{1, \ldots, \bar{m}\}$, $X = \Omega_1$, $\mathcal{K} = \mathcal{C}_\Omega$ and the sequence $(A_i)_{i \in I} = (\bar{A}_i \,|\, i \in \{1, \ldots, \bar{m}\})$, to obtain a particular sequence $\bar{B} = (\bar{B}_i \,|\, i \in \{1, \ldots, \bar{m}\})$ in $\mathcal{C}_\Omega$ such that

$$\forall i\, (i \in \{1, \ldots, \bar{m}\} \Rightarrow \Omega_1 \cap \bar{B}_i = \bar{A}_i). \tag{11.567}$$

This sequence $\bar{B}$ in turn defines with Corollary 3.238 the unique sequence $\bar{S} = (\Omega_1 \cap \bar{B}_i \,|\, i \in \{1, \ldots, \bar{m}\})$ satisfying

$$(\bar{A}_i \,|\, i \in \{1, \ldots, \bar{m}\}) = (\Omega_1 \cap \bar{B}_i \,|\, i \in \{1, \ldots, \bar{m}\}),$$

i.e. $\bar{A} = \bar{S}$. Consequently, we obtain by means of substitution

$$\begin{aligned} Y &= \bigcap \operatorname{ran}(\bar{S}) \\ &= \bigcap_{i \in \{1, \ldots, \bar{m}\}} (\Omega_1 \cap \bar{B}_i) \\ &= \Omega_1 \cap \bigcap_{i \in \{1, \ldots, \bar{m}\}} \bar{B}_i \\ &= \Omega_1 \cap \bigcap \operatorname{ran}(\bar{B}) \end{aligned}$$

by means of substitution based on the previously established equation $Y = \bigcap \mathrm{ran}(\bar{A})$, the definition of the intersection of a family of sets and the Associative Law (3.809) for families of sets. In view of $\bar{m} \in \mathbb{N}_+$, $\bar{B} : \{1, \ldots, \bar{m}\} \to \mathcal{C}_\Omega$ and the resulting equation $Y = \Omega_1 \cap \bigcap \mathrm{ran}(\bar{B})$, it now follows from the existence of such a sequence with the Generation of a basis for a topology by means of a subbasis that $\bigcap \mathrm{ran}(\bar{B}) \in \mathcal{K}_{\mathcal{C}_\Omega}$ holds. In connection with the preceding equation $\Omega_1 \cap \bigcap \mathrm{ran}(\bar{B}) = Y$, this shows that there exists a basis set $C \in \mathcal{K}_{\mathcal{C}_\Omega}$ for which $\Omega_1 \cap C = Y$ is true, so that (11.562) yields $Y \in \mathcal{K}_{\Omega_1}$, so that the proof of the first part of the equivalence in (11.565) is complete.

Regarding the second part ('$\Leftarrow$') of the equivalence, we now conversely assume $Y \in \mathcal{K}_{\Omega_1}$ to be true. This assumption implies the existence of a particular set $\bar{C} \in \mathcal{K}_{\mathcal{C}_\Omega}$ with $\Omega_1 \cap \bar{C} = Y$, because of (11.562). Recalling the Generation of a basis for a topology by means of a subbasis, we then obtain also a particular number $\bar{n} \in \mathbb{N}_+$ and a particular function $\bar{A} : \{1, \ldots, \bar{n}\} \to \mathcal{C}_\Omega$ for which $\bar{C} = \bigcap \mathrm{ran}(\bar{A})$. Writing $\bar{A}$ in sequence notation as $(\bar{A}_i \,|\, i \in \{1, \ldots, \bar{n}\})$, we may then define the sequence $\bar{B} = (\Omega_1 \cap \bar{A}_i \,|\, i \in \{1, \ldots, \bar{n}\})$, according to Exercise 3.101a). We therefore obtain evidently the equations

$$\begin{aligned}
Y &= \Omega_1 \cap \bar{C} \\
&= \Omega_1 \cap \bigcap \mathrm{ran}(\bar{A}) \\
&= \Omega_1 \cap \bigcap_{i \in \{1, \ldots, \bar{n}\}} \bar{A}_i \\
&= \bigcap_{i \in \{1, \ldots, \bar{n}\}} (\Omega_1 \cap \bar{A}_i) \\
&= \bigcap_{i \in \{1, \ldots, \bar{n}\}} \bar{B}_i \\
&= \bigcap \mathrm{ran}(\bar{B}).
\end{aligned}$$

Let us now verify here that $\mathcal{C}_{\Omega_1}$ is a codomain of that sequence, i.e. that the inclusion $\mathrm{ran}(\bar{B}) \subseteq \mathcal{C}_{\Omega_1}$ holds. For this purpose, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall X \, (X \in \mathrm{ran}(\bar{B}) \Rightarrow X \in \mathcal{C}_{\Omega_1}). \tag{11.568}$$

Letting X be arbitrary in $\mathrm{ran}(\bar{B})$, there is then (by definition of a range) a particular constant $\bar{k}$ such that $(\bar{k}, X) \in \bar{B}$, which we may write in function/sequence notation (recalling the definition of $\bar{B}$) also as $X = \bar{B}_{\bar{k}} = \Omega_1 \cap \bar{A}_{\bar{k}}$; here, $\bar{A}_{\bar{k}}$ is evidently in the codomain $\mathcal{C}_\Omega$ of the sequence $\bar{A}$. These findings show that there is a set $C \in \mathcal{C}_\Omega$ with $\Omega_1 \cap C = X$, so that (11.558) gives $X \in \mathcal{C}_{\Omega_1}$, as desired. Since X is arbitrary, the universal sentence (11.568) follows to be true, with the consequence that $\text{ran}(\bar{B}) \subseteq \mathcal{C}_{\Omega_1}$. By definition of a codomain, we thus have $\bar{B} : \{1, \ldots, \bar{n}\} \to \mathcal{C}_{\Omega_1}$ (with $\bar{n} \in \mathbb{N}_+$), which sequence we previously showed to satisfy $Y = \bigcap \text{ran}(\bar{B})$. The existence of such a sequence implies then the desired consequent $Y \in \mathcal{K}_{\mathcal{C}_{\Omega_1}}$ of the second part of the equivalence in (11.565), which is therefore true. Because Y was arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (11.565) and consequently the truth of the equation (11.564). Applying now a substitution based on this equation to (11.563), we finally obtain (11.559), so that the proof of c) is complete.

Since Ω, $\mathcal{C}_\Omega$ and Ω_1 were arbitrary in the proofs of a) – c), we may conclude that the stated theorem holds. □

Order topologies revisited

Lemma 11.81. *It is true for any linearly ordered set $(\Omega, <_\Omega)$ such that Ω is neither empty nor a singleton*

a) and for any element $a \in \Omega$ that the open and right-unbounded interval beginning in a (with respect to $\leq_\Omega$) is an open set of the order topology on Ω, i.e.

$$(a, +\infty) \in \mathcal{O}_{<_\Omega}. \tag{11.569}$$

b) and for any element $b \in \Omega$ that the open and left-unbounded interval ending in b (with respect to $\leq_\Omega$) is an open set of the order topology on Ω, i.e.

$$(-\infty, b) \in \mathcal{O}_{<_\Omega}. \tag{11.570}$$

Proof. We take arbitrary sets Ω and $<_\Omega$, assuming the ordered pair $(\Omega, <_\Omega)$ to be linearly ordered such that Ω is neither empty nor a singleton.

Concerning a), we take an arbitrary $\bar{a} \in \Omega$ and consider the two cases (1) that the maximum of Ω does not exist and (2) that $\max \Omega$ exists, according to (11.444) and (11.445), respectively.

In the first case that the maximum of Ω does not exist, we prove (11.569) by means of the Characterization of the elements of a topology generated by a basis. To begin with, we see in light of the Axiom of Specification and the Equality Criterion for sets that there exists a unique set $\mathcal{G}$ such that

$$\forall Z \, (Z \in \mathcal{G} \Leftrightarrow [Z \in \{(a,b) : a, b \in \Omega\} \wedge \exists x \, (\bar{a} <_\Omega x \wedge (\bar{a}, x) = Z)]). \tag{11.571}$$

Here, we see that $Z \in \mathcal{G}$ implies in particular $Z \in \{(a,b) : a, b \in \Omega\}$ for any Z, so that the inclusion $\mathcal{G} \subseteq \{(a,b) : a, b \in \Omega\}$ follows to be true by definition of a subset. In addition, $\{(a, b) : a, b \in \Omega\}$ is included in the basis generating the order topology $\mathcal{O}_{<_\Omega}$, according to Proposition 11.65 (irrespective of whether the minimum of Ω does or does not exist). Therefore, the set system $\mathcal{G}$ is included in the basis generating $\mathcal{O}_{<_\Omega}$, as required by the Characterization of the elements of a topology generated by a basis.

Next, we can apply the Equality Criterion for sets to prove that the set (system) $\mathcal{G}$ satisfies

$$(\bar{a}, +\infty) = \bigcup \mathcal{G}, \tag{11.572}$$

by establishing the truth of the universal sentence

$$\forall y \, (y \in (\bar{a}, +\infty) \Leftrightarrow y \in \bigcup \mathcal{G}). \tag{11.573}$$

Letting y be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming

$$y \in (\bar{a}, +\infty) \tag{11.574}$$

to be true. Therefore, the definition of an open and right-unbounded interval in Ω gives $\bar{a} <_\Omega y$ with $y \in \Omega$. Since the current case assumption (11.444) implies with the Negation Law for existential conjunctions

$$\forall m \, (m \in \Omega \Rightarrow \neg \forall \omega \, (\omega \in \Omega \Rightarrow \omega \leq_\Omega m)),$$

it follows from $y \in \Omega$ via the Negation Law for universal implications that there exists a particular element $\bar{\omega} \in \Omega$ with $\neg \bar{\omega} \leq_\Omega y$. This negation in turn implies with the Negation Formula for $\leq$ that $y <_\Omega \bar{\omega}$ holds. Together with the previously established $\bar{a} <_\Omega y$, this further implies $y \in (\bar{a}, \bar{\omega})$ with the definition of an open interval in Ω. This shows on the one hand that

$$(\bar{a}, \bar{\omega}) \in \{(a, b) : a, b \in \Omega\} \tag{11.575}$$

is true; on the other hand, we see that the existential sentence

$$\exists x \, (\bar{a} <_\Omega x \land (\bar{a}, x) = (\bar{a}, \bar{\omega})) \tag{11.576}$$

holds. The conjunction of (11.575) and (11.576) implies then $(\bar{a}, \bar{\omega}) \in \mathcal{G}$ because of (11.571). Next, the conjunction of $(\bar{a}, \bar{\omega}) \in \mathcal{G}$ and $y \in (\bar{a}, \bar{\omega})$ demonstrates the truth of the existential sentence

$$\exists G \, (G \in \mathcal{G} \land y \in G), \tag{11.577}$$

so that the desired consequent

$$y \in \bigcup \mathcal{G} \tag{11.578}$$

of the first part of the equivalence in (11.573) follows to be true by definition of the union of a set system.

To prove the second part ('$\Leftarrow$') directly, we now assume (11.578) to be true, which assumption yields the existential sentence (11.577). Thus, $\bar{G} \in \mathcal{G}$ and $y \in \bar{G}$ are true for a particular set $\bar{G}$. Here, $\bar{G} \in \mathcal{G}$ implies with (11.571) in particular the existence of a particular constant $\bar{x}$ satisfying $\bar{a} <_\Omega \bar{x}$ and $(\bar{a}, \bar{x}) = \bar{G}$. Consequently, $y \in \bar{G}$ gives us $y \in (\bar{a}, \bar{x})$ via substitution and subsequently $\bar{a} <_\Omega y <_\Omega \bar{x}$ (by definition of an open interval). Thus, $\bar{a} <_\Omega y$ is especially true, with the consequence that (11.574) holds (by definition of an open and right-unbounded interval). This finding completes the proof of the equivalence in (11.573), and as y is arbitrary, we may therefore conclude that (11.573) is true.

We may then infer from the truth of that universal sentence the truth of the equation (11.572). We thus proved the existence of a set system $\mathcal{G}$ which is included in the basis generating $\mathcal{O}_{<_\Omega}$ and whose union equals the set $(\bar{a}, +\infty)$, so that the Characterization of the elements of a topology generated by a basis yields $(\bar{a}, +\infty) \in \mathcal{O}_{<_\Omega}$ (for the first case).

Regarding the second case that $\max \Omega$ exists, we obtain

$$(\bar{a}, +\infty) = (\bar{a}, \max \Omega] \tag{11.579}$$

with Proposition 3.139a). Let us now consider the two subcases (2.1) that the minimum of Ω does not exist and (2.2) that $\min \Omega$ exists, according to (11.446) and (11.447), respectively. In the first subcase, we evidently have that $(\bar{a}, \max \Omega]$ is an element of the basis (11.422) generating the order topology on Ω. Since that basis is included $\mathcal{O}_{<_\Omega}$ according to Proposition 11.55, the interval $(\bar{a}, \max \Omega]$ follows by definition of a subset to be also element of $\mathcal{O}_{<_\Omega}$. We therefore obtain $(\bar{a}, +\infty) \in \mathcal{O}_{<_\Omega}$ via substitution based on (11.579). In the second subcase that $\min \Omega$ exists, the interval $(\bar{a}, \max \Omega]$ now turns out to be an element of the basis (11.423) generating the corresponding order topology on Ω. Again, the basis is included in that order topology $\mathcal{O}_{<_\Omega}$, so that $(\bar{a}, \max \Omega] \in \mathcal{O}_{<_\Omega}$ follows to be true, and this yields as in the first subcase the desired $(a, +\infty) \in \mathcal{O}_{<_\Omega}$.

As the latter finding holds in any case, and since Ω, $<_\Omega$ and $\bar{a}$ were initially arbitrary, we may now finally conclude that Part a) of the lemma is true. Part b) is proved analogously. □

Exercise 11.34. Prove Part b) of Lemma 11.81.

Corollary 11.82. *It is true for any linearly ordered set* $(\Omega, <_\Omega)$ *such that* Ω *is neither empty nor a singleton* *a) and for any element $a \in \Omega$ that the left-closed and right-unbounded interval $[a, +\infty)$ is a closed set in Ω with respect to the order topology $\mathcal{O}_{<_\Omega}$.*

b) and for any element $b \in \Omega$ that the left-unbounded and right-closed interval $(-\infty, b]$ is a closed set in Ω with respect to the order topology $\mathcal{O}_{<_\Omega}$.

Proof. Letting $(\Omega, <_\Omega)$ be an arbitrary linearly ordered set such that Ω is neither empty nor a singleton, and letting a as well as b be arbitrary elements in Ω, we have $(-\infty, a) = [a, +\infty)^c$ and $(b, +\infty) = (-\infty, b]^c$ according to (3.447) and (3.451). Since $(-\infty, a)$ and $(b, +\infty)$ are open sets of the order topology $\mathcal{O}_{<_\Omega}$ in view of (11.570) and (11.569), it follows via substitutions that $[a, +\infty)^c$ and $(-\infty, b]^c$ are elements of $\mathcal{O}_{<_\Omega}$. Consequently, $[a, +\infty)$ and $(-\infty, b]$ are closed sets by definition. As $(\Omega, <_\Omega)$, a and b were arbitrary, we may therefore conclude that the corollary holds. □

Note 11.30. In view of Lemma 11.81, any open and left- or right-unbounded interval in $\mathbb{R}$ is an open set of the order topology on $\mathbb{R}$, that is, we have for any real numbers a and b

$$(-\infty, b)_\mathbb{R} \in \mathcal{O}_{<_\mathbb{R}}, \tag{11.580}$$

$$(a, +\infty)_\mathbb{R} \in \mathcal{O}_{<_\mathbb{R}}. \tag{11.581}$$

Corollary 11.82 demonstrates then (for any $a, b \in \mathbb{R}$) that

$$[a, +\infty)_\mathbb{R} \text{ is a closed set in } \Omega, \tag{11.582}$$

$$(-\infty, b]_\mathbb{R} \text{ is a closed set in } \Omega. \tag{11.583}$$

The preceding lemma also shows in light of Proposition 3.139a) and Exercise 3.66c) – in connection with the equations $\min \overline{\mathbb{R}} = -\infty$ and $\max \overline{\mathbb{R}} = +\infty$ established in Corollary 9.13 – that for any extended real numbers a, b

$$(-\infty, b)_{\overline{\mathbb{R}}}, [-\infty, b)_{\overline{\mathbb{R}}} \in \mathcal{O}_{<_{\overline{\mathbb{R}}}}, \tag{11.584}$$

$$(a, +\infty)_{\overline{\mathbb{R}}}, (a, +\infty]_{\overline{\mathbb{R}}} \in \mathcal{O}_{<_{\overline{\mathbb{R}}}}. \tag{11.585}$$

In the present case of extended real numbers, the notation $(-\infty, b)_{\overline{\mathbb{R}}}$ could represent two different types of intervals: Either the open and left-unbounded interval in $\overline{\mathbb{R}}$ ending in b, defined to be the set of all elements $y <_{\overline{\mathbb{R}}} b$ (possibly including $y = -\infty$), or the open interval in $\overline{\mathbb{R}}$ from $-\infty$ to b, specified as the set of all elements $-\infty <_{\overline{\mathbb{R}}} y <_{\overline{\mathbb{R}}} b$ (which excludes $y = -\infty$). We will therefore write in the following $[-\infty, b)_{\overline{\mathbb{R}}}$ for an open and left-unbounded interval and preserve the symbol $(-\infty, b)_{\overline{\mathbb{R}}}$ for the open interval from $-\infty$ to b. A similar ambiguity applies to the symbol $(a,+\infty)_{\overline{\mathbb{R}}}$, which we will use in the sequel exclusively to represent an open interval in $\overline{\mathbb{R}}$ from a to $+\infty$ (and not to represent an open and right-unbounded interval beginning in a, which we will express in the form of $(a,+\infty]_{\overline{\mathbb{R}}}$).

Theorem 11.83 (Hausdorff Property of topological spaces involving the order topology). *It is true for any linearly ordered set* $(\Omega,<_\Omega)$ *such that* Ω *is neither empty nor a singleton that the topological space* $(\Omega,\mathcal{O}_{<_\Omega})$ *is a Hausdorff space.*

Proof. Letting Ω and $<_\Omega$ be arbitrary such that $(\Omega,<_\Omega)$ is linearly ordered and such that Ω is neither empty nor a singleton, we need to establish the universal sentence

$$\begin{aligned}&\forall \omega,\nu\,([\omega,\nu\in\Omega\;\wedge\;\omega\neq\nu]\\&\quad\Rightarrow \exists U,V\,(U,V\in\mathcal{O}_{<_\Omega}\;\wedge\;U\cap V=\emptyset\;\wedge\;\omega\in U\;\wedge\;\nu\in V)).\end{aligned}\tag{11.586}$$

We let $\omega,\nu\in\Omega$ be arbitrary, assuming $\omega\neq\nu$ to hold, so that the disjunction $\omega<_\Omega\nu\;\vee\;\nu<_\Omega\omega$ is true because of the connexity of the linear ordering $<_\Omega$. We now prove the desired existential sentence in (11.586) by cases.

In the first case $\omega<_\Omega\nu$, we observe the truth of the disjunction

$$\exists z\,(\omega<_\Omega z<_\Omega\nu)\;\vee\;\neg\exists z\,(\omega<_\Omega z<_\Omega\nu)\tag{11.587}$$

in light of the Law of the Excluded Middle, which we use to prove the existential sentence in (11.586) via sub-cases. In the first sub-case, we assume the existence of a particular constant $\bar z$ satisfying $\omega<_\Omega\bar z$ and $\bar z<_\Omega\nu$. Thus, the definitions of an open, left-unbounded interval in Ω and of an open and right-unbounded interval in Ω show that

$$\omega\in(-\infty,\bar z)\;\wedge\;\nu\in(\bar z,+\infty)\tag{11.588}$$

is true. Here, we have due to (11.570) – (11.569)

$$(-\infty,\bar z),(\bar z,+\infty)\in\mathcal{O}_{<_\Omega}\tag{11.589}$$

with (2.13) and the definition of a subset. Since $\neg\bar z<_\Omega\bar z$ is true due to the irreflexivity of the linear ordering $<_\Omega$, we furthermore obtain with (3.459)

$$(-\infty,\bar z)\cap(\bar z,+\infty)=\emptyset.\tag{11.590}$$

The findings (11.589), (11.590) and (11.588) now clearly demonstrate the truth of the desired existential sentence in (11.586) in the first sub-case.

In the second sub-case, we assume the negated existential sentence in (11.587) to be true, which implies with the Negation Law for existential sentences

$$\forall z\, (\neg[\omega <_\Omega z \,\wedge\, z <_\Omega \nu]). \tag{11.591}$$

Let us now consider the two intervals $(-\infty, \nu)$ and $(\omega, +\infty)$ in Ω. The current case assumption $\omega <_\Omega \nu$ implies then (by definition of these intervals)

$$\omega \in (-\infty, \nu) \,\wedge\, \nu \in (\omega, +\infty), \tag{11.592}$$

where we have in view of (11.570) – (11.569)

$$(-\infty, \nu), (\omega, +\infty) \in \mathcal{O}_{<_\Omega}. \tag{11.593}$$

It remains for us to prove that $(-\infty, \nu)$ and $(\omega, +\infty)$ are disjoint, which task we accomplish by establishing the universal sentence

$$\forall z\, (\, z \notin (-\infty, \nu) \cap (\omega, +\infty)\,). \tag{11.594}$$

To do this, we take an arbitrary z, for which constant (11.591) gives us with De Morgan's Law for the conjunction $\neg\omega <_\Omega z \vee \neg z <_\Omega \nu$. We now use this true disjunction to prove the negation in (11.594) by cases. On the one hand, if $\neg\omega <_\Omega z$ is true, then we may prove the desired negation by contradiction. For this purpose, we assume the negation of that negation to be true. The Double Negation Law yields then the true sentence $z \in (-\infty, \nu) \cap (\omega, +\infty)$, which in turn implies the truth of $z \in (-\infty, \nu)$ and of $z \in (\omega, +\infty)$, by definition of the intersection of two sets. We therefore obtain evidently $z <_\Omega \nu$ as well as $\omega <_\Omega z$. Since the assumed $\neg\omega <_\Omega z$ implies with the Negation Formula for $<$ that $z \leq_\Omega \omega\ [<_\Omega z]$ holds, it follows with the Transitivity Formula for $\leq$ and $<$ that $z <_\Omega z$ is true. Because $\neg z <_\Omega z$ is also true in view of the irreflexivity of $<_\Omega$, we thus obtained a contradiction, so that the negation in (11.594) holds in case of $\neg\omega <_\Omega z$.

On the other hand, if $\neg z <_\Omega \nu$ holds, then we can establish that negation by finding the contradiction $\nu <_\Omega \nu \wedge \neg\nu <_\Omega \nu$, assuming again the truth of the negation of the negation to be proven. As before, this assumption gives us first $z \in (-\infty, \nu) \cap (\omega, +\infty)$, then the conjunction of $z \in (-\infty, \nu)$ and $z \in (\omega, +\infty)$, and therefore $z <_\Omega \nu$ as well as $\omega <_\Omega z$. The previous assumption $\neg z <_\Omega \nu$ yields now (with the Negation Formula for $<$) $\nu \leq_\Omega z\ [<_\Omega \nu]$, and consequently $\nu <_\Omega \nu$ with the Transitivity Formula for $\leq$ and $<$. As the negation $\neg\nu <_\Omega \nu$ is true as well (due to the irreflexivity of $<_\Omega$), we arrived at the previously stated contradiction, so that the negation in (11.594) holds also in case of $\neg z <_\Omega \nu$. Since z was arbitrary, we may now infer from the truth of that negation, according to the definition of the empty set, that

$$(-\infty, \nu) \cap (\omega, +\infty) = \emptyset \tag{11.595}$$

is true. The findings (11.593), (11.595) and (11.592) demonstrate that the existential sentence in (11.586) to be proven is also true for the second subcase, so that the proof of the first case is now complete.

We can prove the second case by switching the roles that ω and ν play in the proof of the first case.

Since ω and ν are arbitrary, we may therefore conclude that the universal sentence (11.586) holds, which means that the topological space $(\Omega, \mathcal{O}_{<_\Omega})$ is a Hausdorff space, by definition. Because Ω and $<_\Omega$ were initially arbitrary, we may therefore conclude that the stated theorem is indeed true. □

Exercise 11.35. Elaborate the second case in the proof of Theorem 11.83.

Corollary 11.84. *It is true for any linearly ordered set $(\Omega, <_\Omega)$ such that Ω is neither empty nor a singleton that the closed interval $[a, b]$ from an element $a \in \Omega$ to an element $b \in \Omega$ is a closed set in Ω.*

Proof. Letting Ω, $<_\Omega$, a and b be arbitrary such that $(\Omega, <_\Omega)$ is linearly ordered, such that Ω is neither empty nor a singleton, and such that $a, b \in \Omega$ holds, we obtain the equations

$$[a, b]^c = \bigcup\{(-\infty, a), (b, +\infty)\}$$

with (3.462) and the notation for the union of two sets. Here, we have $(-\infty, a) \in \mathcal{O}_{<_\Omega}$ and $(b, +\infty) \in \mathcal{O}_{<_\Omega}$ in view of (11.570) and (11.569), so that we obtain for the preceding union $[a, b]^c \in \mathcal{O}_{<_\Omega}$ with Property 3 of a topology. Thus, the closed interval $[a, b]$ in Ω is a closed set in Ω, by definition. Since Ω, $<_\Omega$, a and b are arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Note 11.31. We see in light of the preceding Corollary 11.84 that

a) any closed interval $[a, b]_{\mathbb{R}}$ is a closed set in $\mathbb{R}$ with respect to the order topology on $\mathbb{R}$, and that

b) any closed interval $[a, b]_{\overline{\mathbb{R}}}$ is a closed set in $\overline{\mathbb{R}}$ with respect to the order topology on $\overline{\mathbb{R}}$.

Theorem 11.85 (Subbasis for the order topology). *The following sentences are true for any linearly ordered set $(\Omega, <_\Omega)$ such that Ω is neither empty nor a singleton.*

a) *The union*

$$\mathcal{C}_{<_\Omega} = \{(a, +\infty) : a \in \Omega\} \cup \{(-\infty, b) : b \in \Omega\} \tag{11.596}$$

of the set of open, right-unbounded and the set of open, left-unbounded intervals in Ω constitutes a subbasis for a topology on Ω.

b) *This subbasis is included in the order topology on Ω, i.e.*

$$\mathcal{C}_{<_\Omega} \subseteq \mathcal{O}_{<_\Omega}. \tag{11.597}$$

c) *Furthermore, this subbasis $\mathcal{C}_{<_\Omega}$ generates the order topology on Ω, i.e.*

$$\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}}) = \mathcal{O}_{<_\Omega}. \tag{11.598}$$

Proof. We take arbitrary sets Ω and $<_\Omega$, assume the ordered pair $(\Omega, <_\Omega)$ to be linearly ordered, and assume moreover that Ω is neither empty nor a singleton, i.e. that $\Omega \neq \emptyset$ as well as $\forall a\,(\Omega \neq \{a\})$ holds.

To prove a), we begin with the verification of Property 1 of subbasis for a topology on Ω. Indeed, since the two inclusions

$$\{(a, +\infty) : a \in \Omega\} \subseteq \mathcal{P}(\Omega)$$
$$\{(-\infty, b) : b \in \Omega\} \subseteq \mathcal{P}(\Omega)$$

are true according to Exercise 3.56f), we obtain the desired inclusion

$$\mathcal{C}_{<_\Omega} \subseteq \mathcal{P}(\Omega) \tag{11.599}$$

by applying (2.252) and then substitution based on the equation (11.596). Regarding Property 2 of a subbasis, we need to establish the equation

$$\bigcup \mathcal{C}_{<_\Omega} = \Omega. \tag{11.600}$$

To do this, we apply the Equality Criterion for sets and prove the equivalent universal sentence

$$\forall \omega\,(\omega \in \bigcup \mathcal{C}_{<_\Omega} \Leftrightarrow \omega \in \Omega), \tag{11.601}$$

letting ω be arbitrary, and assuming first $\omega \in \bigcup \mathcal{C}_{<_\Omega}$ to be true. By definition of the union of a set system, there exists a set, say $\bar{A}$, satisfying both $\bar{A} \in \mathcal{C}_{<_\Omega}$ and $\omega \in \bar{A}$. The former implies now $\bar{A} \in \mathcal{P}(\Omega)$ with (11.599) and the definition of subset, which finding in turn gives $\bar{A} \subseteq \Omega$ with the definition of a power set. Because of this inclusion, the previously found $\omega \in \bar{A}$ implies then $\omega \in \Omega$, as desired.

Assuming now conversely $\omega \in \Omega$ to be true and recalling the initial assumption that Ω is a nonempty non-singleton, there is then a particular element $\bar{c} \in \Omega$ such that $\omega \neq \bar{c}$ holds, according to (2.21). Since the linear ordering $<_\Omega$ is by definition connex, the preceding inequality gives us the true disjunction $\omega <_\Omega \bar{c} \vee \bar{c} <_\Omega \omega$, which we now use to prove $\omega \in \bigcup \mathcal{C}_{<_\Omega}$ by cases.

The first case $\omega <_\Omega \bar{c}$ implies $\omega \in (-\infty, \bar{c})$ with the definition of an open and left-unbounded interval, and this shows in light of $\bar{c} \in \Omega$ that $(-\infty, \bar{c}) \in \{(-\infty, b) : b \in \Omega\}$ holds. As the inclusion $\{(-\infty, b) : b \in \Omega\} \subseteq \mathcal{C}_{<_\Omega}$ is true in view of (11.596) and (2.245), we therefore obtain $(-\infty, \bar{c}) \in \mathcal{C}_{<_\Omega}$ by means of the definition of a subset. In connection with $\omega \in (-\infty, \bar{c})$, this shows us now that there exists a set in $\mathcal{C}_{<_\Omega}$ containing ω, so that the desired consequent $\omega \in \bigcup \mathcal{C}_{<_\Omega}$ follows to be true by definition of the union of a set system.

We may apply similar arguments to establish this result for the second case $\bar{c} <_\Omega \omega$, which implies first $\omega \in (\bar{c}, +\infty)$ with the definition of an open and right-unbounded interval, so that $(\bar{c}, +\infty) \in \{(a, +\infty) : a \in \Omega\}$ evidently holds. Consequently, since the inclusion $\{(a, +\infty) : a \in \Omega\} \subseteq \mathcal{C}_{<_\Omega}$ is now clearly true, we obtain $(\bar{c}, +\infty) \in \mathcal{C}_{<_\Omega}$. Recalling $\omega \in (\bar{c}, +\infty)$, we thus see that there is also in the current second case a set $\mathcal{C}_{<_\Omega}$ containing ω, so that $\omega \in \bigcup \mathcal{C}_{<_\Omega}$ follows again to be true. As this finding holds in any case, the proof of the equivalence in (11.601) is thus complete, and as ω was arbitrary, we may therefore conclude that (11.601) is true. The truth of this universal sentence in turn implies the truth of the equation (11.600), which means that the set system $\mathcal{C}_{<_\Omega}$ satisfies also Property 2 of a subbasis for a topology on Ω, completing the proof of a).

To prove the inclusion b), we apply the definition of subset and demonstrate accordingly the truth of the universal sentence

$$\forall A \, (A \in \mathcal{C}_{<_\Omega} \Rightarrow A \in \mathcal{O}_{<_\Omega}), \tag{11.602}$$

letting $A \in \mathcal{C}_{<_\Omega}$ be arbitrary. This assumption implies with (11.596) and the definition of the union of two sets that $A \in \{(a, +\infty) : a \in \Omega\}$ or $A \in \{(-\infty, b) : b \in \Omega\}$ holds. Let us now prove $A \in \mathcal{O}_{<_\Omega}$ by cases, based on the preceding disjunction. According to Exercise 3.56e), the first case implies the existence of a particular element $\bar{a} \in \Omega$ with $(\bar{a}, +\infty) = A$, so that $A \in \mathcal{O}_{<_\Omega}$ follows to be true with Lemma 11.81a). Similarly, the second case implies the existence of a particular element $\bar{b} \in \Omega$ with $(-\infty, \bar{b}) = A$, so that $A \in \mathcal{O}_{<_\Omega}$ follows again to be true now with Lemma 11.81b). Since A is arbitrary, we may therefore conclude that (11.602) holds, which universal sentence then yields the proposed inclusion (11.597).

Next, we prove the equation of c) via the Axiom of Extension, by establishing the truth of the conjunction

$$\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}}) \subseteq \mathcal{O}_{<\Omega} \ \wedge \ \mathcal{O}_{<\Omega} \subseteq \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}}). \tag{11.603}$$

On the one hand, recalling from b) that $\mathcal{O}_{<\Omega}$ is a topology on Ω that includes $\mathcal{C}_{<\Omega}$, it follows with Proposition 11.79d) that the topology $\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})$ generated by the subbasis $\mathcal{C}_{<\Omega}$ satisfies the first inclusion

$$\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}}) \subseteq \mathcal{O}_{<\Omega}$$

in (11.603). On the other hand, we may apply the analogous Proposition 11.59d) to establish the second inclusion. As a preparation, we verify that the topology $\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})$ on Ω includes the basis $\mathcal{K}_{<\Omega}$ generating the order topology $\mathcal{O}_{<\Omega} = \mathcal{O}(\mathcal{K}_{<\Omega})$. For this purpose, we utilize the definition of a subset and prove the universal sentence

$$\forall B \, (B \in \mathcal{K}_{<\Omega} \Rightarrow B \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})), \tag{11.604}$$

letting $B \in \mathcal{K}_{<\Omega}$ be arbitrary. To prove $B \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})$, we consider the two cases (1) that the maximum of Ω does not exist and (2) that $\max \Omega$ exists. In the first case, we then consider the two subcases (i) that the minimum of Ω does not exist and (ii) that $\min \Omega$ exists.

In the first subcase, the basis $\mathcal{K}_{<\Omega}$ is thus given by the set of open intervals (11.420), according to the definition of an order topology. By definition of the set open intervals, $B \in \mathcal{K}_{<\Omega}$ implies the existence of particular elements $\bar{a}, \bar{b} \in \Omega$ such that $(\bar{a}, \bar{b}) = B$ holds. Consequently, we may write B as the intersection

$$B = (\bar{a}, \bar{b}) = (-\infty, \bar{b}) \cap (\bar{a}, +\infty)$$

using Proposition 3.134. Here, we evidently have

$$(-\infty, \bar{b}) \in \{(-\infty, b) : b \in \Omega\} \quad \left[\subseteq \mathcal{C}_{<\Omega} \subseteq \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})\right], \tag{11.605}$$

$$(\bar{a}, +\infty) \in \{(a, +\infty) : a \in \Omega\} \quad \left[\subseteq \mathcal{C}_{<\Omega} \subseteq \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})\right], \tag{11.606}$$

recalling the truth of the equation (11.596), (2.245) and the fact that $\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})$ is generated by – and includes therefore – the subbasis $\mathcal{C}_{<\Omega}$. We thus see in light of the definition of a subset and the transitivity of subsets (2.13) that the intervals $(-\infty, \bar{b})$ and $(\bar{a}, +\infty)$ are elements of the topology $\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})$. Due to Property 4 of a topology, the intersection B of these two intervals follows now to be also element of $\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<\Omega}})$, as desired.

In the second subcase (ii), in which the maximum of Ω does not exist whereas the minimum of Ω exists, the basis $\mathcal{K}_{<\Omega}$ is now given by the union (11.421), so that $B \in \mathcal{K}_{<\Omega}$ implies with the definition of the union of two sets that $B \in \{(a,b) : a,b \in \Omega\}$ or $B \in \{[\min\Omega, b) : b \in \Omega\}$ holds. We may use this true disjunction to prove $B \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$ by considering two further sub-subcases. On the one hand, if $B \in \{(a,b) : a,b \in \Omega\}$ is true, then we evidently obtain $(\bar{a}, \bar{b}) = B$ for some particular elements $\bar{a}, \bar{b} \in \Omega$, and we may therefore proceed in exactly the same way as in the first subcase (i) to infer from this the desired consequent $B \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$. On the other hand, if $B \in \{[\min\Omega, b) : b \in \Omega\}$ is true, then there is, according to Exercise 11.25a), a particular constant $\bar{b}$ satisfying $B = [\min\Omega, \bar{b})$, which interval is in $\{[a,b) : a,b \in \Omega\}$, so that $\bar{b} \in \Omega$ is true. We may then utilize Exercise 3.66c) to write for that interval $B = (-\infty, \bar{b})$, which turns out to be in $\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$, because (11.605) holds evidently also in the current subcase.

Next, we consider the second case (2) in connection with the first subcase (i), meaning that $\max\Omega$ exists but the minimum of Ω does not. Therefore, the union (11.422) constitutes the basis $\mathcal{K}_{<_\Omega}$ for the order topology on Ω. Consequently, $B \in \mathcal{K}_{<_\Omega}$ yields (by definition of the union of two sets) the disjunction of $B \in \{(a,b) : a,b \in \Omega\}$ and $B \in \{(a, \max\Omega] : a \in \Omega\}$. On the one hand, if $B \in \{(a,b) : a,b \in \Omega\}$ holds, then we clearly obtain the desired consequent $B \in \mathcal{K}_{<_\Omega}$ in exactly the same way as in subcase (i) of case (1). On the other hand, if $B \in \{(a, \max\Omega] : a \in \Omega\}$ holds, then then there is, according to Exercise 11.25b), a particular constant $\bar{a}$ satisfying $B = (\bar{a}, \max\Omega]$, which interval is in $\{(a,b] : a,b \in \Omega\}$, so that $\bar{a} \in \Omega$ is true. We may now make use of Exercise 3.66d) to write that interval as $B = (\bar{a}, +\infty)$, which turns out to be in $\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$, since (11.606) is evidently true in the currently considered subcase.

It now remains for us to inspect the second subcase within case (2). Thus, the maximum and the minimum of Ω exist both, in which case the union (11.423) defines the basis $\mathcal{K}_{<_\Omega}$ for the order topology on Ω. The assumed $B \in \mathcal{K}_{<_\Omega}$ implies, using the definition of the union of two sets twice in connection with the Associative Law for the disjunction, the truth of the multiple disjunction

$$\begin{aligned} B \in \{(a,b) : a,b \in \Omega\} \ &\vee \ B \in \{[\min\Omega, b) : b \in \Omega\} \\ &\vee \ B \in \{(a, \max\Omega] : a \in \Omega\}, \end{aligned}$$

which allows us to prove $B \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$ by considering three corresponding cases. Here, $B \in \{(a,b) : a,b \in \Omega\}$ represents subcase (i) of case (1), $B \in \{[\min\Omega, b) : b \in \Omega\}$ represents the second sub-subcase within subcase (ii) of case (1), and $B \in \{(a, \max\Omega] : a \in \Omega\}$ represents the second sub-subcase of subcase (1) within case (2). In all of these situations, we established $B \in \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$, which finding thus holds in any case.

Having completed the proof of the implication in (11.604), we may now infer from this the truth of the universal sentence (11.604), since B was arbitrary, and moreover the truth of the inclusion $\mathcal{K}_{<_\Omega} \subseteq \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$. Thus, $\mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$ is a topology on Ω that includes $\mathcal{K}_{<_\Omega}$; with this, it follows because of Proposition 11.59d) that the (order) topology $\mathcal{O}_{<_\Omega}$ generated by the basis $\mathcal{K}_{<_\Omega}$ satisfies the second inclusion

$$\mathcal{O}_{<_\Omega} \subseteq \mathcal{O}(\mathcal{K}_{\mathcal{C}_{<_\Omega}})$$

in (11.603). We thus completed the proof the conjunction (11.603), which in turn implies the equation (11.598).

In the proof of a) – c), the sets Ω and $<_\Omega$ were arbitrary, so that the stated theorem follows now to be true. □

Note 11.32. We thus have that the subbasis

$$\mathcal{C}_{<_\mathbb{R}} = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\} \tag{11.607}$$

generates the order topology on $\mathbb{R}$. Furthermore, the subbasis

$$\mathcal{C}_{<_{\overline{\mathbb{R}}}} = \{(a, +\infty] : a \in \overline{\mathbb{R}}\} \cup \{[-\infty, b) : b \in \overline{\mathbb{R}}\} \tag{11.608}$$

generates the order topology on $\overline{\mathbb{R}}$, where we use the equations (3.477) and (3.477) to avoid the ambiguities mentioned in Note 11.30.

Theorem 11.86 (Compatibility of subspace and order topologies). *It is true for any sets Ω and Ω_1 being neither empty nor singletons and for any linear ordering $<_X$ of X such that Ω_1 is convex in Ω with respect to $<_X$ that the subspace topology of the order topology on Ω (with respect to $<_\Omega$) in Ω_1 coincides with the order topology on Ω_1 with respect to the linear ordering $<_{\Omega_1}$ defined according to Theorem 3.68, i.e.*

$$\mathcal{O}_{<_\Omega}|\Omega_1 = \mathcal{O}_{<_{\Omega_1}}. \tag{11.609}$$

Proof. We let Ω, Ω_1 and $<_\Omega$ be arbitrary sets, assuming that $(\Omega, <_\Omega)$ is linearly ordered, that Ω and Ω_1 are neither empty nor singletons, and assuming moreover that Ω_1 is a convex set in Ω with respect to $<_X$. Here, $<_X$ defines the irreflexive partial ordering $<_{\Omega_1}$ of Ω_1 according to Theorem 3.68, which is itself a linear ordering because of Theorem 3.76. Thus, the order topologies $\mathcal{O}_{<_\Omega} = \mathcal{O}(\mathcal{K}_{\mathcal{C}_\Omega})$ and $\mathcal{O}_{<_{\Omega_1}} = \mathcal{O}(\mathcal{K}_{\mathcal{C}_{\Omega_1}})$ are generated by the subbases

$$\mathcal{C}_{<_\Omega} = \{(a, +\infty) : a \in \Omega\} \cup \{(-\infty, b) : b \in \Omega\} \tag{11.610}$$

and

$$\mathcal{C}_{<_{\Omega_1}} = \{(a, +\infty) : a \in \Omega_1\} \cup \{(-\infty, b) : b \in \Omega_1\}, \tag{11.611}$$

according to Theorem 11.85, and the subspace topology $\mathcal{O}_{<_\Omega}|\Omega_1$ is generated by the subbasis $\mathcal{C}_{\Omega_1}$ according to Theorem 11.80. Let us now prove the equation (11.609) via the Axiom of Extension, by establishing the inclusions

$$\mathcal{O}_{<_\Omega}|\Omega_1 \subseteq \mathcal{O}_{<_{\Omega_1}} \wedge \mathcal{O}_{<_{\Omega_1}} \subseteq \mathcal{O}_{<_\Omega}|\Omega_1. \tag{11.612}$$

To prove the first inclusion, we demonstrate first the truth of the inclusion $\mathcal{C}_{\Omega_1} \subseteq \mathcal{O}_{<_{\Omega_1}}$. To do this, we utilize the definition of a subset and prove accordingly the universal sentence

$$\forall X\,(X \in \mathcal{C}_{\Omega_1} \Rightarrow X \in \mathcal{O}_{<_{\Omega_1}}), \tag{11.613}$$

letting X be an arbitrary set and assuming $X \in \mathcal{C}_{\Omega_1}$. This assumption implies with Theorem 11.80b) that there exists a set, say $\bar{C}$, such that $\bar{C} \in \mathcal{C}_{<_\Omega}$ and $\Omega_1 \cap \bar{C} = X$ hold. The former implies then in view of (11.610) and the definition of the union of two sets that the disjunction

$$\bar{C} \in \{(a, +\infty) : a \in \Omega\} \vee \bar{C} \in \{(-\infty, b) : b \in \Omega\}$$

is true. We now prove the desired consequent $X \in \mathcal{O}_{<_{\Omega_1}}$ by cases, based on the preceding disjunction. In the first case $\bar{C} \in \{(a, +\infty) : a \in \Omega\}$, the definition of the set of open and right-unbounded intervals in Ω gives us a particular element $\bar{a} \in \Omega$ satisfying $\bar{C} = (\bar{a}, +\infty)_\Omega$, so that substitution yields $\Omega_1 \cap (\bar{a}, +\infty)_\Omega = X$. Using the fact that the Law of the Excluded Middle gives rise to the true disjunction $\bar{a} \in \Omega_1 \vee \bar{a} \notin \Omega_1$, we prove $X \in \mathcal{O}_{<_{\Omega_1}}$ by considering corresponding sub-cases within the current first case. In the first sub-case $\bar{a} \in \Omega_1$, we obtain

$$[X =] \quad \Omega_1 \cap (\bar{a}, +\infty)_\Omega = (\bar{a}, +\infty)_{\Omega_1} \quad [\in \mathcal{O}_{<_{\Omega_1}}]$$

according to (3.496) and (11.569), consequently $X \in \mathcal{O}_{<_{\Omega_1}}$ as desired. The second sub-case $\bar{a} \notin \Omega_1$ yields with (3.497)

$$[X =] \quad \Omega_1 \cap (\bar{a}, +\infty)_\Omega = \Omega_1 \quad [\in \mathcal{O}_{<_{\Omega_1}}]$$

(recalling Property 2 of a topology on Ω_1) or

$$[X =] \quad \Omega_1 \cap (\bar{a}, +\infty)_\Omega = \emptyset \quad [\in \mathcal{O}_{<_{\Omega_1}}]$$

(using Corollary 11.43); thus, we obtain $X \in \mathcal{O}_{<_{\Omega_1}}$ irrespective of which part of the preceding disjunction is true. Having completed the proof for the first case, we consider now the second case $\bar{C} \in \{(-\infty, b) : b \in \Omega\}$. By definition of the set of open and left-unbounded intervals in Ω, there is then a particular element $\bar{b} \in \Omega$ for which $\bar{C} = (-\infty, \bar{b})$ holds, so that substitution gives now $\Omega_1 \cap (-\infty, \bar{b}) = X$. Similarly to the first case, we use now the true disjunction $\bar{b} \in \Omega_1 \vee \bar{b} \notin \Omega_1$ (provided by the Law of the Excluded Middle) to prove the desired consequent $X \in \mathcal{O}_{<_{\Omega_1}}$ via two sub-cases. The first sub-case $\bar{b} \in \Omega_1$ implies

$$[X =] \quad \Omega_1 \cap (-\infty, \bar{b})_\Omega = (-\infty, \bar{b})_{\Omega_1} \quad [\in \mathcal{O}_{<_{\Omega_1}}]$$

according to (3.502) and (11.570), with the desired consequence $X \in \mathcal{O}_{<_{\Omega_1}}$. Furthermore, the second sub-case $\bar{b} \notin \Omega_1$ evidently implies with (3.503) that

$$[X =] \quad \Omega_1 \cap (-\infty, \bar{b})_\Omega = \Omega_1 \quad [\in \mathcal{O}_{<_{\Omega_1}}]$$

or

$$[X =] \quad \Omega_1 \cap (-\infty, \bar{b})_\Omega = \emptyset \quad [\in \mathcal{O}_{<_{\Omega_1}}]$$

is true; in any of these cases, we obtain $X \in \mathcal{O}_{<_{\Omega_1}}$, so that the proof by cases is complete. As X was arbitrary, we may therefore conclude that the universal sentence (11.613) holds, from which we may then infer the truth of the inclusion $\mathcal{C}_{\Omega_1} \subseteq \mathcal{O}_{<_{\Omega_1}}$. Since $\mathcal{O}_{<_{\Omega_1}}$ is a topology on Ω_1, it follows with Proposition 11.79d) that the topology generated by the subbasis $\mathcal{C}_{\Omega_1}$, i.e. the subspace topology $\mathcal{O}_{<_\Omega}|\Omega_1$, is included in the order topology $\mathcal{O}_{<_{\Omega_1}}$.

To prove the second inclusion in (11.612), we first show that the inclusion $\mathcal{C}_{<_{\Omega_1}} \subseteq \mathcal{O}_{<_\Omega}|\Omega_1$ holds, which we do by proving

$$\forall Y \, (Y \in \mathcal{C}_{<_{\Omega_1}} \Rightarrow Y \in \mathcal{O}_{<_\Omega}|\Omega_1). \tag{11.614}$$

We take an arbitrary set Y and assume $Y \in \mathcal{C}_{<_{\Omega_1}}$. This assumption implies because of (11.611) and the definition of the union of two sets that

$$Y \in \{(a, +\infty) : a \in \Omega_1\} \vee Y \in \{(-\infty, b) : b \in \Omega_1\}$$

holds, and we use this disjunction now to prove $Y \in \mathcal{C}_{\Omega_1}$ by cases. The first case $Y \in \{(a, +\infty) : a \in \Omega_1\}$ evidently implies the existence of a particular element $\bar{a} \in \Omega_1$ with $Y = (\bar{a}, +\infty)_{\Omega_1}$, so that (3.496) yields

$$\Omega_1 \cap (\bar{a}, +\infty)_\Omega = (\bar{a}, +\infty)_{\Omega_1} \quad [= Y].$$

Here, $(\bar{a}, +\infty)_\Omega \in \mathcal{C}_\Omega$ is true according to (11.569), which shows in conjunction with $\Omega_1 \cap (\bar{a}, +\infty)_\Omega = Y$ that there exists a set C such that $C \in \mathcal{C}_\Omega$ and $\Omega_1 \cap C = Y$ are both true. This existential sentence implies with Theorem 11.80b) that $Y \in \mathcal{C}_{\Omega_1}$ holds indeed. Similarly to the first case, the second case $Y \in \{(-\infty, b) : b \in \Omega_1\}$ gives us now a particular element $\bar{b} \in \Omega_1$ with $Y = (-\infty, \bar{b})_{\Omega_1}$. Consequently, (3.502) yields

$$\Omega_1 \cap (-\infty, \bar{b})_\Omega = (-\infty, \bar{b})_{\Omega_1} \quad [= Y],$$

where $(-\infty, \bar{b})_\Omega \in \mathcal{C}_\Omega$ holds according to (11.570). The conjunction of this finding and the equation $\Omega_1 \cap (-\infty, \bar{b})_\Omega = Y$ demonstrates the existence of a set C satisfying both $C \in \mathcal{C}_\Omega$ and $\Omega_1 \cap C = Y$. As in the first case, we therefore obtain $Y \in \mathcal{C}_{\Omega_1}$, so that the proof by cases is now complete. Then, because the subspace topology $\mathcal{O}_{<_\Omega}|\Omega_1$ includes its generating subbasis $\mathcal{C}_{\Omega_1}$ (recall Proposition 11.78), the definition of a subset gives us $Y \in \mathcal{O}_{<_\Omega}|\Omega_1$, completing the proof of the implication in (11.614). Here, Y was arbitrary, so that the universal sentence (11.614) follows to be true, which in turn implies the inclusion $\mathcal{C}_{<_{\Omega_1}} \subseteq \mathcal{O}_{<_\Omega}|\Omega_1$. Together with the fact that $\mathcal{O}_{<_\Omega}|\Omega_1$ is a topology on Ω_1, this inclusion further implies that the topology generated by the subbasis $\mathcal{C}_{<_{\Omega_1}}$, namely the order topology $\mathcal{O}_{<_{\Omega_1}}$, is included in the subspace topology $\mathcal{O}_{<_\Omega}|\Omega_1$.

We thus proved both inclusions in (11.612), which give us now the equation (11.609). Because Ω, Ω_1 and $<_\Omega$ were initially arbitrary sets, we may therefore conclude that the stated universal sentence is true. □

We obtain then immediately the following relationship between the order topologies on $\mathbb{R}$ and on $\overline{\mathbb{R}}$.

Corollary 11.87. *It is true that the subspace topology of the standard topology on $\overline{\mathbb{R}}$ in $\mathbb{R}$ coincides with the standard topology on $\mathbb{R}$, i.e.*

$$\mathcal{O}_{<_{\overline{\mathbb{R}}}}|\mathbb{R} = \mathcal{O}_{<_\mathbb{R}}. \tag{11.615}$$

Proof. On the one hand, $\mathbb{R}$ is convex in $\overline{\mathbb{R}}$ with respect to $<_{\overline{\mathbb{R}}}$ (see Proposition 9.17), which linear ordering defines the order topology $\mathcal{O}_{<_{\overline{\mathbb{R}}}}$. On the other hand, the standard linear ordering $<_\mathbb{R}$, which defines the standard/order topology on $\mathbb{R}$ ($\mathcal{O}_{<_\mathbb{R}}$), is identical with the linear ordering obtained from $<_{\overline{\mathbb{R}}}$ via Theorem 3.68. We may therefore apply the preceding Theorem 11.86 to obtain the proposed equation (11.615). □

Product topologies

We explore now the situation in which the underlying set Ω is given by a Cartesian product of sets, where each individual set has a topology.

Theorem 11.88 (Subbasis for a topology on a Cartesian product of a family of sets). *The following sentences are true for any nonempty set I and any families of sets $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ such that $\mathcal{O}_i$ is a topology on Ω_i for any $i \in I$.*

a) *For any $j \in I$ there exists a unique set (system)*

$$\mathcal{C}^{(j)} = \{\pi_j^{-1}[U] : U \in \mathcal{O}_j\} \tag{11.616}$$

which consists of all the inverse images of open sets in Ω_j under the projection function $\pi_j : \bigtimes_{i\in I} \Omega_i \to \Omega_j$ in the sense of

$$\forall A\,(A \in \mathcal{C}^{(j)} \Leftrightarrow \exists U\,(U \in \mathcal{O}_j \;\wedge\; \pi_j^{-1}[U] = A)), \tag{11.617}$$

and this set system $\mathcal{C}^{(j)}$ is a π-system on $\bigtimes_{i\in I} \Omega_i$ containing $\bigtimes_{i\in I} \Omega_i$.

b) *Furthermore, there also exists a unique family of sets $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ such that*

$$\forall i\,(i \in I \Rightarrow \mathcal{C}_i = \{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}). \tag{11.618}$$

c) *Then, the union*

$$\mathcal{C}_{\times\Omega_i} = \bigcup_{i\in I} \mathcal{C}_i = \bigcup_{i\in I} \{\pi_i^{-1}[U] : U \in \mathcal{O}_i\} \tag{11.619}$$

of the family of sets $\mathcal{C}$ is a subbasis for a topology on the Cartesian product $\Omega = \bigtimes_{i\in I} \Omega_i$.

Proof. We let I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ be arbitrary sets such that $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ are families sets with index set $I \neq \emptyset$, and such $\mathcal{O}_i$ is a topology on Ω_i for all $i \in I$.

Concerning a), we let j be arbitrary and assume $j \in I$ to be true. We then notice in light of Proposition 3.249 that the projection function

$$\pi_j : \bigtimes_{i\in I} \Omega_i \to \Omega_j$$

is specified, whose domain is thus the Cartesian product of the family of sets $(\Omega_i)_{i\in I}$. Next, we may easily verify that the inverse image of any open set U in $\mathcal{O}_j$ is a specified element of the power set of the domain $\bigtimes_{i\in I}\Omega_i$ of π_j, that is,

$$\forall U\,(U \in \mathcal{O}_j \Rightarrow \pi_j^{-1}[U] \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)). \tag{11.620}$$

We let U be an arbitrary set, and we assume $U \in \mathcal{O}_j$ to be true. Since $\mathcal{O}_j$ is by assumption a topology on Ω_j, the inclusion $\mathcal{O}_j \subseteq \mathcal{P}(\Omega_j)$ holds according to Property 1 of a topology. Therefore, the preceding assumption implies $U \in \mathcal{P}(\Omega_j)$ with the definition of a subset, and this finding gives $U \subseteq \Omega_j$ by definition of a power set. Thus, the inverse image of U under π_j is defined, and $\pi_j^{-1}[U]$ is a subset of the domain $\bigtimes_{i\in I}\Omega_i$ of π_j because of Note 3.30, consequently an element of $\mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ by definition of a power set. As U was arbitrary, we may therefore conclude that the universal sentence (11.620) is indeed true. Next, we observe in light of the Axiom of Specification and the Equality Criterion for sets that there exists a unique set $\mathcal{C}^{(j)}$ such that

$$\forall A\,(A \in \mathcal{C}^{(j)} \Leftrightarrow [A \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)\ \wedge\ \exists U\,(U \in \mathcal{O}_j\ \wedge\ \pi_j^{-1}[U] = A)]). \tag{11.621}$$

To demonstrate that this set $\mathcal{C}^{(j)}$ satisfies (11.617), we take an arbitrary set A and assume first $A \in \mathcal{C}^{(j)}$ to be true. Then, the existential sentence in (11.617) follows especially to be true with (11.621), proving the first part ('$\Rightarrow$') of the equivalence. Assuming conversely that existential sentence to be true, there is thus a particular set $\bar{U} \in \mathcal{O}_j$ with $\pi_j^{-1}[\bar{U}] = A$. The former implies $\pi_j^{-1}[\bar{U}] \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ with (11.620), so that a substitution based on the preceding equation yields $A \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$. The conjunction of this finding and the assumed existential sentence in turn implies $A \in \mathcal{C}^{(j)}$ with (11.621), so that the second part of the equivalence in (11.617) also holds. Because A was arbitrary, we may now further conclude that the universal sentence (11.617) is true.

Concerning the assertion that $\mathcal{C}^{(j)}$ is a π-system on $\bigtimes_{i\in I}\Omega_i$ that contains $\bigtimes_{i\in I}\Omega_i$, we firstly note in view of (11.621) that $A \in \mathcal{C}^{(j)}$ implies especially also $A \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ for any A, which fact allows us to infer the truth of $\mathcal{C}^{(j)} \subseteq \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ by means of the definition of a subset. This inclusion shows that $\mathcal{C}^{(j)}$ possesses Property 1 of a π-system on $\bigtimes_{i\in I}\Omega_i$. Secondly, we observe that the inverse image of Ω_j under the projection function $\pi_j : \bigtimes_{i\in I}\Omega_i \to \Omega_j$ is given by

$$\pi_j^{-1}[\Omega_j] = \bigtimes_{i\in I}\Omega_i \tag{11.622}$$

according to (3.746), where $\Omega_j \in \mathcal{O}_j$ holds because of Property 2 of a topology (on Ω_j). Thus, the existential sentence

$$\exists U\,(U \in \mathcal{O}_j \;\wedge\; \pi_j^{-1}[U] = \bigtimes_{i\in I} \Omega_i)$$

is true, which in turn implies $\bigtimes_{i\in I} \Omega_i \in \mathcal{C}^{(j)}$ with (11.617). This finding shows that the set system $\mathcal{C}^{(j)}$ is nonempty, so that it satisfies Property 2 of a π-system. We may thirdly show that the set system $\mathcal{C}^{(j)}$ is closed under pairwise intersections, i.e.

$$\forall A, B\,(A, B \in \mathcal{C}^{(j)} \Leftrightarrow A \cap B \in \mathcal{C}^{(j)}). \tag{11.623}$$

Letting $A, B \in \mathcal{C}^{(j)}$ be arbitrary, there exist then due to (11.617) particular open sets $\bar{V}, \bar{W}$ of $\mathcal{O}_j$ satisfying $\pi_j^{-1}[\bar{V}] = A$ as well as $\pi_j^{-1}[\bar{W}] = B$. Since the topology $\mathcal{O}_j$ is by definition closed under pairwise intersections, $\bar{V}, \bar{W} \in \mathcal{O}_j$ implies $\bar{V} \cap \bar{W} \in \mathcal{O}_j$, so that this intersection is a subset of Ω_j. We then obtain for the inverse image of that intersection under the j-th projection function

$$\pi_j^{-1}[\bar{V} \cap \bar{W}] = \pi_j^{-1}[\bar{V}] \cap \pi_j^{-1}[\bar{W}] = A \cap B$$

by applying (3.760) and substitutions. These findings demonstrate the truth of the existential sentence

$$\exists U\,(U \in \mathcal{O}_j \;\wedge\; \pi_j^{-1}[U] = A \cap B),$$

and this gives us with (11.617) the desired consequent $A \cap B \in \mathcal{C}^{(j)}$ of the implication in (11.623). Since A and B are arbitrary, we may therefore conclude that $\mathcal{C}^{(j)}$ satisfies also Property 3 of a π-system. In summary, $\mathcal{C}^{(j)}$ constitutes a π-system on $\bigtimes_{i\in I} \Omega_i$ containing $\bigtimes_{i\in I} \Omega_i$. Here, j was initially arbitrary, so that the proposed universal sentence a) follows to be true.

Concerning b), we may apply Function definition by replacement to establish the unique existence of a function $\mathcal{C}$ with domain I such that $\mathcal{C}(i) = \mathcal{C}^{(i)}$ holds for all $i \in I$. For this purpose, we prove the universal sentence

$$\forall i\,(i \in I \Rightarrow \exists! \mathcal{Y}\,(\mathcal{Y} = \mathcal{C}^{(i)})), \tag{11.624}$$

letting i be arbitrary and assuming i to be an element of the index set I. Then, the set $\mathcal{C}^{(i)}$ is uniquely determined according to a), so that we may apply (1.109) to infer the truth of the uniquely existential sentence in (11.624). As i was arbitrary, we may therefore conclude that (11.624) is true, so that there exists indeed a unique function $\mathcal{C}$ with domain I such that

$$\forall i\,(i \in I \Rightarrow \mathcal{C}(i) = \mathcal{C}^{(i)}). \tag{11.625}$$

Thus, $\mathcal{C}$ is a family with index set I, which we may write as $(\mathcal{C}_i)_{i\in I}$, and the equation (11.616) shows that the terms of this family satisfy (11.618).

Concerning c), we need to show that the union (11.619) satisfies Property 1 and Property 2 of a subbasis for a topology on $\Omega = \bigtimes_{i\in I}\Omega_i$. Regarding Property 1, i.e. the required inclusion $\mathcal{C}_{\times\Omega_i} \subseteq \mathcal{P}(\Omega)$, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall C\,(C \in \bigcup_{i\in I}\mathcal{C}_i \Rightarrow C \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)). \tag{11.626}$$

We take an arbitrary set C and assume $C \in \bigcup_{i\in I}\mathcal{C}_i$ to be true. According to the Characterization of the union of a family of sets, this assumption implies the existence of a particular index $\bar{k} \in I$ such that $C \in \mathcal{C}_{\bar{k}}$. Here, $\bar{k} \in I$ implies with (11.625) $[\mathcal{C}_{\bar{k}} =]\ \mathcal{C}(\bar{k}) = \mathcal{C}^{(\bar{k})}$, so that substitution gives us $C \in \mathcal{C}^{(\bar{k})}$. Let us observe in light of (11.621) that $A \in \mathcal{C}^{(\bar{k})}$ implies in particular $A \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ for any A, so that the inclusion $\mathcal{C}^{(\bar{k})} \subseteq \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ follows to be true by definition of a subset. With this inclusion, $C \in \mathcal{C}^{(\bar{k})}$ implies now (again by definition of a subset) the desired consequent $C \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ of the implication in (11.626). Here, C is arbitrary, so that the universal sentence (11.626) follows to be true, and this in turn gives us the inclusion $\bigcup_{i\in I}\mathcal{C}_i \subseteq \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$. Thus, $\mathcal{C}_{\times\Omega_i} \subseteq \mathcal{P}(\Omega)$ is indeed true, which means that $\mathcal{C}_{\times\Omega_i}$ possesses Property 1 of a subbasis for a topology on the Cartesian product Ω.

Regarding Property 2, i.e. the required equation $\bigcup \mathcal{C}_{\times\Omega_i} = \Omega$, we apply the Equality Criterion for sets and prove accordingly the universal sentence

$$\forall \omega\,(\omega \in \bigcup \mathcal{C}_{\times\Omega_i} \Leftrightarrow \omega \in \Omega), \tag{11.627}$$

letting ω be arbitrary. We prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\omega \in \bigcup \mathcal{C}_{\times\Omega_i}$. Consequently, the definition of the union of a set system gives us a particular set $\bar{C} \in \mathcal{C}_{\times\Omega_i}$ for which $\omega \in \bar{C}$. We thus have $\bar{C} \in \bigcup_{j\in I}\mathcal{C}_j$ in view of (11.619), so that the Characterization of the union of a family of sets gives us also a particular index $\bar{k} \in I$ for which $\bar{C} \in \mathcal{C}_{\bar{k}}$ $[= \mathcal{C}^{(\bar{k})}]$. This further implies $\bar{C} \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ with (11.621), so that $\bar{C} \subseteq \bigtimes_{i\in I}\Omega_i$ follows to be true by definition of a power set. Using this inclusion, the previously established $\omega \in \bar{C}$ implies now $\omega \in \bigtimes_{i\in I}\Omega_i$ $[= \Omega]$, so that the first part of the equivalence in (11.627) holds.

To prove the second part ('$\Leftarrow$') directly, we assume now $\omega \in \Omega$ $[= \bigtimes_{i\in I}\Omega_i]$. Noting that the initial assumption $I \neq \emptyset$ evidently implies the existence of an element of I, say $\bar{k}$, so that the preceding Cartesian product is the domain of the $\bar{k}$-th projection mapping $\pi_{\bar{k}} : \bigtimes_{i\in I}\Omega_i \to \Omega_{\bar{k}}$, we obtain

$$\pi_{\bar{k}}^{-1}[\Omega_{\bar{k}}] = \bigtimes_{i\in I}\Omega_i$$

with Exercise 3.90b). Furthermore, since $\mathcal{O}_{\bar{k}}$ is by assumption a topology on $\Omega_{\bar{k}}$, we have $\Omega_{\bar{k}} \in \mathcal{O}_{\bar{k}}$ because of Property 2 of a topology. Therefore, we evidently obtain for the preceding inverse image

$$\left[\bigtimes_{i\in I}\Omega_i =\right] \quad \pi_{\bar{k}}^{-1}[\Omega_{\bar{k}}] \in \{\pi_{\bar{k}}^{-1}[U] : U \in \mathcal{O}_{\bar{k}}\} \quad \left[= \mathcal{C}^{(\bar{k})} = \mathcal{C}_{\bar{k}}\right].$$

In connection with $\bar{k} \in I$, the resulting $\bigtimes_{i\in I}\Omega_i \in \mathcal{C}_{\bar{k}}$ shows that there exists a constant j satisfying $j \in I$ as well as $\bigtimes_{i\in I}\Omega_i \in \mathcal{C}_j$, which existential sentence implies with the Characterization of the union of a family of sets

$$\bigtimes_{i\in I}\Omega_i \in \bigcup_{j\in I}\mathcal{C}_j \quad [= \mathcal{C}_{\times\Omega_i}].$$

Then, the conjunction of the resulting $\bigtimes_{i\in I}\Omega_i \in \mathcal{C}_{\times\Omega_i}$ and the previously established $\omega \in \bigtimes_{i\in I}\Omega_i$ clearly implies with the definition of the union of a set system that $\omega \in \bigcup\mathcal{C}_{\times\Omega_i}$ holds. This is the desired consequent of the second part of the equivalence in (11.627), which equivalence is thus true.

Since ω is arbitrary, we may now infer from the truth of this equivalence the truth of the universal sentence (11.627), and therefore the truth of the equation $\bigcup\mathcal{C}_{\times\Omega_i} = \Omega$. This means that $\mathcal{C}_{\times\Omega_i}$ has also Property 2 of a subbasis for a topology on the Cartesian product Ω, so that the proof of c) is complete.

As the sets I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ were initially arbitrary, the theorem follows then to be true. □

Note 11.33. Every subbasis element consists of families in $\bigtimes_{i\in I}\Omega_i$.

Exercise 11.36. Verify the following sentences for any topological spaces $(\Omega_1, \mathcal{O}_1)$ and $(\Omega_2, \mathcal{O}_2)$.

a) There exist unique set (systems)

$$\mathcal{C}^{(1)} = \{\pi_1^{-1}[U] : U \in \mathcal{O}_1\} \tag{11.628}$$

$$\mathcal{C}^{(2)} = \{\pi_2^{-1}[U] : U \in \mathcal{O}_2\} \tag{11.629}$$

such that $\mathcal{C}^{(1)}$ consists of all the inverse images of open sets in Ω_1 under the projection functions $\pi_1 : \Omega_1 \times \Omega_2 \to \Omega_1$ and such that

$\mathcal{C}^{(2)}$ consists of all the inverse images of open sets in Ω_2 under the projection functions $\pi_2 : \Omega_1 \times \Omega_2 \to \Omega_2$, in the sense of

$$\forall A\,(A \in \mathcal{C}^{(1)} \Leftrightarrow \exists U\,(U \in \mathcal{O}_1 \,\wedge\, \pi_1^{-1}[U] = A)), \tag{11.630}$$

$$\forall A\,(A \in \mathcal{C}^{(2)} \Leftrightarrow \exists U\,(U \in \mathcal{O}_2 \,\wedge\, \pi_2^{-1}[U] = A)), \tag{11.631}$$

and these set systems $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ are π-systems on $\Omega_1 \times \Omega_2$ containing $\Omega_1 \times \Omega_2$.

b) The union

$$\mathcal{C}_{\Omega_1\times\Omega_2} = \mathcal{C}^{(1)} \cup \mathcal{C}^{(2)} \tag{11.632}$$

is a subbasis for a topology on the Cartesian product $\Omega = \Omega_1 \times \Omega_2$.

Note 11.34. In the situation of Exercise 11.36, every subbasis element consists now of ordered pairs in $\Omega_1 \times \Omega_2$.

Exercise 11.37. Establish the following sentences for any $I \neq \emptyset$, for any families $(\Omega_i)_{i\in I}$, $(\mathcal{O}_i)_{i\in I}$ such that $(\Omega_i, \mathcal{O}_i)$ is a topological space for any $i \in I$.

a) For any $j \in I$ there exists the unique binary operation

$$\cap_{\mathcal{C}^{(j)}} : \mathcal{C}^{(j)} \times \mathcal{C}^{(j)} \to \mathcal{C}^{(j)}, \quad (A, B) \mapsto A \cap B, \tag{11.633}$$

and the Cartesian product $\bigtimes_{i\in I} \Omega_i$ is the neutral element in $\mathcal{C}^{(j)}$ with respect to $\cap_{\mathcal{C}^{(j)}}$.

b) In addition, the n-fold binary operation

$$\bigcap_{k=1}^{n} : \mathcal{C}_j^{\{1,\ldots,n\}} \to \mathcal{C}_j, \quad (A_k \,|\, k \in \{1, \ldots, n\}) \mapsto \bigcap_{k=1}^{n} A_i \tag{11.634}$$

is uniquely specified for any natural number n, which satisfies for any $n \in \mathbb{N}_+$ and any sequence of set $f = (A_k \,|\, k \in \{1, \ldots, n\})$ in $\mathcal{C}^{(j)}$

$$\bigcap_{k=1}^{n} A_k = \operatorname{ran}(f). \tag{11.635}$$

Notation 11.1. Given the subbasis $\mathcal{C}_{\times \Omega_i}$ for a topology on a Cartesian product $\bigtimes_{i\in I} \Omega_i$, we denote the corresponding basis $\mathcal{K}_{\mathcal{C}_{\times \Omega_i}}$ (according to the Generation of a basis for a topology by means of a subbasis) also by

$$\mathcal{K}_{\times \Omega_i}. \tag{11.636}$$

Similarly, in case that Ω is the Cartesian product of two sets Ω_1 and Ω_2, we will write

$$\mathcal{K}_{\Omega_1\times\Omega_2} \tag{11.637}$$

rather than $\mathcal{K}_{\mathcal{C}_{\Omega_1\times\Omega_2}}$.

Definition 11.30 (Product topology, product topological space). We call

(1) for any topological spaces $(\Omega_1, \mathcal{O}_1)$ and $(\Omega_2, \mathcal{O}_2)$ the topology

$$\mathcal{O}_1 \otimes \mathcal{O}_2 = \mathcal{O}(\mathcal{K}_{\Omega_1\times\Omega_2}) = \mathcal{O}(\mathcal{K}_{\mathcal{C}_{\Omega_1\times\Omega_2}}) \tag{11.638}$$

(generated by the subbasis $\mathcal{C}_{\Omega_1\times\Omega_2}$) the *product topology* of $\mathcal{O}_1$ and $\mathcal{O}_2$ (on $\Omega = \Omega_1 \times \Omega_2$). We then call the topological space

$$(\Omega_1 \times \Omega_2, \mathcal{O}_1 \otimes \mathcal{O}_2) \tag{11.639}$$

the *product topological space* with respect to $(\Omega_1, \mathcal{O}_1)$ and $(\Omega_2, \mathcal{O}_2)$.

(2) for any nonempty index set I, any family of sets $(\Omega_i)_{i\in I}$ and any family of topologies $(\mathcal{O}_i)_{i\in I}$ such that $\mathcal{O}_i$ is a topology on Ω_i for all $i \in I$ the topology

$$\bigotimes_{i\in I} \mathcal{O}_i = \mathcal{O}(\mathcal{K}_{\times \Omega_i}) = \mathcal{O}(\mathcal{K}_{\mathcal{C}_{\times \Omega_i}}) \tag{11.640}$$

(generated by the subbasis $\mathcal{C}_{\times \Omega_i}$) the *product topology* of $(\mathcal{O}_i)_{i\in I}$ (on $\Omega = \bigtimes_{i\in I} \Omega_i$). We then call the topological space

$$\left(\bigtimes_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{O}_i\right) \tag{11.641}$$

the *product topological space* with respect to $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$.

Notation 11.2. In case the index set of a Cartesian product $\bigtimes_{i\in I} \Omega_i$ is given by an initial segment $I = \{1, \ldots, n\}$ of $\mathbb{N}_+$ with $n \neq 0$, we write for a corresponding product topological space

$$\left(\bigtimes_{i\in\{1,\ldots,n\}} \Omega_i, \bigotimes_{i\in\{1,\ldots,n\}} \mathcal{O}_i\right) = \left(\bigtimes_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathcal{O}_i\right) \tag{11.642}$$

$$= (\Omega_1 \times \cdots \times \Omega_n, \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n). \tag{11.643}$$

Furthermore, we write in case of $I = \mathbb{N}_+$

$$\left(\bigtimes_{i\in\mathbb{N}_+} \Omega_i, \bigotimes_{i\in\mathbb{N}_+} \mathcal{O}_i\right) = \left(\bigtimes_{i=1}^{\infty} \Omega_i, \bigotimes_{i=1}^{\infty} \mathcal{O}_i\right) \tag{11.644}$$

$$= (\Omega_1 \times \Omega_2 \cdots, \mathcal{O}_1 \otimes \mathcal{O}_2 \cdots). \tag{11.645}$$

Note 11.35. Taking, respectively, the constant n-tuples (for any $n \in \mathbb{N}_+$) and sequence (on $\mathbb{N}_+$) with "values" $\mathbb{R}$ and $\mathcal{O}_{<_\mathbb{R}}$ yields

$$\left(\mathbb{R}^n, \bigotimes_{i=1}^{n} \mathcal{O}_{<_\mathbb{R}}\right) \tag{11.646}$$

with (4.384) and

$$\left(\omega, \bigotimes_{i=1}^{\infty} \mathcal{O}_{<_\mathbb{R}}\right) \tag{11.647}$$

with (8.138).

Definition 11.31 (Standard topology on $\mathbb{R}^n$, open set of $\mathbb{R}^n$). For any positive natural number n we call the product topology

$$\bigotimes_{i=1}^{n} \mathcal{O}_i = \bigotimes_{i=1}^{n} \mathcal{O}_{<_\mathbb{R}} = \mathcal{O}_{<_\mathbb{R}} \otimes \cdots \otimes \mathcal{O}_{<_\mathbb{R}} \tag{11.648}$$

on $\bigtimes_{i=1}^{n} \Omega_i = \mathbb{R}^n$ the *standard topology* on $\mathbb{R}^n$. We will refer to the elements of this topology as the *open sets of* $\mathbb{R}^n$.

Corollary 11.89. *For any nonempty index set I, any family of sets $(\Omega_i)_{i\in I}$ and any family of sets $(\mathcal{O}_i)_{i\in I}$ such that $\mathcal{O}_i$ is a topology on Ω_i for all $i \in I$, it is true that the product topology of $(\mathcal{O}_i)_{i\in I}$ is the singleton formed by the empty set if $(\Omega_i)_{i\in I}$ has an empty term, i.e.*

$$\exists i\,(i \in I \wedge \Omega_i = \emptyset) \Rightarrow \bigotimes_{i\in I} \mathcal{O}_i = \{\emptyset\}. \tag{11.649}$$

Proof. Letting I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ be arbitrary sets such that $\mathcal{O}_i$ is a topology on Ω_i for all $i \in I$, and assuming the antecedent of the implication (11.649) to be true, we obtain $\bigtimes_{i\in I} \Omega_i = \emptyset$ with the Emptiness Criterion for Cartesian products of families of sets. Since the product topology $\bigotimes_{i\in I} \mathcal{O}_i$ on the preceding empty Cartesian product is defined, and recalling from Proposition 11.45 that $\{\emptyset\}$ is the only topology on the empty set, the desired equation in (11.649) follows to be true. Since the sets I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ were initially arbitrary, we may therefore conclude that the corollary holds indeed. □

Theorem 11.90 (Characterization of the elements of the basis for a product topology). *For any nonempty index set I, any family $(\Omega_i)_{i\in I}$ of nonempty sets and any family of sets $(\mathcal{O}_i)_{i\in I}$ such that $\mathcal{O}_i$ is a topology on Ω_i for any $i \in I$, it is true that every element A of the basis $\mathcal{K}_{\times\Omega_i}$ (corresponding to the subbasis $\mathcal{C}_{\times\Omega_i}$ that generates the product topology on $\bigtimes_{i\in I}\Omega_i$) can be written, for some element U of the Cartesian product of $(\mathcal{O}_i)_{i\in I}$ and some finite subset J of I, as the Cartesian product of the terms of U_i where U_i is identical with Ω_i for all indexes i in $I \setminus J$, i.e.*

$$\forall A\,(A \in \mathcal{K}_{\times\Omega_i} \Rightarrow \exists U, J\,(U \in \bigtimes_{i\in I}\mathcal{O}_i \,\wedge\, A = \bigtimes_{i\in I} U_i \,\wedge\, J \subseteq I \,\wedge\, J \text{ is finite}$$
$$\wedge\, \forall i\,(i \in I \setminus J \Rightarrow U_i = \Omega_i))). \tag{11.650}$$

Proof. We take arbitrary sets I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$, assuming that $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ are families of sets with index set $I \neq \emptyset$ and with $\Omega_i \neq \emptyset$ for all $i \in I$, and assuming furthermore that $\mathcal{O}_i$ is a topology on Ω_i for all $i \in I$. We then take $\mathcal{C}_{\times\Omega_i}$ to be the subbasis generating the product topology on $\bigtimes_{i\in I}\Omega_i$ (according to Theorem 11.88) via the basis $\mathcal{K}_{\times\Omega_i}$ (according to Notation 11.1). Moreover, we take an arbitrary set A and assume A to be an element of that basis. The definition of a basis shows that the inclusion $\mathcal{K}_{\times\Omega_i} \subseteq \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ holds, so that the preceding assumption $A \in \mathcal{K}_{\times\Omega_i}$ implies $A \in \mathcal{P}(\bigtimes_{i\in I}\Omega_i)$ with the definition of a subset, and this in turn implies the inclusion $A \subseteq \bigtimes_{i\in I}\Omega_i$ with the definition of a power set. We thus note that every element ω of A is a family with index set I and $\omega_i \in \Omega_i$ for all $i \in I$. Moreover, $A \in \mathcal{K}_{\times\Omega_i}$ implies, according to the Generation of a basis for a topology by means of a subbasis, that there exists a particular positive natural number M as well as a particular sequence $\bar{B} : \{1,\ldots,M\} \to \mathcal{C}_{\times\Omega_i}$ such that $A = \bigcap \mathrm{ran}(\bar{B})$ holds.

In a first step, we construct a particular function G with domain $\{1,\ldots,M\}$, proving first the universal sentence

$$\forall k\,(k \in \{1,\ldots,M\} \Rightarrow \exists! Y\,(\forall z\,(z \in Y \Leftrightarrow [z \in I \times \mathcal{C}_{\times\Omega_i} \tag{11.651}$$
$$\wedge\, \exists j, V\,(j \in I \,\wedge\, V \in \mathcal{O}_j \,\wedge\, \pi_j^{-1}[V] = \bar{B}_k \,\wedge\, (j, \pi_j^{-1}[V]) = z)]))).$$

Letting $k \in \{1,\ldots,M\}$ be arbitrary, the truth of the uniquely existential sentence follows indeed to be true with the Axiom of Specification and the Equality Criterion for sets. Therefore, according to the Axiom of Replacement, there exists a unique function G with domain $\{1,\ldots,M\}$ such that

$$\forall k\,(k \in \{1,\ldots,M\} \Rightarrow \forall z\,(z \in G(k) \Leftrightarrow [z \in I \times \mathcal{C}_{\times\Omega_i} \tag{11.652}$$
$$\wedge\, \exists j, V\,(j \in I \,\wedge\, V \in \mathcal{O}_j \,\wedge\, \pi_j^{-1}[V] = \bar{B}_k \,\wedge\, (j, \pi_j^{-1}[V]) = z)])).$$

Let us prove now that $\emptyset \notin \operatorname{ran}(G)$ holds, which sentence we may write equivalently in the form

$$\forall Y \, (Y \in \operatorname{ran}(G) \Rightarrow Y \neq \emptyset) \tag{11.653}$$

by means of (2.5). Letting Y be arbitrary and assuming $Y \in \operatorname{ran}(G)$ to be true, the definition of a range gives us then a particular constant $\bar{k}$ with $(\bar{k}, Y) \in G$. Since we established G as a function with domain $\{1, \ldots, M\}$, we may write in function notation $Y = G(\bar{k})$, and we obtain $\bar{k} \in \{1, \ldots, M\}$ with the definition of a domain. Here, $\{1, \ldots, M\}$ is also the domain of $\bar{B} : \{1, \ldots, M\} \to \mathcal{C}_{\times \Omega_i}$, whose term $\bar{B}_{\bar{k}}$ is thus a uniquely determined element of its codomain $\mathcal{C}_{\times \Omega_i}$. Because this subbasis satisfies the equations (11.619), we obtain now via substitution

$$\bar{B}_{\bar{k}} \in \bigcup_{j \in I} \{\pi_j^{-1}[U] : U \in \mathcal{O}_j\}.$$

By definition of the union of a family of sets, there exists then a particular index $\bar{i} \in I$ with

$$\bar{B}_{\bar{k}} \in \{\pi_{\bar{i}}^{-1}[U] : U \in \mathcal{O}_{\bar{i}}\}.$$

According to Theorem 11.88, the preceding sentence implies the existence of a particular set $\bar{V} \in \mathcal{O}_{\bar{i}}$ such that $\pi_{\bar{i}}^{-1}[\bar{V}] = \bar{B}_{\bar{k}}$. Therefore, the previously established $\bar{B}_{\bar{k}} \in \mathcal{C}_{\times \Omega_i}$ implies via substitution $\pi_{\bar{i}}^{-1}[\bar{V}] \in \mathcal{C}_{\times \Omega_i}$, which shows in connection with $\bar{i} \in I$ and in light of the definition of the Cartesian product of two sets that the ordered pair $\bar{z} = (\bar{i}, \pi_{\bar{i}}^{-1}[\bar{V}])$ is an element of $I \times \mathcal{C}_{\times \Omega_i}$. Furthermore, the previous findings demonstrate the truth of the existential sentence

$$\exists j, V \, (j \in I \, \wedge \, V \in \mathcal{O}_j \, \wedge \, \pi_j^{-1}[V] = \bar{B}_k \, \wedge \, (j, \pi_j^{-1}[V]) = z),$$

so that $\bar{z}$ turns out to be an element of $G(\bar{k})$ $[= Y]$ in view of (11.652). Thus, Y is clearly nonempty, and since Y is arbitrary, we may therefore infer from this the truth of the universal sentence (11.653), and consequently the truth of $\emptyset \notin \operatorname{ran}(G)$.

This negation in turn implies with the Axiom of Choice that there exists a particular function $\bar{F} : \operatorname{ran}(G) \to \bigcup \operatorname{ran}(G)$ satisfying

$$\forall K \, (K \in \operatorname{ran}(G) \Rightarrow \bar{F}(K) \in K). \tag{11.654}$$

Consequently, the composition $\bar{F} \circ G$ is a function from $\{1, \ldots, M\}$ to $\bigcup \operatorname{ran}(G)$ because of Proposition 3.178.

Next, we verify that the range of $\bar{F} \circ G$ is included in the Cartesian product $I \times \mathcal{C}_{\times \Omega_i}$, by proving the universal sentence

$$\forall z \, (z \in \operatorname{ran}(\bar{F} \circ G) \Rightarrow z \in I \times \mathcal{C}_{\times \Omega_i}). \tag{11.655}$$

Letting z be arbitrary and assuming $z \in \operatorname{ran}(\bar{F} \circ G)$ to be true, the definition of a range gives us then a particular constant $\bar{k}$ for which $(\bar{k}, z) \in \bar{F} \circ G$ holds. Having already established $\bar{F} \circ G$ as a composition function with domain $\{1, \ldots, M\}$, we may write this in the form $z = \bar{F}(G(\bar{k}))$, and the definition of a domain yields $\bar{k} \in \{1, \ldots, M\}$. Clearly, $G(\bar{k})$ is in the range of G, so that we obtain $\bar{F}(G(\bar{k})) \in G(\bar{k})$ with (11.654), thus $z \in G(\bar{k})$ via substitution. Together with $\bar{k} \in \{1, \ldots, M\}$, this further implies $z \in I \times \mathcal{C}_{\times \Omega_i}$ due to (11.652), as desired. Here, z was arbitrary, so that the universal sentence (11.655) follows to be true. As this implies the inclusion $\operatorname{ran}(\bar{F} \circ G) \subseteq I \times \mathcal{C}_{\times \Omega_i}$ by definition of a subset, we now see on the one hand that this Cartesian product is indeed a codomain of the composition $\bar{F} \circ G$, and on the other hand that $\operatorname{ran}(\bar{F} \circ G)$ is a binary relation in view of (3.73). The latter finding allows us now to form the domain and the range of that binary relation, consisting of indexes of I and of subbasis elements of $\mathcal{C}_{\times \Omega_i}$, respectively.

Next, we define now a function H with domain $\operatorname{dom}(\operatorname{ran}(\bar{F} \circ G))$ such that

$$\begin{aligned} &\forall i \, (i \in \operatorname{dom}(\operatorname{ran}(\bar{F} \circ G)) \\ &\quad \Rightarrow \forall z \, (z \in H(i) \Leftrightarrow [z \in \operatorname{ran}(\bar{F} \circ G) \, \wedge \, \exists C \, (z = (i, C))])), \end{aligned} \tag{11.656}$$

which thus assigns to each index in I relevant to the expression of A as an intersection of subbasis elements the set of ordered pairs in $\operatorname{ran}(\bar{F} \circ G)$ having that index as first coordinate. To prove the existence of that function via replacement, we establish

$$\begin{aligned} &\forall i \, (i \in \operatorname{dom}(\operatorname{ran}(\bar{F} \circ G)) \\ &\quad \Rightarrow \exists! Y \, (\forall z \, (z \in H(i) \Leftrightarrow [z \in \operatorname{ran}(\bar{F} \circ G) \, \wedge \, \exists C \, (z = (i, C))]))), \end{aligned} \tag{11.657}$$

letting i be arbitrary and assuming the truth of $i \in \operatorname{dom}(\operatorname{ran}(\bar{F} \circ G))$. Then, the uniquely existential sentence follows to be true with the Axiom of Replacement and the Equality Criterion for sets. Since i is arbitrary, we may infer from this the truth of (11.657) and consequently the unique existence of a function H satisfying (11.656). Here, we see for any $z \in \operatorname{ran}(\bar{F} \circ G)$ that $z \in H(i)$ implies in particular $z \in \operatorname{ran}(\bar{F} \circ G)$ for any z, so that $H(i) \subseteq \operatorname{ran}(\bar{F} \circ G)$ follows to be true by definition of a subset. Recalling the previously established inclusion $\operatorname{ran}(\bar{F} \circ G) \subseteq I \times \mathcal{C}_{\times \Omega_i}$, we therefore obtain also the inclusion $H(i) \subseteq I \times \mathcal{C}_{\times \Omega_i}$ with (2.13), which shows that $H(i)$ is a binary relation (for any i in the domain of H). Our next task is to prove for any $i \in \operatorname{dom}(\operatorname{ran}(\bar{F} \circ G))$ that the range of $H(i)$ is included in the particular system $\{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}$ of inverse images under the i-th projection function. To do this, we let $i \in \operatorname{dom}(\operatorname{ran}(\bar{F} \circ G))$ be arbitrary and verify – according to the definition of a subset – the universal sentence

$$\forall C \, (C \in \operatorname{ran}(H(i)) \Leftrightarrow C \in \{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}). \tag{11.658}$$

We take an arbitrary set C and assume $C \in \mathrm{ran}(H(i))$ to be true. This assumption implies now by definition of a range that there is a constant, say $\bar{i}$, such that $(\bar{i}, C) \in H(i)$. This in turn implies with (11.656) $(\bar{i}, C) \in \mathrm{ran}(\bar{F} \circ G)$ and the existence of a particular set $\bar{C}$ satisfying $(\bar{i}, C) = (i, \bar{C})$. According to the Equality Criterion for ordered pairs, $\bar{i} = i$ is then true in particular, so that a substitution based on this equation gives $(i, C) \in \mathrm{ran}(\bar{F} \circ G)$. Another application of the definition of a range then gives us again a constant, say $\bar{k}$, such that $(\bar{k}, (i, C)) \in \bar{F} \circ G$ holds. Let us write this in the form $(i, C) = \bar{F}(G(\bar{k}))$, using the fact that the preceding composition is a function, and let us also observe that $\bar{k} \in \{1, \ldots, M\}$ $[= \mathrm{dom}(\bar{F} \circ G)]$ holds by definition of a domain. Noting here also that $G(\bar{k})$ is in the range of G, we therefore obtain $\bar{F}(G(\bar{k})) \in G(\bar{k})$ with (11.654), so that substitution yields $(i, C) \in G(\bar{k})$. According to (11.652), there are then further constants, say $\bar{j}$ and $\bar{U}$, such that $\bar{j} \in I$, $\bar{V} \in \mathcal{O}_{\bar{j}}$, $\pi_{\bar{j}}^{-1}[\bar{V}] = \bar{B}_{\bar{k}}$ and $(\bar{j}, \pi_{\bar{j}}^{-1}[\bar{V}]) = (i, C)$ hold. The latter implies with the Equality Criterion for ordered pairs $\bar{j} = i$ as well as $\pi_{\bar{j}}^{-1}[\bar{V}] = C$, so that we obtain via substitutions $\bar{V} \in \mathcal{O}_i$ and $\pi_i^{-1}[\bar{V}] = C$. These findings demonstrate that there exists an open set U of $\mathcal{O}_i$ that satisfies $\pi_i^{-1}[U] = C$, so that C turns out to be an element of the system $\{\pi_i^{-1}[U] : U \in \mathcal{O}_i$, according to Theorem 11.88a). We thus completed the proof of the implication in (11.658), in which the set C was arbitrary, so that universal sentence (11.658) and the equivalent inclusion

$$\mathrm{ran}(H(i)) \subseteq \{\pi_i^{-1}[U] : U \in \mathcal{O}_i\} \tag{11.659}$$

follow to be true. Since i was also arbitrary, we may therefore conclude that this inclusion holds for any i.

The idea is now to 'fuse' for any such index i all ordered pairs in $H(i)$ that have i as first coordinate, by forming the intersection of the corresponding second coordinates (i.e., by intersecting the corresponding subbasis elements). We may indeed prove the unique existence of a function h with domain $\mathrm{dom}(\mathrm{ran}(\bar{F} \circ G))$ for which

$$\forall i \, (i \in \mathrm{dom}(\mathrm{ran}(\bar{F} \circ G)) \Rightarrow h(i) = \bigcap \mathrm{ran}(H(i))), \tag{11.660}$$

applying again Function definition by replacement. To do this, we demonstrate the truth of the universal sentence

$$\forall i \, (i \in \mathrm{dom}(\mathrm{ran}(\bar{F} \circ G)) \Rightarrow \exists! y \, (y = \bigcap \mathrm{ran}(H(i)))), \tag{11.661}$$

letting i be arbitrary and assuming that $i \in \mathrm{dom}(\mathrm{ran}(\bar{F} \circ G))$ is true. Because that domain is the domain of H, we have the uniquely determined value $H(i)$, which constitutes a binary relation, as mentioned earlier. Consequently, the range of $H(i)$ is also a uniquely specified set. Let us verify that this range is nonempty, so that its intersection can be formed. The preceding assumption gives us, by definition of a domain, a particular constant $\bar{C}$ such that $(i, \bar{C}) \in \mathrm{ran}(\bar{F} \circ G)$ holds; since the existential sentence $\exists C\,((i, \bar{C}) = (i, C))$ is evidently true as well, we obtain $(i, \bar{C}) \in H(i)$ with (11.656). This finding shows that $H(i)$ is nonempty, which implies that $\mathrm{ran}(H(i))$ is also nonempty according (3.121), so that $\bigcap \mathrm{ran}(H(i))$ constitutes a uniquely specified set. Then, the uniquely existential sentence in (11.661) follows to be true with (1.109), and since i was arbitrary, we may therefore infer from this the truth of the universal sentence (11.661). Consequently, the function h satisfying (11.660) exists indeed uniquely, and we may view this function as a family with index set $J = \mathrm{dom}(h)$.

We are now in a position to show that the set $A = \bigcap \mathrm{ran}(\bar{B})$ can also be expressed as the intersection of (the range of) the family h, i.e. as the intersection $A = \bigcap_{i \in J} h_i$. We will achieve this by means of the Equality Criterion of sets, via the verification of the universal sentence

$$\forall \omega\,(\omega \in A \Leftrightarrow \omega \in \bigcap_{i \in J} h_i). \tag{11.662}$$

We let ω be arbitrary and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\omega \in A$. Since A can be written in the form of the intersections $\bigcap \mathrm{ran}(\bar{B}) = \bigcap_{k=1}^{M} \bar{B}_k$, the preceding assumption implies with the Characterization of the intersection of a family of sets

$$\forall k\,(k \in \{1, \ldots, M\} \Rightarrow \omega \in \bar{B}_k). \tag{11.663}$$

Using the same argument, we may write the desired consequent $\omega \in \bigcap_{i \in J} h_i$ equivalently as

$$\forall i\,(i \in J \Rightarrow \omega \in h_i), \tag{11.664}$$

which universal sentence we prove by letting i be arbitrary and by assuming

$$i \in J\ [= \mathrm{dom}(h) = \mathrm{dom}(\mathrm{ran}(\bar{F} \circ G))]. \tag{11.665}$$

According to (11.660), the corresponding term is given by $h_i = \bigcap \mathrm{ran}(H(i))$, so that the desired consequent $\omega \in h_i$ can be written equivalently as (using again the Characterization of the intersection of a family of sets)

$$\forall C\,(C \in \mathrm{ran}(H(i)) \Rightarrow \omega \in C). \tag{11.666}$$

Here, we take an arbitrary element $C \in \mathrm{ran}(H(i))$, so that there is (by definition of a range) a constant, say $\bar{j}$, for which $(\bar{j}, C) \in H(i)$. Recalling the truth of (11.665), the preceding finding implies especially $(\bar{j}, C) \in \operatorname{ran}(\bar{F} \circ G)$ with (11.657), so that there is (by definition of a range) another constant, say $\bar{k}$, satisfying $(\bar{k}, (\bar{j}, C)) \in \bar{F} \circ G$. Recalling that $\bar{F} \circ G$ is a function/composition with domain $\{1, \ldots, M\}$, we may write $(\bar{j}, C) = \bar{F}(G(\bar{k}))$, and we also see (in light of the definition of a domain) that $\bar{k} \in \{1, \ldots, M\}$ holds. Here, we note on the one hand that the latter implies $\omega \in \bar{B}_{\bar{k}}$ with (11.663), and on the other hand that $G(\bar{k})$ is in the range of G, so that $\bar{F}(G(\bar{k})) \in G(\bar{k})$ follows to be true with (11.654). Therefore, substitution based on the preceding equation gives us $(\bar{j}, C) \in G(\bar{k})$, which in turn implies with (11.652) the existence of a particular index $\bar{j}' \in I$ and of a particular open set $\bar{V} \in \mathcal{O}_{\bar{j}'}$ such that $\pi_{\bar{j}'}^{-1}[\bar{V}] = \bar{B}_{\bar{k}}$ and $(\bar{j}', \pi_{\bar{j}'}^{-1}[\bar{V}]) = (\bar{j}, C)$ are true. The latter equation implies then in particular $C = \pi_{\bar{j}'}^{-1}[\bar{V}]$ $[= \bar{B}_{\bar{k}}]$ with the Equality Criterion for ordered pairs, so that $C = \bar{B}_{\bar{k}}$ holds. With this equation, the previously found $\omega \in \bar{B}_{\bar{k}}$ yields via substitution $\omega \in C$, as desired. Here, C is arbitrary, which is why we may infer from this finding the truth of the universal sentence (11.666), and therefore the truth of the equivalent sentence $\omega \in h_i$. This is the desired consequent of the implication in (11.664), in which J is arbitrary, so that the universal sentence (11.664) follows to be true. Consequently, the equivalent sentence $\omega \in \bigcap_{i \in J} h_i$ holds as well, which completes the proof the implication '$\Rightarrow$' in (11.662).

To prove the second part ('$\Leftarrow$') of the equivalence, we assume $\omega \in \bigcap_{i \in J} h_i$ to be true, so that the equivalent universal sentence (11.664) holds as well. Recalling the equation $A = \bigcap_{k=1}^{M} \bar{B}_k$, we may establish the desired consequent $\omega \in A$ by proving the equivalent sentence (11.663), applying once again the Characterization of the intersection of a family of sets. Letting $k \in \{1, \ldots, M\}$ $[= \operatorname{dom}(\bar{F} \circ G)]$ be arbitrary, there is then (by definition of a domain) a particular set $\bar{z}$ for which $(k, \bar{z}) \in \bar{F} \circ G$. On the one hand, we may evidently write this as $\bar{z} = \bar{F}(G(k))$, where $G(k) \in \operatorname{ran}(G)$ holds, so that $\bar{F}(G(k)) \in G(k)$ follows to be true with (11.654). Consequently, we obtain $\bar{z} \in G(k)$ by means of substitution. On the other hand, $(k, \bar{z}) \in \bar{F} \circ G$ yields (using the definition of a range) $\bar{z} \in \operatorname{ran}(\bar{F} \circ G)$, which range we showed earlier to be included in $I \times \mathcal{C}_{\times \Omega_i}$, so that we also obtain $\bar{z} \in I \times \mathcal{C}_{\times \Omega_i}$ (by definition of a subset). Therefore, there are (by definition of the Cartesian product of two sets) particular elements $\bar{i} \in I$ and $\bar{C} \in \mathcal{C}_{\times \Omega_i}$ with $\bar{z} = (\bar{i}, \bar{C})$. With this equation, the previously established $\bar{z} \in G(k)$ gives us $(\bar{i}, \bar{C}) \in G(k)$ (using substitution). Due to $k \in \{1, \ldots, M\}$, the preceding finding further implies with (11.652) that there are particular elements $\bar{j} \in I$ and $\bar{V} \in \mathcal{O}_{\bar{j}}$ satisfying $\pi_{\bar{j}}^{-1}[\bar{V}] = \bar{B}_k$ as well as $(\bar{j}, \pi_{\bar{j}}^{-1}[\bar{V}]) = (\bar{i}, \bar{C})$. Applying now the Equality Criterion for sets to the latter equation and then a substitution based on the former equation, we get $\bar{C} = \bar{B}_k$, which allows us to write $\bar{z} = (\bar{i}, \bar{C})$ also as $\bar{z} = (\bar{i}, \bar{B}_k)$, and therefore $\bar{z} \in \mathrm{ran}(\bar{F} \circ G)$ as

$$(\bar{i}, \bar{B}_k) \in \mathrm{ran}(\bar{F} \circ G).$$

This demonstrates (in light of the definition of a domain) the truth of

$$\bar{i} \in \mathrm{dom}(\mathrm{ran}(\bar{F} \circ G)) \quad [= \mathrm{dom}(h) = J], \tag{11.667}$$

and evidently also the truth of

$$\exists C\,((\bar{i}, \bar{B}_k) = (\bar{i}, C)).$$

The previous three findings imply then $(\bar{i}, \bar{B}_k) \in H(\bar{i})$ with (11.656), with the evident consequence that $\bar{B}_k \in \mathrm{ran}(H(\bar{i}))$. Now, (11.667) implies $\omega \in h_{\bar{i}}$ with (11.664) and moreover $h_{\bar{i}} = \bigcap \mathrm{ran}(H(\bar{i}))$ because of (11.660), so that substitution gives $\omega \in \bigcap \mathrm{ran}(H(\bar{i}))$. In view of the Characterization of the intersection of a family of sets, we therefore have $\omega \in C$ for any $C \in \mathrm{ran}(H(\bar{i}))$, so that $\bar{B}_k \in \mathrm{ran}(H(\bar{i}))$ implies $\omega \in \bar{B}_k$. We thus completed the proof of the implication in (11.663), and since k was arbitrary, we may therefore conclude that the universal sentence (11.663) is true. Consequently, the equivalent $\omega \in \bigcap_{k=1}^{M} \bar{B}_k$ $[= A]$ also holds, so that the proof of the implication '$\Leftarrow$' in (11.662) is now complete.

Because ω was arbitrary, we may further conclude that the universal sentence (11.662) is true, which in turn implies then the proposed equation $A = \bigcap_{i \in J} h_i$.

Let us now bring out more clearly the fact that J is a subset of I, by proving the universal sentence

$$\forall i\,(i \in J \Rightarrow i \in I). \tag{11.668}$$

We take an arbitrary i and assume $i \in J$ to be true, so that the equations in (11.667) give $i \in \mathrm{dom}(\mathrm{ran}(\bar{F} \circ G))$. By definition of a domain, there exists then a constant, say $\bar{z}$, for which $(i, \bar{z}) \in \mathrm{ran}(\bar{F} \circ G)$ holds. Recalling again the inclusion $\mathrm{ran}(\bar{F} \circ G) \subseteq I \times \mathcal{C}_{\times \Omega_i}$, we therefore obtain by definition of a subset first $(i, \bar{z}) \in I \times \mathcal{C}_{\times \Omega_i}$, and subsequently the desired $i \in I$ by definition of the Cartesian product of two sets. Since i is arbitrary, we may now conclude that (11.668) holds, which universal sentence in turn implies $J \subseteq I$ with the definition of a subset.

Moreover, since any function from its domain to its range is a surjection by definition, we have $\bar{F} \circ G : \{1, \ldots, M\} \twoheadrightarrow \mathrm{ran}(\bar{F} \circ G)$. Recalling now the truth of $M \in \mathbb{N}_+$, the existence of such a natural number and surjection shows in light of the Finiteness Criterion (4.600) that $\mathrm{ran}(\bar{F} \circ G)$ is a finite set. Recalling furthermore that this range is a binary relation, we may now infer from this by means of Proposition 4.128 that its domain $J =$

dom(ran($\bar{F} \circ G$)) is itself a finite set. Moreover, because we established earlier the inclusion $H(i) \subseteq \mathrm{ran}(\bar{F} \circ G)$ for an arbitrary $i \in J$, we see in light of Exercise 4.40 that $H(i)$, as a subset of a finite set, is itself finite. It follows now with Exercise 4.38 that the range of the binary relation $H(i)$ is also finite. By definition of a finite set, there exists then a particular natural number $\bar{m}$ and a particular bijection $\bar{c} : \{1, \ldots, \bar{m}\} \rightleftarrows \mathrm{ran}(H(i))$, which allows us to view $\bar{c}$ as the sequence

$$(\bar{c}_k \,|\, k \in \{1, \ldots, \bar{m}\}) = (H(i)_k \,|\, k \in \{1, \ldots, \bar{m}\}),$$

whose terms $H(i)_1 = \bar{c}_1, \ldots, H(i)_{\bar{m}} = \bar{c}_{\bar{m}}$ make up the entire set ran($H(i)$). This then true for any $i \in J$ (since i was arbitrary), so that we may evidently use this notation to rewrite (11.660) equivalently in the form

$$\forall i \, (i \in J \Rightarrow h(i) = \bigcap \mathrm{ran}(H(i)) = \bigcap_{k=1}^{\bar{m}} H(i)_k). \tag{11.669}$$

In light of the inclusion (11.659), we see here for any $i \in J$ that $(H(i)_k \,|\, k \in \{1, \ldots, \bar{m}\})$ is a sequence in $\{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}$, which system is closed under n-fold intersections, as shown in Exercise 11.37b). Thus, the intersection of the sequence $(H(i)_k \,|\, k \in \{1, \ldots, \bar{m}\})$ is contained in the preceding system. Because i is arbitrary, we evidently obtain then from (11.669) the universal sentence

$$\forall i \, (i \in J \Rightarrow h(i) \in \{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}). \tag{11.670}$$

The previously found equation $A = \bigcap_{i \in J} h_i$ means therefore that the given basis element A can be written as the (finite) union of subbasis elements $\pi_i^{-1}[U]$ based on projections into sets Ω_i with distinct indexes i. Thus, our focus was so far on the identification of distinct indexes $i \in J$ and unique associated inverse images forming the intersection A. We now determine the family of open sets that give rise to the family of these inverse images. Let us verify for this purpose the universal sentence

$$\forall i \, (i \in J \Rightarrow \exists! Y \, (Y = \pi_i[h(i)])), \tag{11.671}$$

letting i be arbitrary and assuming $i \in J$ to be true. This assumption implies with (11.670) $h(i) \in \{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}$, so that there evidently exists an open set of $\mathcal{O}_i$, say $\bar{U}$, for which $\pi_i^{-1}[\bar{U}] = h(i)$ holds. Because we assumed $(\Omega_i)_{i \in I}$ to be a family of nonempty sets, it follows with the Surjectivity of projection functions with nonempty domain that π_i is a surjection from $\bigtimes_{i \in I} \Omega_i$ to Ω_j for any $j \in I$. Consequently, we obtain with (3.749) the equation $\pi_i[\pi_i^{-1}[\bar{U}]] = \bar{U}$ and therefore $\pi_i[h(i)] = \bar{U}$ by means of substitution. Since $\pi_i[h(i)]$ is the given constant $\bar{U}$, we may infer from this fact the truth of the uniquely existential sentence $\exists! Y\,(Y = \pi_i[h(i)])$ by using (1.109). Here, i is arbitrary, so that that (11.671) follows to be true. This universal sentence implies according to Function definition by replacement that there exists a unique function u with domain J such that

$$\forall i\,(i \in J \Rightarrow u(i) = \pi_i[h(i)]). \tag{11.672}$$

We define now also a family on the complement of J with respect to I through replacement, proving first the universal sentence

$$\forall i\,(i \in J^c \Rightarrow \exists! Y\,(Y = \Omega_i)). \tag{11.673}$$

Letting i be arbitrary and assuming $i \in J^c$ to be true (which complement exists in view of the previously establish fact $J \subseteq I$), we thus have $i \in I \setminus J$ and consequently $i \in I$ (by definition of a set difference). Therefore, Ω_i is a unique term of the family $(\Omega_i)_{i \in I}$, allowing us to infer from this the truth of the uniquely existential existence $\exists! Y\,(Y = \Omega_i)$ by means of (1.109). Here, i is arbitrary, so that (11.673) holds, and this universal sentence implies the unique existence of a function o with domain J^c such that

$$\forall i\,(i \in J^c \Rightarrow o(i) = \Omega_i). \tag{11.674}$$

Because the domains J and J^c of the functions u and o, respectively, are disjoint because of (2.135), it follows that u and o are compatible functions, according to Exercise 3.73. Then, we may apply Concatenation of Functions according to Proposition 3.176 in connection with (2.257) to obtain the new function $\bar{U} = u \cup o$ having the domain $J \cup J^c = I$. Thus, $\bar{U} = u \cup o$ constitutes a family with index set I, and we may show now in addition that this family satisfies

$$\forall i\,(i \in I \Rightarrow [(i \in J \Rightarrow \bar{U}_i = \pi_i[h(i)]) \wedge (i \in J^c \Rightarrow \bar{U}_i = \Omega_i)]), \tag{11.675}$$

letting i be arbitrary such that $i \in I$ is true. To establish the first part of the conjunction, we assume $i \in J$ to be true, which implies on the one hand $u(i) = \pi_i[h(i)]$ with (11.672), which equation we may write also as $(i, \pi_i[h(i)]) \in u$. Since $u \in \{u, o\}$ is clearly also true, it follows with the definitions of the union of a set system and of a pair that

$$(i, \pi_i[h(i)]) \in \bigcup \{u, o\}\ [= u \cup o = \bar{U}].$$

Because $\bar{U}$ is a function/family, we may write the resulting $(i, \pi_i[h(i)]) \in \bar{U}$ equivalently in the form of $\pi_i[h(i)] = \bar{U}_i$, which proves the first part of the conjunction in (11.675). Concerning the second part, we observe that the assumption $i \in J^c$ implies with (11.674) $o(i) = \Omega_i$, which we can write in the form $(i, \Omega_i) \in o$. In view of the evident $o\{u, o\}$, we clearly obtain

$$(i, \Omega_i) \in \bigcup\{u, o\} \; [= u \cup o = \bar{U}],$$

and we can write the resulting $(i, \Omega_i) \in \bar{U}$ in the form $\Omega_i = \bar{U}_i$. Thus, the proof of the conjunction in (11.675) is complete, and since i was arbitrary, we may therefore conclude that the suggested universal sentence (11.675) is indeed true. We are now in a position to conveniently prove

$$\forall i \, (i \in I \Rightarrow \bar{U}_i \in \mathcal{O}_i). \tag{11.676}$$

We let $i \in I$ be arbitrary, so that the previously found equation $I = J \cup J^c$ yields $i \in J \cup J^c$, giving rise to the true disjunction $i \in J \vee i \in J^c$ by definition of the union of two sets. Based on this disjunction, we prove now $\bar{U}_i \in \mathcal{O}_i$ by cases. The first case $i \in J$ implies on the one hand $\bar{U}_i = \pi_i[h(i)]$ with (11.675). On the other hand, the current case assumption implies with (11.670) $h(i) \in \{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}$, so that $\pi_i^{-1}[\bar{V}] = h(i)$ holds for some particular open set $\bar{V} \in \mathcal{O}_i$. Combining the previous two equations yields now

$$\bar{U}_i = \pi_i[\pi_i^{-1}[\bar{V}]] = \bar{V},$$

applying again (3.749) in connection with the previously mentioned fact that π_i is a surjection. Therefore, $\bar{V} \in \mathcal{O}_i$ gives the desired consequent $\bar{U}_i \in \mathcal{O}_i$ after substitution. The second case $i \in J^c$ implies with (11.675) $\bar{U}_i = \Omega_i$. Since $\Omega_i \in \mathcal{O}_i$ is true according to Property 2 of a topology, we immediately see that $\bar{U}_i \in \mathcal{O}_i$ holds also in the second case. We thus completed the proof of the implication in (11.676), in which i is arbitrary, so that the universal sentence (11.676) follows to be true. Recalling that $\bar{U}$ is a family with index set I, this universal sentence shows us in light of the definition of the Cartesian product of a family of sets that

$$\bar{U} \in \bigtimes_{i \in I} \mathcal{O}_i. \tag{11.677}$$

We now finally prove that the given basis element A can be written as the Cartesian product of this family $\bar{U} = (\bar{U}_i)_{i \in I}$, that is, we prove the equation $A = \bigtimes_{i \in I} \bar{U}_i$. Let us apply for this purpose the Equality Criterion for sets and prove accordingly the universal sentence

$$\forall \omega \, (\omega \in A \Leftrightarrow \omega \in \bigtimes_{i \in I} \bar{U}_i). \tag{11.678}$$

We let ω be arbitrary and assume first $\omega \in A$ to be true, which implies $\omega \in \bigcap_{i \in J} h_i$ with (11.662). Therefore, the Characterization of the intersection of a family of sets gives us the true universal sentence

$$\forall i\,(i \in J \Rightarrow \omega \in h_i). \tag{11.679}$$

To establish the desired consequent $\omega \in \bigtimes_{i\in I} \bar{U}_i$, we must show that ω is a family with index set I satisfying

$$\forall j\,(j \in I \Rightarrow \omega_j \in \bar{U}_j), \tag{11.680}$$

according to the definition of the Cartesian product of a family of sets. Recalling the previous observation that the inclusion $A \subseteq \bigtimes_{i\in I} \Omega_i$ holds, so that $\omega \in A$ implies $\omega \in \bigtimes_{i\in I} \Omega_i$ by definition of a subset, we thus see (in light of the definition of the Cartesian product of a family of sets) that ω is indeed a family with index set I. Next, we show that ω also satisfies (11.680), letting j be arbitrary and assuming that $j \in I$ holds. Recalling that this implies the truth of the disjunction $j \in J \vee j \in J^c$, we prove now $\omega_j \in \bar{U}_j$ by cases.

The first case $j \in J$ gives us firstly $\omega \in h_i$ with (11.679), secondly $\bar{U}_j = \pi_j[h_j]$ with (11.675), and thirdly $h_j \in \{\pi_j^{-1}[U] : U \in \mathcal{O}_j\}$ with (11.670). The latter implies the existence of a particular open set $\bar{V} \in \mathcal{O}_j$ such that $\pi_j^{-1}[\bar{V}] = h_j$, so that the second one of the preceding three findings yields via substitution $\bar{U}_j = \pi_j[\pi_j^{-1}[\bar{V}]]$ and therefore $\bar{U}_j = \bar{V}$, applying once again (3.749) in connection with the surjectivity of π_j. The latter equation allows us to write the previous finding $\pi_j^{-1}[\bar{V}] = h_j$ equivalently as $h_j = \pi_j^{-1}[\bar{U}_j]$, so that the previously obtained $\omega \in h_i$ yields after substitution $\omega \in \pi_j^{-1}[\bar{U}_j]$. According to the definition of an inverse image, $\pi_j(\omega) \in \bar{U}_j$ is then also true, and this gives us $\omega_j \in \bar{U}_j$ by definition of a projection function, as required.

The second case $j \in J^c$ implies $\bar{U}_j = \Omega_j$ with (11.675). Noting that the projection function $\pi_j : \bigtimes_{i\in I} \Omega_i \to \Omega_j$ gives rise to the inverse image $\pi_j^{-1}[\Omega_j] = \bigtimes_{i\in I} \Omega_i$ according to (3.746), we have that the previously stated $\omega \in \bigtimes_{i\in I} \Omega_i$ can be written after two substitutions as $\omega \in \pi_j^{-1}[\bar{U}_j]$. This in turn implies $\pi_j(\omega) \in \bar{U}_j$ with the definition of an inverse image, and therefore $\omega_j \in \bar{U}_j$ as in the first case. Having thus proved the implication in (11.680), the universal sentence (11.680) follows now to be true, because j was arbitrary. We thus completed the demonstration that ω is in the Cartesian product $\bigtimes_{i\in I} \bar{U}_i$, so that the first part ('$\Rightarrow$') of the equivalence in (11.678) holds.

To establish the second part ('$\Leftarrow$'), we conversely assume $\omega \in \bigtimes_{i\in I} \bar{U}_i$ to be true, so that ω is evidently a family with index set I whose terms satisfy (11.680). Let us now prove the universal sentence (11.679), by taking an arbitrary index $j \in J$. Recalling that J is a subset of I, we also have that $j \in I$ holds, so that $\omega_j \in \bar{U}_j$ follows to be true with (11.680). Here, ω_j

is the value of the projection of ω under π_j, so that we may also write $\pi_j(\omega) \in \bar{U}_j$. Evidently, $\omega \in \pi_j^{-1}[\bar{U}_j]$ holds then as well. In addition, $j \in J$ implies $\bar{U}_j = \pi_j[h_j]$ with (11.675), so that the preceding finding yields $\omega \in \pi_j^{-1}[\pi_j[h_j]]$ $[= h_j]$, therefore $\omega \in h_j$. Here, j is arbitrary, so that the universal sentence (11.679) holds indeed. This means that $\omega \in \bigcap_{i\in J} h_i$ is true, which in turn implies $\omega \in A$ with (11.662).

We thus completed the proof of the equivalence in (11.678). Since A was arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (11.678), and therefore also the truth of the equation

$$A = \bigtimes_{i\in I} \bar{U}_i. \tag{11.681}$$

Having found a family $\bar{U}$ and a finite subset J of I satisfying (11.677), (11.681) and (11.675), we now see that the existential sentence in (11.650) holds. Since the basis element A was arbitrary, the universal sentence (11.650) follows then to be also true. Finally, because I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ were also arbitrary, we may infer from this the truth of the stated theorem. □

In the preceding theorem, we started off with the basis for the product topology of a family of topological spaces. The following theorem deals with case that a family of (individual) bases is given, and shows that a basis for the resulting product topology has a similar structure as the given basis for the product topology in the preceding situation.

Theorem 11.91 (Generation of a basis for a product topology by means of a family of bases). *The following sentences are true for any nonempty index set I and any families $(\Omega_i)_{i\in I}$, $(\mathcal{K}_{\Omega_i})_{i\in I}$ such that $\mathcal{K}_{\Omega_i}$ is a basis for a topology on Ω_i for any $i \in I$.*

a) There exists a unique set (system) $\mathcal{K}'_{\times \Omega_i}$ consisting of all elements A (in the power set of the Cartesian product of $(\Omega_i)_{i\in I}$) for which there is an element U of the Cartesian product of the family $(\mathcal{O}(\mathcal{K}_{\Omega_i}))_{i\in I}$ of generated topologies and a finite subset J of I such that A can be written as the Cartesian product of the terms of U where U_i is a basis element in $\mathcal{K}_{\Omega_j}$ for any i in J and where U_i equals Ω_i for all the other indexes in I, in the sense that

$$\forall A\,(A \in \mathcal{K}'_{\times \Omega_i} \Leftrightarrow [A \in \mathcal{P}(\bigtimes_{i\in I} \Omega_i) \wedge \exists U, J\,(U \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$$
$$\wedge\ A = \bigtimes_{i\in I} U_i \ \wedge\ J \subseteq I \ \wedge\ J \text{ is finite}$$
$$\wedge\ \forall i\,([i \in J \Rightarrow U_i \in \mathcal{K}_{\Omega_i}] \wedge [i \in I \setminus J \Rightarrow U_i = \Omega_i]))]). \tag{11.682}$$

b) The set $\mathcal{K}'_{\times\,\Omega_i}$ *is a basis for a topology on* $\bigtimes_{i\in I}\Omega_i$.

c) The product topology of the family $(\mathcal{O}(\mathcal{K}_{\Omega_i}))_{i\in I}$ *of generated topologies is generated by the basis* $\mathcal{K}'_{\times\,\Omega_i}$, *i.e.*

$$\bigotimes_{i\in I}\mathcal{O}(\mathcal{K}_{\Omega_i}) = \mathcal{O}(\mathcal{K}'_{\times\,\Omega_i}). \tag{11.683}$$

Proof. We let I, $(\Omega_i)_{i\in I}$ and $(\mathcal{K}_{\Omega_i})_{i\in I}$ be arbitrary sets, assuming $I \neq \emptyset$, assuming $(\Omega_i)_{i\in I}$ to be a family of sets, and assuming furthermore $(\mathcal{K}_{\Omega_i})_{i\in I}$ to be a family such that $\mathcal{K}_{\Omega_i}$ is a basis for a topology on Ω_i for any $i \in I$. We then see in light of the Axiom of Specification and the Equality Criterion for sets that there exists indeed a unique set $\mathcal{K}'_{\times\,\Omega_i}$ satisfying (11.682). The idea is now to establish b) and c) jointly by means of the Characterization of the basis generating a given topology, whose application we prepare by verifying the required condition

$$\mathcal{K}'_{\times\,\Omega_i} \subseteq \bigotimes_{i\in I}\mathcal{O}(\mathcal{K}_{\Omega_i}). \tag{11.684}$$

To do this, we use the definition of a subset and establish the truth of the equivalent universal sentence

$$\forall A\,(A \in \mathcal{K}'_{\times\,\Omega_i} \Rightarrow A \in \bigotimes_{i\in I}\mathcal{O}(\mathcal{K}_{\Omega_i})), \tag{11.685}$$

letting A be arbitrary and assuming $A \in \mathcal{K}'_{\times\,\Omega_i}$ to be true. This assumption implies with (11.682) especially the existence of a particular element $\bar{U} \in \bigtimes_{i\in I}\mathcal{O}(\mathcal{K}_{\Omega_i})$ and of a particular finite subset $\bar{J} \subseteq I$ satisfying the equation $A = \bigtimes_{i\in I}\bar{U}_i$ and the universal sentence

$$\forall i\,([i \in \bar{J} \Rightarrow \bar{U}_i \in \mathcal{K}_{\Omega_i}] \wedge [i \in I \setminus \bar{J} \Rightarrow \bar{U}_i = \Omega_i]). \tag{11.686}$$

The latter implies

$$\forall i\,(i \in I \setminus \bar{J} \Rightarrow \bar{U}_i = \Omega_i),$$

noting that an arbitrary i satisfies especially the second part $i \in I \setminus \bar{J} \Rightarrow \bar{U}_i = \Omega_i$ of the conjunction in (11.686), so that the preceding universal sentence follows indeed to be true (since i is arbitrary). These findings show that the existential sentence

$$\exists U, J\,(U \in \bigtimes_{i\in I}\mathcal{O}(\mathcal{K}_{\Omega_i}) \wedge A = \bigtimes_{i\in I}U_i \wedge J \subseteq I \wedge J \text{ is finite}$$
$$\wedge\ \forall i\,(i \in I \setminus J \Rightarrow U_i = \Omega_i))$$

holds, which in turn implies $A \in \mathcal{K}_{\times \Omega_i}$ with the Characterization of the elements of the basis for a product topology. Since the preceding basis generates the product topology, we have the inclusion $\mathcal{K}_{\times \Omega_i} \subseteq \bigotimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$ due to Proposition 11.55. With this, $A \in \mathcal{K}_{\times \Omega_i}$ implies now the desired consequent of the implication in (11.685), in which A is arbitrary, so that the universal sentence (11.685) follows to be true. We thus completed the proof of the required inclusion (11.684), allowing us to proceed with the previously mentioned proof idea. We prove now accordingly the universal sentence

$$\forall B, f\,([B \in \bigotimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i}) \wedge f \in B] \Rightarrow \exists A'\,(A' \in \mathcal{K}'_{\times \Omega_i} \wedge f \in A' \wedge A' \subseteq B)), \tag{11.687}$$

letting B and f be arbitrary and assuming that $B \in \bigotimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$ and $f \in B$ are both true. Because $\bigotimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$ is the unique topology generated by the basis $\mathcal{K}_{\times \Omega_i}$, it follows from $B \in \bigotimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$ and from $f \in B$, according to the Generation of a topology by means of a basis, that there exists a particular basis element $\bar{A} \in \mathcal{K}_{\times \Omega_i}$ satisfying

$$f \in \bar{A} \wedge \bar{A} \subseteq B. \tag{11.688}$$

Here, $\bar{A} \in \mathcal{K}_{\times \Omega_i}$ implies – according to the Characterization of the elements of the basis for a product topology – that there exists a particular element $\bar{U} \in \bigtimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$ and a particular finite set $\bar{J} \subseteq I$ such that $\bar{A} = \bigtimes_{i \in I} \bar{U}_i$ and

$$\forall i\,(i \in I \setminus \bar{J} \Rightarrow \bar{U}_i = \Omega_i) \tag{11.689}$$

are true. Due to the former equation, the first part of the conjunction in (11.688) yields $f \in \bigtimes_{i \in I} \bar{U}_i$ and the second part of that conjunction gives $\bigtimes_{i \in I} \bar{U}_i \subseteq B$ via substitutions.

Let us now observe that the elements $\bar{U} \in \bigtimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$ and $f \in \bigtimes_{i \in I} \bar{U}_i$ constitute, by definition of the Cartesian product of a family of sets, families with index set I and terms

$$\forall i\,(i \in I \Rightarrow \bar{U}_i \in \mathcal{O}(\mathcal{K}_{\Omega_i})), \tag{11.690}$$
$$\forall i\,(i \in I \Rightarrow f_i \in \bar{U}_i). \tag{11.691}$$

We may therefore choose for any $i \in I$ (according to the Generation of a topology by means of a basis) some basis element $\bar{A}_i \in \mathcal{K}_{\Omega_i}$ containing f_i and included in $\bar{U}_i$. Let us now collect all the basis elements that correspond to indexes in $\bar{J}$, by establishing

$$\forall i\,(i \in \bar{J} \Rightarrow \exists! \mathcal{Y}\,(\forall K\,(K \in \mathcal{Y} \Leftrightarrow [K \in \mathcal{K}_{\Omega_i} \wedge f_i \in K \wedge K \subseteq \bar{U}_i]))).$$

Letting i be arbitrary and assuming $i \in \bar{J}$ to be true, the uniquely existential sentence follows then to be true with the Axiom of Specification and the Equality Criterion for sets. According to Function definition by replacement, there exists then a unique function F with domain $\bar{J}$ such that

$$\forall i\,(i \in \bar{J} \Rightarrow \forall K\,(K \in F(i) \Leftrightarrow [K \in \mathcal{K}_{\Omega_i} \wedge f_i \in K \wedge K \subseteq \bar{U}_i])). \quad (11.692)$$

Here, we may readily prove that the range of F does not contain the empty set, by demonstrating the truth of the equivalent universal sentence

$$\forall X\,(X \in \mathrm{ran}(F) \Rightarrow X \neq \emptyset) \quad (11.693)$$

according to (2.5). We take an arbitrary set X and assume $X \in \mathrm{ran}(F)$ to be true, so that the definition of a range gives us a particular constant $\bar{k}$ satisfying $(\bar{k}, X) \in F$. We thus have in function notation $X = F(\bar{k})$, where $\bar{k} \in \bar{J}$ [$= \mathrm{dom}(F)$] holds by definition of a domain. Consequently, we obtain with (11.692) the true universal sentence

$$\forall K\,(K \in F(\bar{k}) \Leftrightarrow [K \in \mathcal{K}_{\Omega_{\bar{k}}} \wedge f_{\bar{k}} \in K \wedge K \subseteq \bar{U}_{\bar{k}}]). \quad (11.694)$$

Furthermore, $\bar{k} \in \bar{J}$ implies in particular $\bar{k} \in I$ with the previously obtained inclusion $\bar{J} \subseteq I$ and the definition of a subset, and this finding further implies $\bar{U}_{\bar{k}} \in \mathcal{O}(\mathcal{K}_{\Omega_{\bar{k}}})$ as well as $f_{\bar{k}} \in \bar{U}_{\bar{k}}$ with (11.690) – (11.691). As indicated earlier, these findings imply (according to the Generation of a topology by means of a basis) the existence of a particular basis element $\bar{A}_{\bar{k}} \in \mathcal{K}_{\Omega_{\bar{k}}}$ containing $f_{\bar{k}}$ and included in $\bar{U}_{\bar{k}}$. Therefore, the basis element $\bar{A}_{\bar{k}}$ follows to be an element of $F(\bar{k})$ due to (11.694), which clearly demonstrates the truth of [$X =$] $F(\bar{k}) \neq \emptyset$. Since X was arbitrary, we may now infer from the truth of the resulting inequality $X \neq \emptyset$ the truth of the universal sentence (11.693), and then also the truth of $\emptyset \notin \mathrm{ran}(F)$. The preceding inequality in turn implies with the Axiom of Choice that there exists a particular function $\bar{g} : \mathrm{ran}(F) \to \bigcup \mathrm{ran}(F)$ such that

$$\forall X\,(X \in \mathrm{ran}(F) \Rightarrow \bar{g}(X) \in X). \quad (11.695)$$

We then obtain also the composition $\bar{g} \circ F : \bar{J} \to \bigcup \mathrm{ran}(F)$ with Proposition 3.178.

Next, we consider the restriction of the family $\bar{U}$ (with index set I) to $I \setminus \bar{J}$, which is then a function/family with index $I \setminus \bar{J}$ due to Proposition 3.164 and the fact that the inclusion $I \setminus \bar{J} \subseteq I$ holds with (2.125). Since $\bar{J}$ and $I \setminus \bar{J}$ are disjoint sets because of (2.111), we now see that the domains of the functions $\bar{g} \circ F$ and $\bar{U} \upharpoonright [I \setminus \bar{J}]$ are disjoint. Therefore, these two functions are compatible according to Exercise 3.73, which finding allows us to apply Concatenation of Functions (as shown by Proposition 3.176) to obtain the new function

$$H = (\bar{g} \circ F) \cup (\bar{U} \upharpoonright [I \setminus \bar{J}]) = \bigcup \{\bar{g} \circ F, \bar{U} \upharpoonright [I \setminus \bar{J}]\}, \tag{11.696}$$

and this function has the domain I, noting that the inclusion $\bar{J} \subseteq I$ implies $\bar{J} \cup (I \setminus \bar{J}) = I$ with (2.263) and the Commutative Law for the union of two sets. This shows that H is a family with index set I, so that we may form the Cartesian product $\bar{A}' = \bigtimes_{i \in I} H_i$. To bring out more clearly the properties of the family H, we prove the universal sentence

$$\begin{aligned} \forall i\, (i \in I \Rightarrow ([i \in \bar{J} \Rightarrow (H_i \in \mathcal{K}_{\Omega_i} \,\wedge\, f_i \in H_i \,\wedge\, H_i \subseteq \bar{U}_i)] \\ \wedge [i \in I \setminus \bar{J} \Rightarrow H_i = \Omega_i])), \end{aligned} \tag{11.697}$$

letting i be arbitrary, assuming $i \in I$ to be true, and assuming (concerning the implication in the first part of the conjunction) furthermore $i \in \bar{J}$ [$=$ $\mathrm{dom}(\bar{g} \circ F) = \mathrm{dom}(F)$] to hold. By definition of a domain, there is then a particular constant $\bar{X}$ for which $(i, \bar{X}) \in \bar{g} \circ F$. Since (11.696) clearly shows that $\bar{g} \circ F$ is an element of the pair/set system $\{\bar{g} \circ F, \bar{U} \upharpoonright [I \setminus \bar{J}]\}$, it follows that $(i, \bar{X})$ is an element also of the union of that set system, i.e. any element of H. We may write this finding in function/sequence notation as $\bar{X} = H(i) = H_i$, and we may therefore write $(i, \bar{X}) \in \bar{g} \circ F$ in the form $H_i = \bar{g}(F(i))$. Recalling that i is also in the domain of F, to which index $F_i = F(i)$ is the associated term/value, we may then write $(i, F_i) \in F$, which demonstrates that F_i is in the range of F. We may therefore apply (11.695) to infer from $F_i \in \mathrm{ran}(F)$ the truth of $[H_i =]\ \bar{g}(F(i)) \in F(i)$. The resulting $H_i \in F_i$ implies now, in view of the current assumption $i \in \bar{J}$, with (11.692) that the conjunction of $H_i \in \mathcal{K}_{\Omega_i}$, $f_i \in H_i$ and $H_i \subseteq \bar{U}_i$ holds, so that the implication in the first part of the conjunction in (11.697) is true. Concerning the second part, we assume $i \in I \setminus \bar{J}$ [$= \mathrm{dom}(\bar{U} \upharpoonright [I \setminus \bar{J}])$], so that we now have $(i, \bar{Y}) \in \bar{U} \upharpoonright [I \setminus \bar{J}]$ for some particular constant $\bar{Y}$. Evidently, that restriction is an element of the pair/set system $\{\bar{g} \circ F, \bar{U} \upharpoonright [I \setminus \bar{J}]\}$, so that the union H of that set system contains $(i, \bar{Y})$. Writing this in the form $\bar{Y} = H(i) = H_i$ and noting that $(i, \bar{Y}) \in \bar{U} \upharpoonright [I \setminus \bar{J}]$ implies $(i, \bar{Y}) \in \bar{U}$ by definition of a restriction, we obtain $H_i = \bar{U}_i$. As the current assumption $i \in I \setminus \bar{J}$ implies $\bar{U}_i = \Omega_i$ with (11.689), we arrive at the desired consequent $H_i = \Omega_i$ of the second part of the conjunction in (11.697), which conjunction is thus true. Here, i was initially arbitrary, so that the universal sentence (11.697) follows now to be true.

We are now in a position to show that the set $\bar{A}'$ contains f, i.e. that $f \in \bigtimes_{i \in I} H_i$ holds. We observed already that f is a family with index set, so that it remains for us only to demonstrate (according to the definition of the Cartesian product of a family of sets) the truth of

$$\forall i\,(i \in I \Rightarrow f_i \in H_i). \tag{11.698}$$

We take an arbitrary index $i \in I$ $[= \bar{J} \cup (I \setminus \bar{J})]$ and note that the equation yields the true disjunction $i \in \bar{J} \vee i \in I \setminus \bar{J}$ by definition of the union of two sets. We use this disjunction to prove the desired consequent $f_i \in H_i$ by cases. This consequent is directly implied by the first case $i \in \bar{J}$ by means of (11.697). The other case $i \in I \setminus \bar{J}$ implies on the one hand $H_i = \Omega_i$ again with (11.697), and on the other hand $\bar{U}_i = \Omega_i$ with (11.689); we note in addition that $i \in I$ implies $f_i \in \bar{U}_i$ with (11.691), so that substitutions give us $f_i \in H_i$, as desired. Thus, the proof by cases is complete, and ss i was arbitrary, we may therefore conclude that (11.698) holds. Consequently, we obtain indeed the suggested sentence

$$f \in \bar{A}' \quad [= \bigtimes_{i \in I} H_i]. \tag{11.699}$$

Our next task is to show that $\bar{A}'$ is included in B, for which purpose we prove first the universal sentence

$$\forall i\,(i \in I \Rightarrow H_i \subseteq \bar{U}_i), \tag{11.700}$$

letting i be an arbitrary index in I and considering the same two cases as before. Then, the first case $i \in \bar{J}$ immediately implies the desired consequent $H_i \subseteq \bar{U}_i$ with (11.697). As before, the second case $i \in I \setminus \bar{J}$ yields $H_i = \Omega_i$ and $\bar{U}_i = \Omega_i$ with, respectively, (11.697) and (11.689). Since $H_i \subseteq H_i$ holds according to (2.10), we get the desired $H_i \subseteq \bar{U}_i$ via substitution. Having thus completed the proof by cases, we may subsequently infer from this the truth of the universal sentence (11.700), which in turn implies the inclusion $\bigtimes_{i \in I} H_i \subseteq \bigtimes_{i \in I} \bar{U}_i$ with Proposition 3.246. Recalling now the previously established inclusion $\bigtimes_{i \in I} \bar{U}_i \subseteq B$, we obtain the proposed inclusion

$$[\bigtimes_{i \in I} H_i =] \quad \bar{A}' \;\subseteq\; B \tag{11.701}$$

by applying (2.13). Finally, we show that $\bar{A}'$ is a basis element of $\mathcal{K}'_{\times \Omega_i}$, which we approach by proving first

$$\forall i\,(i \in I \Rightarrow H_i \subseteq \Omega_i). \tag{11.702}$$

We take an arbitrary $i \in I$, so that $H_i \subseteq \bar{U}_i$ and $\bar{U}_i \in \mathcal{O}(\mathcal{K}_{\Omega_i})$ follow to be true, respectively, with (11.700) and (11.690). According to Property 1 of a topology (on Ω_i), we also have the inclusion $\mathcal{O}(\mathcal{K}_{\Omega_i}) \subseteq \mathcal{P}(\Omega_i)$, so that $\bar{U}_i$ turns out to be an element of the power set $\mathcal{P}(\Omega_i)$ (by definition of a subset), and thus a subset of Ω_i (by definition of a power set). The two inclusions $H_i \subseteq \bar{U}_i$ and $\bar{U}_i \subseteq \Omega_i$ give us then the desired inclusion $H_i \subseteq \Omega_i$ with (2.13). Since i is arbitrary, the universal sentence (11.702) follows to be true, so that another application of Proposition 3.246 yields now the inclusion $\bigtimes_{i\in I} H_i \subseteq \bigtimes_{i\in I} \Omega_i$, which in turn implies (using again the definition of a power set)

$$[\bigtimes_{i\in I} H_i =] \quad \bar{A}' \in \mathcal{P}(\bigtimes_{i\in I} \Omega_i). \tag{11.703}$$

We now show that the family H is an element of the Cartesian product $\bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$, by establishing the universal sentence

$$\forall i\,(i \in I \Rightarrow H_i \in \mathcal{O}(\mathcal{K}_{\Omega_i})). \tag{11.704}$$

Letting $i \in I$ be arbitrary, we prove $H_i \in \mathcal{O}(\mathcal{K}_{\Omega_i})$ by cases, based on the previously derived disjunction $i \in \bar{J} \vee i \in I \setminus \bar{J}$. The first case $i \in \bar{J}$ implies with (11.697) $H_i \in \mathcal{K}_{\Omega_i}$, which basis is included in the topology $\mathcal{O}(\mathcal{K}_{\Omega_i})$ generated by it (recalling Proposition 11.55). Thus, the definition of a subset yields indeed the desired consequent of the implication in (11.704) for the first case. In the second case $i \in I \setminus \bar{J}$, we obtain with (11.697) now $H_i = \Omega_i$, which set is an element of $\mathcal{O}(\mathcal{K}_{\Omega_i})$ according to Property 2 of a topology (on Ω_i). We thus completed the proof by cases, and because i is arbitrary, we may therefore conclude that (11.704) holds. This gives us then

$$H \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i}). \tag{11.705}$$

The last universal sentence to be verified is

$$\forall i\,([i \in \bar{J} \Rightarrow H_i \in \mathcal{K}_{\Omega_i}] \wedge [i \in I \setminus \bar{J} \Rightarrow H_i = \Omega_i]), \tag{11.706}$$

where we let i be arbitrary. Then, the assumption $i \in \bar{J}$ gives us $H_i \in \mathcal{K}_{\Omega_i}$, and the assumption $i \in I \setminus \bar{J}$ implies $H_i = \Omega_i$, according to (11.697). We may therefore conclude that the preceding universal sentence is indeed true.

Our previous findings (11.705), $\bar{A}' = \bigtimes_{i\in I} H_i$, $\bar{J} \subseteq I$, where $\bar{J}$ is finite, and (11.706) clearly demonstrate the truth of the existential sentence

$$\begin{gathered}\exists U, J\,(U \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i}) \wedge \bar{A}' = \bigtimes_{i\in I} U_i \wedge J \subseteq I \wedge J \text{ is finite} \\ \wedge\, \forall i\,([i \in J \Rightarrow U_i \in \mathcal{K}_{\Omega_i}] \wedge [i \in I \setminus J \Rightarrow U_i = \Omega_i])).\end{gathered}$$

In conjunction with (11.703), this implies now

$$\bar{A}' \in \mathcal{K}'_{\times \Omega_i} \tag{11.707}$$

because of (11.682). Then, (11.707) demonstrates in conjunction with (11.699) and (11.701) that the existential sentence in (11.687) holds. As the sets B and f were arbitrary, we may further conclude that the universal sentence (11.687) is true. According to the Characterization of the basis generating a given topology, the truth of that universal sentence implies now the truth of b) and c).

Since I, $(\Omega_i)_{i\in I}$ and $(\mathcal{K}_{\Omega_i})_{i\in I}$ were initially arbitrary sets, we may therefore finally conclude that the stated theorem is true. □

We apply the preceding theorem to establish specific basis elements of the standard topology on $\mathbb{R}^n$.

Proposition 11.92. *It is true for any $n \in \mathbb{N}_+$ that the Cartesian product of any n-tuple of open intervals in $\mathbb{R}$ is contained in the basis $\mathcal{K}'_{\times\mathbb{R}^n}$ that generates the standard topology on $\mathbb{R}^n$, that is,*

$$\forall a,b\,(a,b \in \mathbb{R}^n \Rightarrow [\,(a,b)_{\mathbb{R}^n} \in \mathcal{K}'_{\mathbb{R}^n} \;\wedge\; (a,b)_{\mathbb{R}^n} \in \bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}}]). \tag{11.708}$$

Proof. We let n be an arbitrary positive natural number, define the index set $I = \{1,\ldots,n\}$, and let $a = (a_i \,|\, i \in I)$ as well as $b = (b_i \,|\, i \in I)$ be arbitrary real n-tuples. Thus, the n-tuple $\bar{U} = (\bar{U}_i \,|\, i \in I) = ((a_i,b_i) \,|\, i \in I)$ is defined according to Exercise 4.30, and its Cartesian product

$$A = \bigtimes_{i=1}^{n} \bar{U}_i = \bigtimes_{i=1}^{n} (a_i,b_i) \tag{11.709}$$

is identical with the open interval $(a,b)_{\mathbb{R}^n}$. Let us observe here that the index set I is nonempty, as shown by Corollary 4.54. Since the open interval (a_i,b_i) in $\mathbb{R}$ is evidently a subset of $\mathbb{R}$ for any $i \in I$, the inclusion

$$[A =] \quad \bigtimes_{i\in I}(a_i,b_i) \subseteq \bigtimes_{i\in I} \mathbb{R} \quad [= \mathbb{R}^n]$$

follows to be true with (3.872). By definition of a power set, we thus find

$$A \in \mathcal{P}(\mathbb{R}^n). \tag{11.710}$$

Furthermore, we evidently have for any $i \in I$

$$[\bar{U}_i =] \quad (a_i,b_i) \in \{(a,b) : a,b \in \mathbb{R}\} \quad [\subseteq \mathcal{O}(\{(a,b) : a,b \in \mathbb{R}\})] \tag{11.711}$$

(recalling that the generated topology includes its basis, as shown in Proposition 11.55). We therefore find $\bar{U}_i \in \mathcal{O}(\{(a,b) : a,b \in \mathbb{R}\})$ for any $i \in I$ with the definition of a subset, and consequently

$$\bar{U} \in \bigtimes_{i\in I} \mathcal{O}(\{(a,b) : a,b \in \mathbb{R}\}) \tag{11.712}$$

with the definition of the Cartesian product of a family. Moreover, we have

$$I \subseteq I \tag{11.713}$$

in view of (2.10), and the initial assumption $n \in \mathbb{N}_+$ implies that

$$[I =] \quad \{1, \ldots, n\} \text{ is finite,} \tag{11.714}$$

as shown by Exercise 4.34. Finally, we may demonstrate the truth of the universal sentence

$$\forall i\,([i \in I \Rightarrow \bar{U}_i \in \{(a,b) : a, b \in \mathbb{R}\}] \wedge [i \in I \setminus I \Rightarrow \bar{U}_i = \mathbb{R}]). \tag{11.715}$$

For this purpose, we let i be arbitrary. On the one hand, $i \in I$ implies the desired $\bar{U}_i \in \{(a,b) : a, b \in \mathbb{R}\}$, as shown in (11.711); on the other hand, $i \in I \setminus I$ yields the false sentence $i \in \emptyset$ with (2.104) and the definition of the empty set. Thus, the second implication is also true. Thus, the conjunction holds, in which i is arbitrary, so that (11.715) follows to be true as well. As the sets I and $\bar{U}$ satisfy (11.709), (11.712), (11.713) and (11.714), we see that the set A satisfies the existential sentence in (11.682) under the specification of $\Omega_i = \mathbb{R}$ and $\mathcal{K}_{\Omega_i} = \{(a,b) : a, b \in \mathbb{R}\}$ for every i in the given index set $I = \{1, \ldots, n\}$. In conjunction with (11.710), this implies

$$A \in \mathcal{K}'_{\mathbb{R}^n}$$

with (11.682). Here, $\mathcal{K}'_{\mathbb{R}^n}$ is a basis for a topology on $\mathbb{R}^n$ according to Theorem 11.91b), and this basis is evidently included in the topology $\mathcal{O}(\mathcal{K}'_{\mathbb{R}^n})$. Therefore,

$$A \in \mathcal{O}(\mathcal{K}'_{\mathbb{R}^n}) \quad \left[= \bigotimes_{i \in I} \mathcal{O}(\{(a,b) : a, b \in \mathbb{R}\} = \bigotimes_{i \in I} \mathcal{O}_{<_{\mathbb{R}}}\right]$$

also holds (applying also Theorem 11.91b) and recalling the definition of the standard topology on $\mathbb{R}$). Substitutions of $A = (a,b)_{\mathbb{R}^n}$ and $I = \{1, \ldots, n\}$ give us now also the second part of the conjunction in (11.708). Initially, n, a and b were arbitrary, so that we may infer from these findings that the proposition is true. □

We consider now the special case that the index set I constitutes a nonempty, countable set, which gives rise to the countable set $\mathcal{J}$ of all finite subsets J of I according to Theorem 5.164.

Lemma 11.93. *The following sentences are true for any nonempty, countable index set I and for any families $(\Omega_i)_{i \in I}$, $(\mathcal{K}_{\Omega_i})_{i \in I}$ such that $\mathcal{K}_{\Omega_i}$ is a basis for a topology on Ω_i for any $i \in I$.*

a) There exists a unique family $k = (k_J)_{J \in \mathcal{J}}$ whose domain is the set $\mathcal{J}$ of all finite subsets of I and whose terms are defined by

$$\begin{aligned}&\forall J\,(J \in \mathcal{J} \Rightarrow \forall A\,(A \in k(J) \\ &\quad \Leftrightarrow [A \in \mathcal{K}'_{\times\,\Omega_i} \,\wedge\, \exists U\,(U \in \bigtimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i}) \,\wedge\, A = \bigtimes_{i \in I} U_i \\ &\quad \wedge\, \forall i\,([i \in J \Rightarrow U_i \in \mathcal{K}_{\Omega_i}] \,\wedge\, [i \in I \setminus J \Rightarrow U_i = \Omega_i]))]))).\end{aligned} \tag{11.716}$$

b) The union of the family $k = (k_J)_{J \in \mathcal{J}}$ is identical with the basis $\mathcal{K}'_{\times\,\Omega_i}$ generating the product topology $\bigotimes_{i \in I} \mathcal{O}_i$ of the family of topologies $(\mathcal{O}(\mathcal{K}_{\Omega_i}))_{i \in I}$, that is,

$$\bigcup_{J \in \mathcal{J}} k_J = \mathcal{K}'_{\times\,\Omega_i}. \tag{11.717}$$

c) If the basis $\mathcal{K}_{\Omega_i}$ generating the topology $\mathcal{O}(\mathcal{K}_{\Omega_i})$ is countable for every $i \in I$, then the basis $\mathcal{K}'_{\times\,\Omega_i}$ generating the product topology is also countable.

Proof. We let I, $(\Omega_i)_{i \in I}$ and $(\mathcal{K}_{\Omega_i})_{i \in I}$ be arbitrary sets such that I is a nonempty and countable (index) set, such that $(\Omega_i)_{i \in I}$ and $(\mathcal{K}_{\Omega_i})_{i \in I}$ are families of sets, and such that $\mathcal{K}_{\Omega_i}$ is a basis for a topology on Ω_i for any $i \in I$. The truth of a) and b) can then be established, respectively, by means of Function definition by replacement and by means of the Equality Criterion for sets.

Concerning a), we apply Function definition by replacement and establish accordingly

$$\begin{aligned}&\forall J\,(J \in \mathcal{J} \Rightarrow \exists! \mathcal{Y}\,(\forall A\,(A \in \mathcal{Y} \\ &\quad \Leftrightarrow [A \in \mathcal{K}'_{\times\,\Omega_i} \,\wedge\, \exists U\,(U \in \bigtimes_{i \in I} \mathcal{O}(\mathcal{K}_{\Omega_i}) \,\wedge\, A = \bigtimes_{i \in I} U_i \\ &\quad \wedge\, \forall i\,([i \in J \Rightarrow U_i \in \mathcal{K}_{\Omega_i}] \,\wedge\, [i \in I \setminus J \Rightarrow U_i = \Omega_i]))])))).\end{aligned} \tag{11.718}$$

We take an arbitrary set $J \in \mathcal{J}$ and observe in light of the Axiom of Specification and the Equality Criterion for sets that the uniquely existential sentence is true. Since J is arbitrary, we may therefore conclude that the (11.718) holds, so that there exists a unique function k with domain $\mathcal{J}$, that is, a unique family with index set $\mathcal{J}$ satisfying (11.716).

Concerning b), we apply the Equality Criterion for sets and prove the equivalent universal sentence

$$\forall A\,(A \in \bigcup_{J \in \mathcal{J}} k_J \Leftrightarrow A \in \mathcal{K}'_{\times\,\Omega_i}), \tag{11.719}$$

letting A be an arbitrary set. To establish the first part ('$\Rightarrow$') of the equivalence, we assume $A \in \bigcup_{J\in\mathcal{J}} k_J$, so that the Characterization of the union of a family of sets gives us a particular index $\bar{J}$ in $\mathcal{J}$ with $A \in k_{\bar{J}}$. Then, the desired consequent $A \in \mathcal{K}'_{\times\Omega_i}$ follows especially to be true with (11.716). Regarding the second part ('$\Leftarrow$') of the equivalence, we assume now $A \in \mathcal{K}'_{\times\Omega_i}$ to be true. According to (11.682), there are then sets, say $\bar{U}$ and $\bar{J}$, satisfying

$$\bar{U} \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i}), \tag{11.720}$$

the equation

$$A = \bigtimes_{i\in I} \bar{U}_i \tag{11.721}$$

and the inclusion $\bar{J} \subseteq I$, such that $\bar{J}$ is finite and such that the universal sentence

$$\forall i\,([i \in \bar{J} \Rightarrow U_i \in \mathcal{K}_{\Omega_i}] \,\wedge\, [i \in I \setminus \bar{J} \Rightarrow U_i = \Omega_i]) \tag{11.722}$$

holds. Here, the finding that $\bar{J}$ is a finite subset of I implies evidently $\bar{J} \in \mathcal{J}$. Moreover, having found the particular set $\bar{U}$ satisfying (11.720) – (11.722), the existential sentence

$$\exists U\,(U \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i}) \,\wedge\, A = \bigtimes_{i\in I} U_i$$
$$\wedge\, \forall i\,([i \in \bar{J} \Rightarrow U_i \in \mathcal{K}_{\Omega_i}] \,\wedge\, [i \in I \setminus \bar{J} \Rightarrow U_i = \Omega_i]))$$

turns out to be true. Because of $\bar{J} \in \mathcal{J}$, the conjunction of the assumed $A \in \mathcal{K}'_{\times\Omega_i}$ and the preceding existential sentence implies $A \in k(\bar{J})$ with (11.716). Thus, there exists a constant J for which $J \in \mathcal{J}$ and $A \in k(J)$ are both true, so that the Characterization of the union of a family of sets yields $A \in \bigcup_{J\in\mathcal{J}} k_J$, as desired.

Having completed the proof of the equivalence, we may now infer from this the truth of (11.719), and therefore the truth of the proposed equation (11.717).

Concerning c), we assume that the universal sentence

$$\forall i\,(i \in I \Rightarrow \mathcal{K}_{\Omega_i} \text{ is countable}), \tag{11.723}$$

and we demonstrate that this assumption implies the countability of each of the terms of the family $k = (k_J)_{J\in\mathcal{J}}$, that is,

$$\forall J\,(J \in \mathcal{J} \Rightarrow k_J \text{ is countable}). \tag{11.724}$$

We take an arbitrary set J in $\mathcal{J}$, so that J is a finite subset of I. We consider now on the one hand the restriction of the given sequence $(\mathcal{K}_{\Omega_i})_{i\in I}$ to J, which thus constitutes a function with domain J according to (3.566). Alternatively, we can view this restriction as being the family $(\mathcal{K}_{\Omega_j})_{j\in J}$. On the other hand, we can establish a unique function ω with domain $I \setminus J$ such that

$$\forall i\,(i \in I \setminus J \Rightarrow \omega(i) = \{\Omega_i\}). \tag{11.725}$$

To do this, we apply Function definition by replacement and verify

$$\forall i\,(i \in I \setminus J \Rightarrow \exists! y\,(y = \{\Omega_i\})), \tag{11.726}$$

letting i be arbitrary and assuming $i \in I \setminus J$. Since the inclusion $I \setminus J \subseteq I$ is true in view of (2.125), the preceding assumption implies $i \in I$ by definition of a subset, so that Ω_i represents the corresponding term of the given sequence $(\Omega_i)_{i\in I}$. Then, the singleton $\{\Omega_i\}$ is also a defined constant, giving rise to the truth of the uniquely existential sentence $\exists! y\,(y = \{\Omega_i\})$ by virtue of (1.109). Consequently, the universal sentence (11.726) is true, as i was arbitrary. We thus established the function ω with domain $I \setminus J$, whose values are defined by (11.725). Alternatively, we may view ω as the family $\omega = (\{\Omega_i\})_{i\in I\setminus J}$.

The domain J of the restricted sequence $(\mathcal{K}_{\Omega_j})_{j\in J}$ and the domain $I \setminus J$ of the function ω are disjoint sets according to (2.111), which fact implies that the preceding sequence and function are compatible (see Exercise 3.73). Therefore, the union

$$A = (\mathcal{K}_{\Omega_j})_{j\in J} \cup \omega$$

is a function/family with domain $J \cup (I \setminus J)$ due to Proposition 3.176. Here, the previously stated established inclusion $J \subseteq I$ implies

$$\begin{aligned} I &= (I \setminus J) \cup J \\ &= J \cup (I \setminus J) \end{aligned}$$

with (2.263) and the Commutative Law for the union of two sets, so that A is a function with domain I. Thus, we can write A as the sequence $(A_i)_{i\in I}$. The idea is now to utilize the Countability of the Cartesian product of a family of singletons except for finitely many countable terms, for which purpose we verify

$$\forall i\,([i \in J \Rightarrow A_i \text{ is countable}] \wedge [i \in I \setminus J \Rightarrow A_i = \{\Omega_i\}]). \tag{11.727}$$

We let i be arbitrary and assume first $i \in J$ to be true. Then, $\mathcal{K}_{\Omega_i}$ is the associated term of the restricted sequence $(\mathcal{K}_{\Omega_j})_{j\in J}$, so that the ordered pair $(i, \mathcal{K}_{\Omega_i})$ is evidently an element of sequence $(\mathcal{K}_{\Omega_j})_{j\in J}$. Consequently,

$(i, \mathcal{K}_{\Omega_i})$ is also an element of the union $A = (\mathcal{K}_{\Omega_j})_{j \in J} \cup \omega$, and we can write in function/family notation $\mathcal{K}_{\Omega_i} = A_i$. Recalling that $\mathcal{K}_{\Omega_i}$ is a countable basis by assumption, we obtain through substitution that A_i is countable, which finding proves the first implication in (11.727). Concerning the other implication, we assume now $i \in I \setminus J$ to be true, so that i is in the domain of ω. Thus, the index i is associated with the value $\omega(i) = \{\Omega_i\}$. Writing this equation in the form $(i, \{\Omega_i\}) \in \omega$, we see that the ordered pair $(i, \{\Omega_i\})$ is also in the union $A = (\mathcal{K}_{\Omega_j})_{j \in J} \cup \omega$, so that we can write $\{\Omega_i\} = A_i$. Thus, the second implication in (11.727) is also true, and since i is arbitrary, we may therefore conclude that the universal sentence (11.727) holds.

In summary, we have families $A = (A_i)_{i \in I}$ and $\omega = (\{\Omega_i\})_{i \in I \setminus J}$ such that J is finite, such that $J \subseteq I$ holds, and such that (11.727) is true. These findings allow us to apply Theorem 5.166b) and to infer from them that the Cartesian product $\bigtimes_{i \in I} A_i$ is countable. The proof of (11.727) showed that each of the terms A_i is identical with the basis $\mathcal{K}_{\Omega_i}$ (in case of $i \in J$) or identical with the singleton $\{\Omega_i\}$. According to Corollary 11.53, every basis for a topology is nonempty, and according to Corollary 2.49 every singleton contains an element and is thus also nonempty. Consequently, $A_i \neq \emptyset$ is true for all $i \in I$. Noting that the index set I is nonempty by assumption, we then see in light of the Emptiness Criterion for Cartesian products of families of sets that $\bigtimes_{i \in I} A_i \neq \emptyset$ holds. Therefore, the countability of that Cartesian product implies with the Countability Criterion (4.653) that there exists a surjection from the set of natural numbers to the preceding Cartesian product, say

$$G : \mathbb{N} \twoheadrightarrow \bigtimes_{i \in I} A_i. \tag{11.728}$$

To show that the set $k(J)$ is also countable, we establish now a surjection F from the $\bigtimes_{i \in I} A_i$ to $k(J)$. For this purpose, we use Function definition by replacement and demonstrate the truth of

$$\forall x \, (x \in \bigtimes_{i \in I} A_i \Rightarrow \exists! y \, (y = \bigtimes_{i \in I} x_i)), \tag{11.729}$$

letting x be arbitrary and assuming $x \in \bigtimes_{i \in I} A_i$ to be true. This assumption means that x is a family $x = (x_i)_{i \in I}$ with index set I whose terms satisfy $x_i \in A_i$ for all $i \in I$. Let us verify that x_i is a defined set for all $i \in I$. Letting $i \in I$ be arbitrary and recalling that I is identical with the union $J \cup (I \setminus J)$, we obtain the true disjunction $i \in J \vee i \in I \setminus J$ with the definition of the union of two sets. Then, according to (11.727), the case $i \in J$ yields for A_i a countable set, and the other case $i \in I \setminus J$ gives for A_i the singleton $\{\Omega_i\}$. Since i was arbitrary, we can infer from these findings that $x = (x_i)_{i \in I}$ is a family of sets, so that the Cartesian product

$\bigtimes_{i\in I} x_i$ of that family is indeed a defined set. This fact now gives rise to the truth of the uniquely existential sentence in (11.729) due to (1.109). As x was arbitrary, we therefore conclude that the universal sentence (11.729) is true, so that there exists a unique function F with domain $\bigtimes_{i\in I} A_i$ such that

$$\forall x\,(x \in \bigtimes_{i\in I} A_i \Rightarrow F(x) = \bigtimes_{i\in I} x_i). \tag{11.730}$$

Next, we prove by means of the Equality Criterion for sets that the range of the function F is given by $k(J)$, that is, we prove the universal sentence

$$\forall y\,(y \in \mathrm{ran}(F) \Leftrightarrow y \in k(J)). \tag{11.731}$$

Regarding the first part ('$\Rightarrow$') of the equivalence, we make the assumption $y \in \mathrm{ran}(F)$, which implies with the definition of a range that $(\bar{x}, y) \in F$ holds for some particular constant $\bar{x}$. Here, we see in light of the definition of a domain that $\bar{x} \in \bigtimes_{i\in I} A_i$ [$= \mathrm{dom}(F)$] holds, so that $\bar{x}$ is a family with index set I whose terms satisfy

$$\forall i\,(i \in I \Rightarrow \bar{x}_i \in A_i). \tag{11.732}$$

Moreover, $\bar{x}$ is evidently associated with the function value

$$y = F(\bar{x}) = \bigtimes_{i\in I} \bar{x}_i, \tag{11.733}$$

and we can readily prove that $\bar{x}$ satisfies also

$$\forall i\,([i \in J \Rightarrow \bar{x}_i \in \mathcal{K}_{\Omega_i}] \wedge [i \in I \setminus J \Rightarrow \bar{x}_i = \Omega_i]). \tag{11.734}$$

Letting i be arbitrary, the assumption $i \in J$ gives $\bar{x}_i \in A_i$ with (11.732), and in addition $A_i = \mathcal{K}_{\Omega_i}$ – as shown in the proof of (11.727). Therefore, substitution yields $\bar{x}_i \in \mathcal{K}_{\Omega_i}$, proving the first implication in (11.734). Furthermore, the assumption $i \in I \setminus J$ implies with (11.727) firstly $A_i = \{\Omega_i\}$, and secondly $\bar{x}_i \in A_i$ with (11.732), resulting in $\bar{x}_i \in \{\Omega_i\}$. Then, the desired consequent $\bar{x}_i = \Omega_i$ follows to be true with (2.169), so that the second implication in (11.734) holds, too. As i was arbitrary, we can therefore conclude that the universal sentence (11.734) is true. That sentence allows us to prove in addition

$$\bar{x} \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i}). \tag{11.735}$$

We already mentioned earlier that $\bar{x}$ is a family with index set I, so that it only remains for us to establish

$$\forall i\,(i \in I \Rightarrow \bar{x}_i \in \mathcal{O}(\mathcal{K}_{\Omega_i})). \tag{11.736}$$

We take an arbitrary $i \in I$ and consider as before the two cases $i \in J$ and $i \in I \setminus J$. The first case $i \in J$ implies $\bar{x}_i \in \mathcal{K}_{\Omega_i}$ with (11.734), where we recall that the basis $\mathcal{K}_{\Omega_i}$ is included in the generated topology $\mathcal{O}(\mathcal{K}_{\Omega_i})$ (see Proposition 11.55), so that the definition of a subset gives us the desired consequent $\bar{x}_i \in \mathcal{O}(\mathcal{K}_{\Omega_i})$. The second case $i \in I \setminus J$ implies $\bar{x}_i = \Omega_i$ again with (11.734), where $\Omega_i \in \mathcal{O}(\mathcal{K}_{\Omega_i})$ is true according to Property 2 of a topology (on Ω_i), so that substitution yields $\bar{x}_i \in \mathcal{O}(\mathcal{K}_{\Omega_i})$ also for the second case. Because n was arbitrary, we may now further conclude that (11.736) holds, which means that the family $\bar{x}$ (with index set I) is an element of the Cartesian product $\bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$, as claimed by (11.735).

Having found the particular family $\bar{x}$ satisfying (11.735), (11.733) and (11.734), we thus see that the existential sentence

$$\begin{aligned} \exists U \, (U \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i}) \wedge y = \bigtimes_{i\in I} U_i \\ \wedge \forall i \, ([i \in J \Rightarrow U_i \in \mathcal{K}_{\Omega_i}] \wedge [i \in I \setminus J \Rightarrow U_i = \Omega_i])) \end{aligned} \tag{11.737}$$

is true. Our next task is to establish $y \in \mathcal{K}'_{\times \Omega_i}$ by means of (11.682). First, we observe that (11.737) implies the truth of the existential sentence

$$\begin{aligned} \exists U, J \, (U \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i}) \wedge y = \bigtimes_{i\in I} U_i \wedge J \subseteq I \wedge J \text{ is finite} \\ \wedge \forall i \, ([i \in J \Rightarrow U_i \in \mathcal{K}_{\Omega_i}] \wedge [i \in I \setminus J \Rightarrow U_i = \Omega_i])), \end{aligned} \tag{11.738}$$

so that we only need to prove $y \in \mathcal{P}(\bigtimes_{i\in I} \Omega_i)$. For this purpose, we observe in light of Property 1 of a topology and (11.736) that $\bar{x}_i$ – as an open set of the topology $\mathcal{O}(\mathcal{K}_{\Omega_i})$ on Ω_i – constitutes a subset of Ω_i for every $i \in I$. This universal sentence implies then the inclusion

$$[y =] \quad \bigtimes_{i\in I} \bar{x}_i \subseteq \bigtimes_{i\in I} \Omega_i$$

with Proposition 3.246, recalling (11.733). Consequently, $y \in \mathcal{P}(\bigtimes_{i\in I} \Omega_i)$ is indeed true by definition of a power set. In conjunction with the existential sentence (11.738), this implies now the desired

$$y \in \mathcal{K}'_{\times \Omega_i} \tag{11.739}$$

with (11.682). This sentence implies now in conjunction with the existential sentence (11.737) that $y \in k(J)$ holds, according to the definition of the function k in (11.716). We thus proved the implication '$\Rightarrow$' in (11.731).

We prove the converse implication '$\Leftarrow$' also directly, assuming that $y \in k(J)$ holds. By definition of the function k, this implies the truth of (11.739) and of the existential sentence (11.737). Thus, there exists a particular set $\bar{U}$ which satisfies $\bar{U} \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$, $y = \bigtimes_{i\in I} \bar{U}_i$ and

$$\forall i\,([i \in J \Rightarrow \bar{U}_i \in \mathcal{K}_{\Omega_i}] \wedge [i \in I \setminus J \Rightarrow \bar{U}_i = \Omega_i]). \tag{11.740}$$

Let us verify now $\bar{U} \in \bigtimes_{i\in I} A_i$. On the one hand, the previous finding $\bar{U} \in \bigtimes_{i\in I} \mathcal{O}(\mathcal{K}_{\Omega_i})$ shows that $\bar{U}$ is a family with index set I, as required. On the other hand, we can show that the universal sentence

$$\forall i\,(i \in I \Rightarrow \bar{U}_i \in A_i]) \tag{11.741}$$

holds. Letting $i \in I$ be arbitrary, the case $i \in J$ implies with (11.740) $\bar{U}_i \in \mathcal{K}_{\Omega_i}$, where $\mathcal{K}_{\Omega_i} = A_i$ is true by definition of the family $A = (\mathcal{K}_{\Omega_j})_{j\in J} \cup \omega$. Substitution yields then $\bar{U}_i \in A_i$, as desired. The other case $i \in I \setminus J$ implies first $\bar{U}_i = \Omega_i$ with (11.740) and then evidently $\bar{U}_i \in \{\Omega_i\}$ $[= A_i]$, so that we obtain $\bar{U}_i \in A_i$ again. Having completed the proof by cases, the universal sentence (11.741) follows now to be true, because i was arbitrary. Consequently, the family $\bar{U}$ (having the index set I) constitutes indeed an element of the Cartesian product $\bigtimes_{i\in I} A_i$. This means that $\bar{U}$ is in the domain of the function F defined by (11.730), so that the corresponding value is given by $F(\bar{U}) = \bigtimes_{i\in I} \bar{U}_i$ $[= y]$. Writing the resulting equation $y = F(\bar{U})$ in the form $(\bar{U}, y) \in F$, we see in light of the definition of a range that $y \in \mathrm{ran}(F)$ is true, which proves the second part ('$\Leftarrow$') of the equivalence in (11.731).

That equivalence is therefore true, and since y was arbitrary, we may infer from the truth of that equivalence the truth of the universal sentence (11.731), and therefore the truth of the equality $\mathrm{ran}(F) = k(J)$. Consequently, the function F is a surjection from $\bigtimes_{i\in I} A_i$ to $k(J)$, that is,

$$F : \bigtimes_{i\in I} A_i \twoheadrightarrow k(J). \tag{11.742}$$

Since we already found the surjection $G : \mathbb{N} \twoheadrightarrow \bigtimes_{i\in I} A_i$ in (11.728), we can make use of the Surjectivity of the composition of two surjections in order to establish the new surjection

$$F \circ G : \mathbb{N} \twoheadrightarrow k(J). \tag{11.743}$$

Having thus proved the existence of a surjection from $\mathbb{N}$ to $k(J)$, it follows with the Countability Criterion (4.653) that $k(J)$ is countable, so that the proof of the implication in (11.724) is now complete. Here, J was initially arbitrary, so that the universal sentence (11.724) is then also true. Therefore, $k = (k_J)_{J\in\mathcal{J}}$ is a family of countable sets. Moreover, as the set I was assumed to be countable, it follows with Theorem 5.164 that the set $\mathcal{J}$ of all finite subsets of I is also countable. Thus, $k = (k_J)_{J \in \mathcal{J}}$ is a family having a countable index set and countable terms, so that the union $\bigcup_{J \in \mathcal{J}} k_J = \mathcal{K}'_{\times \Omega_i}$ established in b) turns out to be countable as well because of (5.651). This finding completes the proof of c).

As I, $(\Omega_i)_{i \in I}$ and $(\mathcal{K}_{\Omega_i})_{i \in I}$ were initially arbitrary, we can finally conclude that the lemma is true. □

Theorem 11.94 (Second-countability of countable products of second-countable topological spaces). *It is true for any nonempty, countable index set I and for any families $(\Omega_i)_{i \in I}$, $(\mathcal{O}_i)_{i \in I}$ for which $(\Omega_i, \mathcal{O}_i)$ is a second-countable topological space for every $i \in I$ that*

a) there exists a family $(\mathcal{K}_{\Omega_i})_{i \in I}$ such that $\mathcal{K}_{\Omega_i}$ is a countable basis generating the topology $\mathcal{O}_i$ for any $i \in I$, and moreover that

b) the product topological space $(\bigtimes_{i \in I} \Omega_i, \bigotimes_{i \in I} \mathcal{O}_i)$ is second-countable.

Proof. We take arbitrary sets I, $(\Omega_i)_{i \in I}$ and $(\mathcal{O}_i)_{i \in I}$, assuming I to be a nonempty and countable (index) set, assuming $(\Omega_i)_{i \in I}$ and $(\mathcal{O}_i)_{i \in I}$ to be families of sets, and assuming $(\Omega_i, \mathcal{O}_i)$ to be a second-countable topological space for all $i \in I$. Thus, there exists for any $i \in I$ a basis $\mathcal{K}_{\Omega_i}$ for a topology on Ω_i which generates the topology $\mathcal{O}_i$ and which constitutes a countable set. In the following, we use this fact to establish a unique family $(\mathcal{K}_{\Omega_i})_{i \in I}$ of such countable bases. For this purpose, we establish first the universal sentence

$$\forall i\, (i \in I \Rightarrow \exists! \mathcal{Y}\, \forall K\, (K \in \mathcal{Y} \Leftrightarrow [K \in \mathcal{P}(\mathcal{P}(\Omega_i)) \\ \wedge\ K \text{ is a countable basis generating } \mathcal{O}_i])),$$

letting $i \in I$ be arbitrary and observing the truth of the uniquely existential sentence in light of the Axiom of Specification and the Equality Criterion for sets. According to Function definition by replacement, there exists then a unique function F with domain I such that

$$\forall i\, (i \in I \Rightarrow \forall K\, (K \in F(i) \Leftrightarrow [K \in \mathcal{P}(\mathcal{P}(\Omega_i)) \\ \wedge\ K \text{ is a countable basis generating } \mathcal{O}_i])). \tag{11.744}$$

Because the range of F is included in itself according to (2.10), it is a codomain of F, so that we have

$$F : I \to \operatorname{ran}(F).$$

We demonstrate now that the range of this function does not contain the empty set, by showing that the universal sentence

$$\forall X \, (X \in \mathrm{ran}(F) \Rightarrow X \neq \emptyset) \tag{11.745}$$

holds. We take an arbitrary set X and assume $X \in \mathrm{ran}(F)$ to be true. Consequently, there exists by definition of a range a constant, say $\bar{k}$, satisfying $(\bar{k}, X) \in F$. By definition of a domain, $\bar{k} \in I$ [$= \mathrm{dom}(F)$] is then also true. Thus, the topological space $(\Omega_{\bar{k}}, \mathcal{O}_{\bar{k}})$ is second-countable by assumption, so that there exists a countable basis generating $\mathcal{O}_{\bar{k}}$, say $\bar{\mathcal{K}}_{\Omega_i}$. Due to Property 1 of a basis for a topology (on $\Omega_{\bar{k}}$), the inclusion $\bar{\mathcal{K}}_{\Omega_i} \subseteq \mathcal{P}(\Omega_{\bar{k}})$ holds, and this inclusion implies with the definition of a power set

$$\bar{\mathcal{K}}_{\Omega_i} \in \mathcal{P}(\mathcal{P}(\Omega_{\bar{k}})).$$

Together with the fact that $\bar{\mathcal{K}}_{\Omega_i}$ is a countable basis generating $\mathcal{O}_{\bar{k}}$, this further implies $\bar{\mathcal{K}}_{\Omega_i} \in F(\bar{k})$ by definition of the function F in (11.744). Let us then write the previous finding $(\bar{k}, X) \in F$ in function notation as $X = F(\bar{k})$, so that substitution yields $\bar{\mathcal{K}}_{\Omega_i} \in X$. We now see clearly that the set X is nonempty, which is the desired consequent of the implication in (11.745). As X was arbitrary, we can therefore conclude that the universal sentence (11.745) holds. That sentence in turn implies the suggested negation $\emptyset \notin \mathrm{ran}(F)$ with (2.5), which enables us to apply the Axiom of Choice to infer from this negation the existence of a particular function

$$G : \mathrm{ran}(F) \to \bigcup \mathrm{ran}(F)$$

such that

$$\forall \mathcal{Y} \, (\mathcal{Y} \in \mathrm{ran}(F) \Rightarrow G(\mathcal{Y}) \in \mathcal{Y}). \tag{11.746}$$

We obtain then for the composition of G and F

$$G \circ F : I \to \bigcup \mathrm{ran}(F)$$

in view of Proposition 3.178, so that $G \circ F$ constitutes a family with index set I. Here, we can show that each term $(G \circ F)_i$ of that family is a countable basis generating the $\mathcal{O}_i$. Taking an arbitrary index $i \in I$, we can write $(G \circ F)_i = G(F(i))$, where $F(i)$ is evidently an element of the range of F. Consequently, we obtain $G(F(i)) \in F(i)$ with (11.746), and this implies with (11.744) especially that $G(F(i))$ is a (particular) countable basis generating $\mathcal{O}_i$. Recalling the equation $(G \circ F)_i = G(F(i))$, it follows then by means of substitution that the term $(G \circ F)_i$ is a (particular) countable basis generating $\mathcal{O}_i$, which basis we denote by $\bar{\mathcal{K}}_{\Omega_i}$. Since i is arbitrary, this finding is true for every index i, so that we can write the family $G \circ F$ also in the form $(\bar{\mathcal{K}}_{\Omega_i})_{i \in I}$.

We are now in a position to apply the Generation of a basis for a product topology by means of a family of bases, which gives us the basis $\mathcal{K}'_{\times \Omega_i}$ for a topology on $\bigtimes_{i \in I} \Omega_i$, generating the product topology $\bigotimes_{i \in I} \mathcal{O}_i$. Moreover, as the basis $\bar{\mathcal{K}}_{\Omega_i}$ generating $\mathcal{O}_i$ is countable for every $i \in I$, and since the index set I was assumed to be nonempty and countable, it follows with Lemma 11.93c) that the basis $\mathcal{K}'_{\times \Omega_i}$ is also countable. This proves that the product topology $\bigotimes_{i \in I} \mathcal{O}_i$ has a countable basis, so that the product topological space $(\bigtimes_{i \in I} \Omega_i, \bigotimes_{i \in I} \mathcal{O}_i)$ is second-countable, by definition. Because the sets I, $(\Omega_i)_{i \in I}$ and $(\mathcal{O}_i)_{i \in I}$ were initially arbitrary, we may therefore conclude that the theorem is true. $\square$

Note 11.36. Due to the second-countability of $(\mathbb{R}, \mathcal{O}_{<_\mathbb{R}})$, the previously established product topological spaces

$$\left(\mathbb{R}^n, \bigotimes_{i=1}^{n} \mathcal{O}_{<_\mathbb{R}} \right)$$

(for any $n \in \mathbb{N}_+$) and

$$\left(\omega, \bigotimes_{i=1}^{\infty} \mathcal{O}_{<_\mathbb{R}} \right)$$

are also second-countable according to the preceding theorem.

Corollary 11.95. *It is true for any nonempty countable index set I and for any families $(\Omega_i)_{i \in I}$, $(d_i)_{i \in I}$ for which (Ω_i, d_i) is a separable metric space for every $i \in I$ that the product topological space $(\bigtimes_{i \in I} \Omega_i, \bigotimes_{i \in I} \mathcal{O}_{d_i})$ of the family $(\mathcal{O}_{d_i})_{i \in I}$ of induced metric topologies is second-countable.*

Proof. Letting I be an arbitrary nonempty countable index set and letting $(\Omega_i)_{i \in I}$ and $(d_i)_{i \in I}$ arbitrary families of sets such that (Ω_i, d_i) is a separable metric space for any $i \in I$, we evidently have that the family of metrics induces a corresponding family $(\mathcal{O}_{d_i})_{i \in I}$ of metric topologies. The separability assumption implies then with the Equivalence of separable metric spaces and second-countable topological spaces that the topological space $(\Omega_i, \mathcal{O}_{d_i})$ is second-countable for every $i \in I$. It then follows with the Second-countability of countable products of second-countable topological spaces that $(\bigtimes_{i \in I} \Omega_i, \bigotimes_{i \in I} \mathcal{O}_{d_i})$ is second countable, as claimed. Since I, $(\Omega_i)_{i \in I}$ and $(d_i)_{i \in I}$ were initially arbitrary sets, we may therefore conclude that the corollary holds. $\square$

Note 11.37. Since the metric space $(\mathbb{R}, d_{\mathbb{R}})$ is separable (see Note 11.29), the product topological spaces

$$\left(\mathbb{R}^n, \bigotimes_{i=1}^{n} \mathcal{O}_{d_{\mathbb{R}}}\right) \tag{11.747}$$

(for any $n \in \mathbb{N}_+$) and

$$\left(\omega, \bigotimes_{i=1}^{\infty} \mathcal{O}_{d_{\mathbb{R}}}\right) \tag{11.748}$$

are second-countable in view of the preceding corollary. In view of the equality of the order topology and the Euclidean topology on $\mathbb{R}$, this statement is identical with Note 11.36.

Exercise 11.38. Show for any nonempty index set I and for any families of sets $(\Omega_i)_{i\in I}$, $(\mathcal{O}_i)_{i\in I}$, $(A_i)_{i\in I}$ such that $\mathcal{O}_i$ is a topology on Ω_i and such that A_i is a subset of Ω_i for all $i \in I$ that there exists the unique family $(\mathrm{cl}(A_i))_{i\in I}$ in $\bigtimes_{i\in I} \Omega_i$, where $\mathrm{cl}(A_i)$ is the closure of A_i in Ω_i with respect to $\mathcal{O}_i$.

(Hint: Proceed similarly as in Exercise 3.101.)

Lemma 11.96. *For any nonempty index set I and for any families of sets $(\Omega_i)_{i\in I}$, $(\mathcal{O}_i)_{i\in I}$, $(A_i)_{i\in I}$ such that $\Omega_i \neq \emptyset$, such that $\mathcal{O}_i$ is a topology on Ω_i and such that A_i is a subset of Ω_i for all $i \in I$, it is true that the closure of the Cartesian product of $(A_i)_{i\in I}$ in $\bigtimes_{i\in I} \Omega_i$ with respect to the product topology $\bigotimes_{i\in I} \mathcal{O}_i$ equals the Cartesian product of the family $(\mathrm{cl}(A_i))_{i\in I}$ of closures A_i in Ω_i with respect to $\mathcal{O}_i$, i.e.*

$$\mathrm{cl}(\bigtimes_{i\in I} A_i) = \bigtimes_{i\in I} \mathrm{cl}(A_i). \tag{11.749}$$

Proof. We take arbitrary sets I, $(\Omega_i)_{i\in I}$, $(\mathcal{O}_i)_{i\in I}$ and $(A_i)_{i\in I}$, assuming the latter three to be families of sets with index set $I \neq \emptyset$ such that $(\Omega_i, \mathcal{O}_i)$ is a topological space with $\Omega_i \neq \emptyset$ and such that $A_i \subseteq \Omega_i$ holds for any $i \in I$. To prove the stated equation, we apply the Equality Criterion for sets and establish accordingly universal sentence

$$\forall f \, (f \in \mathrm{cl}(\bigtimes_{i\in I} A_i) \Leftrightarrow f \in \bigtimes_{i\in I} \mathrm{cl}(A_i)), \tag{11.750}$$

letting f be arbitrary. We prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $f \in \mathrm{cl}(\bigtimes_{i\in I} A_i)$ to be true. Let us observe here that the assumption of $A_i \subseteq \Omega_i$ for all $i \in I$ implies the inclusion $\bigtimes_{i\in I} A_i \subseteq \bigtimes_{i\in I} \Omega_i$ with Proposition 3.246, so that the closure of $\bigtimes_{i\in I} A_i$ in $\bigtimes_{i\in I} \Omega_i$ with respect to the product topology $\bigotimes_{i\in I}\mathcal{O}_i$ is indeed defined. Furthermore, this closure is a closed set in $\bigtimes_{i\in I}\Omega_i$ according to Exercise 11.19b) and thus a subset of $\bigtimes_{i\in I}\Omega_i$. Therefore, the assumed $f\in \mathrm{cl}(\bigtimes_{i\in I}A_i)$ implies $f\in\bigtimes_{i\in I}\Omega_i$ with the definition of a subset, so that the Characterization of the elements of a closure gives us the true universal sentence

$$\forall V\,([V\in\bigotimes_{i\in I}\mathcal{O}_i\;\wedge\;f\in V]\Rightarrow\left(\bigtimes_{i\in I}A_i\right)\cap V\neq\emptyset). \tag{11.751}$$

To establish the desired consequent $f\in\bigtimes_{i\in I}\mathrm{cl}(A_i)$, we observe on the one hand that $f\in\bigtimes_{i\in I}\Omega_i$ implies by definition of the Cartesian product of a family of sets that f is a family of sets with index set I, as required. On the other hand, we must show that

$$\forall i\,(i\in I\Rightarrow f_i\in\mathrm{cl}(A_i)) \tag{11.752}$$

holds. To do this, we let $\bar{k}\in I$ be arbitrary, and we demonstrate the truth of $f_i\in\mathrm{cl}(A_i)$ by means of the Characterization of the elements of a closure, by verifying the universal sentence

$$\forall U\,([U\in\mathcal{O}_{\bar{k}}\;\wedge\;f_{\bar{k}}\in U]\Rightarrow A_{\bar{k}}\cap U\neq\emptyset). \tag{11.753}$$

We take an arbitrary open set $\bar{U}\in\mathcal{O}_{\bar{k}}$ satisfying $f_{\bar{k}}\in\bar{U}$, so that we evidently have

$$\pi_{\bar{k}}^{-1}[\bar{U}]\in\{\pi_j^{-1}[U]:U\in\mathcal{O}_j\}$$

according to Theorem 11.88. This in turn shows in light of the definition of the union of a family of sets that

$$\pi_{\bar{k}}^{-1}[\bar{U}]\in\bigcup_{j\in I}\{\pi_j^{-1}[U]:U\in\mathcal{O}_j\}\quad[=\mathcal{C}_{\times\Omega_i}].$$

Since the subbasis $\mathcal{C}_{\times\Omega_i}$ is included in the corresponding generated product topology $\bigotimes_{i\in I}\mathcal{O}_i$ according to Proposition 11.78, we obtain

$$\pi_{\bar{k}}^{-1}[\bar{U}]\in\bigotimes_{i\in I}\mathcal{O}_i \tag{11.754}$$

with the definition of a subset. Furthermore, by definition of a projection function, $\pi_{\bar{k}}:\bigtimes_{i\in I}\Omega_i\to\Omega_{\bar{k}}$ maps the element $f\in\bigtimes_{i\in I}\Omega_i$ to $\pi_{\bar{k}}(f)=f_{\bar{k}}$. Therefore, the previously established $f_{\bar{k}}\in\bar{U}$ yields $\pi_{\bar{k}}(f)\in\bar{U}$ via substitution. Consequently,

$$f\in\pi_{\bar{k}}^{-1}[\bar{U}]$$

is true by definition of an inverse image. In conjunction with (11.754), this finding further implies $\bigtimes_{i\in I} A_i \cap \pi_{\bar{k}}^{-1}[\bar{U}] \neq \emptyset$ with (11.751). This inequality clearly shows that there exists an element in $\bigtimes_{i\in I} A_i \cap \pi_{\bar{k}}^{-1}[\bar{U}]$, say $\bar{g}$, so that $\bar{g} \in \bigtimes_{i\in I} A_i$ and $\bar{g} \in \pi_{\bar{k}}^{-1}[\bar{U}]$ follow to be both true by definition of the intersection of two sets. On the one hand, the former implies with the definition of the Cartesian product of a family of sets that $\bar{g}$ is a family with index set I such that $\bar{g}_i \in A_i$ holds for any $i \in I$; thus, $\bar{k} \in I$ implies $\bar{g}_{\bar{k}} \in A_i$. On the other hand, $\bar{g} \in \pi_{\bar{k}}^{-1}[\bar{U}]$ implies $\pi_{\bar{k}}(\bar{g}) \in \bar{U}$ by definition of an inverse image, where $\pi_{\bar{k}}(\bar{g}) = \bar{g}_{\bar{k}}$ holds according to the definition of a projection function, so that substitution gives $\bar{g}_{\bar{k}} \in \bar{U}$. In conjunction with the preceding finding $\bar{g}_{\bar{k}} \in A_i$, this yields with the definition of the intersection of two sets $\bar{g}_{\bar{k}} \in A_i \cap \bar{U}$. Thus, the intersection $A_i \cap \bar{U}$ is evidently nonempty, which finding proves the implication in (11.753). Since $\bar{U}$ is arbitrary, we may therefore conclude that the universal sentence (11.753) is true, and this implies (with the Characterization of the elements of a closure) $f_i \in \mathrm{cl}(A_i)$, which is the desired consequent of the implication in (11.752). Here, $\bar{k}$ was arbitrary, so that the universal sentence (11.752) follows to be true. Because we already showed that f is a family of sets with index set I, we thus have that f is an element of the Cartesian product $\bigtimes_{i\in I} \mathrm{cl}(A_i)$, and this in turn proves the first implication '$\Rightarrow$'.

To establish the second implication '$\Leftarrow$' in (11.750), we now assume conversely $f \in \bigtimes_{i\in I} \mathrm{cl}(A_i)$ to be true, so that f is evidently a family with index set I satisfying (11.752). We also note (for any $i \in I$) that $\mathrm{cl}(A_i) \subseteq \Omega_i$ holds, because the closure of A_i in Ω_i is a closed set in Ω_i (see Exercise 11.19), and this closed set is thus by definition a subset of Ω_i. Consequently, we have the inclusion $\bigtimes_{i\in I} \mathrm{cl}(A_i) \subseteq \bigtimes_{i\in I} \Omega_i$ (due to Proposition 3.246), so that the assumed antecedent implies (by definition of a subset) $f \in \bigtimes_{i\in I} \Omega_i$. We may therefore utilize the Characterization of the elements of a closure by means of a basis in order to establish the desired consequent $f \in \mathrm{cl}(\bigtimes_{i\in I} A_i)$, via the proof of the universal sentence

$$\forall B\,([B \in \mathcal{K}_{\times\Omega_i} \wedge f \in B] \Rightarrow \left(\bigtimes_{i\in I} A_i\right) \cap B \neq \emptyset). \tag{11.755}$$

For this purpose, we take an arbitrary basis element $B \in \mathcal{K}_{\times\Omega_i}$ satisfying $f \in B$. According to the Characterization of the elements of the basis for a product topology, there is then a particular finite subset $\bar{J} \subseteq I$ and a particular element $\bar{U} \in \bigtimes_{i\in I} \mathcal{O}_i$ with $B = \bigtimes_{i\in I} \bar{U}_i$ such that $\bar{U}_i = \Omega_i$ holds for every index $i \in I \setminus J$. Therefore, $\bar{U}$ is a family with index set I satisfying

$$\forall i\,(i \in I \Rightarrow \bar{U}_i \in \mathcal{O}_i), \tag{11.756}$$

and the previous assumption $f \in B$ gives via substitution $f \in \bigtimes_{i\in I} \bar{U}_i$, so that the terms of the family f satisfy also

$$\forall i\,(i \in I \Rightarrow f_i \in \bar{U}_i). \tag{11.757}$$

We now define by means of replacement a function F with domain I such that each of the terms F_i constitutes the intersection of A_i and $\bar{U}_i$. For this purpose, we verify the universal sentence

$$\forall i\,(i \in I \Rightarrow \exists! Y\,(Y = A_i \cap \bar{U}_i)), \tag{11.758}$$

letting $i \in I$ be arbitrary. Thus, A_i and $\bar{U}_i$ are specific terms of the families $(A_i)_{i\in I}$ and $\bar{U}$, respectively, and the intersection $A_i \cap \bar{U}_i$ of these terms is then a uniquely specified set. We may therefore apply (1.109) to infer from this finding the truth of the uniquely existential sentence $\exists! Y\,(Y = A_i \cap \bar{U}_i)$. Here, i is arbitrary, so that the universal sentence (11.758) holds. Therefore, there exists a unique function F with domain I such that

$$\forall i\,(i \in I \Rightarrow F(i) = A_i \cap \bar{U}_i). \tag{11.759}$$

We now prove that none of the terms of F is empty, i.e. that $\emptyset \notin \mathrm{ran}(F)$ is true. We may express this assertion in the equivalent form of the universal sentence

$$\forall X\,(X \in \mathrm{ran}(F) \Rightarrow X \neq \emptyset) \tag{11.760}$$

because of (2.5). We take an arbitrary set X and assume $X \in \mathrm{ran}(F)$ to be true. By definition of a range, there exists then a constant, say $\bar{k}$, such that $(\bar{k}, X) \in F$ holds. This shows in light of the definition of a domain that $\bar{k} \in I$ [$= \mathrm{dom}(F)$] is true. This implies firstly $\bar{U}_{\bar{k}} \in \mathcal{O}_{\bar{k}}$ with (11.756), secondly $f_{\bar{k}} \in \bar{U}_{\bar{k}}$ with (11.757), and thirdly $f_{\bar{k}} \in \mathrm{cl}(A_{\bar{k}})$ with (11.752) . Because the previously established $f \in \bigtimes_{i\in I} \Omega_i$ shows that the terms of the family f satisfy $f_i \in \Omega_i$ for any $i \in I$, we see in particular that $f_{\bar{k}} \in \Omega_{\bar{k}}$ holds. We may therefore use the Characterization of the elements of a closure to infer from the truth of $\bar{U}_{\bar{k}} \in \mathcal{O}_{\bar{k}}$ and of $f_{\bar{k}} \in \bar{U}_{\bar{k}}$ the truth of $A_{\bar{k}} \cap \bar{U}_{\bar{k}} \neq \emptyset$. Noting that the previously found $(\bar{k}, X) \in F$ can be written in function notation as $X = F_{\bar{k}}$ and noting that $\bar{k} \in I$ implies $F_{\bar{k}} = A_{\bar{k}} \cap \bar{U}_{\bar{k}}$ with (11.759), we obtain via substitution $X \neq \emptyset$, as desired. Here, X was arbitrary, so that the universal sentence (11.760) follows now to be true. As this implies $\emptyset \notin \mathrm{ran}(F)$, we may now apply the Axiom of Choice to infer from this negation the existence of a particular function $\bar{g}$ with domain $\mathrm{ran}(F)$ and codomain $\bigcup \mathrm{ran}(F)$ for which

$$\forall K\,(K \in \mathrm{ran}(F) \Rightarrow \bar{g}(K) \in K) \tag{11.761}$$

holds. We now consider the composition h of $\bar{g} : \mathrm{ran}(F) \to \bigcup \mathrm{ran}(F)$ and $F : I \to \mathrm{ran}(F)$, which is a function from I to $\bigcup \mathrm{ran}(F)$ according to Proposition 3.178. Next, we demonstrate that this composition $h : I \to \bigcup \mathrm{ran}(F)$, which is a family with index set I, satisfies also the two universal sentences

$$\forall i \, (i \in I \Rightarrow h_i \in A_i), \tag{11.762}$$

$$\forall i \, (i \in I \Rightarrow h_i \in \bar{U}_i). \tag{11.763}$$

We let $i \in I$ be arbitrary, which index gives us the term $h_i = (\bar{g} \circ F)(i) = \bar{g}(F(i))$. Here, we may write for the term $F_i = F(i)$ also $(i, F_i) \in F$, which shows in light of the definition of a range that $F_i \in \mathrm{ran}(F)$ is true. Consequently, we obtain with (11.761) $[h_i =]\ \bar{g}(F_i) \in F_i$. Furthermore, $i \in I$ implies $F(i) = A_i \cap \bar{U}_i$, so that the preceding finding $h_i \in F_i$ yields $h_i \in A_i \cap \bar{U}_i$ by means of substitution. Therefore, $h_i \in A_i$ and $h_i \in \bar{U}_i$ are both true by definition of the intersection of two sets. Since i is arbitrary, the universal sentences (11.762) – (11.763) follow evidently both to be true. In view of the fact that h is a family with index set I, we thus see in light of the definition of the Cartesian product of a family of sets that $h \in \bigtimes_{i \in I} A_i$ and $h \in \bigtimes_{i \in I} \bar{U}_i$ $[= B]$ are true. Consequently, h turns out to be an element of the intersection of $\bigtimes_{i \in I} A_i$ and B, which demonstrates the truth of $(\bigtimes_{i \in I} A_i) \cap B \neq \emptyset$. Thus, the proof of the implication in (11.755) is complete, and since B was an arbitrary set, we may infer from the truth of that implication the truth of the universal sentence (11.755), and therefore (having applied the Characterization of the elements of a closure by means of a basis) also the truth of the desired consequent $f \in \mathrm{cl}(\bigtimes_{i \in I} A_i)$ of the implication '$\Leftarrow$' in (11.750).

Having proved both parts of the equivalence, the universal sentence (11.750) follows now to be true, because f was an arbitrary set. This in turn completes the proof of the proposed equation (11.749), and as the sets I, $(\Omega_i)_{i \in I}$, $(\mathcal{O}_i)_{i \in I}$ and $(A_i)_{i \in I}$ were initially arbitrary, we may now finally conclude that the stated lemma is indeed true. $\square$

11.8. Product σ-Algebras

We use the approach to defining a product topology $\bigotimes_{i\in I}\mathcal{O}_i$ by means of a system of inverse images $\pi_i^{-1}[U]$ of open sets U in the corresponding individual topologies $\mathcal{O}_i$ under projection functions onto corresponding sets Ω_i in a similar way to define a product σ-algebra.

Exercise 11.39. Verify the following sentences for any nonempty index set I and any families of sets $(\Omega_i)_{i\in I}$, $(\mathcal{K}_i)_{i\in I}$ such that $\mathcal{K}_i \subseteq \mathcal{P}(\Omega_i)$ holds for any $i \in I$.

a) For any $j \in I$ there exists a unique set (system)

$$\{\pi_j^{-1}[K] : K \in \mathcal{K}_j\} \tag{11.764}$$

which consists of all the inverse images of sets in $\mathcal{K}_j$ under the projection function $\pi_j : \bigtimes_{i\in I}\Omega_i \to \Omega_j$ in the sense of

$$\forall X\,(X \in \{\pi_j^{-1}[K] : K \in \mathcal{K}_j\} \Leftrightarrow \exists K\,(K \in \mathcal{K}_j \wedge \pi_j^{-1}[K] = X)), \tag{11.765}$$

b) The family of sets $(\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})_{i\in I}$ is uniquely determined.

c) Then, the union

$$\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\} \tag{11.766}$$

is a generating system for a σ-algebra on $\Omega = \bigtimes_{i\in I}\Omega_i$.

(Hint: Proceed in analogy to the proof of Theorem 11.88.)

Corollary 11.97. *It is true for any nonempty index set I and any families of sets $(\Omega_i)_{i\in I}$, $(\mathcal{A}_i)_{i\in I}$ such that $\mathcal{A}_i$ is a σ-algebra on Ω_i for any $i \in I$ that the family of sets $(\{\pi_i^{-1}[A] : A \in \mathcal{A}_i\})_{i\in I}$ is uniquely specified and that its union*

$$\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{A}_i\} \tag{11.767}$$

constitutes a generating system for a σ-algebra on $\Omega = \bigtimes_{i\in I}\Omega_i$.

Proof. Letting I, $(\Omega_i)_{i\in I}$ and $(\mathcal{A}_i)_{i\in I}$ be arbitrary sets such that $(\Omega_i)_{i\in I}$ and $(\mathcal{A}_i)_{i\in I}$ are families sets with index set $I \neq \emptyset$, and such that $\mathcal{A}_i$ is a σ-algebra on Ω_i for any $i \in I$, we thus have $\mathcal{A}_i \subseteq \mathcal{P}(\Omega_i)$ for all $i \in I$ according to Property 2 of a σ-algebra. Therefore, the findings of Exercise 11.39 apply directly to the family $(\mathcal{A}_i)_{i\in I}$ in place of $(\mathcal{K}_i)_{i\in I}$. This is then true for any I, any $(\Omega_i)_{i\in I}$ and any $(\mathcal{A}_i)_{i\in I}$. □

Exercise 11.40. Verify for any nonempty index set I and any families of sets $(\Omega_i)_{i\in I}$, $(\mathcal{A}_i)_{i\in I}$ such that $\mathcal{A}_i$ is a σ-algebra on Ω_i for any $i \in I$ that the set system

$$\{\pi_i^{-1}[A] : A \in \mathcal{A}_i\} \tag{11.768}$$

is a π-system on $\bigtimes_{i\in I} \Omega_i$ containing $\bigtimes_{i\in I} \Omega_i$.
(Hint: Recall the proof of Theorem 11.88.)

Definition 11.32 (Product σ-algebra, product measurable space). For any nonempty index set I, any family of sets $(\Omega_i)_{i\in I}$ and any family of sets $(\mathcal{A}_i)_{i\in I}$ such that $\mathcal{A}_i$ is a σ-algebra on Ω_i for all $i \in I$, we call the generated σ-algebra

$$\bigotimes_{i\in I} \mathcal{A}_i = \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{A}_i\}) \tag{11.769}$$

the *product σ-algebra* of $(\mathcal{A}_i)_{i\in I}$ (on $\Omega = \bigtimes_{i\in I} \Omega_i$). We then call the measurable space

$$(\bigtimes_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{A}_i) \tag{11.770}$$

the *product measurable space* with respect to $(\Omega_i)_{i\in I}$ and $(\mathcal{A}_i)_{i\in I}$.

We explore in the following how the product σ-algebra of a family of σ-algebras can equivalently be obtained from a family of set systems generating the individual σ-algebras.

Lemma 11.98. *It is true for any nonempty index set I and any families of sets $(\Omega_i)_{i\in I}$, $(\mathcal{K}_i)_{i\in I}$ with $\mathcal{K}_i \subseteq \mathcal{P}(\Omega_i)$ for all $i \in I$ that, for every $j \in I$, there exists a unique set*

$$\mathcal{X}^{(j)} = \{X : X \in \mathcal{P}(\Omega_j) \wedge \pi_j^{-1}[X] \in \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})\} \tag{11.771}$$

consisting of all subsets of Ω_i whose inverse image under the j-th projection map $\pi_j : \bigtimes_{i\in I} \Omega_i \to \Omega_j$ is in the σ-algebra generated by the union (11.767). This set is for any $j \in I$ a σ-algebra on Ω_j including the σ-algebra generated by $\mathcal{K}_j$, i.e.

$$\mathcal{A}(\mathcal{K}_j) \subseteq \{X : X \in \mathcal{P}(\Omega_j) \wedge \pi_j^{-1}[X] \in \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})\}. \tag{11.772}$$

Proof. We let I, $(\Omega_i)_{i\in I}$ and $(\mathcal{K}_i)_{i\in I}$ be arbitrary sets, assume $(\Omega_i)_{i\in I}$ and $(\mathcal{K}_i)_{i\in I}$ to be families of sets with index set $I \neq \emptyset$, and assume $\mathcal{K}_i \subseteq$

$\mathcal{P}(\Omega_i)$ to be true for any $i \in I$. Letting now also $j \in I$ be arbitrary, we note that the union $\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}$ generates the σ-algebra $\mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})$ on $\bigtimes_{i\in I} \Omega_i$ according to Exercise 11.39. Let us then observe in light of the Axiom of Specification and the Equality Criterion for sets that there indeed exists a unique set $\mathcal{X}^{(j)}$ such that

$$\forall X\,(X \in \mathcal{X}^{(j)} \Leftrightarrow [X \in \mathcal{P}(\Omega_j) \wedge \pi_j^{-1}[X] \in \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})]). \tag{11.773}$$

Thus, we have for any X that $X \in \mathcal{X}^{(j)}$ implies especially $X \in \mathcal{P}(\Omega_j)$, so that

$$\mathcal{X}^{(j)} \subseteq \mathcal{P}(\Omega_j) \tag{11.774}$$

follows to be true by definition of a subset, as required by Property 1 of a σ-algebra on Ω_j.

Secondly, we obtain $\Omega_j \in \mathcal{P}(\Omega_j)$ with (3.15) and

$$\pi_j^{-1}[\Omega_j] = \bigtimes_{i\in I} \Omega_i \quad \left[\in \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})\right]$$

by using (3.746) and Property 2 of a σ-algebra (on $\bigtimes_{i\in I} \Omega_i$). With these findings, $\Omega_j \in \mathcal{X}^{(j)}$ follows to be true with (11.773), which shows that $\mathcal{X}^{(j)}$ satisfies itself Property 2 of a σ-algebra on Ω_j.

Thirdly, we take an arbitrary set A, assuming that A is a sequence from $\mathbb{N}_+$ to $\mathcal{X}^{(j)}$, and we show that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{X}^{(j)}$ is implied. According to the definition of a codomain, we have the inclusion $\mathrm{ran}(A) \subseteq \mathcal{X}^{(j)}$, which implies in conjunction with (11.774) the inclusion $\mathrm{ran}(A) \subseteq \mathcal{P}(\Omega_j)$ because of (2.13). Thus, $\mathcal{P}(\Omega_j)$ is also a codomain of A, which function we may therefore write in the form $A : \mathbb{N}_+ \to \mathcal{P}(\Omega_j)$. We found in Exercise 11.11 the power set of any set to be a σ-algebra on that set, so that the union of the preceding sequence turns out to be an element of $\mathcal{P}(\Omega_j)$ due to Property 3 of a σ-algebra, that is,

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{P}(\Omega_j). \tag{11.775}$$

Moreover, the sequence $(A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{P}(\Omega_j)$ gives rise to the sequence of inverse images $p = (\pi_j^{-1}[A_n])_{n\in\mathbb{N}_+}$ according to Exercise 3.101e), and we obtain the equation

$$\pi_j^{-1}[\bigcup_{n=1}^{\infty} A_n] = \bigcup_{n=1}^{\infty} \pi_j^{-1}[A_n] \tag{11.776}$$

with Proposition 3.242. Let us verify that

$$p : \mathbb{N}_+ \to \mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}), \tag{11.777}$$

i.e. that the range of the sequence p is included in the preceding generated σ-algebra. To do this, we apply again the definition of a subset and prove accordingly the universal sentence

$$\forall Y\, (Y \in \mathrm{ran}(p) \Rightarrow Y \in \mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})), \tag{11.778}$$

letting Y be arbitrary and assuming $Y \in \mathrm{ran}(p)$ to be true. The definition of a range gives us then a particular constant $\bar{n}$ with $(\bar{n}, Y) \in p$, which we may write also in function/sequence notation as $Y = p_{\bar{n}} = \pi_j^{-1}[A_{\bar{n}}]$. Here, $A_{\bar{n}} = A(\bar{n})$ is a term/value of the sequence A, for which we can equivalently write $(\bar{n}, A_{\bar{n}}) \in A$, so that we see in light of the definition of a domain that $A_{\bar{n}} \in \mathrm{ran}(A)$ holds. This finding implies with the previously stated inclusion $\mathrm{ran}(A) \subseteq \mathcal{X}^{(j)}$ (by definition of a subset) that $A_{\bar{n}} \in \mathcal{X}^{(j)}$ is true, which gives us then with (11.773) especially

$$[Y =] \quad \pi_j^{-1}[A_{\bar{n}}] \in \mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}),$$

proving the implication in (11.778). Because Y was arbitrary, we may therefore conclude that the universal sentence (11.778) is also true, so that the range of p is included in the σ-algebra $\mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})$, as claimed. Writing the sequence p accordingly as the function (11.777) it follows by Property 3 of a σ-algebra that the union of that sequence is also an element of that σ-algebra, i.e.

$$\bigcup_{n=1}^{\infty} \pi_j^{-1}[A_n] \in \mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}).$$

Consequently, a substitution based on (11.776) yields

$$\pi_j^{-1}[\bigcup_{n=1}^{\infty} A_n] \in \mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}).$$

Together with (11.775), this implies now $\bigcup_{n=1}^{\infty} A_n \in \mathcal{X}^{(j)}$ with (11.773), completing the demonstration that Property 3 of a σ-algebra is satisfied by the set system $\mathcal{X}^{(j)}$.

Finally, concerning Property 4, we take an arbitrary set $A \in \mathcal{X}^{(j)}$, which evidently yields $A \in \mathcal{P}(\Omega_j)$ as well as $\pi_j^{-1}[A] \in \mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})$.

On the one hand, the previously mentioned fact that $\mathcal{P}(\Omega_j)$ is a σ-algebra gives us $A^c \in \mathcal{P}(\Omega_j)$. On the other hand, we obtain (again by Property 4 of a σ-algebra)

$$\left(\pi_j^{-1}[A]\right)^c \in \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}),$$

where the equation

$$\pi_j^{-1}[A^c] = (\pi_j^{-1}[A])^c$$

holds according to (3.759), noting that $A \in \mathcal{P}(\Omega_j)$ gives $A \subseteq \Omega_j$ with the definition of a power set. The resulting

$$\pi_j^{-1}[A^c] \in \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\})$$

implies then in conjunction with $A^c \in \mathcal{P}(\Omega_j)$ the truth of $A^c \in \mathcal{X}^{(j)}$. Since A is arbitrary, we may infer from this that $\mathcal{X}^{(j)}$ is closed under complements, as required by Property 4 of a σ-algebra.

We thus completed the proof that the set system $\mathcal{X}^{(j)}$ constitutes a σ-algebra on Ω_j, and we show now that it satisfies the inclusion (11.772). For this purpose, we first prove the inclusion $\mathcal{K}_j \subseteq \mathcal{X}^{(j)}$ by means of the definition of a subset, i.e. by demonstrating the truth of the universal sentence

$$\forall K\,(K \in \mathcal{K}_j \Rightarrow K \in \mathcal{X}^{(j)}). \tag{11.779}$$

We take an arbitrary set $\bar{K}$ and assume $\bar{K} \in \mathcal{K}_j$ to be true. Because we initially assumed $\mathcal{K}_j \subseteq \mathcal{P}(\Omega_j)$, the definition of a subset yields $\bar{K} \in \mathcal{P}(\Omega_j)$. We may therefore form the inverse image $\bar{X} = \pi_i^{-1}[\bar{K}]$, which equation shows that there exists an element in $\mathcal{K}_j$ such that $\pi_i^{-1}[K] = \bar{X}$. This existential sentence implies now $\bar{X} \in \{\pi_j^{-1}[K] : K \in \mathcal{K}_j\}$ with (11.765), and this in turn demonstrates the existence of an index $i \in I$ for which $\bar{X} \in \{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}$. This existential sentence gives us now with the Characterization of the union of a family of sets $\bar{X} \in \bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}$. As a generating system, this union is evidently included in the σ-algebra it generates, so that another application of the definition of a subset yields

$$\left[\pi_i^{-1}[\bar{K}] =\right] \quad \bar{X} \in \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[K] : K \in \mathcal{K}_i\}).$$

Alongside the previous finding $\bar{K} \in \mathcal{P}(\Omega_j)$, this implies with (11.773) $\bar{K} \in \mathcal{X}^{(j)}$, which is the desired consequent of the implication in (11.779). Here, $\bar{K}$ was arbitrary, so that the universal sentence (11.779) follows to be true, and this gives us now the proposed inclusion $\mathcal{K}_j \subseteq \mathcal{X}^{(j)}$. Because $\mathcal{A}(\mathcal{K}_j)$ is the smallest σ-algebra including $\mathcal{K}_j$, we then also have the inclusion (11.772), according to Theorem 11.35c).

Since j and moreover the sets I, $(\Omega_i)_{i\in I}$ and $(\mathcal{K}_i)_{i\in I}$ were previously arbitrary, we may now finally conclude that the stated lemma is true. $\square$

Exercise 11.41. Show for any nonempty index set I and any families of sets $(\Omega_i)_{i\in I}$ and $(\mathcal{K}_i)_{i\in I}$ with $\mathcal{K}_i \subseteq \mathcal{P}(\Omega_i)$ for any $i \in I$ that the family $(\mathcal{A}(\mathcal{K}_i))_{i\in I}$ of generated σ-algebras is uniquely determined.

Theorem 11.99 (Generation of product σ-algebras by means of families of generating systems). *It is true for any nonempty index set I and any families of sets $(\Omega_i)_{i\in I}$, $(\mathcal{K}_i)_{i\in I}$ with $\mathcal{K}_i \subseteq \mathcal{P}(\Omega_i)$ for all $i \in I$ that the product σ-algebra of the family $(\mathcal{A}(\mathcal{K}_i))_{i\in I}$ of generated σ-algebras is generated by the union*

$$\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{K}_i\}, \tag{11.780}$$

in the sense that

$$\bigotimes_{i\in I}\mathcal{A}(\mathcal{K}_i) = \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{K}_i\}). \tag{11.781}$$

Proof. We let I, $(\Omega_i)_{i\in I}$ and $(\mathcal{K}_i)_{i\in I}$ be arbitrary sets, assume $(\Omega_i)_{i\in I}$ and $(\mathcal{K}_i)_{i\in I}$ to be families of sets with index set $I \neq \emptyset$, and assume $\mathcal{K}_i \subseteq \mathcal{P}(\Omega_i)$ to be true for any $i \in I$. According to Exercise 11.41, we therefore obtain a corresponding family of sets $(\mathcal{A}(\mathcal{K}_i))_{i\in I}$ such that $\mathcal{A}(\mathcal{K}_i)$ is the σ-algebra generated by $\mathcal{K}_i$ for any $i \in I$. This family gives rise to the product σ-algebra

$$\bigotimes_{i\in I}\mathcal{A}(\mathcal{K}_i) = \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{A}(\mathcal{K}_i)\}). \tag{11.782}$$

To prove the equation (11.771), we establish now the two inclusions

$$\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{A}(\mathcal{K}_i)\} \subseteq \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{K}_i\}), \tag{11.783}$$

$$\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{K}_i\} \subseteq \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{A}(\mathcal{K}_i)\}). \tag{11.784}$$

Concerning (11.773), we apply the definition of a subset and prove the equivalent universal sentence

$$\forall\omega\,(\omega \in \bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{A}(\mathcal{K}_i)\} \Rightarrow \omega \in \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{K}_i\})), \tag{11.785}$$

letting ω be arbitrary and assuming the antecedent to be true. According to the Characterization of the union of a family of sets, there is then a particular index $\bar{k} \in I$ such that $\omega \in \{\pi_{\bar{k}}^{-1}[A] : A \in \mathcal{A}(\mathcal{K}_{\bar{k}})\}$ holds. By definition of this set system (see Exercise 11.39), there is then also a particular set $\bar{A} \in \mathcal{A}(\mathcal{K}_{\bar{k}})$ satisfying $\pi_{\bar{k}}^{-1}[\bar{A}] = \omega$. Here, $\bar{A} \in \mathcal{A}(\mathcal{K}_{\bar{k}})$ implies with the inclusion (11.772) and the definition of a subset

$$\bar{A} \in \{X : X \in \mathcal{P}(\Omega_{\bar{k}}) \wedge \pi_{\bar{k}}^{-1}[X] \in \mathcal{A}(\bigcup_{i \in I} \{\pi_{\bar{k}}^{-1}[K] : K \in \mathcal{K}_i\})\},$$

and this in turn gives $\bar{A} \in \mathcal{P}(\Omega_{\bar{k}})$ as well as

$$[\omega =] \quad \pi_{\bar{k}}^{-1}[\bar{A}] \in \mathcal{A}(\bigcup_{i \in I} \{\pi_{\bar{k}}^{-1}[K] : K \in \mathcal{K}_i\})$$

by definition of that set in (11.771) and (11.773). We thus find the implication in (11.785) to be true, in which ω is arbitrary, so that the universal sentence (11.785) and the equivalent inclusion (11.783) follow to be true.

Regarding the second inclusion (11.784), we establish the truth of

$$\forall \omega \, (\omega \in \bigcup_{i \in I} \{\pi_i^{-1}[A] : A \in \mathcal{K}_i\} \Rightarrow \omega \in \mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[A] : A \in \mathcal{A}(\mathcal{K}_i)\}), \tag{11.786}$$

letting ω be arbitrary and assuming that the antecedent is true. Applying now the Characterization of the union of a family of sets alongside the definition of the set $\{\pi_i^{-1}[A] : A \in \mathcal{K}_i\}$, there is evidently a particular index $\bar{k} \in I$ and a particular set $\bar{A} \in \mathcal{K}_i$ such that $\pi_{\bar{k}}^{-1}[\bar{A}] = \omega$ holds. Now, recalling that the σ-algebra $\mathcal{A}(\mathcal{K}_{\bar{k}})$ includes its generating system $\mathcal{K}_{\bar{k}}$, we clearly see that $\bar{A} \in \mathcal{K}_i$ implies $\bar{A} \in \mathcal{A}(\mathcal{K}_{\bar{k}})$. Thus, there evidently exist an index $i \in I$ and a set A satisfying both $A \in \mathcal{A}(\mathcal{K}_i)$ and $\pi_i^{-1}[A] = \omega$, so that ω turns out to be an element of the union $\bigcup_{i \in I} \{\pi_i^{-1}[A] : A \in \mathcal{A}(\mathcal{K}_i)\}$, which union itself is included in the σ-algebra generated by it. Consequently, the definition of a subset yields the desired consequent of the implication in (11.786), which then holds for any ω (as ω was arbitrary). Therefore, the inclusion (11.784) holds as well.

Finally, the truth of the two inclusions (11.783) – (11.784) implies with the Equality Criterion for generated σ-algebras the equation

$$\mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[A] : A \in \mathcal{A}(\mathcal{K}_i)\}) = \mathcal{A}(\bigcup_{i \in I} \{\pi_i^{-1}[A] : A \in \mathcal{K}_i\}), \tag{11.787}$$

which implies (11.781) via substitution based on (11.782). As the sets I, $(\Omega_i)_{i \in I}$ and $(\mathcal{K}_i)_{i \in I}$ were initially arbitrary, we may infer from the truth of that equation the truth of the stated theorem. □

11.9. Systems of Compact Sets

Definition 11.33 (Separation, disconnected & connected topological space). We say for any topological space $(\Omega, \mathcal{O})$ and for any sets U, V that the ordered pair (U, V) is a *separation* of Ω iff

1. U and V are open sets in X, i.e.

$$U \in \mathcal{O} \ \wedge \ V \in \mathcal{O}, \tag{11.788}$$

2. U and V are nonempty, i.e.

$$U \neq \emptyset \ \wedge \ V \neq \emptyset, \tag{11.789}$$

3. U and V are disjoint, i.e.

$$U \cap V = \emptyset, \tag{11.790}$$

4. the union of U and V is identical with Ω, i.e.

$$U \cup V = \Omega, \tag{11.791}$$

Furthermore, we say that the topological space $(\Omega, \mathcal{O})$ is *disconnected* iff there exists a separation of Ω. Moreover, we call $(\Omega, \mathcal{O})$ a *connected topological space* iff it is not true that $(\Omega, \mathcal{O})$ is disconnected.

Proposition 11.100. *It is true for any disconnected topological space* $(\Omega, \mathcal{O})$ *and for any separation* (U, V) *of* Ω *that* U *is the complement of* V *with respect to* Ω *and vice versa, i.e.*

$$U = V^c \ \wedge \ V = U^c. \tag{11.792}$$

Proof. We let Ω, $\mathcal{O}$, U and V be arbitrary sets such that $(\Omega, \mathcal{O})$ is a disconnected topological space and such that (U, V) is a separation of Ω. In view of Property 1 of a separation, we have $U, V \in \mathcal{O}$, which implies $U, V \in \mathcal{P}(\Omega)$ with Property 1 of a topology on Ω and then $U \subseteq \Omega$ as well as $V \subseteq \Omega$ with the definition of a power set. Thus, the complements $U^c = \Omega \setminus U$ and $V^c = \Omega \setminus V$ are indeed defined. Furthermore, $U \cup V = \Omega$ and $U \cap V = \emptyset$ are both true because of Property 4 and Property 3 of a separation, and these equations imply

$$U = \Omega \setminus V = V^c$$

with (2.262) and the definition of a complement with respect to Ω. In addition, the equations $U \cup V = \Omega$ and $U \cap V = \emptyset$ imply, respectively,

$V \cup U = \Omega$ and $V \cap U = \emptyset$ with the Commutative Laws for the union and for the intersection of two sets. Another application of (2.262) yields then

$$V = \Omega \setminus U = U^c,$$

so that the proof of the conjunction (11.792) is now complete. Since Ω, $\mathcal{O}$, U and V were initially arbitrary, we therefore conclude that the proposed universal sentence is true. $\square$

Exercise 11.42. It is true for any topological space $(\Omega, \mathcal{O})$ and for any separation (U, V) of Ω that an element ω of Ω is element of U iff ω is not in V, and vice versa, i.e.

$$\forall \omega\, (\omega \in \Omega \Rightarrow [\omega \in U \Leftrightarrow \omega \notin V]), \tag{11.793}$$
$$\forall \omega\, (\omega \in \Omega \Rightarrow [\omega \notin U \Leftrightarrow \omega \in V]). \tag{11.794}$$

(Hint: Proceed in analogy to the proofs of Proposition 8.3 and Exercise 8.2.)

Theorem 11.101 (Disconnectedness Criterion). *It is true that a topological space $(\Omega, \mathcal{O})$ is disconnected iff there exists, besides the empty set and besides Ω, a subset of Ω which is both open and closed in Ω, i.e.*

$$\begin{aligned} &(\Omega, \mathcal{O}) \text{ is disconnected} \\ &\quad \Leftrightarrow\ \exists X\, (X \subseteq \Omega \ \wedge\ X \neq \emptyset \ \wedge\ X \neq \Omega \ \wedge\ X \in \mathcal{O} \ \wedge\ X \text{ is closed in } \Omega). \end{aligned} \tag{11.795}$$

Proof. We let Ω and $\mathcal{O}$ be arbitrary sets such that $(\Omega, \mathcal{O})$ constitutes a topological space. Regarding the first part ('$\Rightarrow$') of the equivalence, we assume that $(\Omega, \mathcal{O})$ is disconnected, so that there exists a separation of Ω, say $\bar{s}$. By definition of a separation, $\bar{s}$ constitutes an ordered pair, so that we have $\bar{s} = (\bar{U}, \bar{V})$ for some particular sets $\bar{U}, \bar{V}$. We can show now that $\bar{U}$ is an open and closed set in Ω different from $\emptyset$ and Ω. To begin with, we note that $\bar{U}$ is open in Ω according to Property 1 of a separation, and that $\bar{U} \neq \emptyset$ holds because of Property 2 of a separation. Next, we prove $\bar{U} \neq \Omega$ by contradiction, assuming $\neg \bar{U} \neq \Omega$ to be true, with the consequence that $\bar{U} = \Omega$ is true (due to the Double Negation Law). Now, since $\bar{V}$ is also nonempty (because of Property 2 of a separation), there exists an element in $\bar{V}$, say $\bar{y}$. Then, the disjunction $\bar{y} \in \bar{U} \vee \bar{y} \in \bar{V}$ also holds, which means by definition of the union of two sets that $\bar{y} \in \bar{U} \cup \bar{V}$ is true. In view of Property 4 of a separation, we therefore obtain $\bar{y} \in \Omega$ $[= \bar{U}]$ via substitution, thus $\bar{y} \in \bar{U}$. Here, $\bar{y} \in \Omega$ and $\bar{y} \in \bar{U}$ also imply $\bar{y} \notin \bar{V}$ with (11.793), in contradiction to the previously established $\bar{y} \in \bar{V}$. We thus established the truth of $\bar{U} \neq \Omega$. It remains for us to demonstrate that $\bar{U}$

is closed in Ω. Since (11.792) gives $\bar{V} = \bar{U}^c$ where $\bar{V}$ is open in Ω (by Property 1 of a separation), we have that $\bar{U}^c$ is open in Ω. Consequently, $\bar{U}$ is indeed closed in Ω by definition. Having found a particular subset of Ω which is different from $\emptyset$ and Ω and which is both open and closed in Ω, we completed the proof of the first part ('$\Rightarrow$') of the equivalence.

Concerning the second part ('$\Leftarrow$'), we assume now that there exists a subset of Ω, say $\bar{X}$, such that $\bar{X} \neq \emptyset$ and $\bar{X} \neq \Omega$ hold and such that $\bar{X}$ is both open and closed in Ω. The assumption $\bar{X} \subseteq \Omega$ allows us to form the complement $\bar{X}^c$ with respect to Ω, giving rise to the ordered pair $(\bar{X}, \bar{X}^c)$, which we show in the following to be a separation of Ω. The closedness of $\bar{X}$ implies (by definition of a closed set) that $\bar{X}^c$ is open in Ω, so that $\bar{X}$ and $\bar{X}^c$ are both elements of $\mathcal{O}$, as required by Property 1 of a separation. Regarding Property 2, we prove $\bar{X}^c \neq \emptyset$ by contradiction, assuming $\neg \bar{X}^c \neq \emptyset$ to hold, so that $\bar{X}^c = \emptyset$ is true (by virtue of the Double Negation Law). We obtain then the equations

$$\bar{X} = (\bar{X}^c)^c = \emptyset^c = \Omega$$

by applying (2.136), substitution and (2.133). The resulting equation $\bar{X} = \Omega$ contradicts the previously established negation $\bar{X} \neq \Omega$, so that $\bar{X}^c \neq \emptyset$ holds indeed. Because $\bar{X} \neq \emptyset$ is also true, Property 2 of a separation is also satisfied by $(\bar{X}, \bar{X}^c)$. Observing now the truth of the equations $\bar{X} \cap \bar{X}^c = \emptyset$ and $\bar{X} \cup \bar{X}^c = \Omega$ in light of (2.135) and (2.257), we immediately have that $(\bar{X}, \bar{X}^c)$ satisfies Property 3 as well as Property 4 of a separation. We thus demonstrated the existence of a separation of Ω, which means that the topological space $(\Omega, \mathcal{O})$ is disconnected. The proof of the equivalence is now complete, and as the sets Ω and $\mathcal{O}$ were initially arbitrary, we may therefore conclude that the theorem is true. □

Theorem 11.102 (Equivalence of connectedness and convexity for subsets of linear continua). *It is true for any linear continuum* $(\Omega, <_\Omega)$ *and for any subset* Ω_1 *of* Ω *that the topological subspace* $(\Omega_1, \mathcal{O}_{<_\Omega}|\Omega_1)$ *is connected iff* Ω_1 *is convex in* Ω.

Proof. We take arbitrary sets Ω, $<_\Omega$ and Ω_1, assuming $(\Omega, <_\Omega)$ to be a linear continuum and assuming Ω_1 to be a subset of Ω. The former assumption means that $<_\Omega$ is a densely ordered set having the Supremum Property, in other words that Ω is neither empty nor a singleton and that $<_\Omega$ constitutes a linear ordering of Ω satisfying

$$\forall x, y\,([x, y \in \Omega \,\wedge\, x <_\Omega y] \Rightarrow \exists z\,(z \in \Omega \,\wedge\, x <_\Omega z <_\Omega y)). \tag{11.796}$$

Consequently, $(\Omega, <_\Omega)$ defines the order topology $\mathcal{O}_{<_\Omega}$, so that the assumed inclusion $\Omega_1 \subseteq \Omega$ yields the subspace topology $\mathcal{O}_{<_\Omega}|\Omega_1$, which forms together with Ω_1 the topological subspace $(\Omega_1, \mathcal{O}_{<_\Omega}|\Omega_1)$.

We prove now the first part ('⇒') of the equivalence by contraposition, assuming that Ω_1 is not convex in Ω. According to Exercise 3.67, this assumption implies the truth of the negation

$$\neg\forall a, b\,(a, b \in \Omega_1 \Rightarrow \forall x\,(a <_\Omega x <_\Omega b \Rightarrow x \in \Omega_1)),$$

which in turn implies the truth of the existential sentence

$$\exists a, b\,(a, b \in \Omega_1 \,\wedge\, \neg\forall x\,(a <_\Omega x <_\Omega b \Rightarrow x \in \Omega_1))$$

by means of the Negation Law for universal implications. Thus, there exist elements in Ω_1, say $\bar{a}$ and $\bar{b}$, for which the negation

$$\neg\forall x\,(\bar{a} <_\Omega x <_\Omega \bar{b} \Rightarrow x \in \Omega_1)$$

holds. Another application of the Negation Law for universal implications gives us now the existential sentence

$$\exists x\,(\bar{a} <_\Omega x <_\Omega \bar{b} \,\wedge\, \neg x \in \Omega_1),$$

so that the inequalities $\bar{a} <_\Omega \bar{x} <_\Omega \bar{b}$ and the negation $\neg\bar{x} \in \Omega_1$ are satisfied by some particular constant $\bar{x}$. The inequalities are possible because the previously obtained elements $\bar{a}, \bar{b} \in \Omega_1$ are also elements of Ω by virtue of the initially assumed inclusion $\Omega_1 \subseteq \Omega$. We define now the intersections $U = \Omega_1 \cap (-\infty, \bar{x})$ and $V = \Omega_1 \cap (\bar{x}, +\infty)$, and we demonstrate that the ordered pair (U, V) constitutes a separation of Ω_1.

We begin with the observation that the intervals $(-\infty, \bar{x})$ and $(\bar{x}, +\infty)$ are elements of the order topology $\mathcal{O}_{<_\Omega}$ in view of $\bar{x} \in \Omega$ and Lemma 11.81. According to the specification of the subspace topology on Ω_1 in Theorem 11.51), we obtain therefore

$$\begin{aligned}
[U =] \quad & \Omega_1 \cap (-\infty, \bar{x}) \in \mathcal{O}_{<_\Omega}|\Omega_1, \\
[V =] \quad & \Omega_1 \cap (\bar{x}, +\infty) \in \mathcal{O}_{<_\Omega}|\Omega_1,
\end{aligned}$$

so that (U, V) satisfies Property 1 of a separation of Ω_1 with respect to the topological subspace $(\Omega_1, \mathcal{O}_{<_\Omega}|\Omega_1)$.

Regarding Property 2, we recall the truth of $\bar{a}, \bar{b} \in \Omega_1$, and we observe that the previously established inequalities $\bar{a} <_\Omega \bar{x}$ and $\bar{x} <_\Omega \bar{b}$ imply, respectively, $\bar{a} \in (-\infty, \bar{x})$ with the definition of an open, left-unbounded interval and $\bar{b} \in (\bar{x}, +\infty)$ by definition of an open, right-unbounded interval in Ω. We therefore find with the definition of an intersection of two sets

$$\begin{aligned}
\bar{a} &\in \Omega_1 \cap (-\infty, \bar{x}) \quad [= U], \\
\bar{b} &\in \Omega_1 \cap (\bar{x}, +\infty) \quad [= V].
\end{aligned}$$

Thus, the set U contains $\bar{a}$ and the set V contains $\bar{b}$, so that these sets are evidently nonempty, as required.

Property 3 of a separation requires the verification of the disjointness condition $U \cap V = \emptyset$, which we accomplish by applying the definition of the empty set and by proving accordingly the equivalent universal sentence

$$\forall \omega \, (\omega \notin U \cap V). \tag{11.797}$$

To do this, we take an arbitrary ω, and we prove the $\omega \notin U \cap V$ by contradiction, assuming $\neg \omega \notin U \cap V$ to be true, so that $\omega \in U \cap V$ holds according to the Double Negation Law. By definition of the intersection of two sets, we therefore have $\omega \in U$ and $\omega \in V$, which means by definition of these sets in particular that $\omega \in (-\infty, \bar{x})$ and $\omega \in (\bar{x}, +\infty)$ are both true. Evidently, these findings imply $\omega <_\Omega \bar{x}$ and $\bar{x} <_\Omega \omega$, contradiction to the fact that these inequalities are not simultaneously true in view of the Characterization of comparability with respect to the linear ordering $<_\Omega$. We thus completed the proof of $\omega \notin U \cap V$, in which negation ω is arbitrary, so that $U \cap V = \emptyset$ follows indeed to be true.

It remains for us to establish Property 4 of a separation of Ω_1 for (U, V). Let us observe first the truth of the equations

$$\begin{aligned} U \cup V &= [\Omega_1 \cap (-\infty, \bar{x})] \cup [\Omega_1 \cap (\bar{x}, +\infty)] \\ &= \Omega_1 \cap [(-\infty, \bar{x}) \cup (\bar{x}, +\infty)] \end{aligned} \tag{11.798}$$

in light of the definition of U and V and in light of the Distributive Law for the intersection of two sets. Here, we can show that the inclusion

$$\Omega_1 \subseteq [(-\infty, \bar{x}) \cup (\bar{x}, +\infty)] \tag{11.799}$$

holds. We apply the definition of a subset and prove accordingly the universal sentence

$$\forall \omega \, (\omega \in \Omega_1 \Rightarrow \omega \in (-\infty, \bar{x}) \cup (\bar{x}, +\infty)), \tag{11.800}$$

letting ω be arbitrary and assuming $\omega \in \Omega_1$ to be true. This assumption implies in conjunction with the previously found negation $\neg \bar{x} \in \Omega_1$ that $\omega \neq \bar{x}$ holds, according to (2.4). Then, since the linear ordering $<_\Omega$ is connex, the disjunction $\omega <_\Omega \bar{x} \vee \bar{x} <_\Omega \omega$ follows to be true (recalling that $\bar{x}$ is an element of Ω and noting that $\omega \in \Omega_1$ implies $\omega \in \Omega$ with the assumed inclusion $\Omega_1 \subseteq \Omega$). Due to the definitions of open, left-unbounded and of open, right-unbounded intervals in Ω, the preceding disjunction implies $\omega \in (-\infty, \bar{x}) \vee \omega \in (\bar{x}, +\infty)$, so that the desired consequent $\omega \in (-\infty, \bar{x}) \cup (\bar{x}, +\infty)$ of the implication in (11.799) follows to be true by definition of the union of two sets. Consequently, the suggested inclusion (11.799) holds indeed, and this gives us for the union (11.798)

$$\begin{aligned} U \cup V &= \Omega_1 \cap [(-\infty, \bar{x}) \cup (\bar{x}, +\infty)] \\ &= \Omega_1 \end{aligned}$$

because of (2.77).

We thus demonstrated that (U, V) satisfies has all of the five defining properties of a separation of Ω_1. The existence of such a separation implies now that the topological subspace $(\Omega_1, \mathcal{O}_{<_\Omega}|\Omega_1)$ is disconnected, which completes the proof of the first part ('$\Rightarrow$') of the proposed equivalence.

We prove the second part ('$\Leftarrow$') of the equivalence by contradiction, assuming this time that the set Ω_1 is convex in Ω and in addition that the given topological subspace is disconnected. To establish the contradiction, we observe in light of the Characterization of the supremum that it is true for any subset A of Ω, for any set S being the supremum of A (with respect to $\leq_\Omega$) that S is an upper bound for A and less than or equal to any upper bound S' for A (with respect to $\leq_\Omega$). We prove in the sequel that the negation of this universal sentence is also true, that is, we prove (according to the Negation Law for universal implications and the Negation Formula for $\leq$) the existence of a subset A of Ω and of a supremum S of A such that S is not an upper bound for A or such that there exists an upper bound S' for A being less than the supremum S.

To be begin with, the assumed disconnectedness implies that there exists a separation of Ω_1, say s. As a separation, s constitutes an ordered pair $(\bar{U}, \bar{V})$, where $\bar{U}$ and $\bar{V}$ are open, nonempty, disjoint sets of $\mathcal{O}_{<_\Omega}|\Omega_1$ with $\bar{U} \cup \bar{V} = \Omega_1$. The nonemptiness of $\bar{U}$ and $\bar{V}$ implies then clearly the existence of particular elements $\bar{\omega} \in \bar{U}$ and $\bar{\nu} \in \bar{V}$. Because the sets $\bar{U}$ and $\bar{V}$ are open with respect to a topology on Ω_1, they evidently constitute subsets of Ω_1 (as a consequence of Property 1 of a topology). We obtain therefore $\bar{\omega}, \bar{\nu} \in \Omega_1$ with the definition of a subset.

Here, we can prove by contradiction that $\bar{\omega} \neq \bar{\nu}$ holds. Assuming for this purpose $\neg \bar{\omega} \neq \bar{\nu}$ to be true, we obtain $\bar{\omega} = \bar{\nu}$ with the Double Negation Law, then $\bar{\omega} \in \bar{U} \wedge \bar{\omega} \in \bar{V}$ by means of substitution, subsequently $\bar{\omega} \in \bar{U} \cap \bar{V}$ by definition of the intersection of two sets, and finally $\bar{\omega} \in \emptyset$ via substitution based on the disjointness of $\bar{U}$ and $\bar{V}$. Because the negation $\neg \bar{\omega} \in \emptyset$ is also true by definition of the empty set, we obtained a contradiction, so that $\bar{\omega} \neq \bar{\nu}$ holds indeed. The existence of two distinct elements in Ω_1 implies that this set is neither empty nor a singleton (see Exercise 2.21). As we assumed Ω_1 to be convex in Ω, we find then with the Compatibility of subspace and order topologies

$$\mathcal{O}_{<_\Omega}|\Omega_1 = \mathcal{O}_{<_{\Omega_1}}, \tag{11.801}$$

where the linear ordering $<_{\Omega_1}$ defined according to Theorem 3.68.

Furthermore, $\bar{\omega}, \bar{\nu} \in \Omega_1$ implies $\bar{\omega}, \bar{\nu} \in \Omega$ with the assumed inclusion $\Omega_1 \subseteq \Omega$. Then, as the linear ordering $<_\Omega$ is connex, it follows from $\bar{\omega} \neq \bar{\nu}$ that the disjunction $\bar{\omega} <_\Omega \bar{\nu} \vee \bar{\nu} <_\Omega \bar{\omega}$ is true, which we use now to carry out a proof by cases.

In case of $\bar{\omega} <_\Omega \bar{\nu}$, we use the previous finding $\bar{\omega}, \bar{\nu} \in \Omega$ to define the open interval $(\bar{\omega}, \bar{\nu})_\Omega$. We observe now in light of the assumed convexity of Ω_1 in Ω that $\bar{\omega}, \bar{\nu} \in \Omega_1$ implies the inclusion $(\bar{\omega}, \bar{\nu})_\Omega \subseteq \Omega_1$. The current case assumption gives us also

$$\begin{aligned} [\bar{\omega}, \bar{\nu}]_\Omega &= [\bar{\omega}, \bar{\nu})_\Omega \cup \{\bar{\nu}\} \\ &= [(\bar{\omega}, \bar{\nu})_\Omega \cup \{\bar{\omega}\}] \cup \{\bar{\nu}\} \end{aligned} \tag{11.802}$$

by means of (3.405) and (3.402), noting that $\bar{\omega} <_\Omega \bar{\nu}$ implies $\bar{\omega} <_\Omega \bar{\nu} \vee \bar{\omega} = \bar{\nu}$ and therefore $\bar{\omega} \leq_\Omega \bar{\nu}$ according to Definition 3.20(3). Furthermore, $\bar{\omega} \in \Omega_1$ and $\bar{\nu} \in \Omega_1$ give us, respectively, $\{\bar{\omega}\} \subseteq \Omega_1$ and $\{\bar{\nu}\} \subseteq \Omega_1$ with (2.169). Since the inclusion $(\bar{\omega}, \bar{\nu})_\Omega \subseteq \Omega_1$ also holds, we therefore obtain $[\bar{\omega}, \bar{\nu}]_\Omega \subseteq \Omega_1$ by applying (2.252) successively to the unions in (11.802). Let us verify that $[\bar{\omega}, \bar{\nu}]_\Omega$ satisfies also Property 2 of a convex set Ω_1 (with respect to $<_{\Omega_1}$), that is,

$$\forall a, b\, (a, b \in [\bar{\omega}, \bar{\nu}]_\Omega \Rightarrow (a, b)_{\Omega_1} \subseteq [\bar{\omega}, \bar{\nu}]_\Omega). \tag{11.803}$$

We let $a, b \in [\bar{\omega}, \bar{\nu}]_\Omega$ be arbitrary, and we apply the definition of a subset to establish the desired inclusion, by proving accordingly the universal sentence

$$\forall x\, (x \in (a, b)_{\Omega_1} \Rightarrow x \in [\bar{\omega}, \bar{\nu}]_\Omega). \tag{11.804}$$

Letting x be arbitrary and assuming $x \in (a, b)_{\Omega_1}$ to be true, we obtain the inequality $a <_{\Omega_1} x <_{\Omega_1} b$ with the definition of an open interval in Ω_1. Since the linear ordering $<_{\Omega_1}$ was defined by the linear ordering $<_\Omega$ with respect to the subset Ω_1 of Ω, we can rewrite the preceding inequalities as $a <_\Omega x <_\Omega b$. Evidently, these inequalities imply now $a \leq_\Omega x$ and $x \leq_\Omega b$; as the assumed antecedent $a, b \in [\bar{\omega}, \bar{\nu}]_\Omega$ gives in particular $\bar{\omega} \leq_\Omega a$ and $b \leq_\Omega \bar{\nu}$ by definition of a closed interval in Ω, we can apply the transitivity of the total ordering $\leq_\Omega$ to these inequalities in order to obtain $\bar{\omega} \leq_\Omega x$ and $x \leq_\Omega \bar{\nu}$. Consequently, we find $x \in [\bar{\omega}, \bar{\nu}]_\Omega$ to be true, so that the implication in (11.804) holds. Here, x is arbitrary, so that the universal sentence (11.804) follows to be true. This completes the proof of the inclusion in (11.803), and as a and b were arbitrary, we therefore conclude that the universal sentence (11.803) also holds. In view of the previously established inclusion $[\bar{\omega}, \bar{\nu}]_\Omega \subseteq \Omega_1$, this means that the set $[\bar{\omega}, \bar{\nu}]_\Omega$ is convex in Ω_1 (with respect to $<_{\Omega_1}$) by definition.

As this closed interval contains its minimum $\bar{\omega}$ and its maximum $\bar{\nu}$ (with respect to $\leq_\Omega$) in view of Corollary 3.118 and the definition of a minimum/maximum, we see that $[\bar{\omega}, \bar{\nu}]_\Omega$ is a set containing two distinct elements. We thus showed that $[\bar{\omega}, \bar{\nu}]_\Omega$ is a nonempty and non-singleton convex set in Ω_1, so that we obtain for the corresponding subspace topology by virtue of the Compatibility of subspace and order topologies (in connection with the definition of an order topology)

$$\begin{aligned}\mathcal{O}_{<_{\Omega_1}}|[\bar{\omega}, \bar{\nu}]_\Omega &= \mathcal{O}_{<_{[\bar{\omega}, \bar{\nu}]_\Omega}} &(11.805)\\ &= \mathcal{O}(\{(a,b) : a, b \in [\bar{\omega}, \bar{\nu}]_\Omega\} \cup \{[\bar{\omega}, b) : b \in [\bar{\omega}, \bar{\nu}]_\Omega\} \\ &\qquad \cup \{(a, \bar{\nu}] : a \in [\bar{\omega}, \bar{\nu}]_\Omega\}), &(11.806)\end{aligned}$$

where $([\bar{\omega}, \bar{\nu}]_\Omega, <_{[\bar{\omega}, \bar{\nu}]_\Omega})$ constitutes a linearly ordered set according to Theorem 3.68. Next, we define the intersections

$$\bar{A} = [\bar{\omega}, \bar{\nu}]_\Omega \cap \bar{U}, \tag{11.807}$$
$$\bar{B} = [\bar{\omega}, \bar{\nu}]_\Omega \cap \bar{V}. \tag{11.808}$$

Observing now in light of (11.801) that the open sets $\bar{U}$ and $\bar{V}$ of the subspace topology $\mathcal{O}_{<_\Omega}|\Omega_1$ are elements of $\mathcal{O}_{<_{\Omega_1}}$, we obtain from the preceding equations with the definition of a subspace topology (of $\mathcal{O}_{<_{\Omega_1}}$ in $[\bar{\omega}, \bar{\nu}]_\Omega$)

$$\bar{A} \in \mathcal{O}_{<_{\Omega_1}}|[\bar{\omega}, \bar{\nu}]_\Omega \ \wedge \ \bar{B} \in \mathcal{O}_{<_{\Omega_1}}|[\bar{\omega}, \bar{\nu}]_\Omega,$$

and subsequently through substitutions based on (11.805)

$$\bar{A} \in \mathcal{O}_{<_{[\bar{\omega}, \bar{\nu}]_\Omega}} \ \wedge \ \bar{B} \in \mathcal{O}_{<_{[\bar{\omega}, \bar{\nu}]_\Omega}}.$$

Because $([\bar{\omega}, \bar{\nu}]_\Omega, \mathcal{O}_{<_{[\bar{\omega}, \bar{\nu}]_\Omega}})$ is a topological space, this conjunction demonstrates that the ordered pair $(\bar{A}, \bar{B})$ satisfies Property 1 of a separation of $[\bar{\omega}, \bar{\nu}]_\Omega$. Furthermore, we noted previously that $\bar{\omega}$ and $\bar{\nu}$ are elements of the closed interval $[\bar{\omega}, \bar{\nu}]_\Omega$, and we earlier found $\bar{\omega} \in \bar{U}$ and $\bar{\nu} \in \bar{V}$, so that the conjunctions $\bar{\omega} \in [\bar{\omega}, \bar{\nu}]_\Omega \wedge \bar{\omega} \in \bar{U}$ and $\bar{\nu} \in [\bar{\omega}, \bar{\nu}]_\Omega \wedge \bar{\nu} \in \bar{V}$ hold. Consequently, we obtain $\bar{\omega} \in \bar{A}$ as well as $\bar{\nu} \in \bar{B}$ with the definition of the intersection of two sets and the definitions of the sets $\bar{A}$ and $\bar{B}$. These sets are therefore nonempty, that is,

$$\bar{A} \neq \emptyset \ \wedge \ \bar{B} \neq \emptyset,$$

so that Property 2 of a separation is also satisfied by $(\bar{A}, \bar{B})$. Moreover, we can derive the equations

$$\begin{aligned}\bar{A} \cap \bar{B} &= ([\bar{\omega}, \bar{\nu}]_\Omega \cap \bar{U}) \cap ([\bar{\omega}, \bar{\nu}]_\Omega \cap \bar{V}) \\ &= ([\bar{\omega}, \bar{\nu}]_\Omega \cap [\bar{\omega}, \bar{\nu}]_\Omega) \cap (\bar{U} \cap \bar{V}) \\ &= [\bar{\omega}, \bar{\nu}]_\Omega \cap \emptyset \\ &= \emptyset &(11.809)\end{aligned}$$

by applying substitutions, then the Commutative and the Associative Law for the intersection of two sets, the Idempotent Law for the intersection of two sets alongside the previously established disjointness of $\bar{U}$ and $\bar{V}$, and finally (2.62). Thus, the ordered pair $(\bar{A}, \bar{B})$ has Property 3 of a separation. In addition, we get (as required by Property 4 of a separation of $[\bar{\omega}, \bar{\nu}]_\Omega$)

$$\begin{aligned} \bar{A} \cup \bar{B} &= ([\bar{\omega}, \bar{\nu}]_\Omega \cap \bar{U}) \cup ([\bar{\omega}, \bar{\nu}]_\Omega \cap \bar{V}) \\ &= [\bar{\omega}, \bar{\nu}]_\Omega \cap (\bar{U} \cup \bar{V}) \\ &= [\bar{\omega}, \bar{\nu}]_\Omega \cap \Omega_1 \\ &= [\bar{\omega}, \bar{\nu}]_\Omega \end{aligned} \tag{11.810}$$

by means of the Distributive Law for the intersection of two sets, the previously found equation $\bar{U} \cup \bar{V} = \Omega_1$ and (2.77), noting that the closed interval $[\bar{\omega}, \bar{\nu}]_\Omega$ in Ω_1 is a subset of Ω_1 according to Proposition 3.117b). We thus proved that the ordered pair $(\bar{A}, \bar{B})$ constitutes a separation of $[\bar{\omega}, \bar{\nu}]_\Omega$.

We mentioned before that $[\bar{\omega}, \bar{\nu}]_\Omega$ has the maximum $\bar{\nu}$ (with respect to $\leq_\Omega$). Now, since $\bar{A}$ is an open set of the subspace topology $\mathcal{O}_{<_{[\bar{\omega}, \bar{\nu}]_\Omega}}$ in $[\bar{\omega}, \bar{\nu}]_\Omega$, it constitutes a subset of $[\bar{\omega}, \bar{\nu}]_\Omega$ because of Property 1 of a topology (in $[\bar{\omega}, \bar{\nu}]_\Omega$). Thus, $\bar{A}$ is a subset of the bounded-from-above subset $[\bar{\omega}, \bar{\nu}]_\Omega$ of Ω, so that $\bar{A}$ has the same upper bound $\bar{\nu}$ (with respect to $\leq_\Omega$) as $[\bar{\omega}, \bar{\nu}]_\Omega$ due to Proposition 3.94. We thus showed that $\bar{A}$ is a nonempty and bounded-from-above subset of Ω, which allows us to apply the Supremum Property with respect to the linear continuum $(\Omega, <_\Omega)$ to infer the existence of the supremum of $\bar{A}$ (with respect to $\leq_\Omega$). As the least upper bound for $\bar{A}$, the supremum is less than or equal to the upper bound $\bar{\nu}$, that is, $\sup^{\leq_\Omega} \bar{A} \leq_\Omega \bar{\nu}$. For convenience, we denote this supremum in the following by $\bar{S}$. We now show that this supremum satisfies also the inequality $\bar{\omega} \leq_\Omega \bar{S}$. To begin with, the nonemptiness of $\bar{A}$ implies that this set has some element, say $\bar{y}$. Because the supremum $\bar{S}$ is an upper bound for $\bar{A}$, we have $\bar{y} \leq_\Omega \bar{S}$. Moreover, $\bar{y} \in \bar{A}$ implies with the aforementioned inclusion $\bar{A} \subseteq [\bar{\omega}, \bar{\nu}]_\Omega$ also $\bar{y} \in [\bar{\omega}, \bar{\nu}]_\Omega$, with the evident consequence that $\bar{\omega} \leq_\Omega \bar{y}$. In conjunction with $\bar{y} \leq_\Omega \bar{S}$, this yields indeed $\bar{\omega} \leq_\Omega \bar{S}$ with the transitivity of the total ordering $\leq_\Omega$. We thus found the two inequalities

$$\bar{\omega} \ \leq_\Omega \ \bar{S} \ \leq_\Omega \ \bar{\nu}, \tag{11.811}$$

which we can state also in the form $\bar{S} \in [\bar{\omega}, \bar{\nu}]_\Omega$ (using the definition of a closed interval in Ω). In view of (11.810), we therefore have $\bar{S} \in \bar{A} \cup \bar{B}$, and the definition of the union of two sets gives us then also the true disjunction $\bar{S} \in \bar{A} \vee \bar{S} \in \bar{B}$. We consider now two corresponding sub-cases. Let us recall first that $\bar{A}$ and $\bar{B}$ are open sets of the order topology $\mathcal{O}_{<_{[\bar{\omega}, \bar{\nu}]_\Omega}}$, so that we have

$$\bar{A}, \bar{B} \in \mathcal{O}(\{(a,b) : a,b \in [\bar{\omega},\bar{\nu}]_\Omega\} \cup \{[\bar{\omega},b) : b \in [\bar{\omega},\bar{\nu}]_\Omega\} \cup \{(a,\bar{\nu}] : a \in [\bar{\omega},\bar{\nu}]_\Omega\}) \tag{11.812}$$

in view of (11.805) – (11.806).

Assuming $\bar{S} \in \bar{A}$ to be true in the first sub-case, it follows from (11.812) by virtue of the Generation of a topology by means of a basis that there exists a particular basis element

$$\bar{K} \in \{(a,b) : a,b \in [\bar{\omega},\bar{\nu}]_\Omega\} \cup \{[\bar{\omega},b) : b \in [\bar{\omega},\bar{\nu}]_\Omega\} \cup \{(a,\bar{\nu}] : a \in [\bar{\omega},\bar{\nu}]_\Omega\}$$

which satisfies $\bar{S} \in \bar{K}$ and $\bar{K} \subseteq \bar{A}$. According to the definition of the union of two sets, we thus have the true disjunctions

$$\begin{aligned} \bar{K} \in \{(a,b) : a,b \in [\bar{\omega},\bar{\nu}]_\Omega\} \,\vee\, & \bar{K} \in \{[\bar{\omega},b) : b \in [\bar{\omega},\bar{\nu}]_\Omega\} \\ \vee\, & \bar{K} \in \{(a,\bar{\nu}] : a \in [\bar{\omega},\bar{\nu}]_\Omega\}. \end{aligned} \tag{11.813}$$

In the following, we prove by contradiction that the third part of this multiple disjunction is false, in other words that $\bar{K} \notin \{(a,\bar{\nu}] : a \in [\bar{\omega},\bar{\nu}]_\Omega\}$ is true. To do this, we assume that the negation of the preceding negation is true, which assumption implies then with the Double Negation Law that $\bar{K} \in \{(a,\bar{\nu}] : a \in [\bar{\omega},\bar{\nu}]_\Omega\}$ holds. According to the definition of this set system (see Exercise 11.25), there exists then a constant, say $\bar{a}$, which is element of $[\bar{\omega},\bar{\nu}]_\Omega$ and for which $\bar{K}$ becomes identical with the interval $(\bar{a},\bar{\nu}]$. Consequently, we can write for $\bar{S} \in \bar{K}$ after substitution $\bar{S} \in (\bar{a},\bar{\nu}]$, and this means according to the definition of a left-open and right-closed interval in $[\bar{\omega},\bar{\nu}]_\Omega$ that

$$\bar{a} \,<_{[\bar{\omega},\bar{\nu}]_\Omega}\, \bar{S} \,\leq_{[\bar{\omega},\bar{\nu}]_\Omega}\, \bar{\nu}.$$

These inequalities imply $\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{\nu}$ with the Transitivity Formula for $<$ and $\leq$, and since $\bar{\nu} \leq_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{\nu}$ also holds because of the reflexivity of the total ordering $\leq_{[\bar{\omega},\bar{\nu}]_\Omega}$, it follows that $\bar{\nu} \in (\bar{a},\bar{\nu}]$ is true (again by definition of a left-open and right-closed interval). Then, the equality $\bar{K} = (\bar{a},\bar{\nu}]$ gives $\bar{\nu} \in \bar{K}$ via substitution, so that $\bar{\nu} \in \bar{A}$ follows to be true with the previously established inclusion $\bar{K} \subseteq \bar{A}$. Since $\bar{\nu} \in \bar{B}$ was previously found to be true as well, we obtain (with the definition of the intersection of two sets) $\bar{\nu} \in \bar{A} \cap \bar{B}$, which clearly demonstrates the truth of $\bar{A} \cap \bar{B} \neq \emptyset$. We now see in light of (11.809) that we arrived at a contradiction, so that the the third part of the multiple disjunction (11.813) is indeed false. Thus, the first or the second part must be true, which disjunction we use to carry out a proof by cases in order to establish the existential sentence

$$\exists b\,(b \in [\bar{\omega},\bar{\nu}]_\Omega \,\wedge\, \bar{S} <_{[\bar{\omega},\bar{\nu}]_\Omega} b \,\wedge\, [\bar{S},b) \subseteq \bar{A}). \tag{11.814}$$

In the first case $\bar{K} \in \{(a,b) : a,b \in [\bar{\omega},\bar{\nu}]_\Omega\}$, there are constants, say $\bar{a}$ and $\bar{b}$, such that $\bar{a},\bar{b} \in [\bar{\omega},\bar{\nu}]_\Omega$ and $\bar{K} = (\bar{a},\bar{b})$ hold. With this equation, we can write $\bar{S} \in \bar{K}$ as $\bar{S} \in (\bar{a},\bar{b})$, so that the definition of an open interval in $[\bar{\omega},\bar{\nu}]_\Omega$ yields the inequalities

$$\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{S} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{b}. \tag{11.815}$$

Based on these inequalities, we establish now the inclusion $[\bar{S},\bar{b}) \subseteq (\bar{a},\bar{b})$ by means of the definition of a subset. We take an arbitrary constant y, we assume that $y \in [\bar{S},\bar{b})$ is true, and we notice that this assumption yields the inequalities

$$\bar{S} \leq_{[\bar{\omega},\bar{\nu}]_\Omega} y <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{b}. \tag{11.816}$$

The conjunction of the first inequality in (11.815) and the first inequality in (11.816) further implies $\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} y$ with the Transitivity Formula for $<$ and $\leq$. This inequality and the second inequality in (11.816) gives then $y \in (\bar{a},\bar{b})$ by means of the definition of an open interval. As y was arbitrary, we may therefore conclude that the proposed inclusion $[\bar{S},\bar{b}) \subseteq (\bar{a},\bar{b})$ is indeed true. Observe that $\bar{K} \subseteq \bar{A}$ and $\bar{K} = (\bar{a},\bar{b})$ imply also the inclusion $(\bar{a},\bar{b}) \subseteq \bar{A}$, we can use the transitivity property (2.13) of $\subseteq$ to infer from these inclusions the new inclusion $[\bar{S},\bar{b}) \subseteq \bar{A}$. In view of $\bar{b} \in [\bar{\omega},\bar{\nu}]_\Omega$ and $\bar{S} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{b}$ in (11.815), these findings show us now that the existential sentence (11.814) holds for the first case.

In the second case $\bar{K} \in \{[\bar{\omega},b) : b \in [\bar{\omega},\bar{\nu}]_\Omega\}$, there exists a particular element $\bar{b} \in [\bar{\omega},\bar{\nu}]_\Omega$ for which $\bar{K} = [\bar{\omega},\bar{b})$. Applying substitution to $\bar{S} \in \bar{K}$ based on this equation, we get first $\bar{S} \in [\bar{\omega},\bar{b})$ and then also the inequalities

$$\bar{\omega} \leq_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{S} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{b} \tag{11.817}$$

by means of the definition of a left-closed and right-open interval. We use these inequalities in connection with the definition of a subset to verify the inclusion $[\bar{S},\bar{b}) \subseteq [\bar{\omega},\bar{b})$. Letting y be arbitrary and assuming $y \in [\bar{S},\bar{b})$ to be true, the inequalities (11.816) follow now to be true as in the first case. The first inequality in (11.817) implies in conjunction with the first inequality in (11.816) that $\bar{\omega} \leq_{[\bar{\omega},\bar{\nu}]_\Omega} y$ holds (exploiting the transitivity of the total ordering $\leq_{[\bar{\omega},\bar{\nu}]_\Omega}$). Alongside the second inequality in (11.816), this implies $y \in [\bar{\omega},\bar{b}]$, and since y was arbitrary in $[\bar{S},\bar{b})$, we can therefore conclude that the inclusion $[\bar{S},\bar{b}) \subseteq [\bar{\omega},\bar{b})$ holds indeed. Now, $\bar{K} \subseteq \bar{A}$ and $\bar{K} = [\bar{\omega},\bar{b})$ give us in addition the inclusion $[\bar{\omega},\bar{b}) \subseteq \bar{A}$, so that another application of (2.13) yields also the inclusion $[\bar{S},\bar{b}) \subseteq \bar{A}$. Noting that $\bar{b} \in [\bar{\omega},\bar{\nu}]_\Omega$ and $\bar{S} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{b}$ in (11.817) hold as well, we find the existential sentence (11.814) to be true also for the second case.

Having thus completed the proof by cases, we take some particular constant $\bar{b} \in [\bar{\omega},\bar{\nu}]_\Omega$ for which $\bar{S} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{b}$ and $[\bar{S},\bar{b}) \subseteq \bar{A}$ are satisfied. Due to the inclusions $[\bar{\omega}, \bar{\nu}]_\Omega \subseteq \Omega_1 \subseteq \Omega$, the element $\bar{b}$ evidently follows to be element also of Ω. As the supremum $\bar{S}$ of $\bar{A}$ with respect to $\leq_\Omega$ is itself an element of Ω, we can write the preceding inequality also as $\bar{S} <_\Omega \bar{b}$. We now use the initial assumption that $(\Omega, <_\Omega)$ constitutes a linear continuum and thus in particular a densely ordered in order to infer from the preceding inequality the existence of a particular element $\bar{z} \in \Omega$ lying strictly between the supremum and $\bar{b}$, that is, $\bar{S} <_\Omega \bar{z} <_\Omega \bar{b}$. Here, the inequality $\bar{S} <_\Omega \bar{z}$ implies the disjunction $\bar{S} <_\Omega \bar{z} \lor \bar{S} = \bar{z}$ and therefore the inequality $\bar{S} \leq_\Omega \bar{z}$, by definition of an induced reflexive partial ordering. Together with the other inequality $\bar{z} <_\Omega \bar{b}$, this further implies $\bar{z} \in [\bar{S}, \bar{b})$ (by definition of a left-closed and right-open interval), and then also $\bar{z} \in \bar{A}$ with the previously established inclusion $[\bar{S}, \bar{b}) \subseteq \bar{A}$. Writing $\bar{S} <_\Omega \bar{z}$ equivalently as $\neg \bar{z} \leq_\Omega \bar{S}$, we thus showed that there exists an element x of $\bar{A}$ with $\neg x \leq_\Omega \bar{S}$. We can write this existential sentence also in the form of the negated universal sentence (applying the Negation Law for universal implications)

$$\neg \forall x \, (x \in \bar{A} \Rightarrow x \leq_\Omega \bar{S}),$$

which means that $\bar{S}$ is not an upper bound for $\bar{A}$. This finding demonstrates the existence of a subset $A \subseteq \Omega$ such that its supremum S exists and such that S is not an upper bound for A. It is therefore evidently false that, for any set $A \subseteq \Omega$ and for any set S being the supremum of A, S is an upper bound for A (and less than or equal to any upper bound S' for A). This contradicts the Characterization of the supremum, which states the truth of the preceding universal sentence. Thus, the proof of the first subcase $\bar{S} \in \bar{A}$ is now complete.

In the second sub-case $\bar{S} \in \bar{B}$, (11.812) implies (according to the Generation of a topology by means of a basis) the existence of a particular basis element

$$\bar{K} \in \{(a,b) : a, b \in [\bar{\omega}, \bar{\nu}]_\Omega\} \cup \{[\bar{\omega}, b) : b \in [\bar{\omega}, \bar{\nu}]_\Omega\} \cup \{(a, \bar{\nu}] : a \in [\bar{\omega}, \bar{\nu}]_\Omega\}$$

such that $\bar{S} \in \bar{K}$ and $\bar{K} \subseteq \bar{B}$ hold. Therefore, we find (by definition of the union of two sets) the disjunctions

$$\begin{aligned} \bar{K} \in \{(a,b) : a, b \in [\bar{\omega}, \bar{\nu}]_\Omega\} \, &\lor \, \bar{K} \in \{[\bar{\omega}, b) : b \in [\bar{\omega}, \bar{\nu}]_\Omega\} \\ &\lor \, \bar{K} \in \{(a, \bar{\nu}] : a \in [\bar{\omega}, \bar{\nu}]_\Omega\} \end{aligned} \tag{11.818}$$

to be true. Here, we can prove by contradiction that the second part of the multiple disjunction is false, i.e. that $\bar{K} \notin \{[\bar{\omega}, b) : b \in [\bar{\omega}, \bar{\nu}]_\Omega\}$ is true. For this purpose, we assume the negation of this negation to be true, so that the Double Negation Law yields $\bar{K} \in \{[\bar{\omega}, b) : b \in [\bar{\omega}, \bar{\nu}]_\Omega\}$. By definition of that set system (see Exercise 11.25), there is then a particular element

$\bar{b} \in [\bar{\omega}, \bar{\nu}]_\Omega$ such that $\bar{K} = [\bar{\omega}, \bar{b})$. Thus, substitution yields $\bar{S} \in [\bar{\omega}, \bar{b})$ and subsequently

$$\bar{\omega} \ \leq_{[\bar{\omega},\bar{\nu}]_\Omega} \ \bar{S} \ <_{[\bar{\omega},\bar{\nu}]_\Omega} \ \bar{b}$$

by definition of a left-closed and right-open interval in $[\bar{\omega}, \bar{\nu}]_\Omega$. As these inequalities imply $\bar{\omega} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{b}$ with the Transitivity Formula for $\leq$ and $<$, and since $\bar{\omega} \leq_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{\omega}$ is also true according to the reflexivity of the total ordering $\leq_{[\bar{\omega},\bar{\nu}]_\Omega}$, we find $\bar{\omega} \in [\bar{\omega}, \bar{b})$ to be true (using again the definition of a left-closed and right-open interval). Substitution yields then $\bar{\omega} \in \bar{K}$, which in turn implies $\bar{\omega} \in \bar{B}$ with the previously established inclusion $\bar{K} \subseteq \bar{B}$. Because $\bar{\omega} \in \bar{A}$ is also true, we evidently obtain $\bar{\omega} \in \bar{A} \cap \bar{B}$ and therefore $\bar{A} \cap \bar{B} \neq \emptyset$, in contradiction to (11.809). This completes the proof of the assertion that the the second part of the multiple disjunction (11.818) is false. Let us now use the disjunction of the first and the third part to prove the existential sentence

$$\exists a \, (a \in [\bar{\omega}, \bar{\nu}]_\Omega \, \wedge \, a <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{S} \, \wedge \, (a, \bar{S}] \subseteq \bar{B}) \tag{11.819}$$

by cases. The first case $\bar{K} \in \{(a,b) : a, b \in [\bar{\omega}, \bar{\nu}]_\Omega\}$ implies $\bar{K} = (\bar{a}, \bar{b})$ for some particular elements $\bar{a}, \bar{b} \in [\bar{\omega}, \bar{\nu}]_\Omega$, so that $\bar{S} \in \bar{K}$ yields $\bar{S} \in (\bar{a}, \bar{b})$, and the definition of an open interval in $[\bar{\omega}, \bar{\nu}]_\Omega$ gives the inequalities

$$\bar{a} \ <_{[\bar{\omega},\bar{\nu}]_\Omega} \ \bar{S} \ <_{[\bar{\omega},\bar{\nu}]_\Omega} \ \bar{b}. \tag{11.820}$$

We now apply the definition of a subset to show that these inequalities imply the inclusion $(\bar{a}, \bar{S}] \subseteq (\bar{a}, \bar{b})$. Letting y be arbitrary and assuming $y \in (\bar{a}, \bar{S}]$ to be true, we evidently get from this the inequalities

$$\bar{a} \ <_{[\bar{\omega},\bar{\nu}]_\Omega} \ y \ \leq_{[\bar{\omega},\bar{\nu}]_\Omega} \ \bar{S}. \tag{11.821}$$

The second inequality implies now in conjunction with the second inequality in (11.820) that $y <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{b}$ holds (using the Transitivity Formula for $\leq$ and $<$). In conjunction with the preceding inequality $\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} y$, this gives us the desired consequent $y \in (\bar{a}, \bar{b})$, so that the suggested inclusion $(\bar{a}, \bar{S}] \subseteq (\bar{a}, \bar{b})$ follows to be true indeed (since y was arbitrary). Because $\bar{K} \subseteq \bar{B}$ and $\bar{K} = (\bar{a}, \bar{b})$ imply the truth also of $(\bar{a}, \bar{b}) \subseteq \bar{B}$, we now obtain from these inclusions $(\bar{a}, \bar{S}] \subseteq \bar{B}$ with the transitivity property (2.13) of $\subseteq$. Recalling that $\bar{a} \in [\bar{\omega}, \bar{\nu}]_\Omega$ and $\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{S}$ hold, we now see that the existential sentence (11.819) is true in the first case.

In the second case $\bar{K} \in \{(a, \bar{\nu}] : a \in [\bar{\omega}, \bar{\nu}]_\Omega\}$, we have $\bar{K} = (\bar{a}, \bar{\nu}]$ for a particular $\bar{a} \in [\bar{\omega}, \bar{\nu}]_\Omega$. Consequently, $\bar{S} \in \bar{K}$ yields $\bar{S} \in (\bar{a}, \bar{\nu}]$, resulting in the inequalities

$$\bar{a} \ <_{[\bar{\omega},\bar{\nu}]_\Omega} \ \bar{S} \ \leq_{[\bar{\omega},\bar{\nu}]_\Omega} \ \bar{\nu} \tag{11.822}$$

(due to the definition of a left-open and right-closed interval). It is now possible to establish the inclusion $(\bar{a}, \bar{S}] \subseteq (\bar{a}, \bar{\nu}]$, by applying the definition of a subset. We take an arbitrary y, and we assume $y \in (\bar{a}, \bar{S}]$ to be true, so that the inequalities (11.821) turn out to be true again. The second one of these implies in conjunction with the second inequality in (11.822) $y \leq_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{\nu}$ because of the transitivity of the total ordering $\leq_{[\bar{\omega},\bar{\nu}]_\Omega}$. In connection with the first inequality in (11.821), this yields $y \in (\bar{a}, \bar{\nu}]$ as desired. Since y was arbitrary, we may infer from this indeed the truth of the inclusion $(\bar{a}, \bar{S}] \subseteq (\bar{a}, \bar{\nu}]$. As $\bar{K} \subseteq \bar{B}$ and $\bar{K} = (\bar{a}, \bar{\nu}]$ give rise also to the truth of $(\bar{a}, \bar{\nu}] \subseteq \bar{B}$, we arrive at the inclusion $(\bar{a}, \bar{S}] \subseteq \bar{B}$ by means of (2.13). In view of $\bar{a} \in [\bar{\omega}, \bar{\nu}]_\Omega$ and $\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{S}$, the existential sentence (11.819) therefore turns out to be true also in the second case, so that its proof by cases is complete.

We now take a particular element $\bar{a} \in [\bar{\omega}, \bar{\nu}]_\Omega$ that satisfies $\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{S}$ and the inclusion $(\bar{a}, \bar{S}] \subseteq \bar{B}$, and we demonstrate in the following that $\bar{a}$ is an upper bound for $\bar{A}$, that is, we prove the universal sentence

$$\forall y\,(y \in \bar{A} \Rightarrow y \leq_\Omega \bar{a}). \tag{11.823}$$

Letting y be arbitrary, we prove the implication by contradiction, assuming $y \in \bar{A}$ and the negation $\neg y \leq_\Omega \bar{a}$ to be both true. Here, we note that $\bar{a}$ is indeed an element of Ω because of the inclusions $[\bar{\omega}, \bar{\nu}]_\Omega \subseteq \Omega_1 \subseteq \Omega$. Recalling now that $\bar{S}$ is an upper bound for $\bar{A}$ (with respect to $\leq_\Omega$), we can derive from $y \in \bar{A}$ the inequality $y \leq_\Omega \bar{S}$. In addition, the previously assumed negation implies $\bar{a} <_\Omega y$ with the negation Formula for $\leq$. Due to the inclusion $\bar{A} \subseteq [\bar{\omega}, \bar{\nu}]_\Omega$, the element y of $\bar{A}$ is also in $[\bar{\omega}, \bar{\nu}]_\Omega$. As the supremum $\bar{S}$ is contained in $[\bar{\omega}, \bar{\nu}]_\Omega$ as well, we can rewrite the preceding inequalities in the equivalent form

$$\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} y \leq_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{S},$$

with the evident consequence that $y \in (\bar{a}, \bar{S}]$ is true. This yields $y \in \bar{B}$ with the inclusion $(\bar{a}, \bar{S}] \subseteq \bar{B}$, so that the conjunction $y \in \bar{A} \wedge y \in \bar{B}$ holds. Clearly, this implies that the intersection $\bar{A} \cap \bar{B}$ is nonempty, which is in contradiction to (11.809). We thus completed the proof of the implication in (11.823), in which y was arbitrary, so that $\bar{a}$ follows indeed to be an upper bound for $\bar{A}$. In view of $\bar{a} <_{[\bar{\omega},\bar{\nu}]_\Omega} \bar{S}$, we thus proved the existence of a subset $A \subseteq \Omega$ such that its supremum S exists and such that there exists an upper bound a for A which is less than the supremum. We then see in light of the Negation Law for universal implications that it is false that, for any set $A \subseteq \Omega$ and for any set S being the supremum of A, S is (an upper bound for A) and less than or equal to any upper bound S' for A. Because the Characterization of the supremum actually expresses the truth of the preceding universal sentence, we obtained a contradiction, so that the second sub-case $\bar{S} \in \bar{B}$ holds too. This also completes the proof for the first case $\bar{\omega} <_\Omega \bar{\nu}$.

The second case $\bar{\nu} <_\Omega \bar{\omega}$ can be proved in analogy to the first case by interchanging $\bar{U}, \bar{\omega}, \bar{A}$ and $\bar{V}, \bar{\nu}, \bar{B}$.

Having obtained the same contradiction for both cases (and for both corresponding sub-cases), we arrived at the end of the proof of the second part ('$\Leftarrow$') of the equivalence, which is thus true. Since Ω, $<_\Omega$ and Ω_1 were initially arbitrary sets, we may now finally conclude that the theorem is true. $\square$

Exercise 11.43. Provide a detailed proof of the second case $\bar{\nu} <_\Omega \bar{\omega}$ within the proof of Theorem 11.102.

Applying the preceding theorem to the linear continuum of real numbers and combining the resulting equivalence of connectedness and convexity for subsets of $\mathbb{R}$ with the Equivalence of intervals and convex sets in $\mathbb{R}$, we immediately obtain the following equivalence

Corollary 11.103 (Equivalence of intervals in $\mathbb{R}$ and connected topological subspaces of $(\mathbb{R}, \mathcal{O}_{<_\mathbb{R}})$). *It is true for any subset I of $\mathbb{R}$ that the topological subspace $(I, \mathcal{O}_{<_\mathbb{R}}|I)$ is connected iff I is an interval in $\mathbb{R}$, that is, iff*

$$\begin{aligned}
& I \in \{[a,b] : a,b \in \mathbb{R}\} && \lor\, I \in \{(a,b) : a,b \in \mathbb{R}\} \\
\lor\, & I \in \{(a,b] : a,b \in \mathbb{R}\} && \lor\, I \in \{[a,b) : a,b \in \mathbb{R}\} \\
\lor\, & I \in \{(a,+\infty) : a \in \mathbb{R}\} && \lor\, I \in \{[a,+\infty) : a \in \mathbb{R}\} \\
\lor\, & I \in \{(-\infty,b) : b \in \mathbb{R}\} && \lor\, I \in \{(-\infty,b] : b \in \mathbb{R}\} \\
\lor\, & I = \emptyset \\
\lor\, & I = \mathbb{R}.
\end{aligned}$$

Note 11.38. Since $\mathbb{R}$ is a subset of itself in view of (2.10) and since the topology $\mathcal{O}_{<_\mathbb{R}}$ is identical with its subspace topology in $\mathbb{R}$ according to Proposition 11.52, we see in light of the preceding Corollary 11.103 that the topological space $(\mathbb{R}, \mathcal{O}_{<_\mathbb{R}})$ is connected. Therefore, as indicated by the Disconnectedness Criterion (in connection with the Law of Contraposition), there do not exist any subsets of $\mathbb{R}$ aside from $\emptyset$ and $\mathbb{R}$ that are simultaneously open and closed.

Lemma 11.104. *The following intervals in $\mathbb{R}$ are <u>not</u> closed sets in $\mathbb{R}$:*

a) the open interval (a,b) for any $a,b \in \mathbb{R}$ with $a <_\mathbb{R} b$, *b) the open and left-unbounded interval* $(-\infty, b)$ *for any* $b \in \mathbb{R}$*,*

c) the open and right-unbounded interval $(a, +\infty)$ *for any* $a \in \mathbb{R}$*,*

d) the left-closed and right-open interval $[a, b)$ *for any* $a <_{\mathbb{R}} b$*,*

e) the left-open and right-closed interval $(a, b]$ *for any* $a <_{\mathbb{R}} b$*.*

Proof. Concerning a), we let $a, b \in \mathbb{R}$ be arbitrary such that $a <_{\mathbb{R}} b$ holds, and we observe the truth of $(a, b) \in \mathcal{O}_{<_{\mathbb{R}}}$ in light of (11.457). Then, $a <_{\mathbb{R}} b$ implies $\neg(\neg a <_{\mathbb{R}} b)$ with the Double Negation, which further implies $(a, b) \neq \emptyset$ with (3.382) and the Law of Contraposition. Moreover, the irreflexivity of the standard linear ordering $<_{\mathbb{R}}$ yields $\neg a <_{\mathbb{R}} a$; then, the disjunction $(\neg a <_{\mathbb{R}} a) \vee (\neg a <_{\mathbb{R}} b)$ holds as well, and De Morgan's Law for the conjunction gives us therefore $\neg(a <_{\mathbb{R}} a \wedge a <_{\mathbb{R}} b)$, with the consequence that $a \notin (a, b)$ holds (by definition of an open interval in $\mathbb{R}$). Thus, the conjunction $a \in \mathbb{R} \wedge a \notin (a, b)$ holds, and the existential sentence

$$\exists y\,([y \in \mathbb{R} \wedge y \notin (a, b)] \vee [y \in (a, b) \wedge y \notin \mathbb{R}])$$

is then evidently also true, so that we obtain $(a, b) \neq \mathbb{R}$ with (2.23). The findings $(a, b) \neq \emptyset$ and $(a, b) \neq \mathbb{R}$ imply now in view of Note 11.38 that the open set (a, b) in $\mathbb{R}$ cannot simultaneously be a closed set in $\mathbb{R}$. This is then true for any $a, b \in \mathbb{R}$ with $a <_{\mathbb{R}} b$.

The proofs of b) and c) are accomplished by applying similar arguments.

Concerning d), we take arbitrary $a, b \in \mathbb{R}$ and assume $a <_{\mathbb{R}} b$ to be true. To prove that $[a, b)$ is not closed in $\mathbb{R}$, we notice the truth of the equation

$$[a, b)^c = (-\infty, a) \cup [b, +\infty) \tag{11.824}$$

in view of (3.463), and we prove by contradiction that this union is not open in $\mathbb{R}$. Assuming the negation of this negation to be true, it follows with the Double Negation Law that $(-\infty, a) \cup [b, +\infty)$ is open in $\mathbb{R}$. As the reflexivity of the standard total ordering $\leq_{\mathbb{R}}$ yields $b \leq_{\mathbb{R}} b$, we have $b \in [b, +\infty)$ by definition of a left-closed and right-unbounded interval in $\mathbb{R}$. Then, the disjunction $b \in (-\infty, a) \vee b \in [b, +\infty)$ also holds, so that $b \in (-\infty, a) \cup [b, +\infty)$ follows to be true with the definition of the union of two sets. This implies in conjunction with the openness of the preceding union with respect to $\mathcal{O}_{<_{\mathbb{R}}}$, according to the Generation of a topology by means of a basis and according to Note 11.23, that there exists an element in the basis $\{(a, b) : a, b \in \mathbb{R}\}$ of $\mathcal{O}_{<_{\mathbb{R}}}$, say $\bar{A}$, with $b \in \bar{A}$ and $\bar{A} \subseteq (-\infty, a) \cup [b, +\infty)$. Thus, there are real numbers, say $\bar{c}$ and $\bar{d}$, for which $\bar{A} = (\bar{c}, \bar{d})$, so that the preceding findings give us via substitutions $b \in (\bar{c}, \bar{d})$ and

$$(\bar{c}, \bar{d}) \subseteq (-\infty, a) \cup [b, +\infty). \tag{11.825}$$

The former implies $\bar{c} <_{\mathbb{R}} b <_{\mathbb{R}} \bar{d}$ by definition of an open interval in $\mathbb{R}$, and the latter means by definition of a subset

$$\forall y\,(y \in (\bar{c}, \bar{d}) \Rightarrow y \in (-\infty, a) \cup [b, +\infty)\,). \tag{11.826}$$

We establish now the inclusion $(\bar{c}, \bar{d}) \subseteq [b, +\infty)$ by proving (according to the definition of a subset)

$$\forall y\,(y \in (\bar{c}, \bar{d}) \Rightarrow y \in [b, +\infty)\,). \tag{11.827}$$

We let y be arbitrary, and we assume $y \in (\bar{c}, \bar{d})$ to be true, so that (11.826) yields $y \in (-\infty, a) \cup [b, +\infty)$. Consequently, the disjunction

$$y \in (-\infty, a) \,\vee\, y \in [b, +\infty) \tag{11.828}$$

holds by definition of the union of two sets. Here, we can prove by contradiction that the first part $y \in (-\infty, a)$ is false, i.e. that the negation $\neg y \in (-\infty, a)$ is true. For this purpose, we assume the negation of that negation to be true, so that $y \in (-\infty, a)$ follows to be true with the Double Negation Law. This implies with the definition of an open and left-unbounded interval $y <_{\mathbb{R}} a$, and we note that the assumed $y \in (\bar{c}, \bar{d})$ gives $\bar{c} <_{\mathbb{R}} y <_{\mathbb{R}} \bar{d}$. As $(\mathbb{R}, <_{\mathbb{R}})$ is densely ordered (see Corollary 8.12), the assumed inequality $a <_{\mathbb{R}} b$ implies that there exists a real number strictly between a and b, say $a <_{\mathbb{R}} \bar{z} <_{\mathbb{R}} b$. Recalling the truth of the inequality $b <_{\mathbb{R}} \bar{d}$, we thus have

$$\bar{c} <_{\mathbb{R}} y <_{\mathbb{R}} a <_{\mathbb{R}} \bar{z} <_{\mathbb{R}} b <_{\mathbb{R}} \bar{d}.$$

The transitivity of the standard linear ordering $<_{\mathbb{R}}$ gives us now $\bar{c} <_{\mathbb{R}} \bar{z} <_{\mathbb{R}} \bar{d}$ and therefore evidently $\bar{z} \in (\bar{c}, \bar{d})$. This in turn implies $\bar{z} \in (-\infty, a) \cup [b, +\infty)$ with (11.826), which we now refute. We begin with the observation that the inequality $a <_{\mathbb{R}} \bar{z}$ gives rise to the disjunction $a <_{\mathbb{R}} \bar{z} \vee a = \bar{z}$, so that $a \leq_{\mathbb{R}} \bar{z}$ is true by definition of an induced irreflexive partial ordering. In conjunction with $\bar{z} <_{\mathbb{R}} b$, this evidently yields $\bar{z} \in [a, b)$, consequently $\bar{z} \notin [a, b)^c$ due to (2.132), and then also $\bar{z} \notin (-\infty, a) \cup [b, +\infty)$ with (11.824). This negation contradicts the previous finding $\bar{z} \in (-\infty, a) \cup [b, +\infty)$, so that the proof of $\neg y \in (-\infty, a)$ is complete. Thus, the second part $y \in [b, +\infty)$ of the disjunction (11.828) must be true, which in turn proves the implication in (11.827). As y was arbitrary, we therefore conclude that the universal sentence (11.827) holds, so that the inclusion $(\bar{c}, \bar{d}) \subseteq [b, +\infty)$ is indeed true.

Now, since $(\mathbb{R}, <_{\mathbb{R}})$ is densely ordered, the previously found inequality $\bar{c} <_{\mathbb{R}} b$ implies the existence of a particular intermediate value $\bar{\bar{z}}$ of $\bar{c}$ and b, that is,

$$\bar{c} <_{\mathbb{R}} \bar{\bar{z}} <_{\mathbb{R}} b \;[<_{\mathbb{R}} \bar{d}].$$

On the one hand, $\bar{\bar{z}} <_{\mathbb{R}} b$ implies $\neg b \leq_{\mathbb{R}} \bar{\bar{z}}$ with the Negation Formula for $\leq$, which inequality in turn implies $\bar{\bar{z}} \notin [b, +\infty)$. On the other hand, the transitivity of $<_{\mathbb{R}}$ yields $\bar{c} <_{\mathbb{R}} \bar{\bar{z}} <_{\mathbb{R}} \bar{d}$ and therefore $\bar{\bar{z}} \in (\bar{c}, \bar{d})$. This implies now with the previously established inclusion $(\bar{c}, \bar{d}) \subseteq [b, +\infty)$ that $\bar{\bar{z}} \in [b, +\infty)$ is true, which contradicts the preceding negation $\bar{\bar{z}} \notin [b, +\infty)$. We thus proved that the union (11.824) is not open in $\mathbb{R}$. Applying now the Law of Contraposition to the definition of a closed set, we therefore find that the interval $[a, b)$ is not closed in $\mathbb{R}$. Because a and b were arbitrary, d) follows to be true, too.

The proof of e) is similar. □

Exercise 11.44. Prove Lemma 11.104b,c,e).

Theorem 11.105 (Characterization of nonempty closed intervals in $\mathbb{R}$). *It is true that a subset $I \subseteq \mathbb{R}$ constitutes a closed interval $[a, b]$ in $\mathbb{R}$ for some $a, b \in \mathbb{R}$ with $a \leq_{\mathbb{R}} b$ iff*

1. *I is nonempty,*
2. *I is bounded-from-below and bounded-from-above,*
3. *I is a closed set in $\mathbb{R}$ with respect to the standard topology $\mathcal{O}_{<_{\mathbb{R}}}$,*
4. *the topological subspace $(I, \mathcal{O}_{<_{\mathbb{R}}}|I)$ is connected.*

Proof. We let I be an arbitrary set, assuming the inclusion $I \subseteq \mathbb{R}$ to be true. To establish the first part ('$\Rightarrow$') of the equivalence, we assume that there exist particular real numbers $\bar{a}$ and $\bar{b}$ satisfying $\bar{a} \leq_{\mathbb{R}} \bar{b}$ and $I = [\bar{a}, \bar{b}]$. Firstly, the assumed inequality implies $[\bar{a}, \bar{b}] \neq \emptyset$ with (3.354), so that I is nonempty. Secondly, Corollary 3.118 shows that the closed interval $I = [\bar{a}, \bar{b}]$ is bounded from below and bounded from above. Thirdly, $I = [\bar{a}, \bar{b}]$ is a closed set in $\mathbb{R}$ with respect to the order topology $\mathcal{O}_{<_{\mathbb{R}}}$ according to Corollary 11.84. Fourthly, we have $I \in \{[a, b] : a, b \in \mathbb{R}\}$, so that the topological subspace $(I, \mathcal{O}_{<_{\mathbb{R}}}|I)$ is connected in view of Corollary 11.103.

To prove the second part ('$\Leftarrow$') of the equivalence, we assume now that I has the four listed properties. Because of Property 4 and Corollary 11.103, it is true that I is an interval in $\mathbb{R}$. Due to Property 1, we have

$$I \neq \emptyset,$$

and Property 2 evidently yields

$$I \notin \{[a, +\infty) : a \in \mathbb{R}\}, \quad I \notin \{(-\infty, b] : b \in \mathbb{R}\}, \quad I \neq \mathbb{R}$$

because intervals of this kind are not bounded from below and from above (see Proposition 3.137, Exercise 3.64 and Corollary 8.14). Finally, Property 3 yields in connection with Lemma 11.104

$$I \notin \{(a, +\infty) : a \in \mathbb{R}\}, \quad I \notin \{(-\infty, b) : b \in \mathbb{R}\}.$$

Thus, we are left with the possibilities

$$\begin{aligned} & I \in \{[a,b] : a, b \in \mathbb{R}\} \ \vee\ I \in \{(a,b) : a, b \in \mathbb{R}\} \\ \vee\, & I \in \{(a,b] : a, b \in \mathbb{R}\} \ \vee\ I \in \{[a,b) : a, b \in \mathbb{R}\}. \end{aligned}$$

First, we prove

$$I \notin \{(a,b) : a, b \in \mathbb{R}\}$$

by contradiction, assuming the negation of this negation to be true. This assumption implies with the Double Negation Law $I \in \{(a,b) : a, b \in \mathbb{R}\}$, so that we have $I = (\bar{a}, \bar{b})$ for some particular real numbers. We consider the two cases $\bar{a} <_{\mathbb{R}} \bar{b}$ and $\neg \bar{a} <_{\mathbb{R}} \bar{b}$. In the first case $\bar{a} <_{\mathbb{R}} \bar{b}$, Lemma 11.104 shows that interval $(\bar{a}, \bar{b})$ is not a closed set, in contradiction to the assumed Property 3 of I. The other case $\neg \bar{a} <_{\mathbb{R}} \bar{b}$ implies $I = (\bar{a}, \bar{b}) = \emptyset$ with (3.382), which contradicts the assumed Property 1 of I, completing the proof by cases.

We can apply essentially the same arguments to prove now also

$$I \notin \{(a,b] : a, b \in \mathbb{R}\} \tag{11.829}$$

and

$$I \notin \{[a,b) : a, b \in \mathbb{R}\} \tag{11.830}$$

by contradiction, using (3.383) and (3.384). Consequently, the first part $I \in \{[a,b] : a, b \in \mathbb{R}\}$ of the preceding multiple disjunction must be true. This means that there are real numbers, say $\bar{a}$ and $\bar{b}$, for which $I = [\bar{a}, \bar{b}]$. We prove now by contradiction that $\bar{a} \leq_{\mathbb{R}} \bar{b}$ is true, assuming $\neg \bar{a} \leq_{\mathbb{R}} \bar{b}$ to hold. This assumption implies $\neg [\bar{a}, \bar{b}] \neq \emptyset$ with (3.354) and the Law of Contraposition, and another application of the Double Negation Law gives us then $I = [\bar{a}, \bar{b}] = \emptyset$, again in contradiction to the assumed Property 1 for I. We thus completed the proof of the equivalence, and since I was initially arbitrary, we infer from the truth of this equivalence the truth of the theorem. □

Exercise 11.45. Establish (11.829) and (11.830).

The closed intervals forming an α-cut of a fuzzy number are in the following characterized by the topological concept of 'compactness'.

Definition 11.34 (Open covering). We say for any topological space $(\Omega, \mathcal{O})$ that a covering $\mathcal{C}$ of Ω is *open* (with respect to $\mathcal{O}$) iff $\mathcal{C}$ is included in $\mathcal{O}$, i.e. iff

$$\mathcal{C} \subseteq \mathcal{O}. \tag{11.831}$$

We establish now a first example of an open covering.

Proposition 11.106. *It is true that*

a) the sequence $s = ((-n, n)_{\mathbb{R}})_{n \in \mathbb{N}}$ exists uniquely,

b) the range of this sequence constitutes an open covering of $\mathbb{R}$ with respect to the standard topology $\mathcal{O}_{<_{\mathbb{R}}}$ on $\mathbb{R}$,

c) the sequence s is a bijection from $\mathbb{N}$ to its range.

Proof. Concerning a), let us apply Function definition by replacement to establish a unique function/sequence s with domain/index set $\mathbb{N}$ such that

$$\forall n \, (n \in \mathbb{N} \Rightarrow s(n) = (-n, n) \,). \tag{11.832}$$

This task requires the verification of the universal sentence

$$\forall n \, (n \in \mathbb{N} \Rightarrow \exists! y \, (y = (-n, n) \,)). \tag{11.833}$$

Letting n be an arbitrary natural number, we may evidently view n as an integer, form then its negative $-n$, and form subsequently the open interval $(-n, n)$ in $\mathbb{R}$ (viewing $-n$ and n now as real numbers). Then, the uniquely existential sentence is true according to (1.109). As n was arbitrary, we therefore conclude that the universal sentence (11.833) is true, so that there exists indeed a unique function s satisfying (11.832) and having the domain $\mathbb{N}$. Thus, we may write s as the sequence $(s_n)_{n \in \mathbb{N}}$ with terms $s_n = (-n, n)$. Next, we demonstrate that the range of s is included in the power set of $\mathbb{R}$, that is, $\operatorname{ran}(s) \subseteq \mathcal{P}(\mathbb{R})$. Letting x be arbitrary and assuming $x \in \operatorname{ran}(s)$ to be true, it follows by definition of a range and by definition of a domain that there exists a particular element $\bar{n} \in \mathbb{N}$ with $(\bar{n}, x) \in s$. This yields with the definition of the function/sequence s in (11.832) $x = s_{\bar{n}} = (-\bar{n}, \bar{n})$. This open interval x in $\mathbb{R}$ is a subset of $\mathbb{R}$ according to (3.376) and consequently an element of $\mathcal{P}(\mathbb{R})$ by definition of a power set. We thus showed that $x \in \operatorname{ran}(s)$ implies $x \in \mathcal{P}(\mathbb{R})$, in which implication x is arbitrary, so that we may infer from this the truth of the inclusion $\operatorname{ran}(s) \subseteq \mathcal{P}(\mathbb{R})$ by using the definition of a subset. This finding shows that the range of s satisfies Property 1 of a covering of $\mathbb{R}$. We establish now also Property 2, that is, the equation $\bigcup \operatorname{ran}(s) = \mathbb{R}$. For this purpose, we apply the Equality Criterion for sets and prove the equivalent universal sentence

$$\forall x \, (x \in \bigcup \operatorname{ran}(s) \Leftrightarrow x \in \mathbb{R}). \tag{11.834}$$

Letting x be arbitrary, we prove the first implication '$\Rightarrow$' directly, assuming $x \in \bigcup \mathrm{ran}(s)$ to be true. Since s is a sequence with index set $\mathbb{N}$, we can also write $x \in \bigcup_{n=0}^{\infty} s_n$. According to the Characterization of the union of a sequence of sets, there exists then a particular index $\bar{n} \in \mathbb{N}$ such that $x \in s_{\bar{n}}$. The definition of the function s gives us then $x \in (-\bar{n}, \bar{n})$, so that the aforementioned inclusion $(-\bar{n}, \bar{n}) \subseteq \mathbb{R}$ yields the desired consequent $x \in \mathbb{R}$ by means of the definition of a subset. Noting that the totality of $\leq_{\mathbb{R}}$ gives rise to the true disjunction $x \leq_{\mathbb{R}} 0 \vee 0 \leq_{\mathbb{R}} x$, we prove

$$\exists n \, (n \in \mathbb{N} \wedge x \in (-n, n)\,) \tag{11.835}$$

by cases. The first case $x \leq_{\mathbb{R}} 0$ implies on the one hand $0 \leq_{\mathbb{R}} -x$ with the Monotony Law for $+$ and $\leq$, on the other hand $x +_{\mathbb{R}} x \leq_{\mathbb{R}} 0$ with the Additivity of $\leq$-inequalities. The latter implies $x \leq_{\mathbb{R}} -x$ (again with the Monotony Law for $+$ and $\leq$), and the former implies with the Archimedean Property that there exists a particular positive natural number $\bar{n}$ (which is a natural number) with $-x <_{\mathbb{R}} \bar{n}$. The latter in turn implies $-\bar{n} <_{\mathbb{R}} x$ (now with the Monotony Law for $+$ and $<$), and the conjunction of $x \leq_{\mathbb{R}} -x$ and $-x <_{\mathbb{R}} \bar{n}$ yields $x <_{\mathbb{R}} \bar{n}$ with the Transitivity Formula for $\leq$ and $<$. We thus found the inequalities $-\bar{n} <_{\mathbb{R}} x <_{\mathbb{R}} \bar{n}$, which imply $x \in (-\bar{n}, -\bar{n})$ with the definition of an open interval in $\mathbb{R}$. This finding demonstrates in view of $\bar{n} \in \mathbb{N}$ the truth of the existential sentence (11.835) for the first case. In the second case $0 \leq_{\mathbb{R}} x$, we obtain on the one hand with the Archimedean Property a particular $\bar{n} \in \mathbb{N}_+$ with $x <_{\mathbb{R}} \bar{n}$; thus, $\bar{n} \in \mathbb{N}$. On the other hand, the Additivity of $\leq$-inequalities yields $0 \leq_{\mathbb{R}} x +_{\mathbb{R}} x$, so that $-x \leq_{\mathbb{R}} x$ turns out to be true according to the Monotony Law for $+$ and $\leq$. Combining the previous two inequalities via the Transitivity Formula for $\leq$ and $<$, we get $-x <_{\mathbb{R}} \bar{n}$ and subsequently $-\bar{n} <_{\mathbb{R}} x$ (using now the Monotony Law for $+$ and $<$). In connection with $x <_{\mathbb{R}} \bar{n}$, this implies now $x \in (-\bar{n}, -\bar{n})$ by definition of an open interval, so that the existential sentence (11.835) holds also in the second case. Applying now the Characterization of the union of a family of sets, we obtain $x \in \bigcup_{n=0}^{\infty}(-n, n)$, which we can write also in the form $x \in \bigcup \mathrm{ran}(s)$. This completes the proof of the equivalence in (11.834), in which x is arbitrary, so that the universal sentence (11.834) follows to be true. Thus, the equality $\bigcup \mathrm{ran}(s) = \mathbb{R}$ holds indeed, so that the range of s has both of the defining properties of a covering of $\mathbb{R}$.

With respect to b), we verify now that $\mathrm{ran}(s)$ is an open covering of $\mathbb{R}$ (with respect to $\mathcal{O}_{<_{\mathbb{R}}}$), i.e. that the covering $\mathrm{ran}(s)$ is included in the topology $\mathcal{O}_{<_{\mathbb{R}}}$. To do this, we take an arbitrary set U, and we assume $U \in \mathrm{ran}(s)$ to be true. Evidently, we have then $U = s_{\bar{n}} = (-\bar{n}, \bar{n})$ for some particular index $\bar{n} \in \mathbb{N}$. This open interval in $\mathbb{R}$ is indeed an open set in $\mathbb{R}$ as shown in (11.457), so that $U \in \mathrm{ran}(s)$ implies $U \in \mathcal{O}_{<_{\mathbb{R}}}$. Because U is arbitrary, it follows from this implication, by definition of a subset, that the proposed inclusion $\mathrm{ran}(s) \subseteq \mathcal{O}_{<_{\mathbb{R}}}$ holds. Therefore, $\mathrm{ran}(s)$ is indeed an open covering of $\mathbb{R}$.

Regarding c), we first note that $s : \mathbb{N} \to \mathrm{ran}(s)$ is a surjection by definition. Next, we apply the Injection Criterion, letting $n, n' \in \mathbb{N}$ be arbitrary and assuming $n \neq n'$ to be true. Because the standard linear ordering $<_{\mathbb{N}}$ is connex, the disjunction $n <_{\mathbb{N}} n' \vee n' <_{\mathbb{N}} n$ is then true. Viewing the natural numbers n and n' as real numbers, we can write this disjunction also as $n <_{\mathbb{R}} n' \vee n' <_{\mathbb{R}} n$, and we use the latter disjunction to prove now the $(-n, n) \neq (-n', n')$ by cases.

The first case $n <_{\mathbb{R}} n'$ implies with the fact that $(\mathbb{R}, <_{\mathbb{R}})$ is densely ordered (see Corollary 8.12) that there exists a particular real number $\bar{z}$ strictly between n and n', which means $n <_{\mathbb{R}} \bar{z} <_{\mathbb{R}} n'$. The natural numbers n and n' evidently satisfy $0 \leq_{\mathbb{R}} n$ and $0 \leq_{\mathbb{R}} n'$, so that we obtain $-n \leq_{\mathbb{R}} 0$ and $-n' \leq_{\mathbb{R}} 0$ with the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$. We thus have

$$-n' \ \leq_{\mathbb{R}} \ 0 \ \leq_{\mathbb{R}} \ n \ <_{\mathbb{R}} \ \bar{z} \ <_{\mathbb{R}} \ n',$$

and these inequalities imply $-n' <_{\mathbb{R}} \bar{z} <_{\mathbb{R}} n'$ with the Transitivity Formula for $\leq$ and $<$. Consequently, $\bar{z} \in (-n', n')$ holds by definition of an open interval in $\mathbb{R}$. Furthermore, $n <_{\mathbb{R}} \bar{z}$ implies the truth of the negation $\neg \bar{z} <_{\mathbb{R}} n$ with the Characterization of comparability with respect to the linear ordering $<_{\mathbb{R}}$. Then, the disjunction $\neg -n <_{\mathbb{R}} \bar{z} \vee \neg \bar{z} <_{\mathbb{R}} n$ also holds, so that De Morgan's Law for the conjunction gives us $\neg(-n <_{\mathbb{R}} \bar{z} <_{\mathbb{R}} n)$ and consequently $\bar{z} \notin (-n, n)$. This proves in conjunction with $\bar{z} \in (-n', n')$ the existential sentence

$$\exists y\,([y \in (-n, n) \,\wedge\, y \notin (-n', n')] \,\vee\, [y \in (-n', n') \,\wedge\, y \notin (-n, n)]),$$

so that $(-n, n) \neq (-n', n')$ follows to be true with (2.23).

In the second case $n' <_{\mathbb{R}} n$, we can apply essentially the same arguments as in the first case to infer the truth of the preceding negation. Now, since the elements $n, n' \in \mathbb{N}$ give rise to the terms $s_n = (-n, n)$ and $s_{n'} = (-n', n')$, we obtain through substitutions $s_n \neq s_{n'}$. As n and n' were arbitrary, we may therefore conclude that the (surjective) function $s : \mathbb{N} \to \mathrm{ran}(s)$ is an injection, thus a bijection. $\square$

Exercise 11.46. Establish the second case within the proof of Proposition 11.106c).

The idea of an open covering plays a central role in the next definition.

Definition 11.35 (Compact space, compact set/subset). We say that a topological space $(\Omega, \mathcal{O})$ is a *compact space* iff any open covering of Ω (with respect to $\mathcal{O}$) includes a finite set (system) that covers Ω, i.e. iff

$$\forall \mathcal{C} \, ([\mathcal{C} \subseteq \mathcal{P}(\Omega) \wedge \bigcup \mathcal{C} = \Omega \wedge \mathcal{C} \subseteq \mathcal{O}] \tag{11.836}$$
$$\Rightarrow \exists \mathcal{S} \, (\mathcal{S} \subseteq \mathcal{C} \wedge \mathcal{S} \text{ is a finite set } \wedge \mathcal{S} \subseteq \mathcal{P}(\Omega) \wedge \bigcup \mathcal{S} = \Omega)).$$

We then say that a subset $\Omega_1 \subseteq \Omega$ is a *compact set* in Ω or a *compact subset* of Ω (with respect to $\mathcal{O}$) iff the topological subspace $(\Omega_1, \mathcal{O}|\Omega_1)$ is compact.

Note 11.39. For every topological space $(\Omega, \mathcal{O})$, we may apply the Axiom of Specification and the Equality Criterion for sets to establish

$$\exists! \mathcal{K}_\Omega^\mathcal{O} \, \forall K \, (K \in \mathcal{K}_\Omega^\mathcal{O} \Leftrightarrow [K \in \mathcal{P}(\Omega) \wedge K \text{ is a compact subset of } \Omega \text{ w.r.t. } \mathcal{O}]).$$

Definition 11.36 (System of compact sets). For any topological space $(\Omega, \mathcal{O})$, we call the set

$$\mathcal{K}_\Omega^\mathcal{O} \tag{11.837}$$

the *system of compact sets* (in Ω with respect to $\mathcal{O}$).

Theorem 11.107 (Characterization of compact sets). *It is true for any topological space $(\Omega, \mathcal{O})$ and for any set $\Omega_1 \subseteq \Omega$ that Ω_1 is compact iff, for every set $\mathcal{C}_1$ which is included in the topology $\mathcal{O}$ and whose union includes Ω_1, there is a finite subset $\mathcal{S}_1$ of $\mathcal{C}_1$ whose union also includes Ω_1, i.e.*

$$\forall \mathcal{C}_1 \, ([\mathcal{C}_1 \subseteq \mathcal{O} \wedge \Omega_1 \subseteq \bigcup \mathcal{C}_1] \Rightarrow \exists \mathcal{S}_1 \, (\mathcal{S}_1 \subseteq \mathcal{C}_1 \wedge \mathcal{S}_1 \text{ is finite } \wedge \Omega_1 \subseteq \bigcup \mathcal{S}_1)). \tag{11.838}$$

Proof. We let Ω, $\mathcal{O}$ and Ω_1 be arbitrary sets, assuming $(\Omega, \mathcal{O})$ to be a topological space and assuming Ω_1 to be a subset of Ω. Thus, the topological subspace $(\Omega_1, \mathcal{O}|\Omega_1)$ is defined. To prove the first part ('$\Rightarrow$') of the equivalence, we assume that Ω_1 is compact, which means that the topological subspace $(\Omega_1, \mathcal{O}|\Omega_1)$ is compact, so that this subspace satisfies

$$\forall \mathcal{C} \, ([\mathcal{C} \subseteq \mathcal{P}(\Omega_1) \wedge \bigcup \mathcal{C} = \Omega_1 \wedge \mathcal{C} \subseteq \mathcal{O}|\Omega_1] \tag{11.839}$$
$$\Rightarrow \exists \mathcal{S} \, (\mathcal{S} \subseteq \mathcal{C} \wedge \mathcal{S} \text{ is a finite set } \wedge \mathcal{S} \subseteq \mathcal{P}(\Omega_1) \wedge \bigcup \mathcal{S} = \Omega_1)).$$

In order to establish the desired consequent (11.838), we let $\mathcal{C}_1$ be an arbitrary set (system), and we assume the inclusions

$$\mathcal{C}_1 \subseteq \mathcal{O} \wedge \Omega_1 \subseteq \bigcup \mathcal{C}_1 \tag{11.840}$$

to be true. Then, we can evidently apply the Axiom of Specification in connection with the Equality Criterion for sets to establish the unique existence of a set (system) $\bar{\mathcal{C}}$ consisting of all subsets of Ω_1 which can be written as the intersection of Ω_1 and some set A in the given set system $\mathcal{C}_1$, in the sense that

$$\forall U\,(U \in \bar{\mathcal{C}} \Leftrightarrow [U \in \mathcal{P}(\Omega_1) \,\wedge\, \exists A\,(A \in \mathcal{C}_1 \,\wedge\, U = \Omega_1 \cap A)]). \tag{11.841}$$

Here, we see that $U \in \bar{\mathcal{C}}$ implies especially $U \in \mathcal{P}(\Omega_1)$ for any U, so that the inclusion

$$\bar{\mathcal{C}} \subseteq \mathcal{P}(\Omega_1) \tag{11.842}$$

follows to be true by definition of a subset. Next, we apply the Equality Criterion for sets to establish the equation

$$\bigcup \bar{\mathcal{C}} = \Omega_1, \tag{11.843}$$

by verifying the equivalent universal sentence

$$\forall \omega\,(\omega \in \bigcup \bar{\mathcal{C}} \Leftrightarrow \omega \in \Omega_1). \tag{11.844}$$

We let ω be arbitrary, and we assume first $\omega \in \bigcup \bar{\mathcal{C}}$ to be true. By definition of the union of a set system, there exists then a set in $\bar{\mathcal{C}}$, say $\bar{X}$, which contains ω. Here, $\bar{X} \in \bar{\mathcal{C}}$ implies $\bar{X} \in \mathcal{P}(\Omega_1)$ with the inclusion (11.842), so that $\bar{X}$ is a subset of Ω_1 by definition of a power set. Due to this inclusion, $\omega \in \bar{X}$ implies now $\omega \in \Omega_1$, which finding proves the implication '$\Rightarrow$' in (11.844). Regarding the reverse implication '$\Leftarrow$', we assume $\omega \in \Omega_1$ to be true, which assumption gives us $\omega \in \bigcup \mathcal{C}_1$ with the second inclusion in (11.840). The definition of the union of a set system gives us then a particular set $\bar{A} \in \mathcal{C}_1$ with $\omega \in \bar{A}$. Thus, $\omega \in \Omega_1 \cap \bar{A}$ is true according to the definition of the intersection of two sets. Let us denote this intersection by $\bar{U}$, so that $\omega \in \bar{U}$ holds. Since $[\bar{U} =]\ \Omega_1 \cap \bar{A} \subseteq \Omega_1$ is true according to (2.74), we get $\bar{U} \in \mathcal{P}(\Omega_1)$ (using the definition of a power set). Furthermore, the preceding finding $\bar{A} \in \mathcal{C}_1$ and $\bar{U} = \Omega_1 \cap \bar{A}$ demonstrate the truth of the existential sentence $\exists A\,(A \in \mathcal{C}_1 \,\wedge\, \bar{U} = \Omega_1 \cap A)$. Consequently, $\bar{U} \in \bar{\mathcal{C}}$ turns out to be true by definition of the set $\bar{\mathcal{C}}$ in (11.841). In conjunction with $\omega \in \bar{U}$, this gives us now $\omega \in \bigcup \bar{\mathcal{C}}$ (again by means of the definition of a set system). This completes the proof of the equivalence in (11.844), in which ω is arbitrary, so that the equality (11.843) follows to be true indeed.

Our next task is to prove the inclusion

$$\bar{\mathcal{C}} \subseteq \mathcal{O}|\Omega_1, \tag{11.845}$$

which we accomplish by using the definition of a subset. Letting U be arbitrary and assuming $U \in \bar{\mathcal{C}}$, we see in light of (11.841) that there exists a set in $\mathcal{C}_1$, say $\bar{A}$, for which $U = \Omega_1 \cap \bar{A}$. Because of the first inclusion in (11.840), $\bar{A} \in \mathcal{C}_1$ yields $\bar{A} \in \mathcal{O}$. In connection with the preceding equation, this shows that the existential sentence $\exists A\,(A \in \mathcal{O} \wedge \Omega_1 \cap A = U)$ is true, so that $U \in \mathcal{O}|\Omega_1$ holds by definition of a subspace topology. As U was arbitrary in $\bar{\mathcal{C}}$, we may therefore conclude that the inclusion (11.845) holds, as claimed.

Now, the conjunction of (11.842), (11.843) and (11.845) implies because of (11.839) that there is some particular finite subset $\bar{\mathcal{S}} \subseteq \bar{\mathcal{C}}$ which satisfies $\bar{\mathcal{S}} \subseteq \mathcal{P}(\Omega_1)$ and $\bigcup \bar{\mathcal{S}} = \Omega_1$, i.e. which constitutes a covering of Ω_1. Then, the finiteness of $\bar{\mathcal{S}}$ implies by definition the existence of a particular natural number $\bar{m}$ and of a particular bijection $\bar{S} : \{1, \ldots, \bar{m}\} \rightleftarrows \bar{\mathcal{S}}$. Let us establish now the truth of the universal sentence

$$\forall i\,(i \in \{1, \ldots, \bar{m}\} \Rightarrow \exists! \mathcal{Y}\,(\forall Y\,(Y \in \mathcal{Y} \Leftrightarrow [Y \in \mathcal{C}_1 \wedge \bar{S}_i = \Omega_1 \cap Y]))).$$

Indeed, letting i be arbitrary and assuming $i \in \{1, \ldots, \bar{m}\}$ to be true, the uniquely existential sentence follows to be true by virtue of the Axiom of Specification and the Equality Criterion for sets. Since i is arbitrary, the preceding universal sentence follows therefore to be true. According to Function definition by replacement, there exists then a unique function G with domain $\{1, \ldots, \bar{m}\}$ such that

$$\forall i\,(i \in \{1, \ldots, \bar{m}\} \Rightarrow \forall Y\,(Y \in G(i) \Leftrightarrow [Y \in \mathcal{C}_1 \wedge \bar{S}_i = \Omega_1 \cap Y])). \quad (11.846)$$

Since the range of this function G is included in itself according to (2.10, we have $G : \{1, \ldots, \bar{m}\} \twoheadrightarrow \mathrm{ran}(G)$ by definition of a surjection. We demonstrate in the following that the range of G does not contain the empty set, which assertion is equivalent to the universal sentence

$$\forall \mathcal{Y}\,(\mathcal{Y} \in \mathrm{ran}(G) \Rightarrow \mathcal{Y} \neq \emptyset). \quad (11.847)$$

because of (2.5). Letting $\mathcal{Y}$ be an arbitrary set and assuming $\mathcal{Y} \in \mathrm{ran}(G)$ to hold, it follows with the definitions of a range and of a domain that there exists a particular element $\bar{k} \in \{1, \ldots, \bar{m}\}$ [$= \mathrm{dom}(G)$] for which $(\bar{k}, \mathcal{Y}) \in G$ holds. We can write the latter in function notation also as $\mathcal{Y} = G(\bar{k})$. Thus, $\bar{k}$ is also in the domain of $\bar{S} : \{1, \ldots, \bar{m}\} \rightleftarrows \bar{\mathcal{S}}$, giving rise to the value $\bar{S}_{\bar{k}} \in \bar{\mathcal{S}}$. Due to the inclusion $\bar{\mathcal{S}} \subseteq \bar{\mathcal{C}}$, we therefore obtain $\bar{S}_{\bar{k}} \in \bar{\mathcal{C}}$, which finding implies with (11.841) that there exists some particular element $\bar{A} \in \mathcal{C}_1$ with $\bar{S}_{\bar{k}} = \Omega_1 \cap \bar{A}$. Consequently, we obtain $\bar{A} \in G(\bar{k})$ [$= \mathcal{Y}$] with (11.846), with the further consequence that $\bar{A} \in \mathcal{Y}$ is true. Thus, the set $\mathcal{Y}$ is clearly nonempty, so that the implication in (11.847) holds. Since $\mathcal{Y}$ is arbitrary, we can now infer from the truth of that implication the truth of the universal sentence (11.847) and therefore the truth of the equivalent negation $\emptyset \notin \mathcal{Y}$. According to the Axiom of Choice, there exists then a particular function $F : \mathrm{ran}(G) \to \bigcup \mathrm{ran}(G)$ such that

$$\forall \mathcal{Y} \, (\mathcal{Y} \in \mathrm{ran}(G) \Rightarrow F(\mathcal{Y}) \in \mathcal{Y}). \tag{11.848}$$

The composition of F and $G : \{1, \ldots, \bar{m}\} \twoheadrightarrow \mathrm{ran}(G)$ constitutes then the function $F \circ G : \{1, \ldots, \bar{m}\} \to \bigcup \mathrm{ran}(G)$ in view of (3.604).

Let us establish for the range of this composition firstly the inclusion

$$\mathrm{ran}(F \circ G) \subseteq \mathcal{C}_1. \tag{11.849}$$

To do this, we take an arbitrary set X, assuming that $X \in \mathrm{ran}(F \circ G)$ holds. Then, the definitions of a range and of a domain give us a particular element $\bar{k}$ of the domain $\{1, \ldots, \bar{m}\}$ of the composition such that $(\bar{k}, X) \in F \circ G$ is satisfied. Using the notation for functions and function compositions, we can write the latter in the form $X = F(G(\bar{k}))$, where $G(\bar{k})$ is evidently an element of the range of G. Therefore, (11.848) yields $[X =]\ F(G(\bar{k})) \in G(\bar{k})$ and thus $X \in G(\bar{k})$. This finding implies now with (11.846) especially the desired consequent $X \in \mathcal{C}_1$. As X was arbitrary in $\mathrm{ran}(F \circ G)$, we may infer from this the truth of the suggested inclusion (11.849) by means of the definition of a subset.

Secondly, let us observe that the domain $\{1, \ldots, \bar{m}\}$ of the function $F \circ G$ is finite (see Exercise 4.34), so that

$$\mathrm{ran}(F \circ G) \text{ is finite} \tag{11.850}$$

as shown in Exercise 4.38.

Thirdly, we can establish the inclusion

$$\Omega_1 \subseteq \mathrm{ran}(F \circ G). \tag{11.851}$$

Taking an arbitrary ω and assuming $\omega \in \Omega_1$ to be true, we get $\omega \in \bigcup \bar{\mathcal{S}}$ by means of substitution based on the previously established equation $\bigcup \bar{\mathcal{S}} = \Omega_1$. Then, there evidently exists a set in $\bar{\mathcal{S}}$ containing ω, say $\bar{X}$. Recalling that $\bar{\mathcal{S}}$ is the range of the bijection $\bar{S} : \{1, \ldots, \bar{m}\} \rightleftarrows \bar{\mathcal{S}}$, there evidently exists a particular element $\bar{k}$ of the domain $\{1, \ldots, \bar{m}\}$ satisfying $(\bar{k}, \bar{X}) \in \bar{S}$. Rewriting the latter in function/sequence notation as $\bar{X} = \bar{S}_{\bar{k}}$, we thus obtain from $\omega \in \bar{X}$ via substitution $\omega \in \bar{S}_{\bar{k}}$. The element $\bar{k}$ is in the domain $\{1, \ldots, \bar{m}\}$ of the composition $F \circ G$ and thus associated with the value $F(G(\bar{k})) \in \mathrm{ran}(F \circ G)$. Clearly, $G(\bar{k}) \in \mathrm{ran}(G)$ is true, so that $F(G(\bar{k})) \in G(\bar{k})$ also holds according to (11.848). This further implies $\bar{S}_{\bar{k}} = \Omega_1 \cap F(G(\bar{k}))$ with (11.846), so that the previously found $\omega \in \bar{S}_{\bar{k}}$ yields $\omega \in \Omega_1 \cap F(G(\bar{k}))$ through substitution. Thus, $\omega \in \Omega_1 \cap F(G(\bar{k}))$ is true in particular (by definition of the intersection of two sets), and this implies in conjunction with the previously obtained $F(G(\bar{k})) \in \mathrm{ran}(F \circ G)$ the truth of $\omega \in \bigcup \mathrm{ran}(F \circ G)$ (by definition of the union of a set system). We thus showed that $\omega \in \Omega_1$ implies $\omega \in \bigcup \mathrm{ran}(F \circ G)$, in which implication ω is arbitrary, so that the inclusion (11.851) follows now to be true (by definition of a subset).

Having found the particular set $\mathrm{ran}(F \circ G)$ with the properties (11.849) – (11.851), we see now that the existential sentence in (11.838) is true. As $\mathcal{C}_1$ was arbitrary, we therefore conclude that the first part '$\Rightarrow$' of the proposed equivalence holds.

We prove the second part ('$\Leftarrow$') of the equivalence also directly, by assuming now the universal sentence (11.838) to be true and by demonstrating that the universal sentence (11.839) holds, too. For this purpose, we take an arbitrary set (system) $\mathcal{C}$, assuming

$$\mathcal{C} \subseteq \mathcal{P}(\Omega_1) \,\wedge\, \bigcup \mathcal{C} = \Omega_1 \,\wedge\, \mathcal{C} \subseteq \mathcal{O}|\Omega_1 \tag{11.852}$$

to be true. We define first for every set A in $\mathcal{C}$ a unique set system that contains precisely every open set in Ω whose intersection with Ω_1 equals A, that is,

$$\forall A\,(A \in \mathcal{C} \Rightarrow \exists! \mathcal{Z}\,(\forall U\,(U \in \mathcal{Z} \,\wedge\, [U \in \mathcal{O} \,\wedge\, A = \Omega_1 \cap U]))).$$

Letting A be arbitrary and assuming $A \in \mathcal{C}$, the uniquely existential sentence can indeed by proven by means of the Axiom of Specification and the Equality Criterion for sets. Because A is arbitrary, we can therefore conclude that the preceding universal sentence holds. We can then apply Function definition by replacement and infer from the truth of the preceding universal sentence the existence of a unique function H with domain $\mathcal{C}$ and values satisfying

$$\forall A\,(A \in \mathcal{C} \Rightarrow \forall U\,(U \in H(A) \,\wedge\, [U \in \mathcal{O} \,\wedge\, A = \Omega_1 \cap U])). \tag{11.853}$$

We verify now that the empty set is not an element of the range of this function H. For this purpose, we establish equivalently

$$\forall Y\,(Y \in \mathrm{ran}(H) \Rightarrow Y \neq \emptyset), \tag{11.854}$$

letting Y be arbitrary and assuming $Y \in \mathrm{ran}(H)$. This assumption implies $\bar{A} \in \mathcal{C}$ [$= \mathrm{dom}(H)$] and $(\bar{A}, Y) \in H$ for some particular set $\bar{A}$ (in view of the definition of a range and the definition of a domain). The latter reads in function notation $Y = H(\bar{A})$, and the former gives $\bar{A} \in \mathcal{O}|\Omega_1$ with the third part of the multiple conjunction in (11.852). By definition of a subspace topology, $\bar{A}$ can then be written as the intersection $\bar{A} = \Omega_1 \cap \bar{U}$

for some particular open set $\bar{U} \in \mathcal{O}$. Consequently, $\bar{U}$ is an element of $Y = H(\bar{A})$, according to the definition of the function H in (11.853). Thus, Y is evidently not an empty set, and as the set Y was arbitrary, we may therefore conclude that (11.854) holds. The truth of that universal sentence implies then the truth of $\emptyset \notin \mathrm{ran}(H)$ by means of (2.5). The Axiom of Choice gives us therefore a particular function $\bar{F} : \mathrm{ran}(H) \to \bigcup \mathrm{ran}(H)$ satisfying

$$\forall Y\, (Y \in \mathrm{ran}(H) \Rightarrow \bar{F}(Y) \in Y), \tag{11.855}$$

and the composition of $\bar{F}$ with H is then the function $\bar{F} \circ H : \mathcal{C} \to \bigcup \mathrm{ran}(H)$, according to (3.604). Evidently, we can also take the range of this function as a codomain, and this yields the surjection $\bar{F} \circ H : \mathcal{C} \twoheadrightarrow \mathrm{ran}(\bar{F} \circ H)$. We establish in the following for this composition the conjunction

$$\mathrm{ran}(\bar{F} \circ H) \subseteq \mathcal{O} \;\wedge\; \Omega_1 \subseteq \bigcup \mathrm{ran}(\bar{F} \circ H). \tag{11.856}$$

Regarding the first inclusion, we see that the assumption of $V \in \mathrm{ran}(\bar{F} \circ H)$ for an arbitrary V implies the existence of a particular $\bar{A} \in \mathrm{dom}(\bar{F} \circ H)$ $[= \mathcal{C}]$ with $(\bar{A}, V) \in \bar{F} \circ H$. The latter can be written in the form $V = \bar{F}(H(\bar{A}))$, where $H(\bar{A}) \in \mathrm{ran}(H)$ holds, so that $[\bar{F}(H(\bar{A})) =]\; V \in H(\bar{A})$ follows to be true with (11.855). This in turn implies $V \in \mathcal{O}$ with (11.853), recalling that $\bar{A} \in \mathcal{C}$ is also true. We may infer from this that $V \in \mathrm{ran}(\bar{F} \circ H)$ implies $V \in \mathcal{O}$ for any V, which means that $\mathrm{ran}(\bar{F} \circ H)$ is indeed a subset of $\mathcal{O}$.

Concerning the other inclusion in (11.856), we now let ω be arbitrary, and we assume that $\omega \in \Omega_1$ is true. In view of the assumed equation in (11.852), we thus have $\omega \in \bigcup \mathcal{C}$, so that $\mathcal{C}$ contains a particular set $\bar{A}$ which contains ω. Being in the domain of $\bar{F} \circ H$, the set $\bar{A}$ is associated with the value $(\bar{F} \circ H)(\bar{A}) \in \mathrm{ran}(\bar{F} \circ H)$, for which we can also write $\bar{F}(H(\bar{A})) \in \mathrm{ran}(\bar{F} \circ H)$. Here, $\bar{F}(H(\bar{A})) \in H(\bar{A})$ holds due to (11.855), with the consequence that $\bar{A} = \Omega_1 \cap \bar{F}(H(\bar{A}))$ is true, because of (11.853). Applying now a substitution to the previously established $\omega \in \bar{A}$ based on that equation, we get $\omega \in \Omega_1 \cap \bar{F}(H(\bar{A}))$ and therefore especially $\omega \in \bar{F}(H(\bar{A}))$. Since $\bar{F}(H(\bar{A})) \in \mathrm{ran}(\bar{F} \circ H)$ is also true, we obtain now evidently $\omega \in \bigcup \mathrm{ran}(\bar{F} \circ H)$. This finding completes the proof that Ω_1 is a subset of $\bigcup \mathrm{ran}(\bar{F} \circ H)$, because ω was arbitrary.

We thus proved the conjunction (11.856), so that there now exists – by virtue of the assumed antecedent (11.838) – a particular finite subset $\bar{\mathcal{S}}_1 \subseteq \mathrm{ran}(\bar{F} \circ H)$ with $\Omega_1 \subseteq \bigcup \bar{\mathcal{S}}_1$. By definition of a finite set, there exists then a natural number, say $\bar{n}$, and a particular bijection $\bar{S}' : \{1, \ldots, \bar{n}\} \rightleftarrows \bar{\mathcal{S}}_1$. Based on this sequence, we establish the universal sentence

$$\forall i\, (i \in \{1, \ldots, \bar{n}\} \Rightarrow \exists! \mathcal{Y}\, (\forall Y\, (Y \in \mathcal{Y} \Leftrightarrow [Y \in \mathcal{C} \;\wedge\; Y = \Omega_1 \cap \bar{S}'_i]))).$$

Taking an arbitrary i and assuming $i \in \{1, \ldots, \bar{n}\}$, we see in light of the Axiom of Specification and the Equality Criterion for sets that the uniquely existential sentence is true. As i was arbitrary, we therefore conclude that the universal sentence is true, and the method of Function definition by replacement gives us then a unique function K with domain $\{1, \ldots, \bar{n}\}$ whose values satisfy

$$\forall i\,(i \in \{1, \ldots, \bar{n}\} \Rightarrow \forall Y\,(Y \in K(i) \Leftrightarrow [Y \in \mathcal{C} \wedge Y = \Omega_1 \cap \bar{S}'_i])). \quad (11.857)$$

Taking the range of K to be its codomain, we thus have the mapping $K : \{1, \ldots, \bar{n}\} \to \mathrm{ran}(K)$. We prove in the sequel that this range does not have $\emptyset$ as an element, by demonstrating the truth of

$$\forall \mathcal{X}\,(\mathcal{X} \in \mathrm{ran}(K) \Rightarrow \mathcal{X} \neq \emptyset). \quad (11.858)$$

We let $\mathcal{X}$ be an arbitrary set, assuming $\mathcal{X} \in \mathrm{ran}(K)$ to be true, so that there is an index within the index set $\{1, \ldots, \bar{n}\}$ of K, say $\bar{k}$, for which $(\bar{k}, \mathcal{X}) \in K$ holds. Again, we use function notation and write $\mathcal{X} = K(\bar{k})$ instead. We also see that $\bar{k}$ is in the domain of the bijection $\bar{S}' : \{1, \ldots, \bar{n}\} \rightleftarrows \bar{\mathcal{S}}_1$, so that we obtain for the corresponding value $\bar{S}'_{\bar{k}} \in \bar{\mathcal{S}}_1$. Recalling now the truth of the inclusion $\bar{\mathcal{S}}_1 \subseteq \mathrm{ran}(\bar{F} \circ H)$, we therefore obtain $\bar{S}'_{\bar{k}} \in \mathrm{ran}(\bar{F} \circ H)$. In analogy to the proof of the first inclusion in (11.856), this yields a particular set $\bar{A} \in \mathcal{C}$ with $\bar{S}'_{\bar{k}} \in H(\bar{A})$, and $\bar{A} = \Omega_1 \cap \bar{S}'_{\bar{k}}$ follows then also to be true by virtue of the definition of the function H in (11.853). These findings furthermore imply $\bar{A} \in K(\bar{K})$ $[= \mathcal{X}]$ with (11.857), where the resulting $\bar{A} \in \mathcal{X}$ demonstrates the nonemptiness of the set $\mathcal{X}$. As that set was initially arbitrary, we therefore conclude that (11.858) holds, which universal sentence implies then the truth of the desired negation $\emptyset \notin \mathrm{ran}(K)$, using once again (2.5). This enables us to apply the Axiom of Choice in order to obtain a particular function $\bar{f} : \mathrm{ran}(K) \to \bigcup \mathrm{ran}(K)$ such that

$$\forall \mathcal{X}\,(\mathcal{X} \in \mathrm{ran}(K) \Rightarrow \bar{f}(\mathcal{X}) \in \mathcal{X}). \quad (11.859)$$

Next, we form the composition $\bar{f} \circ K : \{1, \ldots, \bar{n}\} \to \bigcup \mathrm{ran}(K)$ by means of (3.604), and we write this function in its surjective form $\bar{f} \circ K : \{1, \ldots, \bar{n}\} \twoheadrightarrow \mathrm{ran}(\bar{f} \circ K)$. Our final task will now be to verify the multiple conjunction

$$\mathrm{ran}(\bar{f} \circ K) \subseteq \mathcal{C} \wedge \mathrm{ran}(\bar{f} \circ K) \text{ is a finite set } \wedge \mathrm{ran}(\bar{f} \circ K) \subseteq \mathcal{P}(\Omega_1)$$
$$\wedge \bigcup \mathrm{ran}(\bar{f} \circ K) = \Omega_1. \quad (11.860)$$

Firstly, we prove the inclusion $\mathrm{ran}(\bar{f} \circ K) \subseteq \mathcal{C}$ by taking an arbitrary set B and by assuming $B \in \mathrm{ran}(\bar{f} \circ K)$ to be true. We therefore find $(\bar{k}, B) \in \bar{f} \circ K$ for some particular element $\bar{k} \in \mathrm{dom}(\bar{f} \circ K)$, which domain is given by $\{1, \dots, \bar{n}\}$. Since the preceding composition is a function, we can write $B = \bar{f}(K(\bar{k}))$, where the value $K(\bar{k})$ of K is evidently an element of the range of K, so that (11.859) yields $[B =]\ \bar{f}(K(\bar{k})) \in K(\bar{k})$. The resulting $B \in K(\bar{k})$ gives us then $B \in \mathcal{C}$ by definition of the function K in (11.857). Since B was an arbitrary set in $\operatorname{ran}(\bar{f} \circ K)$, we can conclude that the inclusion $\operatorname{ran}(\bar{f} \circ K) \subseteq \mathcal{C}$ is indeed true.

Secondly, we see that the domain of the function $\bar{f} \circ K : \{1, \dots, \bar{n}\} \twoheadrightarrow \operatorname{ran}(\bar{f} \circ K)$ is finite (recall the aforementioned Exercise 4.34), which observation gives us the true assertion that

$$\operatorname{ran}(\bar{f} \circ K) \text{ is finite} \tag{11.861}$$

(recalling also the previously used Exercise 4.38).

Thirdly, we have the true inclusions

$$\operatorname{ran}(\bar{f} \circ K) \ \subseteq\ \mathcal{C} \ \subseteq\ \mathcal{P}(\Omega_1)$$

in view of the already established first part of the multiple conjunction (11.860) and in view of the first inclusion in (11.852). Consequently, we obtain $\operatorname{ran}(\bar{f} \circ K) \subseteq \mathcal{P}(\Omega_1)$ by applying (2.13).

Fourthly, we apply the Equality Criterion for sets to establish the essential covering property $\bigcup \operatorname{ran}(\bar{f} \circ K) = \Omega_1$, by proving accordingly

$$\forall \omega\, (\omega \in \bigcup \operatorname{ran}(\bar{f} \circ K) \Leftrightarrow \omega \in \Omega_1). \tag{11.862}$$

To establish the first part ('$\Rightarrow$') of the equivalence, we assume $\omega \in \bigcup \operatorname{ran}(\bar{f} \circ K)$ to be true. By definition of the union of a set system, ω is then element of some set in $\operatorname{ran}(\bar{f} \circ K)$, say of $\bar{B}$. Due to the preceding inclusion $\operatorname{ran}(\bar{f} \circ K) \subseteq \mathcal{P}(\Omega_1)$, it is true that $\bar{B} \in \operatorname{ran}(\bar{f} \circ K)$ implies $\bar{B} \in \mathcal{P}(\Omega_1)$, which means by definition of a power set that $\bar{B}$ is a subset of Ω_1. With this inclusion, $\omega \in \bar{B}$ implies then $\omega \in \Omega_1$, as desired.

We prove the second part ('$\Leftarrow$') of the equivalence in (11.862) also directly, assuming conversely $\omega \in \Omega_1$ to be true. Since the previously established set system $\bar{\mathcal{S}}_1$ satisfies the inclusion $\Omega_1 \subseteq \bigcup \bar{\mathcal{S}}_1$, we therefore obtain $\omega \in \bigcup \bar{\mathcal{S}}_1$. This means that there exists a set in $\bar{\mathcal{S}}_1$ which contains ω, say, the set $\bar{X} \in \bar{\mathcal{S}}_1$. Here, $\bar{\mathcal{S}}_1$ is the range of the bijection $\bar{S}' : \{1, \dots, \bar{n}\} \rightleftarrows \bar{\mathcal{S}}_1$, so that we have $\bar{X} \in \operatorname{ran}(\bar{S}')$. Applying now once again the definition of a range in connection with the definition of a domain, we obtain an element of $\operatorname{dom}(\bar{S}') = \{1, \dots, \bar{n}\}$, say $\bar{k}$, such that $(\bar{k}, \bar{X}) \in \bar{S}'$. On the one hand, we use now sequence notation to write the latter finding as $\bar{X} = \bar{S}'_{\bar{k}}$, so that we can apply substitution to $\omega \in \bar{X}$ in order to obtain $\omega \in \bar{S}'_{\bar{k}}$. Because of the assumption $\omega \in \Omega_1$, we thus have $\omega \in \Omega_1 \cap \bar{S}'_{\bar{k}}$ (by definition of the intersection of two sets). On the other hand, $\bar{k} \in \{1, \dots, \bar{n}\}$

gives us the corresponding value $\bar{f}(K(\bar{k})) \in \mathrm{ran}(\bar{f} \circ K)$ of the composition $\bar{f} \circ K : \{1, \ldots, \bar{n}\} \twoheadrightarrow \mathrm{ran}(\bar{f} \circ K)$. Since $K(\bar{k})$ is in the range of K, it follows that $\bar{f}(K(\bar{k})) \in K(\bar{k})$ is true, according to (11.859). Consequently, the definition of the function K in (11.857) yields in particular $\bar{f}(K(\bar{k})) = \Omega_1 \cap \bar{S}'_{\bar{k}}$. Then, substitution in the previously established $\omega \in \Omega_1 \cap \bar{S}'_{\bar{k}}$ gives us $\omega \in \bar{f}(K(\bar{k}))$, which in turn implies in conjunction with $\bar{f}(K(\bar{k})) \in \mathrm{ran}(\bar{f} \circ K)$ that $\omega \in \bigcup \mathrm{ran}(\bar{f} \circ K)$ holds. This finding completes the proof of the equivalence in (11.862), in which ω is arbitrary, so that the universal sentence (11.862) holds as well.

We thus proved the equality $\bigcup \mathrm{ran}(\bar{f} \circ K) = \Omega_1$, so that the proof of the multiple conjunction (11.860) is now complete. This conjunction demonstrates the truth of the existential sentence in (11.839), where the set system $\mathcal{C}$ was arbitrary. We may therefore conclude that the universal sentence (11.839) is true, which means by definition that the topological subspace $(\Omega_1, \mathcal{O}|\Omega_1)$ is compact, or equivalently that Ω_1 is compact. Thus, the second part '$\Leftarrow$' of the proposed equivalence holds, too. Because the sets Ω, $\mathcal{O}$ and Ω_1 were initially arbitrary, it follows that the theorem is indeed true. □

Corollary 11.108. *It is true for any topological space $(\Omega, \mathcal{O})$ that $\emptyset$ is a compact set in Ω (with respect to the topology $\mathcal{O}$).*

Proof. Taking arbitrary sets Ω and $\mathcal{O}$ such that $(\Omega, \mathcal{O})$ is a topological space, we use the Characterization of compact sets to show that $\emptyset$ is a compact set in Ω. For this purpose, we let $\mathcal{C}_1$ be an arbitrary set such that the inclusions $\mathcal{C}_1 \subseteq \mathcal{O}$ and $\emptyset \subseteq \bigcup \mathcal{C}_1$ are satisfied. We choose the set $\bar{\mathcal{S}}_1 = \emptyset$, which is included in $\mathcal{C}_1$ and satisfies $\emptyset \subseteq \bigcup \bar{\mathcal{S}}_1$ according to (2.43). Furthermore, $\bar{\mathcal{S}}_1 = \emptyset\ [= 0]$ is finite due to (4.464). We thus showed that there exists a set $\mathcal{S}_1$ such that

$$\mathcal{S}_1 \subseteq \mathcal{C}_1 \ \wedge \ \mathcal{S}_1 \text{ is finite} \ \wedge \ \emptyset \subseteq \bigcup \mathcal{S}_1.$$

As $\mathcal{C}_1$ was arbitrary, we may therefore conclude that $\emptyset$ is compact. Since Ω and $\mathcal{O}$ were initially also arbitrary, we further conclude that the corollary is true. □

Theorem 11.109 (Closedness of compact subspaces of Hausdorff spaces). *It is true for any Hausdorff space $(\Omega, \mathcal{O})$ and any compact set Ω_1 in Ω with respect to $\mathcal{O}$ that Ω_1 is closed in Ω.*

Proof. Letting Ω, $\mathcal{O}$ and Ω_1 be arbitrary sets, we assume $(\Omega, \mathcal{O})$ to be a Hausdorff space and $(\Omega_1, \mathcal{O}|\Omega_1)$ to be a compact space. Thus, Ω_1 is included in Ω, and $\mathcal{O}|\Omega_1$ constitutes the subspace topology of $\mathcal{O}$ in Ω_1. To prove that Ω_1 is closed in Ω, we may show equivalently that Ω_1^c is open in Ω (according to the definition of a closed set). Since the complement is taken with respect to Ω, we have (by definition of a complement) $\Omega_1^c = \Omega \setminus \Omega_1$. We apply now the Characterization of open sets to establish $\Omega \setminus \Omega_1 \in \mathcal{O}$, by verifying the universal sentence

$$\forall \omega\, (\omega \in \Omega \setminus \Omega_1 \Rightarrow \exists V\, (V \in \mathcal{O} \,\wedge\, \omega \in V \,\wedge\, V \subseteq \Omega \setminus \Omega_1)). \tag{11.863}$$

For this purpose, we let $\bar{\omega}$ be arbitrary, and we assume $\bar{\omega} \in \Omega \setminus \Omega_1$ to be true. Thus, the definition of a set difference gives us $\bar{\omega} \in \Omega$ and $\bar{\omega} \notin \Omega_1$. Because $(\Omega, \mathcal{O})$ constitutes a Hausdorff space, we obtain in view of $\bar{\omega} \in \Omega$ the true universal sentence

$$\forall \nu\, ([\nu \in \Omega \,\wedge\, \bar{\omega} \neq \nu] \Rightarrow \exists U, V\, (U, V \in \mathcal{O} \,\wedge\, U \cap V = \emptyset \,\wedge\, \bar{\omega} \in U \,\wedge\, \nu \in V)). \tag{11.864}$$

Let us now verify the universal sentence

$$\begin{aligned} &\forall \nu\, (\nu \in \Omega_1 \Rightarrow \exists! \mathcal{Y}\, (\forall Y\, (Y \in \mathcal{Y} \Leftrightarrow [Y \in \mathcal{O} \times \mathcal{O} \\ &\quad \wedge\, \exists U, V\, (U, V \in \mathcal{O} \,\wedge\, U \cap V = \emptyset \,\wedge\, \bar{\omega} \in U \,\wedge\, \nu \in V \,\wedge\, (V, U) = Y)])))). \end{aligned} \tag{11.865}$$

Letting ν be arbitrary and assuming $\nu \in \Omega_1$ to be true, the uniquely existential sentence follows to be true with the Axiom of Specification and the Equality Criterion for sets. Then, since ν is arbitrary, we may indeed infer from this the truth of the universal sentence (11.865). According to Function definition by replacement, there exists then a unique function with domain Ω_1, say $\bar{f}$, whose values satisfy

$$\begin{aligned} &\forall \nu\, (\nu \in \Omega_1 \Rightarrow \forall Y\, (Y \in \bar{f}(\nu) \Leftrightarrow [Y \in \mathcal{O} \times \mathcal{O} \\ &\quad \wedge\, \exists U, V\, (U, V \in \mathcal{O} \,\wedge\, U \cap V = \emptyset \,\wedge\, \bar{\omega} \in U \,\wedge\, \nu \in V \,\wedge\, (V, U) = Y)])). \end{aligned} \tag{11.866}$$

Next, we prove $\emptyset \notin \operatorname{ran}(\bar{f})$ by demonstrating the truth of

$$\forall \mathcal{Y}\, (\mathcal{Y} \in \operatorname{ran}(\bar{f}) \Rightarrow \mathcal{Y} \neq \emptyset), \tag{11.867}$$

based on (2.5). Letting $\mathcal{Y}$ be arbitrary and assuming that $\mathcal{Y} \in \operatorname{ran}(\bar{f})$ holds, it follows with the definitions of a range and of a domain that there exists a particular element $\bar{\nu} \in \operatorname{dom}(\bar{f})$ $[= \Omega_1]$ such that $(\bar{\nu}, \mathcal{Y}) \in \bar{f}$, which reads in function notation $\mathcal{Y} = \bar{f}(\bar{\nu})$. Here, $\bar{\nu} \in \Omega_1$ implies in conjunction with the previously established $\bar{\omega} \notin \Omega_1$ the truth of $\bar{\nu} \neq \bar{\omega}$, using (2.4). Furthermore, $\bar{\nu} \in \Omega_1$ implies with the aforementioned inclusion $\Omega_1 \subseteq \Omega$ (by definition of a subset) that $\bar{\nu} \in \Omega$ holds. Together with $\bar{\omega} \neq \bar{\nu}$, this gives us now two particular disjoint open sets $\bar{U}$ and $\bar{V}$ of the topology with $\bar{\omega} \in \bar{U}$ and $\bar{\nu} \in \bar{V}$, according to (11.864). Forming the ordered pair $\bar{Y} = (\bar{V}, \bar{U})$, it follows by definition of the Cartesian product of two sets that $\bar{Y} \in \mathcal{O} \times \mathcal{O}$ holds, and we also find the existential sentence

$$\exists U, V\,(U, V \in \mathcal{O} \wedge U \cap V = \emptyset \wedge \bar{\omega} \in U \wedge \bar{\nu} \in V \wedge (V, U) = \bar{Y}) \quad (11.868)$$

to be true. Consequently, we obtain $\bar{Y} \in \bar{f}(\bar{\nu})$ $[= \mathcal{Y}]$ by definition of the function $\bar{f}$ in (11.866). This finding clearly shows that the desired consequent $\mathcal{Y} \neq \emptyset$ of the implication in (11.867) is true, and since $\mathcal{Y}$ was arbitrary, we may infer from this the truth of the universal sentence (11.867) and therefore the truth of the suggested negation $\emptyset \notin \operatorname{ran}(\bar{f})$. Then, according to the Axiom of Choice, there exists a particular function $\bar{g} : \operatorname{ran}(\bar{f}) \to \bigcup \operatorname{ran}(\bar{f})$ such that

$$\forall \mathcal{Y}\,(\mathcal{Y} \in \operatorname{ran}(\bar{f}) \Rightarrow \bar{g}(\mathcal{Y}) \in \mathcal{Y}). \quad (11.869)$$

We obtain now for the composition of $\bar{g}$ and $\bar{f}$ the function $\bar{g} \circ \bar{f} : \Omega_1 \to \bigcup \operatorname{ran}(\bar{f})$ due to Proposition 3.178. Evidently, we can write this function also as the surjection $\bar{g} \circ \bar{f} : \Omega_1 \twoheadrightarrow \operatorname{ran}(\bar{g} \circ \bar{f})$. Let us bring out more clearly that the preceding range is a binary relation included in $\mathcal{O} \times \mathcal{O}$. Letting Y be arbitrary and assuming $Y \in \operatorname{ran}(\bar{g} \circ \bar{f})$, there exists (according to the definition of a range) a constant, say $\bar{y}$, with $(\bar{y}, Y) \in \bar{g} \circ \bar{f}$. On the one hand, we obtain then (using the definition of a domain) $\bar{y} \in \Omega_1$ $[= \operatorname{dom}(\bar{g} \circ \bar{f})]$, and on the other hand we can write (using the notations for composed functions) $Y = \bar{g}(\bar{f}(\bar{y}))$. Here, we evidently have $\bar{f}(\bar{y}) \in \operatorname{ran}(\bar{f})$, so that $Y \in \bar{f}(\bar{y})$ follows to be true with (11.869). This in turn implies the desired consequent $Y \in \mathcal{O} \times \mathcal{O}$ with (11.866), where Y was arbitrary, so that the inclusion

$$\operatorname{ran}(\bar{g} \circ \bar{f}) \subseteq \mathcal{O} \times \mathcal{O} \quad (11.870)$$

turns out to be true indeed by definition of a subset. We consider now the domain and the range of this binary relation relation. First, we prove that the range satisfies the conjunction

$$\operatorname{dom}(\operatorname{ran}(\bar{g} \circ \bar{f})) \subseteq \mathcal{O} \wedge \Omega_1 \subseteq \bigcup \operatorname{dom}(\operatorname{ran}(\bar{g} \circ \bar{f})). \quad (11.871)$$

Regarding the first inclusion, we let $\bar{V}$ be arbitrary and assume that $\bar{V} \in \operatorname{dom}(\operatorname{ran}(\bar{g} \circ \bar{f}))$ is true. By definition of a domain, there exists then a particular set $\bar{U}$ with $(\bar{V}, \bar{U}) \in \operatorname{ran}(\bar{g} \circ \bar{f})$. This implies $(\bar{V}, \bar{U}) \in \mathcal{O} \times \mathcal{O}$ with the inclusion (11.870), and this further implies $\bar{V} \in \mathcal{O}$ with the definition of the Cartesian product of two sets. As $\bar{V}$ was arbitrary in $\operatorname{dom}(\operatorname{ran}(\bar{g} \circ \bar{f}))$, we may therefore conclude that the first inclusion $\operatorname{dom}(\operatorname{ran}(\bar{g} \circ \bar{f})) \subseteq \mathcal{O}$ is true (in view of the definition of a subset).

Regarding the other inclusion, we take now an arbitrary x, assuming $x \in \Omega_1$ to be true. Thus, x is in the domain of the function $\bar{f}$, giving rise to the value $\bar{f}(x)$ in the range of $\bar{f}$, so that $\bar{g}(\bar{f}(x)) \in \bar{f}(x)$ holds according to (11.869). This implies with (11.866) that $x \in \bar{V}$ and $\bar{g}(\bar{f}(x)) = (\bar{V}, \bar{U})$ hold for some particular (disjoint and open) sets $\bar{U}, \bar{V}$. We can write the preceding equation in the form $(\bar{V}, \bar{U}) = \bar{g} \circ \bar{f}(x)$, and then also as $(x, (\bar{V}, \bar{U})) \in \bar{g} \circ \bar{f}$. Thus, the ordered pair $(\bar{V}, \bar{U})$ is evidently in the range of $\bar{g} \circ \bar{f}$. Recalling that $\mathrm{ran}(\bar{g} \circ \bar{f})$ is itself a binary relation, we now see that $\bar{V}$ is in the domain of that range. This finding $\bar{V} \in \mathrm{dom}(\mathrm{ran}(\bar{g} \circ \bar{f}))$ implies now in conjunction with the previously established $x \in \bar{V}$ that x is in the union of the set system $\mathrm{dom}(\mathrm{ran}(\bar{g} \circ \bar{f}))$. Because x was arbitrary in Ω_1, we therefore conclude that the second inclusion $\Omega_1 \subseteq \bigcup \mathrm{dom}(\mathrm{ran}(\bar{g} \circ \bar{f}))$ in (11.871) holds, too.

Since the set Ω_1 is by assumption compact, it follows from the conjunction (11.871) – according to the Characterization of compact sets – that there is a set system, say $\bar{\mathcal{S}}_1$, such that $\bar{\mathcal{S}}_1$ is a finite subset of $\mathrm{dom}(\mathrm{ran}(\bar{g} \circ \bar{f}))$ whose union includes Ω_1. We form now the restricted binary relation $\mathrm{ran}(\bar{g} \circ \bar{f}) \restriction \bar{\mathcal{S}}_1$, and we observe that the preceding inclusion implies

$$\mathrm{dom}(\mathrm{ran}(\bar{g} \circ \bar{f}) \restriction \bar{\mathcal{S}}_1) = \bar{\mathcal{S}}_1$$

with (3.106). Consequently, there exists a particular function $\bar{F}$ satisfying the inclusion

$$\bar{F} \subseteq \mathrm{ran}(\bar{g} \circ \bar{f}) \restriction \bar{\mathcal{S}}_1 \tag{11.872}$$

and having the domain $\mathrm{dom}(\bar{F}) = \bar{\mathcal{S}}_1$, according to (3.695). As this domain $\bar{\mathcal{S}}_1$ is a finite set, it is also true that the range of $\bar{F}$ is finite (see Exercise 4.39). This means that there exists a particular natural number $\bar{m}$ as well as a particular bijection $\bar{d} : \{1, \ldots, \bar{m}\} \rightleftarrows \mathrm{ran}(\bar{F})$.

We verify now that this bijection is a sequence in the given topology $\mathcal{O}$, i.e. that the inclusion $\mathrm{ran}(\bar{F}) \subseteq \mathcal{O}$ holds. Letting $\bar{U}$ be arbitrary and assuming $\bar{U} \in \mathrm{ran}(\bar{F})$ to be true, we find $(\bar{V}, \bar{U}) \in \bar{F}$ for some particular constant $\bar{V}$, according to the definition of a range. This in turn implies $(\bar{V}, \bar{U}) \in \mathrm{ran}(\bar{g} \circ \bar{f}) \restriction \bar{\mathcal{S}}_1$ with the inclusion (11.872), with the consequence that $(\bar{V}, \bar{U}) \in \mathrm{ran}(\bar{g} \circ \bar{f})$ holds (by definition of a restriction). Then, $(\bar{V}, \bar{U}) \in \mathcal{O} \times \mathcal{O}$ follows to be true with the inclusion (11.870), so that we obtain $\bar{U} \in \mathcal{O}$ with the definition of the cartesian product of two sets. Because $\bar{U}$ was arbitrary in $\mathrm{ran}(\bar{F})$, we may therefore conclude that the inclusion $\mathrm{ran}(\bar{F}) \subseteq \mathcal{O}$ holds indeed (using the definition of a subset). Thus, $\mathcal{O}$ constitutes a codomain of $\bar{d}$, so that we have $\bar{d} \in \mathcal{O}^{\{1,\ldots,\bar{m}\}}$. Writing $\bar{d}$ as the sequence $(\bar{d}_i \,|\, i \in \{1, \ldots, \bar{m}\})$, the $\bar{m}$-fold repeated binary intersection operation with respect to the topology $\mathcal{O}$ gives us now

$$\bigcap_{i=1}^{\bar{m}} \bar{d}_i \in \mathcal{O}, \tag{11.873}$$

according to (11.327). In view of the existential sentence in (11.863) to be proven, the next task is to establish

$$\bar{\omega} \in \bigcap_{i=1}^{\bar{m}} \bar{d}_i, \tag{11.874}$$

which we can write equivalently as (using the Characterization of the intersection of a family of sets)

$$\forall i\,(i \in \{1, \ldots, \bar{m}\} \Rightarrow \bar{\omega} \in \bar{d}_i). \tag{11.875}$$

We let i be arbitrary, and we assume $i \in \{1, \ldots, \bar{m}\}$ to be true. Being in the domain of the sequence $\bar{d} : \{1, \ldots, \bar{m}\} \rightleftarrows \operatorname{ran}(\bar{F})$, this index is associated with the term $\bar{d}_i \in \operatorname{ran}(\bar{F})$, according to the Function Criterion. Then, the definition of a range gives us a particular set $\bar{A}$ for which $(\bar{A}, \bar{d}_i) \in \bar{F}$ is satisfied. In view of the inclusion (11.872), this gives us $(\bar{A}, \bar{d}_i) \in \operatorname{ran}(\bar{g} \circ \bar{f}) \upharpoonright \bar{\mathcal{S}}_1$, so that $(\bar{A}, \bar{d}_i) \in \operatorname{ran}(\bar{g} \circ \bar{f})$ holds by definition of a restriction. The definition of a range and the definition of a domain give us now also a particular element $\bar{y} \in \Omega_1$ $[= \operatorname{dom}(\bar{g} \circ \bar{f})]$ such that $(\bar{y}, (\bar{A}, \bar{d}_i)) \in \bar{g} \circ \bar{f}$. Using the notation for composed functions, we can write this in the form $(\bar{A}, \bar{d}_i) = \bar{g}(\bar{f}(\bar{y}))$, where $\bar{f}(\bar{y})$ is clearly an element of the range of $\bar{f}$. Consequently, $\bar{g}(\bar{f}(\bar{y})) \in \bar{f}(\bar{y})$ is true because of (11.869), and substitution gives us therefore $(\bar{A}, \bar{d}_i) \in \bar{f}(\bar{y})$. Due to (11.866), there exist then particular open, disjoint sets $\bar{U}$ and $\bar{V}$ satisfying $\bar{\omega} \in \bar{U}$, $\bar{y} \in \bar{V}$ and $(\bar{V}, \bar{U}) = (\bar{A}, \bar{d}_i)$. This equation implies with the Equality Criterion for ordered pairs especially $\bar{U} = \bar{d}_i$, so that $\bar{\omega} \in \bar{U}$ yields via substitution $\bar{\omega} \in \bar{d}_i$, as desired. Since i was arbitrary, we may therefore conclude that (11.875) is true, and the truth of this universal sentence implies the truth of the suggested equivalent sentence (11.874).

Regarding the existential sentence in (11.863), we prove now also the inclusion

$$\bigcap_{i=1}^{\bar{m}} \bar{d}_i \;\subseteq\; \Omega \setminus \Omega_1. \tag{11.876}$$

According to the definition of a subset, we let ω be arbitrary, we assume $\omega \in \bigcap_{i=1}^{\bar{m}} \bar{d}_i$ to be true, and we show that $\omega \in \Omega \setminus \Omega_1$ is implied. In view of the definition of a set difference, this sentence is equivalent to the conjunction $\omega \in \Omega \wedge \omega \notin \Omega_1$. Concerning the first part $\omega \in \Omega$, we notice that the inclusion $\mathcal{O} \subseteq \mathcal{P}(\Omega)$ is true because of Property 1 of a topology (on Ω). Therefore, the true sentence (11.873) implies $\bigcap_{i=1}^{\bar{m}} \bar{d}_i \in \mathcal{P}(\Omega)$ with the definition of a subset, and then also $\bigcap_{i=1}^{\bar{m}} \bar{d}_i \subseteq \Omega$ by definition of a power set. With this inclusion, the assumed $\omega \in \bigcap_{i=1}^{\bar{m}} \bar{d}_i$ implies $\omega \in \Omega$, as desired. It now remains for us to establish $\omega \notin \Omega_1$. For this purpose, we prove first the negation $\omega \notin \bigcup \bar{\mathcal{S}}_1$ by contradiction, assuming the negation of that negation to be true. We thus obtain the true sentence $\omega \in \bigcup \bar{\mathcal{S}}_1$ with the Double Negation Law, so that there exists – by definition of the union of a set system – a particular set $\bar{B} \in \bar{\mathcal{S}}_1$ with $\omega \in \bar{B}$. Recalling that $\bar{\mathcal{S}}_1$ is the domain of the function $\bar{F}$, we find also a particular set $\bar{A}$ such that $(\bar{B}, \bar{A}) \in \bar{F}$ holds. Consequently, $\bar{A}$ is in the range of $\bar{F}$, which is also the range of the sequence $\bar{d} : \{1, \ldots, \bar{m}\} \rightleftarrows \operatorname{ran}(\bar{F})$, noting that this bijection is a surjection. We thus have $\bar{A} \in \operatorname{ran}(\bar{d})$, so that there evidently exists a particular element $\bar{k}$ of the domain $\{1, \ldots, \bar{m}\}$ of $\bar{d}$ with $(\bar{k}, \bar{A}) \in \bar{d}$. Since we can write the assumed $\omega \in \bigcap_{i=1}^{\bar{m}} \bar{d}_i$ also as the universal sentence

$$\forall i \, (i \in \{1, \ldots, \bar{m}\} \Rightarrow \omega \in \bar{d}_i) \tag{11.877}$$

by applying the Characterization of the intersection of a family of sets, it follows from $\bar{k} \in \{1, \ldots, \bar{m}\}$ that $\omega \in \bar{d}_{\bar{k}}$ is true. Moreover, we can write $(\bar{k}, \bar{A}) \in \bar{d}$ in sequence notation as $\bar{A} = \bar{d}_{\bar{k}}$, so that we obtain through substitution $\omega \in \bar{A}$. Furthermore, the previously established $(\bar{B}, \bar{A}) \in \bar{F}$ evidently implies with the inclusion (11.872) $(\bar{B}, \bar{A}) \in \operatorname{ran}(\bar{g} \circ \bar{f}) \upharpoonright \bar{\mathcal{S}}_1$ and therefore $(\bar{B}, \bar{A}) \in \operatorname{ran}(\bar{g} \circ \bar{f})$. Clearly, this implies the existence of a particular element $\bar{y}$ of the domain Ω_1 of $\bar{g} \circ \bar{f}$ such that $(\bar{y}, (\bar{B}, \bar{A})) \in \bar{g} \circ \bar{f}$, which we can also write as $(\bar{B}, \bar{A}) = \bar{g}(\bar{f}(\bar{y}))$. This evidently implies $(\bar{B}, \bar{A}) \in \bar{f}(\bar{y})$ with (11.869), so that (11.866) gives us two particular open and disjoint sets $\bar{U}, \bar{V}$ satisfying in particular $(\bar{V}, \bar{U}) = (\bar{B}, \bar{A})$. Here, the Equality Criterion for ordered pairs yields $\bar{V} = \bar{B}$ and $\bar{U} = \bar{A}$, and these equations allow us to carry out substitutions to infer from the disjointness of $\bar{U}, \bar{V}$ the disjointness of $\bar{A}, \bar{B}$, that is, $\bar{A} \cap \bar{B} = \emptyset$. By definition of the empty set, this means that $y \notin \bar{A} \cap \bar{B}$ holds for any y, thus in particular for the constant ω. Then, $\omega \notin \bar{A} \cap \bar{B}$ implies $\neg(\omega \in \bar{A} \wedge \omega \in \bar{B})$ with the definition of the intersection of two sets. Consequently, we obtain the true disjunction $\omega \notin \bar{A} \vee \omega \notin \bar{B}$ by means of De Morgan's Law for the conjunction. Because we previously found $\omega \in \bar{B}$ to be true, this evidently means that the second part of this disjunction is false, so that its first part $\omega \notin \bar{A}$ must be true. This however contradicts our previous finding $\omega \in \bar{A}$, so that the proof of the negation $\omega \notin \bigcup \bar{\mathcal{S}}_1$ is now complete. Recalling now the truth of the inclusion $\Omega_1 \subseteq \bigcup \bar{\mathcal{S}}_1$, we see in light on (2.9) that the preceding negation implies $\omega \notin \Omega_1$. Thus, the proof of the conjunction $\omega \in \Omega \wedge \omega \notin \Omega_1$ is complete, so that the equivalent $\omega \in \Omega \setminus \Omega_1$ is true. Since ω was an arbitrary element of $\bigcap_{i=1}^{\bar{m}} \bar{d}_i$, we may infer from this finding the truth of the inclusion (11.876).

Since the particular set $\bigcap_{i=1}^{\bar{m}} \bar{d}_i$ satisfies the conjunction of (11.873), (11.874) and (11.876), the existential sentence in (11.863) is true. Here, $\bar{\omega}$ was arbitrary, so that the universal sentence (11.863) follows to be true, too. We thus showed that $\Omega_1^c = \Omega \setminus \Omega_1$ is an open set of the topology $\mathcal{O}$, so that the set Ω_1 is closed in Ω. Consequently, as Ω, $\mathcal{O}$ and Ω_1 were initially arbitrary sets, we may now finally conclude that the stated theorem is true. □

Lemma 11.110. *It is true that*

a) the maximum of any nonempty, closed and bounded-from-above subset $A \subseteq \mathbb{R}$ is given by the supremum of A.

b) the minimum of any nonempty, closed and bounded-from-below subset $A \subseteq \mathbb{R}$ is given by the infimum of A.

Proof. Concerning a), we let A be an arbitrary set, assuming A to be a subset of $\mathbb{R}$, assuming $A \neq \emptyset$, assuming A to be closed in $\mathbb{R}$, and assuming A to be bounded from above. As $(\mathbb{R}, <_{\mathbb{R}})$ is a linear continuum (see Corollary 8.12), it is by definition true that $(\mathbb{R}, <_{\mathbb{R}})$ has the Supremum Property, so that $\sup A$ exists (in $\mathbb{R}$). Thus, $\sup A$ is an upper bound for A by definition. Next, we prove $\sup A \in A$ by contradiction, assuming the negation $\sup A \notin A$ to be true. For this purpose, we show that there exists an upper bound for A which is less than the supremum $\sup A$, in contradiction to the Characterization of the supremum. First, we observe that $\sup A \in \mathbb{R}$ and $\sup A \notin A$ imply $\sup A \in \mathbb{R} \setminus A$ by definition of a set difference. Now, since A is a closed set (in $\mathbb{R}$), it is by definition true that $A^c = \mathbb{R} \setminus A$ is an open set (of the order topology on $\mathbb{R}$), recalling the definition of a complement. According to the Generation of a topology by means of a basis, it follows from $\mathbb{R} \setminus A \in \mathcal{O}_{<_{\mathbb{R}}}$ and from $\sup A \in \mathbb{R} \setminus A$ that there is a particular set $\bar{B}$ in the basis $\{(a, b) : a, b \in \mathbb{R}\}$ (which generates the order topology on $\mathbb{R}$, as shown in Note 11.23), such that $\sup A \in \bar{B}$ and $\bar{B} \subseteq \mathbb{R} \setminus A$ are satisfied. Thus, $\bar{B}$ constitutes an open interval $(\bar{a}, \bar{b})$ for some particular real numbers $\bar{a}, \bar{b}$. Substitutions give us then $\sup A \in (\bar{a}, \bar{b})$ and $(\bar{a}, \bar{b}) \subseteq \mathbb{R} \setminus A$. Here, the former finding implies $\bar{a} <_{\mathbb{R}} \sup A <_{\mathbb{R}} \bar{b}$ by definition of an open interval. Next, we prove that $\bar{a}$ constitutes an upper bound for A, that is,

$$\forall x \, (x \in A \Rightarrow x \leq_{\mathbb{R}} \bar{a}). \tag{11.878}$$

Letting x be arbitrary, we prove the implication by establishing the contradiction $x \in A \wedge x \notin A$, assuming $x \in A$ and $\neg x \leq_{\mathbb{R}} \bar{a}$ to be true. On the one hand, the assumed negation yields $\bar{a} <_{\mathbb{R}} x$ with the Negation Formula for $\leq$. On the other hand, the former assumption $x \in A$ implies $x \leq_{\mathbb{R}} \sup A$ because $\sup A$ is an upper bound for A. Since the inequality $\sup A <_{\mathbb{R}} \bar{b}$ also holds, we obtain $x <_{\mathbb{R}} \bar{b}$ with the Transitivity Formula for $\leq$ and $<$. In conjunction with $\bar{a} <_{\mathbb{R}} x$, this gives us $x \in (\bar{a}, \bar{b})$ by means of the definition of an open interval. Consequently, we obtain $x \in \mathbb{R} \setminus A$ by virtue of the previously established inclusion $(\bar{a}, \bar{b}) \subseteq \mathbb{R} \setminus A$. According to the definition of a set difference, this means in particular that $x \notin A$ is true, in contradiction to $x \in A$. We thus completed the proof of the implication in (11.878), in which x is arbitrary, so that the universal sentence (11.878) follows to be true. This means that $\bar{a}$ is indeed an upper bound for A. Because the previously found $\bar{a} <_{\mathbb{R}} \sup A$ implies $\neg \sup A \leq_{\mathbb{R}} \bar{a}$ with the Negation Formula for $\leq$, we thus find (denoting $S = \sup A$) the existential sentence

$$\exists S' \,(\forall x \,(x \in A \Rightarrow x \leq_{\mathbb{R}} S') \Rightarrow \neg S \leq_{\mathbb{R}} S')$$

to be true. The Negation Law for universal implications gives us the true negation

$$\neg \forall S' \,(\forall x \,(x \in A \Rightarrow x \leq_{\mathbb{R}} S') \Rightarrow S \leq_{\mathbb{R}} S'),$$

and the disjunction

$$\neg \forall x \,(x \in A \Rightarrow x \leq_{\mathbb{R}} S) \;\vee\; \neg \forall S' \,(\forall x \,(x \in A \Rightarrow x \leq_{\mathbb{R}} S') \Rightarrow S \leq_{\mathbb{R}} S'),$$

is then also true. Applying now De Morgan's Law for the conjunction, we arrive at the desired contradiction with respect to the Characterization of the supremum (of A). This completes the proof of $\sup A \in A$, so that $\sup A$ is an upper bound for A contained in A. This means that $\sup A$ is the maximum of A, by definition. As the set A was initially arbitrary, we may therefore conclude that Part a) of the lemma holds.

Part b) can be proved in analogy to Part a). □

Exercise 11.47. Establish Part b) of Lemma 11.110.

Lemma 11.111. *It is true for every open set U of the order topology $\mathcal{O}_{<_{\mathbb{R}}}$ and for every element z of U that U includes some closed interval $[x, y]$ such that z is strictly between x and y, that is,*

$$\forall U, z \,([U \in \mathcal{O}_{<_{\mathbb{R}}} \;\wedge\; z \in U] \Rightarrow \exists x, y \,(x <_{\mathbb{R}} z <_{\mathbb{R}} y \;\wedge\; [x, y] \subseteq U)). \quad (11.879)$$

Proof. We let U and z be arbitrary, and we assume $U \in \mathcal{O}_{<_{\mathbb{R}}}$ as well as $z \in U$ to be true. These assumptions allow us to utilize the Generation of a topology by means of a basis and to infer the existence of a particular element $\bar{B}$ of the basis $\{(a, b) : a, b \in \mathbb{R}\}$ generating $\mathcal{O}_{<_{\mathbb{R}}}$ (see Note 11.23), such that this basis element satisfies $z \in \bar{B}$ and $\bar{B} \subseteq U$. As this basis element can then be written as the open interval $(\bar{a}, \bar{b})$ for some particular real numbers $\bar{a}$ and $\bar{b}$, it satisfies equivalently $z \in (\bar{a}, \bar{b})$ and $(\bar{a}, \bar{b}) \subseteq U$. Here, $z \in (\bar{a}, \bar{b})$ implies now the truth of the inequalities $\bar{a} <_{\mathbb{R}} z <_{\mathbb{R}} \bar{b}$ with the definition of an open interval. Because $(\mathbb{R}, <_{\mathbb{R}})$ is densely ordered (see Corollary 8.12), there exist then intermediate values $\bar{x}$ and $\bar{y}$ satisfying $\bar{a} <_{\mathbb{R}} \bar{x} <_{\mathbb{R}} z$ and $z <_{\mathbb{R}} \bar{y} <_{\mathbb{R}} \bar{b}$. We thus found the inequalities

$$\bar{x} \ <_{\mathbb{R}} \ z \ <_{\mathbb{R}} \ \bar{y}. \tag{11.880}$$

Next, we form the closed interval $[\bar{x}, \bar{y}]$ in $\mathbb{R}$ and establish the inclusion

$$[\bar{x}, \bar{y}] \ \subseteq \ U. \tag{11.881}$$

We apply the definition of a subset and let accordingly $c \in [\bar{x}, \bar{y}]$ be arbitrary. By definition of a closed interval, the inequalities $\bar{x} \leq_{\mathbb{R}} c \leq_{\mathbb{R}} \bar{y}$ are then true. On the one hand, the conjunction of the previously established $\bar{a} <_{\mathbb{R}} \bar{x}$ and $\bar{x} \leq_{\mathbb{R}} c$ implies $\bar{a} <_{\mathbb{R}} c$ with the Transitivity Formula for $<$ and $\leq$. On the other hand, the conjunction of $c \leq_{\mathbb{R}} \bar{y}$ and the previously found $\bar{y} <_{\mathbb{R}} \bar{b}$ yields $c <_{\mathbb{R}} \bar{b}$ with the Transitivity Formula for $\leq$ and $<$. We thus have $\bar{a} <_{\mathbb{R}} c <_{\mathbb{R}} \bar{b}$, so that c follows to be an element of the open interval $(\bar{a}, \bar{b})$. Then, $c \in (\bar{a}, \bar{b})$ implies $c \in U$ with the previously obtained inclusion $(\bar{a}, \bar{b}) \subseteq U$. We thus showed that $c \in [\bar{x}, \bar{y}]$ implies $c \in U$, and since c is arbitrary, we may infer now from the truth of this implication the truth of the inclusion (11.881). ¡in view of (11.880) and (11.881), the existential sentence in (11.879) is true, so that the proposed universal sentence follows to be true, since U and z were initially arbitrary. □

Theorem 11.112 (Heine-Borel Theorem for $\mathbb{R}$). *It is true that a subset A of $\mathbb{R}$ is a compact set (with respect to the order topology $\mathcal{O}_{<_{\mathbb{R}}}$) iff A is closed in $\mathbb{R}$, bounded from below and bounded from above.*

Proof. We take an arbitrary set A and assume A to be included in $\mathbb{R}$. Therefore, the subspace topology $\mathcal{O}_{<_{\mathbb{R}}}|A$ of the standard topology on $\mathbb{R}$ is defined, and we may assume A to be a compact set in $\mathbb{R}$ (with respect to $\mathcal{O}_{<_{\mathbb{R}}}$) to establish the first part ('$\Rightarrow$') of the equivalence. Due to the Hausdorff Property of topological spaces involving the order topology, it is true that $(\mathbb{R}, \mathcal{O}_{<_{\mathbb{R}}})$ constitutes a Hausdorff space. Therefore, the compact set A is closed in $\mathbb{R}$ because of the Closedness of compact subspaces of Hausdorff spaces. Next, we prove that A is both bounded from below and bounded from above. Recalling from Proposition 11.106 that the sequence $s = ((-n, n)_{\mathbb{R}})_{n \in \mathbb{N}}$ constitutes an open covering of $\mathbb{R}$ with respect to $\mathcal{O}_{<_{\mathbb{R}}}$, we thus have $\mathrm{ran}(s) \subseteq \mathcal{P}(\mathbb{R})$, $\bigcup \mathrm{ran}(s) = \mathbb{R}$ and

$$\mathrm{ran}(s) \subseteq \mathcal{O}_{<_{\mathbb{R}}}. \tag{11.882}$$

Because of the preceding equation, the initial assumption $A \subseteq \mathbb{R}$ gives us also

$$A \subseteq \bigcup \mathrm{ran}(s) \quad \left[= \bigcup_{n=0}^{\infty} (-n, n) \right]. \tag{11.883}$$

According to the Characterization of compact sets, it follows from (11.882) and (11.883) that there exists a particular finite subset $\bar{\mathcal{S}}_1 \subseteq \mathrm{ran}(s)$ with $A \subseteq \bigcup \bar{\mathcal{S}}_1$. Let us recall now from Proposition 11.106c) that s is the bijection $s : \mathbb{N} \rightleftarrows \mathrm{ran}(s)$, giving rise to the inverse bijection $s^{-1} : \mathrm{ran}(s) \rightleftarrows \mathbb{N}$ in view of the Bijectivity of inverse functions. Noting that this bijection is in particular an injection $s^{-1} : \mathrm{ran}(s) \hookrightarrow \mathbb{N}$, the inclusion $\bar{\mathcal{S}}_1 \subseteq \mathrm{ran}(s)$ gives us the restricted injection $s^{-1} \upharpoonright \bar{\mathcal{S}}_1 : \bar{\mathcal{S}}_1 \hookrightarrow \mathbb{N}$ with (3.624). Consequently, we obtain the bijection

$$s^{-1} \upharpoonright \bar{\mathcal{S}}_1 : \bar{\mathcal{S}}_1 \rightleftarrows \mathrm{ran}(s^{-1} \upharpoonright \bar{\mathcal{S}}_1)$$

with (3.664). Now, since the domain $\bar{\mathcal{S}}_1$ of $s^{-1} \upharpoonright \bar{\mathcal{S}}_1$ is finite, the range of this function is also finite according to Exercise 4.39. Because $\mathbb{N}$ is a codomain of $s^{-1} \upharpoonright \bar{\mathcal{S}}_1$, the inclusion $\mathrm{ran}(s^{-1} \upharpoonright \bar{\mathcal{S}}_1) \subseteq \mathbb{N}$ holds, so that this range constitutes a finite subset of $\mathbb{N}$. Consequently, the set $\mathrm{ran}(s^{-1} \upharpoonright \bar{\mathcal{S}}_1)$ is bounded from above (with respect to $\leq_{\mathbb{N}}$) because of Corollary 4.118, which means that there exists an upper bound for that range, say $\bar{u}$. Thus, $\bar{u}$ is a natural number satisfying

$$\forall n\,(n \in \mathrm{ran}(s^{-1} \upharpoonright \bar{\mathcal{S}}_1) \Rightarrow n \leq_{\mathbb{N}} \bar{u}). \tag{11.884}$$

Here, $\bar{u} \in \mathbb{N}$ is an index of the sequence s, so that $\bar{u}$ is associated with the term $s_{\bar{u}} = (-\bar{u}, \bar{u})$. Exercise 3.53c) shows that this open interval is bounded from below and bounded from above (with respect to $\leq_{\mathbb{R}}$). We show now in the following that A is a subset of this interval $(-\bar{u}, \bar{u})$, by verifying accordingly

$$\forall y\,(y \in A \Rightarrow y \in (-\bar{u}, \bar{u})\,). \tag{11.885}$$

Letting y be arbitrary and assuming $y \in A$ to be true, we obtain $y \in \bigcup \bar{\mathcal{S}}_1$ with the previously established inclusion $A \subseteq \bigcup \bar{\mathcal{S}}_1$. By definition of the union of a set system, there exists then a particular set $\bar{X} \in \bar{\mathcal{S}}_1$ with $y \in \bar{X}$. Since $\bar{X}$ is in the domain of the restriction $s^{-1} \upharpoonright \bar{\mathcal{S}}_1 : \bar{\mathcal{S}}_1 \hookrightarrow \mathbb{N}$, it is associated with the value $(s^{-1} \upharpoonright \bar{\mathcal{S}}_1)(\bar{X}) \in \mathbb{N}$, according to the Function Criterion. Denoting this natural number by m, it is clearly true that the value m is also in the range of that restriction, so that (11.884) yields $m \leq_{\mathbb{N}} \bar{u}$. Viewing m and $\bar{u}$ as real numbers, we can write the preceding inequality also as $m \leq_{\mathbb{R}} \bar{u}$, and this gives us with the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ also the inequality $-\bar{u} \leq_{\mathbb{R}} -m$. This implies now in conjunction with $m \leq_{\mathbb{R}} \bar{u}$ that the inclusion

$$(-m, m) \subseteq (-\bar{u}, \bar{u}) \tag{11.886}$$

is true. Let us observe next that $\bar{\mathcal{S}}_1 \subseteq \mathrm{ran}(s)$ and $\bar{X} \in \bar{\mathcal{S}}_1$ imply

$$[m =] \quad (s^{-1} \upharpoonright \bar{\mathcal{S}}_1)(\bar{X}) = s^{-1}(\bar{X})$$

with (3.567), and moreover that the resulting equation $m = s^{-1}(\bar{X})$ implies

$$\bar{X} = s(m) = (-m, m)$$

with the Characterization of the function values of an inverse function and with the definition of the sequence s. Therefore, the previously found $y \in \bar{X}$ yields via substitution $y \in (-m, m)$, which in turn implies $y \in (-\bar{u}, \bar{u})$ with (11.886), by definition of a subset. We thus proved the implication in (11.885), in which y is arbitrary, so that the universal sentence (11.885) follows now to be true. This means by definition of a subset that the inclusion $A \subseteq (-\bar{u}, \bar{u})$ holds indeed. Recalling that $(-\bar{u}, \bar{u})$ is bounded from below and bounded from above (with respect to $\leq_{\mathbb{R}}$), we now see in light of Exercise 3.38 and Proposition 3.94 that the subset A is itself bounded from below and bounded from above (with respect to $\leq_{\mathbb{R}}$). Thus, the implication '⇒' of the proposed equivalence holds.

Concerning the second part ('⇐') of the equivalence, we assume now A to be closed in $\mathbb{R}$, to be bounded from above, and to be bounded from above. We consider the two cases $A = \emptyset$ and $A \neq \emptyset$, noting that $\emptyset$ is closed, bounded from above and bounded from below according to Corollary 11.43, Proposition 3.90 and Exercise 3.34, respectively. In the first case $A = \emptyset$, it is true that A is compact by virtue of Corollary 11.108. In the second case $A \neq \emptyset$, we recall that $(\mathbb{R}, <_{\mathbb{R}})$ has both the Infimum Property and the Supremum Property (see Corollary 8.12), so that the preceding assumptions imply the existence of the infimum $\inf A$ and of the supremum $\sup A$. Moreover, we see in light of Lemma 11.110 that $\inf A$ constitutes the minimum and $\sup A$ the maximum of A.

Now, to prove that A is compact, we apply the Characterization of compact sets and let accordingly $\mathcal{C}_1$ be arbitrary, assuming the two inclusions $\mathcal{C}_1 \subseteq \mathcal{O}_{<_{\mathbb{R}}}$ and $A \subseteq \bigcup \mathcal{C}_1$ to be true. Our task is now to demonstrate the truth of the existential sentence

$$\exists \mathcal{S}_1 \, (\mathcal{S}_1 \subseteq \mathcal{C}_1 \,\wedge\, \mathcal{S}_1 \text{ is finite} \,\wedge\, A \subseteq \bigcup \mathcal{S}_1). \tag{11.887}$$

Let us verify first the universal sentence

$$\forall x \, (x \in \mathbb{R} \Rightarrow \exists! Y \, (Y = A \cap (-\infty, x]\,)).$$

Letting $x \in \mathbb{R}$ be arbitrary, we see that the interval $(-\infty, x]$ and then also the intersection $A \cap (-\infty, x]$ are specified sets. Therefore, the uniquely existential sentence is true in view of (1.109), and as x was arbitrary, the preceding universal sentence follows indeed to be true. According to Function definition by replacement, there exists then a unique function F with domain $\mathbb{R}$ such that

$$\forall x \, (x \in \mathbb{R} \Rightarrow F(x) = A \cap (-\infty, x]\,). \tag{11.888}$$

Next, we observe in light of the Axiom of Specification and the Equality Criterion for sets that there exists a unique set K satisfying

$$\forall x\,(x \in K \Leftrightarrow [x \in \mathbb{R} \,\wedge\, \exists \mathcal{S}_1\,(\mathcal{S}_1 \subseteq \mathcal{C}_1 \,\wedge\, \mathcal{S}_1 \text{ is finite} \,\wedge\, F(x) \subseteq \bigcup \mathcal{S}_1)]). \tag{11.889}$$

Here, we see that $x \in K$ implies especially $x \in \mathbb{R}$ for any x, so that K is a subset of $\mathbb{R}$. We can also prove that K is nonempty, by showing that $\min A$ is an element of K. To do this, we note that the real number $\min A$ gives rise to the value $F(\min A) = A \cap (-\infty, \min A]$ by virtue of (11.888), and we verify in the following that this value is identical with $\{\min A\}$. For this purpose, we apply the Equality Criterion for sets and establish accordingly the universal sentence

$$\forall y\,(y \in F(\min A) \Leftrightarrow y \in \{\min A\}). \tag{11.890}$$

Letting y be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence by assuming $y \in F(\min A)$ to hold. We thus have $y \in A \cap (-\infty, \min A]$, so that we obtain $y \in A$ and $y \in (-\infty, \min A]$ by definition of the intersection of two sets. On the one hand, $y \in A$ yields $\min A \leq_{\mathbb{R}} y$ with the fact that $\min A$ constitutes a lower bound for A. On the other hand, $y \in (-\infty, \min A]$ implies $y \leq_{\mathbb{R}} \min A$ by definition of a left-unbounded and right-closed interval. The conjunction of these two inequalities implies now $y = \min A$ with the antisymmetry of the standard total ordering $\leq_{\mathbb{R}}$. Therefore, the desired consequent $y \in \{\min A\}$ follows to be true with (2.169). Regarding the second part ('$\Leftarrow$') of the equivalence, we assume conversely $y \in \{\min A\}$ to be true, so that we find the equation $y = \min A$ to be true. Then, the fact that the minimum of A is by definition contained in A gives us via substitution $y \in A$. Furthermore, the inequality $\min A \leq_{\mathbb{R}} \min A$ holds because of the reflexivity of $\leq_{\mathbb{R}}$, with the consequence that $[\min A =]\ y \in (-\infty, \min A]$ holds. Consequently, y is in the intersection of A and $(-\infty, \min A]$, which is given by $F(\min A)$, so that $y \in F(\min A)$ is true, as desired. Having thus established the truth of the equivalence, we may now infer from this the truth of (11.890), since y is arbitrary, and in addition the truth of the equivalent equality $F(\min A) = \{\min A\}$.

Now, the aforementioned $\min A \in A$ implies $\min A \in \bigcup \mathcal{C}_1$ with the assumed inclusion $A \subseteq \bigcup \mathcal{C}_1$, so that there exists a particular set $\bar{Y} \in \mathcal{C}_1$ with $\min A \in \bar{Y}$ (in view of the definition of the union of a set system). We therefore obtain the two inclusions $\{\bar{Y}\} \subseteq \mathcal{C}_1$ and $[F(\min A) =]\ \{\min A\} \subseteq \bar{Y}$ with (2.184). Here, the singleton $\{\bar{Y}\}$ is a finite set due to (4.470), and the union of that singleton is given by $\bigcup\{\bar{Y}\} = \bar{Y}$, as shown in (2.199). The preceding inclusion yields then $F(\min A) \subseteq \bigcup\{\bar{Y}\}$ by means of substitution.

The previous findings show us that the existential sentence

$$\exists \mathcal{S}_1 \, (\mathcal{S}_1 \subseteq \mathcal{C}_1 \, \wedge \, \mathcal{S}_1 \text{ is finite} \, \wedge \, F(\min A) \subseteq \bigcup \mathcal{S}_1)$$

is true, so that the real number $\min A$ turns out to be an element of K, according to the specification of that set in (11.889). Thus, the set K is clearly nonempty.

We prove in the following by contradiction that K is not bounded from above, assuming the negation of that negation to be true. Thus, K is bounded from above by virtue of the Double Negation Law. Due to the Supremum Property of $(\mathbb{R}, <_{\mathbb{R}})$, the nonempty and bounded-from-above subset $K \subseteq \mathbb{R}$ has then a supremum. We consider now (within the current second case $A \neq \emptyset$) the two sub-cases $\sup K \in A$ and $\sup K \notin A$ (to prove that K is not bounded from above).

The first sub-case $\sup K \in A$ implies $\sup K \in \bigcup \mathcal{C}_1$ with the assumed inclusion $A \subseteq \bigcup \mathcal{C}_1$, so that there exists – by definition of the union of a set system – a set, say $\bar{U}$, such that $\bar{U} \in \mathcal{C}_1$ and $\sup K \in \bar{U}$ hold. The other assumed inclusion $\mathcal{C}_1 \subseteq \mathcal{O}_{<_{\mathbb{R}}}$ gives us then $\bar{U} \in \mathcal{O}_{<_{\mathbb{R}}}$. In connection with $\sup K \in \bar{U}$, this implies by virtue of Lemma 11.879 the existence of particular real numbers $\bar{a}$ and $\bar{b}$ satisfying $\bar{a} <_{\mathbb{R}} \sup K <_{\mathbb{R}} \bar{b}$ and $[\bar{a}, \bar{b}] \subseteq \bar{U}$. Due to the Supremum Criterion, $\bar{a} <_{\mathbb{R}} \sup K$ implies then the existence of a particular element $\bar{x} \in K$ with $\bar{a} <_{\mathbb{R}} \bar{x}$. According to the specification of the set K in (11.889), it follows now from $\bar{x} \in K$ that there exists a particular set (system) $\bar{\mathcal{S}}_1$ such that

$$\bar{\mathcal{S}}_1 \subseteq \mathcal{C}_1 \, \wedge \, \bar{\mathcal{S}}_1 \text{ is finite} \, \wedge \, F(\bar{x}) \subseteq \bigcup \bar{\mathcal{S}}_1. \tag{11.891}$$

Based on the third part of this multiple conjunction, we can demonstrate that the value $F(\bar{a}) = A \cap (-\infty, \bar{a}]$, associated with the real number $\bar{a}$, is also included in the union $\bigcup \bar{\mathcal{S}}_1$. We apply the definition of a subset and let accordingly y be arbitrary, where we assume that $y \in F(\bar{a})$ is true. We therefore obtain with the definition of the intersection of two sets $y \in A$ and $y \in (-\infty, \bar{a}]$, so that $y \leq_{\mathbb{R}} \bar{a}$ holds by definition of a left-unbounded and right-closed interval. As $\bar{a} <_{\mathbb{R}} \bar{x}$ holds, too, we get $y <_{\mathbb{R}} \bar{x}$ with the Transitivity Formula for $\leq$ and $<$. Then, the disjunction $y <_{\mathbb{R}} \bar{x} \vee y = \bar{x}$ is true as well, which yields $y \leq_{\mathbb{R}} \bar{x}$ by definition of an induced irreflexive partial ordering. This means that y is an element of the interval $(-\infty, \bar{x}]$, and thus an element of $F(\bar{x})$. In view of the second inclusion (11.891), this gives us $y \in \bigcup \bar{\mathcal{S}}_1$, as desired. Since y was arbitrary in $F(\bar{a})$, we may therefore conclude that $F(\bar{a})$ is indeed a subset of $\bigcup \bar{\mathcal{S}}_1$. We consider now the set $\bar{\mathcal{S}}_1 \cup \{\bar{U}\}$. Recalling the truth of $\bar{U} \in \mathcal{C}_1$, we see that $\{\bar{U}\} \subseteq \mathcal{C}_1$ holds because of (2.184). In conjunction with the first inclusion $\bar{\mathcal{S}}_1 \subseteq \mathcal{C}_1$ in

(11.891), this gives us

$$\bar{\mathcal{S}}_1 \cup \{\bar{U}\} \subseteq \mathcal{C}_1 \tag{11.892}$$

with (2.252). In addition, the second part of the multiple conjunction (11.891) states that $\bar{\mathcal{S}}_1$ is finite. Consequently, it follows that

$$\bar{\mathcal{S}}_1 \cup \{\bar{U}\} \text{ is finite,} \tag{11.893}$$

according to (4.467). We demonstrate now that the inclusion

$$F(\bar{b}) \subseteq \bigcup(\bar{\mathcal{S}}_1 \cup \{\bar{U}\}) \tag{11.894}$$

also holds, by applying the definition of a subset and by verifying accordingly the universal sentence

$$\forall y\,(y \in F(\bar{b}) \Rightarrow y \in \bigcup\,(\bar{\mathcal{S}}_1 \cup \{\bar{U}\})). \tag{11.895}$$

We let y be arbitrary and assume $y \in F(\bar{b})$ to be true. The definition of the function F in (11.888) gives us here $F(\bar{b}) = A \cap (-\infty, \bar{b}]$, so that $y \in A$ and $y \in (-\infty, \bar{b}]$ follow to be true by means of substitution and the definition of the intersection of two sets. Consequently, $y \leq_{\mathbb{R}} \bar{b}$ according to the definition of a left-unbounded and right-closed interval. We consider now within the present first sub-case $\sup K \in A$ the further cases $y \leq_{\mathbb{R}} \bar{x}$ and $\neg y \leq_{\mathbb{R}} \bar{x}$. In case of $y \leq_{\mathbb{R}} \bar{x}$, we evidently find $y \in (-\infty, \bar{x}]$ to be true, and this implies in conjunction with $y \in A$ the truth of $y \in A \cap (-\infty, \bar{x}]$, thus $y \in F(\bar{x})$. Because of the inclusion $F(\bar{x}) \subseteq \bigcup \bar{\mathcal{S}}_1$ in (11.891), this gives us $y \in \bigcup \bar{\mathcal{S}}_1$. By definition of the union of a set system, there exists therefore an element of the set system $\bar{\mathcal{S}}_1$ containing y, say $\bar{X}$. Then, the disjunction $\bar{X} \in \bar{\mathcal{S}}_1 \vee \bar{X} \in \{\bar{U}\}$ is also true, so that $\bar{X} \in \bar{\mathcal{S}}_1 \cup \{\bar{U}\}$ holds by definition of the union of two sets. In conjunction with $y \in \bar{X}$, this implies now evidently $y \in \bigcup(\bar{\mathcal{S}}_1 \cup \{\bar{U}\})$, as desired. In the other case $\neg y \leq_{\mathbb{R}} \bar{x}$, we obtain $\bar{x} <_{\mathbb{R}} y$ with the negation Formula for $\leq$. This yields in connection with the previously established inequality $\bar{a} <_{\mathbb{R}} \bar{x}$, because of the transitivity of the standard linear ordering $<_{\mathbb{R}}$, the true inequality $\bar{a} <_{\mathbb{R}} y$. The disjunction $\bar{a} <_{\mathbb{R}} y \vee \bar{a} = y$ is then also true, so that $\bar{a} \leq_{\mathbb{R}} y$ holds by definition of an induced irreflexive partial ordering. Together with the previous finding $y \leq_{\mathbb{R}} \bar{b}$, this implies with the definition of a closed interval $y \in [\bar{a}, \bar{b}]$. Since we earlier found this interval to be a subset of $\bar{U}$, we obtain now $y \in \bar{U}$. Let us observe at this point the truth of $\bar{U} \in \{\bar{U}\}$, and then also the truth of the disjunction $\bar{U} \in \bar{\mathcal{S}}_1 \vee \bar{U} \in \{\bar{U}\}$. Thus, $\bar{U} \in \bar{\mathcal{S}}_1 \cup \{\bar{U}\}$ holds by definition of the union of two sets. Since $y \in \bar{U}$ also holds, it follows that y is again element of the union $\bigcup(\bar{\mathcal{S}}_1 \cup \{\bar{U}\})$. We thus completed the proof of the implication in (11.895), in which y is arbitrary, so the universal sentence (11.895) and subsequently the equivalent inclusion (11.894) follow to be true. The conjunction of (11.893), (11.894) and (11.895) shows now in light of the definition of the set K in (11.889) that $\bar{b}$ is an element of K. Since $\sup K <_{\mathbb{R}} \bar{b}$ also holds, this means that we found an element in K which is greater than $\sup K$. Thus, $\sup K$ is evidently not an upper bound for K, which is in contradiction to the definition of the supremum as the least upper bound (for K). We thus proved for the first sub-case that K is not bounded from above.

The second sub-case $\sup K \notin A$ implies in conjunction with the fact $\sup K \in \mathbb{R}$ that $\sup K \in \mathbb{R} \setminus A$ is true, that is, $\sup K \in A^c$. Since A is closed in $\mathbb{R}$ by assumption, it follows with the Characterization of closed sets in $\mathbb{R}$ that there exists a particular real number $\bar{\varepsilon} >_{\mathbb{R}} 0$ with $V_{\bar{\varepsilon}}(\sup K) \cap A = \emptyset$. This means by definition of an ε-neighborhood that

$$(\sup K - \bar{\varepsilon}, \sup K + \bar{\varepsilon}) \cap A = \emptyset \tag{11.896}$$

holds. Next, we observe the truth of $0 <_{\mathbb{R}} \frac{1}{2} <_{\mathbb{R}} 1$, so that an application of the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$ yields $0 <_{\mathbb{R}} \frac{\bar{\varepsilon}}{2} <_{\mathbb{R}} \bar{\varepsilon}$. Then, the $\bar{\varepsilon}/2$-neighborhood of $\sup K$ is included in the $\bar{\varepsilon}$-neighborhood of $\sup K$ in view of (11.463), that is,

$$(\sup K - \frac{\bar{\varepsilon}}{2}, \sup K + \frac{\bar{\varepsilon}}{2}) \subseteq (\sup K - \bar{\varepsilon}, \sup K + \bar{\varepsilon}). \tag{11.897}$$

Applying now (2.67) in connection with the Commutative Law for the intersection of two sets, we can infer from (11.896) and (11.897)

$$(\sup K - \frac{\bar{\varepsilon}}{2}, \sup K + \frac{\bar{\varepsilon}}{2}) \cap A = \emptyset. \tag{11.898}$$

We notice that $\sup K$ is an element of the $\bar{\varepsilon}/2$-neighborhood of $\sup K$ (see Exercise 11.28), so that the inequalities

$$\sup K - \frac{\bar{\varepsilon}}{2} <_{\mathbb{R}} \sup K <_{\mathbb{R}} \sup K + \frac{\bar{\varepsilon}}{2} \tag{11.899}$$

hold by definition of an open interval. Here, the first inequality $\sup K - \bar{\varepsilon}/2 <_{\mathbb{R}} \sup K$ implies with the Supremum Criterion that there exists an element of K, say $\bar{\bar{x}}$, such that $\sup K - \bar{\varepsilon}/2 <_{\mathbb{R}} \bar{\bar{x}}$. Consequently, the definition of the set K in (11.889) gives us a particular set system $\bar{\bar{\mathcal{S}}}_1$ such that

$$\bar{\bar{\mathcal{S}}}_1 \subseteq \mathcal{C}_1 \;\wedge\; \bar{\bar{\mathcal{S}}}_1 \text{ is finite} \;\wedge\; F(\bar{\bar{x}}) \subseteq \bar{\bar{\mathcal{S}}}_1. \tag{11.900}$$

Let us verify that the inclusion

$$F(\sup K - \frac{\bar{\varepsilon}}{2}) \subseteq \bar{\bar{\mathcal{S}}}_1 \tag{11.901}$$

holds, too. Applying the definition of a subset, we take an arbitrary y and assume $y \in F(\sup K - \bar{\varepsilon}/2)$ to be true. Thus, $y \in A \cap (-\infty, \sup K - \bar{\varepsilon}/2]$ holds by definition of the function F in (11.888), and the definition of the intersection of two sets gives us therefore $y \in A$ and $y \in (-\infty, \sup K - \bar{\varepsilon}/2]$. Consequently, $y \leq_{\mathbb{R}} \sup K - \bar{\varepsilon}/2$ holds by definition of a left-unbounded and right-closed interval. In conjunction with the previously established inequality $\sup K - \bar{\varepsilon}/2 <_X \bar{\bar{x}}$, this gives us $y <_{\mathbb{R}} \bar{\bar{x}}$ by means of the Transitivity Formula for $\leq$ and $<$. Then, the disjunction $y <_{\mathbb{R}} \bar{\bar{x}} \vee y = \bar{\bar{x}}$ holds as well, and this implies $y \leq_{\mathbb{R}} \bar{\bar{x}}$ with the definition of an induced irreflexive partial ordering. The preceding interval definition yields therefore $y \in (-\infty, \bar{\bar{x}}]$. In conjunction with the true $y \in A$, this evidently implies $y \in A \cap (-\infty, \bar{\bar{x}}]$ and then $y \in F(\bar{\bar{x}})$. This finding gives us now $y \in \bar{\bar{\mathcal{S}}}_1$ with the inclusion $F(\bar{\bar{x}}) \subseteq \bar{\bar{\mathcal{S}}}_1$ in (11.900). Then, as y was an arbitrary element $F(\sup K - \bar{\varepsilon}/2)$, we may conclude that (11.901) is indeed a true inclusion.

We establish now the equation

$$F(\sup K - \frac{\bar{\varepsilon}}{2}) = F(\sup K + \frac{\bar{\varepsilon}}{2}). \tag{11.902}$$

To do this, we apply the Equality Criterion for sets and prove accordingly the universal sentence

$$\forall x \, (x \in F(\sup K - \frac{\bar{\varepsilon}}{2}) \Leftrightarrow x \in F(\sup K + \frac{\bar{\varepsilon}}{2}) \,). \tag{11.903}$$

We let x be arbitrary, and we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $x \in F(\sup K - \bar{\varepsilon}/2)$ to be true. By definition of the function F in (11.888), this means that $x \in A \cap (-\infty, \sup K - \bar{\varepsilon}/2]$ is true. Consequently, $x \in A$ and $x \in (-\infty, \sup K - \bar{\varepsilon}/2]$ are both true (by definition of the intersection of two sets), with the consequence that the inequality $x \leq_{\mathbb{R}} \sup K - \bar{\varepsilon}/2$ holds by definition of a left-unbounded and right-closed interval in $\mathbb{R}$. This inequality implies now in conjunction with the inequalities (11.899) that $x <_{\mathbb{R}} \sup K + \bar{\varepsilon}/2$ is true (applying the transitivity of the standard linear ordering $<_{\mathbb{R}}$ and then the Transitivity Formula for $\leq$ and $<$). Then, the disjunction of the preceding inequality and $x = \sup K + \bar{\varepsilon}/2$ holds as well, so that $x \leq_{\mathbb{R}} \sup K + \bar{\varepsilon}/2$ follows to be true by definition of an induced irreflexive partial ordering. Evidently, this inequality gives us now $x \in (-\infty, \sup K + \bar{\varepsilon}/2]$, and since $x \in A$ also holds, we thus obtain $x \in A \cap (-\infty, \sup K + \bar{\varepsilon}/2]$, and subsequently the desired $x \in F(\sup K + \bar{\varepsilon}/2)$.

Regarding the second part ('$\Leftarrow$') of the equivalence in (11.903), we assume now $x \in F(\sup K + \bar{\varepsilon}/2)$ to be true. This assumption evidently implies the truth of $x \in A$ and $x \leq_{\mathbb{R}} \sup K + \bar{\varepsilon}/2$ (using the definition of the function

F, the definition of the intersection of two sets, and the definition of a left-unbounded, right-closed interval). Because of (11.898) and the definition of the empty set, we also find

$$x \notin (\sup K - \frac{\bar{\varepsilon}}{2}, \sup K + \frac{\bar{\varepsilon}}{2}) \cap A$$

to be true. This negation implies with the definition of the intersection of two sets and with De Morgan's Law for the conjunction that the disjunction

$$x \notin (\sup K - \frac{\bar{\varepsilon}}{2}, \sup K + \frac{\bar{\varepsilon}}{2}) \ \vee \ x \notin A$$

is true. Since we previously found $x \in A$ to be true, the second part $x \notin A$ of this disjunction is false. Thus, its first part is true, which we can write also as

$$\neg \left(\sup K - \frac{\bar{\varepsilon}}{2} <_{\mathbb{R}} x \wedge x <_{\mathbb{R}} \sup K + \frac{\bar{\varepsilon}}{2}\right),$$

using the definition of an open interval in $\mathbb{R}$. Another application of De Morgan's Law for the conjunction gives us then the true disjunction

$$x \leq_{\mathbb{R}} \sup K - \frac{\bar{\varepsilon}}{2} \ \vee \ \sup K + \frac{\bar{\varepsilon}}{2} \leq_{\mathbb{R}} x, \tag{11.904}$$

where we used also the Negation Formula for $<$. Here, we can prove by contradiction that the second part of this disjunction is false, in other words, that the negation of its second part is true. Assuming the negation of that negation to be true, we thus have $\sup K + \bar{\varepsilon}/2 \leq_{\mathbb{R}} x$ according to the Double Negation Law. Recalling that $x \leq_{\mathbb{R}} \sup K + \bar{\varepsilon}/2$ also holds, we obtain then the equation

$$x = \sup K + \bar{\varepsilon}/2 \tag{11.905}$$

by virtue of the antisymmetry of the standard total ordering of $\mathbb{R}$. Since the previously established inequality $\bar{\varepsilon}/2 <_{\mathbb{R}} \bar{\varepsilon}$ implies $\sup K + \bar{\varepsilon}/2 <_{\mathbb{R}} \sup K + \bar{\varepsilon}$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$, we obtain after substitution based on the preceding equation also

$$x <_{\mathbb{R}} \sup K + \bar{\varepsilon}. \tag{11.906}$$

Furthermore, the inequalities (11.899) imply $\sup K - \bar{\varepsilon}/2 <_{\mathbb{R}} \sup K + \bar{\varepsilon}/2$ with the transitivity of $<_{\mathbb{R}}$, so that substitution based on (11.905) yields $\sup K - \bar{\varepsilon}/2 <_{\mathbb{R}} x$. Now, the aforementioned inequality $\bar{\varepsilon}/2 <_{\mathbb{R}} \bar{\varepsilon}$ implies with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$ also $-\bar{\varepsilon} <_{\mathbb{R}} -\bar{\varepsilon}/2$ and subsequently $\sup K - \bar{\varepsilon} <_{\mathbb{R}} \sup K - \bar{\varepsilon}/2$, so that

$$\sup K - \bar{\varepsilon} <_{\mathbb{R}} x \tag{11.907}$$

follows to be true because of the transitivity of $<_{\mathbb{R}}$. The conjunction of (11.907) and (11.906) implies then $x \in (\sup K - \bar{\varepsilon}, \sup K + \bar{\varepsilon})$ with the definition of an open interval, and as $x \in A$ is also true, we have therefore $x \in (\sup K - \bar{\varepsilon}, \sup K + \bar{\varepsilon}) \cap A$, in contradiction to the previous finding (11.896) that the intersection is empty. We thus completed the proof of the falseness of the second part of the disjunction (11.904), so that its first part $x \leq_{\mathbb{R}} \sup K - \bar{\varepsilon}/2$ must be true. This inequality gives us now evidently $x \in (-\infty, \sup K - \bar{\varepsilon}/2]$, and therefore $x \in A \cap (-\infty, \sup K - \bar{\varepsilon}/2]$ in view of the true $x \in A$. Thus, the desired consequent $x \in F(\sup K - \bar{\varepsilon}/2)$ of the implication '$\Leftarrow$' in (11.903) turns out to be true, so that the proof of the equivalence is now complete. Because x is arbitrary, we may then infer from the truth of that equivalence the truth of the universal sentence (11.903), and consequently the truth of the equality (11.902).

Applying now substitution to the inclusion (11.901) based on that equation, we get

$$F(\sup K + \frac{\bar{\varepsilon}}{2}) \subseteq \bar{\bar{\mathcal{S}}}_1. \tag{11.908}$$

The conjunction of the first two parts of (11.900) and (11.908) implies now (by definition of the set K) that $\sup K + \bar{\varepsilon}/2 \in K$ is true. In view of the inequality $\sup K <_{\mathbb{R}} \sup K + \bar{\varepsilon}/2$ in (11.899), we thus found an element of K which is greater than the supremum of K. This finding evidently implies that $\sup K$ is not an upper bound for K, but this contradicts the fact that $\sup K$ constitutes an upper bound for K by definition. Thus, it is also true for the second sub-case that K is not bounded from above, which means by definition that the negation

$$\neg \exists u \, (u \in \mathbb{R} \wedge \forall x \, (x \in K \Rightarrow x \leq_{\mathbb{R}} u))$$

holds. Due to the Negation Law for existential conjunction, this implies the truth of the universal sentence

$$\forall u \, (u \in \mathbb{R} \Rightarrow \neg \forall x \, (x \in K \Rightarrow x \leq_{\mathbb{R}} u)).$$

Recalling that $\max A$ exists, which constitutes an element of $\mathbb{R}$ by definition of a maximum, we obtain then the true negation

$$\neg \forall x \, (x \in K \Rightarrow x \leq_{\mathbb{R}} \max A).$$

Because of the Negation Law for universal implications, there exists then an element of K, say x', such that the negation $\neg x' \leq_{\mathbb{R}} \max A$ is satisfied. Consequently, we find the inequality $\max A <_{\mathbb{R}} x'$ by means of the Negation Formula for $\leq$. Moreover, $x' \in K$ implies (according to the specification of that set) the existence of a particular set system $\mathcal{S}'$ satisfying

$$\mathcal{S}_1' \subseteq \mathcal{C}_1 \;\wedge\; \mathcal{S}_1' \text{ is finite} \;\wedge\; F(x') \subseteq \bigcup \mathcal{S}_1'. \tag{11.909}$$

Here, we have $F(x') = A \cap (-\infty, x']$ according to the definition of the function F, and we can show that this value is identical with the set A. For this purpose, we apply the Equality Criterion for sets, verifying

$$\forall x\,(x \in F(x') \Leftrightarrow x \in A). \tag{11.910}$$

We let x be arbitrary and assume first $x \in F(x')$ to hold. We therefore obtain evidently $x \in A \cap (-\infty, x']$ and then $x \in A \wedge x \in (-\infty, x']$. Thus, the first part $x \in A$ of that conjunction proves the first part ('$\Rightarrow$') of the equivalence. To prove the second part ('$\Leftarrow$'), we conversely assume $x \in A$ to be true. Since $\max A$ is an upper bound for A, we therefore have $x \leq_{\mathbb{R}} \max A$. This yields in connection with the previously established $\max A <_{\mathbb{R}} x'$ also the inequality $x <_{\mathbb{R}} x'$ (using the Transitivity Formula for $\leq$ and $<$). Clearly, the inequality $x \leq_{\mathbb{R}} x'$ is then true as well, so that x turns out to be an element of the left-unbounded and right-closed interval $(-\infty, x']$. Due to the assumption $x \in A$, we therefore find $x \in A \cap (-\infty, x']$, thus $x \in F(x')$ as desired. As x was arbitrary, we may infer from these findings the truth of the universal sentence (11.910) and consequently the truth of the proposed equality $F(x') = A$. Substitution into (11.909) gives us subsequently

$$\mathcal{S}_1' \subseteq \mathcal{C}_1 \ \wedge \ \mathcal{S}_1' \text{ is finite } \wedge \ A \subseteq \bigcup \mathcal{S}_1',$$

which demonstrates the truth of the desired existential sentence (11.887). Because $\mathcal{C}_1$ was initially arbitrary, we may therefore conclude that A is compact. Since A was also arbitrary, we are now finally in a position to conclude that the theorem is true. □

The Heine-Borel Theorem for $\mathbb{R}$ allows us to directly reformulate the Characterization of nonempty closed intervals in $\mathbb{R}$ in terms of compactness rather than in terms of boundedness and closedness.

Corollary 11.113. *It is true that a subset $I \subseteq \mathbb{R}$ constitutes a closed interval $[a, b]$ in $\mathbb{R}$ for some $a, b \in \mathbb{R}$ with $a \leq_{\mathbb{R}} b$ iff*

1. *I is nonempty,*
2. *I is a compact set in $\mathbb{R}$ with respect to the standard topology $\mathcal{O}_{<_{\mathbb{R}}}$,*
3. *the topological subspace $(I, \mathcal{O}_{<_{\mathbb{R}}}|I)$ is connected.*

Theorem 11.114 (Compactness of closed sets in compact topological spaces). *It is true for any compact topological space $(\Omega, \mathcal{O})$ and for any closed set A in Ω that A is a compact set in Ω (with respect to the topology $\mathcal{O}$).*

Proof. We take arbitrary sets Ω, $\mathcal{O}$ and A, assuming that $(\Omega, \mathcal{O})$ is a compact topological space and assuming A to be closed in Ω with respect to the topology $\mathcal{O}$. To prove that A is also compact in Ω with respect to $\mathcal{O}$, we note that A, as a closed set in Ω, constitutes a subset of Ω, and we apply now the Characterization of compact sets, letting $\mathcal{C}_1$ be an arbitrary set such that the inclusions

$$\mathcal{C}_1 \subseteq \mathcal{O} \;\wedge\; A \subseteq \bigcup \mathcal{C}_1 \tag{11.911}$$

are satisfied. To establish the required existential sentence

$$\exists \mathcal{S}_1 \, (\mathcal{S}_1 \subseteq \mathcal{C}_1 \;\wedge\; \mathcal{S}_1 \text{ is finite } \;\wedge\; A \subseteq \bigcup \mathcal{S}_1), \tag{11.912}$$

we consider the union $\mathcal{C}_1 \cup \{A^c\}$ and prove that this set is an open covering of Ω, that is,

$$\mathcal{C}_1 \cup \{A^c\} \subseteq \mathcal{P}(\Omega) \;\wedge\; \bigcup(\mathcal{C}_1 \cup \{A^c\}) = \Omega \;\wedge\; \mathcal{C}_1 \cup \{A^c\} \subseteq \mathcal{O}. \tag{11.913}$$

Addressing the first inclusion, we apply the definition of a subset and let accordingly $X \in \mathcal{C}_1 \cup \{A^c\}$ be arbitrary, so that the definition of the union of two sets yields the true disjunction $X \in \mathcal{C}_1 \vee X \in \{A^c\}$. This disjunction allows us to prove $X \in \mathcal{P}(\Omega)$ by cases. The first case $X \in \mathcal{C}_1$ evidently implies with the first inclusion $\mathcal{C}_1 \subseteq \mathcal{O}$ in (11.911) $X \in \mathcal{O}$, which in turn implies the desired $X \in \mathcal{P}(\Omega)$ with Property 1 of a topology on Ω. The second case $X \in \{A^c\}$ implies $X = A^c$ with (2.169), where A^c is a subset of Ω because of $A \subseteq \Omega$ and (2.137). Consequently, X turns out to be an element of the power set of Ω, by definition. We thus showed that $X \in \mathcal{C}_1 \cup \{A^c\}$ implies $X \in \mathcal{P}(\Omega)$ in any case, and since X is arbitrary, we may now infer from the truth of that implication the truth of the inclusion $\mathcal{C}_1 \cup \{A^c\} \subseteq \mathcal{P}(\Omega)$.

Regarding the equation $\bigcup(\mathcal{C}_1 \cup \{A^c\}) = \Omega$ in (11.913), we apply the Axiom of Extension and establish the two inclusions

$$\bigcup(\mathcal{C}_1 \cup \{A^c\}) \subseteq \Omega \;\wedge\; \Omega \subseteq \bigcup(\mathcal{C}_1 \cup \{A^c\}). \tag{11.914}$$

Since the already established inclusion $\mathcal{C}_1 \cup \{A^c\} \subseteq \mathcal{P}(\Omega)$ implies $\bigcup(\mathcal{C}_1 \cup \{A^c\}) \in \mathcal{P}(\Omega)$ with (3.22), we obtain $\bigcup(\mathcal{C}_1 \cup \{A^c\}) \subseteq \Omega$ with the definition of a power set. Concerning the second inclusion in (11.914), we let ω be arbitrary, we assume $\omega \in \Omega$ to be true, and we show that $\omega \in \bigcup(\mathcal{C}_1 \cup \{A^c\}$ is implied. Observing the truth of $\Omega = A \cup A^c$ in light of (2.257), we obtain via substitution $\omega \in A \cup A^c$ and therefore evidently $\omega \in A \vee \omega \in A^c$. In case of $\omega \in A$, the second inclusion $A \subseteq \bigcup \mathcal{C}_1$ in (11.911) gives us $\omega \in \bigcup \mathcal{C}_1$, with the consequence that $\omega \in \bar{Y}$ is true for some particular set $\bar{Y} \in \mathcal{C}_1$.

Then, the disjunction $\bar{Y} \in \mathcal{C}_1 \vee \bar{Y} \in \{A^c\}$ holds as well, so that $\bar{Y}$ turns out to be in the union of the two sets $\mathcal{C}_1$ and $\{A^c\}$. The conjunction of this finding $\bar{Y} \in \mathcal{C}_1 \cup \{A^c\}$ and $\omega \in \bar{Y}$ implies subsequently $\omega \in \bigcup(\mathcal{C}_1 \cup \{A^c\})$ with the definition of the union of a set system, as desired. In the other case of $\omega \in A^c$, we note that the evident fact $A^c \in \{A^c\}$ implies the truth of the disjunction $A^c \in \mathcal{C}_1 \vee A^c \in \{A^c\}$, with the consequence that $A^c \in \mathcal{C}_1 \cup \{A^c\}$. In conjunction with $\omega \in A^c$, this gives us again the desired consequent $\omega \in \bigcup(\mathcal{C}_1 \cup \{A^c\})$. Since ω was arbitrary, we may therefore conclude that the inclusion $\Omega \subseteq \bigcup(\mathcal{C}_1 \cup \{A^c\})$ also holds, so that the proof of the conjunction (11.914) is now complete. Consequently, the equality $\bigcup(\mathcal{C}_1 \cup \{A^c\}) = \Omega$ is true, and it remains for us to establish the inclusion $\mathcal{C}_1 \cup \{A^c\} \subseteq \mathcal{O}$ in (11.913).

Because A is by assumption a closed set in Ω, it is by definition true that A^c is an open set in Ω, that is, $A^c \in \mathcal{O}$. Consequently, the inclusion $\{A^c\} \subseteq \mathcal{O}$ holds according to (2.184). In connection with the true inclusion $\mathcal{C}_1 \subseteq \mathcal{O}$, this gives us indeed $\mathcal{C}_1 \cup \{A^c\} \subseteq \mathcal{O}$ by virtue of (2.252), which finding completes the verification of (11.914). The fact that $\mathcal{C}_1 \cup \{A^c\}$ constitutes thus an open covering of Ω implies now in connection with the assumption of $(\Omega, \mathcal{O})$ as a compact topological space that there exists a particular finite covering $\bar{\mathcal{S}}$ of Ω included in $\mathcal{C}_1 \cup \{A^c\}$, that is,

$$\bar{\mathcal{S}} \subseteq \mathcal{C}_1 \cup \{A^c\} \ \wedge\ \bar{\mathcal{S}} \text{ is finite} \ \wedge\ \bar{\mathcal{S}} \subseteq \mathcal{P}(\Omega) \ \wedge\ \bigcup \bar{\mathcal{S}} = \Omega. \tag{11.915}$$

We consider in the following the two cases $A^c \in \bar{\mathcal{S}}$ and $A^c \notin \bar{\mathcal{S}}$ to complete the proof of the desired existential sentence (11.912). In the first case $A^c \in \bar{\mathcal{S}}$, we show that the set difference $\bar{\mathcal{S}} \setminus \{A^c\}$ has the required properties

$$\bar{\mathcal{S}} \setminus \{A^c\} \subseteq \mathcal{C}_1 \ \wedge\ \bar{\mathcal{S}} \setminus \{A^c\} \text{ is finite} \ \wedge\ A \subseteq \bigcup(\bar{\mathcal{S}} \setminus \{A^c\}). \tag{11.916}$$

We prove the inclusion $\bar{\mathcal{S}}_1 \subseteq \mathcal{C}_1$ by taking an arbitrary set X and by assuming $X \in \bar{\mathcal{S}} \setminus \{A^c\}$ to be true. By definition of a set difference, we therefore find $X \in \bar{\mathcal{S}}$ and $X \notin \{A^c\}$ to be true. The former finding further implies $X \in \mathcal{C}_1 \cup \{A^c\}$ with the first inclusion in (11.915), so that the disjunction $X \in \mathcal{C}_1 \vee X \in \{A^c\}$ holds (by definition of the union of two sets). Because of the preceding finding $X \notin \{A^c\}$, the second part of the disjunction is false, so that its first part $X \in \mathcal{C}_1$ is true. As X was an arbitrary element of $\bar{\mathcal{S}} \setminus \{A^c\}$, we may therefore conclude that $\bar{\mathcal{S}} \setminus \{A^c\}$ is indeed a subset of $\mathcal{C}_1$. Next, we note that the set difference $\bar{\mathcal{S}} \setminus \{A^c\}$ is a subset of the finite set $\bar{\mathcal{S}}$ in view of (2.125) and (11.915), so that the set difference $\bar{\mathcal{S}} \setminus \{A^c\}$ is itself finite according to (4.610). Concerning the third part $A \subseteq \bigcup(\bar{\mathcal{S}} \setminus \{A^c\})$ of the multiple conjunction (11.916), we let now ω be arbitrary, assuming that $\omega \in A$ holds. This yields $\omega \in \Omega$ due to the inclusion $A \subseteq \Omega$, and since Ω

is identical with $\bigcup \bar{\mathcal{S}}$ according to (11.915), we thus have $\omega \in \bigcup \bar{\mathcal{S}}$. Then, there exists (by definition of the union of a set system) a particular element $\bar{Z} \in \bar{\mathcal{S}}$ with $\omega \in \bar{Z}$. Let us observe here that $\omega \in A$ implies also $\omega \in (A^c)^c$ with (2.136) and subsequently (in connection with $A \subseteq \Omega$ and $\omega \in \Omega$) also $\omega \notin A^c$ because of (2.132). The previous two findings $\omega \in \bar{Z}$ and $\omega \notin A^c$ gives us now evidently $\bar{Z} \neq A^c$ with (2.23), so that $\bar{Z} \notin \{A^c\}$ follows to be true with (2.169) and the Law of Contraposition. This yields in conjunction with the previously obtained $\bar{Z} \in \bar{\mathcal{S}}$ the true sentence $\bar{Z} \in \bar{\mathcal{S}} \setminus \{A^c\}$ by means of the definition of a subset. Then, because of $\omega \in \bar{Z}$, we finally arrive at $\omega \in \bigcup(\bar{\mathcal{S}} \setminus \{A^c\})$ (applying again the definition of the union of a set system). As ω was arbitrary in A, the inclusion $A \subseteq \bigcup(\bar{\mathcal{S}} \setminus \{A^c\})$ follows therefore to be true, so that the proof of the multiple conjunction (11.916) is complete. Having found the particular set $\bar{\mathcal{S}} \setminus \{A^c\}$ with these properties, we see then that the existential sentence (11.912) is true for the current first case.

In the second case $A^c \notin \bar{\mathcal{S}}$, we can show that $\bar{\mathcal{S}}$ itself satisfies

$$\bar{\mathcal{S}} \subseteq \mathcal{C}_1 \ \wedge \ \bar{\mathcal{S}} \text{ is finite} \ \wedge \ A \subseteq \bigcup \bar{\mathcal{S}}. \tag{11.917}$$

First we let X be an arbitrary set such that $X \in \bar{\mathcal{S}}$ is true. The first inclusion in (11.915) gives us then $X \in \mathcal{C}_1 \cup \{A^c\}$, with the evident consequence that the disjunction $X \in \mathcal{C}_1 \vee X \in \{A^c\}$ holds. Now, $X \in \bar{\mathcal{S}}$ and the current case assumption $A^c \notin \bar{\mathcal{S}}$ imply $X \neq A^c$ with (2.4), so that $X \notin \{A^c\}$ is clearly true. This means that the second part of the preceding disjunction is false, and therefore its first part $X \in \mathcal{C}_1$ must be true. Because this finding was implied by $X \in \bar{\mathcal{S}}$ for any arbitrary set X, we may infer from this that the first inclusion $\bar{\mathcal{S}} \subseteq \mathcal{C}_1$ in (11.917) is true. Moreover, we already know from (11.915) that $\bar{\mathcal{S}}$ is a finite set, so that the second part of the multiple conjunction (11.917) also holds. Finally, letting ω be arbitrary and assuming $\omega \in A$ to be true, we obtain as in the first case $\omega \in \Omega$ $[= \bigcup \bar{\mathcal{S}}]$, which shows that $\omega \in \bigcup \bar{\mathcal{S}}$ is implied by $\omega \in A$. Since ω is arbitrary, we therefore conclude that the inclusion $A \subseteq \bigcup \bar{\mathcal{S}}$ holds, too. Thus, $\bar{\mathcal{S}}$ satisfies indeed (11.917), demonstrating the truth of the existential sentence (11.912) for the second case.

As the set $\mathcal{C}_1$ was arbitrary, it follows now that A is a compact set in Ω. Because Ω, $\mathcal{O}$ and A were initially also arbitrary, we therefore conclude that the theorem is true. □

Theorem 11.115 (Compactness of closed sets in compact topological subspaces). *It is true for any topological space $(\Omega, \mathcal{O})$, any compact topological subspace $(\Omega_1, \mathcal{O}|\Omega_1)$ of $(\Omega, \mathcal{O})$ and any closed set $A \subseteq \Omega_1$ in Ω with respect to the topology $\mathcal{O}$ that A is a compact set in Ω with respect to the topology $\mathcal{O}$.*

Exercise 11.48. Prove Theorem 11.115.

(Hint: Proceed similarly as in the proof of Theorem 11.114, replacing (11.913) by a condition based on the Characterization of compact sets.

Corollary 11.116. *It is true for any Hausdorff space $(\Omega, \mathcal{O})$ that the intersection of any closed set A in Ω and any compact set B in Ω is itself compact in Ω (with respect to the topology $\mathcal{O}$).*

Proof. We let Ω, $\mathcal{O}$, A and B be arbitrary sets such that $(\Omega, \mathcal{O})$ is a Hausdorff space, such that A is a closed set in Ω (with respect to $\mathcal{O}$), and such that B is a compact set in Ω (with respect to $\mathcal{O}$). Due to the Closedness of compact subspaces of Hausdorff spaces, it is then true that B is closed in Ω. Furthermore, since the pair $\{A, B\}$ evidently constitutes a nonempty set system, it follows from the closedness of A and B that the intersection $\bigcap\{A, B\} = A \cap B$ is also closed in Ω, according to Proposition 11.47. Since the inclusion $A \subseteq B$ is true according to (2.74) and since the compact set B in Ω gives rise to the compact topological subspace $(B, \mathcal{O}|B)$ of $(\Omega, \mathcal{O})$ by definition, it follows with Theorem 11.115 that $A \cap B$ is a compact set in Ω with respect to the topology $\mathcal{O}$, as claimed. As the sets Ω, $\mathcal{O}$, A and B were initially arbitrary, we therefore conclude that the corollary holds. $\square$

11.10. Borel σ-Algebras

Definition 11.37 (Borel σ-algebra, Borel set, Borel measurable space). For any topological space $(\Omega, \mathcal{O})$ we say that a set $\mathcal{B}(\Omega)$ is the *Borel σ-algebra* on Ω with respect to $\mathcal{O}$ iff $\mathcal{B}(\Omega)$ is the σ-algebra generated by the topology $\mathcal{O}$, i.e. iff

$$\mathcal{B}(\Omega) = \mathcal{A}(\mathcal{O}). \tag{11.918}$$

We then call every element of $\mathcal{B}$ a *Borel set* of Ω with respect to $\mathcal{O}$ and the measurable space

$$(\Omega, \mathcal{B}(\Omega)) = (\Omega, \mathcal{A}(\mathcal{O})) \tag{11.919}$$

the *Borel measurable space* with respect to $\mathcal{O}$.

Note 11.40. Using a topology $\mathcal{O}$ on a set Ω as a system to generate the Borel σ-algebra $\mathcal{B}(\Omega) = \mathcal{A}(\mathcal{O})$, we thus have by definition of a generated σ-algebra that $\mathcal{B}(\Omega)$ is a σ-algebra on Ω. Therefore, by Property 2 of a σ-algebra,

$$\Omega \in \mathcal{B}(\Omega). \tag{11.920}$$

Furthermore,, since any σ-algebra – being a semiring of sets according to Proposition 11.30 – contains the empty set, we have for any topological space $(\Omega, \mathcal{O})$

$$\emptyset \in \mathcal{B}(\Omega). \tag{11.921}$$

Lemma 11.117. *It is true for any second-countable topological space* $(\Omega, \mathcal{O})$ *that every countable basis* $\mathcal{K}_\Omega^{(c)}$ *that generates the topology* $\mathcal{O}$ *generates also the Borel* σ*-algebra on* Ω *with respect to* $\mathcal{O}$*, i.e.*

$$\forall \mathcal{K}_\Omega^{(c)} \left(\left[\mathcal{K}_\Omega^{(c)} \text{ is a countable basis } \wedge \mathcal{O} = \mathcal{O}(\mathcal{K}_\Omega^{(c)}) \right] \Rightarrow \mathcal{B}(\Omega) = \mathcal{A}(\mathcal{K}_\Omega^{(c)}) \right). \tag{11.922}$$

Proof. We take arbitrary sets Ω and $\mathcal{O}$, assuming that $(\Omega, \mathcal{O})$ constitutes a second-countable topological space. This means that there exists a countable basis $\mathcal{K}_\Omega^{(c)}$ for a topology on Ω which generates the topology $\mathcal{O}$. We now take an arbitrary countable basis $\mathcal{K}_\Omega^{(c)}$ generating $\mathcal{O}$, and we prove the equation $\mathcal{B}(\Omega) = \mathcal{A}(\mathcal{K}_\Omega^{(c)})$ by means of the Axiom of Extension, establishing the truth of the inclusions

$$\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{K}_\Omega^{(c)}) \wedge \mathcal{A}(\mathcal{K}_\Omega^{(c)}) \subseteq \mathcal{A}(\mathcal{O}). \tag{11.923}$$

Concerning the first part of this conjunction, we verify first the inclusion

$$\mathcal{O} \subseteq \mathcal{A}(\mathcal{K}_\Omega^{(c)}). \tag{11.924}$$

For this purpose, we apply the definition of a subset and demonstrate the truth of the equivalent universal sentence

$$\forall U \, (U \in \mathcal{O} \Rightarrow U \in \mathcal{A}(\mathcal{K}_\Omega^{(c)})), \tag{11.925}$$

letting $U \in \mathcal{O}$ be arbitrary. We consider now the two cases $U = \emptyset$ and $U \neq \emptyset$. In the first case $U = \emptyset$, we observe that the σ-algebra $\mathcal{A}(\mathcal{K}_\Omega^{(c)})$ constitutes a semiring of sets (see Proposition 11.30), so that the empty set U is contained in it. In the second case $U \neq \emptyset$, the open set U in the topology $\mathcal{O}$ can be written as the union $\bigcup_{n=1}^\infty B_n$ of some sequence $B = (B_n)_{n \in \mathbb{N}_+}$ of basis elements in $\mathcal{K}_\Omega^{(c)}$ (see Exercise 11.32). Letting $\bar{B} = (\bar{B}_n)_{n \in \mathbb{N}_+}$ be such a sequence, we can view it as the function $\bar{B} : \mathbb{N}_+ \to \mathcal{K}_\Omega^{(c)}$. Here, the basis is by definition included in the generated σ-algebra $\mathcal{A}(\mathcal{K}_\Omega^{(c)})$, so that the latter is also a codomain of the function $\bar{B}$ due to (2.13). We thus have $\bar{B} : \mathbb{N}_+ \to \mathcal{A}(\mathcal{K}_\Omega^{(c)})$, which implies with Property 3 of a σ-algebra

$$[U =] \quad \bigcup_{n=1}^\infty \bar{B}_n \in \mathcal{A}(\mathcal{K}_\Omega^{(c)}),$$

and therefore the desired consequent of the implication in (11.925). Because U was arbitrary, the universal sentence (11.925) follows then to be true, so that the inclusion (11.924) holds indeed. Thus, $\mathcal{A}(\mathcal{K}_\Omega^{(c)})$ is a σ-algebra that includes the set system $\mathcal{O}$; as $\mathcal{A}(\mathcal{O})$ is the smallest σ-algebra that includes $\mathcal{O}$ (in the sense of Theorem 11.35), the inclusion given by the first part of the conjunction (11.923) turns out to be true.

Concerning the second inclusion, we note that the topology $\mathcal{O}$ includes its generating basis $\mathcal{K}_\Omega^{(c)}$. Together with the previously found inclusion $\mathcal{O} \subseteq \mathcal{A}(\mathcal{O})$, this yields the new inclusion $\mathcal{K}_\Omega^{(c)}\mathcal{A}(\mathcal{O})$ with (2.13). Because $\mathcal{A}(\mathcal{K}_\Omega^{(c)})$ is the smallest σ-algebra including the set system $\mathcal{K}_\Omega^{(c)}$, we obtain now also the inclusion given by the second part of the conjunction (11.923).

Consequently, that conjunction yields the equation

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{K}_\Omega^{(c)}),$$

where $\mathcal{B}(\Omega) = \mathcal{A}(\mathcal{O})$ holds by definition of the Borel σ-algebra on Ω. Then, the implication in (11.922) follows to be true by means of substitution. Here, $\mathcal{K}_\Omega^{(c)}$, Ω and $\mathcal{O}$ were arbitrary, allowing us to infer from the truth of that implication the truth of the stated lemma. □

Proposition 11.118. *It is true for any topological space* $(\Omega, \mathcal{O})$ *that any open and any closed set in* Ω *with respect to the topology* $\mathcal{O}$ *is contained in the Borel* σ*-algebra on* Ω *with respect to* $\mathcal{O}$*.*

Proof. First we let Ω and $\mathcal{O}$ be arbitrary sets such that $(\Omega, \mathcal{O})$ constitutes a topological space. Next, we take an arbitrary open set $U \in \mathcal{O}$ and observe the truth of

$$\mathcal{O} \subseteq \mathcal{A}(\mathcal{O}) \quad [= \mathcal{B}(\Omega)], \tag{11.926}$$

in light of the definition of a generated σ-algebra and the definition of the Borel σ-algebra. We therefore obtain by definition of a subset $U \in \mathcal{B}(\Omega)$, and as U was arbitrary, we may infer from this finding the truth of the first stated universal sentence.

Letting now A be an arbitrary closed set in Ω with respect to $\mathcal{O}$, we thus have by definition $A^c \in \mathcal{O}$. By virtue of the inclusion in (11.926), we obtain $A^c \in \mathcal{B}(\Omega)$ by means of the definition of a subset and consequently

$$[A =]\ (A^c)^c \in \mathcal{B}(\Omega)$$

with (2.136) and Property 4 of a σ-algebra. This gives us the desired $A \in \mathcal{B}(\Omega)$, and since A is arbitrary, we may therefore conclude that the second proposed universal sentence holds, too. □

Theorem 11.119 (Compatibility of subspace topologies and trace σ-algebras). *It is true for any topological space* $(\Omega, \mathcal{O})$ *and any subset*

$\Omega_1 \subseteq \Omega$ that the trace of the Borel σ-algebra on Ω with respect to $\mathcal{O}$ in Ω_1 is the σ-algebra generated by subspace topology of $\mathcal{O}$ in Ω_1, i.e.

$$\mathcal{A}(\mathcal{O})|\Omega_1 = \mathcal{A}(\mathcal{O}|\Omega_1). \tag{11.927}$$

Proof. We let Ω, $\mathcal{O}$ and Ω_1 be arbitrary sets, assuming $(\Omega, \mathcal{O})$ to be a topological space and Ω_1 to be included in Ω. Thus, the Borel σ-algebra $\mathcal{A}(\mathcal{O})$ on Ω, the trace σ-algebra $\mathcal{A}(\mathcal{O})|\Omega_1$ on Ω_1, the subspace topology $\mathcal{O}|\Omega_1$ on Ω_1, and the generated σ-algebra $\mathcal{A}(\mathcal{O}|\Omega_1)$ on Ω_1 are all defined. We now prove the stated equation via the Axiom of Extension, by establishing the conjunction of inclusions

$$\mathcal{A}(\mathcal{O})|\Omega_1 \subseteq \mathcal{A}(\mathcal{O}|\Omega_1) \;\wedge\; \mathcal{A}(\mathcal{O}|\Omega_1) \subseteq \mathcal{A}(\mathcal{O})|\Omega_1. \tag{11.928}$$

We prove the first inclusion by means of the definition of a subset, i.e. by demonstrating the truth of the universal sentence

$$\forall X\,(X \in \mathcal{A}(\mathcal{O})|\Omega_1 \Rightarrow X \in \mathcal{A}(\mathcal{O}|\Omega_1)). \tag{11.929}$$

We take an arbitrary set $\bar{X} \in \mathcal{A}(\mathcal{O})|\Omega_1$, so that there exists – by definition of a trace σ-algebra – a particular set $\bar{A} \in \mathcal{A}(\mathcal{O})$ with $\Omega_1 \cap \bar{A} = \bar{X}$. Let us now apply Proposition 11.32 to define the σ-algebra $\mathcal{A}_\Omega$ on Ω consisting of all subsets of Ω whose intersections with Ω_1 are in $\mathcal{A}(\mathcal{O}|\Omega_1)$, in the sense that

$$\forall X\,(X \in \mathcal{A}_\Omega \Leftrightarrow [X \in \mathcal{P}(\Omega) \;\wedge\; \Omega_1 \cap X \in \mathcal{A}(\mathcal{O}|\Omega_1)]). \tag{11.930}$$

We now establish the inclusion $\mathcal{O} \subseteq \mathcal{A}_\Omega$ by applying the definition of a subset, i.e. by proving the equivalent universal sentence

$$\forall U\,(U \in \mathcal{O} \Rightarrow U \in \mathcal{A}_\Omega). \tag{11.931}$$

Letting $\bar{U} \in \mathcal{O}$ be arbitrary, we observe on the one hand the truth of $\mathcal{O} \subseteq \mathcal{P}(\Omega)$ in light of Property 1 of a topology on Ω. Then, $\bar{U} \in \mathcal{P}(\Omega)$ follows to be true by definition of a subset. On the other hand, we notice the truth of the equation $\Omega_1 \cap \bar{U} = \Omega_1 \cap \bar{U}$, which demonstrates in conjunction with the assumed $\bar{U} \in \mathcal{O}$ the truth of the existential sentence

$$\exists U\,(U \in \mathcal{O} \;\wedge\; \Omega_1 \cap U = \Omega_1 \cap \bar{U}).$$

This existential sentence implies now $\Omega_1 \cap \bar{U} \in \mathcal{O}|\Omega_1$ by definition of a subspace topology. Because the inclusion $\mathcal{O}|\Omega_1 \subseteq \mathcal{A}(\mathcal{O}|\Omega_1)$ holds by definition of a generated σ-algebra, the preceding finding gives us then $\Omega_1 \cap \bar{U} \in \mathcal{A}(\mathcal{O}|\Omega_1)$ with the definition of a subset. In connection with the previously obtained $\bar{U} \in \mathcal{P}(\Omega)$, this further implies $\Omega_1 \cap \bar{U} \in \mathcal{A}_\Omega$ with (11.930), which proves the implication in (11.931). Here, $\bar{U}$ was arbitrary, so that the universal sentence (11.931) follows to be true, completing the proof of the desired inclusion $\mathcal{O} \subseteq \mathcal{A}_\Omega$. Thus, $\mathcal{A}_\Omega$ is a σ-algebra on Ω that includes $\mathcal{O}$, which set system generates the σ-algebra $\mathcal{A}(\mathcal{O})$. We therefore obtain the inclusion $\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}_\Omega$ with Theorem 11.35c), so that the previously established $\bar{A} \in \mathcal{A}(\mathcal{O})$ implies $\bar{A} \in \mathcal{A}_\Omega$ with the definition of a subset. This finding in turns implies $\Omega_1 \cap \bar{A} \in \mathcal{A}(\mathcal{O}|\Omega_1)$ with (11.930) and then $\bar{X} \in \mathcal{A}(\mathcal{O}|\Omega_1)$ via substitution based on the previously found equation $\Omega_1 \cap \bar{A} = \bar{X}$. Thus, the proof of the implication in (11.929) is complete, and since $\bar{X}$ was arbitrary, we may now infer from the truth of that implication the truth of the universal sentence (11.929), and consequently the truth of the first inclusion in (11.928).

Regarding the second inclusion, we establish first $\mathcal{O}|\Omega_1 \subseteq \mathcal{A}(\mathcal{O})|\Omega_1$, by proving the equivalent universal sentence

$$\forall V \, (V \in \mathcal{O}|\Omega_1 \Rightarrow V \in \mathcal{A}(\mathcal{O})|\Omega_1). \tag{11.932}$$

We let $V \in \mathcal{O}|\Omega_1$ be arbitrary, so that the definition of a subspace topology gives us a particular set $\bar{U} \in \mathcal{O}$ such that $\Omega_1 \cap \bar{U} = V$. Since the σ-algebra $\mathcal{A}(\mathcal{O})$ includes its generating system $\mathcal{O}$, we obtain $\bar{U} \in \mathcal{A}(\mathcal{O})$. These findings show us that the existential sentence

$$\exists A \, (A \in \mathcal{A}(\mathcal{O}) \, \wedge \, \Omega_1 \cap A = V)$$

is true, so that the desired consequent of the implication in (11.932) follows to be true by definition of a trace σ-algebra. Because V is arbitrary, we may therefore conclude that the universal sentence (11.932) holds, and this further implies the truth of the inclusion $\mathcal{O}|\Omega_1 \subseteq \mathcal{A}(\mathcal{O})|\Omega_1$. Thus, $\mathcal{A}(\mathcal{O})|\Omega_1$ is a σ-algebra on Ω_1 which includes $\mathcal{O}|\Omega_1$, where the latter set system generates the σ-algebra $\mathcal{A}(\mathcal{O}|\Omega_1)$. Consequently, we obtain the second inclusion in (11.928) with Theorem 11.35c).

Then, the truth of these inclusions implies the truth of the equation (11.927) and therefore the truth of the stated theorem, since Ω, $\mathcal{O}$ and Ω_1 were initially arbitrary. □

We now consider a straightforward application of the Generation of product σ-algebras by means of families of generating system to the present situation that topologies act as generating systems for the considered σ-algebras.

Lemma 11.120. *It is true for any nonempty index set I and for any families of sets $(\Omega_i)_{i \in I}$ and $(\mathcal{O}_i)_{i \in I}$ such that $\mathcal{O}_i$ is a topology on Ω_i for any $i \in I$*

that the product σ-algebra of $(\mathcal{B}(\Omega_i))_{i\in I}$ is included in the Borel σ-algebra on $\bigtimes_{i\in I}\Omega_i$ with respect to the product topology of $(\mathcal{O}_i)_{i\in I}$, i.e.

$$\bigotimes_{i\in I}\mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\bigtimes_{i\in I}\Omega_i). \tag{11.933}$$

Proof. We let I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ be arbitrary sets, we assume that $I \neq \emptyset$ holds, and we assume $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ to be families such that $\mathcal{O}_i$ is a topology on Ω_i for any $i \in I$. According to Theorem 11.99, we then obtain the equation

$$\bigotimes_{i\in I}\mathcal{B}(\Omega_i) = \bigotimes_{i\in I}\mathcal{A}(\mathcal{O}_i) = \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{O}_i\}),$$

noting that the terms of the family $(\mathcal{O}_i)_{i\in I}$ of topologies constitute generating systems for the corresponding Borel σ-algebras, which give rise to the family $(\mathcal{B}(\Omega_i))_{i\in I}$. The right-hand side of the preceding equation shows that $\bigotimes_{i\in I}\mathcal{B}(\Omega_i)$ is generated by subbasis $\mathcal{C}_{\times\Omega_i}$ of the product topology $\bigotimes_{i\in I}\mathcal{O}_i$, that is,

$$\bigotimes_{i\in I}\mathcal{B}(\Omega_i) = \mathcal{A}(\mathcal{C}_{\times\Omega_i}). \tag{11.934}$$

Observing now the truth of the inclusions

$$\mathcal{C}_{\times\Omega_i} \subseteq \bigotimes_{i\in I}\mathcal{O}_i \subseteq \mathcal{A}(\bigotimes_{i\in I}\mathcal{O}_i)$$

due to the fact that subbasis is included in the corresponding generated topology (see Proposition 11.78) and due to the definition of a generated σ-algebra, we obtain also the inclusion

$$\mathcal{C}_{\times\Omega_i} \subseteq \mathcal{A}(\bigotimes_{i\in I}\mathcal{O}_i) \tag{11.935}$$

with (2.13). Thus, $\mathcal{A}(\bigotimes_{i\in I}\mathcal{O}_i)$ is a σ-algebra on $\bigtimes\Omega_i$ that includes $\mathcal{C}_{\times\Omega_i}$. Because the σ-algebra $\mathcal{A}(\mathcal{C}_{\times\Omega_i})$ (generated by $\mathcal{C}_{\times\Omega_i}$) is the smallest σ-algebra on $\bigtimes\Omega_i$ that includes $\mathcal{C}_{\times\Omega_i}$, we obtain

$$\mathcal{A}(\mathcal{C}_{\times\Omega_i}) \subseteq \mathcal{A}(\bigotimes_{i\in I}\mathcal{O}_i) \quad \left[= \mathcal{B}(\bigtimes_{i\in I}\Omega_i)\right] \tag{11.936}$$

with Theorem 11.35c). Thus, substitution based on (11.934) in yields the suggested inclusion (11.933). Here, the sets I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ were initially arbitrary, so that the lemma follows to be true indeed. $\square$

Lemma 11.121. *It is true for any nonempty, countable index set I and for any families of sets $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ such that $(\Omega_i, \mathcal{O}_i)$ is a second-countable topological space for every $i \in I$ that there exists a family $(\mathcal{K}^{(c)}_{\Omega_i})_{i\in I}$ such that $\mathcal{K}^{(c)}_{\Omega_i}$ is a countable basis generating the topology $\mathcal{O}_i$ (for any $i \in I$) and such that*

$$\bigotimes_{i\in I} \mathcal{B}(\Omega_i) = \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \mathcal{K}^{(c)}_{\Omega_i}\}). \tag{11.937}$$

Proof. We let I be an arbitrary nonempty, countable (index) set and $(\Omega_i)_{i\in I}$, $(\mathcal{O}_i)_{i\in I}$ arbitrary families of sets such that $(\Omega_i, \mathcal{O}_i)$ is a second-countable topological space for every $i \in I$. According to Theorem 11.94a), there exists then a particular family $(\bar{\mathcal{K}}^{(c)}_{\Omega_i})_{i\in I}$ for which $\bar{\mathcal{K}}^{(c)}_{\Omega_i}$ is a countable basis generating the topology $\mathcal{O}_i$ for any $i \in I$. Because of Lemma 11.117, each of these bases $\bar{\mathcal{K}}^{(c)}_{\Omega_i}$ generates then also the Borel σ-algebra, in the sense that

$$\mathcal{B}(\Omega_i) = \mathcal{A}(\bar{\mathcal{K}}^{(c)}_{\Omega_i}).$$

Since Property 1 of a basis for a topology evidently yields the inclusion $\bar{\mathcal{K}}^{(c)}_{\Omega_i} \subseteq \mathcal{P}(\Omega_i)$ for all $i \in I$, these (generated) Borel σ-algebras define the family $(\mathcal{A}(\bar{\mathcal{K}}^{(c)}_{\Omega_i}))_{i\in I} = (\mathcal{B}(\Omega_i))_{i\in I}$ (see Exercise 11.41), and the product σ-algebra of that family is given by

$$\left[\bigotimes_{i\in I} \mathcal{B}(\Omega_i) =\right] \quad \bigotimes_{i\in I} \mathcal{A}(\bar{\mathcal{K}}^{(c)}_{\Omega_i}) = \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \bar{\mathcal{K}}^{(c)}_{\Omega_i}\}) \tag{11.938}$$

in view of Theorem 11.99. Thus, the equation (11.937) is true, and since I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ were initially arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Theorem 11.122 (Compatibility of product Borel σ-algebras and product topologies for second-countable topological spaces). *It is true for any nonempty, countable set I and any families $(\Omega_i)_{i\in I}$, $(\mathcal{O}_i)_{i\in I}$ for which $(\Omega_i, \mathcal{O}_i)$ is a second-countable topological space for every $i \in I$ that the product σ-algebra of $(\mathcal{B}(\Omega_i))_{i\in I}$ constitutes the Borel σ-algebra on $\bigtimes_{i\in I} \Omega_i$ with respect to the product topology $\bigotimes_{i\in I} \mathcal{O}_i$, i.e.*

$$\bigotimes_{i\in I} \mathcal{B}(\Omega_i) = \mathcal{B}(\bigtimes_{i\in I} \Omega_i). \tag{11.939}$$

Proof. We take arbitrary sets I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$, assuming I to be a nonempty and countable and assuming $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ to be families with index set I such that $(\Omega_i, \mathcal{O}_i)$ is a second-countable topological space for any $i \in I$. These assumptions imply the existence of a particular family $(\bar{\mathcal{K}}^{(c)}_{\Omega_i})_{i\in I}$ such that $\bar{\mathcal{K}}^{(c)}_{\Omega_i}$ is a countable basis generating the topology $\mathcal{O}_i$ for every $i \in I$ (see Theorem 11.94a)). Then, the basis $\bar{\mathcal{K}}^{(c)}_{\Omega_i}$ generates (for every $i \in I$) the Borel σ-algebra

$$\mathcal{B}(\Omega_i) = \mathcal{A}(\bar{\mathcal{K}}^{(c)}_{\Omega_i})$$

(see Lemma 11.117), and the corresponding product σ-algebra is generated through

$$\bigotimes_{i\in I} \mathcal{B}(\Omega_i) = \mathcal{A}(\bigcup_{i\in I}\{\pi_i^{-1}[A] : A \in \bar{\mathcal{K}}^{(c)}_{\Omega_i}\}). \tag{11.940}$$

(in view of the preceding Lemma 11.121). Furthermore, the family $(\bar{\mathcal{K}}^{(c)}_{\Omega_i})_{i\in I}$ determines the basis $\mathcal{K}'_{\times \Omega_i}$ generating the product topology on $\bigtimes_{i\in I} \Omega_i$ via

$$\bigotimes_{i\in I} \mathcal{O}_i = \mathcal{O}(\mathcal{K}'_{\times \Omega_i})$$

according to Theorem 11.91. Here, the basis $\mathcal{K}'_{\times \Omega_i}$ is also countable by virtue of Lemma 11.93, so that it gives us (again with Lemma 11.117)

$$\mathcal{B}(\bigtimes_{i\in I} \Omega_i) = \mathcal{A}(\mathcal{K}'_{\times \Omega_i}). \tag{11.941}$$

We are now in a position to prove the equation (11.939). Let us observe first that the inclusion

$$\bigotimes_{i\in I} \mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\bigtimes_{i\in I} \Omega_i) \tag{11.942}$$

is already true because of Lemma 11.120, so that it suffices to prove the reverse inclusion

$$\mathcal{B}(\bigtimes_{i\in I} \Omega_i) \subseteq \bigotimes_{i\in I} \mathcal{B}(\Omega_i), \tag{11.943}$$

noting that the Axiom of Extension will allow us to infer from the truth of these two inclusions the truth of the desired equation. The idea is now to show that the σ-algebra $\bigotimes_{i\in I} \mathcal{B}(\Omega_i)$ includes the set system $\mathcal{K}'_{\times \Omega_i}$, that is,

$$\mathcal{K}'_{\times \Omega_i} \subseteq \bigotimes_{i\in I} \mathcal{B}(\Omega_i), \tag{11.944}$$

because this inclusion will imply

$$\left[\mathcal{B}(\bigtimes_{i\in I} \Omega_i) =\right] \quad \mathcal{A}(\mathcal{K}'_{\times \Omega_i}) \subseteq \bigotimes_{i\in I} \mathcal{B}(\Omega_i) \tag{11.945}$$

with (11.941) and the fact that the generated $\mathcal{A}(\mathcal{K}'_{\times \Omega_i})$ is the smallest σ-algebra that includes the set system $\mathcal{K}'_{\times \Omega_i}$. To accomplish this task, we apply the definition of a subset and verify

$$\forall B \,(B \in \mathcal{K}'_{\times \Omega_i} \Rightarrow B \in \bigotimes_{i\in I} \mathcal{B}(\Omega_i)), \tag{11.946}$$

letting B be an arbitrary basis element in $\mathcal{K}'_{\times \Omega_i}$. According to the Generation of a basis for a product topology by means of a family of bases, this assumption implies especially the existence of particular sets $\bar{U}$ and $\bar{J}$ such that

$$\bar{U} \in \bigtimes_{i\in I} \mathcal{O}_i, \tag{11.947}$$

such that

$$B = \bigtimes_{i\in I} \bar{U}_i, \tag{11.948}$$

such that $\bar{J}$ is a finite subset of I, and such that

$$\forall i \,([i \in \bar{J} \Rightarrow \bar{U}_i \in \bar{\mathcal{K}}^{(c)}_{\Omega_i}] \wedge [i \in I \setminus \bar{J} \Rightarrow \bar{U}_i = \Omega_i]). \tag{11.949}$$

Based on the fact that the Law of the Excluded Middle gives rise to the truth disjunction $\bar{J} = \emptyset \vee \bar{J} \neq \emptyset$, we carry out a proof by cases to establish the desired consequent $B \in \bigotimes_{i\in I} \mathcal{B}(\Omega_i)$.

The first case $\bar{J} = \emptyset$ yields $I \setminus \bar{J} = I \setminus \emptyset = I$ with (2.102), so that the universal sentence (11.949) implies especially

$$\forall i \,(i \in I \Rightarrow \bar{U}_i = \Omega_i)$$

by means of the Distributive Law for quantification (1.74) and substitution. The preceding universal sentence in turn implies with the Equality Criterion for Cartesian products of families of sets

$$\bigtimes_{i\in I} \bar{U}_i = \bigtimes_{i\in I} \Omega_i,$$

so that substitution based on (11.948) gives us $B = \bigtimes_{i\in I} \Omega_i$. Because $\bigotimes_{i\in I} \mathcal{B}(\Omega_i)$ constitutes a σ-algebra on $\bigtimes_{i\in I} \Omega_i$, it contains the latter set by Property 2 of a σ-algebra, so that the desired consequent $B \in \bigotimes_{i\in I} \mathcal{B}(\Omega_i)$ follows to be true via substitution for the first case.

In the second case $\bar{J} \neq \emptyset$, we show that the Cartesian product B in (11.948) can be written as the intersection

$$B = \bigcap_{j\in \bar{J}} \pi_j^{-1}[\bar{U}_j]. \tag{11.950}$$

To do this, we apply the Equality Criterion for sets and prove the equivalent universal sentence

$$\forall \omega \, (\omega \in B \Leftrightarrow \omega \in \bigcap_{j \in \bar{J}} \pi_j^{-1}[\bar{U}_j]). \tag{11.951}$$

Letting ω be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $\omega \in B$ $[= \bigtimes_{i \in I} \bar{U}_i]$ is true. By definition of the Cartesian product of a family of sets, ω is therefore a family with index set I whose terms satisfy

$$\forall i \, (i \in I \Rightarrow \omega_i \in \bar{U}_i). \tag{11.952}$$

To show that this implies $\omega \in \bigcap_{j \in \bar{J}} \pi_j^{-1}[\bar{U}_j]$, we use the Characterization of the intersection of a family of sets and verify

$$\forall j \, (j \in \bar{J} \Rightarrow \omega \in \pi_j^{-1}[\bar{U}_j]), \tag{11.953}$$

letting $j \in \bar{J}$ be arbitrary. Recalling the truth of the inclusion $\bar{J} \subseteq I$, we obtain by definition of a subset $j \in I$, which then further implies $\omega_j \in \bar{U}_j$ with (11.952). Let us observe here in light of (11.947) that $\bar{U}_i$ is (for any $i \in I$) an open set of the topology $\mathcal{O}_i$ on Ω_i, and thus evidently a subset of Ω_i. Consequently, the inclusion

$$[B =] \quad \bigtimes_{i \in I} \bar{U}_i \subseteq \bigtimes_{i \in I} \Omega_i$$

follows to be true with Proposition 3.246, so that the previous assumption $\omega \in B$ implies $\omega \in \bigtimes_{i \in I} \Omega_i$. Being in the domain of the projection function $\pi_j : \bigtimes_{i \in I} \Omega_i \to \Omega_j$, the element ω is associated with the value $\pi_j(\omega) = \omega_j$. The previous finding $\omega_j \in \bar{U}_j$ yields then $\pi_j(\omega) \in \bar{U}_j$ through substitution, and this in turn implies $\omega \in \pi_j^{-1}[\bar{U}_j]$ with the definition of an inverse image. We thus completed the proof of the implication in (11.953), and as j was arbitrary, we can infer from this the truth of the universal sentence (11.953). Therefore, the desired consequent of the implication '$\Rightarrow$' in (11.951) is also true.

Next, we prove the second part ('$\Leftarrow$') of the equivalence also directly, assuming now $\omega \in \bigcap_{j \in \bar{J}} \pi_j^{-1}[\bar{U}_j]$ to be true. This means according to the Characterization of the intersection of a family of sets that the universal sentence (11.953) holds. To establish the desired $\omega \in B$ $[= \bigtimes_{i \in I} \bar{U}_i]$, we recall our previous finding $\omega \in \bigtimes_{i \in I} \Omega_i$, which shows that ω is a family with index set I. It remains for us to show that the terms of ω satisfy (11.952). We take an arbitrary index $i \in I$, where $I = (I \setminus \bar{J}) \cup \bar{J}$ is implied by assumed inclusion $\bar{J} \subseteq I$ by virtue of (2.263), so that the disjunction

$i \in I \setminus \bar{J} \lor i \in \bar{J}$ follows to be true by definition of the union of two sets. We use this disjunction in the following to prove $\omega_i \in \bar{U}_i$ by cases.

The first case $i \in I \setminus \bar{J}$ implies with (11.949) $\bar{U}_i = \Omega_i$, resulting in the equations (using also Exercise 3.90)

$$\bigtimes_{i \in I} \Omega_i = \pi_i^{-1}[\Omega_i] = \pi_i^{-1}[\bar{U}_i]. \tag{11.954}$$

Furthermore, as we assumed $\bar{J}$ to be nonempty, there exists an element in that set, say $\bar{k}$. This finding $\bar{k} \in \bar{J}$ implies now with the true sentence (11.953) that $\omega \in \pi_{\bar{k}}^{-1}[\bar{U}_{\bar{k}}]$ holds. This inverse image is included in the domain $\bigtimes_{i \in I} \Omega_i$ of the function $\pi_{\bar{k}}$ (see Note 3.30), so that the definition of a subset yields $\omega \in \bigtimes_{i \in I} \Omega_i$, and then $\omega \in \pi_i^{-1}[\bar{U}_i]$ with (11.954). This means $\pi_i(\omega) \in \bar{U}_i$ by definition of an inverse image, and we obtain therefore $\omega_i \in \bar{U}_i$ with the definition of a projection function.

The second case $i \in \bar{J}$ gives immediately $\omega \in \pi_i^{-1}[\bar{U}_i]$ with (11.953), with the consequence that $\pi_i(\omega) \in \bar{U}_i$ and $\omega_i \in \bar{U}_i$ hold, as in the first case. Being true in any case, the finding $\omega_i \in \bar{U}_i$ proves the implication in (11.952), in which i is arbitrary, so that the universal sentence (11.952) holds indeed. Thus, the family ω (having the index set I) is an element of the Cartesian product $\bigtimes_{i \in I} \bar{U}_i$ $[= B]$, that is, $\omega \in B$. This in turn proves the implication '$\Leftarrow$' in (11.951), completing the proof of the equivalence.

Since ω was arbitrary, we may now infer from the truth of this equivalence the truth of (11.951), and this universal sentence in turn implies the truth of the equality (11.950). By definition of the intersection of a family of sets, we can write for this equation also

$$B = \bigcap \operatorname{ran}(p), \tag{11.955}$$

where $p = (\pi_j^{-1}[\bar{U}_j])_{j \in \bar{J}}$. Here, we may view the family p as being the surjection

$$p : \bar{J} \twoheadrightarrow \operatorname{ran}(p).$$

As $\bar{J}$ is a finite set, there exists also a particular bijection $\bar{c} : \{1, \ldots, \bar{n}\} \rightleftarrows \bar{J}$ with $\bar{n} \in \mathbb{N}$. Since the range $\bar{J}$ is nonempty (as currently assumed), the domain $\{1, \ldots, \bar{n}\}$ of that bijection is also nonempty due to (3.119). By definition of an initial segment of $\mathbb{N}_+$, we thus find $\bar{n} \neq 0$ to be true, so that $\bar{n} \in \mathbb{N}_+$ holds by definition of the set of positive natural numbers. Furthermore, the bijection $\bar{c}$ is by definition also a surjection, that is,

$$\bar{c} : \{1, \ldots, \bar{n}\} \twoheadrightarrow \bar{J}$$

which allows us to form the surjective composition

$$p \circ \bar{c} : \{1, \ldots, \bar{n}\} \twoheadrightarrow \operatorname{ran}(p) \tag{11.956}$$

according to Theorem 3.199. This composition constitutes thus a sequence $(p(\bar{c}(i)) \,|\, i \in \{1,\ldots,\bar{n}\})$ with range $\mathrm{ran}(p \circ \bar{c}) = \mathrm{ran}(p)$, and we can use these findings to write (11.955) as

$$B = \bigcap \mathrm{ran}(p \circ \bar{c}) = \bigcap_{i=1}^{\bar{n}} p(\bar{c}(i)) = \bigcap_{i=1}^{\bar{n}} \pi_{\bar{c}(i)}^{-1}[\bar{U}_{\bar{c}(i)}]. \tag{11.957}$$

Next, we verify that $\mathrm{ran}(p)$ is included in the σ-algebra $\bigotimes_{i\in I} \mathcal{B}(\Omega_i)$. For this purpose, we prove

$$\forall X\, (X \in \mathrm{ran}(p) \Rightarrow X \in \bigotimes_{i\in I} \mathcal{B}(\Omega_i)), \tag{11.958}$$

letting X be an arbitrary set and assuming $X \in \mathrm{ran}(p)$ to be true. By definition of a range, there is then a constant, say $\bar{k}$, such that $(\bar{k}, X) \in p$ holds. Therefore, $\bar{k} \in \bar{J}$ $[= \mathrm{dom}(p)]$ is true by definition of a domain, and we can write $(\bar{k}, X) \in p$ in function/family notation as

$$X = p(\bar{k}) = \pi_{\bar{k}}^{-1}[\bar{U}_{\bar{k}}].$$

We notice also that $\bar{k} \in \bar{J}$ implies $\bar{U}_{\bar{k}} \in \bar{\mathcal{K}}_{\Omega_{\bar{k}}}^{(c)}$ with (11.949), so that we evidently have

$$\pi_{\bar{k}}^{-1}[\bar{U}_{\bar{k}}] \in \{\pi_i^{-1}[A] : A \in \bar{\mathcal{K}}_{\Omega_i}^{(c)}\}.$$

Furthermore, $\bar{k} \in \bar{J}$ implies $\bar{k} \in I$ because of the inclusion $\bar{J} \subseteq I$. According to the Characterization of the union of a family of sets, we obtain then also

$$[X =] \quad \pi_{\bar{k}}^{-1}[\bar{U}_{\bar{k}}] \in \bigcup_{i\in I} \{\pi_i^{-1}[A] : A \in \bar{\mathcal{K}}_{\Omega_i}^{(c)}\}. \tag{11.959}$$

Observing now the truth of the inclusion

$$\bigcup_{i\in I} \{\pi_i^{-1}[A] : A \in \bar{\mathcal{K}}_{\Omega_i}^{(c)}\} \subseteq \mathcal{A}(\bigcup_{i\in I} \{\pi_i^{-1}[A] : A \in \bar{\mathcal{K}}_{\Omega_i}^{(c)}\})$$

in light of the definition of a generated σ-algebra, and recalling that this generated σ-algebra is identical with $\bigotimes_{i\in I} \mathcal{B}(\Omega_i)$ in view of (11.940), we can use the definition of a subset to infer from (11.959) the truth of the desired consequent $X \in \bigotimes_{i\in I} \mathcal{B}(\Omega_i)$. As X was arbitrary, we may therefore conclude that (11.958) is indeed true, so that the definition of a subset yields the true inclusion

$$\mathrm{ran}(p) \subseteq \bigotimes_{i\in I} \mathcal{B}(\Omega_i).$$

This shows us that $\bigotimes_{i\in I}\mathcal{B}(\Omega_i)$ is also a codomain of the function (11.956), which we can therefore write also as

$$p\circ\bar{c}:\{1,\ldots,\bar{n}\}\twoheadrightarrow\bigotimes_{i\in I}\mathcal{B}(\Omega_i).$$

Being a σ-algebra, $\bigotimes_{i\in I}\mathcal{B}(\Omega_i)$ allows for the definition of the n-fold binary intersection operation $\bigcap_{i=1}^{n}$ on it (see Note 11.15), so that the intersection $B=\bigcap_{i=1}^{\bar{n}}\pi_{\bar{c}(i)}^{-1}[\bar{U}_{\bar{c}(i)}]$ of the sequence $p\circ\bar{c}$ established in (11.957) is an element of that σ-algebra.

We thus found $B\in\bigotimes_{i\in I}\mathcal{B}(\Omega_i)$ to be true both in case of $\bar{J}=\emptyset$ and in case of $\bar{J}\neq\emptyset$, so that the proof of the implication in (11.946) is now complete. Here, B was arbitrary, so that the universal sentence (11.946) follows to be true as well. Consequently, the equivalent inclusion (11.944) holds, which then further implies the truth of the inclusion (11.945) by means of Theorem 11.35c). As mentioned before, the resulting inclusion (11.943) implies in conjunction with the inclusion (11.942) that the equation (11.939) holds. Then, as I, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ were initially arbitrary, we may finally conclude that the stated theorem is true. □

Corollary 11.123 (Compatibility of product Borel σ-algebras and product topologies for separable metric spaces). *It is true for any nonempty, countable set I and any families $(\Omega_i)_{i\in I}$, $(d_i)_{i\in I}$ for which (Ω_i,d_i) is a separable metric space for every $i\in I$ that the product σ-algebra of the family $(\mathcal{B}(\Omega_i))_{i\in I}$ of Borel σ-algebras (with respect to the family $(\mathcal{O}_{d_i})_{i\in I}$ of induced metric topologies) constitutes the Borel σ-algebra on $\bigtimes_{i\in I}\Omega_i$ with respect to the product topology $\bigotimes_{i\in I}\mathcal{O}_{d_i}$, i.e.*

$$\bigotimes_{i\in I}\mathcal{B}(\Omega_i)=\mathcal{B}(\bigtimes_{i\in I}\Omega_i). \tag{11.960}$$

Proof. We let I, $(\Omega_i)_{i\in I}$ and $(d_i)_{i\in I}$ be arbitrary sets, where we assume that I is a nonempty and countable, and where we also assume that $(\Omega_i)_{i\in I}$ and $(d_i)_{i\in I}$ are families such that (Ω_i,d_i) is a separable metric space for all $i\in I$. Thus, for every $i\in I$, the metric d_i induces the metric topology $\mathcal{O}_{d_i}$ on Ω_i, and the topological space $(\Omega_i,\mathcal{O}_{d_i})$ is then second-countable as a consequence of the Equivalence of separable metric spaces and second-countable topological spaces. The families $(\Omega_i)_{i\in I}$ and $(d_i)_{i\in I}$ evidently define the family of metric topologies $(\mathcal{O}_{d_i})_{i\in I}$ and the family of corresponding Borel σ-algebras $(\mathcal{B}(\Omega_i))_{i\in I}$. These families in turn give rise, respectively, to the product topology $\bigotimes_{i\in I}\mathcal{O}_{d_i}$ and to the product σ-algebra $\bigotimes_{i\in I}\mathcal{B}(\Omega_i)$. We can now apply the preceding Theorem 11.122 to infer the truth of the equation (11.960). □

11.10.1. The real Borel measurable space $(\mathbb{R}, \mathcal{B})$

In this section, we explore on the one hand the kinds of sets that are contained in the σ-algebra generated by the semiring of left-closed and right-open sets in $\mathbb{R}$ as well as in the Borel σ-algebra on $\mathbb{R}$. On the other hand, we identify some of the set systems that generate these σ-algebras.

Definition 11.38 (Borel σ-algebra on $\mathbb{R}$, real Borel set, real Borel measurable space). We say that a set $\mathcal{B}$ is the *Borel σ-algebra on* $\mathbb{R}$ iff $\mathcal{B}$ is the σ-algebra generated by the order topology on $\mathbb{R}$, symbolically

$$\mathcal{B} = \mathcal{B}(\mathbb{R}) = \mathcal{A}(\mathcal{O}_{<_{\mathbb{R}}}). \tag{11.961}$$

We then call every element of $\mathcal{B}$ a *real Borel set* and

$$(\mathbb{R}, \mathcal{B}) \tag{11.962}$$

the *real Borel measurable space.*

Note 11.41. In light of (11.920) – (11.921), we see immediately that the Borel σ-algebra on $\mathbb{R}$ contains both the empty set and the set of real numbers, that is,

$$\emptyset, \mathbb{R} \in \mathcal{B}. \tag{11.963}$$

In view of Proposition 11.118, any open and any closed set in $\mathbb{R}$ with respect to the order topology on $\mathbb{R}$ are elements of $\mathcal{B}$ as well. Then, any compact set in $\mathbb{R}$ with respect to the order topology on $\mathbb{R}$ is also element of $\mathcal{B}$ (recalling that every compact set in $\mathbb{R}$ with respect to the order topology on $\mathbb{R}$ is closed; cf. the Heine-Borel Theorem for $\mathbb{R}$).

Proposition 11.124. *It is true that the σ-algebra generated by the set of open intervals in $\mathbb{R}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,*

$$\mathcal{A}(\{(a,b) : a, b \in \mathbb{R}\}) = \mathcal{B}. \tag{11.964}$$

Proof. We prove the equivalent equation

$$\mathcal{A}(\{(a,b) : a, b \in \mathbb{R}\}) = \mathcal{A}(\mathcal{O}_{<_{\mathbb{R}}}). \tag{11.965}$$

by means of the Equality Criterion for generated σ-algebras, i.e. by establishing the two inclusions

$$\{(a,b) : a, b \in \mathbb{R}\} \subseteq \mathcal{A}(\mathcal{O}_{<_{\mathbb{R}}}) \ \wedge \ \mathcal{O}_{<_{\mathbb{R}}} \subseteq \mathcal{A}(\{(a,b) : a, b \in \mathbb{R}\}). \tag{11.966}$$

Since the inclusions

$$\{(a,b) : a, b \in \mathbb{R}\} \subseteq \mathcal{O}_{<_{\mathbb{R}}} \subseteq \mathcal{A}(\mathcal{O}_{<_{\mathbb{R}}})$$

hold according to (11.449) and the definition of a generated σ-algebra, we obtain the first inclusion in (11.966) by means of (2.13). To establish the other inclusion, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall A\,(A \in \mathcal{O}_{<_{\mathbb{R}}} \Rightarrow A \in \mathcal{A}(\{(a,b) : a,b \in \mathbb{R}\})), \tag{11.967}$$

letting $A \in \mathcal{O}_{<_{\mathbb{R}}}$ be arbitrary. Because of (11.481), the open set A in the order topology on $\mathbb{R}$ can then be written as the countable union of some sequence of open intervals in the set of rational numbers, that is, there exists then a particular function $\bar{B} : \mathbb{N}_+ \to \{(a,b) : a,b \in \mathbb{Q}\}$ with

$$A = \bigcup_{n=1}^{\infty} \bar{B}_n. \tag{11.968}$$

As we may now obtain the true inclusions

$$\mathrm{ran}(\bar{B}) \subseteq \{(a,b) : a,b \in \mathbb{Q}\} \subseteq \{(a,b) : a,b \in \mathbb{R}\} \subseteq \mathcal{A}(\{(a,b) : a,b \in \mathbb{R}\})$$

by using the definition of a codomain, (11.474), and the definition of a generated σ-algebra, so that another application of (2.13) yields the inclusion

$$\mathrm{ran}(\bar{B}) \subseteq \mathcal{A}(\{(a,b) : a,b \in \mathbb{R}\}),$$

we thus have that the preceding σ-algebra is also a codomain of $\bar{B}$, i.e.

$$\bar{B} : \mathbb{N}_+ \to \mathcal{A}(\{(a,b) : a,b \in \mathbb{R}\}).$$

Consequently, the union A in (11.968) turns out to be an element of $\mathcal{A}(\{(a,b) : a,b \in \mathbb{R}\})$ with Property 3 of a σ-algebra, so that the implication in (11.967) holds. Because A is arbitrary, we may now infer from the truth of this implication the truth of the universal sentence (11.967), therefore the truth of the second inclusion in (11.966), and moreover the truth of the equation (11.965). Thus, the proof of the proposed equation (11.964) is complete. □

Exercise 11.49. Prove that the σ-algebra generated by the set of open intervals in $\mathbb{Q}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,

$$\mathcal{A}(\{(a,b) : a,b \in \mathbb{Q}\}) = \mathcal{B}. \tag{11.969}$$

(Hint: Slightly rearrange the arguments used in the proof of Proposition 11.124.)

Proposition 11.125. *It is true for any $a,b \in \mathbb{R}$ that the left-closed and right-open interval from a to b is an element of the Borel σ-algebra on $\mathbb{R}$, that is,*

$$[a,b) \in \mathcal{B}. \tag{11.970}$$

Proof. We let a and b be arbitrary real numbers, and we apply Function definition by replacement to establish the unique existence of a function s with domain $\mathbb{N}_+$ such that every positive natural number n is mapped to the open interval $(a - \frac{1}{n}, b)$ in $\mathbb{R}$, i.e. such that

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow s(n) = (a - \frac{1}{n}, b)), \tag{11.971}$$

where we form the differences and fractions with respect to $\mathbb{R}$. For this purpose, we verify the universal sentence

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \exists! y\,(y = (a - \frac{1}{n}, b))), \tag{11.972}$$

letting $n \in \mathbb{N}_+$ be arbitrary and observing that $a - \frac{1}{n}$ is a real number, so that $(a - \frac{1}{n}, b)$ is a uniquely specified open interval in $\mathbb{R}$. We therefore see in light of (1.109) that the uniquely existential sentence in (11.972) is true. Since n is arbitrary, we may then further conclude that the universal sentence (11.972) holds, which now yields the unique function s with domain $\mathbb{N}_+$ such that (11.971). Thus, s constitutes a sequence which we may write in the form $s = ((a - \frac{1}{n}, b))_{n \in \mathbb{N}_+}$. Let us also check that the Borel σ-algebra on $\mathbb{R}$ is a codomain of s, i.e. that the inclusion

$$\mathrm{ran}(s) \subseteq \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\}) \tag{11.973}$$

holds. To do this we apply the definition of a subset, i.e. we prove the universal sentence

$$\forall A\,(A \in \mathrm{ran}(s) \Rightarrow A \in \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\})), \tag{11.974}$$

letting A be arbitrary and assuming $A \in \mathrm{ran}(s)$ to be true. We then obtain with the definition of a range a particular constant $\bar{n}$ with $(\bar{n}, A) \in s$. On the one hand, this shows in light of the definition of a domain that $\bar{n} \in \mathbb{N}_+$ $[= \mathrm{dom}(s)]$ holds. On the other hand, we may write in function notation $A = s(\bar{n}) = (a - \frac{1}{\bar{n}}, b)$, using also (11.971). Thus, A is an open interval in $\mathbb{R}$, which is clearly an element of $\{(a, b) : a, b \in \mathbb{R}\}$. Then, since the σ-algebra $\mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\})$ clearly includes its generating system $\{(a, b) : a, b \in \mathbb{R}\}$, we have that $A \in \{(a, b) : a, b \in \mathbb{R}\}$ implies $A \in \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\})$ by definition of a subset. Since A is arbitrary, we may now infer from this the truth of the universal sentence (11.974) and consequently the truth of the inclusion (11.973). We thus constructed the function

$$s : \mathbb{N}_+ \to \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\}), \qquad n \mapsto (a - \frac{1}{n}, b),$$

and this gives us with Proposition 11.31

$$\bigcap_{n=1}^{\infty}(a-\frac{1}{n},b)\in\mathcal{A}(\{(a,b):a,b\in\mathbb{R}\}). \tag{11.975}$$

The final step of the proof consists in the verification of the equation

$$[a,b)=\bigcap_{n=1}^{\infty}(a-\frac{1}{n},b), \tag{11.976}$$

which we do by applying the Equality Criterion for sets, i.e. by proving the universal sentence

$$\forall\omega\,(\,\omega\in[a,b)\Leftrightarrow\omega\in\bigcap_{n=1}^{\infty}(a-\frac{1}{n},b)\,). \tag{11.977}$$

We let ω be arbitrary and observe first that the equivalence to be proven is equivalent to

$$a\leq_{\mathbb{R}}\omega<_{\mathbb{R}}b\;\Leftrightarrow\;\forall n\,(\,n\in\mathbb{N}_{+}\Rightarrow\omega\in(a-\frac{1}{n},b)\,) \tag{11.978}$$

because of the definition of a left-closed and right-open interval in $\mathbb{R}$ and the Characterization of the intersection of a family of sets. We prove the first part ('$\Rightarrow$') of the preceding equivalence directly, assuming $a\leq_{\mathbb{R}}\omega$ and $\omega<_{\mathbb{R}}b$ to be true, and letting then $n\in\mathbb{N}_{+}$ be arbitrary. To establish the desired consequent $\omega\in(a-\frac{1}{n},b)$, we apply the definition of an open interval in $\mathbb{R}$ and demonstrate the truth of the equivalent inequalities $a-\frac{1}{n}<_{\mathbb{R}}\omega$ and $\omega<_{\mathbb{R}}b$. Here, the latter inequality is already true by assumption. To verify the other inequality $a-\frac{1}{n}<_{\mathbb{R}}\omega$, we observe that the assumption $n\in\mathbb{N}_{+}$ implies evidently $0<_{\mathbb{R}}n$ and therefore $0<_{\mathbb{R}}\frac{1}{n}$ according to (7.101). This finding in turn implies $-\frac{1}{n}<_{\mathbb{R}}0$ and then $a-\frac{1}{n}<_{\mathbb{R}}a$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$; the latter, together with the assumed $a\leq_{\mathbb{R}}\omega$, then implies $a-\frac{1}{n}<_{\mathbb{R}}\omega$ with Transitivity Formula for $<$ and $\leq$. The conjunction of this inequality and the previously established inequality $\omega<_{\mathbb{R}}b$ yields then $\omega\in(a-\frac{1}{n},b)$ by definition of an open interval in $\mathbb{R}$. As n was arbitrary, we may therefore conclude that the universal sentence in (11.978) is true, so that the proof of the first part of the equivalence is complete.

We prove the second part of the equivalence (11.978) by contraposition, assuming the negation $\neg\omega\in[a,b)$ to be true and showing that this assumption implies the truth of the negated universal sentence

$$\neg\forall n\,(\,n\in\mathbb{N}_{+}\Rightarrow\omega\in(a-\frac{1}{n},b)\,), \tag{11.979}$$

which we may write equivalently as

$$\exists n\,(n \in \mathbb{N}_+ \,\wedge\, \neg\omega \in (a - \frac{1}{n}, b)\,) \tag{11.980}$$

by means of the Negation Law for universal implications. Now, the initial assumption gives us with the definition of a left-closed and right-open interval in $\mathbb{R}$

$$\neg(a \leq_{\mathbb{R}} \omega \,\wedge\, \omega <_{\mathbb{R}} b)$$

and subsequently with De Morgan's Law for the conjunction and the Negation Formulas for $\leq$ & $<$

$$\omega <_{\mathbb{R}} a \,\vee\, b \leq_{\mathbb{R}} \omega.$$

We now use this true disjunction to prove the desired existential sentence (11.980) by cases. The first case $\omega <_{\mathbb{R}} a$ gives $0 <_{\mathbb{R}} a - \omega$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$, consequently $a - \omega \in \mathbb{R}_+$ by definition of the set of positive real numbers. Thus, there exists a positive natural number, say $\bar{n}$, such that $\frac{1}{\bar{n}} <_{\mathbb{R}} a - \omega$ holds, according to (8.364). This finding implies evidently $\omega <_{\mathbb{R}} a - \frac{1}{\bar{n}}$, and the disjunction

$$\omega <_{\mathbb{R}} a - \frac{1}{\bar{n}} \,\vee\, \omega = a - \frac{1}{\bar{n}}$$

is then also true, with the consequence that $\omega \leq_{\mathbb{R}} a - \frac{1}{\bar{n}}$ holds (using the definition of an induced reflexive partial ordering). This inequality in turn gives us $\neg a - \frac{1}{\bar{n}} <_{\mathbb{R}} \omega$ with the Negation Formula for $<$, so that the disjunction

$$\neg a - \frac{1}{\bar{n}} <_{\mathbb{R}} \omega \,\vee\, \neg\omega <_{\mathbb{R}} b$$

is then also true. We therefore obtain with De Morgan's Law for the conjunction

$$\neg(a - \frac{1}{\bar{n}} <_{\mathbb{R}} \omega \,\wedge\, \omega <_{\mathbb{R}} b)$$

and then $\neg\omega \in (a - \frac{1}{\bar{n}}, b)$ by definition of an open interval in $\mathbb{R}$. This finding demonstrates the truth of the existential sentence (11.980), so that the proof is complete for the first case.

The second case $b \leq_{\mathbb{R}} \omega$ implies $\neg\omega <_{\mathbb{R}} b$ with the Negation Formula for $<$, leading to the true disjunction

$$\neg a - \frac{1}{1} <_{\mathbb{R}} \omega \,\vee\, \neg\omega <_{\mathbb{R}} b.$$

Similarly to the first case, an application of De Morgan's Law for the conjunction in connection with the definition of an open interval in $\mathbb{R}$ yields first

$$\neg(a - \frac{1}{1} <_{\mathbb{R}} \omega \wedge \omega <_{\mathbb{R}} b)$$

and then $\neg\omega \in (a - \frac{1}{1}, b)$. Because of $1 \in \mathbb{N}_+$, we thus see that the existential sentence (11.980) is true also in the second case.

Thus, the proof of the negated universal sentence (11.979) is complete, so that the second part of the equivalence (11.978) holds as well. Since ω was arbitrary, we may now infer from the truth of that equivalence the truth of (11.977), which universal sentence in turn gives us the equation (11.976). Applying now a substitution based on that equation to (11.975), we arrive at

$$[a, b) \in \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\}), \tag{11.981}$$

which yields (11.970) via substitution based on (11.964). Here, a and b were arbitrary, so that the proposition follows now to be true. □

Note 11.42. Since (11.976) shows that any left-closed and right-open interval $[a, b)$ in $\mathbb{R}$ is a 'non-finite' intersection of open sets in $\mathbb{R}$, we cannot establish this type of interval as an element of the order topology on $\mathbb{R}$, because a topology is closed only under n-ary (thus 'finite') intersections.

Exercise 11.50. Prove for any $a, b \in \mathbb{R}$ that the open interval from a to b is an element of the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$, that is,

$$(a, b) \in \mathcal{A}(\mathcal{I}). \tag{11.982}$$

(Hint: Proceed similarly as in the proof of Proposition 11.125, replacing the proof by contraposition by a direct proof.)

Proposition 11.126. *It is true that the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,*

$$\mathcal{A}(\mathcal{I}) = \mathcal{B}. \tag{11.983}$$

Proof. Let us establish first the conjunction

$$\mathcal{I} \subseteq \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\}) \ \wedge \ \{(a, b) : a, b \in \mathbb{R}\} \subseteq \mathcal{A}(\mathcal{I}), \tag{11.984}$$

by verifying the equivalent universal sentences (using the definition of a subset and (11.17))

$$\forall A\, (A \in \{[a, b) : a, b \in \mathbb{R}\} \Rightarrow A \in \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\})), \tag{11.985}$$

$$\forall A\, (A \in \{(a, b) : a, b \in \mathbb{R}\} \Rightarrow A \in \mathcal{A}(\mathcal{I})). \tag{11.986}$$

Letting first $A \in \{[a, b) : a, b \in \mathbb{R}\}$ be arbitrary, so that $A = [\bar{a}, \bar{b})$ holds for particular $\bar{a}, \bar{b} \in \mathbb{R}$ by definition of the set of left-closed and right-open intervals in $\mathbb{R}$, we obtain $A \in \mathcal{B}$ with (11.970) and therefore $A \in \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\})$ with (11.964). Since A is arbitrary, we may infer from the truth of this finding the truth of (11.985) and thus the truth of the first inclusion in (11.984). Next, we let $A \in \{(a, b) : a, b \in \mathbb{R}\}$ be arbitrary, so that there are particular real numbers $\bar{a}$ and $\bar{b}$ with $A = (\bar{a}, \bar{b})$. Consequently, we obtain $A \in \mathcal{A}(\mathcal{I})$ with (11.982), which is the desired consequent of the implication in (11.986). Because A is arbitrary, we may further conclude that the universal sentence (11.986) holds, which in turn yields the second inclusion in (11.984). Then, the conjunction (11.984) gives us with the Equality Criterion for generated σ-algebras and (11.964) the equations

$$\mathcal{A}(\mathcal{I}) = \mathcal{A}(\{(a, b) : a, b \in \mathbb{R}\}) = \mathcal{B}.$$

Thus, the proof of (11.983) is complete. □

Substitutions in Note 11.41 based on the equation (11.983) immediately gives us the following result.

Corollary 11.127. *It is true that any open and any closed set in $\mathbb{R}$ with respect to the order topology on $\mathbb{R}$ is contained in the σ-algebra $\mathcal{A}(\mathcal{I})$ generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$.*

Exercise 11.51. Establish the following sentences.

a) For any $a, b \in \mathbb{R}$, the left-open and right-closed interval from a real number a to a real number b is an element of the Borel σ-algebra on $\mathbb{R}$, that is,
$$(a, b] \in \mathcal{B}. \tag{11.987}$$

b) The open interval from a real number a to a real number b is an element of the σ-algebra generated by the set of left-open and right-closed intervals in $\mathbb{R}$, that is,
$$(a, b) \in \mathcal{A}(\{(a, b] : a, b \in \mathbb{R}\}). \tag{11.988}$$

c) The σ-algebra generated by the set of left-open and right-closed intervals in $\mathbb{R}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,
$$\mathcal{A}(\{(a, b] : a, b \in \mathbb{R}\}) = \mathcal{B}. \tag{11.989}$$

(Hint: Proceed as in Proposition 11.125, Exercise 11.50 and Proposition 11.126.)

Part a) of the preceding Exercise and (11.983) give us immediately the following elements of $\mathcal{A}(\mathcal{I})$.

Corollary 11.128. *It is true for any $a, b \in \mathbb{R}$ that the left-open and right-closed interval from a to b is an element of the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$, that is,*

$$(a, b] \in \mathcal{A}(\mathcal{I}). \tag{11.990}$$

Proposition 11.129. *It is true that the σ-algebra generated by the set of open and left-unbounded intervals in $\mathbb{R}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,*

$$\mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\}) = \mathcal{B}. \tag{11.991}$$

Proof. We begin with the proof of the conjunction

$$\{(-\infty, b) : b \in \mathbb{R}\} \subseteq \mathcal{A}(\mathcal{I}) \ \wedge \ \mathcal{I} \subseteq \mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\}), \tag{11.992}$$

which we may write equivalently as the conjunction of the universal sentences (using the definition of a subset and (11.17))

$$\forall A \, (A \in \{(-\infty, b) : b \in \mathbb{R}\} \Rightarrow A \in \mathcal{A}(\{[a, b) : a, b \in \mathbb{R}\})), \tag{11.993}$$
$$\forall A \, (A \in \{[a, b) : a, b \in \mathbb{R}\} \Rightarrow A \in \mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\}). \tag{11.994}$$

Regarding (11.993), we let $A \in \{(-\infty, b) : b \in \mathbb{R}\}$ be arbitrary, so that there exists (by definition of the set of open and left-unbounded intervals in $\mathbb{R}$) a particular number $\bar{b} \in \mathbb{R}$ with $A = (-\infty, \bar{b})$. Then, we obtain $A \in \mathcal{O}_{<_\mathbb{R}}$ with (11.580), where the inclusion and equations

$$\mathcal{O}_{<_\mathbb{R}} \subseteq \mathcal{A}(\mathcal{O}_{<_\mathbb{R}}) \quad [= \mathcal{B} = \mathcal{A}(\mathcal{I}) = \mathcal{A}(\{[a, b) : a, b \in \mathbb{R}\})]$$

hold by definition of a generated σ-algebra, due to (11.983) and with (11.17). Consequently, the definition of a subset yields $A \in \mathcal{A}(\{[a, b) : a, b \in \mathbb{R}\})$, proving the implication in (11.993); here, A is arbitrary, so that the first universal sentence follows to be true. Regarding (11.994), we now let $A \in \{[a, b) : a, b \in \mathbb{R}\}$ be arbitrary, which assumption gives us evidently a particular left-closed and right-open interval $A = [\bar{a}, \bar{b})$ in $\mathbb{R}$, for which we may write

$$[A =] \quad [\bar{a}, \bar{b}) = (-\infty, \bar{b}) \setminus (-\infty, \bar{a}) \tag{11.995}$$

because of (3.466). Here, we have

$$(-\infty, \bar{b}), (-\infty, \bar{a}) \in \{(-\infty, b) : b \in \mathbb{R}\} \quad [\subseteq \mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\})]$$

by definition of the set of open, left-unbounded intervals in $\mathbb{R}$ and with the definition of a generated σ-algebra, so that the definition of a subset gives

$$(-\infty, \bar{b}), (-\infty, \bar{a}) \in \mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\}). \tag{11.996}$$

Being a ring of sets according to Proposition 11.30c), the generated σ-algebra $\mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\})$ is closed under set differences, which is why (11.996) implies

$$(-\infty, \bar{b}) \setminus (-\infty, \bar{a}) \in \mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\}).$$

Consequently, substitution based on (11.995) yields the desired consequent $A \in \mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\})$ of the implication in (11.994). Since A is arbitrary, the second universal sentence follows now to be true as well.

The resulting truth of the conjunction of (11.993) and (11.994) implies then the truth of the conjunction (11.992), which further implies

$$\mathcal{A}(\{(-\infty, b) : b \in \mathbb{R}\}) = \mathcal{A}(\mathcal{I})$$

with the Equality Criterion for generated σ-algebras. Applying now a substitution based on the equation (11.983), we finally obtain the proposed equation (11.991). □

Note 11.43. The preceding proof shows that any open and left-unbounded interval in $\mathbb{R}$ is an element of the Borel σ-algebra on $\mathbb{R}$ and thus an element of the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$, that is,

$$(-\infty, b) \in \mathcal{B} \tag{11.997}$$
$$(-\infty, b) \in \mathcal{A}(\mathcal{I}) \tag{11.998}$$

are true for any $b \in \mathbb{R}$.

Exercise 11.52. Prove that the σ-algebra generated by the set of open and right-unbounded intervals in $\mathbb{R}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,

$$\mathcal{A}(\{(a, +\infty) : a \in \mathbb{R}\}) = \mathcal{B}. \tag{11.999}$$

(Hint: Carry out a proof similar to that of Proposition 11.129, using now (11.989) and (3.469).)

Note 11.44. The proof of the preceding exercise demonstrates that any open and right-unbounded interval in $\mathbb{R}$ is in the Borel σ-algebra on $\mathbb{R}$ and thus in the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$, that is,

$$(a,+\infty) \in \mathcal{B} \tag{11.1000}$$
$$(a,+\infty) \in \mathcal{A}(\mathcal{I}) \tag{11.1001}$$

hold for any $a \in \mathbb{R}$.

Proposition 11.130. *It is true that the σ-algebra generated by the set of left-unbounded and right-closed intervals in $\mathbb{R}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,*

$$\mathcal{A}(\{(-\infty,b] : b \in \mathbb{R}\}) = \mathcal{B}. \tag{11.1002}$$

Proof. We exploit the fact that the intervals $(-\infty,b]$ and $(b,+\infty)$ are related via complementation, under which operation a σ-algebra is closed. Let us prove then

$$\{(-\infty,b] : b \in \mathbb{R}\} \subseteq \mathcal{A}(\{(a,+\infty) : a \in \mathbb{R}\}) \tag{11.1003}$$
$$\wedge\ \{(a,+\infty) : a \in \mathbb{R}\} \subseteq \mathcal{A}(\{(-\infty,b] : b \in \mathbb{R}\}). \tag{11.1004}$$

The first inclusion can be written explicitly in the form

$$\forall A\,(A \in \{(-\infty,b] : b \in \mathbb{R}\} \Rightarrow A \in \mathcal{A}(\{(a,+\infty) : a \in \mathbb{R}\})), \tag{11.1005}$$

which universal sentence we prove directly by letting A be arbitrary and assuming $A \in \{(-\infty,b] : b \in \mathbb{R}\}$ to hold. By definition of the preceding set system of left-unbounded and right-closed intervals in $\mathbb{R}$, there is then a particular real number $\bar{b}$ with $A = (-\infty,\bar{b}]$, which we may write also as $A = (\bar{b},+\infty)^c$ by virtue of (3.450). Here, we evidently have $(\bar{b},+\infty) \in \{(a,+\infty) : a \in \mathbb{R}\}$ by definition of the set of open and right-unbounded intervals in $\mathbb{R}$, which set system is included in $\mathcal{A}(\{(a,+\infty) : a \in \mathbb{R}\})$ due to the definition of a generated σ-algebra. Therefore, the definition of a subset gives us $(\bar{b},+\infty) \in \mathcal{A}(\{(a,+\infty) : a \in \mathbb{R}\})$, which in turn implies with Property 4 of a σ-algebra

$$[A =] \quad (\bar{b},+\infty)^c \in \mathcal{A}(\{(a,+\infty) : a \in \mathbb{R}\}).$$

Thus, the desired consequent of the implication in (11.1005) is true. Since A is arbitrary, we may now infer from this the truth of the universal sentence (11.1005) and consequently the truth of the corresponding inclusion (11.1003).

We use similar arguments to establish the second inclusion (11.1004) via the proof of the universal sentence

$$\forall A\,(A \in \{(a,+\infty) : a \in \mathbb{R}\} \Rightarrow A \in \mathcal{A}(\{(-\infty,b] : b \in \mathbb{R}\})). \tag{11.1006}$$

We let again A be arbitrary, assuming now $A \in \{(a, +\infty) : a \in \mathbb{R}\}$ to hold. Clearly, there is then a particular real number $\bar{a}$ such that $A = (\bar{a}, +\infty)$. We may then apply (3.451) to express the set A in the form of the complement $A = (-\infty, \bar{a}]^c$, where we observe that $(-\infty, \bar{a}] \in \{(-\infty, b] : b \in \mathbb{R}\}$ holds. Because the preceding interval system generates the σ-algebra $\mathcal{A}(\{(-\infty, b] : b \in \mathbb{R}\}$, the interval system is included in the σ-algebra, so that the definition of a subset yields $(-\infty, \bar{a}] \in \mathcal{A}(\{(-\infty, b] : b \in \mathbb{R}\})$. This further implies (again by Property 4 of a σ-algebra)

$$[A =] \quad (-\infty, \bar{a}]^c \in \mathcal{A}(\{(-\infty, b] : b \in \mathbb{R}\}),$$

proving the implication in (11.1006). As the set A was arbitrary, the universal sentence (11.1006) follows to be true as well, and this finding gives us the inclusion in (11.1004) with the definition of a subset.

In conjunction with the other inclusion (11.1003), this implies the equation

$$\mathcal{A}(\{(-\infty, b] : b \in \mathbb{R}\}) = \mathcal{A}(\{(a, +\infty) : a \in \mathbb{R}\})$$

with the Equality Criterion for generated σ-algebras, where the σ-algebra on the right-hand side of the equation is identical with the Borel σ-algebra $\mathcal{B}$ according to (11.999). Consequently, the proposed equation (11.1002) holds as well. □

Note 11.45. Thus, any left-unbounded, right-closed interval in $\mathbb{R}$ is in the Borel σ-algebra on $\mathbb{R}$ and therefore in the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$, that is,

$$(-\infty, b] \in \mathcal{B} \tag{11.1007}$$

$$(-\infty, b] \in \mathcal{A}(\mathcal{I}) \tag{11.1008}$$

hold for any $b \in \mathbb{R}$.

Exercise 11.53. Prove that the σ-algebra generated by the set of left-closed and right-unbounded intervals in $\mathbb{R}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,

$$\mathcal{A}(\{[a, +\infty) : a \in \mathbb{R}\}) = \mathcal{B}. \tag{11.1009}$$

(Hint: Proceed similarly as in the proof of Proposition 11.130, using now (3.449), (3.447) and (11.991).)

Note 11.46. This shows that any left-closed, right-unbounded interval in $\mathbb{R}$ is in the Borel σ-algebra on $\mathbb{R}$ and consequently in the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$, that is,

$$[a, +\infty) \in \mathcal{B} \tag{11.1010}$$

$$[a, +\infty) \in \mathcal{A}(\mathcal{I}) \tag{11.1011}$$

are true for all $a \in \mathbb{R}$.

Proposition 11.131. *It is true for any $a, b \in \mathbb{R}$ that the closed interval from a to b is element both of the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$ and of the Borel σ-algebra on $\mathbb{R}$, i.e.*

$$[a, b] \in \mathcal{A}(\mathcal{I}), \tag{11.1012}$$

$$[a, b] \in \mathcal{B}. \tag{11.1013}$$

Proof. We take arbitrary real numbers a and b, so that we obtain the equations

$$[a, b] = (\, [a, b]^c \,)^c = (\, (-\infty, a) \,\cup\, (b, +\infty) \,)^c$$

by using (2.136) and (3.462). Here, we have $(-\infty, a), (b, +\infty) \in \mathcal{A}(\mathcal{I})$ in view of (11.1001) and (11.998), so that

$$(-\infty, a) \cup (b, +\infty) \in \mathcal{A}(\mathcal{I})$$

follows to be true with the fact that the σ-algebra $\mathcal{A}(\mathcal{I})$ is closed under pairwise unions (see Note 11.13). Because of Property 4 of a σ-algebra, we obtain furthermore

$$(\, (-\infty, a) \,\cup\, (b, +\infty) \,)^c \in \mathcal{A}(\mathcal{I}),$$

so that substitution based on the previously obtained equations gives the desired (11.1012). Then, another substitution based on (11.983) yields also (11.1013). As a and b are arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Corollary 11.132. *It is true that the singleton formed by any $a \in \mathbb{R}$ is contained both in the σ-algebra generated by the semiring of left-closed and right-open intervals in $\mathbb{R}$ and in the Borel σ-algebra on $\mathbb{R}$, i.e.*

$$\{a\} \in \mathcal{A}(\mathcal{I}), \tag{11.1014}$$

$$\{a\} \in \mathcal{B}. \tag{11.1015}$$

Proof. Letting $a \in \mathbb{R}$ be arbitrary, we have $[a, a] = \{a\}$ according to (3.363), so that we may evidently apply substitutions to (11.1012) and (11.1013) in order to obtain (11.1014) as well as (11.1015). Since a is arbitrary, the stated universal sentence follows then to be true. □

11.10.2. The Borel measurable space $(\mathbb{R}^n, \mathcal{B}^n)$

Definition 11.39 (Borel σ-algebra on $\mathbb{R}^n$, Borel set in $\mathbb{R}^n$, n-dimensional real Borel measurable space). We call for any $n \in \mathbb{N}_+$

$$\mathcal{B}(\mathbb{R}^n) = \mathcal{A}\left(\bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}}\right). \tag{11.1016}$$

the *Borel σ-algebra on* $\mathbb{R}^n$. We then call every element of $\mathcal{B}^n$ a *Borel set in* $\mathbb{R}^n$ and

$$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \tag{11.1017}$$

the *Borel measurable space on* $\mathbb{R}^n$.

Note 11.47. By definition of a generated σ-algebra, we thus have the inclusion

$$\bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}} \subseteq \mathcal{B}(\mathbb{R}^n). \tag{11.1018}$$

Note 11.48. Due to the Compatibility of product Borel σ-algebras and product topologies for separable metric spaces, we have

$$\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^{n} \mathcal{B}(\mathbb{R}) = \bigotimes_{i=1}^{n} \mathcal{B}, \tag{11.1019}$$

where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$ and where $\mathcal{B}(\mathbb{R}^n)$ constitutes the Borel σ-algebra on $\mathbb{R}^n$ with respect to the product topology $\bigotimes_{i=1}^{n} \mathcal{O}_{d_{\mathbb{R}}} = \bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}}$.

Note 11.49. Note 11.40 shows that the Borel σ-algebra on $\mathbb{R}^n$ contains both the empty set and the set of real n-tuples, that is,

$$\emptyset, \mathbb{R}^n \in \mathcal{B}(\mathbb{R}^n). \tag{11.1020}$$

Moreover, all open and all closed sets in $\mathbb{R}^n$ with respect to the product topology $\bigotimes_{i=1}^{n} \mathcal{O}_{d_{\mathbb{R}}} = \bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}}$ on $\mathbb{R}^n$ are also contained in $\mathcal{B}(\mathbb{R}^n)$ due to Proposition 11.118.

Corollary 11.133. *It is true for any $n \in \mathbb{N}_+$ that every open interval in $\mathbb{R}^n$ is contained in the Borel σ-algebra on $\mathbb{R}^n$, that is,*

$$\forall a, b\, (a, b \in \mathbb{R}^n \Rightarrow (a, b)_{\mathbb{R}^n} \in \mathcal{B}(\mathbb{R}^n)). \tag{11.1021}$$

Proof. Letting $n \in \mathbb{N}_+$ and $a, b \in \mathbb{R}^n$ be arbitrary, we find $(a, b)_{\mathbb{R}^n} \in \bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}}$ with (11.708). Due to the inclusion (11.1018), we therefore obtain $(a, b)_{\mathbb{R}^n} \in \mathcal{B}(\mathbb{R}^n)$ with the definition of a subset. Since n, a and b were initially arbitrary, we therefore conclude that the stated universal sentence holds. □

to be expanded!

11.10.3. The extended real Borel measurable space $(\overline{\mathbb{R}}, \overline{\mathcal{B}})$

Similarly to the case of real numbers, we may define a Borel σ-algebra based on the order topology on $\overline{\mathcal{B}}$ mentioned in Note 11.23.

Definition 11.40 (Borel σ-algebra on $\overline{\mathbb{R}}$, extended real Borel set, extended real Borel measurable space). We say that a set $\mathcal{B}$ is the *Borel σ-algebra on* $\mathbb{R}$ iff $\overline{\mathcal{B}}$ is the σ-algebra generated by the order topology on $\overline{\mathbb{R}}$, symbolically

$$\overline{\mathcal{B}} = \mathcal{B}(\overline{\mathbb{R}}) = \mathcal{A}(\mathcal{O}_{<_{\overline{\mathbb{R}}}}). \tag{11.1022}$$

We then call every element of $\overline{\mathcal{B}}$ an *extended real Borel set* and

$$(\overline{\mathbb{R}}, \overline{\mathcal{B}}) \tag{11.1023}$$

the *extended real Borel measurable space.*

Note 11.50. Recalling (11.921) – (11.921), it follows that the Borel σ-algebra on $\overline{\mathbb{R}}$ contains both the empty set and the set of extended real numbers, that is,

$$\emptyset, \overline{\mathbb{R}} \in \overline{\mathcal{B}}. \tag{11.1024}$$

Furthermore, Proposition 11.118 shows that every open and every closed set in $\overline{\mathbb{R}}$ with respect to the order topology on $\overline{\mathbb{R}}$ is also in $\overline{\mathcal{B}}$.

Corollary 11.134. *The set of real numbers is an extended real Borel set, that is,*

$$\mathbb{R} \in \overline{\mathcal{B}}. \tag{11.1025}$$

Proof. Since $\mathbb{R} \in \mathcal{O}_{<_{\overline{\mathbb{R}}}}$ is true according to (11.460) and since the Borel σ-algebra $\overline{\mathcal{B}}$ is generated by $\mathcal{O}_{<_{\overline{\mathbb{R}}}}$ by definition, so that the inclusion

$$\mathcal{O}_{<_{\overline{\mathbb{R}}}} \subseteq \overline{\mathcal{B}} \tag{11.1026}$$

holds by definition of a generated σ-algebra, we obtain (11.1025) with the definition of a subset. □

The preceding inclusion gives in connection with (11.584) – (11.585) the following elements of the Borel σ-algebra on $\overline{\mathbb{R}}$.

Corollary 11.135. *It is true*

a) *for any $b \in \overline{\mathbb{R}}$ that the open and left-unbounded interval ending in b, or equivalently the left-closed and right-open interval from $-\infty$ to b, is contained in the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.*

$$[-\infty, b)_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}}, \tag{11.1027}$$

and

b) *for any $a \in \overline{\mathbb{R}}$ that the open and right-unbounded interval beginning in a, or equivalently the left-open and right-closed interval from a to $+\infty$, is contained in the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.*

$$(a, +\infty]_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}}. \tag{11.1028}$$

Corollary 11.136. *It is true that any left-closed and right-open as well as any left-open and right-closed interval in $\overline{\mathbb{R}}$ is contained in the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.*

$$[a, b)_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}} \tag{11.1029}$$

$$(a, b]_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}} \tag{11.1030}$$

hold for any $a, b \in \overline{\mathbb{R}}$.

Proof. Letting a and b be arbitrary extended real numbers, we obtain with (3.487) and (3.489) – in connection with (9.52) as well as (9.53) – the equations

$$[a, b)_{\overline{\mathbb{R}}} = [-\infty, b)_{\overline{\mathbb{R}}} \setminus [-\infty, a)_{\overline{\mathbb{R}}}, \tag{11.1031}$$

$$(a, b]_{\overline{\mathbb{R}}} = (a, +\infty]_{\overline{\mathbb{R}}} \setminus (b, +\infty]_{\overline{\mathbb{R}}}. \tag{11.1032}$$

In view of (11.1027) and (11.1028), we have here

$$[-\infty, b)_{\overline{\mathbb{R}}}, [-\infty, a)_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}},$$
$$(a, +\infty]_{\overline{\mathbb{R}}}, (b, +\infty]_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}}.$$

Then, since the σ-algebra $\overline{\mathcal{B}}$ – being a ring of sets according to Proposition 11.30c) – is closed under set differences, the particular set differences (11.1031) and (11.1032) follow to be elements of $\overline{\mathcal{B}}$. As a and b are arbitrary, we may therefore conclude that (11.1029) and (11.1030) are universally true. □

Exercise 11.54. Show that the closed interval from an $a \in \overline{\mathbb{R}}$ to a $b \in \overline{\mathbb{R}}$ is an element of the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.

$$[a, b]_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}}. \tag{11.1033}$$

(Hint: Use (3.486), (9.52), (9.53)), (2.136), Note 11.13, and Property 4 of a σ-algebra.)

Corollary 11.137. *It is true that the singleton formed by any $a \in \overline{\mathbb{R}}$ is contained in the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.*

$$\{a\} \in \overline{\mathcal{B}}. \tag{11.1034}$$

Proof. Letting $a \in \overline{\mathbb{R}}$ be arbitrary, we have $[a,a]_{\overline{\mathbb{R}}} = \{a\}$ according to (3.363), so that a substitution based on (11.1033) yields immediately (11.1034). Because a is arbitrary, we may therefore conclude that the stated universal sentence is true. □

Corollary 11.138. *The trace of the Borel σ-algebra on $\overline{\mathbb{R}}$ in $\mathbb{R}$ is identical with the Borel σ-algebra on $\mathbb{R}$, that is,*

$$\overline{\mathcal{B}}|\mathbb{R} = \mathcal{B}. \tag{11.1035}$$

Proof. We obtain the equations

$$\overline{\mathcal{B}}|\mathbb{R} = \mathcal{A}(\mathcal{O}_{<_{\overline{\mathbb{R}}}})|\mathbb{R} = \mathcal{A}(\mathcal{O}_{<_{\overline{\mathbb{R}}}}|\mathbb{R}) = \mathcal{A}(\mathcal{O}_{<_{\mathbb{R}}}) = \mathcal{B}$$

by applying substitution based on the definition of the Borel σ-algebra on $\overline{\mathbb{R}}$, using the Compatibility of subspace topologies and trace σ-algebras, applying then a substitution based on (11.615), and using finally the definition of the Borel σ-algebra on $\mathbb{R}$, which equations give us then (11.1035). □

Corollary 11.139. *It is true that the intersection of the set of real numbers and an extended real Borel set constitutes a real Borel set, that is,*

$$\forall A\,(A \in \overline{\mathcal{B}} \Rightarrow \mathbb{R} \cap A \in \mathcal{B}). \tag{11.1036}$$

Proof. Letting $\bar{A} \in \overline{\mathcal{B}}$ be arbitrary, we observe the truth of the equation $\mathbb{R} \cap \bar{A} = \mathbb{R} \cap \bar{A}$ and then the truth of the existential sentence

$$\exists A\,(A \in \overline{\mathcal{B}} \wedge \mathbb{R} \cap A = \mathbb{R} \cap \bar{A}),$$

which implies with the definition of a trace σ-algebra $\mathbb{R} \cap \bar{A} \in \overline{\mathcal{B}}|\mathbb{R}$. A substitution based on the equation (11.1035) yields now $\mathbb{R} \cap \bar{A} \in \mathcal{B}$, proving the implication in (11.1036). Here, $\bar{A}$ was arbitrary, so that the proposed universal sentence follows to be true. □

Corollary 11.140. *The Borel σ-algebra on $\mathbb{R}$ is included in the Borel σ-algebra on $\overline{\mathbb{R}}$, that is,*

$$\mathcal{B} \subseteq \overline{\mathcal{B}} \tag{11.1037}$$

Proof. Since $\mathbb{R}$ is a subset of $\overline{\mathbb{R}}$ and moreover a measurable set of $\overline{\mathcal{B}}$ according to (11.1025), it follows with Corollary 11.42 that the trace of $\overline{\mathcal{B}}$ in $\mathbb{R}$ is included in $\overline{\mathcal{B}}$, i.e. $\overline{\mathcal{B}}|\mathbb{R} \subseteq \overline{\mathcal{B}}$. This inclusion yields then the desired equation (11.1037) via substitution based on (11.1035). □

Note 11.51. In light of the inclusion (11.1037), the definition of a subset and the findings of Section 11.10.1, we now see that

a) any open and any closed set in $\mathbb{R}$ with respect to the order topology on $\mathbb{R}$ is an element of $\overline{\mathcal{B}}$.

b) any bounded interval in $\mathbb{R}$ is in the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.

$$(a,b)_{\mathbb{R}},\ [a,b]_{\mathbb{R}},\ [a,b)_{\mathbb{R}},\ (a,b]_{\mathbb{R}} \in \overline{\mathcal{B}} \tag{11.1038}$$

hold for any $a,b \in \mathbb{R}$.

c) any unbounded interval in $\mathbb{R}$ is in the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.

$$(a,+\infty)_{\mathbb{R}},\ (-\infty,b)_{\mathbb{R}},\ [a,+\infty)_{\mathbb{R}},\ (-\infty,b]_{\mathbb{R}} \in \overline{\mathcal{B}} \tag{11.1039}$$

are true for all $a,b \in \mathbb{R}$.

Corollary 11.141. *It is true for any $a,b \in \overline{\mathbb{R}}$ that the open interval from a to b is element of the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.*

$$(a,b)_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}}. \tag{11.1040}$$

Proof. We see in (11.450) that the union

$$\mathcal{K}_{<_{\overline{\mathbb{R}}}} = \bigcup \{\{(a,b) : a,b \in \overline{\mathbb{R}}\}, \{[-\infty,b) : b \in \overline{\mathbb{R}}\}, \{(a,+\infty] : a \in \overline{\mathbb{R}}\}\} \tag{11.1041}$$

constitutes the basis for the order topology $\mathcal{O}_{<_{\overline{\mathbb{R}}}}$. Therefore, each of the sets of intervals in the triple is included in the union $\mathcal{K}_{<_{\overline{\mathbb{R}}}}$ according to (2.201). This basis in turn is included in the generated topology $\mathcal{O}_{<_{\overline{\mathbb{R}}}}$ due to Proposition 11.55. In addition, that order topology is included in the Borel σ-algebra $\overline{\mathcal{B}}$, as shown in (11.1026). Applying now (2.13) to the previous inclusions, we obtain then

$$\{(a,b) : a,b \in \overline{\mathbb{R}}\} \subseteq \overline{\mathcal{B}}. \tag{11.1042}$$

Letting now $\bar{a}$ and $\bar{b}$ be arbitrary extended real numbers, we clearly have $(\bar{a},\bar{b}) \in \{(a,b) : a,b \in \overline{\mathbb{R}}\}$, so that the definition of a subset yields (11.1040). This finding is then universally true, because a and b were arbitrary. □

Proposition 11.142. *It is true for any $a \in \overline{\mathbb{R}}$ that the closed interval from a to $+\infty$ is element of the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.*

$$[a,+\infty]_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}}. \tag{11.1043}$$

Proof. We let a be an arbitrary extended real number, y an arbitrary constant, and observe the truth of the equivalences

$$\begin{aligned} y \in [a,+\infty]_{\overline{\mathbb{R}}} &\Leftrightarrow y \in \overline{\mathbb{R}} \wedge a \leq_{\overline{\mathbb{R}}} y \\ &\Leftrightarrow y \in \overline{\mathbb{R}} \wedge \neg y <_{\overline{\mathbb{R}}} a \\ &\Leftrightarrow y \in \overline{\mathbb{R}} \wedge \neg y \in [-\infty,a) \\ &\Leftrightarrow y \in \overline{\mathbb{R}} \setminus [-\infty,a) \end{aligned}$$

in light of the specification of a closed and right-unbounded interval in $\overline{\mathbb{R}}$ in connection with Proposition 3.139b) and (9.53), the Negation Formula for $<$, the definition of a left-unbounded and right-open interval in $\overline{\mathbb{R}}$ in connection with Exercise 3.66c) and (9.52), and the definition of a set difference. Since y is arbitrary, we may therefore infer from the truth of these equivalences the truth of the equation

$$[a, +\infty]_{\overline{\mathbb{R}}} = \overline{\mathbb{R}} \setminus [-\infty, a) \tag{11.1044}$$

by means of the Equality Criterion for sets. Here, we have $\overline{\mathbb{R}}, [-\infty, a) \in \overline{\mathcal{B}}$ because of (11.1025) and (11.1027), so that we obtain for the difference of these sets $\overline{\mathbb{R}} \setminus [-\infty, a) \in \overline{\mathcal{B}}$ with the fact that any σ-algebra – being a ring of sets as shown by Proposition 11.30c) – is closed under set differences. Then, substitution based on the equation (11.1044) yields (11.1043). Consequently, the proposed universal sentence follows to be true since a was arbitrary. □

Exercise 11.55. Prove for any $b \in \overline{\mathbb{R}}$ that the closed interval from $-\infty$ to b is element of the Borel σ-algebra on $\overline{\mathbb{R}}$, i.e.

$$[-\infty, b]_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}}. \tag{11.1045}$$

(Hint: Apply a similar proof as in Proposition 11.142.)

Exercise 11.56. Show that there are unique sequences

- $s_1 = ([-\infty, -n])_{n \in \mathbb{N}_+}$,
- $s_2 = ([-\infty, n]^c)_{n \in \mathbb{N}_+}$,
- $s_3 = ([-\infty, -n))_{n \in \mathbb{N}_+}$,
- $s_4 = ([-\infty, n)^c)_{n \in \mathbb{N}_+}$,
- $s_5 = ((n, +\infty])_{n \in \mathbb{N}_+}$,
- $s_6 = ((-n, +\infty]^c)_{n \in \mathbb{N}_+}$,
- $s_7 = ([n, +\infty])_{n \in \mathbb{N}_+}$,
- $s_8 = ([-n, +\infty]^c)_{n \in \mathbb{N}_+}$.

in $\overline{\mathcal{B}}$. Establish then the equations $s_1 = s_6$, $s_3 = s_8$, $s_5 = s_2$ and $s_7 = s_4$.

Proposition 11.143. *It is true that the singleton formed by $-\infty$ can be written as the countable intersection of left-closed, right-open intervals in $\overline{\mathbb{R}}$ beginning in $-\infty$, that is,*

$$\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n). \tag{11.1046}$$

Proof. We accomplish the proof by means of the Equality Criterion for sets, i.e. by proving

$$\forall \omega \, (\, \omega \in \{-\infty\} \Leftrightarrow \omega \in \bigcap_{n=1}^{\infty} [-\infty, -n) \,). \tag{11.1047}$$

We take an arbitrary constant ω and prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\omega \in \{-\infty\}$. We therefore obtain $\omega = -\infty$ with (2.169), which shows in light of the definition of the set of extended real numbers that $\omega \in \overline{\mathbb{R}}$ is true. Furthermore, we may write the desired consequent in the form of the universal sentence

$$\forall n \, (\, n \in \mathbb{N}_+ \Rightarrow \omega \in [-\infty, -n) \,), \tag{11.1048}$$

using the Characterization of the intersection of a family of sets. To prove it, we let $n \in \mathbb{N}_+$ be arbitrary, so that n is evidently an extended real number. Consequently, we also have $-n \in \overline{\mathbb{R}}$, so that the definition of the standard ordering of $\overline{\mathbb{R}}$ gives $(-\infty, -n) \in <_{\overline{\mathbb{R}}}$, which we may write also as $-\infty <_{\overline{\mathbb{R}}} -n$. A substitution based on the previously established equation $\omega = -\infty$ yields now $\omega <_{\overline{\mathbb{R}}} -n$. Furthermore, $\omega \in \overline{\mathbb{R}}$ implies $-\infty \leq_{\overline{\mathbb{R}}} \omega$ with (9.50), so that $\omega \in [-\infty, -n)$ follows to be true by definition of a left-closed and right-open interval in $\overline{\mathbb{R}}$. Since n is arbitrary, we may therefore conclude that the universal sentence (11.1048) holds, which finding completes the proof of the first part of the equivalence in (11.1047).

We prove the second part ('$\Leftarrow$') of the equivalence also directly, assuming $\omega \in \bigcap_{n=1}^{\infty} [-\infty, -n)$ to be true, which we write again in the form of the universal sentence (11.1048). Since the range of the sequence $([-\infty, -n))_{n \in \mathbb{N}_+}$ contains for instance $[-\infty, -1)$, the intersection $\bigcap_{n=1}^{\infty} [-\infty, -n)$ is included in $[-\infty, -1)$ according to (2.92). As this interval in $\overline{\mathbb{R}}$ is evidently included in $\overline{\mathbb{R}}$, we obtain the inclusion $\bigcap_{n=1}^{\infty} [-\infty, -n) \subseteq \overline{\mathbb{R}}$ with (2.13). With this inclusion, the assumed antecedent implies $\omega \in \overline{\mathbb{R}}$ by definition of a subset. Thus, the multiple disjunction

$$\omega = -\infty \, \vee \, \omega \in \mathbb{R} \, \vee \, \omega = +\infty \tag{11.1049}$$

is true according to (9.17), whose second and third part we now prove to be wrong via two proofs by contradiction. To establish the negation $\neg \omega \in \mathbb{R}$, we assume its negation to be true, so that the Double Negation Law yields the true sentence $\omega \in \mathbb{R}$. Then, $-\omega \in \mathbb{R}$ evidently holds as well, so that there exists – according to the Archimedean Property – a positive natural number, say $\bar{n}$, such that $-\omega <_{\mathbb{R}} \bar{n}$. Using the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$, we may write this inequality as $-\bar{n} <_{\mathbb{R}} \omega$, and then also as $-\bar{n} <_{\overline{\mathbb{R}}} \omega$ by virtue of (9.32) and the evident fact that $\bar{n}$ is also a real number (besides

ω). Now, $\bar{n} \in \mathbb{N}_+$ implies with (11.1048) also $\omega \in [-\infty, -\bar{n})$, which yields $\omega <_{\overline{\mathbb{R}}} -\bar{n}$. In view of the previously established $-\bar{n} <_{\overline{\mathbb{R}}} \omega$, the transitivity of the linear ordering $<_{\overline{\mathbb{R}}}$ gives us $\omega <_{\overline{\mathbb{R}}} \omega$. Since the negation $\neg \omega <_{\overline{\mathbb{R}}} \omega$ is also true due to the irreflexivity of $<_{\overline{\mathbb{R}}}$, we arrived at a contradiction, so that the proof of $\neg \omega \in \mathbb{R}$ is complete.

Next, we prove also $\neg \omega = +\infty$ by contradiction, assuming $\neg\neg \omega = +\infty$, so that evidently $\omega = +\infty$. Then, the positive natural number 1 gives us with (11.1048) $\omega \in [-\infty, -1)$, with the consequence that $\omega <_{\overline{\mathbb{R}}} -1$. Recalling the equation $\omega = +\infty$, we thus find $+\infty <_{\overline{\mathbb{R}}} -1$ in contradiction to the fact $\neg +\infty <_{\overline{\mathbb{R}}} -1$ provided by (9.31). This completes the proof of $\neg \omega = +\infty$, so that the first part $\omega = -\infty$ of the true disjunction (11.1049) must be true. Because this finding implies $\omega \in \{-\infty\}$, the second part of the equivalence in (11.1047) holds then, too.

Here, ω was arbitrary, so that the universal sentence (11.1047) follows to be true, which in turn implies the truth of the equation (11.1046) by means of the Equality Criterion for sets. □

Corollary 11.144. *It is true that the singleton formed by $-\infty$ is an element of the Borel σ-algebra on $\overline{\mathbb{R}}$, that is,*

$$\{-\infty\} \in \overline{\mathcal{B}}. \tag{11.1050}$$

Proof. Because the σ-algebra $\overline{\mathcal{B}}$ satisfies Proposition 11.31, we obtain for the sequence $s_1 = ([-\infty, -n))_{n \in \mathbb{N}_+}$ in $\overline{\mathcal{B}}$ (established in Exercise 11.56)

$$\bigcap_{n=1}^{\infty} [-\infty, -n) \in \overline{\mathcal{B}}. \tag{11.1051}$$

Then, applying a substitution to (11.1051) based on the equation (11.1046), we obtain the proposed sentence (11.1050). □

Corollary 11.145. *Prove that the singleton formed by $-\infty$ can also be written as the countable intersection of complements of closed intervals in $\overline{\mathbb{R}}$ ending in $+\infty$, that is,*

$$\{-\infty\} = \bigcap_{n=1}^{\infty} [-n, +\infty]^c. \tag{11.1052}$$

Proof. We obtain the equations

$$\begin{aligned}
\{-\infty\} &= \bigcap_{n=1}^{\infty} [-\infty, -n) \\
&= \bigcap \operatorname{ran}(s_3) \\
&= \bigcap \operatorname{ran}(s_8) \\
&= \bigcap_{n=1}^{\infty} [-n, +\infty]^c
\end{aligned}$$

by using (11.1046) and the definition of the intersection of a family of sets in connection with the findings of Exercise 11.56. □

Exercise 11.57. Prove that the singleton formed by $-\infty$ can be written as the countable intersection of closed intervals in $\overline{\mathbb{R}}$ beginning in $-\infty$, i.e.

$$\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n] \tag{11.1053}$$

$$= \bigcap_{n=1}^{\infty} (-n, +\infty]^c \tag{11.1054}$$

(Hint: Carry out a proof similarly to that of Proposition 11.143, using the definition of an induced reflexive/total ordering and then the findings of Exercise 11.56.)

Exercise 11.58. Establish the equations

$$\{+\infty\} = \bigcap_{n=1}^{\infty} (n, +\infty], \tag{11.1055}$$

$$= \bigcap_{n=1}^{\infty} [-\infty, n]^c, \tag{11.1056}$$

$$= \bigcap_{n=1}^{\infty} [n, +\infty], \tag{11.1057}$$

$$= \bigcap_{n=1}^{\infty} [-\infty, n)^c, \tag{11.1058}$$

and show then that the singleton formed by $+\infty$ is an element of the Borel σ-algebra on $\overline{\mathbb{R}}$, that is,

$$\{+\infty\} \in \overline{\mathcal{B}}. \tag{11.1059}$$

(Hint: Proceed in analogy to Proposition 11.143, Exercise 11.57, Corollary 11.145 and Corollary 11.144.)

Corollary 11.146. *It is true that the pair formed by* $-\infty$ *and* $+\infty$ *is an extended real Borel set, that is,*

$$\{-\infty,+\infty\} \in \overline{\mathcal{B}}. \tag{11.1060}$$

Proof. Since $\{-\infty\}, \{+\infty\} \in \overline{\mathcal{B}}$ holds in view of (11.1050) and (11.1059), we obtain $\{-\infty\} \cup \{+\infty\} \in \overline{\mathcal{B}}$ with the fact that any σ-algebra is closed under pairwise unions (see Note 11.13). Because (2.226) gives $\{-\infty\} \cup \{+\infty\} = \{-\infty,+\infty\}$, substitution yields (11.1060). □

Proposition 11.147. *The intersection of the set of real numbers and an extended real Borel set* A *is identical with the difference of* A *and the pair formed by* $-\infty$ *and* $+\infty$*, that is,*

$$\forall A\,(A \in \overline{\mathcal{B}} \Rightarrow \mathbb{R} \cap A = A \setminus \{-\infty,+\infty\}). \tag{11.1061}$$

Proof. Letting $A \in \overline{\mathcal{B}}$ be arbitrary, we apply now the Equality Criterion for sets and verify the universal sentence

$$\forall y\,(y \in \mathbb{R} \cap A \Leftrightarrow y \in A \setminus \{-\infty,+\infty\}). \tag{11.1062}$$

We let y be arbitrary and assume first $y \in \mathbb{R} \cap A$ to be true, so that the definition of the intersection of two sets gives us $y \in \mathbb{R}$ and $y \in A$. Since we also have $-\infty \notin \mathbb{R}$ and $+\infty \notin \mathbb{R}$ by definition of the set of extended real numbers, we may apply (2.4) to infer on the one hand from $y \in \mathbb{R}$ and $-\infty \notin \mathbb{R}$ that $y \neq -\infty$ holds; on the other hand, $y \in \mathbb{R}$ and $+\infty \notin \mathbb{R}$ yields $y \neq +\infty$. We then obtain the true equivalences

$$\begin{aligned}
\neg y = -\infty \wedge \neg y = +\infty &\Leftrightarrow \neg y \in \{-\infty\} \wedge \neg y \in \{+\infty\} \\
&\Leftrightarrow \neg(y \in \{-\infty\} \vee y \in \{+\infty\}) \\
&\Leftrightarrow \neg y \in \{-\infty\} \cup \{+\infty\} \\
&\Leftrightarrow \neg y \in \{-\infty,+\infty\}
\end{aligned}$$

by means of (2.169), De Morgan's Law for the disjunction, the definition of the union of two sets, and (2.226). Thus, the previously established $y \neq -\infty$ and $y \neq +\infty$ imply $\neg y \in \{-\infty,+\infty\}$, which further implies in conjunction with the true $y \in A$ that $y \in A \setminus \{-\infty,+\infty\}$ holds, by virtue of the definition of a set difference. We thus proved the first part ('⇒') of the equivalence in (11.1062).

Regarding the second part ('⇐'), we now assume $y \in A \setminus \{-\infty,+\infty\}$ to be true, so that the definition of a set difference gives us $y \in A$ and $\neg y \in$

$\{-\infty, +\infty\}$. The latter implies in view of the four equivalences established in the proof of ('$\Rightarrow$') that $y \neq -\infty$ and $y \neq +\infty$ hold. Since the inclusion $\overline{\mathcal{B}} \subseteq \mathcal{P}(\overline{\mathbb{R}})$ is true according to Property 1 of a σ-algebra (recalling that $\overline{\mathcal{B}}$ represents the Borel σ-algebra on $\overline{\mathbb{R}}$), the initial assumption $A \in \overline{\mathcal{B}}$ implies $A \in \mathcal{P}(\overline{\mathbb{R}})$ with the definition of a subset and therefore $A \subseteq \overline{\mathbb{R}}$ with the definition of a power set. Consequently, $y \in A$ implies $y \in \overline{\mathbb{R}}$ (using again the definition of a subset), so that the disjunction $y = -\infty \vee y \in \mathbb{R} \vee y = +\infty$ holds according to (9.17). As we proved earlier that $y \neq -\infty$ and $y \neq +\infty$ are true, the second part $y \in \mathbb{R}$ of that disjunction must be true. In conjunction with $y \in A$, this implies now $y \in \mathbb{R} \cap A$ with the definition of the intersection of two sets, proving the second part of the equivalence in (11.1062). Since y was arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (11.1062), and therefore the truth of the equation $\mathbb{R} \cap A = A \setminus \{-\infty, +\infty\}$. As the set A was initially arbitrary, we may then further conclude that the proposed universal sentence holds. $\square$

The following sentences specializes the finding of the preceding Proposition to cases where an extended real Borel set does not contain $-\infty$ or $+\infty$.

Exercise 11.59. Establish the following universal sentences in analogy to (11.1061).

$$\forall A\,(A \in \overline{\mathcal{B}} \Rightarrow [+\infty \notin A \Rightarrow \mathbb{R} \cap A = A \setminus \{-\infty\}]), \tag{11.1063}$$

$$\forall A\,(A \in \overline{\mathcal{B}} \Rightarrow [-\infty \notin A \Rightarrow \mathbb{R} \cap A = A \setminus \{+\infty\}]), \tag{11.1064}$$

$$\forall A\,(A \in \overline{\mathcal{B}} \Rightarrow [(-\infty \notin A \wedge +\infty \notin A) \Rightarrow \mathbb{R} \cap A = A]). \tag{11.1065}$$

After this preparation, we are now in a position to prove that every extended real Borel set can be decomposed into a real Borel set and singleton or pair formed by the non-real elements $-\infty$ and $+\infty$.

Theorem 11.148 (Characterization of extended real Borel sets). *It is true that any extended real Borel set is a real Borel set, the union of some real Borel set with $\{-\infty\}$, the union of some real Borel set with $\{+\infty\}$, or the union of some real Borel set with $\{-\infty, +\infty\}$, that is,*

$$\begin{aligned} \forall A\,(A \in \overline{\mathcal{B}} \Leftrightarrow [A \in \mathcal{B} \vee{} & \exists B\,(B \in \mathcal{B} \wedge A = B \cup \{-\infty\}) && (11.1066)\\ \vee{} & \exists B\,(B \in \mathcal{B} \wedge A = B \cup \{+\infty\}) && (11.1067)\\ \vee{} & \exists B\,(B \in \mathcal{B} \wedge A = B \cup \{-\infty, +\infty\})]). && (11.1068) \end{aligned}$$

Proof. Letting A be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming A to be an extended real Borel set. We therefore have $\mathbb{R} \cap A \in \mathcal{B}$ due to (11.1036). In the following, we prove the multiple disjunction by considering the two cases $-\infty \in A$ and $-\infty \notin A$, and in each of these cases the two sub-cases $+\infty \in A$ and $+\infty \notin A$.

In the first case $-\infty \in A$ and the first sub-case $+\infty \in A$, we observe the truth of the equation

$$\mathbb{R} \cap A = A \setminus \{-\infty, +\infty\}$$

in (11.1061). In view of $\mathbb{R} \cap A \in \mathcal{B}$, a substitution based on that equation yields

$$A \setminus \{-\infty, +\infty\} \in \mathcal{B}. \tag{11.1069}$$

Let us observe now that the assumed $-\infty, +\infty \in A$ implies $\{-\infty, +\infty\} \subseteq A$ with (2.164). Therefore, an application of (2.263) gives us

$$A = [A \setminus \{-\infty, +\infty\}] \cup \{-\infty, +\infty\}. \tag{11.1070}$$

Having found the particular set $\bar{B} = A \setminus \{-\infty, +\infty\}$ satisfying $\bar{B} \in \overline{\mathcal{B}}$ and $A = \bar{B} \cup \{-\infty, +\infty\}$ according to (11.1069) and (11.1070), we thus see that the existential sentence in (11.1068) holds, and the multiple disjunction to be proven is then also true.

In the second sub-case $+\infty \notin A$ (of the current first case $-\infty \in A$), we obtain with (11.1063)

$$\mathbb{R} \cap A = A \setminus \{-\infty\}.$$

Recalling $\mathbb{R} \cap A \in \mathcal{B}$, the preceding equation yields

$$A \setminus \{-\infty\} \in \mathcal{B}. \tag{11.1071}$$

Because the current case assumption $-\infty \in A$ implies $\{-\infty\} \subseteq A$ with (2.184), we may apply again (2.263) to obtain the equation

$$A = [A \setminus \{-\infty\}] \cup \{-\infty\}. \tag{11.1072}$$

We thus found the set $\bar{B} = A \setminus \{-\infty\}$ for which $\bar{B} \in \overline{\mathcal{B}}$ and $A = \bar{B} \cup \{-\infty\}$ are satisfied, as shown by (11.1071) and (11.1072). This finding demonstrates the truth of the existential sentence in (11.1066), with the consequence that the desired multiple disjunction holds again. Thus, the proof of the first case is complete.

Regarding the second case, we assume $-\infty \notin A$, and we consider once again the first sub-case $+\infty \in A$. Therefore,

$$\mathbb{R} \cap A = A \setminus \{+\infty\}.$$

is true according to (11.1064), and combining this equation with $\mathbb{R} \cap A \in \mathcal{B}$ gives rise to the true sentence

$$A \setminus \{+\infty\} \in \mathcal{B}. \tag{11.1073}$$

Furthermore, the current sub-case assumption $+\infty \in A$ implies $\{+\infty\} \subseteq A$ with (2.184), so that another application of (2.263) gives us

$$A = [A \setminus \{+\infty\}] \cup \{+\infty\}. \tag{11.1074}$$

The set $\bar{B} = A \setminus \{+\infty\}$ satisfies then $\bar{B} \in \overline{\mathcal{B}}$ and $A = \bar{B} \cup \{+\infty\}$ by virtue of (11.1073) – (11.1074), establishing thus the truth of the existential sentence in (11.1067). Consequently, the multiple disjunction to be proven is again true.

Finally, the second sub-case $+\infty \notin A$ within the second case $-\infty \notin A$ gives us $\mathbb{R} \cap A = A$ with (11.1065) and therefore $A \in \mathcal{B}$ via substitution based on the previously established $\mathbb{R} \cap A \in \mathcal{B}$. This finding immediately implies the truth of the multiple disjunction, which thus holds in any case.

With this, the proof of the implication '$\Rightarrow$' in (11.1066) – (11.1067) is complete.

To establish conversely the implication '$\Leftarrow$', we assume the multiple disjunction to be true. We now apply the Distributive Law for quantification (1.75) twice to write it in the form of the simple disjunction

$$\begin{aligned} A \in \mathcal{B} \vee \exists B\,([B \in \mathcal{B} \,\wedge\, A = B \cup \{-\infty\}] \quad &(11.1075)\\ \vee\ [B \in \mathcal{B} \,\wedge\, A = B \cup \{+\infty\}] \quad &(11.1076)\\ \vee\ [B \in \mathcal{B} \,\wedge\, A = B \cup \{-\infty, +\infty\}]), \quad &(11.1077) \end{aligned}$$

which we use to prove the desired consequent $A \in \overline{\mathcal{B}}$ by cases. The first case $A \in \mathcal{B}$ immediately implies $A \in \overline{\mathcal{B}}$ because of the inclusion $\mathcal{B} \subseteq \overline{\mathcal{B}}$ in (11.1037) and the definition of a subset. In the second case, there exists a set, say $\bar{B}$, satisfying the threefold disjunction

$$\begin{aligned} [\bar{B} \in \mathcal{B} \,\wedge\, A = \bar{B} \cup \{-\infty\}] \vee [\bar{B} \in \mathcal{B} \,\wedge\, A = \bar{B} \cup \{+\infty\}] \\ \vee\ [\bar{B} \in \mathcal{B} \,\wedge\, A = \bar{B} \cup \{-\infty, +\infty\}]. \end{aligned}$$

The Distributivity of the conjunction over the disjunction allows us evidently to write the preceding sentence in the form of

$$\bar{B} \in \mathcal{B} \,\wedge\, [A = \bar{B} \cup \{-\infty\} \,\vee\, A = \bar{B} \cup \{+\infty\} \,\vee\, A = \bar{B} \cup \{-\infty, +\infty\}]. \tag{11.1078}$$

This shows on the one hand that $\bar{B} \in \mathcal{B}$ is true, and this clearly implies $\bar{B} \in \overline{\mathcal{B}}$ with the previously mentioned inclusion $\mathcal{B} \subseteq \overline{\mathcal{B}}$. On the other hand, we may use the threefold disjunction in (11.1078) to prove the desired $A \in \overline{\mathcal{B}}$ via three sub-cases.

In the first sub-case, we have $A = \bar{B} \cup \{-\infty\}$, where $\bar{B}$ and $\{-\infty\}$ are both elements of $\overline{\mathcal{B}}$ in view of (11.1050). Consequently, their union $A = \bar{B} \cup \{-\infty\}$ is also in $\overline{\mathcal{B}}$, recalling that any σ-algebra is closed under pairwise unions (see Note 11.13).

Similarly, the second sub-case $A = \bar{B} \cup \{+\infty\}$ implies $A \in \overline{\mathcal{B}}$ with the fact that $\bar{B}, \{+\infty\} \in \overline{\mathcal{B}}$ holds due to (11.1059), so that the union $\bar{B} \cup \{+\infty\}$ is also contained in $\overline{\mathcal{B}}$.

Finally, we see in the third case $A = \bar{B} \cup \{-\infty, +\infty\}$ in light of (11.1060) that $\bar{B}, \{-\infty, +\infty\} \in \overline{\mathcal{B}}$ is true. The consequence is that the union $\bar{B} \cup \{-\infty, +\infty\}$ is in $\overline{\mathcal{B}}$ as well, so that the desired consequent $A \in \overline{\mathcal{B}}$ turns out to be true once again, completing the proof by sub-cases and thus the proof of the second case.

Since $A \in \overline{\mathcal{B}}$ is true in any case, the second part of the equivalence in (11.1066) – (11.1067) holds, so that this equivalence is true. As the set A was initially arbitrary, the proposed universal sentence follows therefore to be true. □

We now consider four different systems of intervals in the set of extended real numbers and prove subsequently that all of them generate the Borel σ-algebra on $\overline{\mathbb{R}}$.

Exercise 11.60. Prove the following sentences.

a) There exist unique sets $\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}$, $\{(a, +\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\}$, $\{[a, +\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\}$ and $\{[-\infty, b]_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}$ such that

$$\forall Z\,(Z \in \{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\} \Leftrightarrow \exists a\,(b \in \mathbb{R} \wedge [-\infty, b)_{\overline{\mathbb{R}}} = Z)), \tag{11.1079}$$

$$\forall Z\,(Z \in \{(a, +\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\} \Leftrightarrow \exists a\,(a \in \mathbb{R} \wedge (a, +\infty]_{\overline{\mathbb{R}}} = Z)), \tag{11.1080}$$

$$\forall Z\,(Z \in \{[a, +\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\} \Leftrightarrow \exists b\,(a \in \mathbb{R} \wedge [a, +\infty]_{\overline{\mathbb{R}}} = Z)), \tag{11.1081}$$

$$\forall Z\,(Z \in \{[-\infty, b]_{\overline{\mathbb{R}}} : b \in \mathbb{R}\} \Leftrightarrow \exists b\,(b \in \mathbb{R} \wedge [-\infty, b]_{\overline{\mathbb{R}}} = Z)). \tag{11.1082}$$

(Hint: Proceed as in Exercise 3.56e).)

b) All of these sets are included in the power set of $\overline{\mathbb{R}}$, i.e.

$$\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\} \subseteq \mathcal{P}(\overline{\mathbb{R}}), \tag{11.1083}$$
$$\{(a, +\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\} \subseteq \mathcal{P}(\overline{\mathbb{R}}), \tag{11.1084}$$
$$\{[a, +\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\} \subseteq \mathcal{P}(\overline{\mathbb{R}}), \tag{11.1085}$$
$$\{[-\infty, b]_{\overline{\mathbb{R}}} : b \in \mathbb{R}\} \subseteq \mathcal{P}(\overline{\mathbb{R}}). \tag{11.1086}$$

(Hint: Proceed as in Exercise 3.56f).)

c) Furthermore, the sequences
   - c1) $([-\infty, -n])_{n\in\mathbb{N}_+}$ and $([-\infty, n]^c)_{n\in\mathbb{N}_+}$ are in $\{[-\infty, b]_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}$,
   - c2) $([-\infty, -n))_{n\in\mathbb{N}_+}$ and $([-\infty, n)^c)_{n\in\mathbb{N}_+}$ are in $\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}$,
   - c3) $((n, +\infty])_{n\in\mathbb{N}_+}$ and $((-n, +\infty]^c)_{n\in\mathbb{N}_+}$ are in $\{(a, +\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\}$,
   - c4) $([n, +\infty])_{n\in\mathbb{N}_+}$ and $([-n, +\infty]^c)_{n\in\mathbb{N}_+}$ are in $\{[a, +\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\}$.

Note 11.52. The previous findings (11.1031) – (11.1032) in connection with (9.81) show that

$$[a, b)_{\mathbb{R}} = [-\infty, b)_{\overline{\mathbb{R}}} \setminus [-\infty, a)_{\overline{\mathbb{R}}} \tag{11.1087}$$
$$(a, b]_{\mathbb{R}} = (a, +\infty]_{\overline{\mathbb{R}}} \setminus (b, +\infty]_{\overline{\mathbb{R}}} \tag{11.1088}$$

are true for any $a, b \in \mathbb{R}$.

Proposition 11.149. *It is true that the σ-algebra generated by the set of left-closed and right-open intervals in $\overline{\mathbb{R}}$ from $-\infty$ to a real number b is identical with the Borel σ-algebra on $\overline{\mathbb{R}}$, that is,*

$$\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}) = \overline{\mathcal{B}}. \tag{11.1089}$$

Proof. We prove the proposed equation by means of the Axiom of Extension, verifying the two inclusions

$$\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}) \subseteq \overline{\mathcal{B}}, \tag{11.1090}$$
$$\overline{\mathcal{B}} \subseteq \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}). \tag{11.1091}$$

To establish the first inclusion (11.1090), we demonstrate first the truth of the inclusion

$$\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\} \subseteq \overline{\mathcal{B}} \tag{11.1092}$$

by means of the definition of a subset, i.e. by proving the equivalent universal sentence

$$\forall A \, (A \in \{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\} \Rightarrow A \in \overline{\mathcal{B}}). \tag{11.1093}$$

For this purpose, we take an arbitrary set A and assume $A \in \{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}$ to be true, so that there exists a particular real number $\bar{b}$ with $A = [-\infty, \bar{b})_{\overline{\mathbb{R}}}$, according to Exercise 11.60a). Here, $\bar{b} \in \mathbb{R}$ implies $\bar{b} \in \overline{\mathbb{R}}$ due to the inclusion (9.13), so that Corollary 11.135 gives $[-\infty, \bar{b})_{\overline{\mathbb{R}}} \in \overline{\mathcal{B}}$ and consequently $A \in \overline{\mathcal{B}}$ via substitution. Since A is arbitrary, we may therefore conclude that (11.1093) holds, so that the inclusion (11.1092) follows indeed to be true. Let us observe now that the set system $\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}$ is a subset of the power set $\mathcal{P}(\overline{\mathbb{R}})$, so that the σ-algebra $\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})$ on $\overline{\mathbb{R}}$ generated by that set system is indeed defined. Then, because $\overline{\mathcal{B}}$ is also a σ-algebra on $\overline{\mathbb{R}}$ that includes the generating system, we obtain the desired first inclusion (11.1090) with Theorem 11.35d).

We prove the second inclusion (11.1091) by means of the definition of a subset, i.e. by proving the equivalent universal sentence

$$\forall A\,(A \in \overline{\mathcal{B}} \Rightarrow A \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})), \tag{11.1094}$$

letting $\bar{A}$ be an arbitrary extended real Borel set. Therefore, we have according to the Characterization of extended real Borel sets that $\bar{A}$ is a real Borel set, or that $\bar{A} = B \cup \{-\infty\}$ for some real Borel set B, or that $\bar{A} = B \cup \{+\infty\}$ for some real Borel set B, or that $\bar{A} = B \cup \{-\infty, +\infty\}$ for some real Borel set B. Let us apply the Distributive Law for quantification (1.75) to infer from this multiple disjunction the disjunction

$$\begin{aligned} \bar{A} \in \mathcal{B} \vee \exists B\,([B \in \mathcal{B} \wedge \bar{A} = B \cup \{-\infty\}] & \qquad (11.1095)\\ \vee\ [B \in \mathcal{B} \wedge \bar{A} = B \cup \{+\infty\}] & \qquad (11.1096)\\ \vee\ [B \in \mathcal{B} \wedge \bar{A} = B \cup \{-\infty, +\infty\}]), & \qquad (11.1097) \end{aligned}$$

which we will consider later on for a proof by cases. As a preparation, we prove first the inclusion

$$\mathcal{B} \ \subseteq\ \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}), \tag{11.1098}$$

which we accomplish via the preparatory proof of the inclusion

$$\mathcal{I} \ \subseteq\ \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})|\mathbb{R}, \tag{11.1099}$$

recalling that $\mathcal{I}$ represents the semiring of left-closed and right-open intervals in $\mathbb{R}$, which generates the Borel σ-algebra on $\mathbb{R}$ according to (11.983). We apply again the definition of a subset and verify accordingly

$$\forall B\,(B \in \mathcal{I} \Rightarrow B \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})|\mathbb{R}), \tag{11.1100}$$

letting $\bar{B} \in \mathcal{I}$ be arbitrary. Then, by definition of the set of left-closed and right-open intervals in $\mathbb{R}$, there are particular real numbers $\bar{a}$ and $\bar{b}$ such that $\bar{B} = [\bar{a}, \bar{b})_{\mathbb{R}}$ holds. In view of (11.1087), we may write this interval in the form of a set difference as

$$\bar{B} = [-\infty, \bar{b})_{\overline{\mathbb{R}}} \setminus [-\infty, \bar{a})_{\overline{\mathbb{R}}}.$$

Because of $\bar{a} \in \mathbb{R}$ and $\bar{b} \in \mathbb{R}$, both intervals forming the set difference are elements of $\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}$. Recalling that any generated σ-algebra includes its generating system, we have that the preceding set system is included in the generated σ-algebra $\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})$, so that the intervals $[-\infty, \bar{b})_{\overline{\mathbb{R}}}$ and $[-\infty, \bar{a})_{\overline{\mathbb{R}}}$ follow to be elements of that σ-algebra by virtue of the definition of a subset. Because every σ-algebra is a ring of sets (see Proposition 11.30c)) and thus closed under set differences, we now see that the set difference $\bar{B}$ is also an element of the σ-algebra $\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})$, i.e.

$$\bar{B} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}). \tag{11.1101}$$

Due to the already proven inclusion (11.1090), $\bar{B}$ turns out to be an element of $\overline{\mathcal{B}}$, too. Recalling that $\bar{B}$ represents also the interval $[\bar{a}, \bar{b})_{\mathbb{R}}$, which is a subset of $\mathbb{R}$ according to Exercise 3.53b), we thus have evidently $-\infty \notin \bar{B}$ and $+\infty \notin \bar{B}$, so that (11.1065) yields the equation $\mathbb{R} \cap \bar{B} = \bar{B}$. In light of (11.1101), we thus see that there is a set X in $\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})$ for which $\mathbb{R} \cap X = \bar{B}$ is true. This existential sentence in turn implies with the definition of a trace σ-algebra that $\bar{B}$ is in $\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})|\mathbb{R}$. This finding completes the proof of the implication in (11.1100), from which we may now infer the truth of the universal sentence (11.1100), since $\bar{B}$ was arbitrary. Consequently the inclusion (11.1099) holds. Here, we have that $\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})|\mathbb{R}$ is a σ-algebra on $\mathbb{R}$ (by definition of a trace σ-algebra), which we just showed to include the generating system $\mathcal{I}$ of the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$. Another application of Theorem 11.35d) gives us then the inclusion

$$\mathcal{B} \subseteq \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})|\mathbb{R}. \tag{11.1102}$$

In order to complete the proof of the inclusion (11.1098), we establish now the inclusion

$$\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})|\mathbb{R} \subseteq \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}) \tag{11.1103}$$

via the verification of the universal sentence

$$\forall B\, (B \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})|\mathbb{R} \Rightarrow B \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})). \tag{11.1104}$$

We let $B \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})|\mathbb{R}$ be arbitrary, so that the definition of a trace σ-algebra gives a particular set

$$\bar{X} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}) \tag{11.1105}$$

for which $\mathbb{R} \cap \bar{X} = B$ holds.

Here, we may show that $\mathbb{R} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})$ is also true. To begin with, we recall that $\mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})$ is a σ-algebra on $\overline{\mathbb{R}}$, so that

$$\overline{\mathbb{R}} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}) \tag{11.1106}$$

holds according to Property 2 of a σ-algebra (on $\overline{\mathbb{R}}$). Let us now take the sequences $s_3 = ([-\infty, -n))_{n\in\mathbb{N}_+}$ and $s_4 = ([-\infty, n)^c)_{n\in\mathbb{N}_+}$ established in Exercise 11.56, whose intersections are the singletons $\{-\infty\} = \bigcap_{n=1}^{\infty}[-\infty, -n)$ and $\{+\infty\} = \bigcap_{n=1}^{\infty}[-\infty, n)^c$ according to (11.1046) and (11.1058). Because these sequences are in $\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}$ according to Exercise 11.60, we have the inclusions

$$\begin{aligned}
\operatorname{ran}(s_3) \ &\subseteq \ \{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\} \ \subseteq \ \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}) \\
\operatorname{ran}(s_4) \ &\subseteq \ \{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\} \ \subseteq \ \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})
\end{aligned}$$

by definition of a codomain and by definition of a generated σ-algebra. Because the (generated) σ-algebra is closed under countable intersections in view of Proposition 11.31, we obtain

$$\left[\bigcap_{n=1}^{\infty}[-\infty, -n) =\right] \quad \{-\infty\} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}), \tag{11.1107}$$

$$\left[\bigcap_{n=1}^{\infty}[-\infty, -)^c =\right] \quad \{+\infty\} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}). \tag{11.1108}$$

Then, since the (generated) σ-algebra is closed under pairwise unions as mentioned in Note 11.13, the previous two findings imply

$$[\{-\infty\} \cup \{+\infty\} =] \quad \{-\infty, +\infty\} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}), \tag{11.1109}$$

using also (2.226). As mentioned before, a σ-algebra is closed under set differences, so that (11.1106) and the preceding finding (11.1109) imply

$$[\mathbb{R} =] \quad \overline{\mathbb{R}} \setminus \{-\infty, +\infty\} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}),$$

recalling the truth of the equation (9.14). Together with (11.1105), this further implies

$$[B =] \quad \mathbb{R} \cap \bar{X} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})$$

with the fact that the (generated) σ-algebra is closed under pairwise intersections (see again Note 11.13). This gives us the desired consequent of the implication in (11.1104), and as the set B was arbitrary, we may therefore conclude that the universal sentence (11.1104) holds. Consequently, the inclusion (11.1103) is true by definition of a subset. In conjunction with the already established inclusion (11.1102), this gives us the inclusion (11.1098) because of (2.13).

We now consider finally the true disjunction (11.1095) – (11.1097) and prove the desired consequent

$$\bar{A} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}) \tag{11.1110}$$

of the implication in (11.1094) by cases. The first case $\bar{A} \in \mathcal{B}$ implies it with the inclusion (11.1098) immediately by definition of a subset. In the second case, there exists a set, say $\bar{B}$, such that the multiple disjunction

$$\begin{aligned}[\bar{B} \in \mathcal{B} \wedge \bar{A} = \bar{B} \cup \{-\infty\}] \vee [\bar{B} \in \mathcal{B} \wedge \bar{A} = \bar{B} \cup \{+\infty\}] \\ \vee [\bar{B} \in \mathcal{B} \wedge \bar{A} = \bar{B} \cup \{-\infty, +\infty\}])\end{aligned}$$

is true. By virtue of the Distributivity of the conjunction over the disjunction, we may write it also in the slightly simpler form

$$\bar{B} \in \mathcal{B} \wedge [\bar{A} = \bar{B} \cup \{-\infty\}] \vee \bar{A} = \bar{B} \cup \{+\infty\}] \vee \bar{A} = \bar{B} \cup \{-\infty, +\infty\}] \tag{11.1111}$$

Thus, $\bar{B} \in \mathcal{B}$ is especially true, which implies

$$\bar{B} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}) \tag{11.1112}$$

with the inclusion (11.1098). In addition, the multiple disjunction in (11.1111) holds, which we use now to prove the sentence (11.1110) by (sub-)cases. In the first sub-case $\bar{A} = \bar{B} \cup \{-\infty\}$, we observe that the conjunction of (11.1112) and (11.1107) implies

$$[\bar{A} =] \quad \bar{B} \cup \{-\infty\} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\})$$

with the already used fact that the σ-algebra is closed under pairwise unions. The same fact gives us in the second sub-case $\bar{A} = \bar{B} \cup \{+\infty\}$ in view of the true conjunction of (11.1112) and (11.1108)

$$[\bar{A} =] \quad \bar{B} \cup \{+\infty\} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}).$$

In the third sub-case $\bar{A} = \bar{B} \cup \{-\infty, +\infty\}$, we obtain for the same reason from (11.1112) and (11.1109)

$$[\bar{A} =] \quad \bar{B} \cup \{-\infty, +\infty\} \in \mathcal{A}(\{[-\infty, b)_{\overline{\mathbb{R}}} : b \in \mathbb{R}\}),$$

so that (11.1110) holds in each of the three sub-cases, completing thus the proof for the second case. Having proved the desired consequent of the implication in (11.1094) by cases, the universal sentence (11.1094) follows also to be true, because $\bar{A}$ was arbitrary. Consequently, the inclusion (11.1091) holds as well (by definition of a subset), which yields then in conjunction with the already established inclusion (11.1090) the proposed equation (11.1089), according to the Axiom of Extension. □

Exercise 11.61. Prove that the σ-algebra generated by the set of left-open and right-closed intervals in $\overline{\mathbb{R}}$ from a real number a to $+\infty$ is identical with the Borel σ-algebra on $\overline{\mathbb{R}}$, that is,

$$\mathcal{A}(\{(a,+\infty]_{\overline{\mathbb{R}}} : a \in \mathbb{R}\}) = \overline{\mathcal{B}}. \tag{11.1113}$$

(Hint: Proceed as in the proof of Proposition 11.149, using now (11.989), (11.1088).)

Exercise 11.62. Show that the Borel σ-algebra on $\overline{\mathbb{R}}$ is generated also by

a) the set of closed intervals in $\overline{\mathbb{R}}$ from a real number to $+\infty$, i.e.

$$\mathcal{A}(\{[a,+\infty] : a \in \mathbb{R}\}) = \overline{\mathcal{B}}. \tag{11.1114}$$

b) the set of closed intervals in $\overline{\mathbb{R}}$ from $-\infty$ to a real number, i.e.

$$\mathcal{A}(\{[-\infty,b] : b \in \mathbb{R}\}) = \overline{\mathcal{B}}. \tag{11.1115}$$

(Hint: Carry out proofs in analogy to those of Proposition 11.130 and Exercise 11.53.)

Chapter 12.

Measure Spaces

12.1. Measure Spaces $(\Omega, \mathcal{A}, \mu)$

Definition 12.1 (Set function). We say that a function $\mu : \mathcal{K} \to Y$ is a *set function* iff the domain $\mathcal{K}$ is a set system.

Definition 12.2 (Null set). For any set function $\mu : \mathcal{K} \to Y$ we say that an element A of the domain $\mathcal{K}$ is a *null set* with respect to μ iff

$$\mu(A) = 0. \tag{12.1}$$

Note 12.1. As every n-tuple $s = (A_i \,|\, i \in \{1, \ldots, n\})$ of sets in a set system $\mathcal{K}$ is a function $s : \{1, \ldots, n\} \to \mathcal{K}$, the composition of any set function $\mu : \mathcal{K} \to Y$ and such a sequence yields for the composition $\mu \circ s : \{1, \ldots, n\} \to Y$ by virtue of (3.604), which thus constitutes the n-tuple $(\mu(A_i) \,|\, i \in \{1, \ldots, n\})$ in Y. Similarly, the composition of a set function $\mu : \mathcal{K} \to Y$ and any sequence $s = (A_n)_{n \in \mathbb{N}_+}$ of sets in $\mathcal{K}$ yields the composition $\mu \circ s : \mathbb{N}_+ \to Y$ and thus a sequence $(\mu(A_n))_{n \in \mathbb{N}_+}$ in Y. In the following, we use the addition $+_{\overline{\mathbb{R}}^0_+}$ to form sums $\sum_{i=1}^n$ and series $(\sum_{i=1}^n D_i)_{n \in \mathbb{N}_+}$ (which are increasingly convergent with respect to $\leq_{\overline{\mathbb{R}}^0_+}$), as described in Note 9.11.

Definition 12.3 (Finitely additive function, subadditive function, σ-additive function, σ-subadditive function). For any set Ω and any set system $\mathcal{K} \subseteq \mathcal{P}(\Omega)$, we say that a nonnegative numerical set function $\mu : \mathcal{K} \to \overline{\mathbb{R}}^0_+$ is

(1) *finitely additive* iff the following equation holds for any n-tuple $(A_i \,|\, i \in \{1, \ldots, n\})$ of disjoint sets in $\mathcal{K}$ with $\bigcup_{i=1}^n A_i \in \mathcal{K}$:

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i). \tag{12.2}$$

(2) *finitely subadditive* iff the following equation holds for any n-tuple $(A_i \,|\, i \in \{1, \ldots, n\})$ of disjoint sets in $\mathcal{K}$ with $\bigcup_{i=1}^n A_i \in \mathcal{K}$:

$$\mu(\bigcup_{i=1}^{n} A_i) \leq_{\overline{\mathbb{R}}_+^0} \sum_{i=1}^{n} \mu(A_i). \tag{12.3}$$

(3) *σ-additive* iff the following equation holds for any sequence $(A_n)_{n \in \mathbb{N}_+}$ of disjoint sets in $\mathcal{K}$ with $\bigcup_{n=1}^\infty A_n \in \mathcal{K}$:

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{i=1}^{\infty} \mu(A_i) \quad \left[\overset{\leq_{\overline{\mathbb{R}}_+^0}}{\lim_{n\to\infty}} \sum_{i=1}^{n} \mu(A_i) \right]. \tag{12.4}$$

(4) *σ-subadditive* iff the following equation holds for any sequence $(A_n)_{n \in \mathbb{N}_+}$ of disjoint sets in $\mathcal{K}$ with $\bigcup_{n=1}^\infty A_n \in \mathcal{K}$:

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq_{\overline{\mathbb{R}}_+^0} \sum_{i=1}^{\infty} \mu(A_i) \quad \left[\overset{\leq_{\overline{\mathbb{R}}_+^0}}{\lim_{n\to\infty}} \sum_{i=1}^{n} \mu(A_i) \right]. \tag{12.5}$$

Definition 12.4 (Content, finite content). For any set Ω and any set system $\mathcal{K}$ consisting of subsets of Ω, we say that a nonnegative numerical set function $\mu : \mathcal{K} \to \overline{\mathbb{R}}_+^0$ is a *content* on $\mathcal{K}$ iff

1. $\emptyset$ is a null set with respect to μ and
2. μ is finitely additive.

Furthermore, we say that a content $\mu : \mathcal{K} \to \overline{\mathbb{R}}_+^0$ is *finite* iff every value of μ is less than $+\infty$, that is,

$$\forall A \, (A \in \mathcal{K} \Rightarrow \mu(A) <_{\overline{\mathbb{R}}_+^0} +\infty). \tag{12.6}$$

Corollary 12.1. *Every finite content $\mu : \mathcal{K} \to \overline{\mathbb{R}}_+^0$ is a nonnegative real function.*

Proof. Letting $\mathcal{K}$ be an arbitrary set system and μ an arbitrary finite content on $\mathcal{K}$, we establish the inclusion $\mathrm{ran}(\mu) \subseteq \mathbb{R}_+^0$. For this purpose, we apply the definition of a subset, letting first $y \in \mathrm{ran}(\mu)$ be arbitrary. By definition of a range, there exist then a constant, say $\bar{A}$, such that $(\bar{A}, y) \in \mu$, which we may write also as $y = \mu(\bar{A})$ by using the notation for functions. Here, $\bar{A} \in \mathcal{K}$ [$= \mathrm{dom}(\mu)$] holds by definition of a domain. Then, as the finite content μ satisfies (12.6) by definition, the preceding finding implies $\mu(\bar{A}) <_{\overline{\mathbb{R}}^0_+} +\infty$. Consequently, $\mu(\bar{A}) \neq +\infty$ is true by virtue of the Characterization of comparability with respect to the linear ordering $<_{\overline{\mathbb{R}}^0_+}$. Since $\overline{\mathbb{R}}^0_+$ is a codomain of μ, the range of μ is included in that set, so that $y \in \mathrm{ran}(\mu)$ implies $y \in \overline{\mathbb{R}}^0_+$ by definition of subset. The latter in turn implies the truth of the disjunction $y \in \mathbb{R}^0_+ \vee y = +\infty$ with (9.73), whose second part $y = \mu(\bar{A}) = +\infty$ we previously found to be false. Thus, the first part $y \in \mathbb{R}^0_+$ of that disjunction is true. As y was initially arbitrary, we may therefore conclude that $\mathrm{ran}(\mu)$ is indeed a subset of $\mathbb{R}^0_+$, so that $\mathbb{R}^0_+$ is also a codomain of μ. This means that μ is a function from $\mathcal{K}$ to $\mathbb{R}^0_+$. Since $\mathcal{K}$ and μ were arbitrary, we may now further conclude that the stated universal sentence holds. □

Proposition 12.2. *Every content is isotone.*

Proof. We let Ω be an arbitrary set, $\mathcal{K}$ an arbitrary set system consisting of subsets of Ω, and μ an arbitrary content on $\mathcal{K}$. To prove that μ is isotone, we apply Proposition 3.257 in connection with the Reflexive partial ordering of inclusion of $\mathcal{K}$ and the reflexive partial ordering $\leq_{\overline{\mathbb{R}}^0_+}$. To do this, we show that

$$\forall A, B\,([A, B \in \mathcal{K} \wedge A \subseteq B] \Rightarrow \mu(A) \leq_{\overline{\mathbb{R}}^0_+} \mu(B)) \tag{12.7}$$

holds, letting A and B be arbitrary elements of $\mathcal{K}$ with $A \subseteq B$. The latter implies $(B \setminus A) \cup A = B$ with (2.263). Here, we observe that $B \setminus A$ and A are disjoint in view of (2.107). Therefore, we obtain

$$\mu(B) = \mu((B \setminus A) \cup A) = \mu(B \setminus A) +_{\overline{\mathbb{R}}^0_+} \mu(A)$$

with the finite additivity of the content μ. Now, as $0 \leq_{\overline{\mathbb{R}}^0_+} \mu(B \setminus A)$ holds according to (9.98), we obtain

$$[\mu(A) =] \quad 0 +_{\overline{\mathbb{R}}^0_+} \mu(A) \leq_{\overline{\mathbb{R}}^0_+} \mu(B \setminus A) +_{\overline{\mathbb{R}}^0_+} \mu(A) \quad [= \mu(B)]$$

with the monotony law (9.148), so that $\mu(A) \leq_{\overline{\mathbb{R}}^0_+} \mu(B)$. Since A and B were arbitrary, we may therefore conclude that the universal sentence (12.7) holds, which shows that μ is indeed isotone. □

Proposition 12.3. *It is true for any content $\mu : \mathcal{K} \to \overline{\mathbb{R}}^0_+$ and any isotone sequence of sets $(A_n)_{n \in \mathbb{N}_+}$ in $\mathcal{K}$ that the sequence $(\mu(A_n))_{n \in \mathbb{N}_+}$ is increasing with respect to $\leq_{\overline{\mathbb{R}}^0_+}$.*

Proof. We let Ω be an arbitrary set, $\mathcal{K}$ an arbitrary system of subsets of Ω, μ an arbitrary content on $\mathcal{K}$, and $(A_n)_{n\in\mathbb{N}_+}$ an arbitrary isotone sequence of sets in $\mathcal{K}$. According to the Monotony Criteria for sequences of sets, the latter means

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow A_n \subseteq A_{n+1}). \tag{12.8}$$

We now apply the Monotony Criterion for increasing sequences and show that

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \mu(A_n) \leq_{\overline{\mathbb{R}}^0_+} \mu(A_{n+1})). \tag{12.9}$$

also holds. Letting $n \in \mathbb{N}_+$ be arbitrary, we obtain $A_n \subseteq A_{n+1}$ with (12.8). Observing that the content μ is isotone according to Proposition 12.2, the preceding inclusion implies $\mu(A_n) \leq_{\overline{\mathbb{R}}^0_+} \mu(A_{n+1})$ with (12.7). This is the desired consequent of the implication in (12.9), in which n is arbitrary, so that the universal sentence (12.9) follows to be true. Thus, the sequence $(\mu(A_n))_{n\in\mathbb{N}_+}$ is indeed increasing with respect to $\leq_{\overline{\mathbb{R}}^0_+}$. Since Ω, $\mathcal{K}$, μ and $(A_n)_{n\in\mathbb{N}_+}$ were initially arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Exercise 12.1. Show for any content $\mu : \mathcal{K} \to \overline{\mathbb{R}}^0_+$ and any antitone sequence of sets $(A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{K}$ that the sequence $(\mu(A_n))_{n\in\mathbb{N}_+}$ is decreasing with respect to $\leq_{\overline{\mathbb{R}}^0_+}$.

Note 12.2. For any content $\mu : \mathcal{K} \to \overline{\mathbb{R}}^0_+$, every isotone/antitone sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in a set system $\mathcal{K}$ gives rise to the increasingly/decreasingly convergent sequence $(\mu(A_n))_{n\in\mathbb{N}_+}$ in $\overline{\mathbb{R}}^0_+$ in view of Proposition 12.3, Exercise 12.1 and Corollary 9.29.

Definition 12.5 (Continuous-from-below & -above content, $\emptyset$-continuous content). For any set Ω and any set system $\mathcal{K}$ consisting of subsets of Ω we say that a content μ on $\mathcal{K}$ is

(1) *continuous from below* iff the content of the limit of any isotonely convergent sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in $\mathcal{K}$ is identical with the limit of the sequence $(\mu(A_n))_{n\in\mathbb{N}_+}$, symbolically

$$\overset{\leq_{\overline{\mathbb{R}}^0_+}}{\lim_{n\to\infty}} \mu(A_n) = \mu(\overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n). \tag{12.10}$$

(2) *continuous from above* iff the content of the limit of any antitonely convergent sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in $\mathcal{K}$ with $\mu(A_n) <_{\overline{\mathbb{R}}^0_+} +\infty$ for

every $n \in \mathbb{N}_+$ is identical with the limit of the sequence $(\mu(A_n))_{n\in\mathbb{N}_+}$, symbolically

$$\overset{\leq_{\overline{\mathbb{R}}_+^0}}{\lim_{n\to\infty}} \mu(A_n) = \mu(\overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n). \tag{12.11}$$

(3) $\emptyset$-*continuous* or *null-continuous* iff

$$\overset{\leq_{\overline{\mathbb{R}}_+^0}}{\lim_{n\to\infty}} \mu(A_n) = 0 \tag{12.12}$$

holds for any sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in $\mathcal{K}$ converging antitonely to the empty set, where $\mu(A_n) <_{\overline{\mathbb{R}}_+^0} +\infty$ for every $n \in \mathbb{N}_+$.

Proposition 12.4. *The following implication holds for any set* Ω*, any ring of sets* $\mathcal{R}$ *on* Ω*, and any content* μ *on* $\mathcal{R}$*:*

$$\mu \text{ is } \sigma\text{−additive} \;\Rightarrow\; \mu \text{ is continuous from below.} \tag{12.13}$$

Proof. We let Ω be an arbitrary set, $\mathcal{R}$ an arbitrary ring of sets on Ω, and μ an arbitrary content on $\mathcal{R}$. To prove the implication directly, we assume that μ is σ-additive, i.e., that (12.4) holds for any sequence $(A_n)_{n\in\mathbb{N}_+}$ of disjoint sets in $\mathcal{R}$ with $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$. To show that μ is continuous from below, we let $(A_n)_{n\in\mathbb{N}_+}$ be an arbitrary isotonely convergent sequence of sets in $\mathcal{R}$. We obtain then for the limit of that sequence

$$L = \overset{\subseteq_{\mathcal{K}}}{\lim_{n\to\infty}} A_n = \overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \tag{12.14}$$

with Proposition 4.73, Corollary 4.72a) and Exercise 11.6, where $(B_n)_{n\in\mathbb{N}_+}$ is a sequence of disjoint sets in $\mathcal{R}$ with $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for every $n \neq 1$, and where $A_n = \bigcup_{i=1}^{n} B_i$ holds for every $n \in \mathbb{N}_+$. Consequently, we obtain

$$\begin{aligned}\mu(L) &= \mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) = \overset{\leq_{\overline{\mathbb{R}}_+^0}}{\lim_{n\to\infty}} \sum_{i=1}^{n} \mu(B_i) = \overset{\leq_{\overline{\mathbb{R}}_+^0}}{\lim_{n\to\infty}} \mu(\bigcup_{i=1}^{n} B_i) \\ &= \overset{\leq_{\overline{\mathbb{R}}_+^0}}{\lim_{n\to\infty}} \mu(A_n)\end{aligned}$$

using (12.14), the σ-additivity of μ, the definition of a series, Property 2 of a content, and finally the previously mentioned equation $A_n = \bigcup_{i=1}^{n} B_i$. In view of (12.14), we thus find the equation (12.10), and since $(A_n)_{n\in\mathbb{N}_+}$ was arbitrary, we may therefore conclude that μ is continuous from below. As Ω, $\mathcal{K}$ and μ were initially also arbitrary, we may finally conclude that the proposition holds. □

Definition 12.6 (Measure). For any set Ω and any set system $\mathcal{K}$ included in $\mathcal{P}(\Omega)$ we say that a nonnegative numerical set function μ on $\mathcal{K}$ is a *measure* iff

1. $\emptyset$ is a null set with respect to μ and
2. μ is *σ-additive*.

Definition 12.7 (Measure space). For any measurable space $(\Omega, \mathcal{A})$ and any measure $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+^0$, we call the ordered triple

$$(\Omega, \mathcal{A}, \mu) \tag{12.15}$$

a *measure space*.

Note 12.3. Within a measure space, the domain of the measure is a σ-algebra, so that the measure assigns a nonnegative extended real number to every event.

To be expanded!

12.2. Probability Spaces $(\Omega, \mathcal{A}, P)$

under construction!

12.3. The Lebesgue Measure Space $(\mathbb{R}, \mathcal{B}, \lambda)$

Definition 12.8 (Outer measure, Carathéodory-measurable set). For any set Ω we say that a function $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$ is an *outer measure* iff

1. the empty set is a null set with respect to μ^*, that is,
$$\mu^*(\emptyset) = 0, \tag{12.16}$$
2. μ^* is isotone, that is,
$$\forall A, B\,([A, B \in \mathcal{P}(\Omega) \wedge A \subseteq B] \Rightarrow \mu^*(A) \leq_{\overline{\mathbb{D}}} \mu^*(B)), \tag{12.17}$$
and
3. μ^* is *σ-subadditive*, that is,
$$\forall f\,(f = (A_n)_{n \in \mathbb{N}_+} \in \mathcal{P}(\Omega)^{\mathbb{N}_+} \Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) \leq_{\overline{\mathbb{D}}} \sum_{i=1}^{\infty} \mu^*(A_i)). \tag{12.18}$$

We then say that an element A of $\mathcal{P}(\Omega)$ is *Carathéodory-measurable* by μ^* (or alternatively μ^*-measurable) iff
$$\forall X\,(X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X)). \tag{12.19}$$

Note 12.4. We stated the isotony property (12.17) in the sense of Proposition 3.257.

Note 12.5. Since $f = (A_n)_{n \in \mathbb{N}_+} \in \mathcal{P}(\Omega)^{\mathbb{N}_+}$ represents a function from $\mathbb{N}_+$ to $\mathcal{P}(\Omega)$ and μ^* a function from $\mathcal{P}(\Omega)$ to $\overline{\mathbb{D}}$, the composition $\mu^* \circ f$ represents a function from $\mathbb{N}_+$ to $\overline{\mathbb{D}}$, which may thus be written as the sequence $(\mu^*(A_n))_{n \in \mathbb{N}_+}$ in $\overline{\mathbb{D}}$. Then, the series $(\sum_{i=1}^{n} \mu^*(A_i))_{n \in \mathbb{N}_+}$ is defined and converges increasingly to $\sum_{i=1}^{\infty} \mu^*(A_i)$, so that all of the expressions in the preceding definition are evidently well defined.

Corollary 12.5. *For any set Ω and any outer measure $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$, it is true that there exists a unique set (system) $\mathcal{A}^*$ consisting of all μ^*-measurable sets in $\mathcal{P}(\Omega)$.*

Proof. Letting Ω be an arbitrary set, we may evidently apply the Axiom of Specification in connection with the Equality Criterion for sets to directly establish the unique existence of a set $\mathcal{A}^*$ satisfying
$$\begin{aligned}&\forall A\,(A \in \mathcal{A}^* \Leftrightarrow [A \in \mathcal{P}(\Omega) \\ &\quad \wedge \forall X\,(X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X))]).\end{aligned} \tag{12.20}$$
□

Lemma 12.6 (Criterion for Carathéodory measurability). *For any set Ω and any outer measure $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$, it is true that an element A of $\mathcal{P}(\Omega)$ is μ^*-measurable iff A satisfies*

$$\forall X\,(X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X) \leq_{\overline{\mathbb{D}}} \mu^*(X)). \tag{12.21}$$

Proof. We take arbitrary sets Ω, μ^* and A, assuming that μ^* is an outer measure with domain $\mathcal{P}(\Omega)$ and assuming that $A \in \mathcal{P}(\Omega)$ holds. To prove the first part of the stated equivalence, we assume that A is Carathéodory-measurable by μ^*, so that A satisfies (12.19), and we verify that A satisfies also (12.21). To do this, we take an arbitrary set X in the power set $\mathcal{P}(\Omega)$, so that the equation in (12.19) is true, which we may write also as $\mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X) = \mu^*(X)$. Consequently, the disjunction

$$\mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X) <_{\overline{\mathbb{D}}} \mu^*(X) \vee \mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X) = \mu^*(X)$$

also holds, which in turn gives the desired inequality in (12.21) by definition of an induced reflexive partial ordering. As X was arbitrary, we may therefore conclude that the first part ('$\Rightarrow$') of the stated equivalence holds.

To prove the second part ('$\Leftarrow$'), we now assume (12.21) to be true, and we show that (12.19) is implied by this assumption. Letting $X \in \mathcal{P}(\Omega)$ be arbitrary, we obtain with the preceding assumption the true inequality

$$\mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X) \leq_{\overline{\mathbb{D}}} \mu^*(X) \tag{12.22}$$

We now establish the reversed inequality

$$\mu^*(X) \leq_{\overline{\mathbb{D}}} \mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X), \tag{12.23}$$

(which will imply – together with (12.22) – the desired equation in (12.19) because $\leq_{\overline{\mathbb{D}}}$ is total). For this purpose, we observe that $A \in \mathcal{P}(\Omega)$ implies $A \subseteq \Omega$ and therefore $A^c \subseteq \Omega$ with (2.137); consequently, the intersections $A \cap X$ and $A^c \cap X$ are both subsets of Ω because of (2.85), and therefore elements of the power set $\mathcal{P}(\Omega)$. Since $\emptyset$ is also included in Ω according to (2.43) and thus contained in $\mathcal{P}(\Omega)$, we may define the sequence $f = (A_n)_{n \in \mathbb{N}_+}$ in $\mathcal{P}(\Omega)$ with terms $A_1 = A \cap X$, $A_2 = A^c \cap X$, $A_3 = \emptyset$, etc., according to 5.362. We then obtain the equations

$$\begin{aligned}
\bigcup_{n=1}^{\infty} A_n &= (A \cap X) \cup (A^c \cap X) \\
&= (X \cap A) \cup (X \cap A^c) \\
&= X \cap (A \cup A^c) \\
&= X \cap \Omega \\
&= X
\end{aligned}$$

by applying (5.363), then the Commutative Law for the intersection of two sets, the Distributive Law for the intersection of two sets, (2.257), and finally (2.77). Based on the resulting equation $\bigcup_{n=1}^{\infty} A_n = X$, we obtain via substitution

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \mu^*(X). \tag{12.24}$$

Now, since the sequence $(\mu^*(A_n))_{n\in\mathbb{N}_+}$ has the terms

$$\mu^*(A_n) = \begin{cases} \mu^*(A_1) & \text{if } n = 1 \\ \mu^*(A_2) & \text{if } n = 2 \\ \mu^*(\emptyset) = 0 & \text{if } n > 2 \end{cases}$$

(using Property 1 of an outer measure) it is a sequence as established in Exercise 5.40. As $(\overline{\mathbb{D}}, +_{\overline{\mathbb{D}}}, \cdot_{\overline{\mathbb{D}}})$ is a semiring containing the zero element 0 and as $(\overline{\mathbb{D}}, <_{\overline{\mathbb{D}}})$ is a partially/linearly ordered set satisfying both $0 \leq_{\overline{\mathbb{D}}} a$ for any $a \in \overline{\mathbb{D}}$ (see Proposition **??**) and the Monotony Law for $+$ and $\leq$ (see Exercise **??**), we may apply Proposition 5.135 to obtain for the limit of the series $(\sum_{i=1}^{n} \mu^*(A_i))_{n\in\mathbb{N}_+}$

$$\begin{aligned}\sum_{i=1}^{\infty} \mu^*(A_i) &= \mu^*(A_1) +_{\overline{\mathbb{D}}} \mu^*(A_2) \\ &= \mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(A^c \cap X).\end{aligned}$$

Then, since μ^* is by assumption an outer measure, the sequence $(A_n)_{n\in\mathbb{N}_+}$ in $\mathcal{P}(\Omega)$ satisfies (12.18), where the inequality yields then via substitutions based on (12.24) and the preceding equations the desired result (12.23). This completes the proof of the second part of the proposed equivalence.

As Ω, μ^* and A were initially arbitrary, we may now finally infer from the truth of the previously established equivalence the truth of the stated lemma. □

Lemma 12.7. *For any set Ω and any outer measure $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$, it is true that the set system $\mathcal{A}^*$ consisting of all μ^*-measurable sets in $\mathcal{P}(\Omega)$ is an algebra of sets.*

Proof. We take arbitrary sets Ω and μ^* such that μ^* is a function from $\mathcal{P}(\Omega)$ to $\overline{\mathbb{D}}$ with the properties of an outer measure. Then, the set system $\mathcal{A}^*$ that satisfies (12.20) is defined. Here, we see in particular that $A \in \mathcal{A}^*$ implies $A \in \mathcal{P}(\Omega)$ for any set A, so that $\mathcal{A}^* \subseteq \mathcal{P}(\Omega)$ follows to be true with the definition of a subset; thus, the set system $\mathcal{A}^*$ satisfies Property 1 of an algebra of sets.

Concerning Property 2 of an algebra of sets, we apply (12.20) to establish $\emptyset \in \mathcal{A}^*$. To do this, we verify the truth of the conjunction

$$\emptyset \in \mathcal{P}(\Omega) \wedge \forall X\,(X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \mu^*(\emptyset \cap X) +_{\overline{\mathbb{D}}} \mu^*(\emptyset^c \cap X)). \quad (12.25)$$

Because the empty set is included in every set according to (2.43), and is thus in particular a subset of Ω, we obtain $\emptyset \in \mathcal{P}(\Omega)$ by definition of a power set. To show that the second part of the conjunction (12.25) also holds, we take an arbitrary set X in $\mathcal{P}(\Omega)$ and derive the equations

$$\begin{aligned}\mu^*(\emptyset \cap X) +_{\overline{\mathbb{D}}} \mu^*(\emptyset^c \cap X) &= \mu^*(\emptyset) +_{\overline{\mathbb{D}}} \mu^*(\Omega \cap X)\\ &= 0 +_{\overline{\mathbb{D}}} \mu^*(X \cap \Omega)\\ &= \mu^*(X)\end{aligned}$$

by applying (2.62), (2.136) in connection with the previously mentioned $\emptyset \subseteq \Omega$, then Property 1 of an outer measure alongside the Commutative Law for the intersection of two sets, and finally the definition of a neutral element together with (2.77) based on the fact that the assumed $X \in \mathcal{P}(\Omega)$ gives $x \subseteq \Omega$ (by definition of a power set). As X was arbitrary, we may therefore conclude that the universal sentence in (12.25) is true, so that the proof of the conjunction is complete. This finding in turn implies the truth of $\emptyset \in \mathcal{A}^*$ with (12.20), so that $\mathcal{A}^*$ satisfies indeed also Property 2 of an algebra of sets.

Concerning Property 3, we verify the universal sentence

$$\forall A, B\,(A, B \in \mathcal{A}^* \Rightarrow A \cup B \in \mathcal{A}^*). \quad (12.26)$$

We let $\bar{A}, \bar{B} \in \mathcal{A}^*$ be arbitrary and show that $\bar{A} \cup \bar{B} \in \mathcal{A}^*$ follows to be true. For this purpose, we use (12.20) and verify the equivalent conjunction

$$\begin{aligned}&\bar{A} \cup \bar{B} \in \mathcal{P}(\Omega) \quad (12.27)\\ &\quad \wedge \forall X\,(X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \mu^*([\bar{A} \cup \bar{B}] \cap X) +_{\overline{\mathbb{D}}} \mu^*([\bar{A} \cup \bar{B}]^c \cap X)).\end{aligned}$$

Regarding the first part of this conjunction, we observe that $\bar{A}, \bar{B} \in \mathcal{A}^*$ implies in particular $\bar{A} \subseteq \Omega$ and $\bar{B} \subseteq \Omega$ because of (12.20), so that we obtain $\bar{A} \cup \bar{B} \subseteq \Omega$ with (2.252) and then $\bar{A} \cup \bar{B} \in \mathcal{P}(\Omega)$ with the definition of a power set. To establish the second part of the conjunction (12.27), we first prove the universal sentence

$$\begin{aligned}\forall Z\,(Z \in \mathcal{P}(\Omega) \Rightarrow \mu^*(Z) &= \mu^*(\bar{B} \cap \bar{A} \cap Z) +_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A} \cap Z) \quad (12.28)\\ &\quad +_{\overline{\mathbb{D}}} \mu^*(\bar{B} \cap \bar{A}^c \cap Z) +_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A}^c \cap Z)),\end{aligned}$$

letting $Z \in \mathcal{P}(\Omega)$ be arbitrary. Thus, $Z \subseteq \Omega$ is true, which gives the inclusions

$$\bar{A} \cap Z \ \subseteq \ Z \ \subseteq \ \Omega$$
$$\bar{A}^c \cap Z \ \subseteq \ Z \ \subseteq \ \Omega$$

with (2.74), and they in turn imply

$$\bar{A} \cap Z \subseteq \Omega$$
$$\bar{A}^c \cap Z \subseteq \Omega$$

with (2.13), therefore $\bar{A} \cap Z, \bar{A}^c \cap Z \in \mathcal{P}(\Omega)$ by definition of a power set.

Let us now observe that the previously assumed $\bar{A}, \bar{B} \in \mathcal{A}^*$ implies with (12.20) especially the truth of the universal sentences

$$\forall X\, (X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \mu^*(\bar{A} \cap X) +_{\mathbb{D}} \mu^*(\bar{A}^c \cap X)) \tag{12.29}$$
$$\forall X\, (X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \mu^*(\bar{B} \cap X) +_{\mathbb{D}} \mu^*(\bar{B}^c \cap X)) \tag{12.30}$$

Then, the previously established $\bar{A} \cap Z \in \mathcal{P}(\Omega)$ and $\bar{A}^c \cap Z \in \mathcal{P}(\Omega)$ yield in view of (12.30) the equations

$$\mu^*(\bar{A} \cap Z) = \mu^*(\bar{B} \cap [\bar{A} \cap Z]) +_{\mathbb{D}} \mu^*(\bar{B}^c \cap [\bar{A} \cap Z]),$$
$$\mu^*(\bar{A}^c \cap Z) = \mu^*(\bar{B} \cap [\bar{A}^c \cap Z]) +_{\mathbb{D}} \mu^*(\bar{B}^c \cap [\bar{A}^c \cap Z]),$$

so that the assumed $Z \in \mathcal{P}(\Omega)$ gives with (12.29) via substitutions

$$\begin{aligned} \mu^*(Z) &= \mu^*(\bar{A} \cap Z) +_{\mathbb{D}} \mu^*(\bar{A}^c \cap Z) \\ &= \mu^*(\bar{B} \cap [\bar{A} \cap Z]) +_{\mathbb{D}} \mu^*(\bar{B}^c \cap [\bar{A} \cap Z]) \\ &\quad +_{\mathbb{D}} \mu^*(\bar{B} \cap [\bar{A}^c \cap Z]) +_{\mathbb{D}} \mu^*(\bar{B}^c \cap [\bar{A}^c \cap Z]). \end{aligned}$$

In the latter equation, we may apply the Associative Law for the intersection of two sets, which allows us to omit the squared brackets. As Z was arbitrary, we may therefore conclude that the universal sentence (12.28) is indeed true.

We now return to the proof of the second part of the conjunction (12.27) and let $\bar{X} \in \mathcal{P}(\Omega)$ be arbitrary, so that (12.28) gives

$$\begin{aligned} \mu^*(\bar{X}) = \mu^*(\bar{B} \cap \bar{A} \cap \bar{X}) +_{\mathbb{D}} \mu^*(\bar{B}^c \cap \bar{A} \cap \bar{X}) +_{\mathbb{D}} \mu^*(\bar{B} \cap \bar{A}^c \cap \bar{X}) \\ +_{\mathbb{D}} \mu^*(\bar{B}^c \cap \bar{A}^c \cap \bar{X}), \end{aligned} \tag{12.31}$$

Let us next consider the set $\bar{Z} = \bar{X} \cap (\bar{A} \cup \bar{B})$. Since we already showed that $\bar{A} \cup \bar{B} \subseteq \Omega$ holds, we obtain with (2.74) first

$$[\bar{Z} =] \quad \bar{X} \cap (\bar{A} \cup \bar{B}) \ \subseteq \ \bar{A} \cup \bar{B} \ \subseteq \ \Omega$$

and then $\bar{Z} \subseteq \Omega$ with (2.13), so that the definition of a power set yields $\bar{Z} \in \mathcal{P}(\Omega)$. This further implies with (12.28)

$$\begin{aligned}
\mu^*(\bar{X} \cap [\bar{A} \cup \bar{B}]) = \mu^*(&\bar{B} \cap \bar{A} \cap \bar{X} \cap [\bar{A} \cup \bar{B}]) && (12.32)\\
&+_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A} \cap \bar{X} \cap [\bar{A} \cup \bar{B}]) && (12.33)\\
&+_{\overline{\mathbb{D}}} \mu^*(\bar{B} \cap \bar{A}^c \cap \bar{X} \cap [\bar{A} \cup \bar{B}]) && (12.34)\\
&+_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A}^c \cap \bar{X} \cap [\bar{A} \cup \bar{B}]). && (12.35)
\end{aligned}$$

The argument of the first summand can evidently be written as

$$\begin{aligned}
[\bar{B} \cap \bar{A} \cap \bar{X}] \cap (\bar{A} \cup \bar{B}) &= (\bar{B} \cap \bar{A} \cap \bar{X} \cap \bar{A}) \cup (\bar{B} \cap \bar{A} \cap \bar{X} \cap \bar{B})\\
&= (\bar{B} \cap [\bar{A} \cap \bar{A}] \cap \bar{X}) \cup ([\bar{B} \cap \bar{B}] \cap \bar{A} \cap \bar{X})\\
&= (\bar{B} \cap \bar{A} \cap \bar{X}) \cup (\bar{B} \cap \bar{A} \cap \bar{X})\\
&= \bar{B} \cap \bar{A} \cap \bar{X}
\end{aligned}$$

by using the Distributive, Associative and Commutative Law for the intersection of two sets, the Idempotent Law for the intersection of two sets and (2.250). The argument of the second term (12.33) can be simplified to

$$\begin{aligned}
[\bar{B}^c \cap \bar{A} \cap \bar{X}] \cap (\bar{A} \cup \bar{B}) &= (\bar{B}^c \cap \bar{A} \cap \bar{X} \cap \bar{A}) \cup (\bar{B}^c \cap \bar{A} \cap \bar{X} \cap \bar{B})\\
&= (\bar{B}^c \cap [\bar{A} \cap \bar{A}] \cap \bar{X}) \cup ([\bar{B}^c \cap \bar{B}] \cap \bar{A} \cap \bar{X})\\
&= (\bar{B}^c \cap \bar{A} \cap \bar{X}) \cup (\emptyset \cap \bar{A} \cap \bar{X})\\
&= (\bar{B}^c \cap \bar{A} \cap \bar{X}) \cup \emptyset\\
&= \bar{B}^c \cap \bar{A} \cap \bar{X}
\end{aligned}$$

by applying first the previous three Laws, the Idempotent Law for the intersection of two sets, (2.135), (2.62), and (2.216). Using exactly the same arguments, we evidently obtain for the argument of the third summand (12.34)

$$\begin{aligned}
[\bar{B} \cap \bar{A}^c \cap \bar{X}] \cap (\bar{A} \cup \bar{B}) &= (\bar{B} \cap \bar{A}^c \cap \bar{X} \cap \bar{A}) \cup (\bar{B} \cap \bar{A}^c \cap \bar{X} \cap \bar{B})\\
&= (\bar{B} \cap [\bar{A}^c \cap \bar{A}] \cap \bar{X}) \cup ([\bar{B} \cap \bar{B}] \cap \bar{A}^c \cap \bar{X})\\
&= (\bar{B} \cap \emptyset \cap \bar{X}) \cup (\bar{B} \cap \bar{A}^c \cap \bar{X})\\
&= \emptyset \cup (\bar{B} \cap \bar{A}^c \cap \bar{X})\\
&= \bar{B} \cap \bar{A}^c \cap \bar{X}.
\end{aligned}$$

For the same reasons, we see that the argument of the last summand (12.35) reduces to

$$\begin{aligned}[\bar{B}^c \cap \bar{A}^c \cap \bar{X}] \cap (\bar{A} \cup \bar{B}) &= (\bar{B}^c \cap \bar{A}^c \cap \bar{X} \cap \bar{A}) \cup (\bar{B}^c \cap \bar{A}^c \cap \bar{X} \cap \bar{B}) \\ &= (\bar{B}^c \cap [\bar{A}^c \cap \bar{A}] \cap \bar{X}) \cup ([\bar{B}^c \cap \bar{B}] \cap \bar{A}^c \cap \bar{X}) \\ &= (\bar{B}^c \cap \emptyset \cap \bar{X}) \cup (\emptyset \cap \bar{A}^c \cap \bar{X}) \\ &= \emptyset \cup \emptyset \\ &= \emptyset.\end{aligned}$$

With these findings, the equation (12.32) – (12.35) yields

$$\begin{aligned}\mu^*(\bar{Z}) &= \mu^*(\bar{B} \cap \bar{A} \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A} \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B} \cap \bar{A}^c \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\emptyset) \\ &= \mu^*(\bar{B} \cap \bar{A} \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A} \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B} \cap \bar{A}^c \cap \bar{X}) +_{\overline{\mathbb{D}}} 0 \\ &= \mu^*(\bar{B} \cap \bar{A} \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A} \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B} \cap \bar{A}^c \cap \bar{X}) \quad (12.36)\end{aligned}$$

by means of substitution, Property 1 of an outer measure, and the definition of a neutral element. Thus, we may substitute $\mu^*(\bar{Z})$ for the first three summands in (12.31) and then the definition $\bar{Z} = \bar{X} \cap (\bar{A} \cup \bar{B})$ to obtain the equations

$$\begin{aligned}\mu^*(\bar{X}) &= \mu^*(\bar{Z}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A}^c \cap \bar{X}) \\ &= \mu^*(\bar{X} \cap [\bar{A} \cup \bar{B}]) +_{\overline{\mathbb{D}}} \mu^*(\bar{B}^c \cap \bar{A}^c \cap \bar{X})\end{aligned}$$

Rearranging the terms within the arguments by means of the Commutative Law for the intersection of two sets and by means of De Morgan's Law for the union of two sets, we furthermore obtain

$$\begin{aligned}\mu^*(\bar{X}) &= \mu^*([\bar{A} \cup \bar{B}] \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*([\bar{A}^c \cap \bar{B}^c] \cap \bar{X}) \\ &= \mu^*([\bar{A} \cup \bar{B}] \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*([\bar{A} \cup \bar{B}]^c \cap \bar{X})\end{aligned}$$

Thus, the proof of the implication in (12.27) is finally complete, and as $\bar{X}$ was arbitrary, we may therefore conclude that the second part of the conjunction (12.27) holds, besides the already established first part. Then, the truth of that conjunction implies the truth of $\bar{A} \cup \bar{B} \in \mathcal{A}^*$ because of (12.20). Since $\bar{A}$ and $\bar{B}$ were also arbitrary, we may now further conclude that the universal sentence a) holds, so that $\mathcal{A}^*$ satisfies indeed Property 3 of an algebra of sets.

To verify Property 4, we establish

$$\forall A\, (A \in \mathcal{A}^* \Rightarrow A^c \in \mathcal{A}^*), \tag{12.37}$$

letting $\bar{A} \in \mathcal{A}^*$ be arbitrary. To show that this implies $\bar{A}^c \in \mathcal{A}^*$, we need to prove the universal sentence

$$\bar{A}^c \in \mathcal{P}(\Omega) \wedge \forall X\, (X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \mu^*(\bar{A}^c \cap X) +_{\overline{\mathbb{D}}} \mu^*([\bar{A}^c]^c \cap X)).$$

Since $\bar{A} \in \mathcal{A}^*$ evidently implies $\bar{A} \in \mathcal{P}(\Omega)$ and therefore $\bar{A} \subseteq \Omega$, it follows that $\bar{A}^c \subseteq \Omega$ also holds, according to (2.137). Consequently, $\bar{A}^c \in \mathcal{P}(\Omega)$ is true, which proves the first part of the preceding conjunction. To prove its second part, we take an arbitrary set $X \in \mathcal{P}(\Omega)$, so that (12.20) yields

$$\mu^*(X) = \mu^*(\bar{A} \cap X) +_{\overline{\mathbb{D}}} \mu^*(\bar{A}^c \cap X). \tag{12.38}$$

We then obtain also the equations

$$\begin{aligned}\mu^*(\bar{A}^c \cap X) +_{\overline{\mathbb{D}}} \mu^*([\bar{A}^c]^c \cap X) &= \mu^*(\bar{A}^c \cap X) +_{\overline{\mathbb{D}}} \mu^*(\bar{A} \cap X) \\ &= \mu^*(\bar{A} \cap X) +_{\overline{\mathbb{D}}} \mu^*(\bar{A}^c \cap X) \\ &= \mu^*(X)\end{aligned}$$

with (2.136), the commutativity of $+_{\overline{\mathbb{D}}}$, and (12.38). As X is arbitrary, we may infer from this finding the truth of the desired conjunction, which evidently implies $\bar{A}^c \in \mathcal{A}^*$. Since $\bar{A}$ is also arbitrary, the universal sentence (12.37) follows to be true, which shows that $\mathcal{A}^*$ satisfies also Property 4 (and thus all properties) of an algebra of sets. □

Note 12.6. For any set Ω and any outer measure $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$, we can define for the algebra $\mathcal{A}^*$ of μ^*-measurable sets in $\mathcal{P}(\Omega)$ the function

$${}^c_{\mathcal{A}^*} : \mathcal{A}^* \to \mathcal{A}^*, \quad A \mapsto A^c. \tag{12.39}$$

according to Exercise 11.9. Then, Proposition 11.23 shows that

a) $\mathcal{A}^*$ is a ring of sets on Ω.

b) $\mathcal{A}^*$ is a semiring of sets on Ω.

c) $\mathcal{A}^*$ is a π-system on Ω.

Furthermore, we see in light of Note 11.11 that

a) $(\mathcal{A}^*, \cup_{\mathcal{A}^*})$ is a commutative semigroup with neutral element $\emptyset$ and idempotent $\cup_{\mathcal{A}^*}$,

b) $(\mathcal{A}^*, \Delta_{\mathcal{A}^*})$ is a commutative group with neutral element $\emptyset$,

c) $(\mathcal{A}^*, \cap_{\mathcal{A}^*})$ is a commutative semigroup with neutral element Ω and idempotent $\cap_{\mathcal{A}^*}$.

d) $(\mathcal{A}^*, \Delta_{\mathcal{A}^*}, \cap_{\mathcal{A}^*}, \Delta_{\mathcal{A}^*})$ is a commutative ring with zero element $\emptyset$ and unity element Ω.

e) $(\mathcal{A}^*, \Delta_{\mathcal{A}^*}, \cap_{\mathcal{A}^*}, \Delta_{\mathcal{A}^*})$ is a Boolean ring with unity element Ω.

Moreover, we can then define according to Note 11.12 for any natural number n the n-fold binary operations

$$\bigcup_{i=1}^{n} : (\mathcal{A}^*)^{\{1,\dots,n\}} \to \mathcal{A}^*, \quad (A_i \,|\, i \in \{1,\dots,n\}) \mapsto \bigcup_{i=1}^{n} A_i, \tag{12.40}$$

$$\bigcap_{i=1}^{n} : (\mathcal{A}^*)^{\{1,\dots,n\}} \to \mathcal{A}^*, \quad (A_i \,|\, i \in \{1,\dots,n\}) \mapsto \bigcap_{i=1}^{n} A_i, \tag{12.41}$$

which equal, respectively, the union and the intersection of (the range of) the mapped sequence.

Corollary 12.8. *For any set Ω and any outer measure $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$, it is true that the algebra of sets $\mathcal{A}^*$ consisting of all μ^*-measurable sets in $\mathcal{P}(\Omega)$ satisfies for any $A, B \in \mathcal{A}^*$ with $A \cap B = \emptyset$ and for any $X \in \mathcal{P}(\Omega)$ the equation*

$$\mu^*(X \cap (A \cup B)) = \mu^*(A \cap X) +_{\overline{\mathbb{D}}} \mu^*(B \cap X). \tag{12.42}$$

Proof. We let Ω, μ^*, $\bar{A}$, $\bar{B}$ and $\bar{X}$ be arbitrary sets, assume that μ^* is an outer measure with domain $\mathcal{P}(\Omega)$, assume $\bar{A}$ and $\bar{B}$ to be disjoint sets in $\mathcal{A}^*$, and assume moreover that $\bar{X} \in \mathcal{P}(\Omega)$ holds. Let us first observe that the assumption of the arbitrary $\bar{A}, \bar{B} \in \mathcal{A}^*$ implies the truth of (12.28). As shown also in the proof of Lemma 12.7, we therefore obtain for the arbitrary element $\bar{X} \in \mathcal{P}(\Omega)$ the true equation (12.36) with $\bar{Z} = \bar{X} \cap (\bar{A} \cup \bar{B})$, which we may write also as

$$\begin{aligned}
\mu^*(\bar{X} \cap (\bar{A} \cup \bar{B})) &= \mu^*([\bar{A} \cap \bar{B}] \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*([\bar{A} \cap \bar{B}^c] \cap \bar{X}) \\
&\quad +_{\overline{\mathbb{D}}} \mu^*([\bar{B} \cap \bar{A}^c] \cap \bar{X}) \\
&= \mu^*(\emptyset \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{A} \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B} \cap \bar{X}) \\
&= \mu^*(\emptyset) +_{\overline{\mathbb{D}}} \mu^*(\bar{A} \cap \bar{X}) +_{\overline{\mathbb{D}}} \mu^*(\bar{B} \cap \bar{X})
\end{aligned}$$

using the Commutative Law and the Associative Law for the intersection of two sets, then the disjointness assumption $\bar{A} \cap \bar{B} = \emptyset$ together with (2.141) – (2.143) and (2.77), and finally (2.62). We now see that the first summand satisfies $\mu^*(\emptyset) = 0$ due to Property 1 of an outer measure, so that we obtain the desired equation (12.42) with the definition of a neutral element. Since Ω, μ^*, $\bar{A}$, $\bar{B}$ and $\bar{X}$ were initially arbitrary sets, we may therefore conclude that the corollary is indeed true. □

Proposition 12.9. *For any set Ω, any outer measure $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$, any $X \in \mathcal{P}(\Omega)$, any $n \in \mathbb{N}$ and any sequence of sets $f = (A_i \,|\, i \in \{1,\dots,n\})$ in*

$\mathcal{A}^*$ with $A_j \cap A_k = \emptyset$ for all $j, k \in \{1, \ldots, n\}$ with $j \neq k$ that

$$\mu^*(X \cap \bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu^*(A_i \cap X). \tag{12.43}$$

Proof. First we let Ω, μ^* and X be arbitrary sets such that μ^* is an outer measure with domain $\mathcal{P}(\Omega)$ and such that X is an element of $\mathcal{P}(\Omega)$. Next, we apply a proof by mathematical induction. To establish the base case ($n = 0$), we take the empty sequence of sets $(A_i \,|\, i \in \{1, \ldots, 0\})$ as well as $(\mu^*(A_i \cap X) \,|\, i \in \{1, \ldots, 0\})$, for which we obtain

$$\mu^*(X \cap \bigcup_{i=1}^{0} A_i) = \mu^*(X \cap \emptyset) = \mu^*(\emptyset) = 0 = \sum_{i=1}^{0} \mu^*(A_i \cap X)$$

by applying (5.389) with the fact that $\emptyset$ is the neutral element with respect to $\cup_{\mathcal{A}^*}$, then (2.62), Property 1 of an outer measure, and (5.389) with the fact that 0 is the neutral element with respect to $+_{\overline{\mathbb{D}}}$. Thus, the proposed equation (12.43) holds in the base case.

Regarding the induction step, we let n be an arbitrary natural number and make the induction assumption that (12.43) holds for any sequence of pairwise disjoint sets $(A_i \,|\, i \in \{1, \ldots, n\})$ in $\mathcal{A}^*$. We take an arbitrary set f such that f is a sequence of pairwise disjoint sets $(A_i \,|\, i \in \{1, \ldots, n+1\})$ in $\mathcal{A}^*$, which defines then evidently the sequence $(A_i \cap X \,|\, i \in \{1, \ldots, n+1\})$ in $\mathcal{P}(\Omega)$ and thus the sequence $(\mu^*(A_i \cap X) \,|\, i \in \{1, \ldots, n+1\})$ in $\overline{\mathbb{D}}$. We then obtain the equations

$$\begin{aligned}
\mu^*(X \cap \bigcup_{i=1}^{n+1} A_i) &= \mu^*(X \cap \left[\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right]) \\
&= \mu^*(\left(\bigcup_{i=1}^{n} A_i\right) \cap X) +_{\overline{\mathbb{D}}} \mu^*(A_{n+1} \cap X) \\
&= \mu^*(X \cap \bigcup_{i=1}^{n} A_i) +_{\overline{\mathbb{D}}} \mu^*(A_{n+1} \cap X) \\
&= \sum_{i=1}^{n} \mu^*(A_i \cap X) +_{\overline{\mathbb{D}}} \mu^*(A_{n+1} \cap X) \\
&= \sum_{i=1}^{n+1} \mu^*(A_i \cap X)
\end{aligned}$$

using (5.394) with respect to the n-fold union on $\mathcal{A}^*$, (12.42) with the fact that the sets $\bigcup_{i=1}^{n} A_i$ and A_{n+1} are disjoint because of Proposition 11.11

(which we may apply here because the algebra of sets $\mathcal{A}^*$ on $\Omega$is a ring of sets on Ω according to Proposition 11.23), the Commutative Law for the intersection of two sets, the induction assumption, and finally (5.394) with respect to the n-fold addition on $\overline{\mathbb{D}}$. As f and n were arbitrary, we may therefore conclude that the induction step holds, alongside the base case, so that the proof by mathematical induction is complete. Because Ω, μ^* and X were initially arbitrary sets, we may then further conclude that the proposition holds, as claimed. □

Lemma 12.10 (Carathéodory's construction of measures by outer measures). *The following sentences are true for any set Ω and any outer measure $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$.*

a) The algebra $\mathcal{A}^*$ of μ^*-measurable sets in $\mathcal{P}(\Omega)$ is a σ-algebra on Ω.

b) The restriction $\mu^* \upharpoonright \mathcal{A}^*$ is a measure.

Proof. We let Ω and μ^* be arbitrary sets and assume that μ^* is an outer measure with domain $\mathcal{P}(\Omega)$. Concerning a), we will apply the π-λ Characterization of a σ-algebra, for which purpose we first prove that the system $\mathcal{A}^*$ of μ^*-measurable sets in $\mathcal{P}(\Omega)$ is a λ-system on Ω. Since $\mathcal{A}^*$ is an algebra of sets on Ω according to Lemma 12.7, this set system clearly satisfies Property 1, Property 2 and Property 4 of a λ-system. To demonstrate that $\mathcal{A}^*$ satisfies also Property 3, we verify the universal sentence

$$\forall A\,([A : \mathbb{N}_+ \to \mathcal{A}^* \,\wedge\, \forall m, n\,([m, n \in \mathbb{N}_+ \,\wedge\, m \neq n] \Rightarrow A_m \cap A_n = \emptyset)] \\ \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}^*). \tag{12.44}$$

To do this, we take an arbitrary set A, assume that A is a sequence $(A_n)_{n\in\mathbb{N}_+}$ of pairwise disjoint sets in $\mathcal{A}^*$, and show that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}^*$ follows to be true, i.e. we show that $\bigcup_{n=1}^{\infty} A_n$ is a μ^*-measurable set in $\mathcal{P}(\Omega)$. Let us first check that $\bigcup_{n=1}^{\infty} A_n$ is an element of $\mathcal{P}(\Omega)$. To begin with, we see that the inclusion $\operatorname{ran}(A) \subseteq \mathcal{A}^*$ is true because we assumed $\mathcal{A}^*$ to be a codomain of the sequence A, and the inclusion $\mathcal{A}^* \subseteq \mathcal{P}(\Omega)$ holds because of Property 1 of the algebra of sets $\mathcal{A}^*$. These two inclusions give $\operatorname{ran}(A) \subseteq \mathcal{P}(\Omega)$ with (2.13), so that the definition of the union of a family of sets and Proposition 3.100b) give

$$\bigcup_{n=1}^{\infty} A_n = \bigcup \operatorname{ran}(A) = \sup \operatorname{ran}(A),$$

where the supremum is taken with respect to the reflexive partial ordering of inclusion $\subseteq_{\mathcal{P}(\Omega)}$. Consequently, $\bigcup_{n=1}^{\infty} A_n$ is an element of $\mathcal{P}(\Omega)$ by definition of a supremum. To show now that $\bigcup_{n=1}^{\infty} A_n$ is μ^*-measurable, we apply the Criterion for Carathéodory measurability and verify accordingly

$$\forall X\,(X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap X) +_{\overline{\mathbb{D}}} \mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right]^c \cap X) \leq_{\overline{\mathbb{D}}} \mu^*(X)), \tag{12.45}$$

letting $X \in \mathcal{P}(\Omega)$ be arbitrary. To establish the inequality, let us first define the sequence of sets

$$U = (U_n)_{n \in \mathbb{N}_+} = (\bigcup_{i=1}^{n} A_n)_{n \in \mathbb{N}_+},$$

in $\mathcal{A}^*$, which converges isotonely to

$$[\sup \operatorname{ran}(U) =]\ \lim_{n \to \infty} U_n = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} A_n\ [= \sup \operatorname{ran}(A)] \tag{12.46}$$

with respect to $\subseteq_{\mathcal{P}(\Omega)}$ according to Exercise 11.5 (recalling from Note 12.6 that $\mathcal{A}^*$ is a ring of sets on Ω). Then, the sequence $f = (X \cap A_n)_{n \in \mathbb{N}_+}$ is also uniquely specified in view of Proposition 3.236, and we may show that $\mathcal{P}(\Omega)$ is a codomain of that sequence, i.e. that the range of that sequence is included in $\mathcal{P}(\Omega)$. Letting Y be an arbitrary element of $\operatorname{ran}(f)$, it follows by definition of a range that there is a particular constant $\bar{n}$ with $(\bar{n}, Y) \in f$, which we may write also as $Y = f(\bar{n}) = X \cap A_{\bar{n}}$. Here, the evidently true $A_{\bar{n}} \in \operatorname{ran}(A)$ implies $A_{\bar{n}} \in \mathcal{P}(\Omega)$ with the previously established $\operatorname{ran}(A) \subseteq \mathcal{P}(\Omega)$ by definition of a subset; this and the assumed $X \in \mathcal{P}(\Omega)$, imply $X \subseteq \Omega$ as well as $A_{\bar{n}} \subseteq \Omega$ with the definition of a power set, and these two inclusions in turn imply $X \cap A_{\bar{n}} \subseteq \Omega$ with (2.85), consequently $[Y =]\ X \cap A_{\bar{n}} \in \mathcal{P}(\Omega)$ again by definition of a power set. We thus showed that $Y \in \operatorname{ran}(f)$ implies $Y \in \mathcal{P}(\Omega)$, and as Y was arbitrary, we may infer from this implication the truth of the inlusion $\operatorname{ran}(f) \subseteq \mathcal{P}(\Omega)$. This means that $\mathcal{P}(\Omega)$ is indeed a codomain of the sequence $(X \cap A_n)_{n \in \mathbb{N}_+}$, which constitutes thus a function $f : \mathbb{N}_+ \to \mathcal{P}(\Omega)$. Since the outer measure is by definition a function $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}$, we may apply Proposition 3.178 to obtain the composition $\mu^* \circ f : \mathbb{N}_+ \to \overline{\mathbb{D}}$, which we may write thus as the sequence $(\mu^*(X \cap A_n))_{n \in \mathbb{N}_+}$. Based on these sequences, we may now establish the universal sentence

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \sum_{i=1}^{n} \mu^*(X \cap A_i) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \leq_{\overline{\mathbb{D}}} \mu^*(X)), \tag{12.47}$$

letting $\bar{n} \in \mathbb{N}_+$ be arbitrary. Because the supremum $\sup \operatorname{ran}(U) = \bigcup_{n=1}^{\infty} A_n$ in (12.47) is an upper bound for the range of U (with respect to $\subseteq_{\mathcal{P}(\Omega)}$), which range consists of the terms of $U = (U_n)_{n\in\mathbb{N}_+}$, we have in particular that $U_{\bar{n}} \subseteq_{\mathcal{P}(\Omega)} \bigcup_{n=1}^{\infty} A_n$ is true, which we may write also as the inclusion

$$\bigcup_{i=1}^{\bar{n}} A_i \subseteq \bigcup_{n=1}^{\infty} A_n$$

according to the definition of the sequence U and the specification of a reflexive partial ordering of inclusion. Together with the inclusion $X \subseteq X$ (which holds according to Proposition 2.4), this further implies the inclusion

$$X \setminus \bigcup_{n=1}^{\infty} A_n \ \subseteq \ X \setminus \bigcup_{i=1}^{\bar{n}} A_i \tag{12.48}$$

with (2.122). We already mentioned before the inclusion $X \subseteq \Omega$, which implies both $X \setminus \bigcup_{n=1}^{\infty} A_n \subseteq \Omega$ and $X \setminus \bigcup_{i=1}^{\bar{n}} A_i \subseteq \Omega$ because of (2.126). As these two inclusions give $X \setminus \bigcup_{n=1}^{\infty} A_n \in \mathcal{P}(\Omega)$ and $X \setminus \bigcup_{i=1}^{\bar{n}} A_i \in \mathcal{P}(\Omega)$ with the definition of a power set, so that the two set differences are in the domain of μ^*, we obtain now from (12.48) with Property 2 of an outer measure the inequality

$$\mu^*(X \setminus \bigcup_{n=1}^{\infty} A_n) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X \setminus \bigcup_{i=1}^{\bar{n}} A_i).$$

Let us recall that U is a sequence in $\mathcal{A}^*$ and that $\mathcal{A}^*$ is included in $\mathcal{P}(\Omega)$. Evidently, the element $U_{\bar{n}}$ of the range of U is then also an element of $\mathcal{P}(\Omega)$, so that $U_{\bar{n}} \subseteq \Omega$; alongside the previously established $X \subseteq \Omega$, this yields then $X \cap U_{\bar{n}} \subseteq \Omega$ and therefore $X \cap U_{\bar{n}} \in \mathcal{P}(\Omega)$ $[= \mathrm{dom}(\mu^*)]$. Consequently, we may now apply the Monotony Law for $+_{\overline{\mathbb{D}}}$ and $\leq_{\overline{\mathbb{D}}}$ to obtain also

$$\mu^*(X \setminus \bigcup_{n=1}^{\infty} A_n) +_{\overline{\mathbb{D}}} \mu^*(X \cap U_{\bar{n}}) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X \setminus \bigcup_{i=1}^{\bar{n}} A_i) +_{\overline{\mathbb{D}}} \mu^*(X \cap U_{\bar{n}}),$$

which we may write also as

$$\mu^*(X \cap \bigcup_{i=1}^{\bar{n}} A_i) +_{\overline{\mathbb{D}}} \mu^*(X \setminus \bigcup_{n=1}^{\infty} A_n) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X \cap \bigcup_{i=1}^{\bar{n}} A_i) +_{\overline{\mathbb{D}}} \mu^*(X \setminus \bigcup_{i=1}^{\bar{n}} A_i), \tag{12.49}$$

using the commutativity of $+_{\overline{\mathbb{D}}}$ and the definition of the sequence U. Let us recall next that the terms of the sequence of sets $(A_n)_{n\in\mathbb{N}_+}$ are pairwise disjoint, i.e.

$$\forall m, n\,([m, n \in \mathbb{N}_+ \ \wedge \ m \neq n] \Rightarrow A_m \cap A_n = \emptyset),$$

and let us check that the terms of the restricted sequence $(A_i \mid i \in \{1, \ldots, \bar{n}\})$ in $\mathcal{A}^*$ are also pairwise disjoint, i.e.

$$\forall i, j\,([i, j \in \{1, \ldots, \bar{n}\} \wedge i \neq j] \Rightarrow A_i \cap A_j = \emptyset).$$

Indeed, taking arbitrary $i, j \in \{1, \ldots, \bar{n}\}$ such that $i \neq j$ holds, we evidently have $i, j \in \mathbb{N}_+$ by definition of an initial segment of $\mathbb{N}_+$, which implies together with $i \neq j$ in view of the preceding assumed universal sentence the desired disjointness $A_i \cap A_j = \emptyset$, which is then true for all i and all j. Therefore, we may apply Proposition 12.43 and carry out substitution in (12.49) based on the equation (12.43), with the consequence that

$$\sum_{i=1}^{\bar{n}} \mu^*(A_i \cap X) +_{\overline{\mathbb{D}}} \mu^*(X \setminus \bigcup_{n=1}^{\infty} A_n) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X \cap \bigcup_{i=1}^{\bar{n}} A_i) +_{\overline{\mathbb{D}}} \mu^*(X \setminus \bigcup_{i=1}^{\bar{n}} A_i).$$

Since we previously found all three sets X, $\bigcup_{n=1}^{\infty} A_n$ and $\bigcup_{i=1}^{\bar{n}} A_i$ to be subsets of Ω, we may furthermore use (2.138) to write the two set differences occurring in the preceding inequality as intersections with complements, that is,

$$\sum_{i=1}^{\bar{n}} \mu^*(A_i \cap X) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X \cap \bigcup_{i=1}^{\bar{n}} A_i) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{i=1}^{\bar{n}} A_i\right]^c). \tag{12.50}$$

Because we established $U = (\bigcup_{i=1}^{n} A_i)_{n \in \mathbb{N}_+}$ as a sequence in $\mathcal{A}^*$, we evidently have $\bigcup_{i=1}^{\bar{n}} A_i \in \mathcal{A}^*$, which then implies jointly with the initially assumed $X \in \mathcal{P}(\Omega)$ because of the specification (12.20) of $\mathcal{A}^*$ the equation

$$\mu^*(X) = \mu^*(X \cap \bigcup_{i=1}^{\bar{n}} A_i) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{i=1}^{\bar{n}} A_i\right]^c).$$

We may therefore apply a substitution on the right-hand side of (12.50) and write equivalently

$$\sum_{i=1}^{\bar{n}} \mu^*(A_i \cap X) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X). \tag{12.51}$$

Using finally the Commutative Law for the intersection of two sets on the left-hand side, we obtain the desired inequality in (12.47), and since $\bar{n}$ was arbitrary, we may infer from this finding the truth of the universal sentence (12.47).

Here, we may use the value $\mu^*(X \cap [\bigcup_{n=1}^{\infty} A_n]^c)$ to define the constant function $g = \mathbb{N}_+ \times \{\mu^*(X \cap [\bigcup_{n=1}^{\infty} A_n]^c)\}$ with domain $\mathbb{N}_+$, which function/sequence is a surjection with range $\{\mu^*(X \cap [\bigcup_{n=1}^{\infty} A_n]^c)\}$ due to Proposition 3.193. Since the element of that singleton is in $\overline{\mathbb{D}}$ by definition of an outer measure, the singleton (i.e., the range) is included in $\overline{\mathbb{D}}$ because of (2.184), so that $\overline{\mathbb{D}}$ is a codomain of g. Furthermore, the nonempty constant function g converges increasingly to

$$\lim_{n\to\infty} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) = \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c). \tag{12.52}$$

with respect to the partial/total ordering $\leq_{\overline{\mathbb{D}}}$ in view of Proposition 4.76.

In addition, we may observe in light of Corollary 5.136 that the series $s = (\sum_{i=1}^{n} \mu^*(X \cap A_i))_{n\in\mathbb{N}_+}$ in $\overline{\mathbb{R}}_+^0$ generated by the composed sequence $\mu^* \circ f = (\mu^*(X \cap A_n))_{n\in\mathbb{N}_+}$ is increasingly convergent as well. We may therefore apply Proposition **??** to the increasingly convergent sequences s and g to infer that the sum

$$(\sum_{i=1}^{n} \mu^*(X \cap A_i) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c))_{n\in\mathbb{N}_+} \tag{12.53}$$

of these sequences is also increasingly convergent with limit

$$\begin{aligned} &\lim_{n\to\infty} \left(\sum_{i=1}^{n} \mu^*(X \cap A_i) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c)\right) \qquad (12.54)\\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu^*(X \cap A_i) \ +_{\overline{\mathbb{D}}} \ \lim_{n\to\infty} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c). \end{aligned}$$

Here, we can use the notation for the limit of a series and write the first limit on the right-hand side of the equations also as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mu^*(X \cap A_i) = \sum_{i=1}^{\infty} \mu^*(X \cap A_i). \tag{12.55}$$

As the terms of the sequence (12.53) satisfy (12.47), it follows with Exercise **??** that the limit satisfies

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \mu^*(X \cap A_i) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c)\right) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X).$$

Then, substitutions based on (12.52), (12.54) and yield (12.55) the equivalent inequality

$$\sum_{i=1}^{\infty} \mu^*(X \cap A_i) \ +_{\overline{\mathbb{D}}} \ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X). \tag{12.56}$$

Let us observe here concerning the infinite sum that Property 3 of an outer measure gives the inequality

$$\mu^*(\bigcup_{n=1}^{\infty}(X \cap A_n)) \ \leq_{\overline{\mathbb{D}}} \ \sum_{i=1}^{\infty} \mu^*(X \cap A_i),$$

to which we now apply the Monotony Law for $+_{\overline{\mathbb{D}}}$ and $\leq_{\overline{\mathbb{D}}}$ to obtain

$$\begin{aligned}
&\mu^*(\bigcup_{n=1}^{\infty}(X \cap A_n)) \ +_{\overline{\mathbb{D}}} \ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \\
&\quad \leq_{\overline{\mathbb{D}}} \ \sum_{i=1}^{\infty} \mu^*(X \cap A_i) \ +_{\overline{\mathbb{D}}} \ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c).
\end{aligned} \tag{12.57}$$

Because of the transitivity of the total ordering $\leq_{\overline{\mathbb{D}}}$, the conjunction of this inequality and (12.56) implies

$$\mu^*(\bigcup_{n=1}^{\infty}(X \cap A_n)) \ +_{\overline{\mathbb{D}}} \ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X).$$

Next, we apply now Theorem 3.240b) in connection with the definition of the union of a family of sets to obtain the equation

$$X \cap \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}(X \cap A_n), \tag{12.58}$$

so that the preceding inequality may, after substitution, be written as

$$\mu^*(X \cap \bigcup_{n=1}^{\infty} A_n) \ +_{\overline{\mathbb{D}}} \ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X).$$

In the final step, we use the Commutative Law for the intersection of two sets, which allows us to write this inequality as the desired inequality in (12.45). As X is arbitrary, we may therefore conclude that the universal sentence (12.45) holds, so that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}^*$ true. Since A was also arbitrary, we may infer from this finding the truth of the universal sentence (12.44), which shows that $\mathcal{A}^*$ satisfies Property 3 of a λ-system (alongside Property 1, Property 2 and Property 4). We thus proved that $\mathcal{A}^*$ is a λ-system on Ω, and because the algebra of sets $\mathcal{A}^*$ on Ω is also a π-system on Ω as mentioned in Note 12.6, it follows with the π-λ Characterization of a σ-algebra that $\mathcal{A}^*$ is a σ-algebra on Ω.

Concerning b), we consider now the restriction of the outer measure μ^* :

$\mathcal{P}(\Omega) \to \overline{\mathbb{D}}$ to the σ-algebra $\mathcal{A}^*$ on Ω. Since $\mathcal{A}^* \subseteq \mathcal{P}(\Omega)$ is true according to Property 1 of a σ-algebra, we obtain

$$\mu^* \restriction \mathcal{A}^* : \mathcal{A}^* \to \overline{\mathbb{D}}$$

with Proposition 3.164. Since the algebra of sets $\mathcal{A}^*$ contains $\emptyset$ because of Property 2, we then also obtain the equations

$$\mu^* \restriction \mathcal{A}^*(\emptyset) = \mu^*(\emptyset) = 0$$

with Corollary 3.165 and Property 1 of an outer measure. Thus, the restriction $\mu^* \restriction \mathcal{A}^*$ has Property 1 of a measure. To establish (the σ-additivity) Property 2 of a measure, i.e.

$$\forall A\,([A : \mathbb{N}_+ \to \mathcal{A}^* \wedge \forall m, n\,([m, n \in \mathbb{N}_+ \wedge m \neq n] \Rightarrow A_m \cap A_n = \emptyset)]$$
$$\Rightarrow \mu^* \restriction \mathcal{A}^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{i=1}^{\infty} \mu^* \restriction \mathcal{A}^*(A_i)]),$$

we take an arbitrary set A such that A is a sequence $(A_n)_{n\in\mathbb{N}_+}$ of pairwise disjoint sets in $\mathcal{A}^*$, for which the union $\bigcup_{n=1}^{\infty} A_n$ is then in $\mathcal{A}^*$ according to Property 3 of the σ-algebra $\mathcal{A}^*$, as shown in a). Here, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}^*$ means also that the set $\bigcup_{n=1}^{\infty} A_n$ is μ^*-measurable by definition of the set system $\mathcal{A}^*$. Thus, the union satisfies

$$\forall X\,(X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap X) +_{\overline{\mathbb{D}}} \mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right]^c \cap X)). \tag{12.59}$$

Let us now verify the truth of the universal sentence

$$\forall X\,(X \in \mathcal{P}(\Omega) \Rightarrow \mu^*(X) = \sum_{i=1}^{\infty} \mu^*(X\cap A_i) +_{\overline{\mathbb{D}}} \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c)). \tag{12.60}$$

We take an arbitrary set $X \in \mathcal{P}(\Omega)$ and recall from the proof of a) that the inequalities

$$\mu^*(\bigcup_{n=1}^{\infty}(X \cap A_n)) \ +_{\overline{\mathbb{D}}}\ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c)$$
$$\leq_{\overline{\mathbb{D}}} \sum_{i=1}^{\infty} \mu^*(X \cap A_i) \ +_{\overline{\mathbb{D}}}\ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \ \leq_{\overline{\mathbb{D}}}\ \mu^*(X)$$

hold according to (12.57) and (12.56). Here, the first sum can also be written as

$$\mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap X) \ +_{\overline{\mathbb{D}}} \ \mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right]^c \cap X) = \mu^*(X)$$

by using (12.58), the Commutative Law for the intersection of two sets, and (12.59). With this equation, the preceding inequalities can be written as the conjunction

$$\begin{aligned} \mu^*(X) \ &\leq_{\overline{\mathbb{D}}} \ \sum_{i=1}^{\infty} \mu^*(X \cap A_i) \ +_{\overline{\mathbb{D}}} \ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \\ \wedge \ &\sum_{i=1}^{\infty} \mu^*(X \cap A_i) \ +_{\overline{\mathbb{D}}} \ \mu^*(X \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c) \ \leq_{\overline{\mathbb{D}}} \ \mu^*(X), \end{aligned}$$

which implies the equation in (12.60) because the total/partial ordering $\leq_{\overline{\mathbb{D}}}$ is antisymmetric. As X was arbitrary, we may therefore conclude that the universal sentence (12.60) is true.

Now, because $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}^*$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{P}(\Omega)$ with Property 1 of the σ-algebra $\mathcal{A}*$ and the definition of a subset, we obtain in view of the true universal sentences (12.59) and (12.60) after substitution

$$\begin{aligned} &\mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap \left[\bigcup_{n=1}^{\infty} A_n\right]) \ +_{\overline{\mathbb{D}}} \ \mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right]^c \cap \left[\bigcup_{n=1}^{\infty} A_n\right]) \\ &\quad = \sum_{i=1}^{\infty} \mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap A_i) \ +_{\overline{\mathbb{D}}} \ \mu^*(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap \left[\bigcup_{n=1}^{\infty} A_n\right]^c). \end{aligned}$$

Then, we may write this equation also as

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \ +_{\overline{\mathbb{D}}} \ \mu^*(\emptyset) = \sum_{i=1}^{\infty} \mu^*(\left[\bigcup \operatorname{ran}(A)\right] \cap A_i) \ +_{\overline{\mathbb{D}}} \ \mu^*(\emptyset), \qquad (12.61)$$

by applying the Idempotent Law for the intersection of two sets, (2.135) in connection with the Commutative Law for the intersection of two sets, and the definition of the union of a family of sets. Here, any term A_i of the sequence $A = (A_n)_{n \in \mathbb{N}_+}$ is evidently an element of the range of A, so that $A_i \subseteq \operatorname{ran}(A)$ follows to be true with Proposition 2.67. Consequently, we find any of the intersections $[\bigcup \operatorname{ran}(A)] \cap A_i$ to be identical with A_i by using (2.77) and the Commutative Law for the intersection of two sets. Applying also Property 1 of an outer measure and the definition of a neutral element (here with respect to the addition $+_{\overline{\mathbb{D}}}$) to the equation (12.61), we obtain

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

As we assumed (the terms of) the sequence $A = (A_n)_{n\in\mathbb{N}_+}$ to be in $\mathcal{A}^*$ and as $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}^*$ also holds, the two values of μ^* in the preceding equation are identical with the values of the restriction $\mu^* \upharpoonright \mathcal{A}^*$ for the same arguments (according to Corollary 3.165); thus, substitutions yield the equation

$$\mu^* \upharpoonright \mathcal{A}^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{i=1}^{\infty} \mu^* \upharpoonright \mathcal{A}^*(A_i).$$

Since A is arbitrary, we may therefore conclude that the restriction $\mu^* \upharpoonright \mathcal{A}^*$ satisfies also Property 2 of a measure, which finding completes the proof of b). As the sets Ω and μ^* were initially arbitrary, we may now finally conclude that the stated lemma is true. □

Lemma 12.11 (Carathéodory's construction of outer measures by measures). *The following sentences are true for any set Ω, for any ring of sets $\mathcal{R}$ on Ω such that there is a sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in $\mathcal{R}$ converging isotonely to Ω (with respect to $\subseteq_{\mathcal{P}(\Omega)}$), and for any measure $\mu : \mathcal{R} \to \overline{\mathbb{D}}$.*

a) For any set $B \in \mathcal{P}(\Omega)$, there exists a unique set M_B satisfying

$$\begin{aligned} &\forall x\,(x \in M_B \Leftrightarrow [x \in \overline{\mathbb{D}} \\ &\quad \wedge\ \exists A\,(A : \mathbb{N}_+ \to \mathcal{R} \ \wedge\ B \subseteq \bigcup_{n=1}^{\infty} A_n \ \wedge\ x = \sum_{i=1}^{\infty} \mu(A_i))]). \end{aligned} \tag{12.62}$$

b) Then, there exists the unique function

$$\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{D}}, \qquad B \mapsto \overset{\leq_{\overline{\mathbb{D}}}}{\inf} M_B. \tag{12.63}$$

c) Furthermore, the function μ^* is an outer measure.

Proof. We let Ω, $\mathcal{R}$ and μ be arbitrary sets, we assume that $\mathcal{R}$ is a ring of sets on Ω such that there exists an isotone sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in $\mathcal{R}$ with

$$\overset{\subseteq_{\mathcal{P}(\Omega)}}{\lim_{n\to\infty}} = \bigcup_{n=1}^{\infty} A_n = \Omega, \tag{12.64}$$

and we assume moreover that μ is a measure with domain $\mathcal{R}$.

Concerning a), we take an arbitrary set B and assume furthermore $B \in \mathcal{P}(\Omega)$ to be true. We may then evidently apply the Axiom of Specification and the Equality Criterion for sets to establish the unique existence of a set M_B satisfying (12.62).

Concerning b), we apply now Function definition by replacement to establish μ^*. For this purpose, we verify the universal sentence

$$\forall B\,(B \in \mathcal{P}(\Omega) \Rightarrow \exists! y\,(y = \inf^{\leq_{\overline{\mathbb{D}}}} M_B)), \tag{12.65}$$

letting $B \in \mathcal{P}(\Omega)$ be arbitrary. We notice in light of a) that $x \in M_B$ implies for any x especially $x \in \overline{\mathbb{D}}$, so that $M_B \subseteq \overline{\mathbb{D}}$ follows to be true by definition of a subset. Because $(\overline{\mathbb{D}}, \leq_{\overline{\mathbb{D}}})$ is a complete lattice, the infimum of M_B with respect to $\leq_{\overline{\mathbb{D}}}$ exists, which is thus a uniquely determined set in $\overline{\mathbb{D}}$. We may therefore apply (1.109) to establish the truth of the uniquely existential sentence in (12.65). As B was arbitrary, we may therefore conclude that the universal sentence (12.65) holds, which in turn implies the unique existence of a function μ^* with domain $\mathcal{P}(\Omega)$ such that

$$\forall B\,(B \in \mathcal{P}(\Omega) \Rightarrow \mu^*(B) = \inf^{\leq_{\overline{\mathbb{D}}}} M_B). \tag{12.66}$$

Let us check that $\overline{\mathbb{D}}$ is indeed a codomain of μ^*, i.e. that the range of μ^* is included in $\overline{\mathbb{D}}$. Letting y be arbitrary and assuming $y \in \mathrm{ran}(\mu^*)$ to be true, there exists then by definition of a range a constant, say $\bar{X}$, such that $(\bar{X}, y) \in \mu^*$ holds. We already showed that μ^* is a function, so that we may write the preceding finding also as $y = \mu^*(\bar{X}) = \inf^{\leq_{\overline{\mathbb{D}}}} M_{\bar{X}}$. Thus, the infimum y with respect to $\leq_{\overline{\mathbb{D}}}$ is an element of $\overline{\mathbb{D}}$. Since y was arbitrary, we may apply the definition of a subset to infer from this the truth of the inclusion $y \in \mathrm{ran}(\mu^*) \subseteq \overline{\mathbb{D}}$, which shows that $\overline{\mathbb{D}}$ is a codomain of μ^*. We thus established the function (12.63).

Concerning c), under construction!

□

Exercise 12.2. Verify that any measure on the power set of any set Ω is an outer measure.

Chapter 13.

Sets of Measurable Functions

We begin by defining and characterizing some further fundamental concepts in the context of topological spaces.

13.1. Continuous Functions

Definition 13.1 (Continuous function). We say for any topological spaces $(\Omega, \mathcal{O})$, $(\Omega', \mathcal{O}')$ and for any function $f : \Omega \to \Omega'$

(1) that f is *continuous at* an $\omega \in \Omega$ (with respect to $\mathcal{O}$ and $\mathcal{O}'$) iff every open set of $\mathcal{O}'$ containing the value $f(\omega)$ includes the image of some open set of $\mathcal{O}$ containing ω, i.e. iff

$$\forall B\,([B \in \mathcal{O}' \wedge f(\omega) \in B] \Rightarrow \exists A\,(A \in \mathcal{O} \wedge \omega \in A \wedge f[A] \subseteq B)). \tag{13.1}$$

(2) that f is *continuous* (with respect to $\mathcal{O}$ and $\mathcal{O}'$) iff the inverse image of every open set of $\mathcal{O}'$ is an open set of $\mathcal{O}$, i.e. iff

$$\forall B\,(B \in \mathcal{O}' \Rightarrow f^{-1}[B] \in \mathcal{O}). \tag{13.2}$$

(3) that f is *piecewise continuous* (with respect to $\mathcal{O}$ and $\mathcal{O}'$) iff f is continuous at all but a finite number of elements of its domain, in the sense that

$$\exists A\,(A \subseteq \Omega \wedge A \text{ is finite } \wedge \forall \omega\,(\omega \in \Omega \setminus A \Rightarrow f \text{ is continuous at } \omega)). \tag{13.3}$$

Proposition 13.1. *It is true for any topological spaces* $(\Omega, \mathcal{O})$, $(\Omega', \mathcal{O}')$ *and for any function* $f : \Omega \to \Omega'$ *that* f *is continuous iff* f *is continuous at all elements of its domain, i.e.*

$$f \text{ is continuous } \Leftrightarrow \forall \omega\,(\omega \in \Omega \Rightarrow f \text{ is continuous at } \omega). \tag{13.4}$$

Proof. We let Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f be arbitrary sets such that $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ are topological spaces and such that f is a function from Ω to Ω. To prove the first part ('$\Rightarrow$') of the equivalence, we assume that f is continuous. We then let $\omega \in \Omega$ be arbitrary. To show that f is continuous at ω, we need to verify (13.1). For this purpose, we let B be arbitrary and assume $B \in \mathcal{O}' \wedge f(\omega) \in B$. On the one hand, since f is continuous, the first part of that conjunction implies $f^{-1}[B] \in \mathcal{O}$ with (13.2); on the other hand, the second part of the conjunction implies $\omega \in f^{-1}[B]$ by definition of an inverse image. As an open set of $\mathcal{O}'$, the set B is evidently a subset of Ω' as a consequence of Property 1 of a topology, and this inclusion in turn implies the inclusion $f[f^{-1}[B]] \subseteq B$ with Proposition 3.223. The preceding properties of the particular set $\bar{A} = f^{-1}[B]$ demonstrate the truth of the existential sentence in (13.1), and since B was arbitrary, we may therefore conclude that f satisfies indeed (13.1), which means that f is continuous at ω. Because ω was also arbitrary, it follows that f is continuous at all elements of its domain Ω, so that the first part of the equivalence in (13.4) holds.

To prove the second part ('$\Leftarrow$'), we now assume that f is continuous at all $\omega \in \Omega$, and we show that (13.2) holds. For this purpose, we let $B \in \mathcal{O}'$ be arbitrary and verify $f^{-1}[B] \in \mathcal{O}$. Let us observe in light of the Axiom of Specification and the Equality Criterion for sets that there exists a unique set (system) $\mathcal{K}$ such that

$$\forall A\,(A \in \mathcal{K} \Leftrightarrow [A \in \mathcal{O} \,\wedge\, \omega \in f^{-1}[B] \,\wedge\, \omega \in A \,\wedge\, f[A] \subseteq B]). \tag{13.5}$$

Since $A \in \mathcal{K}$ implies $A \in \mathcal{O}$ for any A, we have by definition of a subset the true inclusion $\mathcal{K} \subseteq \mathcal{O}$. This inclusion implies now $\bigcup \mathcal{K} \in \mathcal{O}$ with Property 3 of a topology, and we can also show that $\bigcup \mathcal{K} = f^{-1}[B]$ is true, by applying the Equality Criterion for sets. To prove accordingly the universal sentence

$$\forall \omega\,(\omega \in \bigcup \mathcal{K} \Leftrightarrow \omega \in f^{-1}[B]), \tag{13.6}$$

we let ω be arbitrary and assume first $\omega \in \bigcup \mathcal{K}$ to be true. By definition of the union of a set system, there then exists an element of $\mathcal{K}$, say $\bar{A}$, such that $\omega \in \bar{A}$ holds. Here, $\bar{A} \in \mathcal{K}$ implies with (13.5) especially the inclusion $f[\bar{A}] \subseteq B$. Consequently, we obtain $\bar{A} \subseteq f^{-1}[B]$ with Proposition 3.224. Due to this inclusion, $\omega \in \bar{A}$ implies $\omega \in f^{-1}[B]$, which is the desired consequent of the implication '$\Rightarrow$' in (13.6). Regarding the implication '$\Leftarrow$', we conversely assume $\omega \in f^{-1}[B]$ to be true, which implies $f(\omega) \in B$ with the definition of an inverse image. Clearly, ω is in the domain Ω of the function f, so that f is continuous at ω, by assumption. Thus, the conjunction of $B \in \mathcal{O}'$ and $f(\omega) \in B$ implies – according to (13.1) – the existence of a particular set $\bar{A} \in \mathcal{O}$ for which $\omega \in \bar{A}$ and $f[\bar{A}] \subseteq B$ hold. Together with the assumed $\omega \in f^{-1}[B]$, these findings imply $\bar{A} \in \mathcal{K}$ with (13.5), and this gives us also the inclusion $\bar{A} \subseteq \bigcup \mathcal{K}$ with (2.201). With this, $\omega \in \bar{A}$ implies $\omega \in \bigcup \mathcal{K}$, as desired. We thus completed the proof of the equivalence in (13.6), in which ω is arbitrary, so that the universal sentence (13.6) follows now to be true. This proves the suggested equality $\bigcup \mathcal{K} = f^{-1}[B]$, so that the previously established $\bigcup \mathcal{K} \in \mathcal{O}$ yields through substitution $f^{-1}[B] \in \mathcal{O}$, as desired. Since B was arbitrary, we therefore conclude that f satisfies the universal sentence (13.2), which means that f is continuous.

Having completed the proof of the equivalence (13.4), we may infer from the truth of this equivalence the truth of the proposition since Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f were initially arbitrary. □

Proposition 13.2. *It is true for any topological spaces* $(\Omega, \mathcal{O})$ *and* $(\Omega', \mathcal{O}')$ *that every continuous function* $f : \Omega \to \Omega'$ *with respect to* $\mathcal{O}$ *and* $\mathcal{O}'$ *is also continuous with respect to* $\mathcal{O}$ *and the subspace topology of* $\mathcal{O}'$ *in the image of* Ω *under* f.

Proof. We take arbitrary sets Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f, assuming $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ to be topological spaces and assuming f to be a continuous function from Ω to Ω' with respect to $\mathcal{O}$ and $\mathcal{O}'$. Observing that Ω is included in itself according to Proposition 2.4, we see that the image of Ω under f is defined and a subset of the codomain Ω', according to Corollary 3.218. Therefore, the topological subspace $(f[\Omega], \mathcal{O}'|f[\Omega])$ of $(\Omega', \mathcal{O}')$ is also defined. Furthermore, the image is identical with the range of f in view of (3.718), so that we can write f as the surjection $f : \Omega \to f[\Omega]$. To show that this function is continuous with respect to $\mathcal{O}$ and the subspace topology $\mathcal{O}'|f[\Omega]$, we establish accordingly the truth of

$$\forall B\,(B \in \mathcal{O}'|f[\Omega] \Rightarrow f^{-1}[B] \in \mathcal{O}). \tag{13.7}$$

We take an arbitrary open set $B \in \mathcal{O}'|f[\Omega]$, so that the definition of a subspace topology gives us a particular open set $\bar{U} \in \mathcal{O}'$ with $f[\Omega] \cap \bar{U} = B$. Since f was assumed to be continuous with respect to $\mathcal{O}$ and $\mathcal{O}'$, we have that $\bar{U} \in \mathcal{O}'$ implies $f^{-1}[\bar{U}] \in \mathcal{O}$. We also obtain the true equations

$$f^{-1}[B] = f^{-1}[f[\Omega] \cap \bar{U}] = f^{-1}[f[\Omega]] \cap f^{-1}[\bar{U}] = \Omega \cap f^{-1}[\bar{U}] = f^{-1}[\bar{U}]$$

by applying substitution, applying then (3.760) to $f : \Omega \to \Omega'$ in connection with the aforementioned $f[\Omega] \subseteq \Omega'$ and the fact that the open set $\bar{U} \in \mathcal{O}'$ is also a subset of Ω' as a consequence of Property 1 of a topology on Ω', applying subsequently (3.746) to $f : \Omega \to f[\Omega]$, and applying finally (2.77) in connection with Note 3.30 (which gives the inclusion $f^{-1}[\bar{U}] \subseteq \Omega$).

Thus, the previous finding $f^{-1}[\bar{U}] \in \mathcal{O}$ yields $f^{-1}[B] \in \mathcal{O}$ by means of substitution, so that the implication in (13.7) holds. Since B is arbitrary, we may therefore conclude that (13.7) is true, which universal sentence implies that $f : \Omega \to f[\Omega]$ is indeed continuous with respect to $\mathcal{O}$ and $\mathcal{O}'|f[\Omega]$. Initially, the set Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f were arbitrary, so that the proposition follows to be true. □

Lemma 13.3 (Characterization of continuity by closure). *It is true for any topological spaces* $(\Omega, \mathcal{O})$, $(\Omega', \mathcal{O}')$ *and for any continuous function* $f : \Omega \to \Omega'$ *that the image of the closure of any subset of* Ω *(under* f*) is included in the closure of the image of that subset (under* f*), that is,*

$$f \text{ is continuous } \Rightarrow \forall A\,(A \subseteq \Omega \Rightarrow f[\mathrm{cl}(A)] \subseteq \mathrm{cl}(f[A])). \tag{13.8}$$

Proof. We let Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f be arbitrary sets, we assume that $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ are topological spaces, and we assume also that f is a function from Ω to Ω'. We may prove the stated implication directly, which we do by assuming f to be continuous, which means that f satisfies

$$\forall B\,(B \in \mathcal{O}' \Rightarrow f^{-1}[B] \in \mathcal{O}). \tag{13.9}$$

To establish the desired consequent, we take an arbitrary set A, we assume A to be included in Ω, and we show that the inclusion $f[\mathrm{cl}(A)] \subseteq \mathrm{cl}(f[A])$ is implied, noting that the closure and the image of A are indeed specified by virtue of the assumption $A \subseteq \Omega$. To prove the desired inclusion, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall \omega'\,(\omega' \in f[\mathrm{cl}(A)] \Rightarrow \omega' \in \mathrm{cl}(f[A])), \tag{13.10}$$

letting ω' be arbitrary and assuming $\omega' \in f[\mathrm{cl}(A)]$ to be true. By definition of an image, this means that $\omega' \in \mathrm{ran}(f \restriction \mathrm{cl}(A))$ is true. Consequently, there exists by definition of a range a constant, say $\bar{\omega}$, such that $(\bar{\omega}, \omega') \in f \restriction \mathrm{cl}(A)$ holds. We thus have by definition of restriction $(\bar{\omega}, \omega') \in f$ and $\bar{\omega} \in \mathrm{cl}(A)$. The former finding reads in function notation $\omega' = f(\bar{\omega})$, and $\bar{\omega} \in \Omega$ $[= \mathrm{dom}(f)]$ holds by definition of a domain. With this, the other finding $\bar{\omega} \in \mathrm{cl}(A)$ implies according to the Characterization of the elements of a closure

$$\forall U\,([U \in \mathcal{O} \wedge \bar{\omega} \in U] \Rightarrow A \cap U \neq \emptyset). \tag{13.11}$$

We are now in a position to establish the truth of the consequent $\omega' \in \mathrm{cl}(f[A])$ of the implication in (13.10), for which task we also use the Characterization of the elements of a closure. We prove accordingly the universal sentence

$$\forall V\,([V \in \mathcal{O}' \wedge \omega' \in V] \Rightarrow f[A] \cap V \neq \emptyset), \tag{13.12}$$

taking an arbitrary open set V of the topology $\mathcal{O}'$ such that $\omega' \in V$ is satisfied. The latter assumption can be written also as $f(\bar{\omega}) \in V$, applying substitution based on the previously found equation $\omega' = f(\bar{\omega})$. Therefore, we obtain $\bar{\omega} \in f^{-1}[V]$ with the definition of an inverse image; as the previous assumption $V \in \mathcal{O}'$ implies $f^{-1}[V] \in \mathcal{O}$ with the continuity assumption (13.9), we obtain now in view of (13.11) the true negation $A \cap f^{-1}[V] \neq \emptyset$. This shows that the intersection $A \cap f^{-1}[V]$ is not empty, so that there is evidently an element, say $\bar{x}$, which is both in A and in $f^{-1}[V]$. Here, $\bar{x} \in f^{-1}[V]$ implies $f(\bar{x}) \in V$ by definition of an inverse image; denoting this value by $\bar{y} = f(\bar{x})$, we can also write $\bar{y} \in V$. Moreover, we can write the preceding equation in the form $(\bar{x}, \bar{y}) \in f$, which implies in conjunction with the previously established $\bar{x} \in A$ that $(\bar{x}, \bar{y}) \in f \upharpoonright A$ holds (due to the definition of a restriction). This shows in light of the definition of a range and the definition of an image that $\bar{y} \in f[A] \quad [= \operatorname{ran}(f \upharpoonright A)]$ is true. In conjunction with the previously obtained $\bar{y} \in V$, this further implies $\bar{y} \in f[A] \cap V$ (by definition of the intersection of two sets), so that the intersection $f[A] \cap V$ is clearly nonempty. This completes the proof of the implication in (13.12), in which V is arbitrary, so that the universal sentence (13.12) follows to be true as well. Recalling $\omega' = f(\bar{\omega})$, we see in light of the Function Criterion that ω' is a value of f in the codomain Ω' of that function. Therefore, the truth of (13.12) implies indeed the truth of $\omega' \in \operatorname{cl}(f[A])$ according to the Characterization of the elements of a closure. This is the desired consequent of the implication in (13.10), and as ω' is here arbitrary, we may conclude that the inclusion $f[\operatorname{cl}(A)] \subseteq \operatorname{cl}(f[A])$. Since A is also arbitrary, the consequent of the implication (13.8) follows to be true, and as the sets Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f were initially all arbitrary, we now finally conclude that the lemma holds. $\square$

Lemma 13.4. *It is true for any topological spaces* $(\Omega, \mathcal{O})$,$(\Omega', \mathcal{O}')$ *and for any function* $f : \Omega \to \Omega'$ *that the inverse image of every closed set* B *in* Ω' *is a closed set in* Ω *if the image of the closure of every subset* A *of* Ω *is included in the closure of the image of* A*, i.e.*

$$\begin{aligned} &\forall A\,(A \subseteq \Omega \Rightarrow f[\operatorname{cl}(A)] \subseteq \operatorname{cl}(f[A])) \\ &\quad \Rightarrow \forall B\,(B \text{ is a closed set in } \Omega' \Rightarrow f^{-1}[B] \text{ is a closed set in } \Omega). \end{aligned} \tag{13.13}$$

Proof. We take arbitrary sets Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f such that $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ are topological spaces, and such that f is a function from Ω to Ω'. Next, we assume also the universal sentence

$$\forall A\,(A \subseteq \Omega \Rightarrow f[\operatorname{cl}(A)] \subseteq \operatorname{cl}(f[A])) \tag{13.14}$$

to be true, and we take an arbitrary set B such that B is closed in Ω'. We thus have $\operatorname{cl}(B) = B$ because of Corollary 11.48, and we show in the following that $\mathrm{cl}(f^{-1}[B]) = f^{-1}[B]$ also holds. For this purpose, we apply the Equality Criterion for sets and prove the equivalent universal sentence

$$\forall \omega \, (\omega \in \mathrm{cl}(f^{-1}[B]) \Leftrightarrow \omega \in f^{-1}[B]). \tag{13.15}$$

We take an arbitrary ω and assume first $\omega \in \mathrm{cl}(f^{-1}[B])$ to hold. This assumption implies

$$f(\omega) \in f[\mathrm{cl}(f^{-1}[B])]$$

with Proposition 3.217, noting that B is a subset of Ω', so that the inverse image $f^{-1}[B]$ and then also its closure $\mathrm{cl}(f^{-1}[B])$ are subsets of the domain Ω of f in view of Note 3.30 and the definition of a closure. Consequently, we obtain

$$f(\omega) \in \mathrm{cl}(f[f^{-1}[B]]) \tag{13.16}$$

with the assumed (13.14). Since $B \subseteq \Omega'$ gives the inclusion $f[f^{-1}[B]] \subseteq B$ with (3.747), it follows with (11.354) that the inclusion

$$\mathrm{cl}(f[f^{-1}[B]]) \subseteq \mathrm{cl}(B)$$

also holds. Therefore, (13.16) implies by definition of a subset

$$f(\omega) \in \mathrm{cl}(B) \quad [= B],$$

and we can write the resulting $f(\omega) \in B$ also as $\omega \in f^{-1}[B]$ by means of the definition of an inverse image. Thus, the first part ('$\Rightarrow$') of the equivalence in (13.15) holds. Regarding the second part ('$\Leftarrow$'), assume $\omega \in f^{-1}[B]$ and recall from the definition of a closure that the inclusion $f^{-1}[B] \subseteq \mathrm{cl}(f^{-1}[B])$ is true, so that the preceding assumption yields already the desired consequent $\omega \in \mathrm{cl}(f^{-1}[B])$ (by definition of a subset). As ω is arbitrary, we may therefore conclude that the universal sentence (13.15) is true, allowing us to infer the truth of the equality $\mathrm{cl}(f^{-1}[B]) = f^{-1}[B]$. This finding demonstrates in light of Exercise 11.19b) that the set $f^{-1}[B]$, being a closure, constitutes a closed set in Ω. Since B and A were arbitrary, we may now further conclude that (13.13) holds, and since the sets Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f were initially also arbitrary, we may finally conclude that the stated lemma is true. □

Theorem 13.5 (Continuity Criterion based on closed sets). *It is true for any topological spaces $(\Omega, \mathcal{O})$,$(\Omega', \mathcal{O}')$ and for any function $f : \Omega \to \Omega'$ that f is continuous iff the inverse image of every closed set in Ω' is a closed set in Ω, i.e.*

$$\begin{aligned} & f \text{ is continuous} \\ \Leftrightarrow \;\; & \forall B \, (B \text{ is a closed set in } \Omega' \Rightarrow f^{-1}[B] \text{ is a closed set in } \Omega). \end{aligned} \tag{13.17}$$

Proof. Letting Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f such that $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ are topological spaces and such that f is a function from Ω to Ω', we assume first that f is continuous. According to the Characterization of continuity by closure, we obtain then the true universal sentence

$$\forall A\,(A \subseteq \Omega \Rightarrow f[\mathrm{cl}(A)] \subseteq \mathrm{cl}(f[A])),$$

which in turn implies the truth of the universal sentence

$$\forall B\,(B \text{ is a closed set in } \Omega' \Rightarrow f^{-1}[B] \text{ is a closed set in } \Omega) \tag{13.18}$$

because of Lemma 13.4, proving the first part ('$\Rightarrow$') of the equivalence in (13.17). Regarding the second part ('$\Leftarrow$'), we assume now the preceding universal sentence to be true, and we show that f satisfies

$$\forall V\,(V \in \mathcal{O}' \Rightarrow f^{-1}[V] \in \mathcal{O}). \tag{13.19}$$

We take an arbitrary open set $V \in \mathcal{O}'$, so that $(V^c)^c$ is also open in view of (2.136). Consequently, V^c is a closed set in Ω' by definition, which implies with the assumed sentence (13.18) that the inverse image $f^{-1}[V^c]$ is a closed set in Ω. Then, $(f^{-1}[V^c])^c$ is open, where the equation $f^{-1}[V^c] = (f^{-1}[V])^c$ holds because of Proposition 3.227, so that substitution yields that $((f^{-1}[V])^c)^c$ is open as well. Another application of (2.136) gives then that $f^{-1}[V]$ is open, completing the proof of the implication in (13.19). Here, V is arbitrary, so that the universal sentence (13.19) follows now to be true. This means by definition that f is continuous, so that the proof of the equivalence (13.17) is also complete. As Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f were initially arbitrary sets, we may now infer from the truth of that equivalence the truth of the theorem. □

Theorem 13.6 (Continuity of constant functions). *For any topological spaces $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ the constant function $f_c : \Omega \to \Omega'$ is continuous for any $c \in \Omega'$.*

Proof. Letting Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and c be arbitrary such that $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ are topological spaces and such that c is an element of Ω', the constant function $f_c = \Omega \times \{c\}$ gives the inverse image $f_c^{-1}[U'] = \Omega$ or $f_c^{-1}[U'] = \emptyset$ for every $U' \subseteq \Omega'$ in view of Proposition 3.225. Let us now verify

$$\forall U'\,(U' \in \mathcal{O}' \Rightarrow f_c^{-1}[U'] \in \mathcal{O}), \tag{13.20}$$

by taking an arbitrary open set U' in $\mathcal{O}'$. Here, the inclusion $\mathcal{O}' \subseteq \mathcal{P}(\Omega')$ is true according to Property 1 of a topology on Ω', so that we obtain $U' \in \mathcal{P}(\Omega')$ with the definition of a subset and then also $U' \subseteq \Omega'$ with the definition of a power set. Consequently, $f_c^{-1}[U'] = \Omega$ or $f_c^{-1}[U'] = \emptyset$ is true, which disjunction we can use to prove the desired consequent $f_c^{-1}[U'] \in \mathcal{O}$. Indeed, the first case $f_c^{-1}[U'] = \Omega$ implies that consequent immediately with Property 2 of a topology on Ω, whereas the second case $f_c^{-1}[U'] = \emptyset$ yields $f_c^{-1}[U'] \in \mathcal{O}$ with (11.324). Having thus completed the proof by cases, we may now conclude that the universal sentence (13.20) holds, because U' was arbitrary. Consequently, f_c is measurable by definition. As Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and c were also arbitrary, the stated theorem follows to be true. □

Lemma 13.7 (Continuity of inclusion functions). *It is true for any topological space* $(\Omega, \mathcal{O})$ *and any subset* Ω_1 *of* Ω *that the inclusion function* $j : \Omega_1 \to \Omega$ *is continuous.*

Proof. We let Ω, $\mathcal{O}$ and Ω_1 be arbitrary sets, assuming $(\Omega, \mathcal{O})$ to be a topological space and assuming the inclusion $\Omega_1 \subseteq \Omega$ to be true. Thus, the topological subspace $(\Omega_1, \mathcal{O}|\Omega_1)$ of $(\Omega, \mathcal{O})$ and moreover the inclusion function $j : \Omega_1 \to \Omega$ are defined. To show that this function is continuous, we verify (13.2) for $f = j$, taking an arbitrary open set $B \in \mathcal{O}$ and demonstrating that $j^{-1}[B] \in \mathcal{O}|\Omega_1$ is implied.

Let us establish now the truth of the equation $j^{-1}[B] = \Omega_1 \cap B$ by means of the Equality Criterion for sets, i.e. by proving

$$\forall \omega \, (\omega \in j^{-1}[B] \Leftrightarrow \omega \in \Omega_1 \cap B). \tag{13.21}$$

We let ω be arbitrary and assume the antecedent $\omega \in j^{-1}[B]$ of the implication '⇒' to be true. Then, the definition of an inverse image gives us $j(\omega) \in B$, where ω is evidently an element of the domain Ω_1 of j, and where the corresponding value is given by $j(\omega) = \omega$, according to the definition of that inclusion function. Thus, substitution yields $\omega \in B$, which implies in conjunction with $\omega \in \Omega_1$ the truth of $\omega \in \Omega_1 \cap B$ by definition of the intersection of two sets, proving the first part ('⇒') of the equivalence in (13.21). Regarding the second part ('⇐'), we assume that $\omega \in \Omega_1 \cap B$ is true, so that we obtain $\omega \in \Omega_1$ and $\omega \in B$ (by definition of the intersection of two sets), where the former implies $j(\omega) = \omega$ (by definition of the inclusion function). Consequently, $\omega \in B$ yields $j(\omega) \in B$ via substitution, and this gives us the desired consequent $\omega \in j^{-1}[B]$ (by definition of an inverse image), so that the proof of the equivalence in (13.21) is now complete. Here, ω is arbitrary, allowing us to infer from the truth of the preceding equivalence the truth of the universal sentence (13.21), and then also the truth of the equality $j^{-1}[B] = \Omega_1 \cap B$.

Because we assumed $B \in \mathcal{O}$ to be true, we thus see that there exists a set U satisfying $U \in \mathcal{O}$ and $\Omega_1 \cap U = j^{-1}[B]$. This existential sentence implies with the definition of a subspace topology that $j^{-1}[B]$ is in $\mathcal{O}|\Omega_1$, and since the set B was arbitrary, we may therefore conclude that j satisfies the universal sentence (13.2). This means that j constitutes a continuous function, and as the sets Ω, $\mathcal{O}$ and Ω_1 were also arbitrary, the stated lemma holds indeed. $\square$

Theorem 13.8 (Continuity of the composition of two continuous functions). *It is true for any topological spaces $(\Omega_1, \mathcal{O}_1)$, $(\Omega_2, \mathcal{O}_2)$, $(\Omega_3, \mathcal{O}_3)$ and any functions $f : \Omega_1 \to \Omega_2$, $g : \Omega_2 \to \Omega_3$ that the composition $g \circ f$ is continuous if f and g are continuous.*

Exercise 13.1. Prove Theorem 13.8.

(Hint: Use Proposition 3.178 and Exercise 3.96.)

Theorem 13.9 (Continuity of the restriction of a continuous function). *It is true for any topological spaces $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$, for any function $f : \Omega \to \Omega'$ and for any subset Ω_1 of Ω that the restriction $f \upharpoonright \Omega_1$ is continuous if f is continuous.*

Exercise 13.2. Prove Theorem 13.9.

(Hint: Apply Proposition 3.164, Proposition 3.181, Lemma 13.7 and Theorem 13.8.)

Proposition 13.10. *It is true for any topological spaces $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$, for any function $f : \Omega \to \Omega'$ and for any family $A = (\Omega_i)_{i \in I}$ of sets in $\mathcal{O}$ whose range covers Ω that f is continuous if the restriction $f \upharpoonright \Omega_i$ is continuous with respect to the topological subspace $(\Omega_i, \mathcal{O}|\Omega_i)$ and the topological space $(\Omega', \mathcal{O}')$ for every $i \in I$.*

Proof. We let Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$, f and A be arbitrary sets such that $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ are topological spaces, such that f is a function from Ω to Ω', and such that $A = (\Omega_i)_{i \in I}$ is a family of sets satisfying $\mathrm{ran}(A) \subseteq \mathcal{O}$, $\mathrm{ran}(A) \subseteq \mathcal{P}(\Omega)$ and

$$\bigcup \mathrm{ran}(A) = \bigcup_{i \in I} \Omega_i = \Omega. \tag{13.22}$$

Here, we note that $\mathcal{O} \subseteq \mathcal{P}(\Omega)$ is true according to Property 1 of a topology (on Ω), so that the assumption $\mathrm{ran}(A) \subseteq \mathcal{O}$ actually implies the covering property $\mathrm{ran}(A) \subseteq \mathcal{P}(\Omega)$ with (2.13). Next, we consider the restriction $f \upharpoonright \Omega_i$ for any $i \in I$. Noting that $\Omega_i = A(i)$ is a term/value of the family A, we can also write $(i, \Omega_i) \in A$, which shows in light of the definition of a range that $\Omega_i \in \mathrm{ran}(A)$ $[\subseteq \mathcal{P}(\Omega)]$ holds. This gives us first $\Omega_i \in \mathcal{P}(\Omega)$ with the definition of a subset and then $\Omega_i \subseteq \Omega$ with the definition of a power set. Consequently, we evidently have for every $i \in I$ that the restriction $f \upharpoonright \Omega_i$ constitutes a function from Ω_i to Ω' by virtue of Proposition 3.164

and in addition that the topological subspace $(\Omega_i, \mathcal{O}|\Omega_i)$ is defined. Next, we assume that $f \upharpoonright \Omega_i$ is continuous for all $i \in I$, and we show that f is continuous, i.e. that f satisfies (13.2). For this purpose, we take an arbitrary open set B of the topology $\mathcal{O}'$ and demonstrate that the inverse image of B under f is an open set of the topology $\mathcal{O}$. Let us observe now that we obtain for this inverse image the equations

$$f^{-1}[B] = f^{-1}[B] \cap \Omega = f^{-1}[B] \cap \bigcup_{i \in I} \Omega_i = \bigcup_{i \in I} (f^{-1}[B] \cap \Omega_i) \tag{13.23}$$

by applying (2.77) with the fact that $f^{-1}[B] \subseteq \Omega$ holds for the given function $f : \Omega \to \Omega'$ according to Note 3.30, then substitution based on (13.22), and subsequently the Distributive Laws for families of sets (3.822). Here, we can show that the range of the family $g = (f^{-1}[B] \cap \Omega_i)_{i \in I}$ is included in the topology $\mathcal{O}$, which we do by means of the definition of a subset, i.e. by verifying

$$\forall X \, (X \in \mathrm{ran}(g) \Rightarrow X \in \mathcal{O}). \tag{13.24}$$

We let X be arbitrary and assume $X \in \mathrm{ran}(g)$ to be true, so that the definition of a range gives us a particular constant $\bar{k}$ with $(\bar{k}, X) \in g$. We then have in family notation $X = g_{\bar{k}} = f^{-1}[B] \cap \Omega_{\bar{k}}$, and the definition of a domain yields $\bar{k} \in I$ [$= \mathrm{dom}(g)$]. Thus, the subspace topology $\mathcal{O}|\Omega_{\bar{k}}$ and the restricted function $f \upharpoonright \Omega_{\bar{k}}$ are defined. Since that restriction is by assumption continuous with respect to the topological subspace $(\Omega_{\bar{k}}, \mathcal{O}|\Omega_{\bar{k}})$ and the topological space $(\Omega', \mathcal{O}')$, it follows from the assumed $B \in \mathcal{O}'$ that

$$(f \upharpoonright \Omega_{\bar{k}})^{-1}[B] \in \mathcal{O}|\Omega_{\bar{k}} \tag{13.25}$$

holds. Because $\Omega_{\bar{k}} = A(\bar{k})$ is a term/value of the family A, we can write $(\bar{k}, \Omega_{\bar{k}}) \in A$, so that we find $\Omega_{\bar{k}} \in \mathrm{ran}(A)$ to be true by definition of a range. Recalling that we assumed this range to be included in the topology $\mathcal{O}$, we obtain now $\Omega_{\bar{k}} \in \mathcal{O}$ with the definition of a subset. This fact that $\Omega_{\bar{k}}$ is an open set of $\mathcal{O}$ allows us to infer from (13.25) especially the truth of

$$(f \upharpoonright \Omega_{\bar{k}})^{-1}[B] \in \mathcal{O} \tag{13.26}$$

by means of Theorem 11.51d). Let us prove here the equation

$$(f \upharpoonright \Omega_{\bar{k}})^{-1}[B] = f^{-1}[B] \cap \Omega_{\bar{k}} \tag{13.27}$$

via the Equality Criterion for sets, that is, by proving the equivalent universal sentence

$$\forall \omega \, (\omega \in (f \upharpoonright \Omega_{\bar{k}})^{-1}[B] \Leftrightarrow \omega \in f^{-1}[B] \cap \Omega_{\bar{k}}). \tag{13.28}$$

We let ω be arbitrary, and we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming that $\omega \in (f \restriction \Omega_{\bar{k}})^{-1}[B]$ holds. On the one hand, the inverse image of B under the restriction $f \restriction \Omega_{\bar{k}}$ (from $\Omega_{\bar{k}}$ to Ω') is a subset of $\Omega_{\bar{k}}$ according to Note 3.30, so that $\omega \in \Omega_{\bar{k}}$ follows to be true by definition of a subset. On the other hand, we obtain with the definition of an inverse image $(f \restriction \Omega_{\bar{k}})(\omega) \in B$, which yields in connection with the preceding finding $(f \restriction \Omega_{\bar{k}})(\omega) = f(\omega)$ with (3.567) and then also $f(\omega) \in B$ by means of substitution. This implies $\omega \in f^{-1}[B]$ (by definition of an inverse image) and consequently $\omega \in f^{-1}[B] \cap \Omega_{\bar{k}}$ (by definition of the intersection of two sets), as desired. Regarding the second part ('$\Leftarrow$') of the equivalence, we assume now $\omega \in f^{-1}[B] \cap \Omega_{\bar{k}}$ to be true, so that we evidently obtain $\omega \in f^{-1}[B]$ and $\omega \in \Omega_{\bar{k}}$. The former finding further implies $f(\omega) \in B$, which gives then with the latter $(f \restriction \Omega_{\bar{k}})(\omega) = f(\omega)$, and therefore through substitution $(f \restriction \Omega_{\bar{k}})(\omega) \in B$. This gives us now the desired consequent $\omega \in (f \restriction \Omega_{\bar{k}})^{-1}[B]$, so that the proof of the equivalence is complete. Since ω is arbitrary, we may infer from this the truth of the universal sentence (13.28) and thus the truth of the proposed equality (13.27).

Combining (13.26) and (13.27) yields then

$$[X =] \quad f^{-1}[B] \cap \Omega_{\bar{k}} \in \mathcal{O}, \tag{13.29}$$

which shows that the intersection X is an open set of $\mathcal{O}$. We thus proved the implication in (13.24), in which X is arbitrary, so that the universal sentence (13.24) follows to be true. This in turn establishes the truth of the suggested inclusion $\operatorname{ran}(g) \subseteq \mathcal{O}$, which gives us now $\bigcup \operatorname{ran}(g) \in \mathcal{O}$ with Property 3 of a topology. That union represents the union of (the range of) the family g, so that we can write

$$\bigcup_{i \in I} (f^{-1}[B] \cap \Omega_i) \in \mathcal{O}.$$

In view of the equations (13.23), we therefore obtain $f^{-1}[B] \in \mathcal{O}$ through substitution, which finding completes the proof that f is continuous (since the set B was arbitrary). As the sets Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$, f and A were initially all arbitrary, we can finally conclude that the proposition holds, as claimed. □

Theorem 13.11 (Continuity of projection functions). *It is true for any nonempty index set I, any index $j \in I$ and any families of sets $(\Omega_i)_{i \in I}$, $(\mathcal{O}_i)_{i \in I}$ such that $\mathcal{O}_i$ is a topology on Ω_i for every $i \in I$ that the j-th projection map π_j on $\bigtimes_{i \in I} \Omega_i$ is continuous with respect to the product topology $\bigotimes_{i \in I} \mathcal{O}_i$ and $\mathcal{O}_j$, i.e.*

$$\pi_j : (\bigtimes_{i \in I} \Omega_i, \bigotimes_{i \in I} \mathcal{O}_i) \to (\Omega_j, \mathcal{O}_j). \tag{13.30}$$

Proof. We let I, j, $(\Omega_i)_{i\in I}$, $(\mathcal{O}_i)_{i\in I}$ be arbitrary such that $I \neq \emptyset$ and $j \in I$ hold, and such that $\mathcal{O}_i$ is a topology on Ω_i for all $i \in I$. Then, the product topology $\bigotimes_{i\in I} \mathcal{O}_i$ is generated by $\bigcup_{i\in I}\{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}$ according to (11.640). To prove (13.30), we apply the definition of a continuous function and establish the truth of

$$\forall B' \, (B' \in \mathcal{O}_j \Rightarrow \pi_j^{-1}[B'] \in \bigotimes_{i\in I} \mathcal{O}_i). \tag{13.31}$$

For this purpose, we let $B' \in \mathcal{O}_j$ be arbitrary, which implies in view of $j \in I$ that

$$\pi_j^{-1}[B'] \in \{\pi_j^{-1}[U] : U \in \mathcal{O}_j\}$$

holds, according to the specification of the preceding set of inverse images in Theorem 11.88. Furthermore, this finding implies in connection with $j \in I$ and the definition of the union of a family of sets that

$$\pi_j^{-1}[B'] \in \bigcup_{i\in I}\{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}$$

is true. Because of Proposition 11.55, we have the inclusion

$$\bigcup_{i\in I}\{\pi_i^{-1}[U] : U \in \mathcal{O}_i\} \;\subseteq\; \bigotimes_{i\in I} \mathcal{O}_i,$$

so that the desired consequent $\pi_j^{-1}[B'] \in \bigotimes_{i\in I} \mathcal{O}_i$ of the implication in (13.31) follows to be true with the definition of a subset. Since B' is arbitrary, we therefore conclude that the universal sentence (13.31) holds, which means that π_j is continuous. Because I, j, $(\Omega_i)_{i\in I}$ and $(\mathcal{O}_i)_{i\in I}$ were initially also arbitrary, we may now further conclude that the stated theorem is true. □

Theorem 13.12 (Basis Criterion for continuity). *For any topological space $(\Omega, \mathcal{O})$, any set Ω' and any basis $\mathcal{K}_{\Omega'}$ for a topology on Ω', generating $\mathcal{O}' = \mathcal{O}(\mathcal{K}_{\Omega'})$, it is true that a function $f : \Omega \to \Omega'$ is continuous iff the inverse image (under f) of every basis set is in $\mathcal{O}$, that is,*

$$\forall f \, (f \in \Omega'^{\Omega} \Rightarrow [f \text{ is continuous } \Leftrightarrow \forall B' \, (B' \in \mathcal{K}_{\Omega'} \Rightarrow f^{-1}[B'] \in \mathcal{O})]). \tag{13.32}$$

Proof. We take arbitrary Ω, $\mathcal{O}$, Ω', $\mathcal{K}_{\Omega'}$ and f such that $(\Omega, \mathcal{O})$ is a topological space, such that $\mathcal{K}_{\Omega'}$ is a basis for a topology on Ω', and such that f is an element of Ω'^{Ω}, that is, a function from Ω to Ω'. The preceding basis generates then the topology $\mathcal{O}' = \mathcal{O}(\mathcal{K}_{\Omega'})$ on Ω'. We prove now the first part ('$\Rightarrow$') of the equivalence in (13.32) directly, assuming f to be continuous. To establish the consequent, we take an arbitrary basis set B' in $\mathcal{K}_{\Omega'}$, and we observe that the inclusion $\mathcal{K}_{\Omega'} \subseteq \mathcal{O}'$ is true in view of Proposition 11.55, so that $B' \in \mathcal{K}_{\Omega'}$ implies $B' \in \mathcal{O}'$ with the definition of a subset. The assumed continuity of f yields then $f^{-1}[B'] \in \mathcal{O}$, and since B' was arbitrary, we may infer from this finding that the first part of the equivalence holds.

We prove the second part ('$\Leftarrow$') of the equivalence in (13.32) also directly, assuming now

$$\forall B' \, (B' \in \mathcal{K}_{\Omega'} \Rightarrow f^{-1}[B'] \in \mathcal{O}) \tag{13.33}$$

to be true, and demonstrating the continuity of f, that is, demonstrating the truth of the universal sentence

$$\forall B' \, (B' \in \mathcal{O}' \Rightarrow f^{-1}[B'] \in \mathcal{O}). \tag{13.34}$$

Letting B' be an arbitrary open set of $\mathcal{O}' = \mathcal{O}(\mathcal{K}_{\Omega'})$, we see in light of the Characterization of the elements of a topology generated by a basis that B' can be written as the union $\bigcup \bar{\mathcal{G}}$ of a particular set system $\bar{\mathcal{G}}$ that is included in the basis $\mathcal{K}_{\Omega'}$. This union $\bigcup \bar{\mathcal{G}}$ in turn can – in view of (3.787) – be written as the union $\bigcup_{i \in \bar{I}} \bar{A}_i$ of a particular family $\bar{A} : \bar{I} \to \bar{\mathcal{G}}$. Here, we note that the inclusions

$$\mathrm{ran}(\bar{A}) \;\subseteq\; \bar{\mathcal{G}} \;\subseteq\; \mathcal{K}_{\Omega'} \;\subseteq\; \mathcal{P}(\Omega')$$

hold due to the definition of a codomain and due to Property 1 of a basis for a topology (on Ω'). Consequently, we obtain the inclusion $\mathrm{ran}(\bar{A}) \subseteq \mathcal{P}(\Omega')$ with (2.13), which shows that $\mathcal{P}(\Omega')$ is also a codomain of $\bar{A}$, which family thus constitutes an element of $\mathcal{P}(\Omega')^{\bar{I}}$. We now find the equations

$$f^{-1}[B'] = f^{-1}[\bigcup \bar{\mathcal{G}}] = f^{-1}[\bigcup_{i \in \bar{I}} \bar{A}_i] = \bigcup_{i \in \bar{I}} f^{-1}[\bar{A}_i] \tag{13.35}$$

by applying substitutions and (3.845) based on $f \in \Omega'^{\Omega}$ and $\bar{A} \in \mathcal{P}(\Omega')^{\bar{I}}$. We prove next that the range of the family $S = (f^{-1}[\bar{A}_i])_{i \in \bar{I}}$ is included in the topology $\mathcal{O}$, by verifying the universal sentence

$$\forall X \, (X \in \mathrm{ran}(S) \Rightarrow X \in \mathcal{O}). \tag{13.36}$$

We let X be an arbitrary set, and we assume $X \in \mathrm{ran}(S)$ to hold. By definition of a range and by definition of a domain, there is then a particular index $\bar{k} \in \bar{I}$ [$= \mathrm{dom}(S)$] for which $(\bar{k}, X) \in S$ is true. By definition of the family S, we can therefore write $X = S_{\bar{k}} = f^{-1}[\bar{A}_{\bar{k}}]$. Because $\bar{k}$ is thus also in the domain of $\bar{A} : \bar{I} \to \bar{\mathcal{G}}$, it follows with the Function Criterion that $\bar{A}_{\bar{k}} \in \bar{\mathcal{G}}$ holds. Recalling now the inclusion $\bar{\mathcal{G}} \subseteq \mathcal{K}_{\Omega'}$, we obtain with the definition of a subset $\bar{A}_{\bar{k}} \in \mathcal{K}_{\Omega'}$, and this finding further implies $[X =]\ f^{-1}[\bar{A}_{\bar{k}}] \in \mathcal{O}$ with the initial assumption (13.33). This shows that X is an open set of $\mathcal{O}$, so that the implication in (13.36) is true. Here, X is arbitrary, allowing us to infer from the truth of the preceding implication the truth of the universal sentence (13.36) and therefore the truth of the inclusion $\mathrm{ran}(S) \subseteq \mathcal{O}$ (by definition of a subset). This inclusion in turn implies with Property 3 of a topology that $\bigcup \mathrm{ran}(S) \in \mathcal{O}$, which we can also write in family notation as

$$\bigcup_{i \in I} f^{-1}[\bar{A}_i] \in \mathcal{O}.$$

Combining this finding with (13.35) yields now $f^{-1}[B'] \in \mathcal{O}$, which is the desired consequent of the implication in (13.34). Since B' is arbitrary, we therefore conclude that the universal sentence (13.34) is true, which means that f is continuous.

Having thus completed the proof of the equivalence in (13.32), the theorem follows finally to be true since the sets Ω, $\mathcal{O}$, Ω', $\mathcal{K}_{\Omega'}$ and f were initially arbitrary. □

As a first application of the Basis Criterion for continuity, we derive the following continuity criterion for situations in which a subbasis for a topology is given.

Theorem 13.13 (Subbasis Criterion for continuity). *For any topological space $(\Omega, \mathcal{O})$, any set Ω' and any subbasis $\mathcal{C}_{\Omega'}$ for a topology on Ω', generating $\mathcal{O}' = \mathcal{O}(\mathcal{K}_{C_{\Omega'}})$, it is true that a function $f : \Omega \to \Omega'$ is continuous iff the inverse image (under f) of every subbasis set is in $\mathcal{O}$, i.e.*

$$\forall f\,(f \in \Omega'^{\Omega} \Rightarrow [f \text{ is continuous } \Leftrightarrow \forall C'\,(C' \in \mathcal{C}_{\Omega'} \Rightarrow f^{-1}[C'] \in \mathcal{O})]). \tag{13.37}$$

Proof. We let Ω, $\mathcal{O}$, Ω', $\mathcal{K}_{\Omega'}$ and f be arbitrary, assuming that $(\Omega, \mathcal{O})$ is a topological space, assuming that $\mathcal{C}_{\Omega'}$ is a subbasis for a topology on Ω', and assuming that f is an element of Ω'^{Ω}, i.e., a function from Ω to Ω'. Here, the given subbasis generates the topology $\mathcal{O}' = \mathcal{O}(\mathcal{K}_{C_{\Omega'}})$ on Ω' through the basis $\mathcal{K}_{C_{\Omega'}}$.

Regarding the first part ('⇒') of the equivalence in (13.37), we assume that f is continuous, take then an arbitrary subbasis element C' in $\mathcal{C}_{\Omega'}$ and observe moreover that the inclusion $\mathcal{C}_{\Omega'} \subseteq \mathcal{O}'$ holds by virtue of Proposition 11.548, so that $C' \in \mathcal{O}'$ follows to be true by definition of a subset. Since f is continuous, we obtain $f^{-1}[C'] \in \mathcal{O}$, and as B' was arbitrary, we may now infer from this the truth of the first part of the equivalence in (13.37).

Regarding the second part ('⇐') of that equivalence, we assume conversely that the universal sentence

$$\forall C' \, (C' \in \mathcal{C}_{\Omega'} \Rightarrow f^{-1}[C'] \in \mathcal{O}) \tag{13.38}$$

is true, and we will establish the continuity of f by means of the Basis Criterion for continuity, by demonstrating the truth of

$$\forall B' \, (B' \in \mathcal{K}_{\mathcal{C}_{\Omega'}} \Rightarrow f^{-1}[B'] \in \mathcal{O}), \tag{13.39}$$

noting that $\mathcal{K}_{\mathcal{C}_{\Omega'}}$ is a basis for a topology on Ω' that generates the topology $\mathcal{O}' = \mathcal{O}(\mathcal{K}_{\mathcal{C}_{\Omega'}})$. We take an arbitrary basis set B' in $\mathcal{K}_{\mathcal{C}_{\Omega'}}$ and observe in light of the Generation of a basis for a topology by means of a subbasis that there exists now a particular positive natural number $\bar{m}$ and a particular function $\bar{A} : \{1, \ldots, \bar{m}\} \to \mathcal{C}_{\Omega'}$ such that

$$B' = \bigcap \operatorname{ran}(\bar{A}) \quad \left[= \bigcap_{i=1}^{\bar{m}} \bar{A}_i \right]. \tag{13.40}$$

Observing that the inclusions

$$\operatorname{ran}(\bar{A}) \;\subseteq\; \mathcal{C}_{\Omega'} \;\subseteq\; \mathcal{P}(\Omega')$$

are true because of the definition of a codomain and because of Property 1 of a subbasis for a topology (on Ω'), we obtain by means of (2.13) also the inclusion $\operatorname{ran}(\bar{A}) \subseteq \mathcal{P}(\Omega')$, which shows that $\bar{A}$ is a family in $\mathcal{P}(\Omega')$ and thus an element of $\mathcal{P}(\Omega')^{\{1,\ldots,\bar{m}\}}$. Here, the fact that $\bar{m}$ is a positive natural number implies that the index set $\{1, \ldots, \bar{m}\}$ is nonempty, according to the definition of an initial segment of $\mathbb{N}_+$. Because of these findings, we obtain for the inverse image of B'

$$f^{-1}[B'] = f^{-1}[\bigcap_{i=1}^{\bar{m}} \bar{A}_i] = \bigcap_{i=1}^{\bar{m}} f^{-1}[\bar{A}_i].$$

by applying substitution based on (13.40), and subsequently (3.854). Considering the sequence $S = (f^{-1}[\bar{A}_i] \,|\, i \in \{1, \ldots, \bar{m}\})$, we can write for the preceding intersection also

$$f^{-1}[B'] = \operatorname{ran}(S), \tag{13.41}$$

using the notation for the intersection of a family of sets. In the following, we show that the range of S is a subset of the topology $\mathcal{O}$, by demonstrating the truth of

$$\forall X \, (X \in \operatorname{ran}(S) \Rightarrow X \in \mathcal{O}). \tag{13.42}$$

For this purpose, we take an arbitrary set X such that $X \in \mathrm{ran}(S)$ holds. The definitions of a range and of a domain give us then a particular index $\bar{k} \in \{1, \ldots, \bar{m}\}$ $[= \mathrm{dom}(S)]$ for which $(\bar{k}, X) \in S$ is true. Recalling the definition of the family S, we can now write $X = S_{\bar{k}} = f^{-1}[\bar{A}_{\bar{k}}]$. Noting that $\bar{k}$ is an element also of the domain of $\bar{A} : \{1, \ldots, \bar{m}\} \to \mathcal{C}_{\Omega'}$, we obtain with the Function Criterion $\bar{A}_{\bar{k}} \in \mathcal{C}_{\Omega'}$. Due to (13.38), this finding further implies $[X =]$ $f^{-1}[\bar{A}_{\bar{k}}] \in \mathcal{O}$, which shows that $X \in \mathcal{O}$ is true and which proves thus the implication in (13.42). As X was arbitrary, we may therefore conclude that (13.42) is true, and this universal sentence implies the truth of the inclusion $\mathrm{ran}(S) \subseteq \mathcal{O}$ by definition of a subset. Thus, $S = (f^{-1}[\bar{A}_i] \,|\, i \in \{1, \ldots, \bar{m}\})$ is a sequence in $\mathcal{O}$. Since the natural number $\bar{m}$ gives rise to the $\bar{m}$-fold repeated binary operation $\bigcap_{i=1}^{\bar{m}}$ on $\mathcal{O}$ according to (11.327), we have then

$$\bigcap_{i=1}^{\bar{m}} f^{-1}[\bar{A}_i] \in \mathcal{O}. \tag{13.43}$$

The intersection $\bigcap_{i=1}^{\bar{m}} f^{-1}[\bar{A}_i]$ is identical with the intersection (13.41) in view of Exercise 11.18, so that substitution gives us the desired consequent $f^{-1}[B'] \in \mathcal{O}$ of the implication in (13.39). As B' was arbitrary, we therefore conclude that (13.34) is true, which universal sentence implies now that f is continuous.

Because the sets Ω, $\mathcal{O}$, Ω', $\mathcal{C}_{\Omega'}$ and f are also arbitrary, we can further conclude that the stated criterion holds indeed. □

Theorem 13.14 (Continuity Criterion for functions to Cartesian products of families of sets). *For any topological space $(\Omega, \mathcal{O})$, any nonempty index set I, any families $(\Omega_i)_{i \in I}$, $(\mathcal{O}_i)_{i \in I}$ such that $(\Omega_i, \mathcal{O}_i)$ is a topological space for every $i \in I$, and any function $f : \Omega \to \bigtimes_{i \in I} \Omega_i$, it is true that f is continuous with respect to $\mathcal{O}$ and $\bigotimes_{i \in I} \mathcal{O}_i$ iff the composition of the j-th projection map π_j on $\bigtimes_{i \in I} \Omega_i$ and f is continuous with respect to $\mathcal{O}$ and $\mathcal{O}_j$ for every $j \in I$.*

Proof. We let Ω, $\mathcal{O}$, I, $(\Omega_i)_{i \in I}$, $(\mathcal{O}_i)_{i \in I}$ and f be arbitrary sets, assuming $(\Omega, \mathcal{O})$ to be a topological space, assuming $I \neq \emptyset$, assuming $(\Omega_i)_{i \in I}$ and $(\mathcal{O}_i)_{i \in I}$ to be families of sets with index set I such that $(\Omega_i, \mathcal{O}_i)$ constitutes a topological space for all $i \in I$, and assuming f to be a function from Ω to the Cartesian product $\bigtimes_{i \in I} \Omega_i$ of the family of sets $(\Omega_i)_{i \in I}$.

To prove the first part ('⇒') of the stated equivalence, we assume that f is continuous with respect to $\mathcal{O}$ and the product topology $\bigotimes_{i \in I} \mathcal{O}_i$, and we let j be an arbitrary index in I. Then, in view of Theorem 13.11, it is true that the projection function $\pi_j : \bigtimes_{i \in I} \Omega_i \to \Omega_j$ is continuous with respect to $\bigotimes_{i\in I}\mathcal{O}_i$ and $\mathcal{O}_j$. It then follows with Theorem 13.8 that the composition $\pi_j \circ f$ is continuous with respect to $\mathcal{O}$ and $\mathcal{O}_j$. As j is arbitrary, we therefore conclude that the first part of the equivalence is true.

To prove the second part ('$\Leftarrow$'), we now assume that the composition $\pi_j \circ f$ is continuous with respect to $\mathcal{O}$ and $\mathcal{O}_j$ for all $j \in I$. To show that f is then continuous with respect to $\mathcal{O}$ and $\bigotimes_{i\in I}\mathcal{O}_i$, we apply the Subbasis Criterion for continuity and prove

$$\forall C'\,(C' \in \mathcal{C}_{\times\,\Omega_i} \Rightarrow f^{-1}[C'] \in \mathcal{O}). \tag{13.44}$$

where the generating subbasis $\mathcal{C}_{\times\,\Omega_i}$ of $\bigotimes_{i\in I}\mathcal{O}_i$ is given by

$$\mathcal{C}_{\times\,\Omega_i} = \bigcup_{i\in I}\{\pi_i^{-1}[U] : U \in \mathcal{O}_i\}$$

according to the definition of a product topology. Letting now $C' \in \mathcal{C}_{\times\,\Omega_i}$ be arbitrary, so that

$$C' \in \bigcup_{i\in I}\{\pi_i^{-1}[U] : U \in \mathcal{O}_i\},$$

it follows with the Characterization of the union of a family of sets that there exists an index in I, say $\bar{k}$, such that

$$C' \in \{\pi_{\bar{k}}^{-1}[U] : U \in \mathcal{O}_{\bar{k}}\}.$$

Then, by definition of the set $\{\pi_{\bar{k}}^{-1}[U] : U \in O_{\bar{k}}\}$ in (11.617), we have that $C' = \pi_{\bar{k}}^{-1}[\bar{U}]$ holds for a particular open set $\bar{U} \in \mathcal{O}_{\bar{k}}$. With the preceding equation and (3.762) we obtain

$$f^{-1}[C'] = f^{-1}[\pi_{\bar{k}}^{-1}[\bar{U}]] = (\pi_{\bar{k}} \circ f)^{-1}[\bar{U}], \tag{13.45}$$

where $\pi_{\bar{k}} \circ f$ is by assumption continuous with respect to $\mathcal{O}$ and $\mathcal{O}_{\bar{k}}$, so that $\bar{U} \in \mathcal{O}_{\bar{k}}$ implies

$$(\pi_{\bar{k}} \circ f)^{-1}[\bar{U}] \in \mathcal{O}$$

with the definition of a continuous function. Thus, (13.45) yields $f^{-1}[C'] \in \mathcal{O}$ via substitution, which is the desired consequent of the implication in (13.44). Since C' was arbitrary, we therefore conclude that the universal sentence (13.44) is true, so that the proof of the second part of the proposed equivalence is complete. As Ω, $\mathcal{O}$, I, $(\Omega_i)_{i\in I}$, $(\mathcal{O}_i)_{i\in I}$ and f were initially also arbitrary, we now further conclude that the theorem is true. □

Definition 13.2 (Bicontinuous function, homeomorphism, homeomorphic topological spaces). We say for any topological spaces $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ that a function $f : \Omega \to \Omega'$ is a *homeomorphism*, symbolically

$$f : (\Omega, \mathcal{O}) \rightleftarrows (\Omega', \mathcal{O}'), \tag{13.46}$$

(alternatively, that a function $f : \Omega \to \Omega'$ is *bicontinuous*) iff

1. f is continuous (with respect to $\mathcal{O}$ and $\mathcal{O}'$),
2. f is a bijection (from Ω to Ω'),
3. f^{-1} is continuous (with respect to $\mathcal{O}'$ and $\mathcal{O}$).

Moreover, we say that $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ are *homeomorphic topological spaces* iff there exists a homeomorphism from $(\Omega, \mathcal{O})$ to $(\Omega', \mathcal{O}')$.

Definition 13.3 (Topological n-manifold). We say that a second-countable Hausdorff space $(\Omega, \mathcal{O})$ is a *topological n-manifold* iff every element ω of Ω is in some open set U of the topology $\mathcal{O}$ such that the topological subspace $(U, \mathcal{O}|U)$ is homeomorphic to the topological subspace $(V, [\bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}}]|V)$ for some open set V of the product topology $\bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}}$, i.e. iff

$$\begin{aligned} &\forall \omega \, (\omega \in \Omega \Rightarrow \exists U, V, f \, (U \in \mathcal{O} \wedge \omega \in U \wedge V \in \bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}} \\ &\quad \wedge f : (U, \mathcal{O}|U) \rightleftarrows (V, \bigotimes_{i=1}^{n} \mathcal{O}_{<_{\mathbb{R}}}|V))). \end{aligned} \tag{13.47}$$

13.2. Measurable Functions

In this section, we will consider functions from a measurable space to another measurable space. We begin with two mechanisms for constructing a σ-algebra by means of a given function.

Theorem 13.15. *It is true for any sets Ω, Ω' and for any function $f : \Omega \to \Omega'$ that the range of the restriction of the preimage function of f to any σ-algebra $\mathcal{A}'$ on Ω' is a σ-algebra on Ω, i.e. the ordered pair*

$$(\Omega, \operatorname{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}')) \tag{13.48}$$

constitutes a measurable space.

Proof. We let Ω, Ω', f and $\mathcal{A}'$ be arbitrary sets, we assume that f is a function from Ω to Ω', and we assume that $\mathcal{A}'$ is a σ-algebra on Ω'. Then, the preimage function $f^{\leftarrow} : \mathcal{P}(\Omega') \to \mathcal{P}(\Omega)$ is defined, where we observe that $\mathcal{A}'$ is a subset of the domain $\mathcal{P}(\Omega')$ according to Property 1 of a σ-algebra (on Ω'), so that the restriction $f^{\leftarrow} \upharpoonright \mathcal{A}'$ is a function from $\mathcal{A}'$ to $\mathcal{P}(\Omega)$ in view of Proposition 3.164. We prove now that the range of this restriction is a σ-algebra on Ω. To begin with, we notice that this range is included in the codomain $\mathcal{P}(\Omega)$ of the restriction, so that it satisfies Property 1 of a σ-algebra on Ω.

Next, we observe the truth of $\Omega' \in \mathcal{A}'$ in light of Property 2 of a σ-algebra (on Ω'), so that we may evaluate the restricted preimage function to obtain

$$(f^{\leftarrow} \upharpoonright \mathcal{A}')(\Omega') = f^{\leftarrow}(\Omega') = f^{-1}[\Omega'] = \Omega$$

by means of (3.567), the definition of the preimage function and (3.746). Writing the resulting equation in the form $(\Omega', \Omega) \in f^{\leftarrow} \upharpoonright \mathcal{A}'$, we see now in light of the definition of a range that

$$\Omega \in \operatorname{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}')$$

holds, as required by Property 2 of a σ-algebra on Ω.

Concerning Property 3, we take an arbitrary set A and assume A to be function from $\mathbb{N}_+$ to $\operatorname{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}')$, that is, to be a sequence $(A_n)_{n\in\mathbb{N}_+}$ of sets in $\operatorname{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}')$. We now choose a related sequence $(B_n)_{n\in\mathbb{N}_+}$ of sets in $\mathcal{A}'$ such that $A_n = f^{-1}[B_n]$ holds for every index n. To do this, we first establish

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \exists! Y\,(\forall C\,(C \in Y \Leftrightarrow [C \in \mathcal{A}' \,\wedge\, f^{-1}[C] = A_n]))), \tag{13.49}$$

letting $n \in \mathbb{N}_+$ be arbitrary and observing that the uniquely existential sentence can be verified by means of the Axiom of Specification and the

Equality Criterion for sets. As n is arbitrary, we therefore conclude that (13.49) holds, and this universal sentence implies – according to Function definition by replacement – that there is a unique function/sequence G with domain $\mathbb{N}_+$ whose terms satisfy

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow \forall C\,(C \in G(n) \Leftrightarrow [C \in \mathcal{A}' \,\wedge\, f^{-1}[C] = A_n])). \tag{13.50}$$

Next, we demonstrate that $\emptyset$ is not an element of the range of this sequence G, by verifying the universal sentence

$$\forall X\,(X \in \mathrm{ran}(G) \Rightarrow X \neq \emptyset). \tag{13.51}$$

Letting X be an arbitrary set and assuming $X \in \mathrm{ran}(G)$ to be true, there is then (by definition of a range and of a domain) a particular element $\bar{n} \in \mathbb{N}_+$ $[= \mathrm{dom}(G)]$ with $(\bar{n}, X) \in G$, which we can write in function notation as $X = G(\bar{n})$. Here, $\bar{n}$ is also in the domain of the sequence A, so that the associated term/value $A_{\bar{n}}$ is a uniquely defined set in the codomain $\mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}')$ of A. Consequently, there exists (by definition of a range) also a particular set $\bar{C}$ with $(\bar{C}, A_{\bar{n}}) \in f^{\leftarrow} \upharpoonright \mathcal{A}'$, and this implies $(\bar{C}, A_{\bar{n}}) \in f^{\leftarrow}$ as well as $\bar{C} \in \mathcal{A}'$ with the definition of a restriction. Writing the former finding in function notation and applying the definition of the preimage function, we obtain now $A_{\bar{n}} = f^{\leftarrow}(\bar{C}) = f^{-1}[\bar{C}]$, and the resulting equation $f^{-1}[\bar{C}] = A_{\bar{n}}$ implies in conjunction with $\bar{C} \in \mathcal{A}'$ that $\bar{C} \in G(\bar{n})$ holds (by definition of the function G). We see now clearly that the set $X = G(\bar{n})$ is nonempty, and since X was initially arbitrary, we may therefore conclude that the universal sentence (13.51) is true. This in turn implies $\emptyset \notin \mathrm{ran}(G)$ with (2.5), which allows us to apply the Axiom of Choice to infer the existence of a particular function $\bar{F} : \mathrm{ran}(G) \to \bigcup \mathrm{ran}(G)$ satisfying

$$\forall Y\,(Y \in \mathrm{ran}(G) \Rightarrow \bar{F}(Y) \in Y). \tag{13.52}$$

Furthermore, we obtain the composed function/sequence $\bar{F} \circ G : \mathbb{N}_+ \to \bigcup \mathrm{ran}(G)$ by means of (3.604), which we can write as the sequence $\bar{B} = (\bar{B}_n)_{n \in \mathbb{N}_+} = \bar{F} \circ G$. We prove now that $\mathcal{A}'$ is also a codomain of this composition $\bar{B}$. Noting that the inclusion $\mathrm{ran}(\bar{B}) \subseteq \bigcup \mathrm{ran}(G)$ already holds for $\bar{F} \circ G$ by definition of a codomain, it suffices to establish the inclusion $\bigcup \mathrm{ran}(G) \subseteq \mathcal{A}'$. To do this, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall C\,(C \in \bigcup \mathrm{ran}(G) \Rightarrow C \in \mathcal{A}'), \tag{13.53}$$

letting C be arbitrary and assuming $C \in \bigcup \mathrm{ran}(G)$ to be true. By definition of the union of a set system, there exists a particular set $\bar{X} \in \mathrm{ran}(G)$ with

$C \in \bar{X}$, where the definition of a range and the definition of a domain give us also a particular index $\bar{n} \in \mathbb{N}_+$ with $(\bar{n}, \bar{X}) \in G$, which we can write also as $\bar{X} = G(\bar{n})$. These findings yield through substitution $C \in G(\bar{n})$, which implies especially $C \in \mathcal{A}'$ with (13.50). Because C was arbitrary, we can infer from this the truth of (13.53), and therefore the truth of the inclusion $\bigcup \operatorname{ran}(G) \subseteq \mathcal{A}'$. In conjunction with the aforementioned inclusion $\operatorname{ran}(\bar{B}) \subseteq \bigcup \operatorname{ran}(G)$, this implies the truth of the inclusion $\operatorname{ran}(\bar{B}) \subseteq \mathcal{A}'$ by virtue of (2.13). We thus have $\bar{B} : \mathbb{N}_+ \to \mathcal{A}'$, so that it remains for us to demonstrate that the values/terms of this function/sequence satisfy

$$\forall n\, (n \in \mathbb{N}_+ \Rightarrow f^{-1}[\bar{B}_n] = A_n). \tag{13.54}$$

We let n be arbitrary in $\mathbb{N}_+$ and consider the corresponding term $\bar{B}_n = (\bar{F} \circ G)(n) = \bar{F}(G(n))$. Here, we have the term $G_n = G(n)$, for which we can write $(n, G_n) \in G$, so that we can observe the truth of $G_n \in \operatorname{ran}(G)$ in light of the definition of a range. Consequently, $\bar{F}(G_n) \in G_n$ follows to be true with (13.52), to which we can apply substitution based on the preceding equation, with the consequence that $\bar{B}_n \in G_n$ is true. This finding yields in particular $f^{-1}[\bar{B}_n] = A_n$ with (13.50), and since n was arbitrary, we can now conclude that the universal sentence (13.54) is indeed true. Let us recall the truth of the inclusions $\operatorname{ran}(\bar{B}) \subseteq \mathcal{A}' \subseteq \mathcal{P}(\Omega')$, which show in light of (2.13) and the definition of a codomain that $(\bar{B}_n)_{n \in \mathbb{N}_+}$ is a sequence in $\mathcal{P}(\Omega')$. In connection with the given function $f : \Omega \to \Omega'$, this yields the unique sequence $S = (f^{-1}[\bar{B}_n])_{n \in \mathbb{N}_+}$ with Exercise 3.101e). Then, the universal sentence (13.54) implies with the Equality Criterion for functions that the sequences S and A are identical. We obtain therefore

$$\bigcup_{n=1}^{\infty} A_n = \bigcup \operatorname{ran}(A) = \bigcup \operatorname{ran}(S) = \bigcup_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} f^{-1}[\bar{B}_n] = f^{-1}[\bigcup_{n=1}^{\infty} \bar{B}_n] \tag{13.55}$$

by means of the definition of the union of a family of sets, substitution, and Proposition 3.242. Here, we find for the chosen function $\bar{B} : \mathbb{N}_+ \to \mathcal{A}'$ with Property 3 of a σ-algebra that $\bigcup_{n=1}^{\infty} \bar{B}_n \in \mathcal{A}'$ holds. Thus, the union is in the domain of the restriction $f^{\leftarrow} \upharpoonright \mathcal{A}'$, so that its value satisfies

$$(f^{\leftarrow} \upharpoonright \mathcal{A}')(\bigcup_{n=1}^{\infty} \bar{B}_n) = f^{\leftarrow}(\bigcup_{n=1}^{\infty} \bar{B}_n) = f^{-1}[\bigcup_{n=1}^{\infty} \bar{B}_n] = \bigcup_{n=1}^{\infty} A_n$$

because of (3.567), the definition of the preimage function and (13.55). We can write for the last equation also

$$(\bigcup_{n=1}^{\infty} \bar{B}_n, \bigcup_{n=1}^{\infty} A_n) \in f^{\leftarrow} \upharpoonright \mathcal{A}',$$

which clearly shows that $\bigcup_{n=1}^{\infty} A_n$ is an element of the range of $f^{\leftarrow} \restriction \mathcal{A}'$. Since the sequence A is arbitrary, we may therefore conclude that the set $\mathrm{ran}(f^{\leftarrow} \restriction \mathcal{A}')$ satisfies Property 3 of a σ-algebra.

Finally, concerning Property 4, we let $A \in \mathrm{ran}(f^{\leftarrow} \restriction \mathcal{A}')$ be arbitrary, so that there is a particular set $\bar{B}$ satisfying $(\bar{B}, A) \in f^{\leftarrow} \restriction \mathcal{A}'$, according to the definition of a range. Then, the definition of a restriction gives us $(\bar{B}, A) \in f^{\leftarrow}$ and $\bar{B} \in \mathcal{A}'$, where the former finding can also be written as

$$A = f^{\leftarrow}(\bar{B}) = f^{-1}[\bar{B}], \tag{13.56}$$

according to the definition of the preimage function. The other finding $\bar{B} \in \mathcal{A}'$ implies $\bar{B}^c \in \mathcal{A}'$ with Property 4 of a σ-algebra, so that $\bar{B}^c$ is in the domain of the restriction $f^{\leftarrow} \restriction \mathcal{A}'$. We obtain now for the corresponding value

$$\begin{aligned}(f^{\leftarrow} \restriction \mathcal{A}')(\bar{B}^c) &= f^{\leftarrow}(\bar{B}^c) = f^{-1}[\bar{B}^c] = (f^{-1}[\bar{B}])^c \\ &= A^c\end{aligned}$$

by using (3.567), the definition of the preimage function, Proposition 3.227, and substitution based on (13.56). Writing the final equation in the form

$$(\bar{B}^c, A^c) \in f^{\leftarrow} \restriction \mathcal{A}',$$

we see in light of the definition of a range that $A^c \in \mathrm{ran}(f^{\leftarrow} \restriction \mathcal{A}')$ is true. Because A was arbitrary, we may infer from the preceding finding that the range $\mathrm{ran}(f^{\leftarrow} \restriction \mathcal{A}')$ satisfies also Property 4 of a σ-algebra.

Initially, the sets Ω, Ω', f and $\mathcal{A}'$ were arbitrary, so that the stated theorem follows now to be true. □

Definition 13.4 (Preimage σ-algebra). For any function $f : \Omega \to \Omega'$ and any σ-algebra $\mathcal{A}'$ on Ω', we call the range

$$\mathrm{ran}(f^{\leftarrow} \restriction \mathcal{A}') \tag{13.57}$$

of the restriction of the preimage function of f to $\mathcal{A}'$ the *preimage σ-algebra* of $\mathcal{A}'$ under f.

Proposition 13.16. *It is true for any function $f : \Omega \to \Omega'$ and any σ-algebra $\mathcal{A}$ on Ω that there exists a unique set $\mathcal{A}'$ consisting of all the subsets A' of Ω' for which $f^{-1}[A'] \in \mathcal{A}$, in the sense that*

$$\forall A' \, (A' \in \mathcal{A}' \Leftrightarrow [A' \in \mathcal{P}(\Omega') \wedge f^{-1}[A'] \in \mathcal{A}]). \tag{13.58}$$

This set $\mathcal{A}'$ is a σ-algebra on Ω'.

Proof. We let Ω, Ω', f and $\mathcal{A}$ be arbitrary sets such that f is a function from Ω to Ω' and such that $\mathcal{A}$ is a σ-algebra on Ω. Noting that the inverse image $f^{-1}[A']$ of any subset $A' \subseteq \Omega'$ and thus of any element $A' \in \mathcal{P}(\Omega')$ is defined, we observe in light of the Axiom of Specification and the Equality Criterion for sets that there exists indeed a unique set $\mathcal{A}'$ that satisfies (13.58).

Because $A' \in \mathcal{A}'$ implies $A' \in \mathcal{P}(\Omega')$ for any set A', it follows with the definition of a subset that the inclusion $\mathcal{A}' \subseteq \mathcal{P}(\Omega')$ holds, so that Property 1 of a σ-algebra on Ω' is satisfied by $\mathcal{A}'$.

Then, we have $\Omega' \in \mathcal{P}(\Omega')$ due to (3.15), and in addition $f^{-1}[\Omega'] = \Omega$ because of (3.746), where $\Omega \in \mathcal{A}$ holds according to Property 2 of a σ-algebra on Ω. Thus, substitution yields $f^{-1}[\Omega'] \in \mathcal{A}$, and this implies in conjunction with $\Omega' \in \mathcal{P}(\Omega')$ the truth of $\Omega' \in \mathcal{A}'$, in view of (13.58). This finding shows us now that Property 2 of a σ-algebra on Ω' is satisfied by $\mathcal{A}'$.

Concerning Property 3, we let A' be an arbitrary set such that A' is a function from $\mathbb{N}_+$ to $\mathcal{A}'$, i.e. a sequence $(A'_n)_{n\in\mathbb{N}_+}$ in $\mathcal{A}'$. We notice here that the inclusions $\operatorname{ran}(A') \subseteq \mathcal{A}' \subseteq \mathcal{P}(\Omega')$ hold, which imply $\operatorname{ran}(A') \subseteq \mathcal{P}(\Omega')$ with (2.13) and therefore $\bigcup \operatorname{ran}(A') \in \mathcal{P}(\Omega')$ due to (3.22). We thus have by definition of a power set that the union $\bigcup_{n=1}^{\infty} A'_n$ is a subset of Ω', giving thus rise to the inverse image

$$f^{-1}[\bigcup_{n=1}^{\infty} A'_n] = \bigcup_{n=1}^{\infty} f^{-1}[A'_n], \tag{13.59}$$

using Proposition 3.242. The union on the right-hand side is with respect to the sequence $S = (f^{-1}[A'_n])_{n\in\mathbb{N}_+}$, whose range we now show to be included in the given σ-algebra $\mathcal{A}$. To do this, we verify the universal sentence

$$\forall X\,(X \in \operatorname{ran}(S) \Rightarrow X \in \mathcal{A}), \tag{13.60}$$

letting $X \in \operatorname{ran}(S)$ be arbitrary, so that $(\bar{n}, X) \in S$ holds for a particular index $\bar{n} \in \mathbb{N}_+$ by definition of a range and by definition of a domain. Writing this finding in function/sequence notation, we have $X = S_{\bar{n}} = f^{-1}[A'_{\bar{n}}]$. Here, the term/value $A'_{\bar{n}}$ is in the codomain $\mathcal{A}'$ of the sequence/function A' according to the Function Criterion, and this fact implies with (13.58) especially the truth of $[X =]\ f^{-1}[A'_{\bar{n}}] \in \mathcal{A}$. The resulting $X \in \mathcal{A}$ thus proves the implication in (13.60), in which X is arbitrary, so that the universal sentence (13.60) follows to be true. Therefore, the inclusion $\operatorname{ran}(S) \subseteq \mathcal{A}$ turns out to be true by definition of a subset, which demonstrates that $\mathcal{A}$ is a codomain of $S = (f^{-1}[A'_n])_{n\in\mathbb{N}_+}$, for which sequence we now write $S : \mathbb{N}_+ \to \mathcal{A}$. By virtue of Property 3 of a σ-algebra, we subsequently find

$\bigcup_{n=1}^{\infty} f^{-1}[A'_n] \in \mathcal{A}$ to be true, which yields in connection with (13.59)

$$f^{-1}[\bigcup_{n=1}^{\infty} A'_n] \in \mathcal{A}. \tag{13.61}$$

Since the previously established $\bigcup \mathrm{ran}(A') \in \mathcal{P}(\Omega')$ can also be written equivalently as

$$\bigcup_{n=1}^{\infty} A'_n \in \mathcal{P}(\Omega'),$$

we can infer from this and (13.61) that $\bigcup_{n=1}^{\infty} A'_n \in \mathcal{A}'$ is true in view of (13.58). Since A' was arbitrary, we may therefore conclude that Property 3 of a σ-algebra is also satisfied by $\mathcal{A}'$.

To establish Property 4, we take an arbitrary set $A' \in \mathcal{A}'$, so that (13.58) yields in particular $f^{-1}[A'] \in \mathcal{A}$. Because $\mathcal{A}$ is a σ-algebra, we obtain then also $(f^{-1}[A'])^c \in \mathcal{A}$. Here, the equation $(f^{-1}[A'])^c = f^{-1}[(A')^c]$ holds because of Proposition 3.227, and substitution gives us therefore

$$f^{-1}[(A')^c] \in \mathcal{A}. \tag{13.62}$$

Due to the already established inclusion $\mathcal{A}' \subseteq \mathcal{P}(\Omega')$, the assumed $A' \in \mathcal{A}'$ implies $A' \in \mathcal{P}(\Omega')$ with the definition of a subset and consequently

$$(A')^c \in \mathcal{P}(\Omega')$$

with (3.25). The conjunction of this and (13.62) implies now the desired $(A')^c \in \mathcal{A}'$ according to (13.58). Since A' was arbitrary, it follows that Property 4 of a σ-algebra is satisfied by $\mathcal{A}'$ as well, which set system thus constitutes a σ-algebra on Ω'.

As the sets Ω, Ω', f and $\mathcal{A}$ were initially arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Definition 13.5 (Measurable function). We say for any measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ that a function $f : \Omega \to \Omega'$ is *measurable* (with respect to $\mathcal{A}$ and $\mathcal{A}'$), symbolically

$$f : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}'), \tag{13.63}$$

iff the inverse image (under f) of every set which is measurable with respect to $\mathcal{A}'$ is measurable with respect to $\mathcal{A}$, that is, iff

$$\forall A' \, (A' \in \mathcal{A}' \Rightarrow f^{-1}[A'] \in \mathcal{A}). \tag{13.64}$$

Corollary 13.17. *It is true for any measurable sets $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ that there exists a unique set $(\Omega', \mathcal{A}')^{(\Omega, \mathcal{A})}$ consisting of all measurable functions $f : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$.*

Proof. Letting Ω, $\mathcal{A}$, Ω' and $\mathcal{A}'$ be arbitrary sets such that $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ are measurable spaces, we can prove by means of the Axiom of Specification and by means of the Equality Criterion for sets that there exists a unique set $(\Omega', \mathcal{A}')^{(\Omega,\mathcal{A})}$ such that

$$\forall f\,(f \in (\Omega', \mathcal{A}')^{(\Omega,\mathcal{A})} \Leftrightarrow [f \in \Omega'^{\Omega} \wedge f : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')]).$$

Since Ω, $\mathcal{A}$, Ω' and $\mathcal{A}'$ are arbitrary, we may therefore conclude that the corollary holds. □

Definition 13.6 (Set of measurable functions). We call for any measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ the set

$$(\Omega', \mathcal{A}')^{(\Omega,\mathcal{A})}, \tag{13.65}$$

consisting of all measurable functions $f : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$ the *set of measurable functions* from $(\Omega, \mathcal{A})$ to $(\Omega', \mathcal{A}')$.

Proposition 13.18. *For any measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$, a function $f : \Omega \to \Omega'$ is measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$ iff the range of the restriction of the preimage function of f to $\mathcal{A}'$ is included in $\mathcal{A}$, i.e.*

$$f : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}') \;\Leftrightarrow\; \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}') \subseteq \mathcal{A}. \tag{13.66}$$

Proof. We let Ω, $\mathcal{A}$, Ω', $\mathcal{A}'$ and f be arbitrary sets, assuming $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ to be measurable spaces and assuming f to be a function from Ω to Ω'. Then, the preimage function $f^{\leftarrow} : \mathcal{P}(\Omega') \to \mathcal{P}(\Omega)$ is defined, where we observe that $\mathcal{A}'$ is a subset of the domain $\mathcal{P}(\Omega')$ according to Property 1 of a σ-algebra (on Ω'), so that the restriction $f^{\leftarrow} \upharpoonright \mathcal{A}'$ is a function from $\mathcal{A}'$ to $\mathcal{P}(\Omega)$ in view of Proposition 3.164.

We prove now the first part ('$\Rightarrow$') of the equivalence directly, assuming f to be measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$, which means that f satisfies the universal sentence (13.64), by definition. To establish the desired consequent, we prove the universal sentence

$$\forall A\,(A \in \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}') \Rightarrow A \in \mathcal{A}), \tag{13.67}$$

letting A be an arbitrary set and assuming $A \in \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}')$ to be true. By definition of a range, there exists then a particular set $\bar{X}$ such that $(\bar{X}, A) \in f^{\leftarrow} \upharpoonright \mathcal{A}'$. This in turn implies with the definition of a restriction $(\bar{X}, A) \in f^{\leftarrow}$ and $\bar{X} \in \mathcal{A}'$. By definition of the preimage function, we thus find $A = f^{\leftarrow}(\bar{X}) = f^{-1}[\bar{X}]$, and $\bar{X} \in \mathcal{A}'$ implies $f^{-1}[\bar{X}] \in \mathcal{A}$ with the assumed (13.64), so that substitution yields $A \in \mathcal{A}$, as desired. Since A was arbitrary, we may therefore conclude that the universal sentence

(13.67) is true, which in turn implies the inclusion $\mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}') \subseteq \mathcal{A}$ with the definition of a subset, proving the implication '$\Rightarrow$'.

We prove the second part ('$\Leftarrow$') of the equivalence (13.66) also directly, assuming the antecedent to be true, so that the universal sentence (13.67) holds by definition of a subset. We now prove the universal sentence (13.64), taking an arbitrary set $A' \in \mathcal{A}'$. Thus, A' is in the domain of the restriction $f^{\leftarrow} \upharpoonright \mathcal{A}'$, and we obtain for the associated value

$$(f^{\leftarrow} \upharpoonright \mathcal{A}')(A') = f^{\leftarrow}(A') = f^{-1}[A']$$

by applying (3.567) and the definition of the preimage function. Writing the resulting equation in the form $(A', f^{-1}[A']) \in f^{\leftarrow} \upharpoonright \mathcal{A}'$, we now see in light of the definition of a range that $f^{-1}[A'] \in \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{A}')$ is true. Because of the true sentence (13.67), this further implies $f^{-1}[A'] \in \mathcal{A}$, proving the implication $A' \in \mathcal{A}' \Rightarrow f^{-1}[A'] \in \mathcal{A}$ and therefore the universal sentence (13.64). This means that f is measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$, so that the proof of the equivalence (13.66) is now complete.

Initially, the sets Ω, $\mathcal{A}$, Ω', $\mathcal{A}'$ and f were arbitrary; consequently, the proposed universal sentence is true. □

Theorem 13.19 (Measurability of constant functions). *For any measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ the constant function $f_c : \Omega \to \Omega'$ is measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$ for any $c \in \Omega'$.*

Exercise 13.3. Establish the Measurability of constant functions in analogy to the Continuity of constant functions, using here Property 2 of a semiring of sets in connection with Proposition 11.30.

Lemma 13.20 (Measurability of inclusion functions). *Show for any measurable space $(\Omega, \mathcal{A})$ and any subset Ω_1 of Ω that the inclusion function $j : \Omega_1 \to \Omega$ is measurable with respect to the trace σ-algebra $\mathcal{A}|\Omega_1$ and $\mathcal{A}$.*

Exercise 13.4. Establish the Measurability of inclusion functions in analogy to the Continuity of inclusion functions.

Theorem 13.21 (Measurability of the composition of two measurable functions). *It is true for any measurable spaces $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(\Omega_3, \mathcal{A}_3)$ and any functions $f : \Omega_1 \to \Omega_2$, $g : \Omega_2 \to \Omega_3$ that the composition $g \circ f$ is measurable if f and g are measurable, in the sense that*

$$\left[f \in (\Omega_2, \mathcal{A}_2)^{(\Omega_1, \mathcal{A}_1)} \wedge g \in (\Omega_3, \mathcal{A}_3)^{(\Omega_2, \mathcal{A}_2)}\right] \Rightarrow g \circ f \in (\Omega_3, \mathcal{A}_3)^{(\Omega_1, \mathcal{A}_1)}. \tag{13.68}$$

Exercise 13.5. Establish the Measurability of the composition of two measurable functions in analogy to the Continuity of the composition of two continuous functions.

Theorem 13.22 (Measurability of the restriction of a measurable function). *It is true for any measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$, for any function $f : \Omega \to \Omega'$, and for any subset Ω_1 of Ω that the restriction of f to Ω_1 is measurable if f is measurable, in the sense that is,*

$$f \in (\Omega', \mathcal{A}')^{(\Omega,\mathcal{A})} \Rightarrow f \upharpoonright \Omega_1 \in (\Omega', \mathcal{A}')^{(\Omega_1,\mathcal{A}|\Omega_1)}. \tag{13.69}$$

Exercise 13.6. Establish the Measurability of the restriction of a measurable function in analogy to the Continuity of the restriction of a measurable function.

Exercise 13.7. Prove for any measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$, for any function $f : \Omega \to \Omega'$ and for any family $A = (\Omega_n)_{n\in\mathbb{N}_+}$ of sets in $\mathcal{A}$ whose range covers Ω that f is measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$ if the restriction $f \upharpoonright \Omega_n$ is measurable with respect to the trace σ-algebra $\mathcal{A}|\Omega_n$ and $\mathcal{A}'$ for every $n \in \mathbb{N}_+$, i.e.

$$\forall n\,(n \in \mathbb{N}_+ \Rightarrow f \upharpoonright \Omega_n \in (\Omega', \mathcal{A}')^{(\Omega_n,\mathcal{A}|\Omega_n)}) \;\Rightarrow\; f \in (\Omega', \mathcal{A}')^{(\Omega,\mathcal{A})}. \tag{13.70}$$

(Hint: Proceed in analogy to the proof of Proposition 13.10.)

Theorem 13.23 (Measurability of projection functions). *It is true for any nonempty index set I, any index $j \in I$ and any families of sets $(\Omega_i)_{i\in I}$, $(\mathcal{A}_i)_{i\in I}$ such that $\mathcal{A}_i$ is a σ-algebra on Ω_i for every $i \in I$ that the j-th projection map π_j on $\bigtimes_{i\in I} \Omega_i$ is measurable with respect to the product σ-algebra $\bigotimes_{i\in I} \mathcal{A}_i$ and $\mathcal{A}_j$, i.e.*

$$\pi_j : (\bigtimes_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{A}_i) \to (\Omega_j, \mathcal{A}_j). \tag{13.71}$$

Exercise 13.8. Establish the Measurability of projection functions in analogy to the Continuity of projection functions, using now the definition of a generated σ-algebra.

To establish measurability, it is not necessary to verify that the inverse image of every set in the σ-algebra $\mathcal{A}'$ is contained in $\mathcal{A}$. We now see that it suffices to inspect the sets in a generating system with respect to $\mathcal{A}'$.

Theorem 13.24 (Measurability Criterion). *For any measurable space $(\Omega, \mathcal{A})$, any set Ω' and any set system $\mathcal{K}' \subseteq \mathcal{P}(\Omega')$, generating the σ-algebra $\mathcal{A}' = \mathcal{A}(\mathcal{K}')$ on Ω', it is true that a function $f : \Omega \to \Omega'$ is measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$ iff the inverse images (under f) of all sets in $\mathcal{K}'$ are in $\mathcal{A}$, that is,*

$$\forall f\,(f \in \Omega'^{\Omega} \Rightarrow [f \in (\Omega', \mathcal{A}')^{(\Omega,\mathcal{A})} \;\Leftrightarrow\; \forall A'\,(A' \in \mathcal{K}' \Rightarrow f^{-1}[A'] \in \mathcal{A})]). \tag{13.72}$$

Proof. We let Ω, $\mathcal{A}$, Ω', $\mathcal{K}'$ and f be arbitrary sets, we assume $(\Omega, \mathcal{A})$ to be a measurable space, assume $\mathcal{K}'$ to be a subset of $\mathcal{P}(\Omega')$, and assume moreover f to be an element of Ω'^{Ω}, that is, to be a function from Ω to Ω'. We consider then the σ-algebra $\mathcal{A}' = \mathcal{A}(\mathcal{K}')$ on Ω' generated by $\mathcal{K}'$. Now, to prove the first part ('$\Rightarrow$') of the stated equivalence, we assume that f is measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$, and we take an arbitrary set A' in $\mathcal{K}'$. Since the inclusion $\mathcal{K}' \subseteq \mathcal{A}'$ is true by definition of a generated σ-algebra, we have that $A' \in \mathcal{K}'$ implies $A' \in \mathcal{A}'$ by definition of a subset. Then, the assumption that f is measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$ implies $f^{-1}[A'] \in \mathcal{A}$. As A' was arbitrary, we may therefore conclude that the implication '$\Rightarrow$' holds.

To prove the second part ('$\Leftarrow$') of the equivalence in (13.72), we now assume

$$\forall A' \, (A' \in \mathcal{K}' \Rightarrow f^{-1}[A'] \in \mathcal{A}) \tag{13.73}$$

and show that this implies $f \in (\Omega', \mathcal{A}')^{(\Omega, \mathcal{A})}$, i.e.

$$\forall A' \, (A' \in \mathcal{A}' \Rightarrow f^{-1}[A'] \in \mathcal{A}). \tag{13.74}$$

Before verifying the previous universal sentence, we observe in light of Proposition 13.72 that the given measurable space $(\Omega, \mathcal{A})$, the given set Ω' and the given function $f : \Omega \to \Omega'$ give rise to a σ-algebra $\overline{\mathcal{A}}'$ on Ω' satisfying

$$\forall A' \, (A' \in \overline{\mathcal{A}}' \Leftrightarrow [A' \in \mathcal{P}(\Omega') \wedge f^{-1}[A'] \in \mathcal{A}]). \tag{13.75}$$

Here, we can show that the generating system $\mathcal{K}'$ is included in that σ-algebra $\overline{\mathcal{A}}'$. To do this, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall K \, (K \in \mathcal{K}' \Rightarrow K \in \overline{\mathcal{A}}'). \tag{13.76}$$

We take an arbitrary set K, assuming $K \in \mathcal{K}'$ to be true, which assumption implies on the one hand $f^{-1}[K] \in \mathcal{A}$ with (13.73), and on the other hand $K \in \mathcal{P}(\Omega')$ with the assumed inclusion $\mathcal{K}' \subseteq \mathcal{P}(\Omega')$. The conjunction of these two findings further implies $K \in \overline{\mathcal{A}}'$ with (13.75), proving the implication in (13.76). Since K is arbitrary, we may now infer from the truth of that implication the truth of the universal sentence (13.76) and therefore the truth of the inclusion $\mathcal{K}' \subseteq \overline{\mathcal{A}}'$. Thus, $\overline{\mathcal{A}}'$ is a σ-algebra (on Ω') which includes the set system $\mathcal{K}'$. Recalling that $\mathcal{A}' = \mathcal{A}(\mathcal{K}')$ is the 'smallest' σ-algebra (on Ω') which includes the set system $\mathcal{K}'$, in the sense of Theorem 11.35c), we obtain the inclusion

$$\mathcal{A}' \subseteq \overline{\mathcal{A}}'. \tag{13.77}$$

We are now in a position to prove the universal sentence (13.74). For this purpose, we let $A' \in \mathcal{A}'$ be arbitrary, so that $A' \in \overline{\mathcal{A}'}$ follows to be true with (13.76) by definition of a subset. This in turn implies $f^{-1}[A'] \in \mathcal{A}$ with (13.75), which finding proves the implication in (13.74). As the set A' was arbitrary, we conclude that the universal sentence (13.74) is true, which means by definition that f is measurable with respect to $\mathcal{A}$ and $\mathcal{A}'$.

We thus completed the proof of the equivalence in (13.72), and since Ω, $\mathcal{A}$, Ω', $\mathcal{K}'$ and f were initially arbitrary sets, the stated theorem follows to be true. □

Theorem 13.25 (Measurability Criterion for functions to Cartesian products of families of sets). *For any measurable space $(\Omega, \mathcal{A})$, any nonempty index set I, any families $(\Omega_i)_{i\in I}$, $(\mathcal{A}_i)_{i\in I}$ such that $(\Omega_i, \mathcal{A}_i)$ is a measurable space for every $i \in I$, and any function $f : \Omega \to \bigtimes_{i\in I} \Omega_i$, it is true that f is measurable with respect to $\mathcal{A}$ and $\bigotimes_{i\in I} \mathcal{A}_i$ iff the composition of the j-th projection map π_j on $\bigtimes_{i\in I} \Omega_i$ and f is measurable with respect to $\mathcal{A}$ and $\mathcal{A}_j$ for every $j \in I$, i.e.*

$$f \in (\bigtimes_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{A}_i)^{(\Omega,\mathcal{A})} \Leftrightarrow \forall j\,(j \in I \Rightarrow \pi_j \circ f \in (\Omega_j, \mathcal{A}_j)^{(\Omega,\mathcal{A})}). \quad (13.78)$$

Exercise 13.9. Prove the Measurability Criterion for functions to Cartesian products of families of sets.

(Hint: Proceed in analogy to the proof of the Continuity Criterion for functions to Cartesian products of families of sets, using now the Measurability Criterion instead of the Subbasis Criterion for continuity.)

The fact that a continuous function is measurable is an immediate consequence of the Measurability Criterion and the definition of continuity.

Theorem 13.26 (Measurability of continuous functions). *It is true for any topological spaces $(\Omega, \mathcal{O})$, $(\Omega', \mathcal{O}')$ and for any function $f : \Omega \to \Omega'$ that f is measurable with respect to the Borel σ-algebra $\mathcal{B}(\Omega)$ and the Borel σ-algebra $\mathcal{B}(\Omega')$ if f is continuous, i.e.*

$$\forall f\,([f \in \Omega'^{\Omega} \wedge f \text{ is continuous}] \Rightarrow f \in (\Omega', \mathcal{B}(\Omega'))^{(\Omega,\mathcal{B}(\Omega))}). \quad (13.79)$$

Proof. We let $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ be arbitrary topological spaces and f an arbitrary element of Ω'^{Ω} such that f is continuous. To prove that f is measurable with respect to the Borel σ-algebra $\mathcal{B}(\Omega) = \mathcal{A}(\mathcal{O})$ and the Borel σ-algebra $\mathcal{B}(\Omega') = \mathcal{A}(\mathcal{O}')$, we apply the Measurability Criterion and verify accordingly

$$\forall A'\,(A' \in \mathcal{O}' \Rightarrow f^{-1}[A'] \in \mathcal{B}(\Omega)). \quad (13.80)$$

For this purpose, we take an arbitrary open set A' in the topology $\mathcal{O}'$ and note that the continuity of the given function $f : \Omega \to \Omega'$ yields then $f^{-1}[A'] \in \mathcal{O}$. Since $\mathcal{A}(\mathcal{O})$ includes its generating system $\mathcal{O}$ by definition of a generated σ-algebra, it follows with the definition of a subset that

$$f^{-1}[A'] \in \mathcal{A}(\mathcal{O}) \quad [= \mathcal{B}(\Omega)]$$

is true. Since A' was arbitrary, we therefore conclude that the universal sentence (13.80) holds, which implies then the truth of the consequent of the implication in (13.79) with the Measurability Criterion (13.72). As $(\Omega, \mathcal{O})$,$(\Omega', \mathcal{O}')$ and f were arbitrary, we may now infer from this finding the truth of the theorem. □

Exercise 13.10. Verify for any measurable space $(\Omega_1, \mathcal{A}_1)$, for any topological spaces $(\Omega_2, \mathcal{O}_2)$, $(\Omega_3, \mathcal{O}_3)$ and for any functions $f : \Omega_1 \to \Omega_2$, $g : \Omega_2 \to \Omega_3$ that the composition of g and f is measurable with respect to $\mathcal{A}_1$ and the Borel σ-algebra $\mathcal{B}(\Omega_3)$ if f is measurable with respect to $\mathcal{A}_1$ and the Borel σ-algebra $\mathcal{B}(\Omega_2)$ and if g is continuous, i.e.

$$\Big[f \in (\Omega_2, \mathcal{B}(\Omega_2))^{(\Omega_1, \mathcal{A}_1)} \wedge g \text{ is continuous}\Big] \Rightarrow g \circ f \in (\Omega_3, \mathcal{B}(\Omega_3))^{(\Omega_1, \mathcal{A}_1)}. \tag{13.81}$$

(Hint: Use (3.762).)

13.3. Vector Spaces of Measurable Real Functions

We begin with a basic observation.

Note 13.1. In view of the Measurability of constant functions, it is true for any measurable set $(\Omega, \mathcal{A})$ and any real number c that the constant function $f_c : \Omega \to \mathbb{R}$ is measurable with respect to $\mathcal{A}$ and the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$.

Considering the measurability of a general real function, we now apply the Measurability Criterion with the fact that $\mathcal{B}$ is generated by various sets of intervals.

Corollary 13.27 (Measurability Criterion for real functions). *It is true for any measurable space $(\Omega, \mathcal{A})$ that a real function f on Ω is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$ iff the inverse images (under f)*

a) of every real right-open interval $(-\infty, b)$, or

b) of every real left-closed and right-unbounded interval $[a, +\infty)$, or

c) of every real left-unbounded and right-closed interval $(-\infty, b]$, or

d) of every real open and right-unbounded interval $(a, +\infty)$

lies in $\mathcal{A}$, that is,

$$f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})} \Leftrightarrow \forall I \, (I \in \{(-\infty, b) : b \in \mathbb{R}\} \Rightarrow f^{-1}[I] \in \mathcal{A}). \quad (13.82)$$

$$f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})} \Leftrightarrow \forall I \, (I \in \{[a, +\infty) : a \in \mathbb{R}\} \Rightarrow f^{-1}[I] \in \mathcal{A}). \quad (13.83)$$

$$f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})} \Leftrightarrow \forall I \, (I \in \{(-\infty, b] : b \in \mathbb{R}\} \Rightarrow f^{-1}[I] \in \mathcal{A}). \quad (13.84)$$

$$f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})} \Leftrightarrow \forall I \, (I \in \{(a, +\infty) : a \in \mathbb{R}\} \Rightarrow f^{-1}[I] \in \mathcal{A}). \quad (13.85)$$

Proof. We take an arbitrary measurable space $(\Omega, \mathcal{A})$ and an arbitrary function $f \in \mathbb{R}^\Omega$. Recalling now the facts (11.991), (11.1009), (11.1002) and (11.999), we obtain the equivalences (13.82) – (13.85) directly with the Measurability Criterion. □

Theorem 13.28 (Measurability of increasing real functions on $\mathbb{R}$). *Any increasing real function on $\mathbb{R}$ is measurable, that is,*

$$\forall f \, ([f \in \mathbb{R}^{\mathbb{R}} \wedge f \text{ is increasing}] \Rightarrow f \in (\mathbb{R}, \mathcal{B})^{(\mathbb{R}, \mathcal{B})}). \quad (13.86)$$

Proof. We let f be an arbitrary increasing real function on $\mathbb{R}$ and prove $f \in (\mathbb{R}, \mathcal{B})^{(\mathbb{R},\mathcal{B})})$ by means of Measurability Criterion b) for real functions, i.e. by verifying

$$\forall I\, (I \in \{[a, +\infty) : a \in \mathbb{R}\} \Rightarrow f^{-1}[I] \in \mathcal{B}). \tag{13.87}$$

Letting $I \in \{[a, +\infty) : a \in \mathbb{R}\}$ be arbitrary, there exists a particular real number a such that $[a, +\infty) = I$, according to (3.432). To prove

$$f^{-1}[[a, +\infty)] \in \mathcal{B}, \tag{13.88}$$

we consider the two exhaustive cases $f^{-1}[[a, +\infty)] = \emptyset$ and $f^{-1}[[a, +\infty)] \neq \emptyset$. Considering the first case, we recall that $\emptyset \in \mathcal{B}$ holds according to (11.963), so that (13.88) is true in this case. Concerning the other case $f^{-1}[[a, +\infty)] \neq \emptyset$, we consider now the two subcases that $f^{-1}[[a, +\infty)]$ is bounded from below or not.

The first subcase that $f^{-1}[[a, +\infty)]$ is bounded from below implies (in conjunction with the non-emptiness of $f^{-1}[[a, +\infty)]$ and the fact that this inverse image of $[a, +\infty)$ under $f : \mathbb{R} \to \mathbb{R}$ is a subset of $\mathbb{R}$ by definition) that the infimum i of $f^{-1}[[a, +\infty)]$ (with respect to $\leq_{\mathbb{R}}$) exists (in $\mathbb{R}$) by virtue of the Infimum Property of $(\mathbb{R}, <_{\mathbb{R}})$. Let us now consider the two sub-subcases $i \in f^{-1}[[a, +\infty)]$ and $i \notin f^{-1}[[a, +\infty)]$.

Regarding the first sub-subcase, we first show by means of the Equality Criterion for sets that

$$f^{-1}[[a, +\infty)] = [i, +\infty) \tag{13.89}$$

holds, by verifying $x \in f^{-1}[[a, +\infty)] \Leftrightarrow x \in [i, +\infty)$ for any x. Letting x be arbitrary, we observe on the one hand that the assumption $x \in f^{-1}[[a, +\infty)]$ implies $i \leq_{\mathbb{R}} x$ since the infimum i is (by definition) a lower bound for $f^{-1}[[a, +\infty)]$ in that set, so that $x \in [i, +\infty)$ holds by definition of a real left-closed and right-unbounded interval. On the other hand, assuming now $x \in [i, +\infty)$ implies $i \leq_{\mathbb{R}} x$. It follows that $f(i) \leq_{\mathbb{R}} f(x)$ holds since f is increasing so that Proposition 3.257 applies here. As $i \in f^{-1}[[a, +\infty)]$ is equivalent to $f(i) \in [a, +\infty)$ by definition of the inverse image, and consequently equivalent also to $a \leq_{\mathbb{R}} f(i)$, we see in light of the transitivity of $\leq_{\mathbb{R}}$ that $a \leq_{\mathbb{R}} f(x)$ is true, i.e. $f(x) \in [a, +\infty)$, thus $x \in f^{-1}[[a, +\infty)]$. This completes the proof of (13.89), since x is arbitrary. As $[i, +\infty) \in \mathcal{B}$ holds in view of (11.1010), we have (13.88) in the first subcase.

Regarding the other sub-subcase $i \notin f^{-1}[[a, +\infty)]$, we now demonstrate that

$$f^{-1}[[a, +\infty)] = (i, +\infty) \tag{13.90}$$

holds (using again the Equality Criterion for sets). We let x be arbitrary and notice on the one hand that the assumption $x \in f^{-1}[[a, +\infty)]$ implies $i \leq_{\mathbb{R}} x$ (since i is a lower bound) and $x \neq i$ (applying Proposition 2.1 to the facts that x is an element of the inverse image and i not). According to Characterization of induced irreflexive partial orderings, these two findings imply $i <_{\mathbb{R}} x$ and therefore $x \in (i, +\infty)$ by definition of a real open and right-unbounded interval. On the other hand, assuming now $x \in (i, +\infty)$, we see that $i <_{\mathbb{R}} x$ holds. Since i is the greatest lower bound for $f^{-1}[[a, +\infty)]$, x cannot be a greater one, i.e., x is not a lower bound for the inverse image. Thus, the negation

$$\neg \forall x' \, (x' \in f^{-1}[[a, +\infty)] \Rightarrow x \leq_{\mathbb{R}} x').$$

Due to the Negation Law for universal implications and the Negation Formula for $\leq$, there exists therefore an element in $f^{-1}[[a, +\infty)]$, say $\bar{x}'$, such that $x >_{\mathbb{R}} \bar{x}'$. As $\bar{x}' \in f^{-1}[[a, +\infty)]$ means $f(\bar{x}') \in [a, +\infty)$, i.e. $a \leq_{\mathbb{R}} f(\bar{x}')$, and since $\bar{x}' <_{\mathbb{R}} x$ implies $f(\bar{x}') \leq_{\mathbb{R}} f(x)$ with the initial assumption that f is increasing, we obtain $a \leq_{\mathbb{R}} f(x)$ with the transitivity of $\leq_{\mathbb{R}}$. Consequently, so that $f(x) \in [a, +\infty)$ holds, so that the desired consequent $x \in f^{-1}[[a, +\infty)]$ follows to be true. Because x is arbitrary, we therefore conclude that (13.90) holds. Thus, we find (13.88) with (11.1000), which completes the proof for the first subcase

In the second subcase, we assume that $f^{-1}[[a, +\infty)]$ is not bounded from below, which means that

$$\neg \exists a_x \, (a_x \in \mathbb{R} \wedge \forall x \, (x \in f^{-1}[[a, +\infty)] \Rightarrow a_x \leq_{\mathbb{R}} x)).$$

By the Negation Law for existential conjunctions and universal implications, we therefore find the true universal sentence

$$\forall a_x \, (a_x \in \mathbb{R} \Rightarrow \exists x \, (x \in f^{-1}[[a, +\infty)] \wedge \neg a_x \leq_{\mathbb{R}} x)). \tag{13.91}$$

We may prove the equality

$$.f^{-1}[[a, +\infty)] = \mathbb{R} \tag{13.92}$$

Letting x be arbitrary, the assumption $x \in f^{-1}[[a, +\infty)]$ clearly implies the desired $x \in \mathbb{R}$ with the definition of an inverse image. The converse assumption $x \in \mathbb{R}$ implies with (13.91) the existence of a particular element $x^* \in f^{-1}[[a, +\infty)$ such that $\neg x \leq_{\mathbb{R}} x^*$. Clearly, this negation implies $x^* <_{\mathbb{R}} x$, and therefore $f(x^*) \leq_{\mathbb{R}} f(x)$ since f is increasing; furthermore, $x^* \in f^{-1}[[a, +\infty)$ evidently means that $a \leq_{\mathbb{R}} f(x^*)$. Combining these two inequalities yields $a \leq_{\mathbb{R}} f(x)$ and subsequently $x \in f^{-1}[[a, +\infty)]$, as desired. As x was arbitrary, we may therefore conclude that the equality

(13.92) holds indeed. Since we know that $\mathbb{R} \in \mathcal{B}$, by recalling (11.963), we find (13.88) also for the second subcase (within the second case).

We thus completed the proof(s) by cases. Since I was initially arbitrary, we may now further conclude that the universal sentence (13.87) also holds. This finding implies now with the Measurability Criterion for real function that f is an element of $(\mathbb{R}, \mathcal{B})^{(\mathbb{R},\mathcal{B})}$, which is the desired consequent of the implication in (13.86). In this implication, f is arbitrary, so that the theorem follows now to be true indeed. □

Proposition 13.29. *For any measurable space $(\Omega, \mathcal{A})$ and any measurable functions f, g from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$ there exists a unique set $\{\omega : f(\omega) < g(\omega)\}$ consisting of all the elements ω in Ω for which $f(\omega)$ is less than $g(\omega)$. For any surjection h from $\mathbb{N}$ to $\mathbb{Q}$, this set satisfies*

$$\{\omega : f(\omega) < g(\omega)\} = \bigcup_{n=1}^{\infty} \left(f^{-1}[(-\infty, h_{n-1})] \cap g^{-1}[(h_{n-1}, +\infty)]\right) \quad (13.93)$$

and is a measurable set in $\mathcal{A}$.

Proof. We let $(\Omega, \mathcal{A})$ be an arbitrary measurable space, f and g arbitrary elements of $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$, and $h : \mathbb{N} \twoheadrightarrow \mathbb{Q}$ an arbitrary surjection (which exists due to the Countable infinity of $\mathbb{Q}$ and the Countability Criteria, using the evident fact that $\mathbb{N}$ is nonempty). We may now evidently apply the Axiom of Specification and the Equality Criterion for sets to establish the truth of the uniquely existential sentence

$$\exists! X \, \forall \omega \, (\omega \in X \Leftrightarrow [\omega \in \Omega \, \wedge \, f(\omega) < g(\omega)]), \quad (13.94)$$

so that X is the desired set. We now verify the equivalence

$$\omega \in \{\omega : f(\omega) < g(\omega)\} \Leftrightarrow \omega \in \bigcup_{n=1}^{\infty} \left(f^{-1}[(-\infty, h_{n-1})] \cap g^{-1}[(h_{n-1}, +\infty)]\right) \quad (13.95)$$

Letting ω be arbitrary, we see that the assumption $\omega \in \{\omega : f(\omega) < g(\omega)\}$ implies $f(\omega) < g(\omega)$. As $(\mathbb{R}, <_{\mathbb{R}})$ is a separably ordered set with respect to $\mathbb{Q}$ in view of Corollary 8.12b), there exists a rational number, say $\bar{q}$, such that $f(\omega) < \bar{q} < g(\omega)$. As h is a surjection, there exists in view of (3.631) a particular constant $\bar{m}$ such that $h_{\bar{m}} = \bar{q}$, so that $f(\omega) < h_{\bar{m}} < g(\omega)$ holds. Clearly, $\bar{m}$ is in the domain $\mathbb{N}$ of h, so that $0 \leq \bar{m}$. Defining now the natural number $\bar{n} = \bar{m} + 1$, it evidently follows from $0 \leq \bar{m}$ that $1 \leq \bar{m} + 1$ $[= \bar{n}]$ holds, and therefore $\bar{n} \in \mathbb{N}_+$. Furthermore, $\bar{n} = \bar{m} + 1$ clearly implies $\bar{m} = \bar{n} - 1$. We then obtain through substitution

$$f(\omega) < h_{\bar{n}-1} < g(\omega), \quad (13.96)$$

and therefore

$$f(\omega) \in (-\infty, h_{\bar{n}-1}) \ \wedge \ g(\omega) \in (h_{\bar{n}-1}, +\infty), \tag{13.97}$$

using the definitions of a real left-unbounded and right-open as well as of a left-open and right-unbounded interval. Recalling that f and g are real functions and observing that the preceding intervals are subsets of $\mathbb{R}$, we find for their inverse images under f and g, respectively,

$$\omega \in f^{-1}[(-\infty, h_{\bar{n}-1})] \ \wedge \ \omega \in g^{-1}[(h_{\bar{n}-1}, +\infty)]. \tag{13.98}$$

Thus,

$$\omega \in f^{-1}[(-\infty, h_{\bar{n}-1})] \cap g^{-1}[(h_{\bar{n}-1}, +\infty)] \tag{13.99}$$

by definition of the intersection of two sets. The previous findings demonstrate the truth of the existential sentence

$$\exists n \, (n \in \mathbb{N}_+ \ \wedge \ \omega \in f^{-1}[(-\infty, h_{n-1})] \cap g^{-1}[(h_{n-1}, +\infty)]), \tag{13.100}$$

which means

$$\omega \in \bigcup_{n=1}^{\infty} \big(f^{-1}[(-\infty, h_{n-1})] \cap g^{-1}[(h_{n-1}, +\infty)]\big)$$

according to the Characterization of the union of a family of sets. Thus, the first part ('$\Rightarrow$') of the equivalence (13.95) holds.

Assuming, conversely, the right-hand side of that equivalence to be true, the existential sentence (13.100) follows to be true with the Characterization of the union of a family of sets. Thus, (13.99) holds indeed for a particular positive natural number $\bar{n}$. Then, (13.98) follows to be definition of the intersection of two sets. That conjunction in turn yields the conjunction (13.97) with the definition of an inverse image. The aforementioned interval definitions gives us then also the two inequalities (13.96), which further imply $f(\omega) < g(\omega)$ with the transitivity of the linear ordering $<_{\mathbb{R}}$. Clearly, ω is an element of the domain Ω of f and g, so that ω turns out to be an element of $\{\omega : f(\omega) < g(\omega)\}$, by definition of that set.

Thus, the proof of the equivalence (13.95) is now complete, in which ω is arbitrary, so that (13.93) follows now to be true with the Equality Criterion for sets.

In addition, the Measurability Criterion for real functions a) and d) show that the initial assumptions $f, g \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ that the inverse images $f^{-1}[(-\infty, h_{n-1})]$ and $g^{-1}[(h_{n-1}, +\infty)]$ are elements of $\mathcal{A}$ for every $n \in \mathbb{N}_+$. Their intersection is also in $\mathcal{A}$, because every σ-algebra, being a π-system (see Proposition 11.30), is closed under such intersections of two sets. Then, in view of Property 3 of a σ-algebra, the union (13.93) is also in $\mathcal{A}$.

Since $(\Omega, \mathcal{A})$, f, g and h were initially all arbitrary, we may therefore conclude that the proposed universal sentence holds. □

Proposition 13.30. *For any measurable space $(\Omega, \mathcal{A})$ and any measurable functions f, g from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$ there exists a unique set $\{\omega : g(\omega) \leq f(\omega)\}$ consisting of all the elements ω in Ω for which $g(\omega)$ is less than or equal to $f(\omega)$, and this set satisfies*

$$\{\omega : g(\omega) \leq f(\omega)\} = \{\omega : f(\omega) < g(\omega)\}^c. \tag{13.101}$$

Proof. We let $(\Omega, \mathcal{A})$ be an arbitrary measurable set and f, g be arbitrary elements of $(\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$. We evidently obtain with the Axiom of Specification and the Equality Criterion for sets the true uniquely existential sentence

$$\exists! Y \, \forall \omega \, (\omega \in X \Leftrightarrow [\omega \in \Omega \, \wedge \, g(\omega) \leq f(\omega)]). \tag{13.102}$$

We now verify that $[Y =] \, \{\omega : g(\omega) \leq f(\omega)\} = \{\omega : f(\omega) < g(\omega)\}^c \, [= X^c]$ by observing the truth of the equivalences (for an arbitrary ω)

$$\begin{aligned}
\omega \in X^c \;&\Leftrightarrow\; \omega \in \Omega \setminus X \\
&\Leftrightarrow\; \omega \in \Omega \, \wedge \, \neg \omega \in X \\
&\Leftrightarrow\; \omega \in \Omega \, \wedge \, \neg(\omega \in \Omega \, \wedge \, f(\omega) < g(\Omega)) \\
&\Leftrightarrow\; \omega \in \Omega \, \wedge \, (\omega \notin \Omega \, \vee \, \neg f(\omega) < g(\Omega)) \\
&\Leftrightarrow\; (\omega \in \Omega \, \wedge \, \omega \notin \Omega) \, \vee \, (\omega \in \Omega \, \wedge \, \neg f(\omega) < g(\Omega)) \\
&\Leftrightarrow\; \omega \in \Omega \, \wedge \, \neg f(\omega) < g(\Omega) \\
&\Leftrightarrow\; \omega \in \Omega \, \wedge \, g(\omega) \leq f(\Omega) \\
&\Leftrightarrow\; \omega \in Y
\end{aligned}$$

using the definition of a complement, the definition of a set difference, the specification of X in (13.94), De Morgan's Law for the conjunction, the Distributivity of the conjunction over the disjunction, (1.15) with the fact that $\omega \in \Omega \wedge \omega \notin \Omega$ is a contradiction as in (1.11), the Negation Formula for $<$, and finally the specification of Y in (13.102). Here, $(\Omega, \mathcal{A})$, f and g were initially all arbitrary, so that the proposition follows to be true. □

Corollary 13.31. *For any measurable space $(\Omega, \mathcal{A})$ and any $f, g \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ the set consisting of all the elements ω in Ω for which $g(\omega)$ is less than or equal to $f(\omega)$ is measurable with respect to $\mathcal{A}$, that is,*

$$\{\omega : g(\omega) \leq f(\omega)\} \in \mathcal{A}. \tag{13.103}$$

Proof. The fact that $\{\omega : f(\omega) < g(\omega)\} \in \mathcal{A}$ (recall Proposition 13.29) implies that $\{\omega : f(\omega) < g(\omega)\}^c \in \mathcal{A}$ with Property 4 of a σ-algebra, and substitution based on (13.30) yields then (13.103). □

In the following lemma, we consider apply the binary operations within a real vector space $(\mathbb{R}^\Omega, +, \cdot)$, a commutative ring $(\mathbb{R}^\Omega, +, \cdot, -)$ and a lattice $(\mathbb{R}^\Omega, \curlyvee, \curlywedge, \preceq)$ of real functions to pairs of measurable functions in $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ and determine the measurability of the results.

Lemma 13.32. *The following sentences are true for any measurable space $(\Omega, \mathcal{A})$.*

a) The scalar multiplication of any $c \in \mathbb{R}$ and any measurable real function $f \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ is also measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, that is,

$$c \cdot f \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}. \tag{13.104}$$

b) The negative of any $f \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ is also measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, that is,

$$-f \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}. \tag{13.105}$$

c) The (pointwise) sum of any $f \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ and any constant real function $g_c \in \mathbb{R}^\Omega$ is also measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, that is,

$$f + g_c \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}. \tag{13.106}$$

d) The (pointwise) sum and difference of any $f, g \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ is also measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, that is,

$$f \pm g \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}. \tag{13.107}$$

e) The (pointwise) product of any $f \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ with itself is also measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, that is,

$$f \cdot f \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}. \tag{13.108}$$

f) The (pointwise) product of any $f, g \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ is also measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, that is,

$$f \cdot g \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}. \tag{13.109}$$

g) The join and meet of any $f, g \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ are also measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, that is,

$$f \curlyvee g, f \curlywedge g \in (\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}. \tag{13.110}$$

Proof. We let $(\Omega, \mathcal{A})$ be an arbitrary measurable space.

Concerning a), we let c be an arbitrary real number and f an arbitrary element of $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$. We will first consider the two cases $c = 0$ and $c \neq 0$ to establish (13.104). In case of $c = 0$, we find $c \cdot f = 0_{\mathbb{R}^\Omega}$ with (10.7), which is the real constant function on Ω with value 0 (see Note 10.9). This function is measurable in view of Note 13.1, i.e., (13.104) holds.

The second case $c \neq 0$ implies that $c >_{\mathbb{R}} 0$ or $c <_{\mathbb{R}} 0$ holds since the linear ordering $<_{\mathbb{R}}$ is connex, we may use this true disjunction to prove

$$\forall I\,(I \in \{(-\infty, b) : b \in \mathbb{R}\} \Rightarrow (c \cdot f)^{-1}[(-\infty, b)] \in \mathcal{A}) \tag{13.111}$$

by (sub)cases. For this purpose, we let I be arbitrary in $\{(-\infty, b) : b \in \mathbb{R}\}$, so that $I = (-\infty, b)$ holds for a particular $b \in \mathbb{R}$. In the first subcase $c >_{\mathbb{R}} 0$, we may establish the equality

$$(c \cdot f)^{-1}[(-\infty, b)] = f^{-1}[(-\infty, b/c)], \tag{13.112}$$

using the Equality Criterion for sets. To do this, we verify

$$\forall \omega\,(\omega \in (c \cdot f)^{-1}[(-\infty, b)] \;\Leftrightarrow\; \omega \in f^{-1}[(-\infty, b/c)]). \tag{13.113}$$

Letting ω be arbitrary, we observe first the truth of the equivalences

$$\begin{aligned} \omega \in (c \cdot f)^{-1}[(-\infty, b)] \;&\Leftrightarrow\; (c \cdot f)(\omega) \in (-\infty, b) && (13.114)\\ &\Leftrightarrow\; c \cdot_{\mathbb{R}} f(\omega) \in (-\infty, b) && (13.115)\\ &\Leftrightarrow\; c \cdot_{\mathbb{R}} f(\omega) <_{\mathbb{R}} b && (13.116) \end{aligned}$$

in light of the definition of an inverse image, the Scalar multiplication for functions and the definition of a real left-unbounded and right-open interval. Now, since $c >_{\mathbb{R}} 0$, we may apply the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$ to obtain the further equivalences

$$\begin{aligned} c \cdot_{\mathbb{R}} f(\omega) <_{\mathbb{R}} b \;&\Leftrightarrow\; f(\omega) <_{\mathbb{R}} b/c\\ &\Leftrightarrow\; f(\omega) \in (-\infty, b/c)\\ &\Leftrightarrow\; \omega \in f^{-1}[(-\infty, b/c)]. \end{aligned}$$

The previous equivalences therefore imply the equivalence in (13.113), in which ω is arbitrary, so that the equality (13.112) follows to be true. Since f is measurable, the right-hand side of (13.112) is in $\mathcal{A}$ by virtue of (13.82); thus, the left-hand side $(c \cdot f)^{-1}[(-\infty, b)]$ is also in $\mathcal{A}$, as desired.

In the second subcase $c <_{\mathbb{R}} 0$, which implies $-c >_{\mathbb{R}} 0$ with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$, we may similarly prove the equality

$$(c \cdot f)^{-1}[(-\infty, b)] = f^{-1}[(b/c, +\infty)]. \tag{13.117}$$

To do this, we demonstrate the truth of

$$\forall \omega\, (\omega \in (c \cdot f)^{-1}[(-\infty, b)] \;\Leftrightarrow\; \omega \in f^{-1}[(b/c, +\infty)]), \tag{13.118}$$

letting ω again be arbitrary. Thus, we obtain the equivalences (13.114) – (13.116) again. Moreover, the definition of a real left-open and right-unbounded interval gives us evidently

$$\begin{aligned} c \cdot_{\mathbb{R}} f(\omega) <_{\mathbb{R}} b \;&\Leftrightarrow\; -f(\omega) <_{\mathbb{R}} -b/c \\ &\Leftrightarrow\; b/c <_{\mathbb{R}} f(\omega) \\ &\Leftrightarrow\; f(\omega) \in (b/c, +\infty) \\ &\Leftrightarrow\; \omega \in f^{-1}[(b/c, +\infty)]. \end{aligned}$$

Combining all these equivalences, we arrive at (13.118), which subsequently yields the equality (13.117) since ω is arbitrary. Here, the right-hand side is in $\mathcal{A}$ now due to (13.85), so that the left-hand side $(c \cdot f)^{-1}[(-\infty, b)]$ is again in $\mathcal{A}$.

Because I was arbitrary, we may infer from these findings that the universal sentence (13.111) is true for both subcases. In view of the Measurability Criterion for real functions, this sentence implies (13.104). Thus, the proof by cases is complete. As c and f were initially arbitrary, we therefore conclude that a) holds.

Concerning b), we let $f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ be arbitrary and observe in light of a) that $(-1) \cdot f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ is then also true, where $(-1) \cdot f = -f$ holds according to Corollary (10.4); thus, $-f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$. Since f is arbitrary, we therefore conclude that b) also holds.

Concerning c), we let $f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ and $c \in \mathbb{R}$ be arbitrary, and we verify

$$\forall I\, (I \in \{(-\infty, b) : b \in \mathbb{R}\} \Rightarrow (f + g_c)^{-1}[(-\infty, b)] \in \mathcal{A}). \tag{13.119}$$

Letting I be arbitrary in $\{(-\infty, b) : b \in \mathbb{R}\}$, we find $I = (-\infty, b)$ for a particular $b \in \mathbb{R}$. Letting now ω be arbitrary, we also find

$$\begin{aligned} \omega \in (f + g_c)^{-1}[(-\infty, b)] \;&\Leftrightarrow\; (f + g_c)(\omega) \in (-\infty, b) \\ &\Leftrightarrow\; f(\omega) +_{\mathbb{R}} g_c(x) \in (-\infty, b) \\ &\Leftrightarrow\; f(\omega) +_{\mathbb{R}} c <_{\mathbb{R}} b \\ &\Leftrightarrow\; f(\omega) <_{\mathbb{R}} b -_{\mathbb{R}} c \\ &\Leftrightarrow\; f(\omega) \in (-\infty, b -_{\mathbb{R}} c) \\ &\Leftrightarrow\; \omega \in f^{-1}[(-\infty, b -_{\mathbb{R}} c)] \end{aligned}$$

using the definition of an inverse image, the Pointwise addition of functions, (3.534) together with the definition of a real left-unbounded and right-open interval, the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$, again the definition of a real left-unbounded and right-open interval, and finally again the definition of an inverse image. Thus,

$$\omega \in (f+g_c)^{-1}[(-\infty,b)] \;\Leftrightarrow\; \omega \in f^{-1}[(-\infty, b -_{\mathbb{R}} c)]$$

holds for any arbitrary ω, so that

$$(f+g_c)^{-1}[(-\infty,b)] = f^{-1}[(-\infty, b -_{\mathbb{R}} c)],$$

follows to be true by virtue of the Equality Criterion for sets. As the right-hand side is in $\mathcal{A}$ due to (13.82), we find $(f+g_c)^{-1}[(-\infty,b)] \in \mathcal{A}$. As I was arbitrary, we therefore conclude that (13.119) holds, and this implies (13.104) with the Measurability Criterion for real functions. Because f and c were also arbitrary, we may infer from this the truth of c).

Regarding d), we let $f, g \in (\mathbb{R},\mathcal{B})^{(\Omega,\mathcal{A})}$ be arbitrary and observe that $f+g$ is then also a function from Ω to $\mathbb{R}$ according to the Pointwise addition of functions. Next, we verify

$$\forall I\,(I \in \{(-\infty,b) : b \in \mathbb{R}\} \Rightarrow (f+g)^{-1}[(-\infty,b)] \in \mathcal{A}), \tag{13.120}$$

letting I be an arbitrary interval $(-\infty,b)$. The particular real number b gives rise to the real constant function g_b on Ω, so that the new function $h = -g + g_b$ is an element of $(\mathbb{R},\mathcal{B})^{(\Omega,\mathcal{A})}$ in view of c) and the fact $-g \in (\mathbb{R},\mathcal{B})^{(\Omega,\mathcal{A})}$ derived from b). Here, $-g + g_b = g_b - g$ holds due to the commutativity of the pointwise addition $+_{\mathbb{R}^\Omega}$; we therefore find $h \in (\mathbb{R},\mathcal{B})^{(\Omega,\mathcal{A})}$. We may establish the truth of the equality

$$(f+g)^{-1}[(-\infty,b)] = \{\omega : f(\omega) < h(\omega)\}. \tag{13.121}$$

Indeed, we obtain for arbitrary ω

$$\begin{aligned}
\omega \in (f+g)^{-1}[(-\infty,b)] \;&\Leftrightarrow\; (f+g)(\omega) \in (-\infty,b) \\
&\Leftrightarrow\; f(\omega) +_{\mathbb{R}} g(\omega) \in (-\infty,b) \\
&\Leftrightarrow\; f(\omega) +_{\mathbb{R}} g(\omega) <_{\mathbb{R}} b \\
&\Leftrightarrow\; f(\omega) +_{\mathbb{R}} g(\omega) <_{\mathbb{R}} g_b(\omega) \\
&\Leftrightarrow\; f(\omega) <_{\mathbb{R}} g_b(\omega) -_{\mathbb{R}} g(\omega) \\
&\Leftrightarrow\; f(\omega) <_{\mathbb{R}} (g_b - g)(\omega) \\
&\Leftrightarrow\; f(\omega) <_{\mathbb{R}} h(\omega) \\
&\Leftrightarrow\; \omega \in \{\omega : f(\omega) < h(\omega)\}
\end{aligned}$$

where we used Proposition 13.29 to obtain the last equivalence. We may therefore conclude that the equality (13.121) holds. According to the preceding proposition, the right-hand side of that equation is an element of $\mathcal{A}$, and so is also the left-hand side $(f+g)^{-1}[(-\infty, b)]$. Since I was initially arbitrary, we may infer from this finding the truth of (13.120), which evidently implies the truth of $f+g \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$. As f and g were also arbitrary, we may therefore conclude that d) holds with respect to the addition.

In addition, we can write for arbitrary $f, g \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ the difference of f and g as $f - g = f + (-g)$. Here, $-g \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ holds according to b), and $f + (-g) \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ is then also true in view of the previous findings in d). Thus, $f - g \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ holds as well. Since f and g are arbitrary, we therefore conclude that d) holds with respect to both the addition and the subtraction.

Regarding e), we let $f \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ be arbitrary. Note that $f \cdot f$ is therefore a function from Ω to $\mathbb{R}$ according to the Pointwise multiplication of functions. Next, we prove

$$\forall I \, (I \in \{(a, +\infty) : a \in \mathbb{R}\} \Rightarrow (f \cdot f)^{-1}[(a, +\infty)] \in \mathcal{A}). \tag{13.122}$$

We take an arbitrary interval $I = (a, +\infty)$, so that the particular real number a gives rise to the two cases $a <_{\mathbb{R}} 0$ and $\neg a <_{\mathbb{R}} 0$. In the first case $a <_{\mathbb{R}} 0$, we can establish the equality

$$(f \cdot f)^{-1}[(a, +\infty)] = \Omega, \tag{13.123}$$

by proving the equivalent universal sentence

$$\forall \omega \, (\omega \in (f \cdot f)^{-1}[(a, +\infty)] \Leftrightarrow \omega \in \Omega). \tag{13.124}$$

Letting ω be arbitrary, the assumption $\omega \in (f \cdot f)^{-1}[(a, +\infty)]$ immediately implies the desired consequent $\omega \in \Omega$ since the given inverse image under $(f \cdot f) : \Omega \to \mathbb{R}$, by definition, consists of elements of Ω. We now conversely assume $\omega \in \Omega$ to be true. Let us observe that the inequalities $a <_{\mathbb{R}} 0 \leq_{\mathbb{R}} f(\omega)^2$ hold due to the current case assumption and then non-negativity of squares (6.243). The Transitivity Formula for $<$ and $\leq$ therefore gives us $a <_{\mathbb{R}} f(\omega)^2$, and this implies $f(\omega)^2 \in (a, +\infty)$ with the definition of a real left-open and right-unbounded interval. In view of the notation for squares, we may write this also as $f(\omega) \cdot_{\mathbb{R}} f(\omega) \in (a, +\infty)$. According to the Pointwise multiplication of functions, this means that $(f \cdot f)(\omega) \in (a, +\infty)$, so that evidently $\omega \in (f \cdot f)^{-1}[(a, +\infty)]$, as desired. We thus completed the proof of the equivalence in (13.124), in which ω is arbitrary, so that the universal sentence (13.124) follows to be true as well. We thus proved the equality (13.123), in which $\Omega \in \mathcal{A}$ holds by virtue of Property 2 of a σ-algebra on Ω. Consequently, we find $(f \cdot f)^{-1}[(a, +\infty)] \in \mathcal{A}$, as desired.

The second case $\neg a <_{\mathbb{R}} 0$ yields the inequality $0 \leq_{\mathbb{R}} a$ with the Negation Formula for $<$. In this case, we can establish the equality

$$(f \cdot f)^{-1}[(a, +\infty)] = f^{-1}[(-\infty, -\sqrt{a})] \cup f^{-1}[(\sqrt{a}, +\infty)]. \tag{13.125}$$

Letting ω be arbitrary, we find the true equivalences

$$\begin{aligned}
\omega \in (f \cdot f)^{-1}[(a,+\infty)] &\Leftrightarrow (f \cdot f)(\omega) \in (a,+\infty) \\
&\Leftrightarrow f(\omega) \cdot_{\mathbb{R}} f(\omega) >_{\mathbb{R}} a \\
&\Leftrightarrow f(\omega)^2 >_{\mathbb{R}} a \\
&\Leftrightarrow \sqrt{f(\omega)^2} >_{\mathbb{R}} \sqrt{a} \\
&\Leftrightarrow |f(\omega)| >_{\mathbb{R}} \sqrt{a} \\
&\Leftrightarrow f(\omega) <_{\mathbb{R}} -\sqrt{a} \vee f(\omega) >_{\mathbb{R}} \sqrt{a} \\
&\Leftrightarrow f(\omega) \in (-\infty, -\sqrt{a}) \vee f(\omega) \in (\sqrt{a}, +\infty) \\
&\Leftrightarrow \omega \in f^{-1}[(-\infty, -\sqrt{a}) \cup (\sqrt{a}, +\infty)] \\
&\Leftrightarrow f^{-1}[(-\infty, -\sqrt{a})] \cup f^{-1}[(\sqrt{a}, +\infty)]
\end{aligned}$$

by using the definition of an inverse image, the Pointwise multiplication of functions in connection with the definition of a real left-open and right-unbounded interval, the notation for squares, (8.428) in connection with the nonnegativity of the square $f(\omega)^2$ and of a due to (6.243) and the current case assumption, (8.429), (8.413), the definitions of a real left-unbounded and right-open and of a real left-open and right-unbounded interval, the definition of the union of two sets, and finally (3.761). Since ω is arbitrary, we may infer from the truth of the preceding inequalities the truth of the equality (13.125).

Since f is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, we obtain

$$f^{-1}[(-\infty, -\sqrt{a})] \in \mathcal{A},$$
$$f^{-1}[(\sqrt{a}, +\infty)] \in \mathcal{A}$$

with (13.82) and (13.85), respectively. As the σ-algebra $\mathcal{A}$ is an algebra of sets (see Proposition 11.30), the union of the two inverse images is then also contained in $\mathcal{A}$. Therefore, the left-hand side of (13.125) is in $\mathcal{A}$ also in the second case. We thus completed the proof of the implication in (13.122); here, I is arbitrary, so that the universal sentence (13.122) holds. According to (13.85), the function $f \cdot f$ is therefore measurable with respect to $\mathcal{A}$ and $\mathcal{B}$. Since f was also arbitrary, we therefore conclude that e) holds, too.

To establish f), we take two arbitrary functions f and g that are measurable with respect to $\mathcal{A}$ and $\mathcal{B}$. We use Proposition 10.18 to write the (pointwise) product $f \cdot g$ in terms of a scalar multiplication and the difference of two Binomial Formulae as

$$f \cdot g = \frac{1}{4} \cdot \left[(f+g)^2 - (f-g)^2\right].$$

Here, $f \pm g$ is in $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ due to d), so that $(f \pm g)^2$ is in $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ by virtue of e). Consequently, the difference $(f+g)^2 - (f-g)^2$ is in $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ because of d). Finally, the scalar multiplication of $\frac{1}{4}$ and that difference is in $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$ due to a). Then, $f \cdot g$ – being identical with that scalar multiplication – is also in $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$. Since f and g were arbitrary, we therefore conclude that e) is true.

Concerning g), we let f, g arbitrary elements of $(\mathbb{R}, \mathcal{B})^{(\Omega,\mathcal{A})}$. To prove that $f \curlyvee g$ is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, we apply the Measurability Criterion for real functions (13.84) and demonstrate accordingly the truth of the universal sentence

$$\forall I\,(I \in \{(-\infty, b] : b \in \mathbb{R}\} \Rightarrow (f \curlyvee f)^{-1}[(-\infty, b]] \in \mathcal{A}). \tag{13.126}$$

Letting $I \in \{(-\infty, b] : b \in \mathbb{R}\}$ be arbitrary, we thus find a particular $b \in \mathbb{R}$ with $I = (-\infty, b]$. Next, we apply the Equality Criterion for sets to prove

$$(f \curlyvee g)^{-1}[(-\infty, b]] = f^{-1}[(-\infty, b]] \cap g^{-1}[(-\infty, b]]. \tag{13.127}$$

Taking an arbitrary ω, we obtain the true equivalences

$$\begin{aligned}
\omega \in (f \curlyvee g)^{-1}[(-\infty, b]] \;&\Leftrightarrow\; (f \curlyvee g)(\omega) \in (-\infty, b] \\
&\Leftrightarrow\; f(\omega) \sqcup_{\mathbb{R}} g(\omega) \in (-\infty, b] \\
&\Leftrightarrow\; \overset{\leq_{\mathbb{R}}}{\sup}\{f(\omega), g(\omega)\} \leq_{\mathbb{R}} b \\
&\Leftrightarrow\; f(\omega) \leq_{\mathbb{R}} b \,\wedge\, g(\omega) \leq_{\mathbb{R}} b \\
&\Leftrightarrow\; \omega \in f^{-1}[(-\infty, b]] \,\wedge\, \omega \in g^{-1}[(-\infty, b]] \\
&\Leftrightarrow\; \omega \in f^{-1}[(-\infty, b]] \cap g^{-1}[(-\infty, b]]
\end{aligned}$$

by applying the definition of an inverse image, the definition of the join (pointwise supremum) $\curlyvee$ on $\mathbb{R}^\Omega$ according to 5.44, the definition of the join $\sqcup$ on $\mathbb{R}$ according to (8.113) together with the definition of a real left-unbounded and right-closed interval, Proposition 3.104, again the definition of a real left-unbounded and right-closed interval as well as of an inverse image, and finally the definition of the intersection of two sets. Since ω is arbitrary, we may infer from these equivalences the truth of the equation (13.127).

Here, since f and g are by assumption measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, we have that the inverse images $f^{-1}[(-\infty, b]]$ and $g^{-1}[(-\infty, b]]$ are both elements of $\mathcal{A}$ (applying again the aforementioned Measurability Criterion for real functions). Consequently, as the σ-algebra $\mathcal{A}$ is a π-system (see Proposition 11.30) and therefore closed under pairwise intersections, the right-hand side of the equation (13.127) – thus also the left-hand side – is in $\mathcal{A}$. As I was arbitrary, we therefore conclude that (13.126) is true, so that $f \curlyvee g$ is indeed measurable with respect to $\mathcal{A}$ and $\mathcal{B}$. The measurability of $f \curlywedge g$ can be established similarly. Since f and g were also arbitrary, we further conclude that g) holds.

In the previous proofs of a) – g), $(\Omega, \mathcal{A})$ was initially arbitrary, so that the entire lemma evidently holds. □

Exercise 13.11. Establish the second part of (13.110).

(Hint: Apply (13.84) and Exercise 3.45.)

Notation 13.1. For brevity of expressions, we introduce the notations

$$\mathcal{M} = \mathcal{M}(\Omega, \mathcal{A}) = (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}. \tag{13.128}$$

Note 13.2. Since every measurable function $f : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is by definition a real function on Ω, the inclusion

$$\mathcal{M} \subseteq \mathbb{R}^{\Omega} \tag{13.129}$$

holds. This subset is nonempty because the proof of Lemma 13.32a) showed that the real constant function with value zero is an element of it. Forming the restrictions of the vector addition and the scalar multiplication on $\mathbb{R}^{\Omega}$ to the Cartesian products $\mathcal{M} \times \mathcal{M}$ and $\mathbb{R} \times \mathcal{M}$, respectively, we may define the ordered triple $(\mathcal{M}, +_{\mathcal{M}}, \cdot_{\mathcal{M}})$. In view of Lemma 13.32a) and d), this triple satisfies the Vector Subspace Criterion and constitutes therefore a vector subspace of $(\mathbb{R}^{\Omega}, +, \cdot)$.

Definition 13.7 (Vector space of measurable real functions). We call

$$(\mathcal{M}, +_{\mathcal{M}}, \cdot_{\mathcal{M}}) \tag{13.130}$$

the *vector space of measurable real functions* on $(\Omega, \mathcal{A})$.

To be expanded!

Chapter 14.

Sets of Lebesgue-Integrable Functions

14.1. Simple Functions

Definition 14.1 (Simple function). We say for any measurable space $(\Omega, \mathcal{A})$ that an element s of $(\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ is a *simple function* on $(\Omega, \mathcal{A})$ iff the range of s is a nonempty and finite set.

Corollary 14.1. *It is true for any measurable space $(\Omega, \mathcal{A})$ that there exists a unique set $\mathcal{S}$ consisting of all the simple functions s on $(\Omega, \mathcal{A})$.*

Proof. Letting $(\Omega, \mathcal{A})$ be an arbitrary measurable space, we may evidently apply the Axiom of Specification and the Equality Criterion for sets to establish the unique existence of a set $\mathcal{S}$ such that

$$\forall s \, (s \in \mathcal{S} \Leftrightarrow [s \in \mathcal{M} \, \wedge \, s \text{ is a simple function}]). \tag{14.1}$$

This is then true for any $(\Omega, \mathcal{A})$. □

Definition 14.2 (Set of simple functions on $(\Omega, \mathcal{A})$). We call the set

$$\mathcal{S}(\Omega, \mathcal{A}) = \mathcal{S} \tag{14.2}$$

the *set of simple functions* on $(\Omega, \mathcal{A})$.

Note 14.1. According to (14.1), $s \in \mathcal{S}$ implies $s \in \mathcal{M}$ for any s, so that the inclusion

$$\mathcal{S}(\Omega, \mathcal{A}) \; \subseteq \; \mathcal{M}(\Omega, \mathcal{A}) \quad [\subseteq \mathcal{R}^{\Omega}]. \tag{14.3}$$

follows to be true by definition of a subset (recalling (13.128)).

Exercise 14.1. Show for any measurable space $(\Omega, \mathcal{A})$ that an element s of $(\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ is a simple function iff

$$\exists n \, (n \in \mathbb{N}_+ \, \wedge \, |\mathrm{ran}(s)| = n). \tag{14.4}$$

(Hint: Use (4.239), (3.118), (2.307) and (4.527).)

Note 14.2. The fact that a simple function s has a (nonempty finite range allows us to use any denumeration $\alpha : \{1, \ldots, n\} \rightleftarrows \operatorname{ran}(s)$ (with $n \in \mathbb{N}_+$) to list the n elements of the range in an ordered form by $(\alpha_1, \ldots, \alpha_n)$.

Note 14.3. According to Note 13.1, any constant function $f_c : \Omega \to \mathbb{R}$ is an element of $(\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$. If Ω is nonempty, then such a constant function is a surjection from Ω to $\{c\}$, as shown by (3.638). Thus, the range of such a function f_c is given by $\{c\}$, which singleton is a finite set with cardinality $|\operatorname{ran}(g_c)| = 1$, according (4.528). Since $1 \in \mathbb{N}_+$ also holds, we now see that every real function with nonempty domain constitutes a simple function by virtue of Exercise 14.1.

Proposition 14.2. *For any sets Ω and any subset $A \subseteq \Omega$, it is true that there exists a unique function $\chi_A : \Omega \to \{0, 1\}$ such that*

$$\forall \omega \, (\omega \in \Omega \Rightarrow ([\omega \in A \Rightarrow \chi_A(\omega) = 1] \wedge [\omega \notin A \Rightarrow \chi_A(\omega) = 0])). \qquad (14.5)$$

Proof. Letting Ω and A be arbitrary sets such that A is included in Ω, we apply Function definition by replacement to establish χ_A. For this purpose, we prove the universal sentence

$$\forall \omega \, (\omega \in \Omega \Rightarrow \exists! y \, ([\omega \in A \Rightarrow y = 1] \wedge [\omega \notin A \Rightarrow y = 0])), \qquad (14.6)$$

letting ω be arbitrary and assuming $\omega \in \Omega$ to be true. Regarding the existential part, we consider the two cases $\omega \in A$ and $\omega \notin A$.

In the first case $\omega \in A$, we notice that the implication $\omega \in A \Rightarrow y = 1$ becomes true when we replace the variable y by the constant 1, because the antecedent and the consequent are then both true. Furthermore, the implication $\omega \notin A \Rightarrow y = 0$ becomes then also true, as its antecedent is false according to the current case assumption. Thus, there exists a constant y such that the conjunction in (14.6) is true.

In the second case $\omega \notin A$, the implication $\omega \in A \Rightarrow y = 1$ becomes true after replacing y by 0, its antecedent being false in view of the current case assumption. In addition, the other implication $\omega \notin A \Rightarrow y = 0$ becomes then also a true sentence, as both the antecedent and the consequent are true in this case. Thus, the existential part holds also in the second case.

Concerning the uniqueness part, we now take arbitrary y and y' such that

$$[\omega \in A \Rightarrow y = 1] \wedge [\omega \notin A \Rightarrow y = 0]$$
$$[\omega \in A \Rightarrow y' = 1] \wedge [\omega \notin A \Rightarrow y' = 0]$$

hold, and we show that $y = y'$ follows to be true. To do this, we consider the same two cases as before. The first case $\omega \in A$ evidently implies $y = 1$ and

$y' = 1$, and the second case $\omega \notin A$ implies $y = 0$ as well as $y' = 0$ with the preceding implications. We thus obtain via substitutions $y = y'$ in any case, so that the proof of the uniquely existential sentence in (14.6) is complete. As ω was arbitrary, we may therefore conclude that the universal sentence (14.6) is true, which in turn implies the unique existence of a function χ_A with domain Ω such that (14.5).

It now remains for us to prove that $\{0,1\}$ is a codomain of χ_A, i.e. that $\mathrm{ran}(\chi_A) \subseteq \{0,1\}$ holds. For this purpose, we verify

$$\forall y\,(y \in \mathrm{ran}(\chi_A) \Rightarrow y \in \{0,1\}), \tag{14.7}$$

letting $\bar{y}$ be arbitrary and assuming $\bar{y} \in \mathrm{ran}(\chi_A)$ to be true. Then, by definition of a range, there exists a constant, say $\bar{\omega}$, such that $(\bar{\omega}, \bar{y}) \in \chi_A$ holds. We already established χ_A as a function (with domain $\mathrm{dom}(\chi_A) = \Omega$), so that we may write the preceding finding also as $\bar{y} = \chi_A(\bar{\omega})$. Furthermore, $(\bar{\omega}, \bar{y}) \in \chi_A$ shows that there exists a constant y with $(\bar{\omega}, y) \in \chi_A$, so that $\bar{\omega} \in \Omega$ $[= \mathrm{dom}(\chi_A)]$ follows to be true by definition of a domain. Then, the first case $\bar{\omega} \in A$ gives $[\bar{y} =]\ \chi_A(\bar{\omega}) = 1$ (so that $y = 0 \vee y = 1$ is also true), and the second case $\bar{\omega} \notin A$ yields $[\bar{y} =]\ \chi_A(\bar{\omega}) = 0$ (so that $y = 0 \vee y = 1$ is again true), according to (14.5). This disjunction implies $y \in \{0,1\}$ by definition of a pair, as desired. Since $\bar{y}$ is arbitrary, we may therefore conclude that the universal sentence (14.7) holds, which then further implies the desired inclusion $\mathrm{ran}(\chi_A) \subseteq \{0,1\}$ by definition of a subset.

We thus established χ_A as a function from Ω to $\{0,1\}$ with values determined by (14.5), and as the sets Ω and A were initially arbitrary, we may infer from this the truth of the stated proposition. □

Note 14.4. The preceding proof clearly shows that every element of the range of a characteristic function is a nonnegative real number, and therefore a real number. Thus, the range is not only included in the pair $\{0,1\}$, but also in $\mathbb{R}_+^0$ and $\mathbb{R}$. Thus, we may write

- $\chi_A : \Omega \to \{0,1\}$, or
- $\chi_A : \Omega \to \mathbb{R}_+^0$, or
- $\chi_A : \Omega \to \mathbb{R}$.

The last possibility is the usual choice.

Definition 14.3 (Characteristic function). For any set Ω and any subset $A \subseteq \Omega$ we call the function

$$\chi_A : \Omega \to \mathbb{R}, \qquad \omega \mapsto \chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \tag{14.8}$$

the *characteristic function* of A (on Ω).

Corollary 14.3. *It is true for any measurable space* $(\Omega, \mathcal{A})$ *that*

a) the characteristic function $\chi_\emptyset$ *of the empty set on* Ω *is the constant function* f_0 *on* Ω *with value* 0.

b) the characteristic function χ_Ω *of* Ω *on* Ω *is the constant function* f_1 *on* Ω *with value* 1.

Proof. We let $(\Omega, \mathcal{A})$ be an arbitrary measurable space. Noting that $\chi_\emptyset$, χ_Ω, f_0 and f_1 are all functions on Ω, we use the Equality Criterion for functions to prove $\chi_\emptyset = f_0$ and $\chi_\Omega = f_1$. Letting $\omega \in \Omega$ be arbitrary and noting that $\omega \notin \emptyset$ holds by definition of the empty set, we find

$$\chi_\emptyset(\omega) = 0 = f_0(\omega)$$
$$\chi_\Omega(\omega) = 1 = f_1(\omega)$$

with the definition of a characteristic function and (3.534). Since ω was arbitrary, we may therefore conclude that the equations $\chi_\emptyset = f_0$ and $\chi_\Omega = f_1$ are indeed true. As $(\Omega, \mathcal{A})$ was initially also arbitrary, we may now further conclude that the proposed universal sentences hold. □

Exercise 14.2. Show for any set Ω and any subset $A \subseteq \Omega$ that

a) any value of the characteristic function of A on Ω is 0 or 1, i.e.

$$\forall \omega \, (\omega \in \Omega \Rightarrow [\chi_A(\omega) = 0 \lor \chi_A(\omega) = 1]). \tag{14.9}$$

(Hint: Apply a proof by cases based according to (14.8).)

b) any value of the characteristic function of A on Ω is 1 iff the argument is in A, i.e.

$$\forall \omega \, (\omega \in \Omega \Rightarrow [\chi_A(\omega) = 1 \Leftrightarrow \omega \in A]). \tag{14.10}$$

(Hint: Apply a proof by contradiction using Corollary 7.27.)

c) any value of the characteristic function of A on Ω is 0 iff the argument is in the complement of A, i.e.

$$\forall \omega \, (\omega \in \Omega \Rightarrow [\chi_A(\omega) = 0 \Leftrightarrow \omega \in A^c]). \tag{14.11}$$

d) the inverse image of the singleton formed by the real number 1 under the characteristic function of A is identical with A, that is,

$$\chi_A^{-1}[\{1\}] = A. \tag{14.12}$$

(Hint: Apply the Equality Criterion for sets with (2.169) and (14.10).)

e) the inverse image of any Borel set B of $\mathbb{R}$ under the characteristic function of A is Ω, A, the complement of A or empty, according to the implications

$$\forall B\,(B \in \mathcal{B} \Rightarrow [(0 \in B \,\wedge\, 1 \in B) \Rightarrow \chi_A^{-1}[B] = \Omega]), \tag{14.13}$$

$$\forall B\,(B \in \mathcal{B} \Rightarrow [(0 \in B \,\wedge\, 1 \notin B) \Rightarrow \chi_A^{-1}[B] = A^c]), \tag{14.14}$$

$$\forall B\,(B \in \mathcal{B} \Rightarrow [(0 \notin B \,\wedge\, 1 \in B) \Rightarrow \chi_A^{-1}[B] = A]), \tag{14.15}$$

$$\forall B\,(B \in \mathcal{B} \Rightarrow [(0 \notin B \,\wedge\, 1 \notin B) \Rightarrow \chi_A^{-1}[B] = \emptyset]). \tag{14.16}$$

Lemma 14.4 (Measurability Criterion for characteristic functions). *For any measurable space $(\Omega, \mathcal{A})$ and any subset $A \subseteq \Omega$, the characteristic function of A is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$ iff A is a measurable set of $\mathcal{A}$, that is,*

$$\chi_A \in (\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})} \;\Leftrightarrow\; A \in \mathcal{A}. \tag{14.17}$$

Proof. We let $(\Omega, \mathcal{A})$ be an arbitrary measurable space and A an arbitrary subset of Ω. To prove the first part ('$\Rightarrow$') of the equivalence, we assume that $\chi_A : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$, which means by definition that $\chi_A^{-1}[B] \in \mathcal{A}$ for any $B \in \mathcal{B}$; since $\{1\} \in \mathcal{B}$ holds in view of (11.1015), we therefore find $\chi_A^{-1}[\{1\}] \in \mathcal{A}$ where $\chi_A^{-1}[\{1\}] = A$ according to (14.12). Thus, $A \in \mathcal{A}$ holds as desired.

To prove '$\Leftarrow$' we now assume $A \in \mathcal{A}$ and show that this implies $\chi^{-1}[B] \in \mathcal{A}$ for any $B \in \mathcal{B}$. Letting $B \in \mathcal{B}$ be arbitrary, we recall that $\mathcal{B}$ is a σ-algebra on $\mathbb{R}$, so that B is a subset of $\mathbb{R}$. We may therefore consider the two cases $0 \in B$ and $0 \notin B$, each with the two further subcases $1 \in B$ and $1 \notin B$. According to (14.13) – (14.16), we obtain then for the inverse images of B under χ_A the corresponding sets Ω, A^c, A and $\emptyset$. All of these sets are elements of $\mathcal{A}$ in view of Property 2 of a σ-algebra on Ω, the current assumption $A \in \mathcal{A}$, Property 4 of a σ-algebra, and (11.255). These findings prove $\chi^{-1}[B] \in \mathcal{A}$, and as B was arbitrary, we may therefore conclude that χ_A is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$.

We thus completed the proof of the equivalence, and since $(\Omega, \mathcal{A})$ and A were initially arbitrary, we may conclude that the lemma holds. □

Lemma 14.5. *It is true for any measurable space $(\Omega, \mathcal{A})$ with $\Omega \neq \emptyset$ and for any measurable set A of $\mathcal{A}$ that the characteristic function of A is simple.*

Proof. We let $(\Omega, \mathcal{A})$ be an arbitrary measurable space and A an arbitrary set such that $\Omega \neq \emptyset$ and $A \in \mathcal{A}$ are satisfied. Thus, χ_A is an element of $(\mathbb{R}, \mathcal{B})^{(\Omega, \mathcal{A})}$ according to Lemma 14.4. We already know from Proposition 14.2 that $\{0, 1\}$ is a codomain of χ_A, which means that the inclusion $\operatorname{ran}(\chi_A) \subseteq \{0, 1\}$ holds. The latter implies with (2.26) that

$\mathrm{ran}(\chi_A) \subset \{0,1\}$ or $\mathrm{ran}(\chi_A) = \{0,1\}$ is true. We now use this disjunction to prove $|\mathrm{ran}(\chi_A)| \leq_{\mathbb{N}} 2$ by cases. In the first case $\mathrm{ran}(\chi_A) \subset \{0,1\}$, we obtain

$$|\mathrm{ran}(\chi_A)| <_{\mathbb{N}} |\{0,1\}| \quad [= 2]$$

with (4.553) and (4.506). Clearly, this implies that $|\mathrm{ran}(\chi_A)| \leq_{\mathbb{N}} 2$ with the Characterization of an induced irreflexive partial ordering. In the second case $\mathrm{ran}(\chi_A) = \{0,1\}$, we find

$$|\mathrm{ran}(\chi_A)| = |\{0,1\}| = 2$$

via substitution, so that $|\mathrm{ran}(\chi_A)| \leq_{\mathbb{N}} 2$ follows evidently to be true again. Recalling that any cardinality is by definition a natural number, this inequality further implies $|\mathrm{ran}(\chi_A)| \in \{0,\ldots,2\}$ with (4.180). Let us observe now that the initial assumption $\Omega \neq \emptyset$ implies $\mathrm{ran}(\chi_A) \neq \emptyset$ with (3.119), and this negation yields $|\mathrm{ran}(\chi_A)| \neq 0$ with (4.527). The latter also gives us $|\mathrm{ran}(\chi_A)| \notin \{0\}$ with (2.169), and therefore in conjunction with the previous finding $|\mathrm{ran}(\chi_A)| \in \{0,\ldots,2\}$

$$|\mathrm{ran}(\chi_A)| \in \{0,\ldots,2\} \setminus \{0\} \quad [= \{1,\ldots,2\} = \{1,2\}]$$

by definition of a set difference, using also (4.244) and (4.242). According to the definition of pair, we thus have $|\mathrm{ran}(\chi_A)| = 1$ or $|\mathrm{ran}(\chi_A)| = 2$. In any case, there exists a positive natural number n such that $|\mathrm{ran}(\chi_A)| = n$ is satisfied, so that χ_A turns out to be a simple function in view of Exercise 14.4. Since $(\Omega, \mathcal{A})$ and A were initially arbitrary, we may therefore conclude that the lemma holds. □

Proposition 14.6. *The following equivalence holds for any set Ω and any sequence $(A_i \,|\, i \in \{1,\ldots,n\})$ of sets in $\mathcal{P}(\Omega)$.*

$$\chi_{\bigcup_{i=1}^n A_i} = \sum_{i=1}^n \chi_{A_i} \;\Leftrightarrow\; \forall j,k\,([j,k \in \{1,\ldots,n\} \,\wedge\, j \neq k] \Rightarrow A_j \cap A_k = \emptyset). \tag{14.18}$$

Proof. We let Ω be an arbitrary set, n an arbitrary positive natural number, and $(A_i \,|\, i \in \{1,\ldots,n\})$ an arbitrary sequence of subsets of Ω. Since χ_{A_i} is the uniquely determined characteristic function of A_i for every $i \in \{1,\ldots,n\}$, we may evidently apply Function definition by replacement of establish the n-tuple $(\chi_{A_i} \,|\, i \in \{1,\ldots,n\})$ with codomain $[\mathbb{R}_+^0]^\Omega$. Since that codomain defines the semiring of nonnegative real functions on Ω, the pointwise addition on $\mathbb{R}^\Omega$ and the (constant) zero function g_0 give rise to the n-fold addition on $[\mathbb{R}_+^0]^\Omega$, so that the sum $\sum_{i=1}^n \chi_{A_i}$ is defined. We also note that the given n-tuple of characteristic functions gives rise to the n-tuple $(\chi_{A_i}(\omega) \mid i \in \{1, \ldots, n\})$ for any $\omega \in \Omega$ in view of Proposition 3.231.

To prove the first part ('$\Rightarrow$') of the equivalence, we assume the stated equation on the left-hand side, we let $j, k \in \{1, \ldots, n\}$ be arbitrary with $j \neq k$, and show that assuming $A_j \cap A_k \neq \emptyset$ implies a contradiction. The latter assumption means, in view of (2.41) that there exists an element of Ω, say $\bar{\omega}$, such that $\bar{\omega} \in A_j \cap A_k$, i.e. $\bar{\omega} \in A_j \wedge \bar{\omega} \in A_k$ with (2.55). On the one hand, $\bar{\omega} \in A_j$ evidently implies $\bar{\omega} \in \bigcup_{i=1}^n A_i$ with the Characterization of the union of a family of sets, so that $\chi_{\bigcup_{i=1}^n A_i}(\bar{\omega}) = 1$ follows to be true by definition of the characteristic function of $\bigcup_{i=1}^n A_i$. On the other hand, $\bar{\omega} \in A_j$ and $\bar{\omega} \in A_k$ imply $\chi_{A_j}(\bar{\omega}) = 1$ as well as $\chi_{A_k}(\bar{\omega}) = 1$ by definition of the characteristic function, and therefore $\chi_{A_j}(\bar{\omega}) +_{\mathbb{R}_+^0} \chi_{A_k}(\bar{\omega}) = 2$. Let us observe here that $\bar{\omega} \in \Omega$ and the assumed equation $\chi_{\bigcup_{i=1}^n A_i} = \sum_{i=1}^n \chi_{A_i}$ give us

$$[1 =] \quad \chi_{\bigcup_{i=1}^n A_i}(\bar{\omega}) = \left(\sum_{i=1}^n \chi_{A_i} \right)(\bar{\omega}) = \sum_{i=1}^n \chi_{A_i}(\bar{\omega}) \tag{14.19}$$

with the Equality Criterion for functions and Proposition 5.112; thus, $\sum_{i=1}^n \chi_{A_i}(\bar{\omega}) = 1$ is true. Recalling that $\mathbb{R}_+^0$ defines an ordered elementary domain, we now apply Proposition 5.120, which yields the inequality

$$[1 <_{\mathbb{R}_+^0} 2 =] \quad \chi_{A_j}(\bar{\omega} +_{\mathbb{R}_+^0} \chi_{A_k}(\bar{\omega} \leq_{\mathbb{R}_+^0} \sum_{i=1}^n \chi_{A_i}(\bar{\omega}),$$

so that $1 <_{\mathbb{R}_+^0} \sum_{i=1}^n \chi_{A_i}(\bar{\omega})$ follows to be true with the Transitivity Formula for $<$ and $\leq$. This inequality in turn implies $\sum_{i=1}^n \chi_{A_i}(\bar{\omega}) \neq 1$ with the Characterization of comparability with respect to the linear ordering $<_{\mathbb{R}_+^0}$, and this negation contradicts the previous finding that the sum equals 1. Since j and k were arbitrary, we may therefore conclude that the first part of the equivalence (14.18) holds.

To prove the second part ('$\Leftarrow$') of the proposed equivalence, we now assume that $A_i \cap A_j = \emptyset$ holds for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and we apply the Equality Criterion for functions to prove the equation on the left-hand side of that equivalence. For this purpose, we let $\omega \in \Omega$ be arbitrary and consider the two cases $\omega \in \bigcup_{i=1}^n A_i$ and $\omega \notin \bigcup_{i=1}^n A_i$ to prove

$$\chi_{\bigcup_{i=1}^n A_i}(\omega) = \sum_{i=1}^n \chi_{A_i}(\omega) \tag{14.20}$$

In the first case that $\omega \in \bigcup_{i=1}^n A_i$ is true, we have on the one hand $\chi_{\bigcup_{i=1}^n A_i}(\omega) = 1$ by definition of the characteristic function; on the other hand, due to the Characterization of the union of a family of sets, there exists a particular constant $I \in \{1, \ldots, n\}$ such that $\omega \in A_I$. The latter implies $\chi_{A_I}(\omega) = 1$ by definition of a characteristic function. We now establish the universal sentence

$$\forall i \, (i \in \{1, \ldots, n\} \setminus \{I\} \Rightarrow \chi_{A_i}(\omega) = 0). \tag{14.21}$$

Letting $i \in \{1, \ldots, n\} \setminus \{I\}$ be arbitrary, we find with the definition of a set difference $i \in \{1, \ldots, n\}$ and $i \notin \{I\}$, where the latter implies $i \neq I$ with (2.169). We therefore obtain $A_i \cap A_I = \emptyset$ with the assumed antecedent of '$\Leftarrow$'. Then, the previously established $\omega \in A_I$ implies $\omega \notin A_i$, since $\neg \omega \notin A_i$ yields with the Double Negation Law $\omega \in A_i$ and therefore $\omega \in A_i \cap A_I$ with the definition of the intersection of two sets, so that evidently $A_i \cap A_I \neq \emptyset$, in contradiction to the fact $A_i \cap A_I = \emptyset$. Now, the finding $\omega \notin A_i$ implies $\chi_{A_i}(\omega) = 0$ (by definition of a characteristic function), which proves the implication in (14.21). As i was arbitrary, we may therefore conclude that the suggested universal sentence (14.21) holds indeed. In conjunction with the previous findings $I \in \{1, \ldots, n\}$ and $\chi_{A_I}(\omega) = 1$ this demonstrates the truth of the existential sentence

$$\exists j \, (j \in \{1, \ldots, n\} \wedge \chi_{A_j}(\omega) = 1 \wedge \forall k \, (k \in \{1, \ldots, n\} \setminus \{j\} \Rightarrow \chi_{A_k}(\omega) = 0)),$$

which implies $\sum_{i=1}^{n} \chi_{A_i}(\omega) = 1 \; [= \chi_{\bigcup_{i=1}^{n} A_i}(\omega)]$ with Exercise 5.48. Thus, (14.20) holds in the first case.

In the other case that $\omega \notin \bigcup_{i=1}^{n} A_i$ is true, we obtain $\chi_{\bigcup_{i=1}^{n} A_i}(\omega) = 0$ as well as $\chi_{A_i}(\omega) = 0$ for all $i \in \{1, \ldots, n\}$ (applying the Characterization of the union of a family of sets in connection with the Negation Law for existential conjunctions). This universal sentence gives $\sum_{i=1}^{n} \chi_{A_i}(\omega) = 0$ with (5.457) and therefore (14.20) via substitution. We Thus completed the proof by cases, and since ω is arbitrary in (14.20), the equation on the left-hand side of the proposed equivalence follows to be true. Thus, the equivalence (14.18) itself is true, and as Ω and $(A_i \,|\, i \in \{1, \ldots, n\})$ were initially arbitrary, we may therefore conclude that the proposition holds. □

Proposition 14.7. *The characteristic function of the intersection of any two subsets A, B of any set Ω equals the pointwise product of the characteristic functions of A and B (on Ω), that is,*

$$\forall \Omega, A, B \, (A, B \subseteq \Omega \Rightarrow \chi_{A \cap B} = \chi_A \cdot_{[\mathbb{R}_+^0]^\Omega} \chi_B). \tag{14.22}$$

Proof. We let Ω be an arbitrary set, A and B two arbitrary subsets of Ω, and we prove the stated equation by means of the Equality Criterion for functions, by showing that

$$\chi_{A\cap B}(\omega) = (\chi_A \cdot_{[\mathbb{R}^0_+]^\Omega} \chi_B)(\omega) \tag{14.23}$$

holds for any $\omega \in \Omega$. For this purpose, we let $\omega \in \Omega$ be arbitrary and consider the two cases $\omega \in A \cap B$ as well as $\omega \notin A \cap B$ to prove first

$$\chi_{A\cap B}(\omega) = \chi_A(\omega) \cdot_{\mathbb{R}^0_+} \chi_B(\omega) \tag{14.24}$$

In the first case, $\omega \in A \cap B$ implies on the one hand $\chi_{A\cap B}(\omega) = 1$ by definition of the characteristic function, and on the other hand the conjunction $\omega \in A \wedge \omega \in B$ by definition of the intersection two sets. The latter evidently results in $\chi_A(\omega) = 1$ and $\chi_A(\omega) = 1$, so that

$$\chi_{A\cap B}(\omega) = 1 = 1 \cdot_{\mathbb{R}^0_+} 1 = \chi_A(\omega) \cdot_{\mathbb{R}^0_+} \chi_B(\omega).$$

Consequently, we obtain (14.24) in the first case.

In the second case, $\omega \notin A \cap B$ implies on the one hand $\chi_{A\cap B}(\omega) = 0$ and on the other hand $\neg(\omega \in A \wedge \omega \in B)$. This negation implies with De Morgan's Law for the conjunction that $\omega \notin A$ or $\omega \notin B$ holds. We may use this disjunction to prove $\chi_A(\omega) \cdot_{\mathbb{R}^0_+} \chi_B(\omega) = 0$ by cases. If $\omega \notin A$, then $\chi_A(\omega) = 0$, so that

$$\chi_A(\omega) \cdot_{\mathbb{R}^0_+} \chi_B(\omega) = 0 \cdot_{\mathbb{R}^0_+} \chi_B(\omega) = 0.$$

If $\omega \notin B$, then $\chi_B(\omega) = 0$, with the consequence that

$$\chi_A(\omega) \cdot_{\mathbb{R}^0_+} \chi_B(\omega) = \chi_A(\omega) \cdot_{\mathbb{R}^0_+} 0 = 0.$$

Thus, the proof by cases is complete, and since $\chi_{A\cap B}(\omega) = 0$ is true, we obtain from the previous equations via substitution (14.24) also for the second case. In view of the definition of a pointwise multiplication of functions, we now find also (14.23), and since ω is arbitrary, we may infer from the truth of that equation the truth of the equation in (14.22). As Ω, A and B were initially also arbitrary, we may therefore conclude that the proposition holds as claimed. □

Proposition 14.8. *The following sentences are true for any measurable space* $(\Omega, \mathcal{A})$*, any simple function* s *on* $(\Omega, \mathcal{A})$*, and any denumeration* $\alpha : \{1, \ldots, n\} \rightleftarrows \mathrm{ran}(s)$ *of the range of* s*.*

a) The n*-tuple*

$$(A_j \,|\, j \in \{1, \ldots, n\}) = (s^{-1}[\{\alpha_j\}] \,|\, j \in \{1, \ldots, n\}) \tag{14.25}$$

of inverse images of the elements $\alpha_1, \ldots, \alpha_n$ *of the range of* s *exists in* $\mathcal{A}$ *and is a partition of* Ω*.*

b) s can be written as the linear combination of the characteristic functions of the inverse images A_1, …, A_n with corresponding coefficients α_1, …, α_n, that is,

$$s = \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j}. \tag{14.26}$$

Proof. We let $s : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ be an arbitrary simple function. As mentioned in Note 14.2, there exists then a particular denumeration $\alpha : \{1, \ldots, n\} \rightleftarrows \operatorname{ran}(s)$ of the (finite) range of s, which constitutes the n-tuple $(\alpha_1, \ldots, \alpha_n)$. In light of the fact that every singleton in $\mathbb{R}$ is also an element of $\mathcal{B}$ due to (11.1015), we see in particular that $\{\alpha_j\} \in \mathcal{B}$ for any $j \in \{1, \ldots, n\}$, so that $[A_j =]\ s^{-1}[\{\alpha_j\}] \in \mathcal{A}$ for all $j \in \{1, \ldots, n\}$ due to (13.64). We now let α be an arbitrary denumeration of the range of s. Concerning a), we may evidently apply Function definition by replacement to establish the n-tuple $(A_j \,|\, j \in \{1, \ldots, n\})$, whose range is then included in $\mathcal{A}$. We now prove that all of the inverse images $A_1, \ldots, A_n$ are pairwise disjoint, i.e.,

$$\forall i, j([i, j \in \{1, \ldots, n\} \wedge i \neq j] \Rightarrow A_i \cap A_j = \emptyset). \tag{14.27}$$

For this purpose, we let I and J be arbitrary sets, choose the contradiction $I = J \wedge I \neq J$, and prove the implication by contradiction, assuming $I, J \in \{1, \ldots, n\}$, $i \neq j$ and $A_i \cap A_j \neq \emptyset$ to be true. The latter implies the existence of a particular constant $\bar{\omega} \in A_I \cap A_J$ because of (2.42). By definition of the intersection of two sets, $\bar{\omega} \in A_I$ and $\bar{\omega} \in A_J$ are then both true. By definition of the n-tuple $(A_j \,|\, j \in \{1, \ldots, n\})$, this means that $\bar{\omega} \in s^{-1}[\{\alpha_I\}]$ and $\bar{\omega} \in s^{-1}[\{\alpha_J\}]$ hold. The definition of an inverse image gives us therefore $s(\bar{\omega}) \in \{\alpha_I\}$ and $s(\bar{\omega}) \in \{\alpha_J\}$. These findings can also be written as $s(\bar{\omega}) = \alpha_I$ and $s(\bar{\omega}) = \alpha_J$ by using (2.169). We thus obtain $\alpha_I = \alpha_J$ via substitution, with the consequence that $I = J$ since the bijection $\alpha : \{1, \ldots, n\} \rightleftarrows \operatorname{ran}(s)$ is by definition an injection. As $I \neq J$ is also true by assumption, we arrived at the desired contradiction. We thus proved that the terms of the family $(A_j)_{j \in \{1, \ldots, n\}}$ are pairwise disjoint.

It remains to show that $\bigcup_{j=1}^{n} A_j = \Omega$ holds. Applying for this purpose the Equality Criterion for sets, we let ω be arbitrary. On the one hand, if $\omega \in \bigcup_{j=1}^{n} A_j$ is assumed, it follows by definition of the union of a family of sets that $\omega \in A_j$ holds for some particular $j \in \{1, \ldots, n\}$. This element of the domain of the n-tuple $(A_j \,|\, j \in \{1, \ldots, n\})$ is associated with the term/inverse image $A_j = s^{-1}[\{\alpha_j\}]$ under the simple function $s : \Omega \to \mathbb{R}$. This inverse image A_j is a subset of Ω in view of Note 3.30. Thus, $\omega \in A_j$ implies $\omega \in \Omega$ with the definition of a subset. On the other hand, if $\omega \in \Omega$ is assumed, we see that ω is in the domain of the simple function s. Therefore, the associated function value $s(\omega)$ is evidently in the range of s. Since α is a function onto $\mathrm{ran}(s)$, that set is identical with range of α due to the definition of a surjection. We thus obtain $s(\omega) \in \mathrm{ran}(\alpha)$ via substitution. By definition of a range, there exists then a particular constant i such that $(i, s(\omega)) \in \alpha$ holds, which we may also write as $s(\omega) = \alpha_i$ by using the notation for functions and the notation for families. Here, $(i, s(\omega)) \in \alpha$ implies that i is an element of the domain $\{1, \ldots, n\}$ of α. As the preceding equation $s(\omega) = \alpha_i$ implies with the definition of an inverse image $\omega \in s^{-1}[\{\alpha_i\}]$ $[= A_i]$, we now see that there exists an element $j \in \{1, \ldots, n\}$ with $\omega \in A_j$. Consequently, $\omega \in \bigcup_{j=1}^{n} A_j$ holds by definition of the union of a family of sets. We thus proved that $\omega \in \bigcup_{j=1}^{m} A_j$ and $\omega \in \Omega$ are equivalent. Here, ω is arbitrary, so that $\bigcup_{j=1}^{m} A_j = \Omega$ follows to be true as well. Together with the pairwise disjointness of the terms of the family $(A_j)_{j \in \{1,\ldots,n\}}$, this shows that this n-tuple is a partition of Ω.

Concerning b), we first note that the characteristic functions $\chi_{A_1}, \ldots, \chi_{A_n}$ constitute elements of $\mathbb{R}^\Omega$, which set defines the real vector space of real functions on Ω. Furthermore, as the range of the simple function $s \in \mathbb{R}^\Omega$ is included in $\mathbb{R}$, the latter constitutes a codomain also of the n-tuple $\alpha = (\alpha_j \,|\, j \in \{1, \ldots, n\})$. Thus, the linear combination $\sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j}$ is defined and constitutes a vector, that is, an element of $\mathbb{R}^\Omega$. We now apply the Equality Criterion for functions to prove the equality of that linear combination and the simple function s. For this purpose, we let $\omega \in \Omega$ be arbitrary, and we show that the equality $s(\omega) = (\sum_{j=1}^{n} \alpha_j \chi_{A_j})(\omega)$ holds. To begin with, we recall from the proof of a) that $\omega \in \Omega$ implies $s(\omega) = \alpha_i$ and $\omega \in A_I$ for some particular constant $I \in \{1, \ldots, n\}$. The latter gives us $\chi_{A_I}(\omega) = 1$ by definition of the characteristic function of A_I, and therefore $\alpha_I \cdot_{\mathbb{R}} \chi_{A_I}(\omega) = \alpha_I$ with the fact that 1 is the neutral element of $\mathbb{R}$ with respect to the multiplication on $\mathbb{R}$. This equation demonstrates the truth of the existential sentence

$$\exists j \, (j \in \{1, \ldots, n\} \wedge \alpha_j \cdot_{\mathbb{R}} \chi_{A_j}(\omega) = \alpha_I). \tag{14.28}$$

Let us now establish the truth of the universal sentence

$$\forall k \, (k \in \{1, \ldots, n\} \setminus \{I\} \Rightarrow \alpha_k \cdot_{\mathbb{R}} \chi_{A_k}(\omega) = 0). \tag{14.29}$$

Letting k be arbitrary and assuming $k \in \{1, \ldots, n\} \setminus \{I\}$ to be true, we find $k \in \{1, \ldots, n\}$ and $k \neq I$ with the definition of a set difference and (2.169). Since $I \in \{1, \ldots, n\}$ also holds, it follows that $A_k \cap A_I = \emptyset$ since the terms of the family $(A_j \,|\, j \in \{1, \ldots, n\})$ are pairwise disjoint, as shown in a). We may now prove by contradiction that $\omega \notin A_k$ is true. Assuming for this purpose the negation of this negation to be true, we obtain by means of the Double Negation Law $\omega \in A_k$, and this implies with the previously found $\omega \in A_I$ and the definition of the intersection of two sets that $\omega \in A_k \cap A_I$ holds. This proves the existence of an element in $A_k \cap A_I$, so that $A_k \cap A_I \neq \emptyset$ is evidently true, in contradiction to the fact $A_k \cap A_I = \emptyset$. We thus completed the proof of $\omega \notin A_k$, and this yields $\chi_{A_k}(\omega) = 0$ by definition of the characteristic function of A_k. This gives us now $\alpha_k \cdot_{\mathbb{R}} \chi_{A_k}(\omega) = \alpha_k \cdot_{\mathbb{R}} 0 = 0$, which proves the implication in (14.29). Here, k is arbitrary, so that the universal sentence (14.29) follows to be true as well. Now, the conjunction of (14.28) and (14.29) further implies

$$[s(\omega) =] \quad \alpha_I = \sum_{j=1}^{n} \alpha_j \cdot_{\mathbb{R}} \chi_{A_j}(\omega) = \sum_{j=1}^{n} (\alpha_j \cdot \chi_{A_j})(\omega) = \left(\sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} \right)(\omega)$$

with Exercise 5.48, the definition of the scalar multiplication for functions and Proposition 5.112. Thus, we obtain the desired equality $s(\omega) = (\sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j})(\omega)$, in which ω is arbitrary, so that the functions s and $\sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j}$ are indeed identical.

Since $(\Omega, \mathcal{A})$ and s were initially arbitrary, we may therefore conclude that the proposition holds. □

The representation (14.26) of a simple function on $(\Omega, \mathcal{A})$ based on some denumeration of its range is not the only one.

Proposition 14.9. *It is true for any measurable space* $(\Omega, \mathcal{A})$*, any simple function* s *on* $(\Omega, \mathcal{A})$ *and any denumeration* $\alpha : \{1, \ldots, n\} \rightleftarrows \mathrm{ran}(s)$ *that there exists a representation of* s *as the linear combination*

$$s = \sum_{j=1}^{m} \beta_j \cdot \chi_{B_j}.$$

of characteristic functions χ_{B_1}, …, χ_{B_m} *for some* $m \in \mathbb{N}_+$ *with coefficients* β_1, …, $\beta_m \in \mathbb{R}$ *such that* $m \neq n$ *and such that* $(B_j \,|\, j \in \{1, \ldots, m\})$ *is a partition of* Ω *in* $\mathcal{A}$.

Proof. We let $(\Omega, \mathcal{A})$ be an arbitrary measurable space, s an arbitrary simple function on $(\Omega, \mathcal{A})$, n an arbitrary positive natural number, and $\alpha : \{1, \ldots, n\} \rightleftarrows \mathrm{ran}(s)$ an arbitrary denumeration of the range of s. We therefore obtain the partition $A = (A_j \,|\, j \in \{1, \ldots, n\})$ of Ω in $\mathcal{A}$ as defined by (14.25) and the representation

$$s = \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} \tag{14.30}$$

as shown by (14.26). Let us define now the unions $B = A \cup (n+1, \emptyset)$ and $\beta = \alpha \cup (n+1, 1)$. Since $n \in \mathbb{N}_+$ implies $n+1 \notin \{1, \ldots, n\}$ with (4.250), the preceding unions are functions with common domain $\{1, \ldots, n\} \cup \{n+1\} = \{1, \ldots, n+1\}$ due to Proposition 3.177 and Proposition 4.51. Thus, B and β are $(n+1)$-tuples evidently with $B_j = A_j$ and $\beta_j = \alpha_j$ for all $j \in \{1, \ldots, n\}$, and having the extra terms $B_{n+1} = \emptyset$ and $\beta_{n+1} = 1$. Therefore, the restricted n-tuple $(\beta_j \cdot \chi_{B_j} \,|\, j \in \{1, \ldots, n\})$ and the n-tuple $(\alpha_j \cdot \chi_{A_j} \,|\, j \in \{1, \ldots, n\})$ are evidently identical, with the consequence that

$$\sum_{j=1}^{n} \beta_j \cdot \chi_{B_j} = \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j}. \tag{14.31}$$

As the coefficients $\alpha_1, \ldots, \alpha_n$ and 1 are real numbers, we have $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{R}$. Furthermore, $n \neq n+1$ holds in view of (4.41), and $n+1 \in \mathbb{N}_+$ is true by virtue of (4.42). Moreover, we obtain the equations

$$\begin{aligned}
\sum_{j=1}^{n+1} \beta_j \cdot \chi_{B_j} &= \sum_{j=1}^{n} \beta_j \cdot \chi_{B_j} +_{\mathbb{R}^\Omega} \beta_{n+1} \cdot \chi_{B_{n+1}} \\
&= \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} +_{\mathbb{R}^\Omega} 1 \cdot \chi_\emptyset \\
&= \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} +_{\mathbb{R}^\Omega} 1 \cdot f_0 \\
&= \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} +_{\mathbb{R}^\Omega} f_0 \\
&= \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} \\
&= s
\end{aligned}$$

using (5.417), (14.31), Corollary 14.3a), the Cancellation Laws for the zero vector (which given here by the constant function with value 0, according to Note 10.9), the fact that the zero vector is the neutral element with respect to the vector addition (by definition of a vector space), and finally (14.30). Clearly, the restricted n-tuple $(B_j \,|\, j \in \{1, \ldots, n\})$ and the n-tuple $(A_j \,|\, j \in \{1, \ldots, n\})$ are also identical, so that

$$\bigcup_{j=1}^{n} B_j = \bigcup_{j=1}^{n} A_j. \tag{14.32}$$

Now, we obtain

$$\bigcup_{j=1}^{n+1} B_j = \left(\bigcup_{j=1}^{n} B_j\right) \cup B_{n+1} = \left(\bigcup_{j=1}^{n} B_j\right) \cup \emptyset = \bigcup_{j=1}^{n} B_j = \bigcup_{j=1}^{n} A_j = \Omega$$

by applying the Recursive evaluation of the union of a sequence of sets on an initial segment of $\mathbb{N}_+$, substitution based on $B_{n+1} = \emptyset$, (2.216), (14.32), and the fact that the n-tuple A constitutes a partition of Ω. It remains to show that the terms of the family B are pairwise disjoint. Letting $i, j \in \{1, \ldots, n+1\}$ be arbitrary such that $i \neq j$, we evidently have $i, j \in \{1, \ldots, n\} \cup \{n+1\}$, so that the disjunctions

$$\begin{aligned} &i \in \{1, \ldots, n\} \;\vee\; i \in \{n+1\} \\ &j \in \{1, \ldots, n\} \;\vee\; j \in \{n+1\} \end{aligned} \tag{14.33}$$

follow to be true by definition of the union of two sets. In case of $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$, we find $B_i \cap B_j = A_i \cap A_j = \emptyset$ by applying substitutions and the fact that the terms of the partition A are pairwise disjoint, by definition. If $j \in \{n+1\}$, we obtain $j = n+1$ with (2.169) and therefore

$$B_i \cap B_j = B_i \cap B_{n+1} = A_i \cap \emptyset = \emptyset$$

by applying substitutions and subsequently (2.62). In the other case $i \in \{n+1\}$, which evidently yields $i = n+1$, we may prove $j \neq n+1$ by contradiction. Assuming the negation of that negation to be true, so that $j = n+1$ follows to be true with the Double Negation Law, we obtain $i = j$ through substitution, in contradiction to the previous assumption $i \neq j$. We thus completed the proof of $j \neq n+1$, and this negation gives us evidently $j \notin \{n+1\}$, so that the first part $j \in \{1, \ldots, n\}$ of the disjunction (14.33) holds. Together with $i = n+1$, this implies now

$$B_i \cap B_j = B_{n+1} \cap B_j = \emptyset \cap A_j = \emptyset.$$

We thus found $B_i \cap B_j = \emptyset$ to be true in any case, and since i and j were initially arbitrary, we may infer from this equation the pairwise disjointness of the terms of B. Thus, $(B_j \,|\, j \in \{1, \ldots, n+1\})$ constitutes a partition of Ω. It remains to verify that $\mathcal{A}$ is a codomain of that $(n+1)$-tuple, i.e., that the range of B is a subset of the given σ-algebra. Letting Y be an arbitrary element of that range, it follows by definition that there exists a particular constant I with $(I, Y) \in B$. Noting that I is then in the domain $\{1, \ldots, n+1\}$ of B, we evidently have $I \in \{1, \ldots, n\}$ or $I = n+1$. On the one hand, $I \in \{1, \ldots, n\}$ gives $[Y =]\ B_I = A_I\ [\in \mathcal{A}]$, recalling that A is an n-tuple in $\mathcal{A}$. On the other hand, $I = n+1$ yields

$[Y =]\ B_I = B_{n+1} = \emptyset\ [\in \mathcal{A}]$, recalling (11.255). We thus have $Y \in \mathcal{A}$ for both cases, and as Y was arbitrary, we may therefore conclude that the range of B is indeed included in the given σ-algebra. Since $(\Omega, \mathcal{A})$ and s were initially also arbitrary, we may infer from the previous findings the truth of the stated proposition. □

Exercise 14.3. Prove for any measurable space $(\Omega, \mathcal{A})$, any simple function s on $(\Omega, \mathcal{A})$, any sequences $(\alpha_j \,|\, j \in \{1, \ldots, m\})$, $(\beta_k \,|\, k \in \{1, \ldots, n\})$ in $\mathbb{R}$ and any partitions $(A_j \,|\, j \in \{1, \ldots, m\})$, $(B_k \,|\, k \in \{1, \ldots, n\})$ of Ω in $\mathcal{A}$ satisfying

$$s = \sum_{j=1}^{m} \alpha_j \cdot \chi_{A_j} = \sum_{k=1}^{n} \beta_k \cdot \chi_{B_k} \tag{14.34}$$

that

$$\forall j, k\, ([j \in \{1, \ldots, m\} \wedge k \in \{1, \ldots, n\} \wedge A_j \cap B_k \neq \emptyset] \Rightarrow \alpha_j = \beta_k). \tag{14.35}$$

(Hint: Use some of the arguments of the proof of Proposition 14.8.)

We consider in the following in particular nonnegative simple functions.

Corollary 14.10. *It is true for any measurable space* $(\Omega, \mathcal{A})$ *that there exists a unique set* $\mathcal{S}_+^0$ *consisting of all the simple functions* s *on* $(\Omega, \mathcal{A})$ *whose range is included in* $\mathbb{R}_+^0$.

Proof. Letting $(\Omega, \mathcal{A})$ be an arbitrary measurable space, we may evidently apply the Axiom of Specification and the Equality Criterion for sets to establish the unique existence of a set $\mathcal{S}_+^0$ such that

$$\forall s\, (s \in \mathcal{S}_+^0 \Leftrightarrow [s \in \mathcal{S} \wedge \operatorname{ran}(s) \subseteq \mathbb{R}_+^0]). \tag{14.36}$$

□

Definition 14.4 (Set of nonnegative simple functions on $(\Omega, \mathcal{A})$). We call the set

$$\mathcal{S}_+^0(\Omega, \mathcal{A}) = \mathcal{S}_+^0 \tag{14.37}$$

the *set of nonnegative simple functions* on $(\Omega, \mathcal{A})$.

Note 14.5. Since $s \in \mathcal{S}_+^0$ implies $s \in \mathcal{S}$ for any s, we find the inclusion

$$\mathcal{S}_+^0(\Omega, \mathcal{A}) \subseteq \mathcal{S}(\Omega, \mathcal{A}) \quad [\subseteq \mathcal{M} \subseteq \mathcal{R}^{\Omega}], \tag{14.38}$$

recalling (14.3) and (13.128). In addition, $s \in \mathcal{S}_+^0$ implies with (14.36) that $\mathbb{R}_+^0$ is a codomain of s, so that $s \in [\mathbb{R}_+^0]^{\Omega}$ holds; thus, the inclusion

$$\mathcal{S}_+^0 \subseteq [\mathbb{R}_+^0]^{\Omega} \tag{14.39}$$

is also true.

Note 14.6. Since Note 14.2 showed that the elements of the range of a simple function can be listed by means of a denumeration $\alpha : \{1, \dots, n\} \rightleftarrows \operatorname{ran}(s)$ with $n \in \mathbb{N}_+$. In case of a nonnegative simple function, the inclusion $\operatorname{ran}(s) \subseteq \mathbb{R}_+^0$ holds, so that $\mathbb{R}_+^0$ is also a codomain of α, for which we can therefore write $\alpha : \{1, \dots, n\} \to \mathbb{R}_+^0$. According to Note 9.5, this constitutes also a nonnegative numerical function, so that we can also write $\alpha : \{1, \dots, n\} \to \overline{\mathbb{R}}_+^0$.

14.2. Lebesgue-Integrable Nonnegative Simple Functions

We now specify and subsequently define the integral of a nonnegative simple function on a given measure space.

Theorem 14.11. *For any measure space $(\Omega, \mathcal{A}, \mu)$ there exists a unique function I_μ on $\mathcal{S}_+^0(\Omega, \mathcal{A})$ such that*

$$\begin{aligned}
&\forall s\, (s \in \mathcal{S}_+^0 \Rightarrow \exists n, \alpha, A\, (n \in \mathbb{N}_+ \wedge \alpha : \{1, \dots, n\} \to \mathbb{R}_+^0 \\
&\quad \wedge A : \{1, \dots, n\} \to \mathcal{A} \wedge A \text{ is a partition of } \Omega \wedge s = \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} \\
&\quad \wedge I_\mu(s) = \sum_{j=1}^{n} \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j)))
\end{aligned} \tag{14.40}$$

Furthermore, I_μ is a function from $\mathcal{S}_+^0(\Omega, \mathcal{A})$ to $\overline{\mathbb{R}}_+^0$.

Proof. We let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary measure space, and we apply Function definition by replacement to establish I_μ. For this purpose, we prove

$$\begin{aligned}
&\forall s\, (s \in \mathcal{S}_+^0 \Rightarrow \exists! y\, (\exists n, \alpha, A\, (n \in \mathbb{N}_+ \wedge \alpha : \{1, \dots, n\} \to \mathbb{R}_+^0 \\
&\quad \wedge A : \{1, \dots, n\} \to \mathcal{A} \wedge A \text{ is a partition of } \Omega \wedge s = \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} \\
&\quad \wedge y = \sum_{j=1}^{n} \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j)))),
\end{aligned} \tag{14.41}$$

letting $s \in \mathcal{S}_+^0$ be arbitrary. We first establish the existential part of the uniquely existential sentence with respect to y. Let us recall from Note 14.6, there is – for some particular $\bar{n} \in \mathbb{N}_+$ – a particular denumeration $\bar{\alpha} : \{1, \dots, \bar{n}\} \to \operatorname{ran}(s)$ that can be written as $\bar{\alpha} : \{1, \dots, \bar{n}\} \to \mathbb{R}_+^0$.

According to Proposition 14.8, the given simple function can therefore be written as $s = \sum_{j=1}^{\bar{n}} \bar{\alpha}_j \chi_{\bar{A}_j}$ where $\bar{A}_1$, …, $\bar{A}_{\bar{n}}$ are the terms of the n-tuple $\bar{A} = (\bar{A}_j \,|\, j \in \{1,\dots,\bar{n}\})$ (in $\mathcal{A}$) of inverse images $\bar{A}_j = s^{-1}[\{\bar{\alpha}_j\}]$, and where $\bar{A}$ constitutes a partition of Ω. As indicated in Note 14.6, $\bar{\alpha}$ may be treated as a nonnegative numerical function, so that $j \in \{1,\dots,\bar{n}\}$ implies $\bar{\alpha}_j \in \overline{\mathbb{R}}_+^0$ for any j. Furthermore, since every measure is a nonnegative numerical function, we also have $\mu(\bar{A}_j) \in \overline{\mathbb{R}}_+^0$ for any $j \in \{1,\dots,\bar{n}\}$. Because the multiplication on $\overline{\mathbb{R}}_+^0$ constitutes a binary operation, the n-tuple $(\alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j) \,|\, j \in \{1,\dots,\bar{n}\})$ is evidently in $\overline{\mathbb{R}}_+^0$ as well. Consequently, the sum $\bar{y} = \sum_{j=1}^{\bar{n}} \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j)$ is defined, as mentioned in Note 9.11. These findings clearly demonstrate the truth of the existential part of the uniquely existential sentence in (14.41).

To establish the uniqueness part, we let y and y' be arbitrary such that there are particular constants $n, n^*, \alpha, \alpha^*, A, A^*$ such that

- $n, n^* \in \mathbb{N}_+$,
- $\alpha : \{1,\dots,n\} \to \mathbb{R}_+^0$ and $\alpha^* : \{1,\dots,n^*\} \to \mathbb{R}_+^0$,
- $A : \{1,\dots,n\} \to \mathcal{A}$ and $A^* : \{1,\dots,n^*\} \to \mathcal{A}$,
- A and A^* are partitions of Ω,
- $s = \sum\limits_{j=1}^{n} \alpha_j \cdot \chi_{A_j}$ and $s = \sum\limits_{j=1}^{n^*} \alpha_j^* \cdot \chi_{A_j^*}$,
- $y = \sum\limits_{j=1}^{n} \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j)$ and $y' = \sum\limits_{j=1}^{n^*} \alpha_j^* \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j^*)$

are all satisfied, and we demonstrate that $y = y'$ follows to be true from the previous assumptions and facts. Let us first establish the universal sentences

$$\forall j \,(j \in \{1,\dots,n\} \Rightarrow \mu(A_j) = \sum_{k=1}^{n^*} \mu(A_j \cap A_k^*)), \tag{14.42}$$

$$\forall k \,(k \in \{1,\dots,n^*\} \Rightarrow \mu(A_k^*) = \sum_{j=1}^{n} \mu(A_j \cap A_k^*)). \tag{14.43}$$

Letting $J \in \{1,\dots,n\}$ and $K \in \{1,\dots,n^*\}$ be arbitrary, we obtain the equations

$$\begin{aligned}\mu(A_J) &= \mu(A_J \cap \Omega) = \mu(A_J \cap \bigcup_{k=1}^{n^*} A_k^*) = \mu(\bigcup_{k=1}^{n^*} (A_J \cap A_k^*)) \\ &= \sum_{k=1}^{n^*} \mu(A_J \cap A_k^*), \\ \mu(A_K^*) &= \mu(A_K^* \cap \Omega) = \mu(A_K^* \cap \bigcup_{j=1}^{n} A_j) = \mu(\bigcup_{j=1}^{n} (A_K^* \cap A_j)) \\ &= \sum_{j=1}^{n} \mu(A_K^* \cap A_j)\end{aligned}$$

using (2.77) with the fact that A and A^* are n-tuples in $\mathcal{A}$ (so that evidently $A_J, A_K^* \in \mathcal{A}$ and therefore $A_J, A_K^* \subseteq \Omega$ by Property 1 of a σ-algebra on Ω), the previous assumptions that A^* and A are partitions of Ω, the Distributive Law for families of sets (3.822), and the finite additivity of the measure/content μ in connection with the fact that the families A and A^* of pairwise disjoint sets give rise to the families $(A_J \cap A_k^* \,|\, k \in \{1, \dots, n^*\})$ and $(A_K^* \cap A_k \,|\, k \in \{1, \dots, n\})$ also of pairwise disjoint sets (see Exercise 3.102). The latter are indeed families in $\mathcal{A}$ since the σ-$\mathcal{A}$, being a π-system according to Proposition 11.30, is closed under intersections of two sets. Thus, we find the equations $\mu(A_J) = \sum_{k=1}^{n^*} \mu(A_J \cap A_k^*)$ and $\mu(A_K^*) = \sum_{j=1}^{n} \mu(A_j \cap A_K^*)$ (applying the Commutative Law for the intersection of two sets). Since J and K were arbitrary, we may therefore conclude that the universal sentences (14.42) – (14.43) are true indeed. Consequently, we obtain

$$\begin{aligned}y &= \sum_{j=1}^{n} \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j) \\ &= \sum_{j=1}^{n} \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \sum_{k=1}^{n^*} \mu(A_j \cap A_k^*) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n^*} \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*)\end{aligned}$$

by means of (14.42) – (14.43) as well as (5.499), and with the same arguments

$$\begin{aligned}
y' &= \sum_{k=1}^{n^*} \alpha_k^* \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_k^*) \\
&= \sum_{k=1}^{n^*} \alpha_k^* \cdot_{\overline{\mathbb{R}}_+^0} \sum_{j=1}^{n} \mu(A_j \cap A_k^*) \\
&= \sum_{k=1}^{n^*} \sum_{j=1}^{n} \alpha_k^* \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n^*} \alpha_k^* \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*),
\end{aligned}$$

applying additionally the Interchange of two nested n-fold sums. The latter can be applied here since we may evidently define the matrix

$$\boldsymbol{X}' : \{1, \ldots, n\} \times \{1, \ldots, n^*\} \to \overline{\mathbb{R}}_+^0, \quad (j,k) \mapsto x'_{j,k} = \alpha_k^* \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*)$$

through Function definition by replacement. Defining also the matrix

$$\boldsymbol{X} : \{1, \ldots, n\} \times \{1, \ldots, n^*\} \to \overline{\mathbb{R}}_+^0, \quad (j,k) \mapsto x_{j,k} = \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*),$$

we may now apply the Equality Criterion for functions to prove that both matrices are identical. Letting $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n^*\}$ be arbitrary, we now consider the two cases $A_j \cap A_k^* = \emptyset$ and $A_j \cap A_k^* \neq \emptyset$. The first case $A_j \cap A_k^* = \emptyset$ implies $\mu(A_j \cap A_k^*) = 0$ with Property 1 of a measure, so that evidently

$$\begin{aligned}
[x_{j,k} =] \quad & \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*) = \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} 0 = 0, \\
[x'_{j,k} =] \quad & \alpha_k^* \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*) = \alpha_k^* \cdot_{\overline{\mathbb{R}}_+^0} 0 = 0,
\end{aligned}$$

and therefore $x_{j,k} = x'_{j,k}$ as desired. In the second case $A_j \cap A_k^* \neq \emptyset$, we observe in light of the initial assumptions concerning the proof of the uniqueness part that

$$s = \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j} = \sum_{j=1}^{n^*} \alpha_j^* \cdot \chi_{A_j^*}$$

holds. We may therefore apply Exercise 14.3 to infer the truth of implies $\alpha_j = \alpha_j^*$. Consequently, we obtain via substitution

$$[x_{j,k} =] \quad \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*) = \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j \cap A_k^*) \quad [= x'_{j,k}],$$

thus $x_{j,k} = x'_{j,k}$ once again. As j and k were arbitrary, we may therefore conclude that $\boldsymbol{X}$ and $\boldsymbol{X}'$ are identical. We obtain then

$$y = \sum_{j=1}^{n} \sum_{k=1}^{n^*} x_{j,k} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \boldsymbol{X} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \boldsymbol{X}' \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \sum_{j=1}^{n} \sum_{k=1}^{n^*} x'_{j,k}$$
$$= y'$$

by applying Theorem 6.29 and substitutions. These equations give us finally $y = y'$, and as y and y' were arbitrary, we may infer from the truth of this equation the truth of the uniqueness part and thus the truth of the uniquely existential sentence in (14.41). Since s was also arbitrary, we may further conclude that the universal sentence (14.41) is true. Consequently, there exists indeed a unique function I_μ with domain $\mathcal{S}_+^0$ satisfying (14.40). It remains to show that the range of I_μ is included in $\overline{\mathbb{R}}_+^0$. Letting $y \in \mathrm{ran}(I_\mu)$ be arbitrary, it follows by definition of a range that there exists a particular constant $\bar{s}$ such that $(\bar{s}, y) \in I_\mu$. Since $\bar{s}$ is therefore an element of the domain $\mathcal{S}_+^0$ of I_μ, the associated value is – according to the specification of that function – the sum $y = \sum_{j=1}^{n} \alpha_j \cdot_{\overline{\mathbb{R}}_+^0} \mu(A_j)$ for some $n \in \mathbb{N}_+$, some $\alpha : \{1, \ldots, n\} \to \mathbb{R}_+^0$ and some partition $A : \{1, \ldots, n\} \to \mathcal{A}$ of Ω satisfying $s = \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j}$. This sum is taken with respect to the addition on $\overline{\mathbb{R}}_+^0$, so that it constitutes itself an element of $\overline{\mathbb{R}}_+^0$. We thus found $y \in \overline{\mathbb{R}}_+^0$, and since y was arbitrary, we may therefore conclude that $\mathrm{ran}(I_\mu)$ is included in $\overline{\mathbb{R}}_+^0$ (by definition of a subset). This means that the latter set is a codomain of I_μ. As $(\Omega, \mathcal{A}, \mu)$ was arbitrary, we may now conclude that the stated theorem is indeed true. □

Definition 14.5 (Integral on a set of nonnegative simple functions, integral of a nonnegative simple functions). For any measure space $(\Omega, \mathcal{A}, \mu)$ we call the function

$$I_\mu : \mathcal{S}_+^0(\Omega, \mathcal{A}) \to \overline{\mathbb{R}}_+^0, \quad s \mapsto \sum_{j=1}^{n} \alpha_j \mu(A_j) \tag{14.44}$$

for some $n \in \mathbb{N}_+$, some $\alpha : \{1, \ldots, n\} \to \mathbb{R}_+^0$ and some partition $A : \{1, \ldots, n\} \to \mathcal{A}$ of Ω satisfying $s = \sum_{j=1}^{n} \alpha_j \cdot \chi_{A_j}$ the *integral on* $\mathcal{S}_+^0$ with respect to μ. We then call for any $s \in \mathcal{S}_+^0(\Omega, \mathcal{A})$

$$\int s \, d\mu = I_\mu(s) \tag{14.45}$$

the *integral of* s with respect to μ.

To be expanded!

Part V.

Fuzzy Observation Data

Chapter 15.

Sets of Fuzzy Numbers

We begin this chapter on fuzzy sets with some basic definitions. Let us consider first the real closed interval from 0 to 1.

Definition 15.1 ((Real) unit interval). We call the real closed interval

$$[0,1] \tag{15.1}$$

the (*real*) *unit interval*.

15.1. Fuzzy Sets $(M, m_{\tilde{A}})$

In the following fundamental definition, we let X be an arbitrary set.

Definition 15.2 (Fuzzy set/class, membership function, degree/ grade of membership). In L.A. Zadeh's words:

> A *fuzzy set* (*class*) A in X is characterized by a *membership* [...] *function* $f_A(x)$ which associates with each point in X a real number in the interval $[0,1]$, with the value of $f_A(x)$ at x representing the "grade of membership" of x in A (Zadeh, 1965).

Set-theoretically, we may also say that a set A is a fuzzy set iff there exist sets X and f_A such that f_A is a function from M to $[0,1]$ and such that A is the ordered pair (A, f_A). More specifically, we say for any set X that a set A is a fuzzy set in M iff there is a set f_A such that $f_A : X \to [0,1]$ and $A = (X, f_A)$. For any $x \in X$, we then call the value $f_A(x)$ also the *degree of membership* of x in the fuzzy set A.

Notation 15.1. We will write fuzzy sets usually with a tilde, for instance $\tilde{A}$ (instead of A as for sets in the sense of Definition 2.1). Moreover, we will write for the underlying set also M (instead of X) and for the membership function also $m_{\tilde{A}} : M \to [0,1]$. Thus, we write for a fuzzy set in M

$$\tilde{A} = (M, m_{\tilde{A}}). \tag{15.2}$$

Proposition 15.1. *For any set M, it is true that the Cartesian product $\{M\} \times [0,1]^M$ consists of all the fuzzy sets in M, in the sense that*

$$\forall Z\,(Z \in \{M\} \times [0,1]^M \Leftrightarrow Z \text{ is a fuzzy set in } M). \tag{15.3}$$

Proof. Letting M and Z be arbitrary sets and assume first that $Z \in \{M\} \times [0,1]^M$ holds. By definition of the Cartesian product of two sets, there are then particular sets $\bar{M}$ and $\bar{f}$ such that $\bar{M} \in \{M\}$, $\bar{f} \in [0,1]^M$ and $Z = (\bar{M}, \bar{f})$ are true. We now obtain $\bar{M} = M$ with (2.169) and therefore $Z = (M, \bar{f})$, where $\bar{f}$ is a function from M to $[0,1]$ (by definition of a set of functions). Thus, Z is a fuzzy set in M, which finding completes the proof of the first part ('$\Rightarrow$') of the equivalence.

Regarding the second part ('$\Leftarrow$'), we now assume conversely Z to be a fuzzy number in M, which is defined to be the ordered pair (M, m_Z) formed by M and by a membership function m_Z. The latter is by definition a function from M to $[0,1]$, so that $M \in \{M\}$ and $m_Z \in [0,1]^M$ are evidently both true. By definition of the Cartesian product of two sets, the ordered pair (M, m_Z) follows then to be an element of $\{M\} \times [0,1]^M$, so that the proof of the equivalence is complete. As Z and M are arbitrary, we may therefore conclude that the proposition holds, as claimed. □

Definition 15.3 (Set of fuzzy sets). We call for any set M the Cartesian product $\{M\} \times [0,1]^M$ the *set of fuzzy sets* in M, which we write also as

$$\mathcal{F}(M). \tag{15.4}$$

Exercise 15.1. Show for any sets M, $m_{\tilde{A}}$ and $m_{\tilde{B}}$ such that the ordered pairs $\tilde{A} = (M, m_{\tilde{A}})$ and $\tilde{B} = (M, m_{\tilde{B}})$ are in $\mathcal{F}(M)$ that the fuzzy sets $\tilde{A}$ and $\tilde{B}$ are identical iff their membership functions are identical.

(Hint: Apply the Equality Criterion for ordered pairs.)

Note 15.1. For any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$, we have for any x and any y that the value $y = m_{\tilde{A}}(x)$ of the membership function of $\tilde{A}$ is in $[0,1]$ (see also (3.517)), i.e.

$$m_{\tilde{A}}(x) \in [0,1], \tag{15.5}$$

so that any value $m_{\tilde{A}}(x)$ is between 0 and 1 according to the definition of a real closed interval, that is,

$$0 \leq_{\mathbb{R}} m_{\tilde{A}}(x) \leq_{\mathbb{R}} 1 \tag{15.6}$$

We state now two simple examples of a fuzzy set/membership function.

Proposition 15.2. *For any set M and any subset $A \subseteq M$, defining the characteristic function $\chi_A : M \to \mathbb{R}$ of A, it is true that the ordered pair (M, χ_A) is a fuzzy set.*

Proof. We let M and A be arbitrary sets and assume $A \subseteq M$ to be true. Let us check that the characteristic function $\chi_A : M \to \mathbb{R}$ is a function from M to the real unit interval $[0,1]$, i.e. that $[0,1]$ is also a codomain of $\chi_A : M \to \mathbb{R}$, i.e. that $\mathrm{ran}(\chi_A) \subseteq [0,1]$ holds. To do this, we prove

$$\forall y\, (y \in \mathrm{ran}(\chi_A) \Leftrightarrow y \in [0,1]), \tag{15.7}$$

letting $\bar{y}$ be arbitrary and assuming $\bar{y} \in \mathrm{ran}(\chi_A)$ to be true. By definition of a range, there exists then a constant, say $\bar{x}$, such that $(\bar{x}, \bar{y}) \in \chi_A$ holds, which we may write also $\bar{y} = \chi_A(\bar{x})$ because χ_A is a function (with domain $\mathrm{dom}(\chi_A) = M$). In addition, $(\bar{x}, \bar{y}) \in \chi_A$ shows that there exists a constant y with $(\bar{x}, y) \in \chi_A$, so that $\bar{x} \in M$ $[= \mathrm{dom}(\chi_A)]$ holds by definition of a domain. To establish the desired consequent $\bar{y} \in [0,1]$, we consider now the two cases $\bar{x} \in A$ and $\bar{x} \notin A$. The first case $\bar{x} \in A$ gives

$$[\bar{y} =] \quad \chi_A(\bar{x}) = 1 \quad [= \max[0,1]],$$

and the second case $\bar{x} \notin A$ yields

$$[\bar{y} =] \quad \chi_A(\bar{x}) = 0 \quad [= \min[0,1]],$$

according to the definition of a characteristic function and Corollary 3.118. Because

$$\min[0,1], \max[0,1] \in [0,1]$$

holds by definition of a minimum/maximum, $\bar{y} \in [0,1]$ is thus true in any case. Since $\bar{y}$ was arbitrary, we may therefore conclude that the universal sentence (15.7) is true, which in turn implies the desired inclusion $\mathrm{ran}(\chi_A) \subseteq [0,1]$ by definition of a subset. We thus showed that $[0,1]$ is indeed a codomain of χ_A, which is therefore a function from M to $[0,1]$. Consequently, the ordered pair (M, χ_A) constitutes a fuzzy set, by definition. Because the sets M and A were initially arbitrary, we may now further conclude that the proposed universal sentence is true. □

Definition 15.4 (Crisp/ordinary set). For any set M and any subset $A \subseteq M$, defining the characteristic function $\chi_A : M \to \mathbb{R}$ of A, we call the fuzzy set

$$\tilde{A} = (M, \chi_A) \tag{15.8}$$

also a *crisp set* or an *ordinary set*.

Proposition 15.3. *It is true for any set $M \neq \emptyset$ that the ordered pair $(M, M \times \{1\})$ formed by M and the constant function on M with value* 1 *is a fuzzy set. Furthermore, this fuzzy set is the crisp set (M, χ_M) with the characteristic function of M on M as the membership function.*

Proof. We let M be an arbitrary set, we assume $M \neq \emptyset$, and we show that $[0,1]$ is a codomain of the constant function $g_1 : M \to \{1\}$, i.e. that $\mathrm{ran}(g_1) \subseteq [0,1]$ holds. Because the assumed $M \neq \emptyset$ implies with (3.638) that g_1 is a surjection from M to $\{1\}$, we have $\mathrm{ran}(g_1) = \{1\}$. Furthermore, $[1 =] \max[0,1] \in [0,1]$ holds according to Corollary 3.118 and the definition of a maximum. Then, $1 \in [0,1]$ implies $\{1\} \subseteq [0,1]$ with (2.184), which gives $\mathrm{ran}(g_1) \subseteq [0,1]$ via substitution, as desired. We thus showed that the constant function $g_1 = M \times \{1\}$ on M with value 1 is a function from from M to $[0,1]$. Thus, the ordered pair $(M, M \times \{1\})$ is evidently a fuzzy set in M by definition. Recalling from Corollary 14.3 that the characteristic function of M on M is identical with the constant function on M with value 1, we thus obtain $(M, M \times \{1\}) = (M, \chi_M)$, so that $(M, M \times \{1\})$ defines indeed a crisp set. Since M was initially arbitrary, we may therefore conclude that the proposed universal sentence is true. □

Definition 15.5 (Fundamental fuzzy set). For any set $M \neq \emptyset$, we call the fuzzy/crisp set

$$\tilde{M} = (M, m_{\tilde{M}}) = (M, \chi_M) = (M, M \times \{1\}) \tag{15.9}$$

the *fundamental fuzzy set.*

Exercise 15.2. Show for any set $M \neq \emptyset$ that the ordered pair $(M, M \times \{0\})$ formed by M and the constant function on M with value 0 is a fuzzy set. Verify also that this fuzzy set is the crisp set $(M, \chi_\emptyset)$ with the characteristic function of $\emptyset$ on M as the membership function.

(Hint: Proceed in analogy to the proof of Proposition 15.3.)

Definition 15.6 (Empty fuzzy set). For any set $M \neq \emptyset$, we call the fuzzy/crisp set

$$\tilde{\emptyset} = (M, m_{\tilde{\emptyset}}) = (M, \chi_\emptyset) = (M, M \times \{0\}) \tag{15.10}$$

the *empty fuzzy set.*

We now come to the concept of 'membership' to fuzzy sets, which extends the concept of an 'element' in a set in Cantor's sense.

Definition 15.7 (Non-member/not-included element, fuzzy/partial member, full member/fully included element). For any sets M, $m_{\tilde{A}}$, $m_{\tilde{B}}$, $\tilde{A}$ and $\tilde{B}$ such that $m_{\tilde{A}}$ and $m_{\tilde{B}}$ are membership functions with domain M and such that $\tilde{A}$ and $\tilde{B}$ are the fuzzy sets $(M, m_{\tilde{A}})$ and $(M, m_{\tilde{B}})$, respectively, we say that an element $x \in M$ is

(1) a *non-member* of $\tilde{A}$ (or *not included* in $\tilde{A}$) iff

$$m_{\tilde{A}}(x) = 0. \tag{15.11}$$

(2) a *partial member* of $\tilde{A}$ (or a *fuzzy member* of $\tilde{A}$) iff

$$0 <_{\mathbb{R}} m_{\tilde{A}}(x) <_{\mathbb{R}} 1. \tag{15.12}$$

(3) a *full member* of $\tilde{A}$ (or *fully included* in $\tilde{A}$) iff

$$m_{\tilde{A}}(x) = 1. \tag{15.13}$$

Note 15.2. We see in light of Corollary 3.154 for any set M that

- all elements of M are full members of the fundamental fuzzy set $\tilde{M}$ in M.
- all elements of M are non-members of the empty fuzzy set $\tilde{\emptyset}$ in M.

Proposition 15.4. *For any set $M \neq \emptyset$ and any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ in M, it is true that the difference of the constant function on M with value 1 and the membership function $m_{\tilde{A}}$ of $\tilde{A}$ is an element of $[0,1]^M$, in the sense that*

$$M \times \{1\} -_{\mathbb{R}^M} m_{\tilde{A}} \in [0,1]^M. \tag{15.14}$$

Proof. We let M and $\tilde{A}$ be arbitrary sets, assume M to be nonempty and $\tilde{A}$ to be a fuzzy set in M, so that there exists a particular membership function $m_{\tilde{A}}$ with $\tilde{A} = (M, m_{\tilde{A}})$. Let us observe now that the constant function $M \times \{1\} : M \to \{1\}$ and the membership function $m_{\tilde{A}} : M \to [0,1]$ are both functions from M to $\mathbb{R}$, since the inclusions $\{1\} \subseteq \mathbb{R}$ and $[0,1] \subseteq \mathbb{R}$ follow to be both true from $0, 1 \in \mathbb{R}$ in view of (2.184) and Proposition 3.117b). Here, we may write the constant function $M \times \{1\}$ also as χ_M, according to Corollary 14.3. Because $(\mathbb{R}^M, +_{\mathbb{R}^M}, \cdot_{\mathbb{R}^M}, -_{\mathbb{R}^M})$ forms a ring of real functions, we obtain $\chi_M -_{\mathbb{R}^M} m_{\tilde{A}} \in \mathbb{R}^M$, which difference is specified according to Corollary 6.21 by

$$\forall x\, (x \in M \Rightarrow (\chi_M -_{\mathbb{R}^M} m_{\tilde{A}})(x) = \chi_M(x) -_{\mathbb{R}} g(x)). \tag{15.15}$$

We thus have $\chi_M -_{\mathbb{R}^M} m_{\tilde{A}} : M \to \mathbb{R}$ by definition of a set of functions. To establish (15.14), we need to show now that $[0,1]$ is also a codomain of that function, i.e. that its range is included in $[0,1]$. For this purpose, we use the definition of a subset and prove the equivalent universal sentence

$$\forall y\, (y \in \mathrm{ran}(\chi_M -_{\mathbb{R}^M} m_{\tilde{A}}) \Rightarrow y \in [0,1]), \tag{15.16}$$

letting $\bar{y}$ be arbitrary and assuming $\bar{y} \in \mathrm{ran}(\chi_M -_{\mathbb{R}^M} m_{\tilde{A}})$ to be true. By definition of a range, there exists then a constant, say $\bar{x}$, with $(\bar{x}, \bar{y}) \in \chi_M -_{\mathbb{R}^M} m_{\tilde{A}}$. On the one hand, this evidently gives $\bar{x} \in M$ [$= \mathrm{dom}(\chi_M -_{\mathbb{R}^M} m_{\tilde{A}})$] with the definition of a domain. On the other hand, we obtain

$$\begin{aligned} \bar{y} &= (\chi_M -_{\mathbb{R}^M} m_{\tilde{A}})(\bar{x}) \\ &= \chi_M(x) -_{\mathbb{R}} m_{\tilde{A}}(\bar{x}) \\ &= 1 -_{\mathbb{R}} m_{\tilde{A}}(\bar{x}) \\ &= 1 +_{\mathbb{R}} [-m_{\tilde{A}}(\bar{x})] \end{aligned} \tag{15.17}$$

by using function notation, (15.15), then (3.534) with the previously mentioned fact that χ_M is the constant function on M with value 1, and finally the definition of a difference. Now, since the inequalities

$$0 \leq_{\mathbb{R}} m_{\tilde{A}}(\bar{x}) \leq_{\mathbb{R}} 1$$

are true according to (15.6), an application of the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ yields

$$0 +_{\mathbb{R}} [-m_{\tilde{A}}(\bar{x})] \leq_{\mathbb{R}} m_{\tilde{A}}(\bar{x}) +_{\mathbb{R}} [-m_{\tilde{A}}(\bar{x})] \leq_{\mathbb{R}} 1 +_{\mathbb{R}} [-m_{\tilde{A}}(\bar{x})]$$

and therefore $-m_{\tilde{A}}(\bar{x}) \leq_{\mathbb{R}} 0$ as well as $0 \leq_{\mathbb{R}} 1 +_{\mathbb{R}} [-m_{\tilde{A}}(\bar{x})]$ with the definition of an inverse element. Whereas the latter inequality shows in light of (15.17) that $0 \leq_{\mathbb{R}} \bar{y}$ holds, the former implies $1 +_{\mathbb{R}} [-m_{\tilde{A}}(\bar{x})] \leq_{\mathbb{R}} 1$ with the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ and therefore $\bar{y} \leq_{\mathbb{R}} 1$. We thus established $0 \leq_{\mathbb{R}} \bar{y} \leq_{\mathbb{R}} 1$, so that the desired consequent $\bar{y} \in [0,1]$ follows to be true by definition of a closed interval. Since $\bar{y}$ was arbitrary, we may infer from this finding the truth of the universal sentence (15.16), so that $M \times \{1\} -_{\mathbb{R}^M} m_{\tilde{A}}$ is indeed a function from M to $[0,1]$. Consequently, (15.14) holds, and as M and $\tilde{A}$ were initially arbitrary sets, we may now finally conclude that the stated proposition is true. □

Note 15.3. The preceding proposition shows for any set $M \neq \emptyset$ and any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ in M that the difference $\chi_M -_{\mathbb{R}^M} m_{\tilde{A}}$ is a function in $[0,1]^M$, where χ_M is the membership function of the fundamental fuzzy set $\tilde{M}$ in M. Therefore, this difference defines the fuzzy set $(M, \chi_M -_{\mathbb{R}^M} m_{\tilde{A}})$ in M.

Definition 15.8 (Fuzzy complement). For any set $M \neq \emptyset$ and any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ in M, we call the fuzzy set

$$(M, \chi_M -_{\mathbb{R}^M} m_{\tilde{A}}) \tag{15.18}$$

the *fuzzy complement* of $\tilde{A}$ in M, which we will symbolize by

$$\tilde{A}^{\tilde{c}} = (M, m_{\tilde{A}^{\tilde{c}}}). \tag{15.19}$$

15.2. Basic Descriptions of Fuzzy Sets

We now define some basic sets (in the sense of Cantor) which describe the 'shape' of a membership function in terms of their values. The first of these sets describes the set of all partial and full members of a fuzzy set.

Exercise 15.3. Specify for any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ the unique set $\mathrm{supp}(\tilde{A})$ consisting of all elements in M which are greater than 0, in the sense that

$$\forall x\,(x \in \mathrm{supp}(\tilde{A}) \Leftrightarrow [x \in M \,\wedge\, m_{\tilde{A}}(x) >_{\mathbb{R}} 0]). \tag{15.20}$$

Definition 15.9 (Support of a fuzzy set). For any set M and any fuzzy set $\tilde{A} \in \mathcal{F}(M)$ we call the set

$$\mathrm{supp}(\tilde{A}) \tag{15.21}$$

the *support* of $\tilde{A}$. We symbolize this set also by

$$\{x \in M : m_{\tilde{A}}(x) >_{\mathbb{R}} 0\}. \tag{15.22}$$

Theorem 15.5 (Characterization of the support of a fuzzy set). *The support of any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ is identical with the inverse image of the real left-open and right-closed interval from 0 to 1 under $m_{\tilde{A}}$, i.e.*

$$\mathrm{supp}(\tilde{A}) = m_{\tilde{A}}^{-1}[(0,1]]. \tag{15.23}$$

Proof. We let M, $m_{\tilde{A}}$ and $\tilde{A}$ be arbitrary sets such that $m_{\tilde{A}} : M \to [0,1]$ and $\tilde{A} = (M, m_{\tilde{A}})$ hold. To prove the stated equation, we apply the Equality Criterion for sets and verify accordingly

$$\forall x\,(x \in \mathrm{supp}(\tilde{A}) \Leftrightarrow x \in m_{\tilde{A}}^{-1}[(0,1]]). \tag{15.24}$$

For this purpose, we take an arbitrary x and establish the equivalences

$$\begin{aligned}
x \in \mathrm{supp}(\tilde{A}) &\Leftrightarrow x \in M \,\wedge\, 0 <_{\mathbb{R}} m_{\tilde{A}}(x) \\
&\Leftrightarrow m_{\tilde{A}}(x) \leq_{\mathbb{R}} 1 \,\wedge\, (x \in M \,\wedge\, 0 <_{\mathbb{R}} m_{\tilde{A}}(x)) \\
&\Leftrightarrow x \in M \,\wedge\, (0 <_{\mathbb{R}} m_{\tilde{A}}(x) \,\wedge\, m_{\tilde{A}}(x) \leq_{\mathbb{R}} 1) \\
&\Leftrightarrow x \in M \,\wedge\, m_{\tilde{A}}(x) \in (0,1] \\
&\Leftrightarrow x \in m_{\tilde{A}}^{-1}[(0,1]]
\end{aligned}$$

by using (15.20), the Tautology Law for the conjunction based on the fact that $m_{\tilde{A}}(x) \leq_{\mathbb{R}} 1$ is always true as shown by the second inequality in (15.6), the Associative as well as the Commutative Law for the conjunction, the definition of a real left-open and right-closed interval, and finally the fact that an inverse image is specified to satisfy (3.743). Since x is arbitrary, we may infer from the truth of the preceding equivalences the truth of the universal sentence (15.24), which in turn implies the truth of the equation (15.23). As M, $m_{\tilde{A}}$ and $\tilde{A}$ were initially arbitrary sets, we may therefore conclude that the theorem holds, as claimed. □

Proposition 15.6. *For any set $M \neq \emptyset$, it is true that the support of the empty fuzzy set $\tilde{\emptyset} = (M, m_{\tilde{\emptyset}})$ is empty, i.e.*

$$\mathrm{supp}(\tilde{\emptyset}) = \emptyset. \tag{15.25}$$

Proof. We let M be an arbitrary set and assume $M \neq \emptyset$ to be true. To prove the stated equation, we verify

$$\forall x\, (x \notin \mathrm{supp}(\tilde{\emptyset})), \tag{15.26}$$

letting x be arbitrary. We may now prove the negation in (15.26) by contradiction, assuming the negation of that negation to be true, so that the Double negation Law yields the true sentence $x \in \mathrm{supp}(\tilde{\emptyset})$. Because of the Characterization of the support of a fuzzy set, we then obtain

$$x \in m_{\tilde{\emptyset}}^{-1}[(0,1]], \tag{15.27}$$

and therefore

$$m_{\tilde{\emptyset}}(x) \in (0,1] \tag{15.28}$$

with the definition of an inverse image. Furthermore, the inverse image $m_{\tilde{\emptyset}}^{-1}[(0,1]]$ is included in the domain M of $m_{\tilde{\emptyset}}$ (see Note 3.30), so that (15.27) implies $x \in M$ with the definition of a subset. Recalling that $m_{\tilde{\emptyset}}$ is the constant function on M with value 0 (by definition of an empty fuzzy set), it follows now from $x \in M$ that

$$m_{\tilde{\emptyset}}(x) = 0$$

holds (see Corollary 3.154), so that (15.28) yields $0 \in (0,1]$. According to the definition of a real left-open and right-closed interval, this implies $0 <_{\mathbb{R}} 0$ $(\leq_{\mathbb{R}} 1)$. Since the irreflexivity of $<_{\mathbb{R}}$ also implies the truth of the negation $\neg 0 <_{\mathbb{R}} 0$, we obtained a contradiction, so that the proof of the negation in (15.26) is complete. As x was arbitrary, we may therefore conclude that the universal sentence (15.26) is true, which then further implies the truth of the equation (15.25) with the definition of the empty set. Because M was initially an arbitrary set, we may infer from this the truth of proposed universal sentence. □

Exercise 15.4. Show that the support of any crisp set $\tilde{A} = (M, \chi_A)$ is identical with A, i.e.

$$\mathrm{supp}(\tilde{A}) = A. \tag{15.29}$$

(Hint: Apply the Equality Criterion for sets via two direct proofs of the implications, using in particular the Characterization of comparability and Exercise 14.2.)

We now generalize the idea of the support of a fuzzy set by considering intervals in the codomain of the membership function which do not necessarily begin with 0.

Exercise 15.5. Specify for any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ and any $\alpha \in [0,1]$

a) the unique set $\tilde{A}_\alpha$ consisting of all elements in M which are greater than or equal to α, in the sense that

$$\forall x \, (x \in \tilde{A}_\alpha \Leftrightarrow [x \in M \ \wedge \ m_{\tilde{A}}(x) \geq_{\mathbb{R}} \alpha]). \tag{15.30}$$

b) the unique set $\tilde{A}_\alpha^>$ consisting of all elements in M which are greater than α, in the sense that

$$\forall x \, (x \in \tilde{A}_\alpha \Leftrightarrow [x \in M \ \wedge \ m_{\tilde{A}}(x) >_{\mathbb{R}} \alpha]). \tag{15.31}$$

Definition 15.10 (α-cut/α-level set, strong α-cut/strong α-level set). For any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ and any $\alpha \in [0,1]$

(1) we call the set

$$\tilde{A}_\alpha \tag{15.32}$$

the *α-cut* or the *α-level set* of $\tilde{A}$. We symbolize this set also by

$$\{x \in M : m_{\tilde{A}}(x) \geq_{\mathbb{R}} \alpha\}. \tag{15.33}$$

(2) we call the set

$$\tilde{A}_\alpha^> \tag{15.34}$$

the *strong α-cut* or the *strong α-level set* of $\tilde{A}$, which is symbolized also by

$$\{x \in M : m_{\tilde{A}}(x) >_{\mathbb{R}} \alpha\}. \tag{15.35}$$

In analogy to the Characterization of the support of a fuzzy set, we may represent any α-cut as an inverse image.

Theorem 15.7 (Characterization of α-cuts). *The following sentences are true for any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ and any $\alpha \in [0,1]$.*

a) The α-cut of $\tilde{A}$ is identical with the inverse image of the closed interval from a to 1 *under $m_{\tilde{A}}$, i.e.*

$$\tilde{A}_\alpha = m_{\tilde{A}}^{-1}[[\alpha, 1]]. \tag{15.36}$$

b) The strong α-cut of $\tilde{A}$ is identical with the inverse image of the left-open and right-closed interval from a to 1 *under $m_{\tilde{A}}$, i.e.*

$$\tilde{A}_\alpha^{>} = m_{\tilde{A}}^{-1}[(\alpha, 1]]. \tag{15.37}$$

Exercise 15.6. Prove Theorem 15.7.

(Hint: Proceed in analogy to the proof of Theorem 15.5.)

We now establish a set of α-cuts which excludes the 0-cut (for later convenience).

Proposition 15.8. *For any set M and any fuzzy set $\tilde{A} \in \mathcal{F}(M)$ there exists a unique set $C_{\tilde{A}}$ containing precisely any α-cut of $\tilde{A}$ with $\alpha \in (0, 1]$, in the sense that*

$$\forall X\, (X \in C_{\tilde{A}} \Leftrightarrow \exists \alpha\, (\alpha \in (0, 1] \wedge X = \tilde{A}_\alpha)). \tag{15.38}$$

Proof. Letting M and $\tilde{A}$ be arbitrary sets and assuming $\tilde{A}$ to be a fuzzy set in M, we may apply evidently the Axiom of Specification and the Equality Criterion for sets to establish the unique existence of a set $C_{\tilde{A}}$ such that

$$\forall X\, (X \in C_{\tilde{A}} \Leftrightarrow [X \in \mathcal{P}(M) \wedge \exists \alpha\, (\alpha \in (0, 1] \wedge X = \tilde{A}_\alpha)]). \tag{15.39}$$

holds (where $\mathcal{P}(M)$ is the power set of M). We now verify that the set $C_{\tilde{A}}$ satisfies also (15.38), letting X be arbitrary. We first assume $X \in C_{\tilde{A}}$ to be true, which implies then in particular the desired existential sentence in (15.38) with (15.39). We now assume conversely that this existential sentence is true, so that there is a particular constant $\bar{\alpha} \in (0, 1]$ with $X = \tilde{A}_{\bar{\alpha}}$. Thus, $0 <_{\mathbb{R}} \bar{\alpha}$ and $\bar{\alpha} \leq_{\mathbb{R}} 1$ hold by definition of a (real) left-open and right-closed interval, where the first inequality yields the true disjunction $0 <_{\mathbb{R}} \bar{\alpha} \vee 0 = \bar{\alpha}$ and therefore $0 \leq_{\mathbb{R}} \bar{\alpha}$ (by definition of an induced reflexive partial ordering); together with $\bar{\alpha} \leq_{\mathbb{R}} 1$, this gives $\bar{\alpha} \in [0, 1]$ with the definition of a (real) closed interval. Therefore, the $\bar{\alpha}$-cut $X = \tilde{A}_{\bar{\alpha}}$ is an inverse image under the membership function $m_{\tilde{A}} : M \to [0, 1]$ of $\tilde{A}$ according to the Characterization of α-cuts, and this inverse image is then a subset of the domain M of $m_{\tilde{A}}$ in view of Note 3.30. Thus, the $\bar{\alpha}$-cut X is a subset of M and consequently an element of the power set $\mathcal{P}(M)$. Thus, $X \in \mathcal{P}(M)$ and the previously established existential sentences imply $X \in C_{\tilde{A}}$ with the assumed (15.39), completing the proof of the equivalence. Because X is arbitrary, we may therefore conclude that the set $C_{\tilde{A}}$ satisfies the universal sentence (15.38). Since M and $\tilde{A}$ are arbitrary as well, we may then further conclude that the proposed universal sentence is true. □

Definition 15.11 (Collection of α-cuts). For any set M and any fuzzy set $\tilde{A} \in \mathcal{F}(M)$, we call the set $C_{\tilde{A}}$ consisting of all the α-cuts of $\tilde{A}$ with $\alpha \in (0,1]$ in the sense of (15.38) the *collection of α-cuts* of $\tilde{A}$.

Proposition 15.9. *For any set M and any fuzzy set $\tilde{A} \in \mathcal{F}(M)$, it is true that the union of the collection $C_{\tilde{A}}$ of α-cuts of $\tilde{A}$ is identical with the support of $\tilde{A}$, that is,*

$$\forall M, \tilde{A}\,(\tilde{A} \in \mathcal{F}(M) \Rightarrow \mathrm{supp}(\tilde{A}) = \bigcup C_{\tilde{A}}). \tag{15.40}$$

Proof. We let M and $\tilde{A}$ be arbitrary sets, and we assume $\tilde{A} \in \mathcal{F}(M)$ to be true. To prove that the sets $\mathrm{supp}(\tilde{A})$ and $\bigcup C_{\tilde{A}}$ are identical, we apply the Equality Criterion for sets and verify accordingly

$$\forall x\,(x \in \mathrm{supp}(\tilde{A}) \Leftrightarrow x \in \bigcup C_{\tilde{A}}), \tag{15.41}$$

letting x be arbitrary. Regarding the first part ('⇒') of the equivalence, we assume $x \in \mathrm{supp}(\tilde{A})$ to be true, so that the definition of the support of a fuzzy set yields $x \in M$ and $m_{\tilde{A}}(x) >_{\mathbb{R}} 0$. Since $m_{\tilde{A}}(x) \leq_{\mathbb{R}} 1$ also holds by definition of a membership function, we have by definition of a (real) left-open and right-closed interval that $m_{\tilde{A}}(x) \in (0,1]$ is true. Let us now introduce the denotation $\bar{\alpha} = m_{\tilde{A}}(x)$. Then, because $m_{\tilde{A}}(x) \geq_{\mathbb{R}} m_{\tilde{A}}(x)$ holds with the reflexivity of $\leq_{\mathbb{R}}$, we obtain $m_{\tilde{A}}(x) \geq_{\mathbb{R}} \bar{\alpha}$ via substitution. Furthermore, as the value $\bar{\alpha}$ of the membership function $m_{\tilde{A}}$ is also in its codomain $[0,1]$, we may apply the definition of an α-cut to infer from the preceding inequality $x \in \tilde{A}_{\bar{\alpha}}$. Using the denotation $\bar{X} = \tilde{A}_{\bar{\alpha}}$, this shows that the existential sentence

$$\exists \alpha\,(\alpha \in (0,1] \wedge \bar{X} = \tilde{A}_{\alpha}) \tag{15.42}$$

is true, which in turn implies $\bar{X} \in C_{\tilde{A}}$ with the specification of the latter set in Proposition 15.38. Moreover, the previously established $x \in \tilde{A}_{\bar{\alpha}}$ and $\bar{X} = \tilde{A}_{\bar{\alpha}}$ give $x \in \bar{X}$ via substitution, which then implies together with $\bar{X} \in C_{\tilde{A}}$ that the existential sentence

$$\exists X\,(X \in C_{\tilde{A}} \wedge x \in X) \tag{15.43}$$

holds, so that the desired consequent $x \in \bigcup C_{\tilde{A}}$ of the first part of the equivalence to be proven follows to be true by definition of the union of a set system.

To establish the second part ('$\Leftarrow$'), we now assume $x \in \bigcup C_{\tilde{A}}$ to be true, so that the equivalent existential sentence (15.43) holds (by definition of the union of a set system). Thus, there exists a particular set $\bar{X}$ such that $\bar{X} \in C_{\tilde{A}}$ and $x \in \bar{X}$ are both true. By definition of the set $C_{\tilde{A}}$, there is then also a particular constant $\bar{\alpha}$ such that $\bar{\alpha} \in (0,1]$ and $\bar{X} = \tilde{A}_{\bar{\alpha}}$ both hold; thus, $x \in \bar{X}$ gives $x \in \tilde{A}_{\bar{\alpha}}$. We therefore obtain $m_{\tilde{A}}(x) \geq_{\mathbb{R}} \bar{\alpha} >_{\mathbb{R}} 0$ with the definition of an α-cut and with the definition of a (real) left-open and right-closed interval, and these two inequalities imply $m_{\tilde{A}}(x) >_{\mathbb{R}} 0$ with the Transitivity Formula for $<$ and $\leq$. Consequently, $x \in \mathrm{supp}(\tilde{A})$ holds by definition of the support of a fuzzy set, which finding completes the proof of the equivalence in (15.41).

Therefore, the sets $\mathrm{supp}(\tilde{A})$ and $\bigcup C_{\tilde{A}}$ are indeed identical, and since the sets M and $\tilde{A}$ were initially arbitrary, we may infer from the truth of that equality the truth of the stated proposition. □

Recalling that 0 is the minimum and 1 the maximum of the real unit interval $[0,1]$ (see Corollary 3.118), we may form the following two particular α-cuts.

Corollary 15.10. *The* 0*-cut of any fuzzy set* $\tilde{A} = (M, m_{\tilde{A}})$ *equals* M*, i.e.*

$$\tilde{A}_0 = M. \tag{15.44}$$

Proof. We let M, $m_{\tilde{A}}$ and $\tilde{A}$ be arbitrary sets and assume that $\tilde{A} = (M, m_{\tilde{A}})$ is a fuzzy set. We then obtain for the 0-cut of $\tilde{A}$

$$\tilde{A}_0 = m_{\tilde{A}}^{-1}[[0,1]] = M$$

with the Characterization of α-cuts and with Exercise 3.90b), using the fact that $[0,1]$ is a codomain and M the domain of $m_{\tilde{A}}$. Then, the resulting equation $\tilde{A}_0 = M$ follows to be true for any M, any $m_{\tilde{A}}$ and any $\tilde{A}$. □

Corollary 15.11. *The* 1*-cut of any fuzzy set* $\tilde{A} = (M, m_{\tilde{A}})$ *is identical with the inverse image of the singleton formed by* 1*, i.e.*

$$\tilde{A}_1 = m_{\tilde{A}}^{-1}[\{1\}]. \tag{15.45}$$

Proof. Letting M, $m_{\tilde{A}}$ and $\tilde{A}$ be arbitrary such that $\tilde{A} = (M, m_{\tilde{A}})$ is a fuzzy set, we obtain the equations

$$\tilde{A}_1 = m_{\tilde{A}}^{-1}[[1,1]] = m_{\tilde{A}}^{-1}[\{1\}]$$

with the Characterization of α-cuts and with (3.363). As M, $m_{\tilde{A}}$ and $\tilde{A}$ are arbitrary, we may therefore infer from the resulting equation $\tilde{A}_1 = m_{\tilde{A}}^{-1}[\{1\}]$ the truth of the corollary. □

Definition 15.12 (Core/kernel of a fuzzy set). For any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ we write for the 1-cut of $\tilde{A}$

$$\operatorname{core}(\tilde{A}) = \tilde{A}_1, \tag{15.46}$$

which we will call the *core* or the *kernel* of $\tilde{A}$.

Note 15.4. We may view the core of any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ as the set of all elements x for which the membership function of $\tilde{A}$ takes the value 1, because we obtain the equations

$$\begin{aligned}\operatorname{core}(\tilde{A}) &= \tilde{A}_1 = m_{\tilde{A}}^{-1}[\{1\}] = \{x : m_{\tilde{A}}(x) \in \{1\}\} \\ &= \{x : m_{\tilde{A}}(x) = 1\}\end{aligned}$$

using the definition of the core of a fuzzy set, (15.45), the notation (3.744) for the inverse image, and (2.169).

Corollary 15.12. *The core of any crisp set $\tilde{A} = (M, \chi_A)$ equals A, i.e.*

$$\operatorname{core}(\tilde{A}) = A. \tag{15.47}$$

Proof. We let M and A be arbitrary sets such that $A \subseteq M$ holds. To establish the stated equation, we apply the equality Criterion for sets and verify accordingly

$$\forall x\, (x \in \operatorname{core}(\tilde{A}) \Leftrightarrow x \in A), \tag{15.48}$$

letting x be arbitrary and observing the truth of the equivalences

$$\begin{aligned} x \in \operatorname{core}(\tilde{A}) &\Leftrightarrow x \in \chi_A^{-1}[\{1\}] \\ &\Leftrightarrow \chi_A(x) \in \{1\} \\ &\Leftrightarrow \chi_A(x) = 1 \\ &\Leftrightarrow x \in A \end{aligned}$$

in light of (15.45), the definition of an inverse image, (2.169), and (14.10). As x is arbitrary, we may infer from the truth of the resulting equivalence $x \in \operatorname{core}(\tilde{A}) \Leftrightarrow x \in A$ the truth of the universal sentence (15.48), which in turn implies the truth of $\operatorname{core}(\tilde{A}) = A$, according to the Equality Criterion for sets. Because M and A were initially arbitrary sets, we may therefore conclude that the proposed universal sentence holds. □

Exercise 15.7. Show for any set $M \neq \emptyset$ that the core of the empty fuzzy set $\tilde{\emptyset} = (M, m_{\tilde{\emptyset}})$ is empty, i.e.

$$\operatorname{core}(\tilde{\emptyset}) = \emptyset. \tag{15.49}$$

We inspect now a first, trivial example of a strong α-cut.

Proposition 15.13. *The strong* 1*-cut of any fuzzy set* $\tilde{A} = (M, m_{\tilde{A}})$ *is empty, i.e.*

$$\tilde{A}_1^{>} = \emptyset. \tag{15.50}$$

Proof. We let M, $m_{\tilde{A}}$ and $\tilde{A}$ be arbitrary sets such that $\tilde{A} = (M, m_{\tilde{A}})$ is a fuzzy set. Letting x be arbitrary, we then obtain

$$\begin{aligned} x \in \tilde{A}_1^{>} \;&\Leftrightarrow\; x \in m_{\tilde{A}}^{-1}[(1,1]] \\ &\Leftrightarrow\; x \in m_{\tilde{A}}^{-1}[\emptyset] \\ &\Leftrightarrow\; x \in \emptyset \end{aligned}$$

using the Characterization of α-cuts, the fact that the irreflexivity of $<_{\mathbb{R}}$ gives $\neg 1 <_{\mathbb{R}} 1$ and therefore $(1,1] = \emptyset$ with (3.383), and finally (3.745). As x was arbitrary, we may infer from the resulting equivalence $x \in \tilde{A}_1^{>} \Leftrightarrow x \in \emptyset$ that $\tilde{A}_1^{>} = \emptyset$ is true, according to the Equality Criterion for sets. Since M, $m_{\tilde{A}}$ and $\tilde{A}$ are also arbitrary, the stated sentence follows then to be true. □

Theorem 15.14 (Equality Criterion for fuzzy sets). *It is true for any set* M *that two fuzzy sets* $\tilde{A}$ *and* $\tilde{B}$ *in* M *are identical iff the* α*-level sets of* $\tilde{A}$ *and* $\tilde{B}$ *are identical for any* α *greater than* 0 *and less than or equal to* 1*, that is,*

$$\forall M, \tilde{A}, \tilde{B}\, (\tilde{A}, \tilde{B} \in \mathcal{F}(M) \Rightarrow \Big[\tilde{A} = \tilde{B} \Leftrightarrow \forall \alpha\, (0 <_{\mathbb{R}} \alpha \leq_{\mathbb{R}} 1 \Rightarrow \tilde{A}_\alpha = \tilde{B}_\alpha)\Big]). \tag{15.51}$$

Proof. We let M, $\tilde{A}$ and $\tilde{B}$ be arbitrary sets, form then the set $\mathcal{F}(M)$ of all fuzzy sets in M, and assume $\tilde{A}, \tilde{B} \in \mathcal{F}(M)$ to be true. To establish the first part ('$\Rightarrow$') of the equivalence, we assume moreover that the equation $\tilde{A} = \tilde{B}$ holds, let α be arbitrary, and assume in addition $0 <_{\mathbb{R}} \alpha \leq_{\mathbb{R}} 1$. Then, the desired equation $\tilde{A}_\alpha = \tilde{B}_\alpha$) follows immediately to be true via substitution based on the assumed $\tilde{A} = \tilde{B}$. Since α is arbitrary, we may therefore conclude that the first part of the equivalence holds. We now prove the second part ('$\Leftarrow$'), under construction! Since M, $\tilde{A}$ and $\tilde{B}$ were arbitrary, we may then infer from the truth of the equivalence the truth of the proposed universal sentence. □

Since the range of the membership function $m_{\tilde{A}} : M \to [0,1]$ of any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ is a subset of $[0,1]$ by definition of a codomain, it follows immediately with the Completeness of the real unit interval lattice and with the definition of a complete lattice that the supremum of that range exists.

Corollary 15.15. *For any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$, it is true that there exists an element S in $[0,1]$ such that S is the supremum of the range of the membership function $m_{\tilde{A}} : M \to [0,1]$ with respect to $\leq [0,1]$, i.e.*

$$\exists S\,(S = \overset{\leq[0,1]}{\sup}\, \mathrm{ran}(m_{\tilde{A}})). \tag{15.52}$$

Besides the support of a fuzzy set, the following concept is also used as a basic description of the shape of a membership function/fuzzy set.

Definition 15.13 (Height of a fuzzy set, normalized/standardized/ regular fuzzy set). For any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$

(1) we call

$$\mathrm{height}(\tilde{A}) = \overset{\leq[0,1]}{\sup}\, \mathrm{ran}(m_{\tilde{A}}) \tag{15.53}$$

the *height* of $\tilde{A}$.

(2) we then say that $\tilde{A}$ is *normalized* (alternatively, *standardized* or *regular* iff the height of $\tilde{A}$ is identical to 1, i.e. iff

$$\mathrm{height}(\tilde{A}) = 1. \tag{15.54}$$

The following result also utilizes the Completeness of the real unit interval lattice.

Theorem 15.16 (α-cut representation of degrees of membership). *The following sentences are true for any set M, any fuzzy set $\tilde{A} \in \mathcal{F}(M)$, and any element $x \in M$.*

a) There exists a unique set

$$Y = \{\alpha \cdot \chi_{\tilde{A}_\alpha}(x) : \alpha \in [0,1]\} \tag{15.55}$$

consisting of all (real) numbers which are products of some $\alpha \in [0,1]$ and the value of the characteristic function of the α-cut at x, in the sense that

$$\forall y\,(y \in Y \Leftrightarrow \exists \alpha\,(\alpha \in [0,1] \,\wedge\, \alpha \cdot_{\mathbb{R}} \chi_{\tilde{A}_\alpha}(x) = y)). \tag{15.56}$$

b) Then, that set is included in the real unit interval, that is,

$$\{\alpha \cdot \chi_{\tilde{A}_\alpha}(x) : \alpha \in [0,1]\} \subseteq [0,1]. \tag{15.57}$$

c) Furthermore, the degree of membership of x in $\tilde{A}$ is the supremum of that set with respect to $\leq_{[0,1]}$, that is,

$$m_{\tilde{A}}(x) = \overset{\leq[0,1]}{\sup}\{\alpha \cdot \chi_{\tilde{A}_\alpha}(x) : \alpha \in [0,1]\}. \tag{15.58}$$

Proof. We let M, $\tilde{A}$ and x be arbitrary such that $\tilde{A}$ is a fuzzy set in M and such that x is an element in M. Then, Part a) may be established by applying first the Axiom of Specification in connection with the Equality Criterion for sets and by using then the fact that $\alpha \in [0,1]$ implies $\alpha \in \mathbb{R}$. In doing this, we observe for an arbitrary $\alpha \in [0,1]$ that the α-cut of the fuzzy set $\tilde{A}$ in M is a subset of M according to the Characterization of α-cuts and Note 3.30, so that we may define the characteristic function $\chi_{\tilde{A}_\alpha} : M \to \mathbb{R}$.

Concerning b), we take an arbitrary y and assume that y is an element of the set (15.55). It then follows with (15.56) that there exists a particular constant $\bar{\alpha}$ such that $\bar{\alpha} \in [0,1]$ and $\bar{\alpha} \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x) = y$ are both true. The former implies

$$0 \leq_{\mathbb{R}} \bar{\alpha} \leq_{\mathbb{R}} 1 \tag{15.59}$$

with the definition of a (real) closed interval. Furthermore, the characteristic function $\chi_{\tilde{A}_{\bar{\alpha}}} : M \to \mathbb{R}$ constitutes the membership function of the crisp set $(M, \chi_{\tilde{A}_{\bar{\alpha}}})$, so that

$$0 \leq_{\mathbb{R}} \chi_{\tilde{A}_{\bar{\alpha}}}(x) \leq_{\mathbb{R}} 1 \tag{15.60}$$

holds according to Note 15.1. Next, we combine the inequalities in (15.59) and (15.60) by means of the Monotony Law for $\cdot_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ to obtain

$$\begin{aligned} [0 =] \quad 0 \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x) &\leq_{\mathbb{R}} \bar{\alpha} \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x), \\ \bar{\alpha} \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x) &\leq_{\mathbb{R}} 1 \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x) \quad [= \chi_{\tilde{A}_{\bar{\alpha}}}(x) \leq_{\mathbb{R}} 1]. \end{aligned}$$

We thus see that the inequalities

$$0 \leq_{\mathbb{R}} \bar{\alpha} \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x) \leq_{\mathbb{R}} 1$$

are true, which give $[y =]\ \chi_{\tilde{A}_{\bar{\alpha}}}(x) \in [0,1]$ with the definition of a (real) closed interval. Since y is arbitrary, we thus showed that $y \in \{\alpha \cdot \chi_{\tilde{A}_\alpha}(x) : \alpha \in [0,1]\}$ implies $y \in [0,1]$ for any y, so that the inclusion (15.57) follows to be true by definition of a subset.

Concerning c), we recall the Completeness of the real unit interval lattice $([0,1], \leq_{[0,1]})$, so that the supremum (with respect to $\leq_{[0,1]}$) of the subset (15.55) of $[0,1]$ exists. To prove (15.58), we apply the Supremum Criterion and establish $m_{\tilde{A}}(x)$ first as an upper bound for the set Y with respect to $\leq_{[0,1]}$. To do this, we prove the universal sentence

$$\forall y\, (y \in \{\alpha \cdot \chi_{\tilde{A}_\alpha}(x) : \alpha \in [0,1]\} \Rightarrow y \leq_{[0,1]} m_{\tilde{A}}(x)), \tag{15.61}$$

letting y be arbitrary and assuming $y \in \{\alpha \cdot \chi_{\tilde{A}_\alpha}(x) : \alpha \in [0,1]\}$ to be true. Due to (15.56), there is then a constant, say $\bar{\alpha}$, such that $\bar{\alpha} \in [0,1]$

and $\bar{\alpha} \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x) = y$ both hold. Observing the truth of the disjunction $x \in \tilde{A}_{\bar{\alpha}} \vee x \notin \tilde{A}_{\bar{\alpha}}$ in light of the Law of the Excluded Middle, we now prove the desired consequent $y \leq_{[0,1]} m_{\tilde{A}}(x)$ by cases. The first case $x \in \tilde{A}_{\bar{\alpha}}$ implies on the one hand with the definition of a characteristic function

$$y = \bar{\alpha} \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x) = \bar{\alpha} \cdot 1 = \bar{\alpha},$$

and on the other hand $m_{\tilde{A}}(x) \geq_{\mathbb{R}} \bar{\alpha}$ with the definition of an α-cut. Consequently, substitution based on the preceding equations yields $y \leq_{\mathbb{R}} m_{\tilde{A}}(x)$. We may use these equations also to substitute y for $\bar{\alpha}$ in the previously established $\bar{\alpha} \in [0,1]$. Note 15.1 gives $0 \leq_{\mathbb{R}} m_{\tilde{A}}(x) \leq_{\mathbb{R}} 1$ and therefore $m_{\tilde{A}}(x) \in [0,1]$ with the definition of a (real) closed interval. Thus, y and $m_{\tilde{A}}(x)$ are both in $[0,1]$, so that we may write $y \leq_{\mathbb{R}} m_{\tilde{A}}(x)$ also as $y \leq_{[0,1]} m_{\tilde{A}}(x)$, recalling the Generation of lattices based on closed intervals in connection with the definition of the real unit interval lattice. The second case $x \notin \tilde{A}_{\bar{\alpha}}$ gives then on the one hand with the definition of a characteristic function

$$y = \bar{\alpha} \cdot \chi_{\tilde{A}_{\bar{\alpha}}}(x) = \bar{\alpha} \cdot 0 = 0.$$

On the other hand, $0 \leq_{\mathbb{R}} m_{\tilde{A}}(x)$ holds as mentioned in the proof of the first case, so that substitution yields $y \leq_{\mathbb{R}} m_{\tilde{A}}(x)$. Clearly, $[0 =]\, y$ and $m_{\tilde{A}}(x)$ are both in $[0,1]$ again, with the consequence that the inequality $y \leq_{[0,1]} m_{\tilde{A}}(x)$ holds as in the first case. As y was arbitrary, we may therefore conclude that the universal sentence (15.61), which shows that $m_{\tilde{A}}(x)$ is indeed an upper bound for $\{\alpha \cdot \chi_{\tilde{A}_{\alpha}}(x) : \alpha \in [0,1]\}$. To show that this upper bound is the least one, we prove, according to the Supremum Criterion,

$$\begin{aligned} &\forall S' \left([S' \in [0,1] \wedge S' <_{[0,1]} m_{\tilde{A}}(x)] \right. \\ &\quad \left. \Rightarrow \exists y \left(y \in \{\alpha \cdot \chi_{\tilde{A}_{\alpha}}(x) : \alpha \in [0,1]\} \wedge S' <_{[0,1]} y \right) \right) \end{aligned} \tag{15.62}$$

letting S' be arbitrary and assuming $S' \in [0,1]$ and $S' <_{[0,1]} m_{\tilde{A}}(x)$ to be both true. To establish the existential sentence, we first define the real number

$$\varepsilon = \frac{1}{2} \cdot (m_{\tilde{A}}(x) - S'), \tag{15.63}$$

and we show that the real number $S' + \varepsilon$ satisfies

$$S' + \varepsilon \in \{\alpha \cdot \chi_{\tilde{A}_{\alpha}}(x) : \alpha \in [0,1]\} \wedge S' <_{[0,1]} S' + \varepsilon. \tag{15.64}$$

To prove the first part of this conjunction, we will demonstrate as a preparation the truth of the conjunction

$$S' + \varepsilon \in [0,1] \wedge (S' + \varepsilon) \cdot \chi_{\tilde{A}_{S'+\varepsilon}}(x) = S' + \varepsilon. \tag{15.65}$$

Let us observe now that the assumed inequality $S' <_{[0,1]} m_{\tilde{A}}(x)$ may be written as $S' <_{\mathbb{R}} m_{\tilde{A}}(x)$, which then implies

$$0 <_{\mathbb{R}} m_{\tilde{A}}(x) - S' \tag{15.66}$$

with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$. Because $\frac{1}{2} <_{\mathbb{R}} 1$ is clearly also true, we may apply the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$ to infer from the previous two inequalities the truth of $\frac{1}{2} \cdot (m_{\tilde{A}}(x) - S') <_{\mathbb{R}} 1 \cdot (m_{\tilde{A}}(x) - S')$, which we may evidently write also as $\varepsilon <_{\mathbb{R}} m_{\tilde{A}}(x) - S'$. We therefore obtain $\varepsilon + S' <_{\mathbb{R}} m_{\tilde{A}}(x)$ (applying again the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$). Then, the disjunction $\varepsilon + S' <_{\mathbb{R}} m_{\tilde{A}}(x) \vee \varepsilon + S' = m_{\tilde{A}}(x)$ also holds, which means that $m_{\tilde{A}}(x) \geq_{\mathbb{R}} S' + \varepsilon$ is true (according to the definition of an induced reflexive partial ordering). By definition of an α-cut, we thus have $x \in \tilde{A}_{S'+\varepsilon}$, and therefore $\chi_{\tilde{A}_{S'+\varepsilon}}(x) = 1$ by definition of a characteristic function. With this equation, we obtain via substitution

$$(S' + \varepsilon) \cdot \chi_{\tilde{A}_{S'+\varepsilon}}(x) = (S' + \varepsilon) \cdot 1 = S' + \varepsilon,$$

so that the second part of the conjunction (15.65) holds. Regarding the first part $S' + \varepsilon \in [0,1]$, we establish $0 \leq_{\mathbb{R}} S' + \varepsilon$ first. Recalling the truth of (15.66) and observing that $0 <_{\mathbb{R}} \frac{1}{2}$ is clearly also true, we may apply the Monotony Law for $\cdot_{\mathbb{R}}$ and $<_{\mathbb{R}}$ to obtain $0 \cdot \frac{1}{2} <_{\mathbb{R}} (m_{\tilde{A}}(x) - S') \cdot \frac{1}{2}$, which evidently gives $0 <_{\mathbb{R}} \varepsilon$ and then

$$[0 \leq_{\mathbb{R}}] \quad S' <_{\mathbb{R}} S' + \varepsilon \tag{15.67}$$

with the initial assumption $S' \in [0,1]$ and with the Monotony Law for $+_{\mathbb{R}}$ and $<_{\mathbb{R}}$. Then, we obtain with the Transitivity Formula for $\leq$ and $<$ the inequality $0 <_{\mathbb{R}} S' + \varepsilon$, so that

$$0 \leq_{\mathbb{R}} S' + \varepsilon \tag{15.68}$$

is then evidently also true. In a next step, establish the truth also of $S' + \varepsilon \leq_{\mathbb{R}} 1$. Recalling that the initial assumption $S' \in [0,1]$ yields $S' \leq_{\mathbb{R}}$ and the definition of a membership function also $m_{\tilde{A}}(x) \leq_{\mathbb{R}} 1$, we may use the Addition of $\leq_{\mathbb{R}}$-inequalities to obtain $m_{\tilde{A}}(x) + S' \leq_{\mathbb{R}} 2$ and therefore $m_{\tilde{A}}(x) \leq_{\mathbb{R}} 2 - S'$ as well as $m_{\tilde{A}}(x) - S' \leq_{\mathbb{R}} 2 - 2S'$ with the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$. We may evidently write the latter inequality as $m_{\tilde{A}}(x) - S' \leq_{\mathbb{R}} 2 \cdot (1 - S')$, which in turn implies $\frac{1}{2} \cdot (m_{\tilde{A}}(x) - S') \leq_{\mathbb{R}} 1 - S'$ with the Monotony Law for $\cdot_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ and then

$$S' + \varepsilon \leq_{\mathbb{R}} 1 \tag{15.69}$$

with the Monotony Law for $+_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ and with (15.62). We may now infer from (15.68) and (15.69) that $S' + \varepsilon \in [0,1]$ holds, which finding completes the proof of the conjunction (15.65). This conjunction shows then that the existential sentence

$$\exists \alpha \, (\alpha \in [0,1] \, \wedge \, \alpha \cdot \chi_{\tilde{A}_\alpha} = S' + \varepsilon)$$

is true, and this existential sentence in turn implies the truth of the first part of the conjunction (15.64), according to a). Regarding the second part, we simply notice that $S' \in [0,1]$ holds by assumption and that $S' + \varepsilon \in [0,1]$ was established in (15.65), so that we may write the previously obtained $<_{\mathbb{R}}$-inequality in (15.67) equivalently as $S' <_{[0,1]} S' + \varepsilon$. Thus, the proof of the conjunction (15.64) is complete as well, which in turn establishes the truth of the existential sentence in (15.62). Since S' was arbitrary, we may therefore conclude that the universal sentence (15.62) holds. Together with the already established fact that $m_{\tilde{A}}(x)$ is an upper bound for the set (15.55) with respect to $\leq_{[0,1]}$, it now follows with the Supremum Criterion that $m_{\tilde{A}}(x)$ is the least upper bound for that set (with respect to $\leq_{[0,1]}$). Symbolically, this means that the proposed equation (15.58) holds.

Because M, $\tilde{A}$ and x were initially arbitrary in the proofs of a) – c), we may infer from the previous findings that the stated theorem is true. □

Exercise 15.8. Prove Part a) of Theorem 15.16.

15.3. Complete Lattices of Fuzzy Sets $(\mathcal{F}(M), \tilde{\subseteq})$

We may directly apply Corollary 5.5 (which is based on the Generation of lattices of functions) to obtain from $([0,1], \leq_{[0,1]})$ the following lattice of functions.

Definition 15.14 (Lattice of membership functions, union & intersection of two fuzzy sets). For any nonempty set M we call

$$([0,1]^M, \preceq) = ([0,1]^M, \curlyvee, \curlywedge, \preceq) = ([0,1]^M, \curlyvee_{[0,1]^M}, \curlywedge_{[0,1]^M}, \preceq_{[0,1]^M}) \tag{15.70}$$

the *lattice of membership functions* on M, where the reflexive partial ordering $\preceq$ of $[0,1]^M$ is specified according to Proposition 3.251, i.e.

$$\forall f, g \, (f, g \in [0,1]^M \Rightarrow [f \preceq g \Leftrightarrow \forall x \, (x \in M \Rightarrow f(x) \leq_{[0,1]} g(x))]), \tag{15.71}$$

and where the join $\curlyvee$ and meet $\curlywedge$ constitute the binary operations

$$\curlyvee_{[0,1]^M} : [0,1]^M \times [0,1]^M \to [0,1]^M, \quad (f,g) \mapsto f \curlyvee g = \overset{\preceq}{\sup}\{f,g\}, \tag{15.72}$$

$$\curlywedge_{[0,1]^M} : [0,1]^M \times [0,1]^M \to [0,1]^M, \quad (f,g) \mapsto f \curlywedge g = \overset{\preceq}{\inf}\{f,g\}. \tag{15.73}$$

For any fuzzy sets $\tilde{A} = (M, m_{\tilde{A}})$, $\tilde{B} = (M, m_{\tilde{B}})$ and $\tilde{C} = (M, m_{\tilde{C}})$, we then say that

(1) $\tilde{C}$ is the *union* of $\tilde{A}$ and $\tilde{B}$, symbolically

$$\tilde{C} = \tilde{A} \tilde{\cup} \tilde{B}, \tag{15.74}$$

iff the membership function of $\tilde{C}$ is the supremum of the pair formed by the membership functions of $\tilde{A}$ and $\tilde{B}$ with respect to $\preceq$, i.e. iff

$$m_{\tilde{C}} = m_{\tilde{A}} \curlyvee m_{\tilde{B}} = \overset{\preceq}{\sup}\{m_{\tilde{A}}, m_{\tilde{B}}\}. \tag{15.75}$$

(2) $\tilde{C}$ is the *intersection* of $\tilde{A}$ and $\tilde{B}$, symbolically

$$\tilde{C} = \tilde{A} \tilde{\cap} \tilde{B}, \tag{15.76}$$

iff the membership function of $\tilde{C}$ is the infimum of the pair formed by the membership functions of $\tilde{A}$ and $\tilde{B}$ with respect to $\preceq$, i.e. iff

$$m_{\tilde{C}} = m_{\tilde{A}} \curlywedge m_{\tilde{B}} = \overset{\preceq}{\inf}\{m_{\tilde{A}}, m_{\tilde{B}}\}. \tag{15.77}$$

We may now apply Proposition 5.33 & 5.23 as well as Exercise 5.13 & 5.25 to a lattice $([0,1]^M, \curlyvee, \curlywedge, \preceq)$ of membership functions to obtain the following properties of the union and intersection of two fuzzy sets, in analogy to the set-theoretical properties of the union and intersection of two non-fuzzy sets.

Corollary 15.17. *For any lattice $([0,1]^M, \curlyvee, \curlywedge, \preceq)$ of membership functions, it is true that*

a) the join $\curlyvee$ on $[0,1]^M$ is an idempotent, commutative and associative binary operation, and the ordered pair $([0,1]^M, \curlyvee)$ is a commutative semigroup.

b) the meet $\curlywedge$ on $[0,1]^M$ is an idempotent, commutative and associative binary operation, and the ordered pair $([0,1]^M, \curlywedge)$ is a commutative semigroup.

Proposition 15.18. *For any lattice $([0,1]^M, \curlyvee, \curlywedge, \preceq)$ of membership functions on M, it is true that the membership function $m_{\tilde{\emptyset}} : M \to [0,1]$ of the empty fuzzy set $\tilde{\emptyset}$ in M is the neutral element of $[0,1]^M$ with respect to the join $\curlyvee$.*

Proof. Letting M, $\curlyvee$, $\curlywedge$ and $\preceq$ be arbitrary sets such that $([0,1]^M, \curlyvee, \curlywedge, \preceq)$ is a lattice of membership functions on M, we observe in light of Note 3.13 that

$$\min[0,1] = 0.$$

Next, we prove that $m_{\tilde{\emptyset}} = M \times \{0\}$ is the bottom of $[0,1]^M$ with respect to the reflexive partial ordering $\preceq$ of $[0,1]^M$. Since the membership function $m_{\tilde{\emptyset}}$ is in $[0,1]^M$ by definition of the latter set, we only need to show that $m_{\tilde{\emptyset}}$ is a lower bound for $[0,1]^M$ with respect to $\preceq$ in order to establish $m_{\tilde{\emptyset}}$ as the bottom of $[0,1]^M$. For this purpose, we verify

$$\forall f \, (f \in [0,1]^M \Rightarrow m_{\tilde{\emptyset}} \preceq f), \tag{15.78}$$

letting $f \in [0,1]^M$ be arbitrary. To demonstrate that this implies $m_{\tilde{\emptyset}} \preceq f$, we prove the universal sentence

$$\forall x \, (x \in M \Rightarrow m_{\tilde{\emptyset}}(x) \leq_{[0,1]} f(x)). \tag{15.79}$$

To do this, we take an arbitrary element $x \in M$, so that we obtain for the corresponding value of the constant (membership) function $m_{\tilde{\emptyset}} = M \times \{0\}$

$$m_{\tilde{\emptyset}}(x) = 0$$

according to Corollary 3.154. Because the value $f(x)$ of the function $f: M \to [0,1]$ is in $[0,1]$ and since $0 = \min[0,1]$ is a lower bound for that interval (with respect to $\leq_{[0,1]}$), we have that the desired inequality $m_{\tilde{\emptyset}}(x) \leq_{[0,1]} f(x)$ is true. As x was arbitrary, we may therefore conclude that the universal sentence (15.79) is true, so that $m_{\tilde{\emptyset}}$ is indeed a lower bound for $[0,1]^M$ and thus the bottom of $[0,1]^M$. It then follows with Proposition 5.14 that $m_{\tilde{\emptyset}}$ is the neutral element of $[0,1]^M$ with respect to the join $\curlyvee$. Because M, $\curlyvee$, $\curlywedge$ and $\preceq$ were initially arbitrary sets, we may infer from this finding the truth of the stated proposition. □

Exercise 15.9. Show for any lattice $([0,1]^M, \curlyvee, \curlywedge, \preceq)$ of membership functions on M that the membership function $m_{\tilde{M}}: M \to [0,1]$ of the fundamental fuzzy set $\tilde{M}$ in M is the neutral element of $[0,1]^M$ with respect to the meet $\curlywedge$.

We now see in light of Theorem 3.253 that the lattice completeness at the level of the codomain $[0,1]$ of a membership function carries over to the level of an entire set of membership functions on a given nonempty set.

Corollary 15.19 (Completeness of lattices of membership functions). *It is true for any set $M \neq \emptyset$ that the lattice of membership functions $([0,1]^M, \preceq)$ is a complete lattice.*

The reflexive partial ordering $\preceq$ of the set of membership functions on a set M is the backbone of the concept of inclusion for fuzzy sets.

Definition 15.15 (Sub-fuzzy set). For any fuzzy sets $\tilde{A} = (M, m_{\tilde{A}})$ and $\tilde{B} = (M, m_{\tilde{B}})$ with $M \neq \emptyset$, we say that $\tilde{A}$ is a *sub-fuzzy set* of $\tilde{B}$, symbolically

$$\tilde{A} \mathrel{\tilde{\subseteq}} \tilde{B}, \tag{15.80}$$

iff the membership function of $\tilde{A}$ is less than or equal to the membership function of $\tilde{B}$, i.e. iff

$$m_{\tilde{A}} \preceq m_{\tilde{B}}. \tag{15.81}$$

Proposition 15.20. *The following sentences are true for any set $M \neq \emptyset$ and any fuzzy sets $\tilde{A} = (M, m_{\tilde{A}})$ and $\tilde{B} = (M, m_{\tilde{B}})$ in M.*

a) If $\tilde{A}$ is a sub-fuzzy set of $\tilde{B}$, then the support of $\tilde{A}$ is included in the support of $\tilde{B}$, i.e.

$$\tilde{A} \mathrel{\tilde{\subseteq}} \tilde{B} \;\Rightarrow\; \mathrm{supp}(\tilde{A}) \subseteq \mathrm{supp}(\tilde{B}). \tag{15.82}$$

b) If $\tilde{A}$ is a sub-fuzzy set of $\tilde{B}$, then the height of $\tilde{A}$ is less than or equal to the height of $\tilde{B}$ with respect to $\leq_{[0,1]}$, i.e.

$$\tilde{A} \mathrel{\tilde{\subseteq}} \tilde{B} \;\Rightarrow\; \mathrm{height}(\tilde{A}) \leq_{[0,1]} \mathrm{height}(\tilde{B}). \tag{15.83}$$

Proof. We let M, $\tilde{A}$, $\tilde{B}$, $m_{\tilde{A}}$ and $m_{\tilde{B}}$ be arbitrary sets such that M is nonempty and such that $\tilde{A} = (M, m_{\tilde{A}})$ and $\tilde{B} = (M, m_{\tilde{B}})$ are fuzzy sets in M. Then, to prove the stated implications directly, we assume that the fuzzy inclusion $\tilde{A} \mathrel{\tilde{\subseteq}} \tilde{B}$ holds. Concerning a), we show that the inclusion $\mathrm{supp}(\tilde{A}) \subseteq \mathrm{supp}(\tilde{B})$ follows to be true. For this purpose, we prove the universal sentence

$$\forall x \, (x \in \mathrm{supp}(\tilde{A}) \Rightarrow x \in \mathrm{supp}(\tilde{B})), \tag{15.84}$$

letting x be arbitrary and assuming $x \in \mathrm{supp}(\tilde{A})$. By definition of the support of a fuzzy set, this assumption implies $x \in M$ and $0 <_{\mathbb{R}} m_{\tilde{A}}(x)$. Let us now observe that the assumed fuzzy inclusion $\tilde{A} \mathrel{\tilde{\subseteq}} \tilde{B}$ implies $m_{\tilde{A}} \preceq m_{\tilde{B}}$, by definition of a sub-fuzzy set. Consequently, we obtain the inequality $m_{\tilde{A}}(x) \leq_{[0,1]} m_{\tilde{B}}(x)$, according to (15.71). As the elements $m_{\tilde{A}}(x)$ and $m_{\tilde{B}}(x)$ are evidently both in $\mathbb{R}$, we may write this inequality also as $m_{\tilde{A}}(x) \leq_{\mathbb{R}} m_{\tilde{B}}(x)$. Together with the previously established inequality $0 <_{\mathbb{R}} m_{\tilde{A}}(x)$, this implies now $0 <_{\mathbb{R}} m_{\tilde{B}}(x)$ with the Transitivity Formula for $<$ and $\leq$. The conjunction of $x \in M$ and the preceding inequality gives us then $x \in \mathrm{supp}(\tilde{B})$ by definition of the support of a fuzzy set, as desired. As x was arbitrary, we may therefore conclude that the universal sentence (15.84) holds, so that the inclusion $\mathrm{supp}(\tilde{A}) \subseteq \mathrm{supp}(\tilde{B})$ follows to be true by definition of a subset. Thus, the stated implication (15.82) is indeed true.

Concerning b), we now seek to establish the consequent $\mathrm{height}(\tilde{A}) \leq_{[0,1]} \mathrm{height}(\tilde{B})$. Since $\mathrm{height}(\tilde{A})$ represents the supremum of the range of $m_{\tilde{A}}$ with respect to $\leq_{[0,1]}$, the idea will be to demonstrate that $\mathrm{height}(\tilde{B})$ is an upper bound for the range of $m_{\tilde{A}}$ with respect to $\leq_{[0,1]}$. We do this by proving the universal sentence

$$\forall y\,(y \in \mathrm{ran}(m_{\tilde{A}}) \Rightarrow y \leq_{[0,1]} \mathrm{height}(\tilde{B})), \tag{15.85}$$

letting $\bar{y}$ be arbitrary such that $\bar{y} \in \mathrm{ran}(m_{\tilde{A}})$ holds. By definition of a range, there is then a constant, say $\bar{x}$, satisfying $(\bar{x}, \bar{y}) \in m_{\tilde{A}}$. On the one hand, we may write this in function notation as $\bar{y} = m_{\tilde{A}}(\bar{x})$; on the other hand, the definition of a domain yields $\bar{x} \in M$ $[= \mathrm{dom}(m_{\tilde{A}})]$. We already mentioned in the proof of a) that the initial assumption $\tilde{A} \tilde{\subseteq} \tilde{B}$ implies $m_{\tilde{A}} \preceq m_{\tilde{B}}$, and therefore $\bar{x} \in M$ yields $[\bar{y} =]\ m_{\tilde{A}}(\bar{x}) \leq_{[0,1]} m_{\tilde{B}}(\bar{x})$; thus, we have $\bar{y} \leq_{[0,1]} m_{\tilde{B}}(\bar{x})$. As $\mathrm{height}(\tilde{B})$ is by definition of the height of a fuzzy set the supremum of the range of $m_{\tilde{B}}$ and thus an upper bound for that range, we obtain the true universal sentence

$$\forall y\,(y \in \mathrm{ran}(m_{\tilde{B}}) \Rightarrow y \leq_{[0,1]} \mathrm{height}(\tilde{B})).$$

Evidently, the value $m_{\tilde{B}}(\bar{x})$ is an element of the range of $m_{\tilde{B}}$, so that the preceding universal sentence gives us the inequality $m_{\tilde{B}}(\bar{x}) \leq_{[0,1]} \mathrm{height}(\tilde{B})$. Together with the previously obtained inequality $\bar{y} \leq_{[0,1]} m_{\tilde{B}}(\bar{x})$, this further implies $\bar{y} \leq_{[0,1]} \mathrm{height}(\tilde{B})$ with the transitivity of the reflexive partial ordering $\leq_{[0,1]}$, proving the implication in (15.85). Since $\bar{y}$ is arbitrary, we may therefore conclude that (15.85) holds, which universal sentence in turn implies that $\mathrm{height}(\tilde{B})$ is an upper bound for the range of $m_{\tilde{A}}$ (with respect to $\leq_{[0,1]}$). Because $\mathrm{height}(\tilde{A})$ is the least upper bound (with respect to $\leq_{[0,1]}$), we then obtain the desired inequality in (15.83) with the Characterization of the supremum. Thus, the implication b) also holds.

As the sets M, $\tilde{A}$, $\tilde{B}$, $m_{\tilde{A}}$ and $m_{\tilde{B}}$ were initially arbitrary, we may infer from the preceding findings the truth of the proposition. □

Theorem 15.21 (Fuzzy Inclusion Criterion). *It is true for any set $M \neq \emptyset$ and any fuzzy sets $\tilde{A}$ and $\tilde{B}$ in M that $\tilde{A}$ is fuzzy included in $\tilde{B}$ iff the α-level set of $\tilde{A}$ is included in the α-level set of $\tilde{A}$ for any α greater* *than* 0 *and less than or equal to* 1*, i.e.*

$$\forall \tilde{A}, \tilde{B} \, (\tilde{A}, \tilde{B} \in \mathcal{F}(M) \Rightarrow \Big[\tilde{A} \mathrel{\tilde{\subseteq}} \tilde{B} \Leftrightarrow \forall \alpha \, (0 <_{\mathbb{R}} \alpha \leq_{\mathbb{R}} 1 \Rightarrow \tilde{A}_\alpha \subseteq \tilde{B}_\alpha)\Big]). \tag{15.86}$$

Proof. We take arbitrary sets M, $\tilde{A}$ and $\tilde{B}$, and we assume $M \neq \emptyset$ as well as $\tilde{A}, \tilde{B} \in \mathcal{F}(M)$ to be true. Thus, there are particular membership functions $m_{\tilde{A}} : M \to [0,1]$ and $m_{\tilde{B}} : M \to [0,1]$ with $\tilde{A} = (M, m_{\tilde{A}})$ as well as $\tilde{B} = (M, m_{\tilde{B}})$. We prove the first part ('$\Rightarrow$') of the equivalence in (15.86) directly, assuming the fuzzy inclusion $\tilde{A} \mathrel{\tilde{\subseteq}} \tilde{B}$ to hold, which we may write also as $m_{\tilde{A}} \preceq m_{\tilde{B}}$ by definition of a sub-fuzzy set. This inequality means in view of the specification of $\preceq$ in (15.71) that the universal sentence

$$\forall x \, (x \in M \Rightarrow m_{\tilde{A}}(x) \leq_{[0,1]} m_{\tilde{B}}(x)) \tag{15.87}$$

is true. Let us now establish the desired consequent

$$\forall \alpha \, (0 <_{\mathbb{R}} \alpha \leq_{\mathbb{R}} 1 \Rightarrow \tilde{A}_\alpha \subseteq \tilde{B}_\alpha), \tag{15.88}$$

letting α be arbitrary and assuming $0 <_{\mathbb{R}} \alpha \leq_{\mathbb{R}} 1$ to hold. To show that this assumption implies the inclusion $\tilde{A}_\alpha \subseteq \tilde{B}_\alpha$, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall x \, (x \in \tilde{A}_\alpha \Rightarrow x \in \tilde{B}_\alpha). \tag{15.89}$$

To do this, we take an arbitrary x, and we assume that $x \in \tilde{A}_\alpha$ is true. Consequently, we obtain $x \in m_{\tilde{A}}^{-1}[[\alpha, 1]]$ with the Characterization of α-cuts, therefore $m_{\tilde{A}}(x) \in [\alpha, 1]$ by definition of an inverse image, and consequently

$$\alpha \leq_{\mathbb{R}} m_{\tilde{A}}(x) \leq_{\mathbb{R}} 1 \tag{15.90}$$

with the definition of a (real) closed interval. Let us observe next that the previous assumption $x \in \tilde{A}_\alpha$ implies also $x \in M$ by definition of an α-cut, so that (15.87) yields the inequality $m_{\tilde{A}}(x) \leq_{[0,1]} m_{\tilde{B}}(x)$. Since $\leq_{\mathbb{R}}$ generates $\leq_{[0,1]}$ and since the values of any membership function are between 0 and 1, we may write the preceding inequality also as

$$m_{\tilde{A}}(x) \leq_{\mathbb{R}} m_{\tilde{B}}(x) \leq_{\mathbb{R}} 1. \tag{15.91}$$

An application of the transitivity of the standard total ordering $\leq_{\mathbb{R}}$ gives then $\alpha \leq_{\mathbb{R}} m_{\tilde{B}}(x) \leq_{\mathbb{R}} 1$, so that $m_{\tilde{B}}(x) \in [\alpha, 1]$ holds (by definition of a closed interval). Consequently, we obtain $x \in m_{\tilde{B}}^{-1}[[\alpha, 1]]$ (by definition of an inverse image), and therefore $x \in \tilde{B}_\alpha$ (with the Characterization of α-cuts). The implication in (15.89) is thus true, and as x was arbitrary, we may infer from this the truth of the desired inclusion $\tilde{A}_\alpha \subseteq \tilde{B}_\alpha$. This in turn proves the implication in (15.88), and since α was arbitrary, we may therefore conclude that the first part of the equivalence in (15.86) holds.

We now prove the second part ('$\Leftarrow$') by contraposition, by assuming the negation of the universal sentence (15.88) to be true and by demonstrating that the negation $\neg \tilde{A} \tilde{\subseteq} \tilde{B}$ also holds. The assumed negation implies with the Negation Law for universal implications that there exists a constant, say $\bar{\alpha}$, such that $0 <_{\mathbb{R}} \bar{\alpha} \leq_{\mathbb{R}} 1$ and the negation $\neg \tilde{A}_\alpha \subseteq \tilde{B}_\alpha$ are both true. This negation implies (with the definition of a subset)

$$\neg \forall x \, (x \in \tilde{A}_{\bar{\alpha}} \Rightarrow x \in \tilde{B}_{\bar{\alpha}}),$$

so that another application of the Negation Law for universal implications yields the existence of a particular constant $\bar{x} \in \tilde{A}_{\bar{\alpha}}$ such that the negation $\neg \bar{x} \in \tilde{B}_{\bar{\alpha}}$ is true. The former implies $\bar{x} \in M$ (by definition of an α-cut) as well as $\bar{x} \in m_{\tilde{A}}^{-1}[[\bar{\alpha}, 1]]$ (according to the Characterization of α-cuts). Consequently, we obtain $m_{\tilde{A}}(\bar{x}) \in [\bar{\alpha}, 1]$ (by definition of an inverse image) and furthermore (by definition of a closed interval)

$$\bar{\alpha} \leq_{\mathbb{R}} m_{\tilde{A}}(\bar{x}) \wedge m_{\tilde{A}}(\bar{x}) \leq_{\mathbb{R}} 1. \tag{15.92}$$

Using the same arguments, the previously established negation $\neg \bar{x} \in \tilde{B}_{\bar{\alpha}}$ implies first $\neg \bar{x} \in m_{\tilde{B}}^{-1}[[\bar{\alpha}, 1]]$, then $\neg m_{\tilde{B}} \in [\bar{\alpha}, 1]$, and subsequently

$$\neg \left[\bar{\alpha} \leq_{\mathbb{R}} m_{\tilde{B}}(\bar{x}) \wedge m_{\tilde{B}}(\bar{x}) \leq_{\mathbb{R}} 1\right],$$

which in turn implies with De Morgan's Law for the conjunction

$$\neg \bar{\alpha} \leq_{\mathbb{R}} m_{\tilde{B}}(\bar{x}) \vee \neg m_{\tilde{B}}(\bar{x}) \leq_{\mathbb{R}} 1. \tag{15.93}$$

Recalling that the values of any membership function are between 0 and 1, we have that $m_{\tilde{B}}(\bar{x}) \leq_{\mathbb{R}} 1$ is true, so that the second part of the disjunction $\neg m_{\tilde{B}}(\bar{x}) \leq_{\mathbb{R}} 1$ is false. Thus, its first part $\neg \bar{\alpha} \leq_{\mathbb{R}} m_{\tilde{B}}(\bar{x})$ is true, and this negation gives $m_{\tilde{B}}(\bar{x}) <_{\mathbb{R}} \bar{\alpha}$ with the Negation Formula for $\leq$. Together with the first part $\bar{\alpha} \leq_{\mathbb{R}} m_{\tilde{A}}(\bar{x})$ of the conjunction (15.92), this inequality implies $m_{\tilde{B}}(\bar{x}) <_{\mathbb{R}} m_{\tilde{A}}(\bar{x})$ with the Transitivity Formula for $<$ and $\leq$, and therefore $\neg m_{\tilde{A}}(\bar{x}) \leq_{\mathbb{R}} m_{\tilde{B}}(\bar{x})$ with the Negation Formula for $\leq$. Because both of the degrees of membership are between 0 and 1 and thus elements of $[0, 1]$ by definition of a closed interval, we may write the preceding negation also in terms of $\leq_{[0,1]}$ as $\neg m_{\tilde{A}}(\bar{x}) \leq_{[0,1]} m_{\tilde{B}}(\bar{x})$. Recalling now that $\bar{x} \in M$ is also true, we thus see that the existential sentence

$$\exists x \, (x \in M \wedge \neg m_{\tilde{A}}(x) \leq_{[0,1]} m_{\tilde{B}}(x))$$

holds, which then evidently implies the truth of the negated universal sentence

$$\neg\forall x\,(x \in M \Rightarrow m_{\tilde{A}}(x) \leq_{[0,1]} m_{\tilde{B}}(x)).$$

Consequently, we obtain $\neg m_{\tilde{A}} \preceq m_{\tilde{B}}$ and therefore also the desired negation $\neg\tilde{A} \mathrel{\tilde{\subseteq}} \tilde{B}$, completing the proof of the equivalence.

Since $\tilde{A}$ and $\tilde{B}$ are arbitrary, we may now infer from the truth of that equivalence the truth of the universal sentence (15.86). Then, as M was initially also arbitrary, we may finally conclude that the stated theorem holds. □

The concept of 'fuzzy inclusion' can be shown to define a reflexive partial ordering of any set of fuzzy sets.

Proposition 15.22. *The following sentences are true for any set $M \neq \emptyset$.*

a) There exists a unique set $\tilde{\subseteq}_{\mathcal{F}(M)}$ consisting of all ordered pairs $(\tilde{A}, \tilde{B})$ that are formed by some fuzzy sets $(M, m_{\tilde{A}})$ and $(M, m_{\tilde{B}})$ in M such that the membership function $m_{\tilde{A}}$ of $\tilde{A}$ is less than or equal to the membership function $m_{\tilde{B}}$ of $\tilde{B}$, in the sense that

$$\forall Z\,(Z \in \tilde{\subseteq}_{\mathcal{F}(M)} \Leftrightarrow [Z \in \mathcal{F}(M) \times \mathcal{F}(M) \\ \wedge\ \exists m_{\tilde{A}}, m_{\tilde{B}}\,(m_{\tilde{A}} \preceq m_{\tilde{B}}\ \wedge\ ((M, m_{\tilde{A}}), (M, m_{\tilde{B}})) = Z)]). \tag{15.94}$$

b) This set $\tilde{\subseteq}_{\mathcal{F}(M)}$ is a binary relation on $\mathcal{F}(M)$, which satisfies also

$$\forall Z\,(Z \in \tilde{\subseteq}_{\mathcal{F}(M)} \Leftrightarrow \exists m_{\tilde{A}}, m_{\tilde{B}}\,(m_{\tilde{A}} \preceq m_{\tilde{B}} \wedge ((M, m_{\tilde{A}}), (M, m_{\tilde{B}})) = Z)). \tag{15.95}$$

c) Furthermore, the binary relation $\tilde{\subseteq}_{\mathcal{F}(M)}$ is a reflexive partial ordering of $\mathcal{F}(M)$.

Proof. We let $M \neq \emptyset$ be an arbitrary and apply the Axiom of Specification in connection with the Equality Criterion for sets to establish the unique existence of a set $\tilde{\subseteq}_{\mathcal{F}(M)}$ such that (15.94) holds. Since $Z \in \tilde{\subseteq}_{\mathcal{F}(M)}$ implies especially $Z \in \mathcal{F}(M) \times \mathcal{F}(M)$ for any Z, it follows with the definition of a subset that $\tilde{\subseteq}_{\mathcal{F}(M)}$ is a binary relation on $\mathcal{F}(M)$. Let us next verify that the set $\tilde{\subseteq}_{\mathcal{F}(M)}$ satisfies also (15.95). Letting Z be arbitrary and assuming first $Z \in \tilde{\subseteq}_{\mathcal{F}(M)}$ to be true, we see that the desired existential sentence

$$\exists m_{\tilde{A}}, m_{\tilde{B}}\,(m_{\tilde{A}} \preceq m_{\tilde{B}}\ \wedge\ ((M, m_{\tilde{A}}), (M, m_{\tilde{B}})) = Z) \tag{15.96}$$

follows especially to be true with (15.94). Assuming now conversely the preceding existential sentence to be true, so that there are particular sets

$\bar{m}_{\tilde{A}}$ and $\bar{m}_{\tilde{B}}$ satisfying $\bar{m}_{\tilde{A}} \preceq \bar{m}_{\tilde{B}}$ and $((M, \bar{m}_{\tilde{A}}), (M, \bar{m}_{\tilde{B}})) = Z$, we may evidently write the inequality also as $(\bar{m}_{\tilde{A}}, \bar{m}_{\tilde{B}}) \in \preceq_{[0,1]^M}$, recalling (15.70). Because the reflexive partial ordering $\preceq_{[0,1]^M}$ of $[0,1]^M$ is by definition a binary relation on $[0,1]^M$ and thus a subset of $[0,1]^M \times [0,1]^M$, it follows with the definition of the Cartesian product of two sets that $\bar{m}_{\tilde{A}}, \bar{m}_{\tilde{B}} \in [0,1]^M$ holds. Thus, defining now the ordered pairs $\tilde{A} = (M, \bar{m}_{\tilde{A}})$ and $\tilde{B} = (M, \bar{m}_{\tilde{B}})$, the existence of the particular functions $\bar{m}_{\tilde{A}} : M \to [0,1]$ and $\bar{m}_{\tilde{B}} : M \to [0,1]$ shows these sets $\tilde{A}$ and $\tilde{B}$ are fuzzy sets in M (by definition) so that $\tilde{A}, \tilde{B} \in \mathcal{F}(M)$ holds (by definition of a set of fuzzy sets). Consequently,

$$[((M, \bar{m}_{\tilde{A}}), (M, \bar{m}_{\tilde{B}})) = (\tilde{A}, \tilde{B}) =] \quad Z \in \mathcal{F}(M) \times \mathcal{F}(M)$$

is true by definition of the Cartesian product of two sets. Together with the assumed existential sentence (15.96), this implies now $Z \in \tilde{\subseteq}_{\mathcal{F}(M)}$, so that the proof of the equivalence (15.95) is complete. As Z was arbitrary, we may therefore conclude that the binary relation $\tilde{\subseteq}_{\mathcal{F}(M)}$ satisfies indeed (15.95).

We now prove that $\tilde{\subseteq}_{\mathcal{F}(M)}$ is reflexive, letting $\tilde{A} \in \mathcal{F}(M)$ be arbitrary. Thus, $\tilde{A}$ is a fuzzy set in M, so that there exists a particular set $\bar{m}_{\tilde{A}}$ with $\bar{m}_{\tilde{A}} : M \to [0,1]$ and $\tilde{A} = (M, \bar{m}_{\tilde{A}})$. Observing now that $\bar{m}_{\tilde{A}} \preceq \bar{m}_{\tilde{A}}$ holds with the reflexivity of $\preceq$ and defining also the set

$$Z = ((M, \bar{m}_{\tilde{A}}), (M, \bar{m}_{\tilde{A}})) \quad [= (\tilde{A}, \tilde{B})],$$

we now see clearly that there exist sets $m_{\tilde{A}}, m_{\tilde{B}}$ with $m_{\tilde{A}} \preceq m_{\tilde{B}}$ and

$$((M, m_{\tilde{A}}), (M, m_{\tilde{B}})) = Z \quad [= ((M, \bar{m}_{\tilde{A}}), (M, \bar{m}_{\tilde{A}}))],$$

so that we find $[(\tilde{A}, \tilde{B}) =]$ $Z \in \tilde{\subseteq}_{\mathcal{F}(M)}$ with (15.96). We already showed that $\tilde{\subseteq}_{\mathcal{F}(M)}$ is a binary relation, so that we may write the preceding finding also as $\tilde{A} \,\tilde{\subseteq}_{\mathcal{F}(M)}\, \tilde{B}$. As $\tilde{A}$ was arbitrary, we may therefore conclude that $\tilde{\subseteq}_{\mathcal{F}(M)}$ is indeed reflexive, by definition.

Next, to prove that $\tilde{\subseteq}_{\mathcal{F}(M)}$ is antisymmetric, we take arbitrary sets $\tilde{A}, \tilde{B} \in \mathcal{F}(M)$, we assume $\tilde{A} \,\tilde{\subseteq}_{\mathcal{F}(M)}\, \tilde{B}$ as well as $\tilde{B} \,\tilde{\subseteq}_{\mathcal{F}(M)}\, \tilde{A}$ to be both true, and we show that $\tilde{A} = \tilde{B}$ is implied. Since we may write these assumptions also as $(\tilde{A}, \tilde{B}) \in \tilde{\subseteq}_{\mathcal{F}(M)}$ and $(\tilde{B}, \tilde{A}) \in \tilde{\subseteq}_{\mathcal{F}(M)}$, there exist then on the one hand particular sets $\bar{m}_{\tilde{A}}, \bar{m}_{\tilde{B}}$ with

$$\bar{m}_{\tilde{A}} \preceq \bar{m}_{\tilde{B}}$$

and

$$((M, \bar{m}_{\tilde{A}}), (M, \bar{m}_{\tilde{B}})) = (\tilde{A}, \tilde{B}),$$

so that the Equality Criterion for ordered pairs yields the two equations

$$(M, \bar{m}_{\tilde{A}}) = \tilde{A},$$
$$(M, \bar{m}_{\tilde{B}}) = \tilde{B}.$$

On the other hand, there are particular sets $\bar{\bar{m}}_{\tilde{B}}, \bar{\bar{m}}_{\tilde{A}}$ such that

$$\bar{\bar{m}}_{\tilde{B}} \preceq \bar{\bar{m}}_{\tilde{A}}$$

and

$$((M, \bar{\bar{m}}_{\tilde{B}}), (M, \bar{\bar{m}}_{\tilde{A}})) = (\tilde{B}, \tilde{A}),$$

so that the Equality Criterion for ordered pairs gives now the two equations

$$(M, \bar{\bar{m}}_{\tilde{B}}) = \tilde{B},$$
$$(M, \bar{\bar{m}}_{\tilde{A}}) = \tilde{A}.$$

Combining now the two equations for $\tilde{A}$ and then also the two equations for $\tilde{B}$, we obtain

$$(M, \bar{m}_{\tilde{A}}) = (M, \bar{\bar{m}}_{\tilde{A}}),$$
$$(M, \bar{m}_{\tilde{B}}) = (M, \bar{\bar{m}}_{\tilde{B}}),$$

so that the Equality Criterion for ordered pairs yields in particular the equations

$$\bar{m}_{\tilde{A}} = \bar{\bar{m}}_{\tilde{A}},$$
$$\bar{m}_{\tilde{B}} = \bar{\bar{m}}_{\tilde{B}}.$$

We may now apply substitutions based on these two equations to infer from the previously established $\bar{\bar{m}}_{\tilde{B}} \preceq \bar{\bar{m}}_{\tilde{A}}$ the truth of $\bar{m}_{\tilde{B}} \preceq \bar{m}_{\tilde{A}}$. Together with the previously obtained $\bar{m}_{\tilde{A}} \preceq \bar{m}_{\tilde{B}}$, this implies now with the antisymmetry of the reflexive partial ordering $\preceq$ the equation $\bar{m}_{\tilde{A}} = \bar{m}_{\tilde{B}}$. We then obtain via substitution

$$\tilde{A} = (M, \bar{m}_{\tilde{A}}) = (M, \bar{m}_{\tilde{B}}) = \tilde{B},$$

so that we arrived at the desired consequent $\tilde{A} = \tilde{B}$. As $\tilde{A}, \tilde{B}$ were arbitrary, we may infer from the truth of this equation that $\tilde{\subseteq}_{\mathcal{F}(M)}$ is also antisymmetric, by definition.

The transitivity of $\tilde{\subseteq}_{\mathcal{F}(M)}$ can be established similarly as the antisymmetry. In summary, we thus showed that the binary relation $\tilde{\subseteq}_{\mathcal{F}(M)}$ is a reflexive partial ordering of $\mathcal{F}(M)$. As M was an arbitrary set in the proofs of a) – c), we may now finally conclude that the stated universal sentences are all true. □

Exercise 15.10. Establish the transitivity of the binary relation $\tilde{\subseteq}_{\mathcal{F}(M)}$.

Corollary 15.23. *For any fuzzy sets $\tilde{A} = (M, m_{\tilde{A}})$ and $\tilde{B} = (M, m_{\tilde{B}})$ with $M \neq \emptyset$, it is true that*

$$\tilde{A} \,\tilde{\subseteq}_{\mathcal{F}(M)}\, \tilde{B} \;\Leftrightarrow\; m_{\tilde{A}} \preceq m_{\tilde{B}}. \tag{15.97}$$

Proof. Letting M, $\tilde{A}$, $\tilde{B}$, $\bar{m}_{\tilde{A}}$, $\bar{m}_{\tilde{B}}$ be arbitrary sets such that $M \neq \emptyset$, $\tilde{A} = (M, \bar{m}_{\tilde{A}})$ and $\tilde{B} = (M, \bar{m}_{\tilde{B}})$ are fuzzy sets (in M), we first assume $\tilde{A} \,\tilde{\subseteq}_{\mathcal{F}(M)}\, \tilde{B}$ to be true. Since we may write the preceding assumption as

$$((M, \bar{m}_{\tilde{A}}), (M, \bar{m}_{\tilde{B}})) \in \tilde{\subseteq}_{\mathcal{F}(M)},$$

we see in light of (15.95) that there exist particular sets $\bar{\bar{m}}_{\tilde{A}}$ and $\bar{\bar{m}}_{\tilde{B}}$ such that $\bar{\bar{m}}_{\tilde{A}} \preceq \bar{\bar{m}}_{\tilde{B}}$ and

$$((M, \bar{\bar{m}}_{\tilde{A}}), (M, \bar{\bar{m}}_{\tilde{B}})) = ((M, \bar{m}_{\tilde{A}}), (M, \bar{m}_{\tilde{B}}))$$

hold. The latter equation implies with the Equality Criterion for ordered pairs first $(M, \bar{\bar{m}}_{\tilde{A}}) = (M, \bar{m}_{\tilde{A}})$ and $(M, \bar{\bar{m}}_{\tilde{B}}) = (M, \bar{m}_{\tilde{B}})$, and then also in particular $\bar{\bar{m}}_{\tilde{A}} = \bar{m}_{\tilde{A}}$ as well as $\bar{\bar{m}}_{\tilde{B}} = \bar{m}_{\tilde{B}}$. Therefore, the previously established $\bar{\bar{m}}_{\tilde{A}} \preceq \bar{\bar{m}}_{\tilde{B}}$ yields via substitution $\bar{m}_{\tilde{A}} \preceq \bar{m}_{\tilde{B}}$, as desired. Let us now conversely assume $\bar{m}_{\tilde{A}} \preceq \bar{m}_{\tilde{B}}$ to be true. Forming now the ordered pair $((M, \bar{m}_{\tilde{A}}), (M, \bar{m}_{\tilde{B}})) = (\tilde{A}, \tilde{B})$, we thus see that there are sets $m_{\tilde{A}}, m_{\tilde{B}}$ satisfying $m_{\tilde{A}} \preceq m_{\tilde{B}}$ and $((M, m_{\tilde{A}}), (M, m_{\tilde{B}})) = (\tilde{A}, \tilde{B})$, so that $(\tilde{A}, \tilde{B})$ follows to be an element of $\tilde{\subseteq}_{\mathcal{F}(M)}$, according to (15.95). Since we may write this finding also as $\tilde{A} \,\tilde{\subseteq}_{\mathcal{F}(M)}$, the proof of the equivalence (15.97) is now complete. As M, $\tilde{A}$, $\tilde{B}$, $\bar{m}_{\tilde{A}}$ and $\bar{m}_{\tilde{B}}$ were initially arbitrary sets, we may therefore conclude that the stated corollary is indeed true. □

Note 15.5. In view of the equivalence (15.97), we see for any fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(M)$ that $\tilde{A}$ is a sub-fuzzy set of $\tilde{B}$ iff $\tilde{A} \,\tilde{\subseteq}_{\mathcal{F}(M)}\, \tilde{B}$. We may therefore write the reflexivity, antisymmetry and transitivity of the reflexive partial ordering $\tilde{\subseteq}_{\mathcal{F}(M)}$, respectively, as

$$\forall \tilde{A}\, (\tilde{A} \in \mathcal{F}(M) \Rightarrow \tilde{A} \,\tilde{\subseteq}\, \tilde{A}), \tag{15.98}$$

$$\forall \tilde{A}, \tilde{B}\, (\tilde{A}, \tilde{B} \in \mathcal{F}(M) \Rightarrow [(\tilde{A} \,\tilde{\subseteq}\, \tilde{B} \wedge \tilde{B} \,\tilde{\subseteq}\, \tilde{A}) \Rightarrow \tilde{A} = \tilde{B}]), \tag{15.99}$$

$$\forall \tilde{A}, \tilde{B}, \tilde{C}\, (\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(M) \Rightarrow [(\tilde{A} \,\tilde{\subseteq}\, \tilde{B} \wedge \tilde{B} \,\tilde{\subseteq}\, \tilde{C}) \Rightarrow \tilde{A} \,\tilde{\subseteq}\, \tilde{C}]). \tag{15.100}$$

Corollary 15.24. *It is true for any set $M \neq \emptyset$ that the empty fuzzy set $\tilde{\emptyset}$ in M is the minimum of the set $\mathcal{F}(M)$ of fuzzy sets in M with respect to $\tilde{\subseteq}_{\mathcal{F}(M)}$, i.e.*

$$\min \mathcal{F}(M) = \tilde{\emptyset} \quad [= (M, m_{\tilde{\emptyset}}) = (M, \chi_{\emptyset}) = (M, M \times \{0\})]. \tag{15.101}$$

Proof. We let $M \neq \emptyset$ and $\tilde{A}$ be arbitrary sets, and we show first that the empty fuzzy set $\tilde{\emptyset} = (M, m_{\tilde{\emptyset}})$ is a lower bound for the set $\mathcal{F}(M)$, i.e.

$$\forall \tilde{A} \, (\tilde{A} \in \mathcal{F}(M) \Rightarrow \tilde{\emptyset} \mathrel{\tilde{\subseteq}_{\mathcal{F}(M)}} \tilde{A}). \tag{15.102}$$

For this purpose, we assume $\tilde{A} \in \mathcal{F}(M)$ to be true, so that $\tilde{A}$ is a fuzzy set in M. Consequently, there is a particular membership function $m_{\tilde{A}} : M \rightarrow$ such that $\tilde{A} = (M, m_{\tilde{A}})$; thus, we have $m_{\tilde{A}} \in [0,1]^M$. Furthermore, the empty fuzzy set $\tilde{\emptyset} = (M, m_{\tilde{\emptyset}})$ is defined by the (constant) membership function $m_{\tilde{\emptyset}} : M \rightarrow [0,1]$, which is a lower bound for $[0,1]^M$, as shown by (15.78). Therefore, $m_{\tilde{A}} \in [0,1]^M$ implies $m_{\tilde{\emptyset}} \preceq m_{\tilde{A}}$, which we may write also as $\tilde{\emptyset} \mathrel{\tilde{\subseteq}_{\mathcal{F}(M)}} \tilde{A}$ in view of Corollary 15.23, proving the implication in (15.101). As $\tilde{A}$ is arbitrary, we may now infer from the truth of this implication the truth of the stated universal sentence, so that $\tilde{\emptyset}$ is indeed a lower bound for $\mathcal{F}(M)$. Moreover, $\tilde{\emptyset} = (M, m_{\tilde{\emptyset}})$ is itself an element of $\mathcal{F}(M)$ by definition of the latter set, so that the lower bound $\tilde{\emptyset}$ for $\mathcal{F}(M)$ is by definition the minimum of that set. Since M was arbitrary, we may therefore conclude that the stated universal sentence is true. □

Note 15.6. Since we may write $\tilde{\emptyset} \mathrel{\tilde{\subseteq}_{\mathcal{F}(M)}} \tilde{A}$ also as $\tilde{\emptyset} \mathrel{\tilde{\subseteq}} \tilde{A}$ for any $\tilde{A} \in \mathcal{F}(M)$, according to the preceding note, we thus see that the empty fuzzy set (in a set M) is 'fuzzy included' in any fuzzy set in M, in analogy to the fact that the empty set $\emptyset$ is included in any set A (in Cantor's sense).

Exercise 15.11. Show for any set $M \neq \emptyset$ that the fundamental fuzzy set $\tilde{M}$ in M is the maximum of the set $\mathcal{F}(M)$ of fuzzy sets in M with respect to $\tilde{\subseteq}_{\mathcal{F}(M)}$, i.e.

$$\max \mathcal{F}(M) = \tilde{M} \quad [= (M, m_{\tilde{M}}) = (M, \chi_M) = (M, M \times \{1\})]. \tag{15.103}$$

Note 15.7. For any nonempty set M, (15.103) means that the fundamental fuzzy set $\tilde{M} \in \mathcal{F}(M)$ is an upper bound for $\mathcal{F}(M)$, satisfying thus

$$\forall \tilde{A} \, (\tilde{A} \in \mathcal{F}(M) \Rightarrow \tilde{A} \mathrel{\tilde{\subseteq}_{\mathcal{F}(M)}} \tilde{M}). \tag{15.104}$$

Exercise 15.12. Show for any set $M \neq \emptyset$ that the partially ordered set $(\mathcal{F}(M), \tilde{\subseteq}_{\mathcal{F}(M)})$ is a complete lattice.

Proof. We let M be an arbitrary set and assume M to be nonempty, so that $(\mathcal{F}(M), \tilde{\subseteq}_{\mathcal{F}(M)})$ is a partially ordered set according to Proposition 15.22. To prove that this partially ordered set is a complete lattice, we need to establish

$$\forall \mathcal{A} \, (\mathcal{A} \subseteq \mathcal{F}(M) \Rightarrow \exists \tilde{S}, \tilde{I} \, (\tilde{S}, \tilde{I} \in \mathcal{F}(M) \wedge \tilde{S} = \overset{\tilde{\subseteq}_{\mathcal{F}(M)}}{\sup} \mathcal{A} \wedge \tilde{I} = \overset{\tilde{\subseteq}_{\mathcal{F}(M)}}{\inf} \mathcal{A})). \tag{15.105}$$

For this purpose, we let $\mathcal{A}$ be an arbitrary set and assume the inclusion $\mathcal{A} \subseteq \mathcal{F}(M)$ to be true, so that

$$\forall \tilde{A}\,(\tilde{A} \in \mathcal{A} \Rightarrow \tilde{A} \in \mathcal{F}(M)) \tag{15.106}$$

holds by definition of a subset. We may now evidently apply the Axiom of Specification in connection with the Equality Criterion for sets to establish the unique existence of a set $\mathcal{M}$ containing precisely every function $m_{\tilde{A}} : M \to [0,1]$ which is membership function of some fuzzy set in $\mathcal{A}$, in the sense that

$$\forall m_{\tilde{A}}\,(m_{\tilde{A}} \in \mathcal{M} \Leftrightarrow [m_{\tilde{A}} \in [0,1]^M \wedge \exists \tilde{A}\,(\tilde{A} \in \mathcal{A} \wedge (M, m_{\tilde{A}}) = \tilde{A})). \tag{15.107}$$

Because $m_{\tilde{A}} \in \mathcal{M}$ implies especially $m_{\tilde{A}} \in [0,1]^M$ for any set $m_{\tilde{A}}$, the inclusion $\mathcal{M} \subseteq [0,1]^M$ follows to be true by definition of a subset. Recalling the Completeness of lattices of membership functions, so that $([0,1]^M, \preceq)$ is a complete lattice, we now see that the preceding inclusion implies the existence of particular sets $\bar{S}, \bar{I} \in [0,1]^M$ such that $\bar{S} = \sup^{\preceq} \mathcal{M}$ and $\bar{I} = \inf^{\preceq} \mathcal{M}$ are true. Then, the functions $\bar{S}$ and $\bar{I}$ define the fuzzy sets $(M, \bar{S})$ and $(M, \bar{I})$ in M, which are thus elements of $\mathcal{F}(M)$. We may now show that $(M, \bar{S})$ is the supremum and $(M, \bar{I})$ the infimum of $\mathcal{A}$ with respect to $\tilde{\subseteq}_{\mathcal{F}(M)}$. To do this, we apply the Characterization of the supremum & infimum, beginning with the proof that $(M, \bar{S})$ is an upper bound and $(M, \bar{I})$ a lower bound for $\mathcal{A}$ with respect to $\tilde{\subseteq}_{\mathcal{F}(M)}$, i.e. with the proof of

$$\forall \tilde{A}\,(\tilde{A} \in \mathcal{A} \Rightarrow \tilde{A} \,\tilde{\subseteq}_{\mathcal{F}(M)}\, (M, \bar{S})), \tag{15.108}$$

$$\forall \tilde{A}\,(\tilde{A} \in \mathcal{A} \Rightarrow (M, \bar{I}) \,\tilde{\subseteq}_{\mathcal{F}(M)}\, \tilde{A}). \tag{15.109}$$

Letting $\tilde{A}$ be arbitrary and assuming $\tilde{A} \in \mathcal{A}$ to be true, we obtain $\tilde{A} \in \mathcal{F}(M)$ with the previously established inclusion $\mathcal{A} \subseteq \mathcal{F}(M)$, so that there is a particular membership function $\bar{m}_{\tilde{A}} : M \to [0,1]$ belonging to the fuzzy set $\tilde{A} = (M, \bar{m}_{\tilde{A}})$ in M. Therefore, we evidently obtain $\bar{m}_{\tilde{A}} \in \mathcal{M}$ with the specification of the set $\mathcal{M}$ in (15.107). Since we previously established $\bar{S}$ as the least upper and $\bar{I}$ as the greatest lower bound for $\mathcal{M}$ with respect to $\preceq$, it follows that $\bar{m}_{\tilde{A}} \preceq \bar{S}$ and $\bar{I} \preceq \bar{m}_{\tilde{A}}$ are both true. Since we may write these inequalities also as $\tilde{A} \,\tilde{\subseteq}_{\mathcal{F}(M)}\, (M, \bar{S})$ and $(M, \bar{I}) \,\tilde{\subseteq}_{\mathcal{F}(M)}\, \tilde{A}$, according to (15.97), we see that the two implications in (15.108) and (15.109) both hold. Because $\tilde{A}$ is arbitrary, we may therefore conclude that $(M, \bar{S})$ is indeed an upper bound and $(M, \bar{I})$ a lower bound for $\mathcal{A}$ with respect to $\tilde{\subseteq}_{\mathcal{F}(M)}$.

Next, we take arbitrary sets $\tilde{S}'$ and $\tilde{I}'$, assuming $\tilde{S}'$ to be an upper bound and $\tilde{I}'$ to be a lower bound for $\mathcal{A}$ with respect to $\tilde{\subseteq}_{\mathcal{F}(M)}$, i.e.

$$\forall \tilde{A}\,(\tilde{A} \in \mathcal{A} \Rightarrow \tilde{A}\,\tilde{\subseteq}_{\mathcal{F}(M)}\,\tilde{S}'), \tag{15.110}$$

$$\forall \tilde{A}\,(\tilde{A} \in \mathcal{A} \Rightarrow \tilde{I}'\,\tilde{\subseteq}_{\mathcal{F}(M)}\,\tilde{A}), \tag{15.111}$$

and we demonstrate that $(M,\bar{S})\,\tilde{\subseteq}_{\mathcal{F}(M)}\,\tilde{S}'$ as well as $\tilde{I}'\,\tilde{\subseteq}_{\mathcal{F}(M)}\,(M,\bar{I})$ ensue. The preceding assumptions show that $\tilde{S}'$ and $\tilde{I}'$ are elements of $\mathcal{F}(M)$, so that there are particular membership functions $\bar{m}_{\tilde{S}'} : M \to [0,1]$ and $\bar{m}_{\tilde{I}'} : M \to [0,1]$ with $\tilde{S}' = (M, \bar{m}_{\tilde{S}'})$ as well as $\tilde{I}' = (M, \bar{m}_{\tilde{I}'})$. Thus, we may write the two fuzzy inclusions to be established equivalently as $\bar{S} \preceq \bar{m}_{\tilde{S}'}$ and $\bar{m}_{\tilde{I}'} \preceq \bar{I}$, using (15.97). We may now verify that $\bar{m}_{\tilde{S}'}$ is an upper bound and that $\bar{m}_{\tilde{I}'}$ is a lower bound for $\mathcal{M}$ with respect to $\preceq$, i.e.

$$\forall m_{\tilde{A}}\,(m_{\tilde{A}} \in \mathcal{M} \Rightarrow m_{\tilde{A}} \preceq \bar{m}_{\tilde{S}'}), \tag{15.112}$$

$$\forall m_{\tilde{A}}\,(m_{\tilde{A}} \in \mathcal{M} \Rightarrow \bar{m}_{\tilde{I}'} \preceq m_{\tilde{A}}). \tag{15.113}$$

Letting $m_{\tilde{A}} \in \mathcal{M}$ be arbitrary, we obtain with (15.107) a particular fuzzy set $\tilde{A} \in \mathcal{A}$ with $(M, m_{\tilde{A}}) = \tilde{A}$. Then, (15.110) and (15.111) give $\tilde{A}\,\tilde{\subseteq}_{\mathcal{F}(M)}\,\tilde{S}'$ and $\tilde{I}'\,\tilde{\subseteq}_{\mathcal{F}(M)}\,\tilde{A}$, respectively. In view of (15.97), we may write these fuzzy inclusions equivalently as $m_{\tilde{A}} \preceq \bar{m}_{\tilde{S}'}$ and $\bar{m}_{\tilde{I}'} \preceq m_{\tilde{A}}$, as desired. As $m_{\tilde{A}}$ was arbitrary, we may therefore conclude that the universal sentences (15.112) – (15.113) are both true, so that $\bar{m}_{\tilde{S}'}$ is indeed an upper bound and $\bar{m}_{\tilde{I}'}$ a lower bound for $\mathcal{M}$ with respect to $\preceq$. Recalling that $\bar{S}$ is the least upper bound and $\bar{I}$ the greatest lower bound for $\mathcal{M}$ with respect to $\preceq$, it follows with the definitions of a supremum and of an infimum that $\bar{S} \preceq \bar{m}_{\tilde{S}'}$ and $\bar{m}_{\tilde{I}'} \preceq \bar{I}$ hold. As mentioned before, these inequalities are equivalent to the fuzzy inclusions $(M,\bar{S})\,\tilde{\subseteq}_{\mathcal{F}(M)}\,\tilde{S}'$ as well as $\tilde{I}'\,\tilde{\subseteq}_{\mathcal{F}(M)}\,(M,\bar{I})$, whose truth establishes $(M,\bar{S})$ as the supremum and $(M,\bar{I})$ as the infimum of $\mathcal{A}$ with respect to $\tilde{\subseteq}_{\mathcal{F}(M)}$. We thus proved the existential sentence in (15.105), and because $\mathcal{A}$ is arbitrary, we may infer from this the truth of the universal sentence (15.105). Furthermore, since M was initially also arbitrary, we may now finally conclude that the proposition holds. □

Definition 15.16 (Complete lattice of fuzzy sets). For any set $M \neq \emptyset$, we call

$$(\mathcal{F}(M), \tilde{\subseteq}) = (\mathcal{F}(M), \tilde{\subseteq}_{\mathcal{F}(M)}) \tag{15.114}$$

the *complete lattice of fuzzy sets* in M.

15.4. Fuzzy Cartesian Products

Let us keep in mind throughout the current section that $([0,1], \leq_{[0,1]}) = ([0,1], \sqcup_{[0,1]}, \sqcap_{[0,1]}, \leq_{[0,1]})$ constitutes a complete lattice in view of the Completeness of the real unit interval lattice.

Definition 15.17 (Triangular norm/t-norm). We say that a binary operation $*_t$ on the real unit interval $[0,1]$ is a *triangular* norm (alternatively, a t-norm) iff

1. $([0,1], *_t)$ is an Abelian semigroup,
2. 1 is the neutral element of $[0,1]$ with respect to $*_t$,
3. $*_t$ satisfies the monotony law

$$\forall x, y, z\,(x, y, z \in [0,1] \Rightarrow [x \leq_{[0,1]} y \Rightarrow x *_t z \leq_{[0,1]} y *_t z]). \quad (15.115)$$

Note 15.8. According to the definition of n-fold binary operations, we thus have for any $n \in \mathbb{N}$ the n-fold t-norm

$$*_{i=1}^{n} : [0,1]^{\{1,\ldots,n\}} \to [0,1], \quad (a_i \,|\, i \in \{1,\ldots,n\}) \mapsto *_{i=1}^{n} a_i. \quad (15.116)$$

Theorem 15.25 (t-norm property of the meet on the real unit interval). *It is true that the meet $\sqcap_{[0,1]}$ is a triangular norm.*

Proof. Since $([0,1], \sqcup_{[0,1]}, \sqcap_{[0,1]}, \leq_{[0,1]})$ is a lattice, $([0,1], \sqcap_{[0,1]})$ is a commutative semigroup due to Exercise 5.25. Furthermore, $\max[0,1] = 1$ holds because of Corollary 3.118, so that 1 is by definition the top element of $[0,1]$. Consequently, we have that 1 is the neutral element of $[0,1]$ with respect to $\sqcap_{[0,1]}$ according to Exercise 5.10. We now demonstrate that $\sqcap_{[0,1]}$ satisfies also the monotony law

$$\forall x, y, z\,(x, y, z \in [0,1] \Rightarrow [x \leq_{[0,1]} y \Rightarrow x \sqcap_{[0,1]} z \leq_{[0,1]} y \sqcap_{[0,1]} z]), \quad (15.117)$$

letting $x, y, z \in [0,1]$ be arbitrary and assuming $x \leq_{[0,1]} y$ to be true. According to the Total ordering of subsets, it is true that the standard total ordering $\leq_{\mathbb{R}}$ gives rise to the totality of the reflexive partial ordering $\leq_{[0,1]}$, noting that the real unit interval is included in the set of real numbers. Therefore, the disjunction $x \leq_{[0,1]} z \vee z \leq_{[0,1]} x$ is true, which we use now to prove the inequality

$$x \sqcap_{[0,1]} z \;\leq_{[0,1]}\; y \sqcap_{[0,1]} z \quad (15.118)$$

by cases. In the first case $x \leq_{[0,1]} z$, we obtain

$$x \sqcap_{[0,1]} z = \inf[x, z] = x \quad (15.119)$$

with Proposition 3.109, and we consider now two further sub-cases, based on the true disjunction $y \leq_{[0,1]} z \vee z \leq_{[0,1]} y$ (which is implied again by the totality of $\leq_{[0,1]}$). The first subcase $y \leq_{[0,1]} z$ implies

$$y \sqcap_{[0,1]} z = \inf[y, z] = y \tag{15.120}$$

again with Proposition 3.109, so that the initial assumption $x \leq_{[0,1]} y$ yields via substitutions based on (15.119) – (15.120) the desired inequality (15.118). Similarly, the second sub-case $z \leq_{[0,1]} y$ implies

$$y \sqcap_{[0,1]} z = \inf[y, z] = z, \tag{15.121}$$

so that the current case assumption $x \leq_{[0,1]} z$ gives the desired inequality (15.118) through substitutions based on (15.119) and (15.121).

In analogy to the first case, the second case $z \leq_{[0,1]} x$ gives us

$$x \sqcap_{[0,1]} z = \inf[x, z] = z.$$

In addition, the conjunction of that case assumption $z \leq_{[0,1]} x$ and the initial assumption $x \leq_{[0,1]} y$ implies $z \leq_{[0,1]} y$ with the transitivity of the total ordering $\leq_{[0,1]}$. The preceding inequality yields now

$$y \sqcap_{[0,1]} z = \inf[y, z] = z,$$

and combining the previous equations for z results in $x \sqcap_{[0,1]} z = y \sqcap_{[0,1]} z$. Then, the disjunction

$$x \sqcap_{[0,1]} z <_{[0,1]} y \sqcap_{[0,1]} z \vee x \sqcap_{[0,1]} z = y \sqcap_{[0,1]} z$$

is also true, which implies the desired inequality (15.118) with the definition of an induced irreflexive partial ordering. We thus completed the proof by cases, and since x, y, z were initially arbitrary, we may therefore conclude that the universal sentence (15.117) is satisfied by $\sqcap_{[0,1]}$.

Thus, the meet $\sqcap_{[0,1]}$ satisfies all three defining properties of a triangular norm. □

Definition 15.18 (Infimum t-norm / minimum t-norm / Gödel t-norm). We call the meet $\sqcap_{[0,1]}$ also the *infimum t-norm* (alternatively, the *minimum t-norm* or the *Gödel t-norm*).

Theorem 15.26 (t-norm property of the multiplication on the real unit interval). *It is true that the restriction*

$$\cdot_{[0,1]} = \cdot_{\mathbb{R}} \restriction ([0,1] \times [0,1]) \tag{15.122}$$

constitutes a triangular norm.

Proof. We begin with the observation that the inclusion $[0,1] \subseteq \mathbb{R}$ holds in view of Proposition 3.117, so that the inclusion

$$[0,1] \times [0,1] \subseteq \mathbb{R} \times \mathbb{R} \tag{15.123}$$

follows to be true by virtue of the Idempotent Law for the conjunction and (3.40). As the multiplication on $\mathbb{R}$ is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, we obtain for its restriction to $[0,1] \times [0,1]$

$$\cdot_{\mathbb{R}} \restriction ([0,1] \times [0,1]) : [0,1] \times [0,1] \to \mathbb{R}. \tag{15.124}$$

in view of (3.566). Next, we verify that $[0,1]$ is also a codomain of that restriction, which means that the range of the restriction is included in $[0,1]$. To accomplish this task, we apply the definition of a subset and establish the truth of the equivalent universal sentence

$$\forall x\,(x \in \operatorname{ran}(\cdot_{\mathbb{R}} \restriction ([0,1] \times [0,1])) \Rightarrow x \in [0,1]). \tag{15.125}$$

We take an arbitrary x and assume the antecedent to be true, so that the definition of a range gives us a particular constant $\bar{z}$ satisfying

$$(\bar{z}, x) \in \cdot_{\mathbb{R}} \restriction ([0,1] \times [0,1]). \tag{15.126}$$

We now see in light of the definition of a domain that $\bar{z} \in [0,1] \times [0,1]$ is true, noting that the preceding Cartesian product constitutes the domain of the restriction. By definition of the Cartesian product of two sets, there exist then particular elements $a \in [0,1]$ and $b \in [0,1]$ with $(a,b) = \bar{z}$. We may therefore apply substitution to write (15.126) in the form

$$((a,b), x) \in \cdot_{\mathbb{R}} \restriction ([0,1] \times [0,1]),$$

which yields $((a,b), x) \in \cdot_{\mathbb{R}}$ with the definition of a restriction and then also

$$x = a \cdot_{\mathbb{R}} b, \tag{15.127}$$

applying the notation for binary operations. Let us observe now that $a \in [0,1]$ and $b \in [0,1]$ give the inequalities, respectively, $0 \leq_{\mathbb{R}} a \leq_{\mathbb{R}} 1$ and $0 \leq_{\mathbb{R}} b \leq_{\mathbb{R}} 1$ with Proposition 3.117. Let us observe now that the Law of the Excluded Middle gives rise to the true disjunction $b = 0 \vee b \neq 0$, which we use in the following to prove the desired consequent $x \in [0,1]$ by cases. In the first case $b = 0$, the equation (15.127) gives us

$$x = a \cdot_{\mathbb{R}} 0 = 0$$

by means of substitution and the Cancellation Law for 0. Since $0 \in [0,1]$ is evidently true by definition of a closed interval, we obtain the desired $x \in [0,1]$ through substitution.

The second case $b \neq 0$ implies in connection with the previously found inequality $0 \leq_{\mathbb{R}} b$ that $0 <_{\mathbb{R}} b$ is true, by definition of an induced irreflexive partial ordering. In conjunction with the inequalities $0 \leq_{\mathbb{R}} a \leq_{\mathbb{R}} 1$, this yields then with the Monotony Law for $\cdot$ and $\leq$

$$0 \cdot_{\mathbb{R}} b \ \leq_{\mathbb{R}} \ a \cdot_{\mathbb{R}} b \ \leq_{\mathbb{R}} \ 1 \cdot_{\mathbb{R}} b.$$

We can simplify these inequalities by means of the Cancellation Law for 0, (15.127) and the definition of the unity element to

$$0 \ \leq_{\mathbb{R}} \ x \ \leq_{\mathbb{R}} \ b \quad [\leq_{\mathbb{R}} 1].$$

We therefore obtain $0 \leq_{\mathbb{R}} x \leq_{\mathbb{R}} 1$ with the transitivity of the total ordering $\leq_{\mathbb{R}}$, with the consequence that $x \in [0,1]$ (according to Proposition 3.117). Thus, $x \in [0,1]$ is true in any case, proving the implication in (15.125). As x was arbitrary, we may infer from the truth of that implication the truth of the universal sentence (15.125) and therefore the truth of the inclusion the restricted multiplication's range in $[0,1]$. We thus showed that this restriction $\cdot_{[0,1]}$ is a binary operation on $[0,1]$, which we can use now to express products of elements of the real unit interval. More specifically, we establish

$$\forall x, y\, (x, y \in [0,1] \Rightarrow x \cdot_{[0,1]} y = x \cdot_{\mathbb{R}} y), \tag{15.128}$$

letting x and y be arbitrary elements of $[0,1]$. The definition of the Cartesian product of two sets gives then $(x,y) \in [0,1] \times [0,1]$, which allows us to derive the equations

$$x \cdot_{[0,1]} y = \cdot_{[0,1]}((x,y)) = [\cdot_{\mathbb{R}} \restriction ([0,1] \times [0,1])]((x,y)) = \cdot_{\mathbb{R}}((x,y)) = x \cdot_{\mathbb{R}} y$$

using the notation for binary operations, (15.26), (3.567) in connection with the inclusion (15.123) and $(x,y) \in [0,1] \times [0,1]$, and finally again the notation for binary operations. Because x and y were arbitrary, we may therefore conclude that (15.128) is indeed true.

We prove now that the multiplication on $[0,1]$ is commutative, that is,

$$\forall x, y\, (x, y \in [0,1] \Rightarrow x \cdot_{[0,1]} y = y \cdot_{[0,1]} x). \tag{15.129}$$

To do this, we let $x, y \in [0,1]$ be arbitrary, and observe the truth of the equations

$$x \cdot_{[0,1]} y = x \cdot_{\mathbb{R}} y = y \cdot_{\mathbb{R}} x = y \cdot_{[0,1]} x$$

in light of (15.128) and the Commutative Law for the multiplication on $\mathbb{R}$. Here, x and y are arbitrary, so that the universal sentence (15.129) follows to be true, which means that $\cdot_{[0,1]}$ is indeed commutative.

Regarding the associativity of $\cdot_{[0,1]}$, we verify

$$\forall x, y, z\,(x, y, z \in [0,1] \Rightarrow (x \cdot_{[0,1]} y) \cdot_{[0,1]} z = x \cdot_{[0,1]} (y \cdot_{[0,1]} z)), \quad (15.130)$$

taking arbitrary elements $x, y, z \in [0,1]$. We obtain then the equations

$$(x \cdot_{[0,1]} y) \cdot_{[0,1]} z = (x \cdot_{\mathbb{R}} y) \cdot_{\mathbb{R}} z = x \cdot_{\mathbb{R}} (y \cdot_{\mathbb{R}} z) = x \cdot_{[0,1]} (y \cdot_{[0,1]} z)$$

with (15.128) and the Associative Law for the multiplication on $\mathbb{R}$. Since x, y and z were arbitrary, we may therefore conclude that (15.130) holds, so that $\cdot_{[0,1]}$ is associative and thus $([0,1], \cdot_{[0,1]})$ a commutative semigroup, as required by Property 1 of t-norm.

Regarding Property 2, we must demonstrate that the real number 1 is the neutral element in $[0,1]$ with respect to the multiplication on $[0,1]$, i.e.

$$\forall x\,(x \in [0,1] \Rightarrow [1 \cdot_{[0,1]} x = x \wedge x \cdot_{[0,1]} 1 = x]). \quad (15.131)$$

Letting $x \in [0,1]$ be arbitrary and noting that $1 \in [0,1]$ holds by definition of a closed interval, we get

$$\begin{aligned} 1 \cdot_{[0,1]} x &= 1 \cdot_{\mathbb{R}} x = x \\ x \cdot_{[0,1]} 1 &= x \cdot_{\mathbb{R}} 1 = x \end{aligned}$$

with (15.128) and the fact that 1 is the neutral element of $\mathbb{R}$ with respect to the multiplication on $\mathbb{R}$. As x was arbitrary, we may infer from the truth of the resulting equations $1 \cdot_{[0,1]} x = x$ and $x \cdot_{[0,1]} 1 = x$ the truth of the universal sentence (15.131), so that 1 is also the neutral element of $[0,1]$ with respect to the multiplication on $[0,1]$.

It remains for us to establish Property 3 of a t-norm for $\cdot_{[0,1]}$, which we accomplish by proving the universal sentence

$$\forall x, y, z\,(x, y, z \in [0,1] \Rightarrow [x \leq_{[0,1]} y \Rightarrow x \cdot_{[0,1]} z \leq_{[0,1]} y \cdot_{[0,1]} z]). \quad (15.132)$$

We take arbitrary elements $x, y, z \in [0,1]$, and we assume $x \leq_{[0,1]} y$ to be true. According to the Reflexive partial ordering of subsets, we can write this assumption also as $x \leq_{\mathbb{R}} y$. Furthermore, the assumed $z \in [0,1]$ evidently implies $0 \leq_{[0,1]} z$ $[\leq_{[0,1]} 1]$ and then $0 <_{[0,1]} z \vee 0 = z$. The case of $0 <_{[0,1]} z$ implies $0 <_{\mathbb{R}} z$ again with the Reflexive partial ordering of subsets, and this inequality allows us to apply the Monotony Law for $\cdot_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ to the previously established inequality $x \leq_{\mathbb{R}} y$ in order to obtain the inequality

$$x \cdot_{\mathbb{R}} z \leq_{\mathbb{R}} y \cdot_{\mathbb{R}} z.$$

Since x, y and z are elements of the real unit interval, we can apply (15.128) and again the Reflexive partial ordering of subsets to rewrite the preceding inequalities in the desired form $x \cdot_{[0,1]} z \leq_{[0,1]} y \cdot_{[0,1]} z$. On the other hand, the second case $0 = z$ yields

$$x \cdot_{[0,1]} z = x \cdot_{\mathbb{R}} 0 = 0$$
$$y \cdot_{[0,1]} z = y \cdot_{\mathbb{R}} 0 = 0$$

by applying substitutions alongside (15.128), and then the Cancellation Law for 0. Since $0 \leq_{[0,1]} 0$ is true because of the reflexivity of $\leq_{[0,1]}$, we obtain now via substitutions $x \cdot_{[0,1]} z \leq_{[0,1]} y \cdot_{[0,1]} z$ also in the second case. As x, y and z were arbitrary, we therefore conclude that the universal sentence (15.132) holds, so that $\cdot_{[0,1]}$ satisfies also the required monotony law. The previous findings thus show that $\cdot_{[0,1]}$ constitutes a t-norm, by definition. □

Definition 15.19 (Product t-norm). We call the multiplication $\cdot_{[0,1]}$ also the *product t-norm*.

We now intend to construct a membership function that characterizes two given fuzzy sets, or more generally, a sequence of given fuzzy sets.

Exercise 15.13. Show for any set $M \neq 0$, for any fuzzy sets $\tilde{A} = (M, m_{\tilde{A}})$ and $\tilde{B} = (M, m_{\tilde{B}})$ and for any t-norm $*_t$ that there is the unique function

$$m_{\tilde{A}\times_t\tilde{B}} : M \times M \to [0,1], \quad (a,b) \mapsto m_{\tilde{A}}(a) *_t m_{\tilde{A}}(b). \tag{15.133}$$

(Hint: Proceed similarly as in the proof of Proposition 5.2.)

Exercise 15.14. Show for any set $M \neq 0$, for any $n \in \mathbb{N}$, for any sequence $m = (m_{\tilde{A}_i} \,|\, i \in \{1,\ldots,n\})$ of membership functions on M and for any sequence $x = (x_i \,|\, i \in \{1,\ldots,n\})$ in M that there is the unique sequence $m_x = (m_{\tilde{A}_i}(x_i) \,|\, i \in \{1,\ldots,n\})$ in $[0,1]$.

Proposition 15.27. *It is true for any set $M \neq 0$, for any $n \in \mathbb{N}$, for any sequence $m = (m_{\tilde{A}_i} \,|\, i \in \{1,\ldots,n\})$ of membership functions on M and for any t-norm $*_t$ that there is the unique function*

$$m_{\times\tilde{A}_i} : M^n \to [0,1], \quad (x_i \,|\, i \in \{1,\ldots,n\}) \mapsto *_{i=1}^n m_{\tilde{A}_i}(x_i). \tag{15.134}$$

Proof. We take arbitrary sets M, n, m and $*_t$, assuming M to be a nonempty set, assuming n to be an element of $\mathbb{N}$, assuming m to be a sequence $(m_{\tilde{A}_i} \,|\, i \in \{1,\ldots,n\})$ of membership functions on M, and assuming $*_t$

to be a t-norm. To establish the desired function with domain M^n, we apply Function definition by replacement and verify

$$\forall x\,(x \in M^n \Rightarrow \exists! y\,(y = *_{i=1}^n m_{\tilde{A}_i}(x_i))). \tag{15.135}$$

For this purpose, we let x be arbitrary, and we assume $x \in M^n$ to be true. By definition of a Cartesian power, x is thus a sequence $(x_i \,|\, i \in 1, \ldots, n\})$ in M. This finding and the previous assumptions imply now with the preceding Exercise 15.14 that there is the unique sequence $m_x = (m_{\tilde{A}_i}(x_i) \,|\, i \in \{1, \ldots, n\})$ in $[0,1]$. Therefore, this sequence is in the domain $[0,1]^{\{1,\ldots,n\}}$ of the n-fold t-norm $*_{i=1}^n$ given by (15.116), which gives the uniquely determined value $*_{i=1}^n m_{\tilde{A}_i}(x_i)$. Therefore, the uniquely existential sentence in (15.135) is true according to (1.109). As x was arbitrary, we may infer from this the truth of (15.135), which universal sentence implies then the unique existence of a function $m_{\times\tilde{A}_i}$ with domain M^n such that

$$\forall x\,(x \in M^n \Rightarrow m_{\times\tilde{A}_i}(x) = *_{i=1}^n m_{\tilde{A}_i}(x_i)). \tag{15.136}$$

It remains for us to show that $[0,1]$ is a codomain of this function, i.e. that the range of $m_{\times\tilde{A}_i}$ is included in the real unit interval. Letting y be arbitrary and assuming y to be an element of that range, it follows with the definitions of a range and of a domain that there exists a particular index $\bar{x} \in M^n$ $[= \mathrm{dom}(m_{\times\tilde{A}_i})]$ such that $(\bar{x}, y) \in m_{\times\tilde{A}_i}$ is satisfied. By definition of the function $m_{\times\tilde{A}_i}$ in (15.136), we obtain therefore the value

$$y = m_{\times\tilde{A}_i}(\bar{x}) = *_{i=1}^n m_{\tilde{A}_i}(\bar{x}_i),$$

which is evidently in the codomain $[0,1]$ of the n-fold binary operation $*_{i=1}^n : [0,1]^{\{1,\ldots,n\}} \to [0,1]$. As y was arbitrary, we may infer from this finding the truth of the inclusion $\mathrm{ran}(m_{\times\tilde{A}_i}) \subseteq [0,1]$ by means of the definition of a subset. Thus, $m_{\times\tilde{A}_i}$ is a function from M^n to $[0,1]$, by definition of a codomain. Initially, the sets M, n, m and $*_t$ were all arbitrary, so that the proposed universal sentence follows then to be true. □

The preceding mechanism of generating a multidimensional membership function by means of a given sequence of (one-dimensional) membership functions and a t-norm gives rise to the following definition.

Definition 15.20 (n-dimensional membership function induced by a t-norm, fuzzy Cartesian product induced by a t-norm). For any set $M \neq \emptyset$, any $n \in \mathbb{N}$, any sequence $(m_{\tilde{A}_i} \,|\, i \in \{1, \ldots, n\})$ of membership functions on M and any t-norm $*_t$ we call

$$m_{\times\tilde{A}_i} : M^n \to [0,1], \quad (x_i \,|\, i \in \{1, \ldots, n\}) \mapsto *_{i=1}^n m_{\tilde{A}_i}(x_i) \tag{15.137}$$

the *n-dimensional membership function* of $\bigtimes\tilde{A}_i$ induced by $*_t$, and we call

$$\tilde{\bigtimes}_{i=1}^{n}\tilde{A}_i = \bigtimes\tilde{A}_i \tag{15.138}$$

the *fuzzy Cartesian product* of the sequence of fuzzy sets $(\tilde{A}_i \,|\, i \in \{1,\ldots,n\})$ induced by $*_t$.

Note 15.9. Using the infimum t-norm $\sqcap_{[0,1]}$, we have for any set $M \neq \emptyset$, any $n \in \mathbb{N}$ and any sequence $(m_{\tilde{A}_i} \,|\, i \in \{1,\ldots,n\})$ of membership functions on M that there is the unique n-dimensional function

$$m_{\bigtimes\tilde{A}_i} : M^n \to [0,1], \quad (x_i \,|\, i \in \{1,\ldots,n\}) \mapsto \sqcap_{i=1}^{n} m_{\tilde{A}_i}(x_i). \tag{15.139}$$

Theorem 15.28 (Characterization of the α-cuts of fuzzy Cartesian products induced by the infimum t-norm). *For any set $M \neq \emptyset$, any $n \in \mathbb{N}$ and any sequence $(m_{\tilde{A}_i} \,|\, i \in \{1,\ldots,n\})$ of membership functions on M, it is true that every α-cut of the induced fuzzy Cartesian product can be written as the Cartesian product of the sequence $([\tilde{A}_i]_\alpha)$ of α-cuts, i.e.*

$$\forall\alpha\,(\alpha \in [0,1] \;\Rightarrow\; [\tilde{\bigtimes}_{i=1}^{n}\tilde{A}_i]_\alpha = \bigtimes_{i=1}^{n}[\tilde{A}_i]_\alpha). \tag{15.140}$$

Proof. We take arbitrary sets M, n, m and α, assuming M to be a nonempty set, n to be an element of $\mathbb{N}$, m to be a sequence $(m_{\tilde{A}_i} \,|\, i \in \{1,\ldots,n\})$ of membership functions on M, and assuming moreover α to be an element of the real unit interval $[0,1]$. To prove the equation in (15.140), we apply the Equality Criterion for sets and verify accordingly the universal sentence

$$\forall x\,(x \in [\tilde{\bigtimes}_{i=1}^{n}\tilde{A}_i]_\alpha \Leftrightarrow x \in \bigtimes_{i=1}^{n}[\tilde{A}_i]_\alpha). \tag{15.141}$$

We let x be an arbitrary set, and we assume first

$$x \in [\tilde{\bigtimes}_{i=1}^{n}\tilde{A}_i]_\alpha \tag{15.142}$$

to be true. Consequently, the definition of an α-cut yields

$$m_{\bigtimes\tilde{A}_i} \geq \alpha, \tag{15.143}$$

which in turn implies with (15.139)

$$\sqcap_{i=1}^{n} m_{\tilde{A}_i}(x_i) \geq \alpha, \tag{15.144}$$

for which we can write also

$$\inf \operatorname{ran}((m_{\tilde{A}_i(x_i)} \,|\, i \in \{1,\ldots,n\})) \;\geq\; \alpha \tag{15.145}$$

because of Proposition 5.107. As the α-cut $[\tilde{\bigtimes}_{i=1}^{n}\tilde{A}_i]_\alpha$ is included in the domain M^n of the membership function $m_{\tilde{\times}\tilde{A}_i}$, it follows from (15.142) by definition of a subset that $x \in M^n$ is true, so that x is a sequence with index set $\{1,\ldots,n\}$ according to the definition of a Cartesian power. We demonstrate now that x satisfies also

$$\forall i\,(i \in \{1,\ldots,n\} \Rightarrow x_i \in [\tilde{A}_i]_\alpha), \tag{15.146}$$

taking an arbitrary i and assuming $i \in \{1,\ldots,n\}$ to hold. Then, the corresponding term $m_{\tilde{A}_i(x_i)}$ is clearly in the range of the sequence $(m_{\tilde{A}_i(x_i)}\,|\,i \in \{1,\ldots,n\})$, and since the infimum of that range is a lower bound for that range, the infimum is less than or equal to the term $m_{\tilde{A}_i(x_i)}$, that is,

$$[\alpha \ \leq] \quad \inf\operatorname{ran}((m_{\tilde{A}_i(x_i)}\,|\,i \in \{1,\ldots,n\})) \ \leq \ m_{\tilde{A}_i(x_i)}.$$

These inequalities give us now $m_{\tilde{A}_i(x_i)} \geq \alpha$ with the transitivity of the standard total ordering $\leq_{\mathbb{R}}$, and this finding further implies $x_i \in [\tilde{A}_i]_\alpha$ by definition of an α-cut, so that the implication in (15.146) holds. Here, i is arbitrary, so that the universal sentence (15.146) is true as well. Recalling that x is a sequence with index set $\{1,\ldots,n\}$, we therefore obtain with the definition of the Cartesian product of a family sets

$$x \in \bigtimes_{i=1}^{n}[\tilde{A}_i]_\alpha, \tag{15.147}$$

which completes the proof of the first part ('$\Rightarrow$') of the equivalence in (15.141).

Regarding the second part ('$\Leftarrow$'), we conversely assume (15.147) to be true, so that x is again a sequence with index set $\{1,\ldots,n\}$ satisfying (15.146). The task is now to establish α as a lower bound for the range of the sequence $(m_{\tilde{A}_i(x_i)}\,|\,i \in \{1,\ldots,n\})$, that is,

$$\forall y\,(y \in \operatorname{ran}((m_{\tilde{A}_i(x_i)}\,|\,i \in \{1,\ldots,n\})) \Rightarrow \alpha \leq y). \tag{15.148}$$

For this purpose, we take an arbitrary set y, and we assume y to be an element of the range of $(m_{\tilde{A}_i(x_i)}\,|\,i \in \{1,\ldots,n\})$. By definition of a range and by definition of a domain, there exists then a particular index $\bar{k} \in \{1,\ldots,n\}$ such that $(\bar{k}, y) \in (m_{\tilde{A}_i(x_i)}\,|\,i \in \{1,\ldots,n\})$ holds, which we can write also in sequence notation as $y = m_{\tilde{A}_{\bar{k}}(x_{\bar{k}})}$. Because x satisfies (15.146), we obtain also $x_{\bar{k}} \in [\tilde{A}_{\bar{k}}]_\alpha$, so that substitution based on the preceding equation yields $y \geq \alpha$, which evidently is the desired consequent of the implication in (15.148). As y was arbitrary, we may infer from the truth of that implication the truth of the universal sentence (15.148), which shows us that α is indeed a lower bound for the range of $(m_{\tilde{A}_i(x_i)} \mid i \in \{1, \ldots, n\})$. Since the infimum of that range is the greatest lower bound, it follows that α is less than or equal to this infimum, that is, the inequality (15.145) holds. This in turn implies the truth of (15.145) by virtue of Proposition 5.107, and then also the truth of (15.144) by definition of the membership function $m_{\times \tilde{A}_i}$ in (15.139). Finally, the definition of an α-cut gives us the desired consequent (15.142) of the second part of the equivalence in (15.141), which is thus true.

Since M, n, m and α were initially all arbitrary, we may therefore conclude that the stated theorem holds. $\square$

15.5. Fuzzy Numbers and Fuzzy Intervals

We define now fuzzy numbers according to Viertl (2011).

Definition 15.21 (Fuzzy number, fuzzy interval). We say that a fuzzy set $\tilde{A}$ is a *fuzzy number* iff

1. the domain of its membership function is identical with the set of real numbers, i.e.
$$\mathrm{dom}(m_{\tilde{A}}) = \mathbb{R}, \tag{15.149}$$
2. every α-cut $\tilde{A}_\alpha$ is nonempty, i.e.
$$\forall \alpha \, (\alpha \in [0,1] \Rightarrow \tilde{A}_\alpha \neq \emptyset), \tag{15.150}$$
3. the α-cut $\tilde{A}_\alpha$ can, for every $\alpha \in (0,1]$, be written as the union of a finite sequence of closed intervals in $\mathbb{R}$, i.e.
$$\begin{aligned} &\forall \alpha \, (\alpha \in (0,1] \Rightarrow \exists n, I \, (n \in \mathbb{N}_+ \wedge I : \{1, \ldots, n\} \to \{[a,b] : a, b \in \mathbb{R}\} \\ &\quad \wedge \tilde{A}_\alpha = \bigcup_{i=1}^{n} I_i)), \end{aligned} \tag{15.151}$$
4. the support of $\tilde{A}$ is bounded from below and bounded from above (with respect to $\leq_\mathbb{R}$), i.e.
$$\exists a \, (a \in \mathbb{R} \wedge \forall x \, (x \in \mathrm{supp}(\tilde{A}) \Rightarrow a \leq_\mathbb{R} x)) \tag{15.152}$$
$$\wedge \exists u \, (u \in \mathbb{R} \wedge \forall x \, (x \in \mathrm{supp}(\tilde{A}) \Rightarrow x \leq_\mathbb{R} u)). \tag{15.153}$$

Furthermore, we say that a fuzzy number $\tilde{A}$ is a *fuzzy interval* iff the α-cut $\tilde{A}_\alpha$ can, for every $\alpha \in (0,1]$, be written as a single closed interval in $\mathbb{R}$, i.e.
$$\forall \alpha \, (\alpha \in (0,1] \Rightarrow \exists a, b \, (a, b \in \mathbb{R} \wedge \tilde{A}_\alpha = [a,b])). \tag{15.154}$$

Theorem 15.29 (Connectedness of the images of connected topological spaces under continuous functions). *It is true for any topological spaces $(\Omega, \mathcal{O})$, $(\Omega', \mathcal{O}')$ and for any continuous function $f : \Omega \to \Omega'$ that the topological subspace $(f[\Omega], \mathcal{O}'|f[\Omega])$ of $(\Omega', \mathcal{O}')$ is connected if $(\Omega, \mathcal{O})$ is connected.*

Proof. We let Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f be arbitrary sets, assuming $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ to be topological spaces, and assuming furthermore that f is a continuous function from Ω to Ω' with respect to $\mathcal{O}$ and $\mathcal{O}'$. Then, because Ω is a subset of itself in view of Proposition 2.4, the image of Ω under f is defined and constitutes a subset of the codomain Ω' according to Corollary 3.218. We thus see that the topological subspace $(f[\Omega], \mathcal{O}'|f[\Omega])$ of $(\Omega', \mathcal{O}')$ is indeed defined. We prove now the stated implication by contradiction, assuming that $(\Omega, \mathcal{O})$ is connected and assuming that $(f[\Omega], \mathcal{O}'|f[\Omega])$ is not connected (i.e., disconnected). The latter assumption means by definition that there exists a separation of $f[\Omega]$, say $\bar{S}_f$. According to the definition of a separation, $\bar{S}_f$ constitutes an ordered pair, so that we can write $\bar{S}_f = (\bar{U}, \bar{V})$ for some particular open sets $\bar{U}, \bar{V} \in \mathcal{O}'|f[\Omega]$. As we assumed the function $f : \Omega \to \Omega'$ to be continuous with respect to $\mathcal{O}$ and $\mathcal{O}'$, we have that $f : \Omega \to f[\Omega]$ is continuous with respect to Ω and the subspace topology $\mathcal{O}'|f[\Omega]$ by virtue of Proposition 13.2. Therefore, the preceding finding $\bar{U}, \bar{V} \in \mathcal{O}'|f[\Omega]$ implies with the definition of a continuous function

$$f^{-1}[\bar{U}] \in \mathcal{O} \ \wedge \ f^{-1}[\bar{V}] \in \mathcal{O},$$

which shows that the ordered pair $(f^{-1}[\bar{U}], f^{-1}[\bar{V}])$ satisfies Property 1 of a separation of Ω.

Observing now the truth of $f[\Omega] = \mathrm{ran}(f)$ in light of (3.718), we also see that $f : \Omega \to f[\Omega]$ constitutes a surjection, by definition. Let us observe in addition that the open sets $\bar{U}, \bar{V} \in \mathcal{O}'|f[\Omega]$ (forming the separation $\bar{S}_f$) are subsets of $f[\Omega]$ according to Property 1 of a topology on $f[\Omega]$, and furthermore that the sets $\bar{U}$ and $\bar{V}$ are nonempty because of Property 2 of a separation. We can then infer from these findings the truth of

$$f^{-1}[\bar{U}] \neq \emptyset \ \wedge \ f^{-1}[\bar{V}] \neq \emptyset$$

by using (3.758). The preceding conjunction shows us that Property 2 of a separation is also satisfied by $(f^{-1}[\bar{U}], f^{-1}[\bar{V}])$.

Next, we note that the separation $\bar{S}_f = (\bar{U}, \bar{V})$ satisfies the disjointness property $\bar{U} \cap \bar{V} = \emptyset$, so that we obtain the equations

$$\begin{aligned} f^{-1}[\bar{U}] \cap f^{-1}[\bar{V}] &= f^{-1}[\bar{U} \cap \bar{V}] \\ &= f^{-1}[\emptyset] \\ &= \emptyset \end{aligned}$$

by applying (3.760), substitution and (3.745). Thus, the ordered pair $(f^{-1}[\bar{U}], f^{-1}[\bar{V}])$ satisfies Property 3 of a separation of Ω.

Finally, Property 4 of a separation of $f[\Omega]$ gives us $\bar{U} \cup \bar{V} = f[\Omega]$, which allows us to derive the equations

$$\begin{aligned} f^{-1}[\bar{U}] \cup f^{-1}[\bar{V}] &= f^{-1}[\bar{U} \cup \bar{V}] \\ &= f^{-1}[f[\Omega]] \\ &= \Omega \end{aligned}$$

by means of (3.761), substitution, and (3.746) with respect to the function $f : \Omega \to f[\Omega]$. The previous findings demonstrate that $(f^{-1}[\bar{U}], f^{-1}[\bar{V}])$ has all of the properties of a separation of Ω, so that we established thereby the existence of a separation of Ω. This means that the topological space $(\Omega, \mathcal{O})$ is disconnected, which is clearly in contradiction to the initial assumption that $(\Omega, \mathcal{O})$ is connected. We thus proved the implication stated within the theorem, and since Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f are arbitrary, we may infer from the truth of this implication the truth of the theorem itself. □

Theorem 15.30 (Compactness of the images of compact spaces under continuous functions). *It is true for any topological spaces* $(\Omega, \mathcal{O})$, $(\Omega', \mathcal{O}')$ *and for any continuous function* $f : \Omega \to \Omega'$ *that the topological subspace* $(f[\Omega], \mathcal{O}'|f[\Omega])$ *of* $(\Omega', \mathcal{O}')$ *is compact if* $(\Omega, \mathcal{O})$ *is compact.*

Proof. We take arbitrary sets Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f, we assume $(\Omega, \mathcal{O})$ and $(\Omega', \mathcal{O}')$ to be topological spaces, and we assume in addition that f is a continuous function from Ω to Ω' with respect to $\mathcal{O}$ and $\mathcal{O}'$. Since Ω is a subset of itself (see Proposition 2.4), the image of Ω under f is defined and constitutes a subset of the codomain Ω' (see Corollary 3.218). Consequently, the topological subspace $(f[\Omega], \mathcal{O}'|f[\Omega])$ of $(\Omega', \mathcal{O}')$ is indeed defined. We assume now that $(\Omega, \mathcal{O})$ is a compact space, so that the universal sentence

$$\forall \mathcal{C}\,([\mathcal{C} \subseteq \mathcal{P}(\Omega) \,\wedge\, \bigcup \mathcal{C} = \Omega \,\wedge\, \mathcal{C} \subseteq \mathcal{O}] \tag{15.155}$$
$$\Rightarrow \exists \mathcal{S}\,(\mathcal{S} \subseteq \mathcal{C} \,\wedge\, \mathcal{S} \text{ is a finite set } \wedge\, \mathcal{S} \subseteq \mathcal{P}(\Omega) \,\wedge\, \bigcup \mathcal{S} = \Omega)).$$

is true by definition. To prove that $(f[\Omega], \mathcal{O}'|f[\Omega])$ of $(\Omega', \mathcal{O}')$ is compact, we must accordingly establish the truth of the universal sentence

$$\forall \mathcal{C}\,([\mathcal{C} \subseteq \mathcal{P}(f[\Omega]) \,\wedge\, \bigcup \mathcal{C} = f[\Omega] \,\wedge\, \mathcal{C} \subseteq \mathcal{O}'|f[\Omega]] \tag{15.156}$$
$$\Rightarrow \exists \mathcal{S}\,(\mathcal{S} \subseteq \mathcal{C} \,\wedge\, \mathcal{S} \text{ is a finite set } \wedge\, \mathcal{S} \subseteq \mathcal{P}(f[\Omega]) \,\wedge\, \bigcup \mathcal{S} = f[\Omega])).$$

For this purpose, we take an arbitrary set $\mathcal{C}$, which we assume to satisfy the inclusion $\mathcal{C} \subseteq \mathcal{P}(f[\Omega])$, the equation $\bigcup \mathcal{C} = f[\Omega]$, and the inclusion $\mathcal{C} \subseteq \mathcal{O}'|f[\Omega]$. Observing that the inverse-image function

$$f^{\leftarrow} : \mathcal{P}(\Omega') \to \mathcal{P}(\Omega), \qquad B \mapsto f^{\leftarrow}(B) = f^{-1}[B]$$

is defined, we consider now its restriction $f^{\leftarrow} \restriction \mathcal{C}$. In view of $\mathcal{C} \subseteq \mathcal{P}(f[\Omega])$ and the fact that the aforementioned inclusion $f[\Omega] \subseteq \Omega'$ implies $\mathcal{P}(f[\Omega]) \subseteq \mathcal{P}(\Omega')$ because of (3.16), we obtain the inclusion $\mathcal{C} \subseteq \mathcal{P}(\Omega')$ with (2.13). Consequently, the preceding restriction is a function from $\mathcal{C}$ to $\mathcal{P}(\Omega)$ due to Proposition 3.164. We thus see in light of the definition of a codomain that

$$\mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C}) \subseteq \mathcal{P}(\Omega) \tag{15.157}$$

holds. Next, we demonstrate the truth of

$$\bigcup \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C}) = \Omega, \tag{15.158}$$

by applying the Equality Criterion for sets and by proving accordingly the universal sentence

$$\forall \omega \, (\omega \in \bigcup \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C}) \Leftrightarrow \omega \in \Omega). \tag{15.159}$$

We let ω be arbitrary and assume first $\omega \in \bigcup \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C})$ to be true. By definition of the union of a set system, there exists then a set, say $\bar{A}$, such that $\bar{A} \in \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C})$ and $\omega \in \bar{A}$ are satisfied. The former implies with the inclusion (15.157) $\bar{A} \in \mathcal{P}(\Omega)$, which gives $\bar{A} \subseteq \Omega$ by definition of a power set. With this inclusion, the previously established $\omega \in \bar{A}$ implies now the desired consequent $\omega \in \Omega$ of the first part ('$\Rightarrow$') of the equivalence in (15.159).

Regarding the second part ('$\Leftarrow$'), we assume conversely $\omega \in \Omega$ to be true. In connection with the fact $\Omega \subseteq \Omega$, this assumption yields $f(\omega) \in f[\Omega]$ $[= \bigcup \mathcal{C}]$ with (3.719). The definition of the union of a set system gives us then a particular set $\bar{Y}$ satisfying $\bar{Y} \in \mathcal{C}$ and $f(\omega) \in \bar{Y}$. The latter gives us $\omega \in f^{-1}[\bar{Y}]$ with the definition of an inverse image, and the former shows us that $\bar{Y}$ is in the domain of the restricted inverse-image function. The corresponding value is therefore given by $(f^{\leftarrow} \upharpoonright \mathcal{C})(\bar{Y}) = f^{-1}[\bar{Y}]$, so that we obtain via substitution $\omega \in (f^{\leftarrow} \upharpoonright \mathcal{C})(\bar{Y})$. Denoting this value by $\bar{X} = (f^{\leftarrow} \upharpoonright \mathcal{C})(\bar{Y})$, we can also write $\omega \in \bar{X}$ and $(\bar{Y}, \bar{X}) \in f^{\leftarrow} \upharpoonright \mathcal{C}$, where the latter shows in light of the definition of a range that $\bar{X} \in \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C})$ holds. In conjunction with $\omega \in \bar{X}$, this finding implies with the definition of the union of a set system $\omega \in \bigcup \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C})$, so that the second part of the equivalence in (15.159) holds, too. Since ω is arbitrary, we may therefore conclude that the universal sentence (15.159) is true, which completes the proof of the equality (15.158). Alongside (15.157), this shows that the range $\mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C})$ is a covering of Ω.

Our next task is to show that this range is included in the topology $\mathcal{O}$, i.e. that all elements of that range constitute open sets in Ω. We use again the definition of subset and prove

$$\forall U \, (U \in \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C}) \Rightarrow U \in \mathcal{O}), \tag{15.160}$$

letting U be arbitrary and assuming $U \in \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C})$ to be true. By definition of a range, $(\bar{V}, U) \in f^{\leftarrow} \upharpoonright \mathcal{C}$ is then satisfied by a particular set $\bar{V}$, and this finding gives us by means of the definition of a restriction $(\bar{V}, U) \in f^{\leftarrow}$ and $\bar{V} \in \mathcal{C}$. Here, we can write the former in function notation as $U = f^{\leftarrow}(\bar{V}) = f^{-1}[\bar{V}]$. Let us recall now the truth of the inclusion $\mathcal{C} \subseteq \mathcal{O}'|f[\Omega]$, so $\bar{V} \in \mathcal{C}$ implies $\bar{V} \in \mathcal{O}'|f[\Omega]$ (using the definition of a subset). Because $f : \Omega \to \Omega'$ is by assumption continuous with respect to $\mathcal{O}$ and $\mathcal{O}'$, it follows with Proposition 13.2 that f, written in the surjective form $f : \Omega \to f[\Omega]$, is continuous with respect to Ω and the subspace topology $\mathcal{O}'|f[\Omega]$. Consequently, $\bar{V} \in \mathcal{O}'|f[\Omega]$ implies (by definition of a continuous function) $[U =]\ f^{-1}[\bar{V}] \in \mathcal{O}$, so that we find the desired consequent $U \in \mathcal{O}$ of the implication in (15.160) to be true. As U was arbitrary, we may now infer from the truth of this implication the truth of the universal sentence (15.160) and thus the truth of the inclusion

$$\operatorname{ran}(f^{\leftarrow} \upharpoonright \mathcal{C}) \subseteq \mathcal{O}. \tag{15.161}$$

The conjunction of (15.157), (15.158) and (15.161) implies now with (15.155) that there exists a set (system), say $\bar{\mathcal{S}}$, such that the inclusion

$$\bar{\mathcal{S}} \subseteq \operatorname{ran}(f^{\leftarrow} \upharpoonright \mathcal{C}) \tag{15.162}$$

holds, such that $\bar{\mathcal{S}}$ is a finite set, and such that $\bar{\mathcal{S}}$ is a covering of Ω, i.e.

$$\bar{\mathcal{S}} \subseteq \mathcal{P}(\Omega) \ \wedge\ \bigcup \bar{\mathcal{S}} = \Omega. \tag{15.163}$$

The finiteness of the set $\bar{\mathcal{S}}$ implies by definition the existence of a particular natural number $\bar{n}$ and of a particular bijection $\bar{S} : \{1, \ldots, \bar{n}\} \rightleftarrows \bar{\mathcal{S}}$, which we can now utilize to define a unique function T with domain $\{1, \ldots, \bar{n}\}$ such that

$$\forall i\,(i \in \{1, \ldots, \bar{n}\} \Rightarrow T(i) = f[\bar{S}(i)]). \tag{15.164}$$

According to Function definition by replacement, we are required to verify accordingly

$$\forall i\,(i \in \{1, \ldots, \bar{n}\} \Rightarrow \exists! Y\,(Y = f[\bar{S}(i)])). \tag{15.165}$$

Letting i be arbitrary and assuming $i \in \{1, \ldots, \bar{n}\}$ to be true, so that i is in the domain of the bijection $\bar{S}$, we obtain with the Function Criterion the corresponding value $\bar{S}(i) \in \bar{\mathcal{S}}$. Due to the inclusion in (15.163), this implies $\bar{S}(i) \in \mathcal{P}(\Omega)$ with the definition of a subset and then $\bar{S}(i) \subseteq \Omega$ with the definition of a power set. Therefore, the image $f[\bar{S}(i)]$ is defined, so that the uniquely existential sentence in (15.165) follows to be true according to (1.109). Since i was arbitrary, we may infer from this finding the truth of the universal sentence (15.165) and consequently the unique existence of a function T with domain $\{1, \ldots, \bar{n}\}$ and values satisfying (15.164).

Let us verify that the range of T satisfies the inclusions

$$\mathrm{ran}(T) \subseteq \mathcal{C}, \tag{15.166}$$
$$\mathrm{ran}(T) \subseteq \mathcal{P}(f[\Omega]), \tag{15.167}$$

taking an arbitrary set Y and assuming $Y \in \mathrm{ran}(T)$ to be true. Then, we have $(\bar{k}, Y) \in T$ for a particular $\bar{k} \in \{1, \ldots, \bar{n}\}$ [$= \mathrm{dom}(T)$] by definition of a range any by definition of a domain, and we obtain $Y = T(\bar{k}) = f[\bar{S}(\bar{k})]$ with the definition of the function T in (15.164). Clearly, $\bar{k}$ is also in the domain of the bijection $\bar{S}$, so that this element is associated with the value $\bar{S}(\bar{k}) \in \bar{\mathcal{S}}$. This implies $\bar{S}(\bar{k}) \in \mathrm{ran}(f^{\leftarrow} \upharpoonright \mathcal{C})$ with the inclusion (15.162), with the consequence that $(\bar{Y}, \bar{S}(\bar{k})) \in f^{\leftarrow} \upharpoonright \mathcal{C}$ holds for some particular set $\bar{Y}$. This means that $(\bar{Y}, \bar{S}(\bar{k})) \in f^{\leftarrow}$ and $\bar{Y} \in \mathcal{C}$ are both true, where the latter yields now $\bar{Y} \in \mathcal{P}(f[\Omega])$ with the assumed inclusion $\mathcal{C} \subseteq \mathcal{P}(f[\Omega])$, and therefore $\bar{Y} \subseteq f[\Omega]$. Applying now (3.749) to this inclusion for the surjection $f : \Omega \twoheadrightarrow f[\Omega]$, we obtain the equation $f[f^{-1}[\bar{Y}]] = \bar{Y}$. On the other hand, the previous finding $(\bar{Y}, \bar{S}(\bar{k})) \in f^{\leftarrow}$ gives us with the definition of the inverse-image function $\bar{S}(\bar{k}) = f^{\leftarrow}(\bar{Y}) = f^{-1}[\bar{Y}]$, and we obtain for the corresponding image

$$Y = f[\bar{S}(\bar{k})] = f[f^{-1}[\bar{Y}]] = \bar{Y}$$

by means of substitutions. Thus, the previously established $\bar{Y} \in \mathcal{C}$ and $\bar{Y} \in \mathcal{P}(f[\Omega])$ gives us through further substitutions $Y \in \mathcal{C}$ as well as $Y \in \mathcal{P}(f[\Omega])$. Since these two findings were implied by $Y \in \mathrm{ran}(T)$ where Y is arbitrary, the inclusions (15.166) and (15.167) follow to be both true.

Next, we observe that the domain $\{1, \ldots, \bar{n}\}$ of the function T is a finite set according to Exercise 4.34), so that

$$\mathrm{ran}(T) \text{ is a finite set} \tag{15.168}$$

as well in view of Exercise 4.39.

Finally, we establish the equation

$$\bigcup \mathrm{ran}(T) = f[\Omega] \tag{15.169}$$

via the Equality Criterion for sets, by proving

$$\forall \omega' \, (\omega' \in \bigcup \mathrm{ran}(T) \Leftrightarrow \omega' \in f[\Omega]). \tag{15.170}$$

Letting ω' be arbitrary, we prove the first part ('$\Rightarrow$') of the equivalence directly, assuming $\omega' \in \bigcup \mathrm{ran}(T)$ to be true. Consequently, there exists (by definition of the union of a set system) a particular set $\bar{Y}$ in the set system $\operatorname{ran}(T)$ with $\omega' \in \bar{Y}$. In view of the inclusion (15.166), it is true that $\bar{Y} \in \operatorname{ran}(T)$ implies $\bar{Y} \in \mathcal{C}$. In conjunction with $\omega' \in \bar{Y}$, this finding further implies $\omega' \in \bigcup \mathcal{C}$ (again by definition of the union of a set system), so that we obtain with assumed equation $\bigcup \mathcal{C} = f[\Omega]$ via substitution $\omega' \in f[\Omega]$, as desired.

We now prove the second part ('$\Leftarrow$') of the equivalence in (15.170) also directly, assuming for this purpose $\omega' \in f[\Omega]$ to be true. Thus, ω' is in the range of the surjection $f : \Omega \twoheadrightarrow f[\Omega]$, so that there is an element of the domain Ω, say $\bar{\omega}$, such that $(\bar{\omega}, \omega') \in f$, which we write in function notation as $\omega' = f(\bar{\omega})$. Here, $\bar{\omega} \in \Omega$ implies with the equation in (15.163) $\bar{\omega} \in \bigcup \bar{\mathcal{S}}$. Since $\bar{\mathcal{S}}$ is the range of the bijection $\bar{S} : \{1, \ldots, \bar{n}\} \rightleftarrows \bar{\mathcal{S}}$, which is a sequence, we can also write $\bar{\omega} \in \bigcup \operatorname{ran}(\bar{S})$ and then $\bar{\omega} \in \bigcup_{i=1}^{\bar{n}} \bar{S}_i$. According to the Characterization of the union of a family of sets, there is therefore a particular index $\bar{k} \in \{1, \ldots, \bar{n}\}$ for which $\bar{\omega} \in \bar{S}_{\bar{k}}$ holds. Thus, $\bar{k}$ is also in the domain of the function T, which gives the value $T(\bar{k}) = f[\bar{S}_{\bar{k}}]$, and $\bar{\omega} \in \bar{S}_{\bar{k}}$ implies

$$[\omega' =] \quad f(\bar{\omega}) \in f[\bar{S}_{\bar{k}}] \quad [= T(\bar{k})]$$

according to (3.719). Noting that T is also a sequence with index set $\{1, \ldots, \bar{n}\}$, the resulting $\omega' \in T_{\bar{k}}$ implies in conjunction with $\bar{k} \in \{1, \ldots, \bar{n}\}$ the truth of $\omega' \in \bigcup_{i=1}^{\bar{n}} T_i$ (using again the Characterization of the union of a family of sets), which we can write also in the form $\omega' \in \bigcup \operatorname{ran}(T)$. We thus completed the proof of the equivalence in (15.170), in which ω' is arbitrary, so that the universal sentence (15.170) follows to be true, and this in turn completes the proof of the equality (15.169).

In view of (15.166), (15.168), (15.167) and (15.169), we see now that the existential sentence in (15.156) is true. Because $\mathcal{C}$ was arbitrary, we may infer from this the truth of the universal sentence (15.156), which means that the topological subspace $(f[\Omega], \mathcal{O}'|f[\Omega])$ of $(\Omega', \mathcal{O}')$ is compact. Since Ω, $\mathcal{O}$, Ω', $\mathcal{O}'$ and f were initially arbitrary sets, the theorem follows therefore to be true. $\square$

For the following characterization of fuzzy sets, we will assume that the domain M of the considered membership functions/fuzzy sets defines a vector space.

Definition 15.22 (Fuzzy convex fuzzy set). We say for any vector space $(M, +_M, \cdot_M)$ over the ordered field of real numbers $(\mathbb{R}, +, \cdot, -, /, <)$ that a fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$ is *fuzzy convex* iff

$$\begin{aligned}&\forall x_1, x_2, t\,([x_1, x_2 \in M \,\wedge\, 0 \leq t \leq 1] \\ &\quad \Rightarrow m_{\tilde{A}}(t \cdot_M x_1 +_M [1-t] \cdot_M x_2) \geq_{[0,1]} m_{\tilde{A}}(x_1) \sqcap_{[0,1]} m_{\tilde{A}}(x_2)).\end{aligned} \quad (15.171)$$

Theorem 15.31 (Characterization of fuzzy convexity). *For any vector space $(M, +_M, \cdot_M)$ over the ordered field of real numbers $(\mathbb{R}, +, \cdot, -, /, <)$ and any fuzzy set $\tilde{A} = (M, m_{\tilde{A}})$, it is true that $\tilde{A}$ is fuzzy convex iff the α-level set $\tilde{A}_\alpha$ is convex for any $\alpha \in (0, 1]$.*

Proof. We let M, $+_M$, $\cdot_M$, $\tilde{A}$ and $m_{\tilde{A}}$ be arbitrary such that $(M, +_M, \cdot_M)$ is a vector space over $\mathbb{R}$, such that $m_{\tilde{A}}$ is a function from M to $[0, 1]$, and such that $\tilde{A}$ is the fuzzy set $(M, m_{\tilde{A}})$. To prove the first part ('$\Rightarrow$') of the equivalence, we assume that $\tilde{A}$ is fuzzy convex, so that $\tilde{A}$ satisfies (15.171), and we show for any $\alpha \in (0, 1]$ that the α-level set $\tilde{A}_\alpha$ is convex. For this purpose, we let $\alpha \in (0, 1]$ be arbitrary, and we prove that $\tilde{A}_\alpha$ satisfies

$$\forall x_1, x_2\, (x_1, x_2 \in \tilde{A}_\alpha \Rightarrow \forall t\, (0 \leq t \leq 1 \Rightarrow t \cdot_M x_1 +_M [1 - t] \cdot_M x_2 \in \tilde{A}_\alpha)), \tag{15.172}$$

letting x_1, x_2 be arbitrary and assuming $x_1, x_2 \in \tilde{A}_\alpha$ to be true. Next, we also let t be arbitrary, and we assume moreover $0 \leq t \leq 1$ to hold, so that the assumed fuzzy convexity (15.171) of $\tilde{A}$ yields with the definition of a meet $\sqcap$ (as a binary infimum operation)

$$m_{\tilde{A}}(t \cdot_M x_1 +_M [1 - t] \cdot_M x_2) \geq_{[0,1]} \overset{\leq_{[0,1]}}{\inf} \{m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2)\}. \tag{15.173}$$

Let us now observe that the assumptions $x_1, x_2 \in \tilde{A}_\alpha$ imply $m_{\tilde{A}}(x_1) \geq \alpha$ and $m_{\tilde{A}}(x_2) \geq \alpha$ by definition of an α-level set. The assumed $\alpha \in (0, 1]$ implies $0 < \alpha$ and $\alpha \leq 1$ with the definition of a real left-open and right-closed interval. Here, the former inequality evidently gives $0 \leq \alpha$, which then implies together with $\alpha \leq 1$ that $\alpha \in [0, 1]$ is true, according to the definition of a closed interval. By definition of a membership function, $m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2) \in [0, 1]$ also holds. Recalling now that the reflexive partial ordering of $\mathbb{R}$ generates $\leq_{[0,1]}$, we may therefore write the previously established inequalities $m_{\tilde{A}}(x_1) \geq \alpha$ and $m_{\tilde{A}}(x_2) \geq \alpha$ also as $\alpha \leq_{[0,1]} m_{\tilde{A}}(x_1)$ and $\alpha \leq_{[0,1]} m_{\tilde{A}}(x_2)$, which show in light of the Characterization of upper & lower bounds for pairs that α is a lower bound for $\{m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2)\}$. We notice that $m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2) \in [0, 1]$ implies that the preceding pair is a subset of $[0, 1]$, in view of (2.164). Consequently, the greatest lower bound for that pair with respect to $\leq_{[0,1]}$ in the complete lattice $([0, 1], \leq_{[0,1]})$ exists. By definition of an infimum, we thus have the inequality

$$\alpha \leq_{[0,1]} \overset{\leq_{[0,1]}}{\inf} \{m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2)\}.$$

We may now apply the transitivity of the reflexive partial ordering $\leq_{[0,1]}$ to combine this inequality with the inequality (15.173) to obtain

$$m_{\tilde{A}}(t \cdot_M x_1 +_M [1 - t] \cdot_M x_2) \geq_{[0,1]} \alpha,$$

which in turn implies evidently the desired consequent

$$t \cdot_M x_1 +_M [1-t] \cdot_M x_2 \in \tilde{A}_\alpha$$

with the definition of an α-level set. Since t and then also x_1, x_2 were arbitrary, we may therefore conclude that the universal sentence (15.172) is true. Because α was also arbitrary, we may now further conclude that the first part of the stated equivalence holds.

To establish the second part ('$\Leftarrow$'), we now assume $\tilde{A}_\alpha$ to be a convex set for all $\alpha \in (0,1]$. To prove that $\tilde{A}$ is then fuzzy convex, we take arbitrary constants x_1, x_2, t and assume $x_1, x_2 \in M$ as well as $0 \leq t \leq 1$ to be true. As in the proof of the first part of the equivalence, the infimum

$$\bar{\alpha} = \overset{\leq_{[0,1]}}{\inf} \{m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2)\} \tag{15.174}$$

exists then. We now prove (15.173) by considering the two cases $\bar{\alpha} = 0$ and $\bar{\alpha} \neq 0$. In the first case $\bar{\alpha} = 0$, we obtain (15.173) immediately via substitution based on the observation that

$$m_{\tilde{A}}(t \cdot_M x_1 +_M [1-t] \cdot_M x_2) \geq 0 \quad \left[= \overset{\leq_{[0,1]}}{\inf} \{m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2)\}\right]$$

holds (recalling that membership functions take values between 0 and 1). In the second case $\bar{\alpha} \neq 0$, we utilize the fact that the infimum $\bar{\alpha}$ is a lower bound for the pair $\{m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2)\}$ with respect to $\leq_{[0,1]}$, so that $\bar{\alpha} \leq_{[0,1]} m_{\tilde{A}}(x_1)$ and $\bar{\alpha} \leq_{[0,1]} m_{\tilde{A}}(x_2)$ follow to be both true (with the Characterization of upper & lower bounds for pairs). Clearly, we may write these inequalities also as $m_{\tilde{A}}(x_1) \geq \alpha$ and $m_{\tilde{A}}(x_2) \geq \alpha$, so that we obtain $x_1, x_2 \in \tilde{A}_{\bar{\alpha}}$ (by definition of an α-level set). Because the infimum $\bar{\alpha}$ with respect to $\leq_{[0,1]}$ is thus in $[0,1]$, so that $0 \leq \bar{\alpha} \wedge \bar{\alpha} \leq 1$ holds, we now see in light of the current case assumption $\bar{\alpha} \neq 0$ that $0 \leq \bar{\alpha}$ yields $0 < \bar{\alpha}$; consequently, $\bar{\alpha} \in (0,1]$ is true. Since we assumed $\tilde{A}_\alpha$ to be a convex set for all $\alpha \in (0,1]$, the set $\tilde{A}_{\bar{\alpha}}$ is convex. Therefore, the previously established $x_1, x_2 \in \tilde{A}_{\bar{\alpha}}$ and the assumed $0 \leq t \leq 1$ give

$$t \cdot_M x_1 +_M [1-t] \cdot_M x_2 \in \tilde{A}_{\bar{\alpha}},$$

which in turn implies (by definition of an α-level set)

$$m_{\tilde{A}}(t \cdot_M x_1 +_M [1-t] \cdot_M x_2) \geq \bar{\alpha} \quad \left[= \overset{\leq_{[0,1]}}{\inf} \{m_{\tilde{A}}(x_1), m_{\tilde{A}}(x_2)\}\right]$$

Since x_1, x_2 and t are arbitrary, we may infer from this finding that $\tilde{A}$ is fuzzy convex, by definition, so that the proof of the equivalence is complete. Because M, $+_M$, $\cdot_M$, $\tilde{A}$ and $m_{\tilde{A}}$ were initially arbitrary sets, we may now finally conclude that the theorem is indeed true. $\square$

Let us now consider in particular the vector space of real numbers $(\mathbb{R}, +, \cdot)$ on the field of real numbers $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}})$. To be expanded!

Chapter 16.

Sets of Fuzzy Functions $\mathcal{F}(Y)^{\mathcal{F}(M)}$

The completeness of the real unit interval lattice enables also the following result theorem, which we will take as the underlying principle for constructing fuzzy functions.

Theorem 16.1 (Zadeh's Extension Principle). *For any nonempty set M, any set Y, any function $f : M \to Y$ and any function $m_{\tilde{A}} : M \to [0,1]$, it is true that there exists a unique function $m_{\tilde{B}} : Y \to [0,1]$ satisfying*

$$\forall y \, (y \in Y \Rightarrow m_{\tilde{B}}(y) = \sup^{\leq_{[0,1]}} m_{\tilde{A}}[f^{-1}[\{y\}]]). \tag{16.1}$$

Proof. We let M, Y, f and $m_{\tilde{A}}$ be arbitrary sets and we assume M to be nonempty, f to be a function from M to Y, and $m_{\tilde{A}}$ to be a function from M to $[0,1]$, so that $\tilde{A} = (M, m_{\tilde{A}})$ is a fuzzy set. Thus, $m_{\tilde{A}}$ is a membership function on M, and $\tilde{A}$ is a fuzzy set. We now apply Function definition by replacement and prove for this purpose the universal sentence

$$\forall y \, (y \in Y \Rightarrow \exists! z \, (z = \sup^{\leq_{[0,1]}} m_{\tilde{A}}[f^{-1}[\{y\}]])), \tag{16.2}$$

letting y be arbitrary and assuming $y \in Y$ to be true. Let us observe now that this assumption implies $\{y\} \subseteq Y$ with (2.184), so that the inverse image of $\{y\}$ under $f : M \to Y$ is defined. Furthermore, this inverse image $f^{-1}[\{y\}]$ is a subset of the domain M of f according to Note 3.30. Therefore, the direct image of $f^{-1}[\{y\}]$ under $m_{\tilde{A}} : M \to [0,1]$ is also defined, and this image is a subset of the codomain $[0,1]$ of $m_{\tilde{A}}$ in view of (3.720). Due to the Completeness of the real unit interval lattice $([0,1], \leq_{[0,1]})$, the supremum of that image exists then with respect to $\leq_{[} 0,1]$; thus, $\sup^{\leq_{[0,1]}} m_{\tilde{A}}[f^{-1}[\{y\}]]$ is an element of $[0,1]$. Consequently, the uniquely existential sentence in (16.2) is true according to (1.109). Because y was arbitrary, we may infer from this the truth of the universal sentence (16.2), which implies then in turn the unique existence of a function $m_{\tilde{B}}$ with domain Y such that (16.1) holds.

It now remains for us check that $[0,1]$ is a codomain of that function, i.e. that its range is included in the unit interval. To do this, we apply the definition of a subset and prove the equivalent universal sentence

$$\forall z\,(z \in \mathrm{ran}(m_{\tilde{B}}) \Rightarrow z \in [0,1]). \tag{16.3}$$

We take an arbitrary $\bar{z}$ and assume $\bar{z} \in \mathrm{ran}(m_{\tilde{B}})$. Consequently, there exists by definition of a range a constant, say $\bar{y}$, with $(\bar{y}, \bar{z}) \in m_{\tilde{B}}$, which we may write in function notation also as $\bar{z} = m_{\tilde{B}}(\bar{y})$. We thus see that there exists a constant z with $(\bar{y}, z) \in m_{\tilde{B}}$, so that $\bar{y}$ follows to be an element of the domain Y of $m_{\tilde{B}}$. Then, $\bar{y} \in Y$ implies with (16.1)

$$[\bar{z} =] \quad m_{\tilde{B}}(\bar{y}) = \overset{\leq_{[0,1]}}{\sup}\, m_{\tilde{A}}[f^{-1}[\{\bar{y}\}]]$$

which supremum $\bar{z}$ with respect to $\leq_{[0,1]}$ is evidently in $[0,1]$. As $\bar{z}$ was arbitrary, we may therefore conclude that the universal sentence (16.3) is true. Thus, $\mathrm{ran}(m_{\tilde{B}}) \subseteq [0,1]$ holds indeed, which means that $[0,1]$ is a codomain of $m_{\tilde{B}}$, i.e. that we obtained the function

$$m_{\tilde{B}} : Y \to [0,1], \quad y \mapsto m_{\tilde{B}}(y) = \overset{\leq_{[0,1]}}{\sup}\, m_{\tilde{A}}[f^{-1}[\{y\}]]. \tag{16.4}$$

Since M, Y, f, $\tilde{A}$ and $m_{\tilde{A}}$ were initially arbitrary sets, we may further conclude that the stated theorem holds. □

Corollary 16.2. *For any set $M \neq \emptyset$, any function $f \in Y^M$ and function $m_{\tilde{A}} \in [0,1]^M$, it is true that every element y of the codomain Y of the function f which is not in the range of f is mapped to 0 by $m_{\tilde{B}}$, i.e.*

$$\forall y\,([y \in Y \,\wedge\, y \notin \mathrm{ran}(f)] \Rightarrow \overset{\leq_{[0,1]}}{\sup}\, m_{\tilde{A}}[f^{-1}[\{y\}]] = 0). \tag{16.5}$$

Proof. Letting M, Y, f, $\tilde{A}$ and $m_{\tilde{A}}$ be arbitrary sets such that M is nonempty, such that f is a function in Y^M, such that $m_{\tilde{A}}$ is a function from M to $[0,1]$, so that $\tilde{A} = (M, m_{\tilde{A}})$ constitutes a fuzzy set, we now take an arbitrary y and assume y to be an element of the codomain Y of f with $y \notin \mathrm{ran}(f)$. In view of the assumptions $y \in Y$ and $f \in Y^M$, the preceding negation implies $f^{-1}[\{y\}] = \emptyset$ with (3.757), so that we obtain

$$\begin{aligned} \overset{\leq_{[0,1]}}{\sup}\, m_{\tilde{A}}[f^{-1}[\{y\}]] &= \overset{\leq_{[0,1]}}{\sup}\, m_{\tilde{A}}[\emptyset] \\ &= \overset{\leq_{[0,1]}}{\sup}\, \emptyset \\ &= 0 \end{aligned}$$

by applying also (3.722) and (8.133). Since y is arbitrary, we may therefore conclude that the universal sentence (16.5) holds, and as the sets M, Y, f, $\tilde{A}$ and $m_{\tilde{A}}$ were initially arbitrary as well, the stated corollary follows then to be true. $\square$

Exercise 16.1. Show for any sets $M \neq \emptyset$ and Y and for any $f \in Y^M$ that there exists a unique function $g : [0,1]^M \to [0,1]^Y$ such that

$$\forall m_{\tilde{A}} \, (m_{\tilde{A}} \in [0,1]^M \Rightarrow g(m_{\tilde{A}}) = m_{\tilde{B}}) \tag{16.6}$$

holds, where $m_{\tilde{B}}$ is defined for any membership function $m_{\tilde{A}}$ by (16.4). (Hint: Define the Function by replacement, using (1.109).)

Lemma 16.3. *It is true for any nonempty set M, for any set Y, for any function $f : M \to Y$, for any fuzzy set $\tilde{A} \in \mathcal{F}(M)$ and for any element α of the real unit interval that the image of the α-cut of $\tilde{A}$ under f is included in the α-cut of the fuzzy set $\tilde{B}$ defined by the membership function $m_{\tilde{B}}$ obtained by means of Zadeh's Extension Principle, i.e.*

$$\forall \alpha \, (\alpha \in [0,1] \Rightarrow f[\tilde{A}_\alpha] \subseteq \tilde{B}_\alpha). \tag{16.7}$$

Proof. We take arbitrary sets M, Y, f, $\tilde{A}$ and α, for which we assume $M \neq \emptyset$, $f : M \to Y$, $\tilde{A} \in \mathcal{F}(M)$ and $\alpha \in [0,1]$ to be true. We thus have $\tilde{A} = (M, m_{\tilde{A}})$ for some membership function $m_{\tilde{A}} : M \to [0,1]$, and Zadeh's Extension Principle gives us then a unique function $m_{\tilde{B}} : M \to [0,1]$, which defines the fuzzy set $\tilde{B} = (M, m_{\tilde{B}})$. Let us observe here that the α-cut $\tilde{A}_\alpha$ is by definition included in the set M underlying the fuzzy set $\tilde{A}$, where M is also the domain of f, so that the image of that α-cut under f is defined to be

$$f[\tilde{A}_\alpha] = \mathrm{ran}(f \restriction \tilde{A}_\alpha). \tag{16.8}$$

To prove that this image is included in the α-cut $\tilde{B}_\alpha$, we apply the definition of a subset and verify the equivalent universal sentence

$$\forall y \, (y \in f[\tilde{A}_\alpha] \Rightarrow y \in \tilde{B}_\alpha). \tag{16.9}$$

We let y be arbitrary and assume $y \in f[\tilde{A}_\alpha]$ to be true. Since the image $f[\tilde{A}_\alpha]$ is included in the codomain Y of f in view of (3.720), the preceding assumption implies $y \in Y$ with the definition of a subset, and this gives us according to Zadeh's Extension Principle

$$m_{\tilde{B}}(y) = \overset{\leq_{[0,1]}}{\sup} m_{\tilde{A}}[f^{-1}[\{y\}]]. \tag{16.10}$$

Furthermore, the assumed antecedent $y \in f[\tilde{A}_\alpha]$ yields with (16.8)

$$y \in \mathrm{ran}(f \restriction \tilde{A}_\alpha),$$

so that the definition of a range gives us a particular constant $\bar{x}$ satisfying $(\bar{x}, y) \in f \restriction \tilde{A}_\alpha$. This finding implies now with the definition of a restriction $(\bar{x}, y) \in f$ and $\bar{x} \in \tilde{A}_\alpha$, where the latter yields

$$m_{\tilde{A}}(\bar{x}) \;\geq_{\mathbb{R}}\; \alpha \tag{16.11}$$

by definition of an α-cut, and where the former can be written in function notation as $y = f(\bar{x})$. We can write this equation also as $f(\bar{x}) \in \{y\}$ because of (2.169), where the inclusion $\{y\} \subseteq Y$ is implied by the previously found $y \in Y$ by virtue of (2.184). Consequently, we obtain with the definition of an inverse image $\bar{x} \in f^{-1}[\{y\}]$. This inverse image is included in the domain M of the function f (according to Note 3.30), where M is also the domain of the membership function $m_{\tilde{A}}$, so that $f^{-1}[\{y\}]$ is a subset of that domain. In conjunction with the previous finding $\bar{x} \in f^{-1}[\{y\}]$, this gives us

$$(m_{\tilde{A}} \restriction f^{-1}[\{y\}])(\bar{x}) = m_{\tilde{A}}(\bar{x})$$

with (3.567), which we can write also in the form

$$(\bar{x}, m_{\tilde{A}}(\bar{x})) \in m_{\tilde{A}} \restriction f^{-1}[\{y\}].$$

We now see in light of the definition of a range that

$$m_{\tilde{A}}(\bar{x}) \in \mathrm{ran}(m_{\tilde{A}} \restriction f^{-1}[\{y\}])$$

is true, which means by definition of an image

$$m_{\tilde{A}}(\bar{x}) \in m_{\tilde{A}}[f^{-1}[\{y\}]]. \tag{16.12}$$

As shown by (16.10), the $m_{\tilde{B}}(y)$ constitutes the supremum of the image $m_{\tilde{A}}[f^{-1}[\{y\}]]$ and thus an upper bound for that set, so that (16.12) implies $m_{\tilde{A}}(\bar{x}) \leq_{[0,1]} m_{\tilde{B}}(y)$. According to the Reflexive partial ordering of subsets, we can write the preceding inequality also as $m_{\tilde{A}}(\bar{x}) \leq_{\mathbb{R}} m_{\tilde{B}}(y)$, and this implies in conjunction with (16.11) the truth of $m_{\tilde{B}}(y) \geq_{\mathbb{R}} \alpha$, using the transitivity of the standard total ordering $\leq_{\mathbb{R}}$. By definition of an α-cut, we thus find the desired consequent $y \in \tilde{B}_\alpha$ of the implication in (16.9) to be true, so that this implication holds. Since α is arbitrary, we may therefore infer from the truth of that implication the truth of the universal sentence (16.9) and then also the truth of the equivalent inclusion $f[\tilde{A}_\alpha] \subseteq \tilde{B}_\alpha$. We thus completed the proof of the implication in (16.7), and as the sets M, Y, f, $\tilde{A}$ and α were initially all arbitrary, the stated lemma follows finally to be true. □

In case we are given a sequence of membership functions on a common set M, we can combine these to a single membership function $m_{\times \tilde{A}_i}$ by means of a t-norm, as shown by Proposition 15.27. This mechanism allows us to write Zadeh's Extension Principle also in the following specific form.

Corollary 16.4 (Zadeh's Extension Principle for sequences of fuzzy sets under functions on Cartesian powers). *For any nonempty set M, for any $n \in \mathbb{N}$, for any set Y, for any function $f : M^n \to Y$, for any sequence $m = (m_{\tilde{A}_i} \,|\, i \in \{1, \ldots, n\})$ of membership functions on M and for any t-norm $*_t$, it is true that there exists a unique function $m_{\tilde{B}} : Y \to [0,1]$ satisfying*

$$\forall y \, (y \in Y \Rightarrow m_{\tilde{B}}(y) = \overset{\leq_{[0,1]}}{\sup} m_{\times \tilde{A}_i}[f^{-1}[\{y\}]]), \tag{16.13}$$

where

$$m_{\times \tilde{A}_i} : M^n \to [0,1], \quad (x_i \,|\, i \in \{1, \ldots, n\}) \mapsto *_{i=1}^{n} m_{\tilde{A}_i}(x_i).$$

Chapter to be expanded!

Bibliography

Arnold SF (1981) *The Theory of Linear Models and Multivariate Analysis.* Wiley, New York

Bandemer H, Näther W (1992) *Fuzzy Data Analysis.* Kluwer Academic Publishers

Bartle RG, Sherbert DR (2010) *Introduction to Real Analysis.* John Wiley & Sons

Bauer H (1992) *Maß- und Integrationstheorie.* Walter de Gruyter & Co.

Cantor G (1895) *Beiträge zur Begründung der transfiniten Mengenlehre.* Mathematische Annalen, 46(4)

Davey BA, Priestley HA (2012) *Lattices and Order.* Second edition. Cambridge University Press

Henze E (1971) *Einführung in die Masstheorie.* Bibliographisches Institut

Hilbert D, Ackermann W (2008) *Principles of Mathematical Logic.* AMS Chelsea Publishing

Hodge JK, Schlicker S, Sundstrom T (2014) *Abstract Algebra.* CRC Press

Hrbacek K, Jech T (1999) *Introduction to Set Theory.* Third edition. CRC Press

Kallenberg O (2002) *Foundations of Modern Probability.* Second Edition. Springer

Kirkham RL (1995) *Theories of Truth: A Critical Introduction.* Massachusetts Institute of Technology

Kreyszig E (1989) *Introductory Functional Analysis with Applications.* John Wiley & Sons

Kunen K (2013) *Set Theory.* Revised edition. College Publications

Kunen K (2012) *The Foundations of Mathematics.* College Publications

Meyer CD (2000) *Matrix Analysis and Applied Linear Algebra.* Society for Industrial and Applied Mathematics (siam)

Munkres J (2014) *Topology.* Second Edition. Prentice Hall

Pinter CC (2013) *A Book of Abstract Algebra.* Second edition. Dover Publications

Rautenberg W (2009) *A Concise Introduction to Mathematical Logic.* Third edition. Springer

Rautenberg W (2007) *Messen und Zählen.* Heldermann Verlag

Rudin W (1976) *Principles of Mathematical Analysis.* Third edition. McGraw-Hill

Stuart A, Ord JK, Arnold SF (1999) *The Advanced Theory of Statistics.* Vol. 2A: Classical inference and the linear model. Arnold, London.

Tarski A (1995) *Introduction to Logic and to the Methodology of Deductive Sciences.* Dover Publications

Velleman DJ (2009) *How to Prove it.* Second edition. Cambridge University Press

Viertl R (2011) *Statistical Methods for Fuzzy Data.* John Wiley and Sons

Zadeh LA (1965) *Fuzzy sets.* Information and Control, Vol. 8, pp. 338-353

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